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For AUT-π we refer to Zucker [2]. If we omit all those features that the languages of the AUTOMATH family have in common (cf. the description of AUT-QE in D. van Daalen [1]), the basic rules are the following (i) - (vii). Two simplifications are made here. First, we use a symbol τ which may be either type or prop. Secondly, we omit all \Pi's in expressions of degree 1, which does not make any essential difference. And we use the notation \((x : α) \vdash \) in order to indicate that something is valid in the context extended by \(x\) (of type \(α\)). As in [1], \([x/A]\%\) means that in \(Z\) we have to replace \(x\) by \(A\).

The rules are

(i)  \[  \perp_\tau  \]

(ii) \[  \vdash α : τ (x : α) \vdash P \]

\[  \vdash [x : α] P  \]

(iii) \[  \vdash α : τ (x : α) \vdash Q : P  \]

\[  \vdash [x : α] Q : [x : α] P  \]

(iv) \[  \vdash A : α : τ \vdash Q : [x : α] P  \]

\[  \vdash [A] Q : [x/A] P  \]

(v) \[  \vdash α : τ \vdash Q : [x : α] τ  \]

\[  \vdash [x : α] Q : τ  \]

(vi) \[  \vdash α : τ (x : α) \vdash R : Q : τ  \]

\[  \vdash [x : α] R : [x : α] Q  \]

(vii) \[  \vdash A : α : τ \vdash R : [x : α] Q  \]

\[  \vdash [A] R : \{A\} Q  \]

We shall now define operators \(e_1, e_2, \ldots, e_m\) acting on Q's with
The symbols are metalinguistic: $\theta_j Q$ is used in the metalanguage to indicate a certain expression in the language, viz.

$$\theta_1 Q = [x_1: \alpha_1] \ldots [x_m: \alpha_m] \Pi [x_{m-1}: \alpha_{m-1}] [x_{m-2}: \ldots [x_1] Q$$

$$\theta_2 Q = [x_1: \alpha_1] \ldots [x_{m-2}: \alpha_{m-2}] \Pi [x_{m-1}: \alpha_{m-1}] \Pi [x_{m-2}: \ldots [x_1] Q$$

$$\theta_{m - 1} Q = [x_1: \alpha_1] \Pi [x_2: \alpha_2] \Pi \ldots [x_{m-1}: \alpha_{m-1}] \Pi [x_{m-2}: \ldots [x_1] Q$$

$$\theta_m Q = \Pi [x_1: \alpha_1] \Pi [x_2: \alpha_2] \Pi \ldots [x_{m-1}: \alpha_{m-1}] \Pi [x_{m-2}: \ldots [x_1] Q$$

Note that $\theta_j$ is built by starting from the expression just given for $\theta_1 Q$ and then omitting the first $m - j$ $\Pi$'s. If $m = 1$ we just have $\theta_1 Q = \Pi Q$. If $m = 2$ then $\theta_1 Q = [x_1: \alpha_1] \Pi [x_1] Q$ and $\theta_2 Q = [x_1: \alpha_1] \Pi [x_1] Q$. If $m = 0$ none of the $\theta_j$'s are defined.

We can now prove the validity of a new rule, viz:

\begin{equation}
\tag{viii}
(\forall \alpha: \tau \quad [x_1: \alpha_1] \ldots [x_m: \alpha_m] \tau
\end{equation}

$$\quad [x_1: \alpha_1] \ldots [x_m: \alpha_m] \tau$$

$$\quad [x_1: \alpha_1] \ldots [x_m: \alpha_m] \tau$$

for $1 \leq j \leq m$. If $j = m - 1$ it is just the old rule (v).

For shortness we shall write $[x_i]$ and $(x_i)$ instead of $[x_i: \alpha_i]$ and $(x_i: \alpha_i)$.

Let us start from

$$\quad [x_1: \alpha_1] \ldots [x_m] \tau$$

Applying (iv) we get

$$(x_1) [x_1] Q : [x_1] \ldots [x_m] \tau$$

and $m - 2$ more applications of the same rule leads to

$$(x_1 \ldots (x_{m-1}) [x_m] \ldots [x_1] Q : [x_m] \tau$$

Next we apply (v):

$$(x_1 \ldots (x_{m-1}) \Pi [x_{m-1}] \ldots [x_1] Q : \tau$$
and by (iii) this gives

\[(x_1)\ldots(x_{m-2}) \overset{\beta}{\to} [x_{m-1}] \Pi \{x_{m-1}\} \ldots \{x_1\} Q : [x_{m-2}] \tau\]  

(2)

Now \(m-2\) further applications of (iii) gives

\[\overset{\beta}{\theta}_1 Q : [x_1] \ldots [x_{m-1}] \tau\]

