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TECHNICAL NOTE

RIGID BODY MOTION CALCULATED FROM SPATIAL CO-ORDINATES OF MARKERS*

INTRODUCTION

Several methods have been developed for the kinematical analysis of bone movements in cadaveric specimens or living subjects (Rodrigues, 1840; Kinzel *et al.*, 1972; Selvik, 1974; Chao, 1978). Methods based on X-ray or light photogrammetry of markers connected to bone are usually relatively accurate as compared to electro-goniometry (Kerschot and Soudan, 1978).

Spatial co-ordinates of markers in or on bone can be calculated from the co-ordinates of projections of these markers in two different directions on one or two planes. These spatial co-ordinates are used to determine kinematical parameters. The object in study is considered rigid and its movement between two subsequent positions is taken to be a screw motion. Such a motion can be described by a rotation matrix and a translation vector or by the position of the screw axis, the angle of rotation about this axis and the translation along this axis.

Rodrigues (1840) needed the spatial co-ordinates of three non-collinear points before and after the movement in order to calculate the direction vector $\mathbf{n}$ of the helical axis and the rotation angle ϕ . If $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are the radius vectors of these points before and after the movement, the equations for $\mathbf{n}$ and ϕ are (Selvik, 1974):

$$\Omega = \mathbf{n} \tan \frac{\phi}{2}$$

$$\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{a}_1 + \mathbf{a}_2 = \Omega^*(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{a}_1 - \mathbf{a}_2)$$

$$\mathbf{p}_1 - \mathbf{p}_3 - \mathbf{a}_1 + \mathbf{a}_3 = \Omega^*(\mathbf{p}_1 - \mathbf{p}_3 + \mathbf{a}_1 - \mathbf{a}_3),$$

where $*$ denotes cross product of vectors. The vector Ω can be solved from the latter two vector equations if these equations are not inconsistent. If they are, a least squares method is needed and the results may be slightly different from the results according to Selvik (1974) or our method.

Kinzel *et al.* (1972) used the co-ordinates of four non-coplanar points in order to calculate the 4×4 matrices that described both rotation and translation.

Chao (1978) calculated the rotation matrix R from two vectors, pointing from one of three markers to the other two. The extension of his method in case of more than three markers was not shown.

Selvik (1974) used a least squares method and minimized

$$\sum_{i=1}^n (\mathbf{p}_i - R\mathbf{a}_i - \mathbf{v})^2,$$

where n ($n \geq 3$) is the number of markers and $\mathbf{v}$ is the translation vector. Variables were the three components of $\mathbf{v}$ and the three Eulerian angles in which R was expressed. Selvik needed an initial approximation for these variables.

Our method comes close to Selvik's method. The ex-

pression to be minimized is the same but the solution process is different. Three or more non-collinear points are used and no initial approximation is needed.

DETERMINATION OF THE ROTATION MATRIX R AND THE TRANSLATIONAL VECTOR $\mathbf{v}$

The movement of a rigid body from a position 1 into another position 2 can be characterized by a translation vector $\mathbf{v}$ and a rotation matrix R . This matrix is orthogonal and therefore satisfies

$$R^T R = I, \quad (1)$$

where I is the 3×3 unit matrix and the superscript T indicates transposition.

Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ denote the radius vectors of n ($n \geq 3$) non-collinear points $P_1, P_2, \dots, P_n$ of the body in position 1, then the radius vectors $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ of these points in position 2 are given by

$$\mathbf{q}_i = R\mathbf{a}_i + \mathbf{v} \quad \text{for } i = 1, 2, \dots, n. \quad (2)$$

R and $\mathbf{v}$ are unknown and must be determined from the measured radius vectors $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ of $P_1, P_2, \dots, P_n$ in position 2. In general these vectors will differ from the exact vectors $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$. An overall measure for this difference is given by the function f of $\mathbf{v}$ and R , defined by:

$$f(\mathbf{v}, R) = \frac{1}{n} \sum_{i=1}^n (R\mathbf{a}_i + \mathbf{v} - \mathbf{p}_i)^T (R\mathbf{a}_i + \mathbf{v} - \mathbf{p}_i). \quad (3)$$

