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PROBABILISTIC ASPECTS OF PHYSICS

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Probabilistic aspects of physics

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Abstract

1 This paper describes 5 examples from physics where probability is used to identify an underlying variational principle.

0 Introduction

0.1 Optimization

Most phenomena in nature follow the principle of optimization, i.e., from all the possible evolutions of a given physical process nature selects the one being optimal, in the sense that some key quantity associated with the process is maximal or minimal. This principle is a Leitmotiv in the mathematical description of physical phenomena.

Examples are:
1. 'survival of the fittest': maximal survival,
2. 'entropy': maximal disorder,
3. 'energy': minimal potential,
4. 'work': minimal friction,
and combinations of these.

0.2 Many-particle systems

Statistical mechanics is an area in physics and mathematics that aims at describing systems consisting of a large number of 'particles' interacting with each other. It tries to build a bridge between, on the one hand, the macroscopic world of (complex) physical phenomena and, on the other hand, the microscopic world of (simple) 'collision rules' through which the particles interact. Since the number of particles is typically extremely large, the notion of probability enters in a natural way. Namely, it makes sense to speak about proportions of particles with a certain property and about how these proportions change with time under the erratic motion of the particles. In this way one is led to a 'statistical' description of the system as an (extremely accurate) approximation of its true behavior. In this description,
the states of the system and (possibly) their evolution are random variables. One therefore
speaks of a statistical ensemble of systems, rather than of a single deterministic system.

Examples are:
1. Gibbs formalism of thermodynamics in equilibrium.
2. Boltzmann equation for relaxation of dilute fluids not in equilibrium.

0.3 Theory of large deviations

The main mathematical instrument used to describe the underlying principle of optimization is the theory of large deviations (the word 'large' should perhaps be omitted). This name stands for an area in probability and statistics that aims at estimating probabilities of deviations from typical behavior in stochastic processes (i.e., probabilities of events that are atypical or even extreme). These probabilities are used to compute the 'most probable' evolution of quantities one is interested in (see below). This approach may be compared with Laplace’s method for integrals of sharply peaked functionals, but reformulated to fit the framework of stochastic processes.

Mathematically, the above remarks may be illustrated by the following. Given a Polish space $E$ and a stochastic process $(X_t)_{t \geq 0}$ taking values in $E$, defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider the empirical measure

$$L_T = \frac{1}{T} \int_0^T \delta_{X_t} \, dt \quad (T > 0)$$  \hspace{1cm} (1)

($\delta_x$ is the measure with unit mass in $x \in E$). This is a random probability measure on $E$, indicating which part of the time the process spends in which state during a given time interval $[0, T]$. In words, $L_T = \mu$ means that a fraction $\mu(S)$ is spent in $S \subset E$, where $\mu \in M(E) = \text{the set of probability measures on } E$ (endowed with the topology of weak convergence).

**Definition** $(L_T)_{T > 0}$ satisfies the large deviation principle (LDP) on $M(E)$ with rate $T$ and rate function $I : M(E) \to [0, \infty]$ if:

(a) $\liminf_{T \to \infty} \frac{1}{T} \log P(L_T \in O) \geq -\inf_{\mu \in O} I(\mu)$ for all $O \subset M(E)$ open.

(b) $\limsup_{T \to \infty} \frac{1}{T} \log P(L_T \in C) \leq -\inf_{\mu \in C} I(\mu)$ for all $C \subset M(E)$ closed.

(c) $I$ is lower semicontinuous and has compact level sets.

Conditions (a)-(b) roughly say that

$$\lim_{T \to \infty} \frac{1}{T} \log P(L_T \approx \mu) = -I(\mu) \quad \text{for all } \mu \in M(E).$$  \hspace{1cm} (2)

The reason to put (a)-(b) and not (2) in the definition of the LDP is topological: they specify what is meant by $\approx$ and at the same time allow for more flexibility by relaxing the equality sign. Think of (a)-(b) as weak convergence of probability measures, but on an exponential scale.

