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Observability of electrical heart activity studied with the singular value decomposition

by

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OBSERVABILITY OF ELECTRICAL HEART ACTIVITY
STUDIED WITH THE SINGULAR VALUE DECOMPOSITION

by

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and

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SUMMARY

The singular value decomposition is a mathematical operation on a matrix. When this procedure is applied to the matrix that is made up of samples of the measured potentials on the skin, a basis of independent components is constructed. About nine of these components define the skin potentials within the noise margin. They contain practically all information available on the skin. When the singular value decomposition is performed on the transfer matrix that relates potentials on the heart surface to potentials on the skin, components are found that can be interpreted as basic potential distributions over the heart surface that cause a potential distribution over the skin. The relative contribution of each basic heart pattern to the total skin potentials is a measure for the observability of that basic distribution over the heart surface. The distributions with the worst observability cause skin potentials that can hardly be measured as they lie mostly under the noise level. For this reason, when calculating inversely the potentials on the heart surface from measured skin potentials, the worst observable distributions must be ignored. In this way it appears to be possible to compute more reliable potentials on the heart surface as a solution for the ill-conditioned inverse problem. It would be interesting to investigate in the future to what extent the nine relevant components of the skin potentials overlap the found 34 potential distributions over the skin caused by good observable basic heart potentials.
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1. INTRODUCTION

Many investigations have already been performed trying to identify from measurements of electrical potentials on the skin (electrocardiogram = ECG) of both human and animal subjects the electrical phenomena that precede the heart contraction. The significance of these investigations is that if the sources of the electrical heart activity would be known, this would present a good indication for the condition of the heart and a help to diagnose diseases of the heart where these electrical sources are disturbed.

The exact calculation of the electrical sources in the heart is hindered by a fundamental lack of information in the measured skin potentials. It is not possible to calculate from the two dimensional potential distribution over the skin a three dimensional distribution of the sources in the heart. This elementary property is illustrated by the electrical field of a point charge and a charged sphere in an unbounded homogeneous medium. The field outside the sphere is the same as the field of the point charge. An observer outside the source area cannot establish the amount of concentration of the source.

A second lack of information is caused by the inevitable noise on the measured skin potentials. Source areas in the heart which produce a potential distribution on the skin with a small amplitude are obscured by the noise.

The problem that a three dimensional source distribution cannot be evaluated from the skin potential measurements leads to two approaches that will be applied in this report:

1. concentration of all sources in a point somewhere in the heart leads to an equivalent source that produces the same skin potentials as the real heart. A multipole series is such an equivalent source. The multipoles will be used in chapter 5.

2. change the aim from calculating a three dimensional source distribution over the heart into calculating epicardial potentials, i.e. into identifying potentials caused by the electrical heart activity on the outer surface of the heart. Then a two dimensional epicardial potential distribution is calculated from the two dimensional skin potential distribution.

It is possible to calculate epicardial potentials via an equivalent source. In that case the simulated or estimated epicardial potentials are restricted by the properties of the equivalent source.

In this report the methods to calculate epicardial potentials will not be described in full detail, as these methods have been employed rather than developed in this study. More details can be found in the literature (see e.g. lit.2, 3, 12, 13).

The epicardial potentials can be calculated if the electrical properties of the body are known. In this report the body will be supposed to be a linear, homogeneous and isotropic volume conductor, having on the outside exactly the geometry of a real torso. The extremities have been ignored in this body model. The inhomogeneities such as lungs, liver, bones, etc., have not been accounted for in the model. Although more reliable and realistic calculated epicardial potentials will be found
if the inhomogeneities are taken into account, they are neglected for the time being to keep the calculations limited and because of the fact that it is hard to measure the size and the electrical properties of these inhomogeneities especially for each individual subject.

In order to use the possibility of numerical calculations on a digital computer, the skin potentials were digitised as a function of the position on the skin and the time. For the position about a hundred points were chosen all over the body (see chapter 2, fig.1) where an ECG was measured by taking samples with an interval time of 3 ms. On the epicardium potentials at 66 points distributed evenly over the surface represent the potentials as a function of the position. This number of points is less than the number of points on the skin as the potentials over the epicardial surface cannot be more detailed than the skin potentials from which they are derived.

The noise on the measured skin potentials will play an important part in this study. A part of the information needed to calculate reliable epicardial potentials is obscured by the noise. When no precautions are taken, the noise will be interpreted as relevant information by the mathematical procedures to calculate epicardial potentials. An example in chapter 7 will indicate that this effect can cause unrealistic epicardial potentials. For this problem the singular value decomposition (abbreviated s.v.d.) may well provide a solution. The s.v.d. is a mathematical operation on a matrix. The decomposition will be treated in chapter 3 as a mathematical technique.

The s.v.d. can be applied to the matrix which contains the samples of the measured skin potentials. Doing so, the skin potentials will be split up in a number of independent components. A part of these components contain almost all relevant information. The rest account for the noise for the greater part. As the noise level is lower, more components will be found that carry relevant information. This explains why a practical measure for the information in the skin potentials will be defined here as the number of components that are found above the noise level. Besides a way is found to separate to some extent the signal from the noise.

The s.v.d. can be used to decompose the transfer matrix, that relates epicardial potentials to skin potentials. This matrix can be calculated from data on the torso geometry. The decomposition results in components that can be interpreted as independent basic potential distributions over the epicardial surface and over the skin surface. The singular values, that result from the s.v.d., will appear to be a measure for the contribution of a basic distribution on the heart to the total skin potentials. Thus a quantification of the observability of the found epicardial potential distribution is possible: the observability of an epicardial potential pattern is defined here as the RMS value of the skin potential distribution resulting from that normalised epicardial potential distribution. As the noise level is standard, an epicardial normalised potential pattern with a smaller observability will result in a lower signal to noise ratio on the skin. When the independent potential distributions on the heart and the skin together with their observability are known, they can be used to reject those
independent distributions on the epicardium that cause skin distributions below the noise level.

The relation between the epicardial potentials and the skin potentials will be described briefly in chapter 5. The multipole model will be used as an illustration to demonstrate a few properties of the transfer function from epicardial potentials to skin potentials. A short chapter (ch. 6) will inform about the results of an experiment employing the s.v.d. on the matrix that consists of calculated samples of the epicardial potentials.
2. DATA ACQUISITION AND DISPLAY

The data that will be used for the calculations provide information about the body geometry and the ECG at a large number of points on the body. These points have been defined in the following way:

The origin of a Cartesian coordinate system is chosen to be in the center of the heart. Define an octahedron in such a way that the angular points all lie on the axes. On each edge of the octahedron points are chosen that divide the edge into \( N \) pieces of equal length. Connecting the points on different edges of one triangle, each triangle of the octahedron is divided into \( N^2 \) congruent triangles. In this way on the octahedron \( 8N^2 \) congruent triangles and \( 4N^2 + 2 \) angular points are defined. The points are numbered, starting at the top vertex, moving spirally down to the bottom vertex. In this report \( N_0 \) will be equal to 5, hence the points 1 through 102 are defined.

The points on the skin where measurements will be made are the points where lines connecting the origin with the points on the octahedron intersect the skin.\(^*\)

This definition of the points implies that the point-density on the skin becomes higher as the points lie closer to the heart. As the ECG is more reliable and has higher space frequencies at points close to the heart, the defined distribution of points is a favourable one.

In spherical coordinates the angles \( \theta \) and \( \phi \) of the skin points are fixed by the definition described before. The geometry of the body is now quantified by measuring the radius of every point by means of a special construction.

The ECG was measured at the points on the skin defined before with an ECG amplifier having a bandfilter with slopes of 6 dB/oct., flat response from 0.1 to 500 Hz.

The ECG signal is supposed to be periodic. Groups of six ECGs were recorded simultaneously on a seven trace analog recorder. One trace was used to record a strong ECG signal as a reference. The signals were read into a computer by a ten bit AD-converter with a sample time of 1 ms. The signals were corrected for baseline drift using the silent periods between the complexes. Finally per measuring point one complex was averaged out of ten complexes and was punched on cards. This signal will be the input for the calculations in this report.