Introducing average vectors $\mathbf{a}$ and $\mathbf{p}$, a matrix M and a scalar quantity f_0 :

$$\mathbf{a} = \frac{1}{n} \sum_{i=1}^n \mathbf{a}_i; \quad \mathbf{p} = \frac{1}{n} \sum_{i=1}^n \mathbf{p}_i; \quad (4)$$

$$M = \frac{1}{n} \sum_{i=1}^n (\mathbf{p}_i \mathbf{a}_i^T) - \mathbf{p} \mathbf{a}^T \quad (5)$$

$$f_0 = \frac{1}{n} \sum_{i=1}^n (\mathbf{a}_i^T \mathbf{a}_i + \mathbf{p}_i^T \mathbf{p}_i) - (\mathbf{a}^T \mathbf{a} + \mathbf{p}^T \mathbf{p}), \quad (6)$$

the expression for $f(\mathbf{v}, R)$ can be written as:

$$f(\mathbf{v}, R) = f_0 + (R\mathbf{a} + \mathbf{v} - \mathbf{p})^T (R\mathbf{a} + \mathbf{v} - \mathbf{p}) - 2 \text{trace}(M^T R). \quad (7)$$

Here $\text{trace}(M^T R)$ is equal to the sum of the components on the main diagonal of the 3×3 matrix $M^T R$.

The Lagrangian multiplier theorem is used to determine the matrix R and the vector $\mathbf{v}$ that minimize f under the constraint condition (1). To use this theorem a 3×3 matrix S of Lagrangian multipliers and a function F of $\mathbf{v}$, R and S are introduced:

$$F(\mathbf{v}, R, S) = f(\mathbf{v}, R) + \text{trace}(S(R^T R - I)). \quad (8)$$

The above mentioned theorem now states that, if $F = F(\mathbf{v}, R$,

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S is stationary for some $\mathbf{v}$, R and S then $f=f(\mathbf{v}, R)$ is stationary and (1) is satisfied for that $\mathbf{v}$ and R . Stationary points of $F(\mathbf{v}, R, S)$ are found by requiring the first variation δF of $F(\mathbf{v}, R, S)$ to be zero for each variation $\delta \mathbf{v}$, δR and δS of $\mathbf{v}$, R and S :

$$\begin{aligned} \delta F = & 2(\mathbf{R}\mathbf{a} + \mathbf{v} - \mathbf{p})^T (\delta \mathbf{R}\mathbf{a} + \delta \mathbf{v}) \\ & - 2 \text{trace}(\mathbf{M}^T \delta \mathbf{R}) + \text{trace}(\delta \mathbf{S}(\mathbf{R}^T \mathbf{R} - \mathbf{I})) \\ & + \text{trace}(\mathbf{S}(\delta \mathbf{R}^T \mathbf{R} + \mathbf{R}^T \delta \mathbf{R})). \end{aligned} \quad (9)$$

Requiring $\delta F = 0$ for each δS results in the constraint condition (1). Next, $\delta F = 0$ for each $\delta \mathbf{v}$ gives an equation for $\mathbf{v}$:

$$\mathbf{v} = \mathbf{p} - \mathbf{R}\mathbf{a}. \quad (10)$$

Finally, using (10) and $\text{trace}(\mathbf{S}\delta \mathbf{R}^T \mathbf{R}) = \text{trace}(\mathbf{S}^T \mathbf{R}^T \delta \mathbf{R})$ the requirement $\delta F = 0$ for each δR leads to the matrix equation

$$\mathbf{M} = \frac{1}{2} \mathbf{R}(\mathbf{S} + \mathbf{S}^T). \quad (11)$$

It is easy to solve equation (11) for the symmetric matrix $\frac{1}{2}(\mathbf{S} + \mathbf{S}^T)$. With

$$\bar{\mathbf{S}} = \frac{1}{2}(\mathbf{S} + \mathbf{S}^T), \quad (12)$$

and with $\mathbf{M} = \mathbf{R}\bar{\mathbf{S}}$ and equation (1) it follows:

$$\bar{\mathbf{S}}^2 = \bar{\mathbf{S}}^T \mathbf{R}^T \bar{\mathbf{S}} = \mathbf{M}^T \mathbf{M}. \quad (13)$$

$\mathbf{M}^T \mathbf{M}$ is a symmetric matrix with eigenvalues $D_{11}^2 \geq D_{22}^2 \geq D_{33}^2 \geq 0$ and a corresponding set of three orthonormal eigenvectors. The eigenvalues are arranged on the principal diagonal of a diagonal matrix D^2 while the eigenvectors are considered as the columns of a 3×3 matrix V . From the definition of eigenvalues and eigenvectors it is seen that

$$\mathbf{M}^T \mathbf{M} = \bar{\mathbf{S}}^2 = \mathbf{V} D^2 \mathbf{V}^T; \quad \mathbf{V} \mathbf{V}^T = \mathbf{I}. \quad (14)$$