The function $I$ depends on the stochastic process under consideration and can take on many different forms. In ‘nice’ cases it has a unique zero $\mu^* \in M(E)$, which corresponds to a typical behavior expressed by

$$\lim_{T \to \infty} P(L_T \approx \mu^*) = 1.$$  \hspace{1cm} (3)
The LDP says that for ‘nice’ \( A \subset M(E) \)
\[
\lim_{T \to \infty} \frac{1}{T} \log P(L_T \in A) = - \inf_{\mu \in A} I(\mu).
\]  
\hspace{1cm} (4)

In other words, all statements of the type ‘\( L_T \) falls in \( A \)’ are expressed in terms of a variational problem w.r.t. \( I \).

The LDP has the following important consequence, called Varadhan’s lemma:

**Lemma** Suppose that \((L_T)_{T>0}\) satisfies the LDP as in the above Definition. Let \( F : M(E) \to \mathbb{R} \) be continuous and bounded from above. Then
\[
\lim_{T \to \infty} \frac{1}{T} \log \int_{M(E)} e^{TF(\mu)} P(L_T \in d\mu) = \sup_{\mu \in M(E)} [F(\mu) - I(\mu)].
\]  
\hspace{1cm} (5)

Thus, if we are interested in quantities that arise as exponential expectations of \( F(L_T) \), with \( F \) some ‘nice’ functional, then we can find their asymptotic behavior from a variational problem w.r.t. \( F - I \). In this way, a single LDP generates a whole family of variational problems, describing different aspects of the stochastic process depending on the choice of \( F \).

It is in the form of (4) and (5) that the principle of optimization, mentioned in Section 0.1, forces itself upon us.

**0.4 Program**

Given a physical system and a pair \((E, (X_t)_{t \geq 0})\) describing the (stochastic) evolution of a certain quantity one wants to study. Then one has the following 3-step program:

I. Prove the LDP.
II. Identify \( I \) in (2) and \( \mu^* \) in (3).
III. Solve the variational problems in (4) and (5).

Depending on the type of problem, these steps can be carried through to a varying degree of detail.

In the remainder of this paper we shall describe 5 examples, illustrating the power as well as the limitations of the theory of large deviations. The goal is to give the reader an impression.

We shall encounter the LDP in various guises. The running variable may be time, as in (2), but may also be system size, noise level, interaction strength, etc. Thus, we shall formulate LDP’s for quantities other than \( L_T \).

**1 Examples 1 and 2: Random dynamics**

1. Flow with noise
Consider the first order differential equation
\[
\frac{dx(t)}{dt} = f(x(t)) \quad (t \geq 0; x(0) = 0)
\]  
\hspace{1cm} (6)
with $f : \mathbb{R}^d \to \mathbb{R}^d$ globally Lipschitz continuous. The solution of (6) is a certain trajectory $x : [0, \infty) \to \mathbb{R}^d$. Suppose now that the motion undergoes a random perturbation, so that (6) is replaced by the stochastic differential equation

$$dx(t) = f(x(t))dt + \epsilon dw(t) \quad (t \geq 0; x(0) = 0).$$

(7)

Here $\epsilon > 0$ is a small parameter and $t \to w(t)$ is (standard) Brownian motion on $\mathbb{R}^d$. Equation (7) describes e.g. the motion of a tiny, electrically charged, latex sphere immersed in water, where $f$ is the drift due to an external electric field and $w(t)$ is the erratic motion caused by the water molecules colliding with the surface of the sphere.

Let $x' : [0, \infty) \to \mathbb{R}^d$ be the random solution of (7). The question we ask is: Depending on $\epsilon$, how close is $x'$ to $x$ and what is the probability of a substantial deviation? We look at deviations in the space of trajectories (starting at 0) over the time interval $[0, T]$ ($T$ is fixed but arbitrary), written $C_0([0, T]; \mathbb{R}^d)$ and endowed with the topology induced by the sup-norm metric.

**Theorem 1** (Ventcel and Freidlin 1970) \{$x'$\}$_{\epsilon > 0}$ satisfies the LDP on $C_0([0, T]; \mathbb{R}^d)$ with rate $1/\epsilon^2$ and rate function

$$I(y) = \frac{1}{2} \int_0^T \left( \frac{dy(t)}{dt} - f(y(t)) \right)^2 dt$$

(8)

($= \infty$ when the integral does not exist).