For a display of the results a two dimensional projection of the three dimensional skin surface is necessary. To do this the octahedron is opened along the four posterior edges. The four back triangles are unfolded until they lie in the same plane as the front triangles to which they are attached. Then all points of the opened octahedron are projected perpendicularly on the \( y-z \) plane (the axes are defined in fig.1, see next page). The octahedron is thus projected into a quadrate. The vertices of the quadrate are the projection of one vertex of the octahedron, viz. the vertex on the back side. In the center of the quadrate lies the pro-

\(^*\) The description given here is simplified. A small correction should be introduced to ensure that the directions are more evenly distributed. See fig.1 on the next page.
Figure 1. The defined points on the torso.

RA and LA right arm side and left arm side.
H head side.
A abdomen.

jection of the octahedron vertex at the front side (see fig.2).

The heart surface is modelled as a sphere tightly around the heart. The radius of this sphere was chosen to be 7 cm. The center of the sphere lies in the origin of the coordinate system. This origin is defined: from the lower end of the sternum: 4 cm upwards, 3 cm to the left shoulder and 7 cm backwards.

On the sphere a distribution of points is defined with an octahedron in the same way as described before for the torso surface. On the heart \( N_0 = 4 \) was chosen, hence there will be \( 4N_0^2 + 2 = 66 \) points distributed evenly over the heart sphere.

For display of the calculated epicardial potentials the same projection via the octahedron on a plane is used as for the skin potentials (see fig.2).

An example for the display of a potential distribution on the skin or the epicardial surface is fig.3. The square is the projection of the
Figure 2. Projection of torso surface and epicardial surface.

B = backside
R = right side
L = left side

CS = coronary sulcus
IS = interventricular sulcus
RA = right atrium
LA = left atrium
RV = right ventricle
LV = left ventricle

closed surface of the torso or the heart. Here isopotential lines are drawn for five equidistant potential levels. As the potential belonging to an isopotential line is not indicated, it is impossible to reconstruct the potential distribution from these plots. The plots of isopotential lines do not give complete information on the potential distribution over the surface.

Figure 3. A potential distribution over the skin surface marked by isopotential lines.
3. THE SINGULAR VALUE DECOMPOSITION

3.1. Introduction of the singular value decomposition (s.v.d.).

For any mxn matrix A the s.v.d. exists, given by:

\[ A = U \Sigma V^T \]  

Where:

- **U** is an mxr matrix consisting of r orthonormal columns \( u_j \),
- **D** is an rxr diagonal matrix:
  
  \[ D = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) \]

  \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0 \)

- **V** is an nxr matrix consisting of r orthonormal columns \( v_j \),

  \[ V^T V = I_r \]

The \( \sigma_j \) are called singular values.

3.2. Proof for the existence of the s.v.d.

The proof that the s.v.d. exists for any matrix A can be formulated as:

\( A^T A \) is a symmetric nxn matrix, hence it has \( n \) real orthonormal eigenvectors \( v_1, v_2, \ldots, v_n \) (see e.g. lit. 4, pp. 35, 36 and 50...54).

For the real eigenvalue \( \lambda \) associated with an eigenvector \( v \) it is found that:

\[ A^T Av = \lambda v, \quad |v| \neq 0 \]

so:

\[ v^T A^T A v = \lambda |v|^2 \]

\[ |Av|^2 = \lambda |v|^2 \]

From which can be concluded that the eigenvalues of \( A^T A \) are positive or zero; they are zero only if:

\[ \lambda = 0 \]

\(*\) The norm of matrix A is defined as:

\[ |A|^2 = \sum_{i,j} a_{i,j}^2 = \text{tr}(A^T A) = \text{tr}(AA^T) \]

The norm of a vector follows from this definition in the case that A is a mx1 matrix.
\[ |A\mathbf{v}|^2 = 0 \quad \text{or:} \quad A\mathbf{v} = 0 \quad \text{(*1)} \]

As the rank of \( A \) is \( r \), the vectors \( \mathbf{v} \) that satisfy \( (*1) \) span a \( n-r \) dimensional space. \( n-r \) orthogonal eigenvectors of \( A^T A \) constitute a basis for this space. From \( (*1) \) it can be seen that the \( n-r \) associated eigenvalues will be zero. The remaining \( r \) eigenvalues of \( A^T A \) must be non zero thus positive.

Proof has been given now that \( A^T A \) can be written in a Jordan canonical form:

\[ A^T A = \mathbf{V} \mathbf{D}^2 \mathbf{V}^T \quad \text{(3)} \]

where:

\[ \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}, \text{ a } n \times r \text{ matrix.} \]

\[ \mathbf{V}^T \mathbf{V} = \mathbf{I}_r \text{ as the eigenvectors are orthonormal.} \]

\[ \mathbf{D} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r), \text{ a } r \times r \text{ matrix.} \]

\[ \sigma_j = \sqrt{\lambda_j} \]

The \( \sigma_i \)s are the square roots of the eigenvalues \( \lambda_j \) of \( A^T A \). They can be ordered in such a way that:

\[ \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0. \]

Now define: \( \mathbf{U} = A \mathbf{V} \mathbf{D}^{-1} \)

Then because of \( (3) \):

\[ \mathbf{U}^T \mathbf{U} = \mathbf{D}^{-1} \mathbf{V}^T A^T A \mathbf{V} \mathbf{D}^{-1} = \mathbf{I}_r. \]

The columns of \( \mathbf{U} \) are orthonormal.

With \( (3) \) it follows that:

\[ A^T A = \mathbf{V} \mathbf{D}^2 \mathbf{V}^T \]

thus:

\[ A^T A (I - \mathbf{V} \mathbf{V}^T) = \mathbf{0}, \quad (*) \]

\[ (I - \mathbf{V} \mathbf{V}^T) A^T A (I - \mathbf{V} \mathbf{V}^T) = \mathbf{0} \]

\[ |A (I - \mathbf{V} \mathbf{V}^T)|^2 = 0 \]

\[ A (I - \mathbf{V} \mathbf{V}^T) = \mathbf{0} \]

\[ (*) \] The null vector will be denoted as \( \mathbf{0} \), the null matrix as \( \mathbf{0} \).
With (4):
\[ A V = U D \]
So:
\[ A = U D V^T \]
by which the existence of (1) has been proven.

3.3. Some properties of the s.v.d.

3.3.1. From:
\[ A^T A = V D^T U^T U D V = V D^T D V = V D^2 V^T \]
and:
\[ A A^T = U D V^T V D^T U^T = U D D^T U^T = U D^2 U^T \]

it follows that the diagonal elements of \( D^2 \) are the eigenvalues (as far as they are unequal to zero) of \( A^T A \) as well as of \( AA^T \). The columns of \( V \) respectively \( U \) are the eigenvectors of \( A^T A \) respectively \( AA^T \), associated with the non-zero eigenvalues.

Note the symmetry of the s.v.d. as may become even more obvious from:
\[ A = U D V^T \iff A^T = V D U^T \]
The derivation given in section 3.2 can just as well start from the eigenvectors \( u_j \) of \( AA^T \) instead of the eigenvectors \( v_j \) of \( A^T A \).

3.3.2. If the \( mxn \) matrix \( A \) (with \( m>n \)) and the \( m \) vector \( z \) are given, the solution for the least squares problem:
\[ \text{find the } n \text{ vector } x \text{ that minimises } |z - Ax| \]
is found to be:
\[ x = A^+ z \]
(see e.g. lit.5). \( A^+ \) is the Moore-Penrose pseudo inverse of \( A \). If the s.v.d. of \( A \) is given by (1), the pseudo inverse \( A^+ \) of \( A \) will be:
\[ A^+ = V D^{-1} U^T \]  \hspace{1cm} (5)
The singular values of \( A^+ \) are: \( \sigma_{r-1}, \sigma_{r-1}, \ldots, \sigma_{r-1} \).

3.3.3. With the s.v.d. and the orthonormality of the columns of \( V \) it follows that:
\[ |A|^2 = \text{tr}(A^T A) = \text{tr}(V D^2 V^T) = \text{tr}(D^2) = \sum_{j=1}^{r} \sigma_j^2. \]  \hspace{1cm} (6)
3.3.4. A least squares fit.
In the next chapter it will appear to be desirable to limit the rank of a matrix. This must be done with a minimal effect on the matrix. So if $A$ is the original $m \times n$ matrix with $\text{rank}(A) = r$ the task is, given an integer $k$ with $0 \leq k \leq r$, to find the $m \times n$ matrix $B$ with $\text{rank}(B) \leq k$ in such a way that $|A-B|$ is minimal.
In other words: find a $B$ with a limited rank and a difference between $A$ and $B$ that is minimised with a least squares criterion.
As before (cf. (1)) the s.v.d. of $A$ is given as:

$$A = U \, D \, V^T,$$
$$D = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r), \quad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0.$$ 

It will be derived in the appendix that the $B$ asked for is found to be:

$$B = U \begin{pmatrix} D_k & 0 \\ 0 & 0 \end{pmatrix} V^T = U_k \, D_k \, V_k^T$$

(7)

$U_k$ contains the first $k$ columns of $U$,
$V_k$ contains the first $k$ columns of $V$,
$D_k = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k)$.

So the $B$ is found by setting the smallest $r-k$ singular values in the s.v.d. of $A$ to zero.
Furthermore it is demonstrated in the appendix that with this $B$ the RMS error is:

$$|A-B| = |E| = \sqrt{\sum_{i,j} E_{i,j}^2} = \sqrt{\sum_{j=k+1}^r \sigma_j^2}$$

(8)

With (6) the relative error is found:

$$\frac{|E|}{|A|} = \sqrt{\sum_{j=k+1}^r \frac{\sigma_j^2}{\sum_{j=1}^r \sigma_j^2}}$$

(9)

3.3.5. The numerical calculation of the s.v.d.
The ALGOL procedure used to calculate the s.v.d. of a matrix was written by Golub and Reinsch (lit.6). The program consists of two parts. The first part reduces the matrix to a bidiagonal form by Householder transformations. The second part uses a QR-type method to calculate the singular values.
The procedure s.v.d. was implemented on the Burroughs B7700 computer of the Eindhoven University of Technology.
4. THE SINGULAR VALUE DECOMPOSITION OF THE SKIN POTENTIALS

Figure 4. The measured skin potentials. The potential at point 91 is zero because this point was used for the reference electrode.

Fig. 4 shows a set of measured skin potentials. In each square a QRS complex is plotted, total duration: 150 ms (51 samples, interval time 3 ms) as measured at one point on the skin. The total enveloping square is the projection of the skin surface, as described in chapter 2. It can be seen that the ECG was not measured at seven points. So the set consists of 95 complexes.

To enable mathematical operations on the skin potentials, they were ordered in a pxt matrix which will be called S. The character p stands for the number of positions on the skin where an ECG has been measured.
and \( t \) is the number of points of time on which a sample is taken. In the example of fig.4 \( p \) equals 95 and \( t \) equals 51.

\[
\begin{array}{c}
\text{position} \\
\downarrow \\
p \quad t
\end{array}
\]

A row of \( S \) gives the potential at one point on the skin as a function of the time. A column of \( S \) gives the potentials at one point of time as a function of the serial number of the points on the skin.

4.1. The interpretation of the s.v.d. of the matrix \( S \).

The s.v.d. of \( S \) yields (cf.(1)):

\[
S_{p \times t} = U_{p \times r} D_{r \times r} V^T_{r \times t}
\]

where:

\[
D = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r),
\]

\[
\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0,
\]

\[
r = \text{rank}(S).
\]

The matrix \( U \) can be regarded as not to depend on the time, but only on the position, whereas \( V \) can be regarded as not to depend on the position but only on the time. A column \( v_j \) of \( V \) can be interpreted as a timesignal with "energy":

\[
\sum_{\tau=1}^{t} v_{\tau j}^2 = 1
\]

because \( V^T V = I_r \).

The timesignals are "uncorrelated" as:

\[
\sum_{\tau=1}^{t} v_{\tau i} v_{\tau j} = 0 \quad \text{for} \quad i \neq j,
\]

again because \( V^T V = I_r \).

In fig.5 the first 16 columns of \( V \) are displayed. The \( r \) orthonormal columns of \( V \) span up a \( r \) dimensional time space. They are a basis for the time space spanned by the \( p \) time signals of \( S \). Each time signal \( v_j \) has its own weighting factor \( \sigma_j \) in \( D \), which is smaller for higher values of the serial number \( j \) (note that when multiplying matrix \( D \) with matrix \( V^T \) each column of \( V \) is multiplied by the corresponding \( \sigma_j \)). The logarithm of each singular value \( \sigma_j \) (\( \log \sigma_j \)) is plotted as a function of \( j \) in fig.6.

Each singular value belongs as well to one corresponding column \( u_j \) of \( U \). A column \( u_j \) can be interpreted as a position signal. A position signal is a function of two variables, viz. the coordinates of the position on the two dimensional skin surface, although a position sig-
Figure 5. The first 16 columns of the matrix $V$ of the s.v.d. of the skin potential matrix of fig.4.

Figure 6. The logarithm of the singular values $\sigma_j$ as a function of the serial number $j$, for the singular values of the skin potentials plotted in fig.4.
Figure 7. Contour lines for the function on the skin given by the first 16 columns of $U$ of the s.v.d. of the skin potentials given in fig.4. Each map gives contour lines at five equidistant levels.

Inal can be given by a one dimensional vector $u_j$, as a result of the serial numbering of the points on the skin. Fig.7 shows contour lines for the first 16 columns $u_j$ of $U$. Each square is a projection of the skin surface. Note that the positions on the skin that were not measured (see fig.4) were not represented in the matrix $S$ and will therefore not be represented in the matrix $U$ either. A column of $U$ gives no values at the not measured points. Consequently the contour lines in fig.7 cannot be reliable around these points, as
they were interpolated by hand.
Alike the time signals the position signals of $U$ are uncorrelated and normalised because $U^T U = I$.
The r orthonormal columns of $U$ span a r dimensional position space. They constitute a basis for the position space spanned by the t position signals of $S$.

4.2. The influence of the noise.

In fig.5 and fig.7 it can be seen that the 16th component has high time respectively high spatial frequencies. This might prompt the idea that this signal contains merely noise. This idea is not altogether wrong, but the noise cannot be discriminated from the signals produced by the heart just on the frequency contents. How can it be understood that signals having a high serial number and thus a small singular value can be supposed to contain noise for a major part?

To answer this question some theory on the s.v.d. be remembered. Suppose the skin potentials would be calculated from a limited basis of the k strongest components. For this purpose define:

$$S_k = U \begin{pmatrix} D_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} V^T = U_k D_k V_k^T$$  \hspace{1cm} (11)

where:

$D_k = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k)$,

$k$ integer, $0 \leq k \leq r$,

$r = \text{rank}(S)$,

$U_k$ respectively $V_k$ consists of the first $k$ columns of $U$ respectively $V$.

As a result of the theory of section 3.3.4 (a least squares fit) it can be concluded here that $S_k$ is the best approximation for $S$ according to a least squares criterion, when the rank of $S_k$ is limited to $k$.

The signals on the skin caused by the heart and measured at one point will correlate strongly to the heart originated signals at the neighbouring points. The noise signals measured at these neighbouring points will not show much correlation. Thus it is supposed that the density of the measuring points on the skin is higher than strictly necessary for the determination of the heart originated skin potentials.

Let $k$ in (11) be equal to one. Then only one position signal $u_1$ and one time signal $v_1$ constitute the best adaptation to the t position signals or the p times signals of $S$. These normalised signals $u_1$ and $v_1$ show the highest possible correlation with all signals in $S$. The influence of the noise of the signals $S$ on $u_1$ and $v_1$ will be small because the influence of the uncorrelated noise signals will average out to a large extend.
If only a small number of \( k \) independent time and position signals is used to approximate \( S \), the noise on \( S \) will have a reduced influence on the independent signals \( u_j \) and \( v_j \) for the same reason. As more position and time signals are included, for growing values of \( k \) in (11), the adaptation becomes more detailed and the noise on \( S \) will be more and more present in the added position and time signals \( u_k \) and \( v_k \).

The smallest orthogonal components \( u_j \) and \( v_j \) of \( S \) in the position and time will contain for a major part only noise signals. These components, that carry hardly any information can be left out by setting the smallest singular values to zero as was done in (11). In section 3.3.4 also an expression for the error was found. The RMS error between \( S_k \) and \( S \) can be stated now as:

\[
\sqrt{\sum_{i,j}^{} (s_{i,j} - s_{k,i,j})^2} = |S - S_k| = \sqrt{\sum_{j=k+1}^r \sigma_j^2}
\]  

(12)

In fig. 6 the function:

\[
e(k) = \sqrt{\sum_{j=k}^r \sigma_j^2}
\]

as derived from the singular values of fig. 6, has been plotted. The measured signals have a maximal signal to noise ratio of about 50 dB. The maximal signal to noise ratio is defined here as the quotient of the maximal possible signal level and the established RMS noise level. The maximal signal level, neglecting the noise, was found to be:

\[
\max(|s_{i,j}|) = 8.16 \times 10^2 \text{ (arbitrary units)}
\]

Since \( S \) is a 95\times 51 matrix the maximum \(|S|_M\) of \(|S|\) will be:

\[
|S|_M = \sqrt{\sum_{i,j}^{} s_{i,j}^2} = \sqrt{95 \times 51 \times (8.16 \times 10^2)^2}
\]

\[
|S|_M = 5.68 \times 10^4
\]

The noise level is 50 dB below the maximum signal level, so for the noise:

\[
|N_s| = 10^{-2.5} \times |S|_M
\]

\[
|N_s| = 1.8 \times 10^2
\]

\[
\log(|N_s|) = 2.26.
\]

In fig. 8 it can be seen that:

\[
e(10) < |N_s|
\]

so for \( k=9 \) expression (12) yields:

*) see lit. 16.
Figure 8. \( \log(e(k)) \) as a function of \( k \).

\[
e(k) = \sqrt{\sum_{j=k}^{\infty} \sigma_j^2}, \quad \text{the error function as:}
\]

\[
|S_k - S| = e(k+1), \quad S_k \text{ defined by (11)}.
\]

\( d_9 \) is a measure for the approximation of \( S \) by 9 components

\[
|S_k - S| < |N_8|
\]

This means that if all singular values except the first nine are set to zero, the error made will be less than the noise level. This fact makes it acceptable to presume that the left out components contain for a major part noise.

The nine remaining independent signals define a basis for the set of ECG signals. Any ECG signal can be calculated as a linear combination of the nine basic position and time signals. It will be clear that the number of independent signals is a measure for the amount of information that can be found above the noise level in a set of ECG signals.

A plot of the ECG signals reconstructed from the nine dominant time and position components does not differ visibly from the original ECG signals in fig.4.
4.3. Relation with principal components.

Observe again the s.v.d. of the skin potential matrix:

\[ S = U D V^T \]  \hspace{1cm} (10)

The rxt matrix:

\[ P = D V^T \]  \hspace{1cm} (14)

contains the same uncorrelated time signals in the rows as \( V \) contains in the columns, with the exception that the "energy" of the time signals in \( P \) is equal to \( \sigma_j^2 \):

\[ P P^T = D V^T V D^T = D^2 \]

so:

\[ \sum_{i=1}^{t} \sigma_i^2 = \sigma_j^2 \]

The time signals given by a row of \( P \) are known in the literature as principal components or intrinsic components of the set of measured signals given by \( S \). The principal components form an orthogonal basis for the time space spanned by the signals \( S \).

Substituting (14) in (10) gives:

\[ S = U P \]  \hspace{1cm} (15)

With the property of s.v.d.: \( U^T U = I \) it is found that:

\[ P = U^T S \]  \hspace{1cm} (16)

With this expression \( P \) can be found without the use of the s.v.d. if \( U \) is known. Section 3.3.1 states that the columns of \( U \) are the eigenvectors of the matrix \( S S^T \). This matrix is known in the literature as the correlation matrix \( H \). Each element of \( H = S S^T \) is equal to the "correlation" between two time signals of the set of time signals \( S \):

\[ h_{i,j} = \sum_{\tau=1}^{t} s_{i,\tau} s_{j,\tau} \]

Articles on principal components of the ECG are: lit.1,8,9 and 10.
5. THE TRANSFER FUNCTION FROM SKIN POTENTIALS TO EPICARDIAL POTENTIALS
BY MEANS OF A MULTIPOLE SERIES.

In this chapter the object will be to demonstrate some principles and properties of the determination of potentials on a sphere that tightly encloses the heart.

As it is permitted to suppose quasi-stationarity of the field produced by the electrical heart activity, (lit.II) all time dependence can be eliminated.

The body is modelled electrically as a linear medium so that the transfer function from epicardial potentials to skin potentials can be given by a matrix, the transfer matrix $A$:

$$\phi_s = A \phi_e$$  \hspace{1cm} (17)

where:

$\phi_s$ is a vector with $p$ elements, the potentials at the $p$ points on the skin.

$\phi_e$ is a vector with $q$ elements, the potentials at the $q$ points on the heart sphere.

$A$ is consequently a $pxq$ matrix.

The matrix $A$ can be evaluated from data on the torso geometry if the body is modelled as a homogeneous, isotropic and linear volume conductor. The evaluation of the matrix can be done with the help of a multipole series (lit. 2, 7) or directly from Green's second identity (lit. 12, 13). The multipole series will be used now to demonstrate some properties of the transfer function which relates epicardial potentials to skin potentials.

If all sources that produce an electrical field can be found in a bounded region, the field outside a sphere, enclosing this region can remain exactly the same when the sources in the region are replaced by a set of multipoles all located at the center of the aforementioned sphere. A multipole series can be used as an equivalent source for the electrical heart activity. The multipoles are supposed in the center of the heart sphere.

The potential $\phi$ produced by a multipole series in an unbounded homogeneous medium with specific conductivity $\sigma$, when all multipoles are located at the origin of a spherical coordinate system $(r, \theta, \phi)$ is:

$$\phi = \frac{1}{4\pi\sigma} \sum_{n=1}^{N} \sum_{m=0}^{n} \frac{1}{r^{n+1}} P_n^m (\cos \theta) (a_n \cos (m \phi) + b_n \sin (m \phi))$$  \hspace{1cm} (18)

where:

$n$ is the order of the multipole,

$N$ is the maximum order of the set of multipoles,
\( a_{nm} \) and \( b_{nm} \) are the multipole coefficients (abbreviated m.p.cs),
\( P^m_n \) are the associated Legendre polynomials.

\[ \begin{array}{ccc}
  a_{10} & a_{11} & b_{11} \\
  \text{Figure 9a.} & \text{Isopotential lines on a spherical surface in the center of which a set of multipoles is placed.} \\
  a_{43} & a_{44} & b_{44} \\
  b_{43} & a_{44} & b_{44} \\
  \text{Figure 9b.} & \text{Isopotential lines on the skin of a torso model, when a set of multipoles is placed in the heart.} \\
  \text{In each square only one m.p.c. is not equal to zero.} 
\end{array} \]
In fig. 9a isopotential lines are drawn for the potential distribution on a sphere (the heart sphere) in the center of which the multipoles are placed. The familiar projection of a spherical surface on a square is used. In each square the multipole has only one m.p.c. unequal to zero. In the top row successively $a_{10}, a_{11}, b_{11}$ are non zero. Then on the next rows in succession $a_{20}, a_{21}, b_{21}, a_{22}, a_{30}$ etcetera until $b_{44}$. In each square contour lines were drawn for five equidistant potential levels. The potential values of the corresponding levels differ, however, from one square in fig. 9 to another.

It can be seen in fig. 9a that the higher order multipoles have higher space frequencies than lower order multipoles have. In fig. 9b in the same way as in fig. 9a equipotential lines are shown for the potential distribution per m.p.c., but now the potentials were calculated on a torso. Comparison of fig. 9a with fig. 9b induces to make some remarks further on in this chapter.

A multipole series can be used to calculate the epicardial potentials from the measured skin potentials using the following strategy. A certain maximum order of the multipole series is chosen on grounds that will be explained at the end of this chapter. Then the m.p.cs are calculated in such a way that the difference between the resulting calculated skin potentials* and the measured skin potentials will be minimal in the least squares sense. The heart sphere potentials can subsequently be calculated using formula (18).

The heart lies eccentrically in the body, closer to the chest than to the back. Hence the strongest ECG potentials can be measured on the chest, in front of the heart. From previous research (lit. 2) a second property can be derived, which will be explained here with the help of some multipole theory.

From formula (18) it follows that the potential is inversely proportional to the radius to the power $n+1$. The field of higher order multipoles decreases faster than the field of lower order multipoles as the distance between the point of observation and the multipoles is larger. As higher order multipoles have higher spatial frequencies, these higher spatial frequencies can only be observed close to the heart, in the precordial region. This can be seen in fig. 9. Low order multipoles produce potential distributions having low space frequencies (fig. 9a) observable all over the skin (fig. 9b), whereas high order multipoles cause skin potentials with high space frequencies only in the precordial region. The backside of the heart lies far away from any point on the skin. The electrical activity of this part of the heart will produce a signal on the skin that has only a small high space frequency contents.

The medium between the heart and the skin acts as a low pass filter for space frequencies. The filter will attenuate the high space frequencies, especially for the regions of the skin far from the heart and the regions of the heart far from any point on the skin. Trying to solve the inverse problem, i.e. the calculation of heart potentials from skin potentials, difficulties arise because of the unavoidable noise on the measured skin potentials. This noise has a high spatial frequency contents. The low pass filter property of the medium between the heart and the skin will, when acting inversely, amplify the high space frequencies and give rise.

*) The effect of the bounded torso is incorporated.
to unrealistic calculated epicardial potentials. The noise on the skin potentials far from the heart is amplified the most. The signal to noise ratio on the backside of the heart will be the worst.

When the multipole model is used to calculate epicardial potentials, the high order multipoles will have very high valued m.p.cs in order to adapt to the potential distribution with high space frequencies due to the noise at points of the skin far from the multipole. These strong high order multipoles, that almost compensate each other at points in the precordial region, produce very unrealistic epicardial potentials, especially on the backside of the heart. Better results can be obtained if the maximum order of the multipole series is limited further to say the order four. In this case, however, there is more information available in the precordial region than is used to evaluate this set of multipoles. Furthermore, the estimation of the set will be biased.

In chapter 7 a method will be proposed to perform a specific filtering action, in order to obtain a better signal to noise ratio while all measurable information of the skin potentials is used.

Equation (17) states that the skin potentials can be calculated from the epicardial potentials by:

$$\phi_s = A \phi_e$$  \hspace{1cm} (17)

Inversely the epicardial potentials can be found in principle from the measured skin potentials. As $A$ is not a square matrix, the inverse of $A$ does not exist. The epicardial potentials can be computed using the Moore-Penrose pseudo inverse $A^+$ (see section 3.3.2) of the transfer matrix $A$:

$$\phi_e = A^+ \phi_s$$  \hspace{1cm} (19)

Introducing again the time dependence of $\phi_e$ and $\phi_s$ results in:

$$E = A^+ S$$  \hspace{1cm} (20)

where:

- $S$ is the pxt matrix of the skin potentials introduced before in chapter 4,
- $E$ is the qxt matrix of the epicardial potentials.

In fig.10 epicardial potentials are plotted that were calculated from measured skin potentials via a multipole series with a maximum order four. A maximum order four implies that the multipole series is given by 24 m.p.cs (see formula (18)). The set of multipoles, used as an equivalent heart generator, is determined by 24 time functions. When the matrix $A$ is evaluated via this multipole series, the rank of $A$ will be limited to 24, as the multipole series has only 24 degrees of freedom.

$\star$) A necessary condition for unbiasedness is a torso shape of a concentric sphere. It will be clear, that the real torso shows a lot of deviations from that sphere.
Figure 10. Epicardial potentials calculated from the skin potentials given in fig. 4 with a multipole series having a maximum order equal to four. The rank of the transfer matrix is therefore limited to 24.
6. THE SINGULAR VALUE DECOMPOSITION OF THE EPICARDIAL POTENTIAL MATRIX.

As a conclusion of the preceding chapter epicardial potentials were calculated via a multipole series of the order four (see fig.10). For these potentials the s.v.d. has been calculated in exactly the same way as has been done for the skin potential matrix in chapter 4. The s.v.d. for the epicardial potential matrix $E$ yields:

$$E_{q 	imes t} = U_{q 	imes r} D_{r 	imes r} V_{r 	imes t}^T$$

(21)

where:

$$D = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$$

$r = \text{rank}(E)$ (The rank of $E$ needs not to be the rank of $S$).

The first 16 columns of $V$ (orthonormal functions of the time) and the first 16 columns of $U$ (orthonormal functions of the position) are displayed in fig. 11 and fig.13 respectively, in the same way as for the skin potentials in chapter 4. The logarithm of the singular values $\sigma_j$ has been plotted as a function of the serial number $j$ in fig.12a.

![Figure 11. The first 16 columns of the matrix V of the s.v.d. of the matrix containing the calculated epicardial potentials plotted in fig.10.](image)

The number of non-zero singular values (see fig.12a) is equal to 24 as the rank of the matrix $E$ is 24 here (see the last paragraph of the preceding chapter, $\text{rank}(A) = 24$ implies that $\text{rank}(A^+) = 24$ and with (20) and $\text{rank}(S) > 24$ it follows that $\text{rank}(E) = 24$).
Figure 12. a. The logarithm of the singular values and b. The logarithm of the error function

\[ e(k) = \sqrt{\sum_{j=k}^{24} \sigma_j^2} \]

for the matrix of the calculated epicardial potentials (fig. 10).

d\(_g\) is a measure for the approximation of S by 9 components.

Analogously to the skin potentials the matrix \( E_k \) is calculated from the k strongest components of E, cf. (11) in chapter 4. Then the RMS error between \( E_k \) and E will be (cf. (12) in chapter 4):

\[ \sqrt{\sum_{i,j} (E_{i,j} - E_{k,i,j})^2} = |E - E_k| = \sqrt{\sum_{j=k+1}^{1} \sigma_j^2} \]
Figure 13. The first 16 columns of the matrix $U$ of the s.v.d. of the matrix containing the calculated epicardial potentials plotted in fig.10.

The error function:

$$e(k) = \sqrt{\sum_{j=k}^{r} \sigma_j^2}$$

which is calculated from the singular values of fig.12a, is plotted in fig.12b.
Again like in chapter 4, the number $k$ of components that are needed to calculate the epicardial potentials $E_k$ so that the error made is less than the noise level, is a measure for the information in the epicardial potentials $E$. It has been explained in chapter 5 that the noise level on estimated signals in the anterior region of the epicardial surface will be less than the noise level on signals in other parts of the epicardial surface. The noise level depends on the position on the epicardial surface, which was not the case for the signals measured on the skin.

The information, i.e. the number of independent components that define the skin or epicardial potentials in the presence of a certain noise level, found for the epicardial potentials cannot be more than the information found in the skin potentials from which they are calculated. Remember that on the skin the information was contained in nine orthogonal components. Compare the error function $e(k)$ for the epicardial potentials in fig. 11b with the same for the skin potential matrix in chapter 4, fig. 8. The approximation of the potentials by the first 9 components, as denoted by $d_9$, is better on the skin than on the epicardium. The remaining deviations are due to the noise and consequently the noise level on the epicardium (fig. 10), is higher than the noise level on the skin (fig. 4). This amplification of the noise has been explained in chapter 5.

Comparison of the orthogonal time signals on the heart (fig. 11) and on the skin (fig. 5) shows clearly that the components one through five on the epicardium are strongly correlated to the respective components on the skin. Note that some of the corresponding components have an opposite sign. Comparing the orthogonal position signals on the epicardial surface fig. 13, and on the skin surface, fig. 7, the correspondence of the first five components is not so evident.
7. THE SINGULAR VALUE DECOMPOSITION OF THE HEART-TO-SKIN TRANSFER MATRIX.

7.1. The interpretation of the s.v.d. of the transfer matrix.

Expression (17) states that the skin potentials at one moment can be calculated from the epicardial potentials via a transfer matrix $A$.

$$\phi_s = A \phi_e \quad (17)$$

where:
- $\phi_s$ is the vector with skin potentials, having $p$ elements,
- $\phi_e$ is the vector with epicardial potentials, having $q$ elements, $q \leq p$.