A solution for the symmetric matrix $\bar{\mathbf{S}}$ is therefore given by

$$\bar{\mathbf{S}} = \mathbf{V} D \mathbf{V}^T, \quad (15)$$

where the signs of the principal diagonal components D_{11} , D_{22} and D_{33} of D are up to now indeterminate. Insertion of this solution into $\mathbf{M} = \mathbf{R}\bar{\mathbf{S}}$ gives:

$$\mathbf{M} = \mathbf{R} \mathbf{V} D \mathbf{V}^T. \quad (16)$$

The signs of D_{11} , D_{22} and D_{33} follow the condition that $f(\mathbf{v}, R)$ must be minimal. With (7), (10) and (16) it follows

$$f = f_0 - 2 \text{trace}(\mathbf{V} D \mathbf{V}^T) = f_0 - 2(D_{11} + D_{22} + D_{33}), \quad (17)$$

and in order to make f minimal D_{11} , D_{22} and D_{33} must be chosen non-negative.

To elucidate the geometrical significance of D_{11} , D_{22} and D_{33} the measured vector $\mathbf{p}_i$ ($i = 1, 2, \dots, n$) is related to the exact vector $\mathbf{q}_i = \mathbf{R}\mathbf{a}_i + \mathbf{v}$ by writing:

$$\mathbf{p}_i = \mathbf{R}\mathbf{a}_i + \mathbf{v} + \delta_i \quad \text{for } i = 1, 2, \dots, n. \quad (18)$$

The error vector δ_i is due to measuring errors and to the fact that the body is not perfectly rigid. Insertion of (18) in (4) and (5) leads to:

$$\mathbf{M} = \mathbf{R}(\mathbf{M}_0 + \mathbf{E}), \quad (19)$$

where $\mathbf{M}_0$ depends only on the radius vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ before the movement while $\mathbf{E}$ depends on $\mathbf{R}, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ and $\delta_1, \delta_2, \dots, \delta_n$:

$$\mathbf{M}_0 = \frac{1}{n} \sum_{i=1}^n \{(\mathbf{a}_i - \mathbf{a})(\mathbf{a}_i - \mathbf{a})^T\};$$

$$\mathbf{E} = \frac{1}{n} \sum_{i=1}^n \{R^T \delta_i (\mathbf{a}_i - \mathbf{a})^T\}. \quad (20)$$

$\mathbf{M}_0$ is a symmetric, (semi) positive definite matrix. Therefore, $\mathbf{M}_0$ can be written as

$$\mathbf{M}_0 = \mathbf{V}_0 D_0 \mathbf{V}_0^T; \quad \mathbf{V}_0 \mathbf{V}_0^T = \mathbf{I}, \quad (21)$$

where $\mathbf{V}_0$ is the matrix of eigenvectors of $\mathbf{M}_0$ and D_0 is a diagonal matrix with the non-negative eigenvalues $D_{01} \geq D_{02} \geq D_{03} \geq 0$ of $\mathbf{M}_0$. It can easily be proved that all eigenvalues are unequal zero if and only if more than three

non-coplanar markers are used. If all markers are lying in one plane then $D_{03} = 0$, whereas $D_{02} = D_{03} = 0$ if all markers are collinear.

With (19) and (21) the eigenvalue problem (14) can be written in a form that is more suitable for further analysis:

$$\mathbf{V} D^2 \mathbf{V}^T = \mathbf{V}_0 D_0^2 \mathbf{V}_0^T + \mathbf{M}_0^T \mathbf{E} + \mathbf{E}^T \mathbf{M}_0 + \mathbf{E}^T \mathbf{E}. \quad (22)$$

From this it is clear that $\mathbf{V} = \mathbf{V}_0$, $D = D_0$ give a solution of (14) if $\mathbf{E} = 0$. In practice, $\mathbf{E}$ will be unequal 0 but very small compared with $\mathbf{M}_0$. Then $\mathbf{V}_0$ and D_0 will be realistic approximations for $\mathbf{V}$ and D . Using (22) and (20) it is possible to establish bounds for the differences between $\mathbf{V}$ and $\mathbf{V}_0$ and between D and D_0 if bounds for the error vectors $\delta_1, \delta_2, \dots, \delta_n$ are given. This will not be analyzed here.