This theorem says e.g. that if we pick $B_\delta(x)$, the sphere of radius $\delta$ around $x$ in the sup-norm, then (recall (4))

$$\lim_{\epsilon \downarrow 0} \epsilon^2 \log P(x' \notin B_\delta(x)) = -\inf_{y \notin B_\delta(x)} I(y) < 0.$$  

(9)

Hence, the probability of a maximal deviation $\delta$ between the two trajectories $x'$ and $x$ (during the time interval $[0, T]$) decays like $\exp[-I(y_\delta)/\epsilon^2]$, where $y_\delta$ is a minimizer of the variational problem in the r.h.s. of (9). (This minimizer need not always exist, nor need it be unique.)

We see that a deviation becomes unlikely as $\epsilon \downarrow 0$, at a rate that is described by the rate function $I$. We also see that a deviation is done in the least unlikely of unlikely ways. Note that $I$ has the solution $x$ of (6) as its unique zero, simply because it carries the unperturbed differential equation (6) in it. Also note that $I$ depends on the type of noise that is used as a perturbation in (7), since it reminds us of $\frac{1}{2} \nabla^2$, which is the generator of the Brownian motion.

2. Magnets in a heat bath

Consider a system consisting of $N$ iron atoms, each of which acts like a microscopic magnet that can point either up (+1) or down (-1). The magnets are called 'spins'. The collection $\sigma = (\sigma_i)_{1 \leq i \leq N}$, with $\sigma_i$ denoting the value of the $i$-th spin, is called the 'spin configuration'. The interaction energy associated with $\sigma$ is given by the Hamiltonian

$$H^N(\sigma) = -\frac{1}{N} \sum_{1 \leq i < j \leq N} \sigma_i \sigma_j.$$  

(10)

This energy is low when most spins are pointing in the same direction, and high otherwise. Therefore (10) models a preference to be parallel, called ferromagnetism, typical for atoms.
in iron and other magnetic materials. All pairs \(i\) and \(j\) contribute with equal strength to the interaction energy. The factor \(1/N\) sees to it that the total interaction per spin is of order 1. This is called the Curie-Weiss model.

Now suppose that we let the spins evolve according to a random dynamics, as follows. Writing \(\sigma(t)\) to denote the configuration at time \(t\), the evolution is given by the equation

\[
P\left(\sigma(t + \Delta t) = \sigma' | \sigma(t) = \sigma\right) = c(i, \sigma) \Delta t + o(\Delta t) \quad (\Delta t \downarrow 0, t \geq 0)
\]

with

\[
c(i, \sigma) = \exp \left[ -\frac{1}{2} \beta \left( H^N(\sigma') - H^N(\sigma) \right) \right].
\]

Here \(\sigma'\) denotes the configuration obtained from \(\sigma\) by flipping the \(i\)-th spin, while \(c(i, \sigma)\) denotes the flip rate of the \(i\)-th spin in configuration \(\sigma\) (i.e., spins flip individually at a rate that depends on the values of the spins at neighboring sites). The parameter \(\beta \in (0, \infty)\) is called the inverse temperature. The random dynamics is a model for the flipping of the iron atoms caused by the 'heat bath' in which they are immersed (through the erratic motion of their surroundings due to temperature). As initial condition we pick \(\sigma(0) \equiv +1\).

Equations (11-12) have the qualitative feature that it is easy to move to a spin configuration with lower energy and hard to move to one with higher energy. The parameter \(\beta\) controls the degree to which this is so. \(^2\) The question we ask is: As \(N \to \infty\), do the spins cooperate by predominantly pointing in the same direction? The answer is 'sometimes yes, sometimes no', as we shall see below.

Let \(\sigma_i(t)\) denote the value of the \(i\)-th spin at time \(t\). Define the total magnetization of the system at time \(t\) by

\[
m^N(t) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i(t).
\]

Using (11-12) together with the observation that

\[
\frac{1}{2} \left( H^N(\sigma'(t)) - H^N(\sigma(t)) \right) = \sigma_i(t) \frac{1}{N} \sum_{j \neq i} \sigma_j(t) = \sigma_i(t) m^N(t) - \frac{1}{N}
\]

\[
\frac{1}{2} [1 \pm m^N(t)]\text{ is the fraction of } \pm 1\text{-spins},
\]

it can be shown that

\[
\lim_{N \to \infty} m^N(t) = m(t) \text{ with probability 1,}
\]

where \(m(t)\) is the deterministic solution of the differential equation

\[
\frac{1}{2} \frac{dm(t)}{dt} = \frac{1}{2} [1 - m(t)] e^{\beta m(t)} - \frac{1}{2} [1 + m(t)] e^{-\beta m(t)}
\]

\[
= \sinh(\beta m(t)) - m(t) \cosh(\beta m(t))
\]

with initial condition \(m(0) = 1\). Eq. (16) plays a role analogous to (6) in the previous example.

We shall be interested in how \(m^N\) deviates from \(m\), this time in the space of right-continuous piecewise-constant trajectories (starting at 1) over the time interval \([0, T]\), written

\(^2\)It is easy to check that (11-12) define a reversible dynamics with an equilibrium measure under which \(\sigma\) occurs with probability proportional to \(\exp[-\beta H^N(\sigma)]\), the Boltzmann weight factor.
and endowed with the topology induced by the Skorohod metric. To formulate the LDP we need the following symbols:

\[ L^+(m(t)) = \frac{1}{2} [1 + m(t)] e^{-\beta m(t)} \]
\[ L^-(m(t)) = \frac{1}{2} [1 - m(t)] e^{\beta m(t)} \]
\[ L = L^+ - L^- \]

and

\[ \Psi(a|b,c) = \sup_{\xi \in \mathbb{R}} \left[ a\xi - b(e^\xi - \xi - 1) - c(e^{-\xi} + \xi - 1) \right] \quad (a \in \mathbb{R}; b, c \in \mathbb{R}^+) \]  

Note that \( \Psi(a|b,c) \geq 0 \) with equality iff \( a = 0 \).

**Theorem 2 (Comets 1987)** \( \{m^N\}_{N \geq 1} \) satisfies the LDP on \( D_1([0,T];[-1,+1]) \) with rate \( N \) and rate function

\[ I(n) = \int_0^T dt \left( \frac{1}{2} \frac{dn(t)}{dt} - L(n(t)) \right) \left( L^+(n(t)) - L^-(n(t)) \right) \]  

\[ (= \infty \text{ when the integral does not exist}). \]

Again we see that that the solution of (16) is the unique zero of \( I \) and that the form of \( I \) is dictated by the evolution equation. However, compared to (8) the expression in (19) is slightly more involved. This is due to the fact that we are dealing now with a more complicated random dynamics, involving a large number of components that are all interacting and all subject to motion. Again a formula of the type in (9) holds, namely

\[ \lim_{N \to \infty} \frac{1}{N} \log P(m^N \notin B_\delta(m)) = - \inf_{n \notin B_\delta(m)} I(n) < 0. \]  

A particularly interesting aspect of (16) is that the stationary solution need not be unique. Indeed, if \( \lim_{t \to \infty} m(t) = m \), then \( m \) satisfies the equation \( m = \tanh(\beta m) \). For \( \beta \leq 1 \) ('high temperature') this equation has only the paramagnetic solution \( m = 0 \), which is globally stable. But for \( \beta > 1 \) ('low temperature') there are two more solutions \( m = \pm m(\beta) \) with \( m(\beta) > 0 \), which correspond to the spins pointing predominantly up resp. down. These ferromagnetic solutions are locally stable, while the paramagnetic solution is now unstable. Since the process starts with all spins up, the solution \( m = +m(\beta) \) globally attracts the trajectory. At \( \beta = 1 \) a phase transition occurs from non-cooperative to cooperative behavior.

## 2 Examples 3 and 4: Transport in random media

### 3. Wiener sausage

Let \( R \) be a random collection of points in \( \mathbb{R}^d \) chosen according to a Poisson process with uniform intensity \( \lambda \in (0, \infty) \). Around each point in \( R \) we place a sphere with radius 1, which gives us the random set

\[ T = \bigcup_{x \in R} B_1(x). \]
Next, let \( t \to w(t) \) be a (standard) Brownian motion on \( \mathbb{R}^d \) starting at 0. Look at the first time when this Brownian motion hits the set \( T \), written

\[
\tau = \inf\{ t \geq 0 : w(t) \in T \}.
\]  

(22)

Think of \( T \) as a trapping set and of \( \tau \) as a killing time. We have then a model for chemical kinetics: a particle of one substance moves between particles of another substance with which it reacts upon encounter.