On the other hand, if we apply (v) to (2) followed by a single application of (iii) we get

\[(x_1)\ldots(x_{m-3}) \overset{\beta}{\to} [x_{m-2}] \Pi [x_{m-1}] \Pi \{x_{m-1}\} \ldots \{x_1\} Q : [x_{m-2}] \tau\]  

(3)

Now \(m-3\) more applications of (iii) lead to

\[\overset{\beta}{\theta}_2 Q : [x_1] \ldots [x_{m-2}] \tau\]

On the other hand, if we apply (v) followed by (iii) to (3) we get

\[(x_1)\ldots(x_{m-4}) \overset{\beta}{\to} [x_{m-3}] \Pi [x_{m-2}] \Pi [x_{m-1}] \Pi \{x_{m-1}\} \ldots \{x_1\} Q : [x_{m-3}] \tau\]

This way we get, indeed

\[\overset{\beta}{\theta}_j Q : [x_1] \ldots [x_{m-j}] \tau\]  

(4)

for all \(j\) (1 \(\leq j \leq m\)).

We shall also show that \(\theta_i \theta_j \overset{\beta}{\to} \theta_{i+j} \). More precisely, if \(\overset{\beta}{\theta} Q : [x_1 : \alpha_1] \ldots [x_m : \alpha_m] \tau\), and if \(i \geq 1, j \geq 1, i+j \leq m\), then \(\theta_i \theta_j Q\) reduces to \(\theta_{i+j} Q\) by means of repeated \(\beta\)-reduction. First we have (4), i.e.

\[\overset{\beta}{\theta} [x_1] \ldots [x_{m-j}] \Pi [x_{m-j+1}] \Pi \ldots \Pi [x_{m-1}] \Pi \{x_{m-1}\} \ldots \{x_1\} Q : [x_1] \ldots [x_{m-j}] \tau\]

Applying \(\theta_1\) to this we get

\[\overset{\beta}{\theta}_1 \theta_j Q : [x_1] \ldots [x_{m-j-1}] \tau\]

and
The sequence \( \{y_{m-j}\} \ldots \{y_1\} \{x_{m-j}\} \ldots \{x_1\} Q \) is annihilated by \( m \) applications of \( \beta \)-reduction. After that, we change the names \( y_1, \ldots, y_{m-j-1} \) into \( x_1, \ldots, x_{m-j-1} \), thus arriving at \( \theta_{i+j} Q \).

Above we extended rule (v) to rule (viii). Similarly, we shall extend rule (vii) to the following rule (ix) for \( m \geq 1 \):

\[
(xix) \quad \vdash A : \alpha_i, \quad \vdash R : \theta_{m-1} Q, \quad \vdash Q : [x_1 : \alpha_1] \ldots [x_m : \alpha_m] \tau
\]

\[\vdash [A] R : \{A\} \theta_{m-1} Q\]

If \( m=1 \) we have \( \theta_m Q = \Pi Q \), and \( \theta_{m-1} Q \) has to be explained as \( Q \) itself (\( \theta_0 \) was not defined before).

Rule (ix) is not hard to derive. Noting that \( \theta_m Q = \Pi \theta_{m-1} Q \), and \( \vdash Q : [x_1 : \alpha_1] \tau \) by (viii), we can apply (vii) with \( Q \) replaced by \( \theta_{m-1} Q \), which leads to \( \vdash [A] R : \{A\} \theta_{m-1} Q \).

We note that in all rules, formulas of the type \( \vdash R : Q \) lead to \( \vdash Q : \tau \). Indeed, in (vi) we have \( \vdash \Pi [x : \alpha] Q : \tau \) by (v), and in (ix) we have \( \vdash [A] \theta_{m-1} Q : \tau \) by (iv), according to the typing of \( \theta_{m-1} Q \) just derived.

Instead of the lower kind of (ix) we may as well get

\[
\vdash [A] R : \theta_{m-1} \{A\} Q
\]

since \( \{A\} \theta_{m-1} Q \) reduces to \( \theta_{m-1} \{A\} Q \) by a single beta reduction.

More generally, we observe that

\[
\{A\} \theta_j Q \text{ reduces to } \theta_j \{A\} Q
\]

by a single beta reduction if \( j < m \).

The symbols \( \theta_j \) also commute with abstraction: if

\[
\vdash Q : [x_1 : \alpha_1] \ldots [x_m : \alpha_m] \tau
\]

then

\[
[y : \beta] \theta_j Q \text{ reduces to } \theta_j [y : \beta] Q
\]

if \( j < m \).

These observations mean that in composite expressions like

\[
\{ \ldots \theta_i [ \ldots \theta_j \{ \ldots \theta_k [ ] \ldots Q \text{ the } \theta's \text{ may all be shifted to the extreme left.}
\]
References:
