A hybrid formulation for determining torsion- and warping constants

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A HYBRID FORMULATION FOR
DETERMINING TORSION- AND WARPING
CONSTANTS

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A HYBRID FORMULATION FOR DETERMINING 
TORSION- AND WARPING CONSTANTS 

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Abstract—Theoretically, the torsion constant of a bar subjected to a torque 
can be calculated irrespective of its cross-sectional shape; either with a 
potential energy (displacement) or with a complementary energy (stress 
function) formulation. The potential energy formulation, however, provides 
the so-called warping constant, too, which may be important in the case of 
thin-walled sections, but the complementary energy formulation does not. On 
the other hand, the complementary energy formulation leads to simple 
analytical expressions for the torsion constant of thin-walled sections, 
while a finite element calculation based on potential energy requires a 
relatively large number of elements to obtain comparable results. 

A general finite element formulation, however, should provide all 
torsional constants with one algorithm and a small number of elements for 
all kinds of cross-section, albeit solid or thin-walled, open or closed. 
Since hybrid formulations often give more accurate results and combine some 
potential energy as well as complementary energy features, a hybrid 
variational formulation was derived for the torsion problem. This was done 
in such a way that both the torsion- and warping constants could be 
calculated consistently with this formulation and could be followed by the 
usual one dimensional response calculation for a bar subjected to a torque. 
The first finite element calculation for rectangular cross-sections resulted 
in an improved warping constant.
NOTATION

a  
\text{half width of rectangle}

A  
\text{column representing cross sectional coordinates in the functional, see eq. (3.14)}

Ae  
\text{cross-sectional area of the bar}

\lambda  
\text{cross sectional area of the e-th element}

b  
\text{half height of rectangle}

C  
\text{constant, polar moment of inertia}

c_{ijkl}  
\text{components of the elastic compliance tensor}

e_{xy}, e_{xz}  
\text{nonzero components of the strain tensor}

G  
\text{shear modulus}

H  
\text{flexibility matrix, see eq. (3.6)}

I_t  
\text{torsion constant}

k  
\text{constant in } I_t

K  
\text{stiffness matrix}

1  
\text{length of bar}

L  
\text{shape functions for } \phi \text{ on the element boundary}

M  
\text{prescribed torque}

n_y, n_z  
\text{components of outer normal}

N_{\beta}  
\text{column with shape functions for } \beta

N_{\psi}  
\text{column with shape functions for } \psi

N_i  
\text{shape function for the } i\text{-th coordinate}

P_i  
\text{i-th component of boundary traction}

P_i  
\text{prescribed boundary traction}

E  
\text{matrix, see eqs. (3.2) and (3.4)}

\mathbf{y}  
\text{Yamada displacement vector [10]}

q  
\text{column with nodal values of } \psi

Q  
\text{column with generalized loads}

s  
\text{coordinate along element boundary}

S^e  
\text{boundary of } \mathcal{A}^e

S^e_p  
\text{portion of the boundary of } \mathcal{V}^e \text{ where the tractions are prescribed}

T  
\text{matrix, see eq. (3.12)}

t_{xy}, t_{xz}  
\text{nonzero components of the stress tensor}

\tau_{xy}, \tau_{xz}  
\text{Lagrange multipliers}
displacements
i-th component of the displacement
volume of the e-th element
boundary of the e-th element
nodal axial displacement
spatial coordinates

twist.
column with nodal values of the stress function
warping constant
dimensionless coordinates
angle of twist
potential energy functional
Hu-Washizu functional
Hellinger-Reissner functional
hybrid functional
column with non-zero shear stresses, see eq. (3.39)
stress function
nodal value of \( \psi(y,z) \)
warping function

components with respect to the i,j,k or l-th coordinate
components with respect to the x,y or z-th coordinate
number of element
i-th smooth side of element
transposed
1. INTRODUCTION

The general linear theory of the uniform torsion of prismatic bars was well-formulated by the pioneers of mechanical science, de Saint-Venant \[1\] and Prandtl \[2\], according to the then main approaches to solid mechanics; an ultimate formulation in either displacements or stresses. A common feature was that a truly three-dimensional problem was split into a two-dimensional problem and a one-dimensional problem with the aid of assumptions such as rigidity of the cross-section in its plane. The two-dimensional problem includes the determination of geometric cross-sectional properties such as the de Saint-Venant torsion constant and the so-called warping constant, while the one-dimensional problem is the response problem of the remaining one-dimensional model of the bar. Analytical solutions were restricted to bars with simple cross-sections. The introduction of the computer, together with the advent of the finite element method enabled the geometric constants to be calculated for cross-sections with complicated shapes, particularly those of extruded rods \[3\]. Although in analytical calculations both the displacement approach and the stress approach were used (with preference for the stress approach in the case of thin-walled sections), in computer calculations, the displacement approach prevails.

Frequently the torsion problem was treated in pre-computer textbooks \[4\], \[5\], \[6\], rarely do publications mention finite element calculations. Perhaps, this is due to the fact that such calculations require the solution of the plane Laplace equation, a beloved subject of introductory books about finite element methods.

From one of the earliest publications \[7\], however, it appears that an accurate determination of the torsion constant requires a relatively large number of elements. The same applies to the warping constant. More striking is the observation that, for some simple thin-walled cross sections (e.g. the narrow rectangle and the slotted tube) where the analytical stress approach provides simple approximations for the de Saint-Venant torsion constant, the displacement based finite element approach, even with 8-node isoparametric elements, requires a relatively large number of elements \[3\].

This prompted us to develop a hybrid torsion formulation, since hybrid formulations often perform more accurately \[8\]. For a consistent formulation of the torsion- and warping constant, its development cannot be based on a
simple transformation of the plane Laplace problem, as suggested by Martin [9], but initially, recourse must be made to the full three-dimensional problem.

As far as we know, only a few hybrid torsion formulations have been published:

Yamada et al. [10] divided the bar under torsion into longitudinal prisms with a triangular cross section. Their element displacement vector:

\[
[q] = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}
\]

contains, apart from the three axial displacements \( w_i \), the angle \( \theta \) over which the whole cross-section rotates; consequently, it is common to all the elements. Since the angle of rotation \( \theta \) and the nodal displacements \( w_i \) are calculated simultaneously, the procedure does not comply with beam- and bar theory where geometric constants have to be determined separately and subsequently are used to calculate the angle of rotation \( \theta \). Moreover, an older hybrid formulation was used that required a priori fulfilment of the dynamic boundary conditions.

In a more recent paper by Tralli [11] that is restricted to thin-walled sections, the Hellinger-Reissner principle [12] was used. The secondary warping of the individual components was taken into account by the a priori known torsional rigidity of the plate-like components. The predominant primary torsional rigidity due to either closed parts (cells) or restrained warping is hidden in the assembled stiffness matrix. As in the aforementioned paper, the angle of rotation of the cross-section as a whole appears in the displacement vector of each individual (bar-)element. Again, axial displacements and the angle of rotation are calculated simultaneously and the torsion- and warping constants are not obtained explicitly (A merit of Tralli's work, however, is that it can deal with variations of the cross-section along the length of the beam).

In the following section a hybrid formulation for arbitrary cross-sections is presented which yields the torsional constants explicitly.
2. DERIVATION OF THE HYBRID FUNCTIONAL

There are several options for deriving the hybrid functional of uniform torsion.

One option is to introduce the classical assumptions of torsion theory into the general hybrid functional that was originally derived by Pian [13], [15];

\[
\Pi_{mc} = \sum_{e} \left[ \int_{V^e} \frac{1}{2} C_{ijkl} t_{ij} t_{kl} \, dV - \int_{\partial V^e} p_i u_i \, ds + \int_{S^e} \bar{p}_i u_i \, ds \right]
\]  

(2.1)

Where: \( \partial V^e \) is the entire boundary of the e-th element having volume \( V^e \). In this functional, the displacements \( u_i \) only appear in the boundary integral and can be viewed as Lagrange multipliers that have been introduced exclusively on the element boundary in order to relax the continuity requirements for the inter-element tractions \( p_i \). In a number of finite element models, such as plate elements, this formulation has proved to be quite advantageous, because for the boundary displacements simple functions of the boundary coordinates can be chosen independently of the implicit internal displacements.

One objective here, however, was to calculate the warping constant \( \Gamma \) using the classical formula:

\[
\Gamma = \int_{A} \psi^2 \, dA
\]  

(2.2)

Where: \( A \) is the cross-sectional area of the bar under torsion, while the so-called warping \( \psi(y,z) \) represents the axial displacements per unit of twist. The coordinates \( y \) and \( z \) are located in the cross-section. Thus, the displacements have to be defined over the whole cross-section and not only along the element boundaries.

That was why another way of constructing the hybrid formulation had to be chosen. It started with the potential energy formulation, adapted to the problem of uniform torsion, then, proceeded to the hybrid formulation along the line of the so-called Friedrich transformation. Apart from making some features of the ultimate formulation clearer, it shows that the (warping-...
displacements occurring exclusively in the boundary integrals, represent the
displacements originally defined within the entire element.

Consider a homogeneous and isotropic prismatic bar of length l (Fig. 1) and introduce Cartesian coordinates in such a way that the x-axis passes through the centroids of the cross-sections. Since determination of both the Saint-Venant torsion constant and the warping constant are based on the situation of uniform torsion, orientation of the y- and z-axis with respect to e.g. principal axes is irrelevant. Moreover, any longitudinal fiber can be chosen as the axis of rotation, in this case, the x-axis was chosen. The displacements in the x, y and z directions are denoted by \( u, v \) and \( w \) respectively, and the twist per unit length is \( \alpha(x) \). When solving the torsion problem, de Saint-Venant assumed that the axial displacement \( u \) was proportional to the constant twist \( \alpha \) and dependent on the coordinates \( y \) and \( z \):

\[
 u = \alpha \psi (y, z) \tag{2.3}
\]
where: $\psi$ is the unknown warping function. The potential energy formulation requires $\psi$ to be continuous. Due to the assumption that a cross-section does not deform in its plane, the other displacements become:

$$v = -u x z$$

$$w = u x y$$

(2.4)

Implicit for these relations is the fact that rotation at $x = 0$ has been suppressed. As a consequence to (2.3) and (2.4), the non-vanishing strains are:

$$e_{xy} = \frac{1}{2} a(\psi_y - z)$$

$$e_{xz} = \frac{1}{2} a(\psi_z + y)$$

(2.5)

A subscript preceded by a comma denotes partial differentiation with respect to the independent variable indicated by the subscript. The pertinent shear stresses are:

$$t_{xy} = 2Ge_{xy}$$

$$t_{xz} = 2Ge_{xz}$$

(2.6)

where: $G$ is the shear modulus of the material. The stresses should satisfy the local equilibrium equation:

$$t_{xy,y} + t_{xz,z} = 0$$

(2.7)

For a bar which is free to warp when loaded with a twisting moment $\bar{M}$ at $x = 1$, the potential energy is:

$$\Pi_p = \frac{1}{2} \int_V \left\{ \gamma G(e_{xy}^2 + e_{xz}^2) \right\} dv - \bar{M}a_1$$

(2.8)
If the bar is subdivided into longitudinal prismatic prisms over its whole length, the elastic energy becomes the sum of the elastic energy of the individual prisms:

$$\Pi_p = \sum_e \left[ \int_{V^e} \frac{1}{2} \left\{ 4G(e_{xy}^2 + e_{xz}^2) \right\} dV \right] - \overline{M}_1$$

(2.9)

where: $V^e$ is the volume of the longitudinal prism whose cross-section contains the element $e$. The summation includes all the elements. In uniform torsion, the twist $\alpha$ is constant; therefore, the continuity requirement for the axial displacement $u(y, z)$ (2.3) implies that the warping function $\psi(y, z)$ must be continuous over inter-element boundaries. Compatibility in the plane of the cross-section is guaranteed by equations (2.4). The stress-hybrid formulation required can be derived by using the general transformation scheme described by Washizu [14] and Pian [15].

In the first step, subsidiary conditions (2.5) are introduced in the functional (2.9) by means of Lagrange multipliers $t^*_{xy}$ and $t^*_{xz}$:

$$\Pi_m = \sum_e \left[ \frac{1}{2} \int_{V^e} \left\{ 4G(e_{xy}^2 + e_{xz}^2) \right\} dV - 2 \int_{V^e} \left\{ t^*_{xy} e_{xy} - \frac{1}{2} u(\psi, y - z) \right\} dV \right] +$$

$$+ t^*_{xz} \left\{ e_{xz} - \frac{1}{2} u(\psi, z + y) \right\} dV \right] - \overline{M}_1$$

(2.10)

In the second step, $\Pi_m$ has to be stationary with respect to the strains, this leads to the relations:

$$t^*_{xy} = 2G e_{xy}$$

$$t^*_{xz} = 2G e_{xz}$$

(2.11)

Comparing (2.11) with (2.6) shows that the multipliers can be identified with the shearing stresses. Replacing the multipliers by these stresses and adopting Hooke's law (2.6) in order to eliminate the strains, leads to the specialized Reissner functional:
The third step requires partial integration in order to generate the local equilibrium equation:

\[
\Pi_{mR1} = \sum_{e} \left[ \int_{V_{e}} \left\{ -\frac{1}{2G}(t_{xy}^2 + t_{xz}^2) + t_{xy}a(\psi_{x}, y - z) + t_{xz}a(\psi_{y}, y - z) \right\} dV \right] - \bar{M}_{e}l
\]  