We shall be interested in the probability of surviving until time \( t \), written \( P(\tau > t) \). Note that this quantity depends on the two types of randomness in the model (motion and medium). It is not hard to show that \( \lim_{t \to \infty} P(\tau > t) = P(\tau = \infty) = 0 \). The question we ask is: How fast does the survival probability decay with \( t \)?

Before we describe the answer to this question, we first reformulate the survival problem in terms of a problem that involves the Brownian motion alone (i.e., in the free space without traps). Define

\[
W(t) = \bigcup_{s \in [0,t]} B_1(w(s)),
\]

(23)

i.e., the set swept out by a unit sphere carried along by the Brownian motion. This set has the shape of a ‘sausage’, which is why \( t \to W(t) \) is called the ‘Wiener sausage’. Now, in terms of \( W(t) \) we have the representation

\[
P(\tau > t) = E\left( \exp\left[-\lambda |W(t)|\right] \right),
\]

(24)

where \( |W(t)| \) is the Lebesgue volume of \( W(t) \) and \( E \) denotes expectation. Indeed, given the path \( \{w(s) : s \in [0,t]\} \), the probability that no sphere lies on this path equals \( P(\mathcal{R} \cap W(t) = \emptyset) = \exp\left[-\lambda |W(t)|\right] \). After averaging over the path we get (24). Note that the r.h.s. of (24) only depends on the free Brownian motion. The effect of the random medium sits all in the exponential.

**Theorem 3** (Donsker and Varadhan 1975) \( t^{-\frac{d}{d+2}} |W(t)| \) satisfies the LDP on \( \mathbb{R}^+ \) with rate \( t^{\frac{d}{d+2}} \) and rate function

\[
I(w) = \text{principal Dirichlet eigenvalue of } -\frac{1}{2} \nabla^2 \text{ on the sphere with volume } w.
\]

(25)

Since, by (24),

\[
P(\tau > t) = \int_{\mathbb{R}^+} e^{-\lambda w} P(|W(t)| \in dw) = \int_{\mathbb{R}^+} e^{-t^{\frac{d}{d+2}} \lambda w} P(t^{-\frac{d}{d+2}} |W(t)| \in dw),
\]

(26)

an immediate consequence of Theorem 3 is that (recall the Lemma in Section 0.3)

\[
\lim_{t \to \infty} t^{-\frac{d}{d+2}} \log P(\tau > t) = \lim_{t \to \infty} t^{-\frac{d}{d+2}} \log \int_{\mathbb{R}^+} dw \ e^{-t^{\frac{d}{d+2}}[\lambda w + I(w)]}
\]

\[
= -\inf_{w \in \mathbb{R}^+}[\lambda w + I(w)].
\]

(27)

The r.h.s. of (27) can easily be computed using the scaling relation \( I(w) = I(1)/w^{\frac{d}{2}} \). The survival probability decays at a subexponential rate which depends on the dimension.
Eq. (27) should be understood as follows. The easiest way for the Wiener sausage to be atypically small is for the Brownian motion to stay 'rolled up' inside a sphere of a certain volume (growing with time like $t^{d+2}$). This explains why the rate function $I$ is given by the principal Dirichlet eigenvalue on a spherical domain. In the trap model this situation corresponds to the following optimal strategy for survival: the Brownian motion stays inside a spherical hole, the traps stay outside this hole. Formulated differently, if one would film the process $10^3$ times, during a time interval of length say $10^2$, and if afterwards one would throw away all the films where the particle does not survive, then the overwhelming majority of the remaining films would show this optimal survival strategy.

4. Diffusion in a random potential
Consider the integer lattice $\mathbb{Z}$ and let

$$p = \{p_x\}_{x \in \mathbb{Z}}$$

(28)

be a sequence of independent random variables taking values in $(0, 1)$ according to a common probability law $\mu$. Given $p$, let

$$X = (X_n)_{n \geq 0} \quad (X_0 = 0)$$

(29)

be the Markov process with the following $p$-dependent transition probabilities:

$$P(X_{n+1} = x + 1|X_n = x) = p_x$$
$$P(X_{n+1} = x - 1|X_n = x) = 1 - p_x \quad (x \in \mathbb{Z}).$$

(30)

The pair $(X, p)$ is called a random walk in random environment. Think of the sites with $p_x < 1/2$ as being 'uphill' for a particle moving to the right and those with $p_x > 1/2$ as being 'downhill'. This models e.g. an electron in copper wire seeing a random potential with hills and valleys. The question we ask is: What is the most probable position for the particle at time $n$ and what is the probability of a substantial deviation from this position?

**Theorem 4** (Greven and den Hollander 1994) $(\frac{1}{n}X_n)_{n \geq 1}$ satisfies the LDP on $[-1, +1]$ with rate $n$ and rate function $I$. The latter is $p$-a.s. constant and is given by (32-35) below.

The LDP says that for any interval $A \subset [-1, +1]$ (compare with (4))

$$\lim_{n \to \infty} \frac{1}{n} \log P\left(\frac{1}{n}X_n \in A\right) = - \inf_{\theta \in A} I(\theta).$$

(31)

Hence, the probability that $X_n \approx \theta n$ decays like $\exp[-I(\theta)n]$.

To describe the rate function, we introduce the following symbols:
1. Write $q_x = (1 - p_x)/p_x$ and $q = \{q_x\}_{x \in \mathbb{Z}}$.
2. For $r \geq 0$, define the infinite continued fraction

$$f(r, q) = \frac{1}{e^r(1 + q_0)} - \frac{q_0}{e^r(1 + q_1)} - \frac{q_1}{e^r(1 + q_2)} - \frac{q_2}{e^r(1 + q_3)} - \cdots$$

(32)

3. Define $\lambda(r)$ by

$$\log \lambda(r) = \int_{\mathbb{R}} \mu(dq) \log f(r, q).$$

(33)
Here the expectation is taken w.r.t. the random environment.

4. For \( \theta \in (0, 1) \), find the unique \( r = r(\theta) \) for which

\[
\frac{d}{dr} \log \lambda(r) = \frac{1}{\theta}
\]

(34)

Then in terms of 1-4 the rate function becomes

\[
I(\theta) = -r(\theta) - \theta \log r(\theta) \quad (\theta \in (0, 1))
\]

(35)

A similar expression holds for \( \theta \in [-1, 0) \), while \( I(0) = \lim_{\theta \to 0} I(\theta) \).

We see that the rate function is rather involved, this time due to the randomness of the environment. However, the above recipe is quite explicit and the continued fraction in (32) is sufficiently nice so that a lot can be said about the shape and the analytical properties of \( I \). For instance, it turns out that if \( \int \mu(dq) \log q_0 = 0 \), then \( I \) is symmetric with a unique zero at \( \theta = 0 \). On the other hand, if \( m_0 = \int \mu(dq) q_0 < 1 \), then \( I \) is zero in the interval \([0, (1 - m_0)/(1 + m_0)]\) and strictly positive outside. In the last case, apparently deviations below a certain critical value decay subexponentially fast with \( n \). One would therefore need a higher order analysis to establish at which rate they actually decay. The typical speed must be in the interval \([0, (1 - m_0)/(1 + m_0)]\) (compare with (3)) and turns out to be the right-most point.

Virtually nothing is known for the model in \( d = 2 \).

3 Example 5: Self-interaction

5. Polymers
A polymer is a long chain of molecules with two characteristic properties: (1) an irregular shape (due to entanglement), (2) a certain stiffness (due to steric hindrance). One way to model a polymer is as follows.

Let \((X_n)_{n \geq 0}\) be simple random walk on \( \mathbb{Z} \), i.e., the process making independent jumps between neighboring lattice sites, choosing left and right with probability \( 1/2 \) \((X_0 = 0) \). Let \( P_n \) be its probability law on \( n \)-step paths. Define a new probability law \( Q^\beta_n \) on \( n \)-step paths by setting

\[
\frac{Q^\beta_n}{dP_n}((X_i)_{i=0}^n) = \frac{1}{Z^\beta_n} \exp \left[ -\beta \sum_{0 \leq i < j \leq n} 1_{\{X_i = X_j\}} \right].
\]

(36)

Here, \( 1_{\{X_i = X_j\}} \) is the indicator of the event that the path hits the same site at times \( i \) and \( j \), \( \beta \in (0, \infty) \) is a parameter, and \( Z^\beta_n \) is the normalizing constant. The exponent in (36) counts the total number of self-intersections of the path until time \( n \). What (36) says is that the new probability measure \( Q^\beta_n \) discourages the path ‘to make turns’ by letting it pay a penalty \( e^{-\beta} \) for every self-intersection.