This transfer matrix can be subjected to the s.v.d. which yields:

$$A_{p \times q} = U_{p \times q} D_{q \times q} V_{q \times q} \quad (21)$$

with:
- $D = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_q)$
- $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_q \geq 0$

Substitution in (17) leads to:

$$\phi_s = U D V^T \phi_e \quad (22)$$

On the basis of this expression an interpretation can be found for the matrices $U$, $D$ and $V$.

The multiplication of $V^T$ and $\phi_e$ gives a vector with $q$ elements:

$$\chi_e = V^T \phi_e \quad (23)$$

Each element $\chi_{ej}$ of $\chi_e$ is a projection of $\phi_e$ on the respective column $v_j$ of $V$. As $V^TV=I$ these columns are orthonormal.

The columns of $V$ can be interpreted as basic, independent potential distributions over the heart sphere. The potential distribution $\phi_e$ is unraveled in its basic distributions. An element of $\chi_e$ gives the strength of the respective basic distribution in the epicardial potentials $\phi_e$.

For the length of the vector $\chi_e$ it is found that:

$$|\chi_e|^2 = \chi_e^T \chi_e = \phi_e^T V^T \phi_e = \phi_e^T \phi_e = |\phi_e|^2$$

as it follows from $V^TV=I$ and the fact that $V$ is a square matrix having full rank that:

$$V^T = V^{-1} \quad \text{thus} \quad V V^T = I_q$$
so:

\[ |\mathbf{x}_e| = |\phi_e| \]  

(24)

\( \phi_e \) and \( \mathbf{x}_e \) are the same vectors in a different coordinate system.

The basic vectors for \( \mathbf{x}_e \) are given by \( \mathbf{v}_j \) in the coordinate system of \( \phi_e \).

The interpretation of the matrix \( U \) may be clear when, with the help of \( U^TU = I \), (22) is written as:

\[ U^T \phi_s = D V^T \phi_e \]  

(25)

\( U^T \) transforms into a vector with \( q \) elements:

\[ \mathbf{x}_s = U^T \phi_s \]  

(26)

The columns \( u_1 \) of \( U \) can be considered to be those basic potential distributions over the skin that can be caused by the heart sources. An element \( \mathbf{x}_{sj} \) of \( \mathbf{x}_s \) gives the strength of the basic distribution over the skin \( u_j \) in the total skin potentials \( \phi_s \).

Substitution of (23) and (26) in (25) yields:

\[ \mathbf{x}_s = D\mathbf{x}_e \]  

(27)

or per element:

\[ \mathbf{x}_{sj} = \sigma_j \mathbf{x}_{ej} \text{ for } j = 1, 2, \ldots, q. \]

From this expression it can be seen that \( \sigma_j \) can be interpreted as a transfer coefficient relating a basic distribution on the epicardial surface with the corresponding basic distribution on the skin.

From (22) and (23) it is derived that:

\[ \phi_s = U D \mathbf{x}_e \]  

(28)

With this relation it follows that:

\[ |\phi_s|^2 = \phi_s^T \phi_s = \mathbf{x}_e^T D^T U^T U D \mathbf{x}_e \]

so:

\[ |\phi_s|^2 = \mathbf{x}_e^T D^2 \mathbf{x}_e \]  

(29)

With (27) it is found that:

\[ |\mathbf{x}_s|^2 = \mathbf{x}_s^T \mathbf{x}_s = \mathbf{x}_e^T D^2 \mathbf{x}_e \]  

(30)

(29) and (30) lead to the conclusion that:

\( \phi_s \) and \( \mathbf{x}_s \) are the same vectors in a different coordinate system.

Suppose that the potential distribution over the heart is equal to a basic distribution \( \mathbf{v}_j \). With the orthonormality of the columns of \( V \) and expression (22) it follows that:

\[ \phi_s = U D V^T \mathbf{v}_j = \sigma_j u_j \]  

(32)
One basic potential distribution over the heart $v_j$ produces via the transfer coefficient $\sigma_j$ a basic potential distribution over the skin surface $u_j$. As the singular values $\sigma_j$ are ordered according to diminishing values, the first basic potential distributions over the heart will give a stronger contribution to the skin potentials than the last ones. The observability of the distributions over the heart on the skin is better as the singular value belonging to this distribution is larger. So a measure for the observability of the potential distributions over the heart is found in the singular value.

An example of the results of the s.v.d. of a transfer matrix is given in figs 14, 15 and 16. The transfer matrix $A$ was evaluated by Vermeulen (lit.13) on the basis of a set of data on the torso geometry and a spherical heart surface. By means of Green's second identity for a two-fold bounded homogeneous, isotropic and linear medium with a specific conductivity $\sigma$, the discretised relation $\psi_s = A \psi_e$ can be obtained.

In fig.14 the basic potential distributions over the heart, given by the columns $v_j$ of $V$, have been drawn. Again each square is the projection of the heart sphere surface. In each square equipotential lines have been drawn for five equidistant potential levels. The potential distributions in fig.14 that have the best observability are the first ones and they have the highest amplitudes on the anterior side of the heart sphere, at points on the heart close to the skin. As the observability decreases, the highest amplitudes will be found more and more outside the region on the anterior side of the heart surface, i.e. outside the center of the square. Besides, the first, best observable distributions have lower space frequencies than the distributions with a higher serial number. These results agree with the properties of the transfer function derived with the multipole model in chapter 5.

The logarithm of the singular values $\sigma_j$ as a function of the serial number $j$ has been plotted in fig.15. The ratio between the largest and the smallest singular value is over 10,000.

Fig.16 shows in the same way as fig.14 the columns $u_j$ of $U$, the basic potential distributions over the skin surface. Note that the first distributions over the skin are predominant in the precordial region and have low space frequencies compared with the rest of the distributions. These subsequent distributions show decreasing amplitudes in the precordial region, though this tendency is less clear than on the heart surface. Even the posterior side of the heart can better be observed precordial then e.g. at sides near to the extremities.

7.2. The calculation of epicardial potentials.

As (19) states the epicardial potentials are calculated from the skin potentials using the Moore-Penrose pseudo inverse $A^+$ of the transfer matrix $A$:

$$\psi_e = A^+ \psi_s$$

In section 3.3.2 $A^+$ was given with the help of the s.v.d. of $A$ (see (21)).
Figure 14.
Maps showing isopotential lines for the basic potential distributions on the epicardial surface as given by the respective columns of the matrix V of the s.v.d. of a transfer matrix.
As \( \sigma_q \neq 0 \) for the transfer matrix used here (see fig. 15), the pseudo inverse \( A^+ \) is:
\[
A^+ = V D^{-1} U^T
\]  
(33)

and so (19) takes the form of:
\[
\phi_e = V D^{-1} U^T \phi_s
\]  
(34)

The potential distributions on the heart \( v_j \) with bad observability, the ones belonging to small singular values \( \sigma_j \), produce on the skin a weak signal \( \sigma_{j,u} \) (cf. (32)). As we have no a priori information, we are
Figure 16.
Maps showing isopotential lines for the basical potential distributions on the skin surface as given by the respective columns of the matrix U of the s.v.d. of a transfer matrix.
obliged to suppose that the different basic distribution \( \psi_j \) have an equal chance of occurrence in the epicardial potentials \( \psi_e \). (The a posteriori conclusion will be that the assumption of an equal chance of occurrence of the different basic distributions is incorrect.) The ratio between the largest and the smallest singular value is more than a factor 10,000. The skin potentials can be measured with a maximal signal to noise ratio of about 50 dB, which equals a factor 300. This means that the components of the skin potentials corresponding to the smallest singular values can never be measured because the noise will be much stronger than these potentials.

Calculating inversely the heart potential distribution of a basic skin potential distribution \( \psi_j \) with (33) leads to:

\[
\psi_e = V D^{-1} U^T \psi_j = \sigma_j^{-1} \psi_j
\]

(35)

As \( \psi_j \) is a relative small number, \( \sigma_j^{-1} \) will be a relative large number. The weak signal on the skin, produced by the distribution on the heart which is not well observable, will be multiplied with a large factor according to (35). As the noise is much stronger than this component of the signal on the skin, the multiplication with a relative large factor in the inverse calculation of the epicardial potentials will cause a high amplification of the noise. This noise will be found at those parts of the heart surface where the basic epicardial potential distributions belonging to small singular values have high amplitudes. In fig.14 it can be seen that these distributions are strong in the region outside the anterior side of the heart surface. The conclusion that calculated epicardial potentials on the backside of the heart surface will be corrupted by noise has been reached before in chapter 5 via the multipole model.

Fig.17 shows the calculated epicardial potentials. They were calculated from the skin potentials displayed in fig.4. In fig.17 the calculated potentials are corrupted by noise especially at the points 66, 65, 60 and 51. Note that the exposed time signals at these points are all similar. Now observe distribution number 66 in fig.14. Notice that this distribution belongs to the smallest singular value (see fig.15: \( \sigma_66 = 10^{-4} \)). So calculating inversely the noise will be amplified the most by \( \sigma_66^{-1} \), and the noise on the heart surface will be the strongest at points where this distribution has the highest amplitude. In distribution 66 in fig.14 the large amplitude in the points 66, 65, 60 and 51 can be recognized.

As the basic distributions belonging to small singular values cause the noise in the calculated epicardial potentials, it can be expected that these potentials will improve if the distributions mentioned will not be added to the calculated epicardial potentials. This can be done by setting:

\[
\sigma_k^{-1}, \sigma_{k+1}^{-1}, \ldots, \sigma_q^{-1}
\]

in (34) to zero. From (29):

\[
|\psi_s|^2 = \chi_e^T D^2 \chi_e
\]

(29)
Figure 17. The calculated epicardial potentials using a transfer matrix evaluated by Vermeulen (lit.13) and the measured skin potentials shown in fig.4. These potentials are clearly corrupted by noise.

It follows that:

\[ |\Phi_e|^2 = \sum_{j=1}^{q} \sigma_j^2 \chi_e^2 \]

We have supposed that all basic distributions \( y_j \) on the heart have an equal chance of occurrence in the epicardial potentials \( \Phi_e \). In statistical terms this can be stated as:

\[ \epsilon(\chi_e) = 0 \quad \epsilon(\chi_e^2) = \sigma \quad \epsilon(\chi_{ei}\chi_{ej}) = 0 \]

where \( \epsilon \) stands for expectation.
then:

\[ \varepsilon(\phi_s^2) = \sum_{j=1}^{q} \sigma_j^2 \]  

(36)

Let \( \phi_{sk} \) be the vector of skin potentials calculated from epicardial potentials \( \phi_e \) via a transfer matrix \( A_k \) derived from the s.v.d. of the transfer matrix \( A \) by setting the smallest singular values to zero. So:

\[ \varepsilon(\phi_s - \phi_{sk}) = \sum_{j=k+1}^{q} \sigma_j^2 \]  

(37)

We want to set the smallest singular values to zero until the difference between \( \phi_s \) and \( \phi_{sk} \) remains just within the noise margin on the measured skin potentials \( \phi_s \). The number of remaining singular values \( k \) must be the smallest number that satisfies:

\[ \varepsilon(\phi_s - \phi_{sk}) < |n_s|^2 \]  

(38)

where \( |n_s| \) stands for the noise level on the measured skin potentials. (36), (37) and (38) result in:

\[ \sum_{j=k+1}^{q} \sigma_j^2 < \frac{|n_s|^2}{\varepsilon(\phi_s^2)} \]  

(39)

where \( k \) is the smallest number that satisfies this inequality.

In chapter 4 it was found that:

\[ |n_s| = 1.8 \times 10^2 \]

For a column \( n_s \) of \( N_s \) this means:

\[ |n_s|^2 = (1.8 \times 10^2)^2 = 6.4 \times 10^2 \]

as \( N_s \) has \( t = 51 \) columns.

With the help of expression (6), chapter 3, it follows from the singular values of the skin potentials that:

\[ |S|^2 = 6.9 \times 10^7 \]

For an average column \( \phi_s \) of \( S \) this results in:

\[ \varepsilon(\phi_s^2) = \frac{6.9 \times 10^7}{51} = 1.35 \times 10^6 \]

Substituting these values for \( |n_s|^2 \) and \( \varepsilon(\phi_s^2) \) and the singular values of the transfer matrix (shown in fig. 15) in (39) yields a value of 34 for \( k \). So \( q-k=66-34=32 \) singular values can be set to zero.
In fig. 18 the results can be seen when 32 basic distributions are left out. Fig. 17 and fig. 18 have a different vertical scale. The vertical sensitivity of the plotter was a factor 17 higher when plotting fig. 18 than when fig. 17 was plotted. As the height of a square is 16.4 mm, the full scale value of fig. 18 (the height of a square) would be about 1 mm on the scale of fig. 17.

The potentials in fig. 18 still do not look realistic, on the basis of what might be expected for a propagating depolarisation front over the ventricles. But a comparison with the epicardial potentials in fig. 17 shows a clear improvement.
We have prohibited that the 32 remaining $x_{e_j}$ were defined by the noise, by setting them to zero. Nevertheless these 32 $x_{e_j}$ can freely be chosen theoretically. A priori information about the depolarisation wave could supply the necessary restrictions here (see also lit.17.)

Even better results than the ones described above were obtained with a transfer matrix according to lit.15. Here leaving out only the least relevant eight basic distributions produces epicardial potentials that have less complexes with an unacceptable relative high amplitude than the ones plotted in fig.18.

It can be concluded that the s.v.d. of the transfer matrix provides us with a tool to filter the measured skin potentials in such a way that the calculated heart potentials will show a better signal to noise ratio. However, more research must be done to investigate how many and which basic distributions must be attenuated or even left out to find an optimal set of basic distributions to calculate epicardial potentials.

In chapter 4 it is concluded that the information, contained in the skin potentials, can practically be given by about nine orthogonal time and position signals. The nine position signals span a subspace of the 66 orthogonal basic distributions over the skin, found in this chapter.

This means not that all 66 distributions can be found in the skin potentials. Based upon our assumption of equal chance for all epicardial basical patterns, we would expect, that the corresponding skin patterns show a variance of 0.7c. Therefore 32 skin basical patterns would be above the noise level. We only found a subspace of dimension 9.

It is of interest to investigate in the future which of the 66 basic distributions are present in the nine dominant orthogonal components of the skin potentials.

We conclude with a plot (fig.19) of the epicardial potentials calculated via the same transfer matrix as used to calculate the potentials shown in fig.17. The skin potentials from which the potentials of fig.19 have been calculated were calculated from the nine dominant orthogonal time and position signals, found in chapter 4. The vertical scale of the plots in fig.19 is equal to the vertical scale in fig.17. From the plots of fig.19 it can be seen that the basic distributions on the skin that have small corresponding singular values are incorporated in the basis consisting of the nine orthogonal skin potential components. For instance: the strong signals, that are much alike, on the points 66, 65, 60 and 51 indicate the presence of the orthogonal pattern number 66 in the nine orthogonal components of the skin potentials.
Figure 19. The epicardial potentials calculated via the same transfer matrix as used to calculate the potentials displayed in fig. 17. Here only the nine dominant orthogonal components of the skin potentials found in chapter 4 have been used.
8. CONCLUSION, SUGGESTIONS FOR FUTURE RESEARCH.

The s.v.d. of the measured skin potentials reveals the functions of the position and the time that carry most information. About nine functions of the time together with nine functions of the position contain practically all relevant information of the measured ECG's, taking the estimated noise level into account.

The s.v.d. of the transfer matrix that relates epicardial potentials to skin potentials, yields a measure for the observability of basic epicardial potential distributions. The skin potential distributions belonging to them are found as well. These results agree qualitatively with conclusions resulting from research with the multipole model as an equivalent heart generator. The merit of the s.v.d. of the transfer matrix lies in the quantification. This permits discrimination between better and worse observable potential distributions on the epicardium. Because of this property the s.v.d. supplies the opportunity for a filter action on the measured skin potentials. The skin potential patterns corresponding to epicardial potential patterns with good observability can be selected. With the transfer matrix used in this report about 34 different potential distributions on the epicardial surface may be observed on the skin above the noise level.

Further research should be carried out to discover to what extent the functions of the position that contain almost all information of the skin potentials (found with the s.v.d. of the skin potential matrix) overlap the potential distributions on the skin caused by epicardial potential patterns with good observability (found by the s.v.d. of the transfer matrix).

Future research on the qualities of the methods proposed in this report should include the use of the string model for the depolarisation wave over the ventricles developed in previous studies (lit. 14).

We are confronted with three numbers respectively 24, 34 and 9. A little comment is desirable.
- Overall observability is given by the maximum order 4 and of an equivalent multipolar series, i.e. 24 heart generators.
- S.V.D. applied to the transfer matrix ultimately leads to a theoretical set of approximately 34 independent, observable, epicardial potential patterns. These patterns are uniquely related to 34 skin patterns.
- S.V.D. applied to the matrix of measurements results in a set of approximately 9 practical components in time or space.

The first number 24 is based upon a theoretical transfer function as well as practical measurements. A derivation of a maximal observable space frequency for all sides of the heart was our goal. As the observability of the posterior parts is very bad, the order 4 is the limit. The fact, that
still 24 independent heart generators could be distinguished, is due to the limitation, that a multipole equivalent is not well adapted to the thoracic boundaries and to the eccentric position of the heart in the thorax. In other words, an equivalent multipole set is not suited to constitute a minimum of observable heart generators.

The 34 independent patterns are found exclusively on the basis of the theoretical transferfunction. Mainly two aspects have to be taken into account therefore:

- It is very likely, that the number of observable patterns increases as the discretisation of the surfaces is improved. It is e.g. very well possible, that the distribution of electrodes on the precordial side is not yet fine enough. So, theoretically, the number 34 may grow as higher space frequencies are allowed.

- It is not to be expected, however, that such high spatial frequencies are present in the epicardial surface. This contradicts our assumption, that all found 34 independent patterns have an equal chance of occurrence. The last set of 9 independent skin patterns shows us in fact, that there isn't such a wide variety of 34 patterns to be found on the epicardium.

Finally, this is the reason, that we are working now to combine the results of the S.V.D. applied to the transfermatrix and the matrix of measurements.
APPENDIX: A LEAST SQUARES FIT ON A MATRIX.

The problem is formulated as:
given a \( m \times n \) matrix \( A \) with rank \( r \) and an integer \( k \), with \( 0 \leq k \leq r \),
find a \( m \times n \) matrix \( B \) with rank \( \leq k \) by minimisation of \( |A - B| \). *)

The solution of the problem will make use of the theory of the orthogonal projector. An orthogonal projector \( E \) is defined by:

\[
E^2 = E \quad \text{and} \quad E^T = E \quad \text{(A0)}
\]

Solution:
For any \( B \) with rank \( \leq k \) an orthogonal projector \( E \) with rank \( k \) exists so that:

\[
\forall x \exists y: Bx = Ey
\]

and so it is valid that: \( EB = B \).
Suppose that \( B \) (rank \( \leq k \)) minimises \( |A - B| \).

Because:

\[
\]

(because of A0 and A1 = \( \text{tr}\{(I - E)A\}^T(I - E)A + (EA - B)^T(EA - B)\)

= \( |(I - E)A|^2 + |EA - B|^2 \)

\[\geq |(I - E)A|^2\]

it follows that the minimum of \( |A - B| \) is greater than \( |(I - E)A| \) or equal to it when \( |EA - B| = 0 \), so \( B = EA \).
On the other hand is \( EA \) (with \( E \) an orthogonal projector with a rank \( \leq k \)) a subset of the set of matrices \( B \) with rank \( B \leq k \). The minimum of \( |(I - E)A| \) can therefore not be less than the minimum of \( |A - B| \).
It must be concluded that:

\[
\min |A - B| = \min |(I - E)A| \quad \text{(A2)}
\]

where \( E \) is an orthogonal projector, rank \( E = k \). If \( E \) minimises the right part of (A2), then \( B = EA \) minimises the left of (A2).

Because:

\[
|A|^2 = |EA + (I - E)A|^2 = \text{tr}\{(EA + (I - E)A)^T(EA + (I - E)A)\}
\]

*) The norm of a matrix is defined by:

\[
|A|^2 = \sum_{i,j} a^2_{i,j} = \text{tr}(A^T A).
\]
\[ |A|^2 = |EA|^2 + |(I - E)A|^2 \]  
(A3)

\[ |(I - E)A|, \text{ and so } |A - E| \text{ as well, will be minimised if } |EA| \text{ is maximised.} \]  
Furthermore (A3) yields:

\[ \min\{|A - B|^2\} = |A|^2 - \max\{|EA|^2\} \]  
(A4)

Suppose \( A \) has the s.v.d. (cf. (1)):

\[ A = U D V^T \]

with:

\[ D = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r), \]

\[ \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0, \]

\[ r = \text{rank}(A). \]

For any orthogonal projector \( E \) it holds that:

\[ |EA|^2 = |E U D V^T|^2 = |E UD|^2 = \sum_{j=1}^{r} |\sigma_j E u_j|^2 \]

\[ |EA|^2 = \sum_{j=1}^{r} \alpha_j \sigma_j^2 \]  
(A5)

where \( \alpha_j = |E u_j|^2 \)  
(A6)

As:

\[ |u_j|^2 = |E u_j + (I - E) u_j|^2 = |E u_j|^2 + |(I - E) u_j|^2 \]

and:

\[ |u_j|^2 = 1 \]

it follows that:

\[ 0 \leq \alpha_j \leq 1 \]  
(A7)

With (A1) it can be derived that:

\[ \sum_{j=1}^{r} \alpha_j = |EU|^2 = |E|^2 = \text{tr}(E^T E) = \text{tr}(E) = \text{rank}(E) \]

\[ \sum_{j=1}^{r} \alpha_j \leq k \]  
(A8)

Putting (A5), (A7) and (A8) together yields:

\[ |EA|^2 = \sum_{j=1}^{r} \alpha_j \sigma_j^2 \ 	ext{with } 0 \leq \alpha_j \leq 1 \ 	ext{and} \sum_{j=1}^{r} \alpha_j \leq k. \]
As in addition $\sigma_j > \sigma_{j+1}$, it is concluded that:

$$\max\{ |EA|^2 \} = \sum_{j=1}^{k} \sigma_j$$

(A9)

This maximum is reached when:

$$\alpha_j = 1 \text{ for } 1 \leq j \leq k \text{ and } \alpha_j = 0 \text{ for } k+1 \leq j \leq r$$

Combining these results with (A6) gives:

$$|Eu_j| = 1 \text{ for } 1 \leq j \leq k$$

Because $|u_j| = 1$ and $E$ is an orthogonal projector it can be stated now that:

$$Eu_j = u_j \text{ for } 1 \leq j \leq k \quad \text{(A10)}$$

Besides:

$$|Eu_j| = 0 \text{ for } k+1 \leq j \leq r$$

and so:

$$Eu_j = 0 \text{ for } k+1 \leq j \leq r \quad \text{(A11)}$$

As $B = EA$ substitution of the s.v.d. of $A$ results in:

$$B = EU \cdot D \cdot VT$$

which can be rewritten using (A10) and (A11) as:

$$B = U^* \cdot D \cdot VT$$

with:

$$U^* = \begin{pmatrix} u_1 & u_2 & \cdots & u_k & 0 & \cdots & 0 \end{pmatrix}$$

This is equal to:

$$B = U \left( \begin{array}{c|c} D_k & \emptyset \\ \hline \emptyset & \emptyset \end{array} \right) VT = U_k \cdot D_k \cdot V_k^T$$

where:

$D_k = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k)$

$U_k$ is a $m \times k$ matrix containing $u_1, u_2, \ldots, u_k$

$V_k$ is a $n \times k$ matrix containing $v_1, v_2, \ldots, v_k$
Equation (6), section 3.3.3, states that:

\[ |A|^2 = \sum_{j=1}^{r} \sigma_j^2 \]  

(6)

Substituting this and result (A9) in expression (A4) produces:

\[ \min\{|A - B|^2\} = \sum_{j=k+1}^{r} \sigma_j^2 \]

This result is known as the theorem of Eckart - Young.
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