(2.12)

Here \( A_{e} \) is the area of the \( e \)-th element, while \( S_{e} \) is the boundary of \( A_{e} \). Now, if local equilibrium is required within each element, and the components of the stress tensor on \( S_{e} \) are expressed in the only relevant component of the stress vector:

\[
p_{x} = t_{xy}n_{y} + t_{xz}n_{z} \quad \text{on} \quad S_{e},
\]  

(2.14)

then the hybrid formulation becomes:

\[
\Pi_{mC} = \sum_{e} \left[ \int_{A_{e}} \left\{ -\frac{1}{2G}(t_{xy}^2 + t_{xz}^2) \right\} dA + al \int_{S_{e}} p_{x} \psi \, ds + \right.
\]

\[
+ \left. ul \int_{A_{e}} (-t_{xy}z + t_{xz}y) \, dA \right] - \bar{M}_{e}l
\]  

(2.15)

Use has been made of the fact that all quantities are independent of \( x \) due to the uniform torsion.

The subsidiary conditions are:
1. continuity of the warping function \( \psi(y, z) \) on interelement boundaries
2. local equilibrium (2.7) within each prism.
The last requirement can be met by introducing the scalar stress function
\( \phi(y,z) \) together with the prescription:

\[ t_{xy} = \phi_z \quad \text{and} \quad t_{xz} = -\phi_y \]  \( \text{(2.16)} \)

The relations which make \( \mathcal{W}_{MC} \) stationary with respect to all the admissible variations are:

1. compatibility within each prism,
2. a continuous stress component \( p_x \) between the prisms,
3. \( p_x = 0 \) on the stress-free longitudinal face of the bar as a whole,
4. the natural boundary condition at \( x = 1 \):

\[ \overline{M} = \sum_e \int_{\Lambda_e} (-t_{xy}z + t_{xz}y) \, d\Lambda \]  \( \text{(2.17)} \)

Apart from its sign, functional \( \mathcal{W}_{(2.15)} \) represents a modified complementary energy functional with continuity of \( p_x \) relaxed by axial displacements \( \omega(s) \) on the longitudinal faces of the prisms and, on the transverse endface, the natural boundary condition \( \text{(2.17)} \) relaxed by rotation \( \varphi_l \).

It is worth noting that, in some hybrid formulations the stress continuity is relaxed by introducing boundary displacements that are only defined on the element boundaries but are independent of the internal displacements [15]. However, the warping function \( \wp(s) \), although it only appears in the contour integral in \( (2.13) \) represents a continuous warping function \( \wp(y,z) \) that is defined over the whole area \( \Lambda \), as shown by the derivation of \( \mathcal{W}_{MC} \). It can thus be used to calculate the warping constant \( \Gamma \) with the classical formula:

\[ \Gamma = \sum_e \int_{\Lambda_e} \wp^2 \, d\Lambda \]  \( \text{(2.18)} \)

Although this formula complies with the potential energy formulation, it may also be used with the hybrid formulation since, if a term accounting for the strain energy due to non-uniform torsion is added to the potential energy \( (2.9) \), this term will remain unaffected by the transformation to the hybrid
formulation. As this term contains (2.18), the classical formula will also remain unaffected.

In the following section, the hybrid formulation will be used to construct a stress-hybrid finite element model. After assembling the elements, it will become clear that determining the torsion- and warping constants can be performed separately and will precede calculating the twist \( \alpha \) as a function of torque \( \dot{M} \), like for the classical displacement formulation.

3. DISCRETISATION OF THE FUNCTIONAL

The finite element discretisation proceeds along the general lines given by Pian [9]. Since the aim was to generate a formulation where the determination of torsional constants could be distinguished from the one-dimensional torsion problem; the end rotation \( \omega \) and the shear modulus \( G \) will be left out of the brackets, when formulating matrices.

We start by assuming a stress function distribution:

\[
\psi(y,z) = N^T(y,z)\beta
\]  

(3.1)

Where column \( \beta \) contains discrete values of the element stress function. Continuity of \( \psi \) between elements is not a requirement. Due to the boundary integral appearing in (2.13), nodal values of \( \psi \) may be advantageous; in which case the so-called shape functions could be taken as components of \( N \).

The pertinent stresses can now be expressed as:

\[
\sigma = \mathcal{P} \beta
\]  

(3.2)

where: \( \sigma \) contains the relevant stresses:

\[
\sigma^T = (t_{xy} \ t_{xz})
\]  

(3.3)

while matrix \( \mathcal{P} \) is defined as:
\[
\mathbf{P} = \begin{bmatrix}
\frac{\partial N^T_x}{\partial x} \\
\frac{\partial N^T_y}{\partial y}
\end{bmatrix}
\]  

(3.4)

The fact that the stresses are related to the derivatives of the stress function will affect (3.1):

If a polynomial shape is chosen for (3.1), the constant value of \( \phi \) can be omitted.

If a description is chosen in nodal values of \( \phi \), column \( \beta \) should contain the appropriate differences between these nodal values; while \( N(y, z) \) should be adapted accordingly. Substituting (3.2) into the first term of (2.13) gives:

\[
- \frac{1}{2G} \int_{A^e} \left( t_{xy}^2 + t_{xz}^2 \right) d\Lambda = - \frac{1}{2G} \beta^T \mathbf{H} \beta
\]

(3.5)

where

\[
\mathbf{H} = \int_{A^e} \mathbf{P}^T \mathbf{P} d\Lambda
\]

(3.6)

with

\[
\mathbf{H} = \mathbf{H}^T
\]

(3.7)

The second term of (2.13) represents the contribution of the prismatic faces to \( \int_{P_i} u_i dS \) of (2.1). It contains the axial shear component \( p_x \) on each individual face. With the aid of (2.14), \( p_x \) can be expressed along each smooth side \( i \) as:

\[
p_x^{(i)} = R^{(i)} (\beta)_x
\]

(3.8)

According to the aim, the warping \( \psi(y, z) \) will be defined over the whole elemental area as:

\[
\psi(y, z) = N^T_{\psi}(y, z) q
\]

(3.9)

This warping distribution appears on the \( i \)-th element boundary as:

\[
\psi^{(i)}(s) = L^{(i)}(s) q
\]

(3.10)
This gives:

\[ \alpha_1 \oint_{S^e} p_x \psi \, ds = \alpha_1 \beta_T^T q \]  
(3.11)

where

\[ T = \sum_{i} \oint_{S^e(i)} R^{(i)} L^{(i)} T \, ds \]  
(3.12)

The third term of (2.13) represents the contribution to the term \( \int p_1 u_1 \, dS \) of (2.1) of the cross-section at \( x = 1 \) which has a surface \( A^e \). Due to the assumption of the rigidity of the cross-section in its plane, the in-plane displacements \( v \) and \( w \) are restrained by (2.4). Together with the stress distribution (3.2), this gives:

\[ \alpha_1 \oint_{A^e} \begin{pmatrix} t_{xy} \\ t_{xz} \end{pmatrix} \begin{pmatrix} -z \\ y \end{pmatrix} \, dA = \alpha_1 \beta_a^T \]  
(3.13)

where:

\[ a = \oint_{A^e} \begin{pmatrix} \frac{E}{\alpha} T^T \\ \frac{E}{\alpha} \end{pmatrix} \begin{pmatrix} -z \\ y \end{pmatrix} \, dA \]  
(3.14)

The individual terms of functional (2.15) have now been discretized so that the total matrix representation becomes:

\[ H_{mc} = \sum_{e} \left[ -\frac{1}{2G} \beta \beta^T + \alpha \beta^T (T q + a) \right] - \bar{\beta} \alpha \]  
(3.15)

For the traction \( p_x \), there is neither a continuity requirement at the inter-elemental boundaries, nor are there requirements at the traction-free longitudinal faces of the bar. Thus, all the stress parameters of \( \beta \) can be varied at element level. If the function (3.15) is required to be stationary with respect to the variations of \( \beta \), this yields for each element:

\[ \delta \beta^T \left[ -\frac{1}{G} \beta \beta^T + \alpha (T q + a) \right] = 0 \]  
(3.16)

and since \( H \) is not singular:

\[ \beta = \alpha \, G H^{-1} (T q + a) \]  
(3.17)
This expression is used to eliminate the stress parameters of $\beta$ from (3.15):

$$ W = \frac{1}{2} \sum_{e} \left[ \frac{1}{2} q^T k_q q + \frac{1}{2} q^T Q + C \right] - \bar{\nu} a l \quad (3.18) $$

where:

$$ k = T^T H^{-1} T \quad (3.19) $$

$$ Q = T^T a \quad (3.20) $$

$$ C = \frac{1}{2} a^T H^{-1} a \quad (3.21) $$

Expression (3.18) has the same appearance as the common displacement formulation. In this special case, the displacement vector $q$ only contains unknown nodal warping values. It does not contain the twist $\alpha$! Contributions to the strain energy of displacements in the cross-sectional plane are contained in $C$; whereas, $q^T Q$ represents the coupling between the warping- and in-plane displacements. Again a plane problem has to be solved in order to find the unknown warping values. Comparing (3.18) with the potential energy expression for a uniformly twisted bar:

$$ W = \frac{1}{2} G t a l^2 - \bar{\nu} a l \quad (3.22) $$

this shows that the torsion constant $I_t$ can be calculated with:

$$ I_t = \sum_{e} \left[ \frac{1}{2} q^T k_q q + \frac{1}{2} q^T Q + C \right] \quad (3.23) $$

Moreover, the warping constant $\Gamma(2.18)$ can now be calculated, because the nodal warping values represent a continuous warping distribution (3.9) over the entire cross-section.
Examples

In order to give a first impression of the hybrid formulation in operation, a simple rectangular element was derived with four nodes.

![Rectangular element diagram]

A bi-linear distribution was chosen for both the stress- and warping functions:

\[
\mathbf{f} = \sum_{i=1}^{4} N_i(\xi, \eta) \mathbf{f}_i
\]

\[
\psi = \sum_{i=1}^{4} N_i(\xi, \eta) \psi_i
\]

Unlike the usual element formulations, the global coordinates \(y\) and \(z\) appeared explicitly in the functional (2.13), because the cross-section is undeformable in its plane. They were treated in the usual way:

\[
y = \sum_{i=1}^{4} N_i(\xi, \eta) y_i
\]

\[
z = \sum_{i=1}^{4} N_i(\xi, \eta) z_i
\]
where: \( y_j \) and \( z_j \) are the coordinates of the \( i \)-th node. In order to avoid singularity of matrix \( H(3,6) \), the column \( \phi \) (3.2) was defined as:

\[
\phi = \begin{bmatrix}
\phi_1 - \phi_4 \\
\phi_2 - \phi_4 \\
\phi_3 - \phi_4
\end{bmatrix}
\]

and \( M(3,1) \) was adapted accordingly. Further formulation of the relevant matrices was straightforward. A test case will now be presented in which a square and a rectangle with aspect ratio 1:10 were involved. Use was made of the double symmetry of these cross-sections.

![Fig. 3. Square cross-section.](image)

The results obtained for the torsion integral \( I_L \) could be expressed with a factor \( k \), defined in:

\[
I_L = k(2a)^\lambda(2b)
\]

These results were compared then with the results obtained from the classical compatible formulation based on a bi-linear warping for each element; also, with the exact values of \( k \) in Table I.
Table I

<table>
<thead>
<tr>
<th>λ</th>
<th>a</th>
<th>nxm</th>
<th>number of elements</th>
<th>compatible</th>
<th>hybrid</th>
<th>exact</th>
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<td>2x2</td>
<td>4</td>
<td>.148</td>
<td>-</td>
<td>.141</td>
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<td>.312</td>
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<td>.314</td>
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<tr>
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<td></td>
<td>5x15</td>
<td>65</td>
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</table>

The results obtained for the warping constant are given in Table II.

Table II

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<th>a</th>
<th>nxm</th>
<th>number of elements</th>
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<th>exact:</th>
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</table>
Conclusions

A hybrid formulation for calculating the torsion- and warping constants for prismatic bars by means of finite elements, can be derived. In the classical formulation, this requires a finite element solution of the two dimensional Laplace equation. When deriving the hybrid formulation, however, the full three dimensional model must be considered. Separating the expression for the torsion constant from the response of the bar to torsion is but possible in the discretized formulation. A procedure for the finite element calculation of the torsion constant, that is consistent with the hybrid formulation, could be developed. The classical expression for the warping constant, which is relevant in cases of non-uniform torsion, is still valid in the hybrid formulation. Calculating this constant requires the warping to be known at every point on the cross-section. In the functional pertaining to the hybrid formulation (2.15) the warping only appears on the element boundaries. The derivation of this functional, however, shows that this warping of the element boundaries represents a warping that was defined originally at every point of the cross-section and thus can be used for calculating the warping constant consistently. A few simple examples showed that the torsion constant calculated with the hybrid method was little better than that calculated by the classical method. The warping constant, however, was more accurate.

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REFERENCES