We expect that, as \( n \to \infty \), the interaction will cause the path to move away from the origin at a positive speed. The question we ask is: What is this speed and how does it depend on \( \beta \)?

\textbf{Theorem 5} (Greven and den Hollander 1993) \((\frac{1}{n} X_n)_{n \geq 1}\) satisfies the LDP on \([-1, +1]\) with rate \( n \) and with rate function given by (37-40) below.
The same formula as in (31) holds.

The recipe for the rate function is as follows ($\beta$ is fixed):
1. For $r \geq 0$, define the $\mathbb{N} \times \mathbb{N}$-matrix
   \[ A_r(i,j) = e^{r(i+j-1) - \beta(i+j-1)^2} P(i,j) \quad (i,j \geq 1) \]
   with $P$ the Markov matrix
   \[ P(i,j) = \left( \begin{array}{c} i+j-2 \\ i-1 \end{array} \right) \left( \frac{1}{2} \right)^{i+j-1}. \]
2. Let $\lambda(r)$ be the principal eigenvalue of $A_r$ in $\ell^2(\mathbb{N})$.
3. For $\theta \in (0,1]$, find the unique $r = r(\theta)$ for which
   \[ \frac{d}{dr} \log \lambda(r) = \frac{1}{\theta}. \]

Then in terms of 1-3 the rate function becomes
\[ I(\theta) = r(\theta) - \theta \log \lambda(r(\theta)). \]

By obvious symmetry $I(\theta) = I(-\theta)$. Moreover, $I(0) = \infty$.

The explicit form of $I$ is not particularly illuminating. The matrix $P$ describes the self-intersections of the free random walk without penalty, while the exponential factor in (37) describes the self-repellence. We find that $I$ has two zeroes, at $\theta = \pm \theta^*$ with $\theta^* \in (0,1)$, which are minima of $I$ and represent the forward resp. backward speed of the polymer. The value of $\theta^*$ is easily deduced from (39-40), namely, $\theta^* = 1/\lambda'(\theta^*)$ with $\theta^*$ the unique solution of $\lambda(r) = 1$.

Concerning the dependence of $\theta^* = \theta^*(\beta)$ on $\beta$, the following is known:
1. $\beta \to \theta^*(\beta)$ is analytic on $(0,\infty)$.
2. $\theta^*(0) = 0$ and $\theta^*(\infty) = 1$.
3. $\theta^*(\beta) \sim C\beta^{1/2}$ as $\beta \downarrow 0$ for some $C > 0$.

Remarkably, the intuitively appealing property that $\beta \to \theta^*(\beta)$ is increasing has not yet been proved (and seems to be difficult to prove).

No results are known so far for the model in $d = 2,3$ or 4. It is believed that
\[ \lim_{n \to \infty} \frac{1}{n^\nu} X_n = X \text{ for some } \nu \in [\frac{1}{2},1), \]
where $X$ is some non-degenerate random variable. The power $\nu$ is believed to be independent of $\beta$ and believed to be equal to $3/4$ ($d = 2$), $0.588 \ldots$ ($d = 3$), $1/2$ ($d \geq 4$). The value in $d = 3$ is obtained via simulations and is well below $3/5$. For $d \geq 5$, on the other hand, expansion techniques can be used to prove (41) (Hara and Slade 1994), but these techniques fail in lower dimensions due to essential divergencies.

4 Epilogue

The above 5 examples illustrate the role of large deviations in the description of physical processes. The rate function $I$ is the 'captain of the ship'. Once we know this function we have very sharp information on both the typical behavior and the fluctuations of the process under consideration. We have seen that $I$ sometimes takes on a simple form and sometimes is quite involved. Still, all 5 examples are 'lucky cases', in the sense that the program sketched in Section 0.4 can be carried through. There are many applications from physics where this is not yet the case.
References