The rotation matrix R can be determined reliably from (16) if and only if at least two of the eigenvalues $D_{11} \geq D_{22} \geq D_{33}$ differ significantly from zero. From the foregoing it is seen that this will be the case if three or more non-collinear points are used. From (16) follows:

$$\mathbf{R} \mathbf{V} D = \mathbf{M} \mathbf{V} = [\mathbf{m}_1 \quad \mathbf{m}_2 \quad \mathbf{m}_3], \quad (23)$$

where $\mathbf{m}_i$ ($i = 1, 2, 3$) represents column i of $\mathbf{M} \mathbf{V}$. As D is a diagonal matrix $\mathbf{m}_i$ is equal to column i of $\mathbf{R} \mathbf{V}$, multiplied by D_{ii} . So, if D_{11} and D_{22} differ significantly from zero (and this will always be true in realistic situations) column 1 and column 2 of $\mathbf{R} \mathbf{V}$ are equal to $(1/D_{11}) \cdot \mathbf{m}_1$ and $(1/D_{22}) \cdot \mathbf{m}_2$ respectively. The third column of $\mathbf{R} \mathbf{V}$ follows from the observation that $\mathbf{R} \mathbf{V}$ is orthogonal. Therefore the third column of $\mathbf{R} \mathbf{V}$ is equal to the cross product $(1/D_{11} \cdot D_{22}) \mathbf{m}_1 \times \mathbf{m}_2$ of the first two columns. So, the final result for R is given by:

$$\mathbf{R} = \left(\frac{1}{D_{11}} \mathbf{m}_1 \quad \frac{1}{D_{22}} \mathbf{m}_2 \quad \frac{1}{D_{11} \cdot D_{22}} \mathbf{m}_1 \times \mathbf{m}_2 \right) \cdot \mathbf{V}^T. \quad (24)$$

For the calculations of R and $\mathbf{v}$ are sufficient: (4) and (5) to determine $\mathbf{M}$, a subroutine for the eigenvalues and eigenvectors of $\mathbf{M}^T \mathbf{M}$, the calculation of the columns $\mathbf{m}_1, \mathbf{m}_2$ and $\mathbf{m}_3$ of $\mathbf{M} \mathbf{V}$ and finally (24) to determine R and (10) for $\mathbf{v}$.

DETERMINATION OF THE HELICAL AXIS

In the foregoing the movement of the body was characterized by the rotation matrix R and the translation vector $\mathbf{v}$. Such a movement can also be considered the result of a rotation through an angle ϕ about the helical axis and a translation t along this axis. Let $\mathbf{n}$ denote a unit vector along the helical axis and let $\mathbf{s}$ be the radius vector of a point on this axis, such that $\mathbf{n}$ and $\mathbf{s}$ are orthogonal:

$$\mathbf{n}^T \mathbf{n} = 1; \quad \mathbf{n}^T \mathbf{s} = 0. \quad (25)$$

The sense of rotation and the direction of $\mathbf{n}$ will correspond with the right-hand screw rule and ϕ will always be non-negative and less than or equal π rad.

The connection between both descriptions of the movement of the body is given by the requirement that

$$\mathbf{R}\mathbf{w} + \mathbf{v} = \mathbf{w} + t\mathbf{n} + (1 - \cos \phi) \mathbf{n} * (\mathbf{n} * (\mathbf{w} - \mathbf{s})) + \sin \phi \mathbf{n} * (\mathbf{w} - \mathbf{s}) \quad (26)$$

must hold for every vector $\mathbf{w}$. Consequently:

$$\mathbf{v} = t\mathbf{n} + (1 - \cos \phi) \mathbf{s} - \sin \phi \mathbf{n} * \mathbf{s}. \quad (27)$$

$$\mathbf{R}\mathbf{w} = \cos \phi \mathbf{w} + (1 - \cos \phi) \mathbf{n} \mathbf{n}^T \mathbf{w} + \sin \phi \mathbf{n} * \mathbf{w} \quad \text{for every } \mathbf{w}, \quad (28)$$

where the last equation is seen to be equivalent with:

$$\frac{1}{2}(\mathbf{R} - \mathbf{R}^T) \mathbf{w} = \sin \phi \mathbf{n} * \mathbf{w} \quad \text{for every } \mathbf{w} \quad (29)$$

$$\frac{1}{2}(\mathbf{R} + \mathbf{R}^T) = \cos \phi \mathbf{I} + (1 - \cos \phi) \mathbf{n} \mathbf{n}^T. \quad (30)$$

The matrix $\frac{1}{2}(\mathbf{R} - \mathbf{R}^T)$ is skew-symmetric and it can easily be shown that $\sin \phi \mathbf{n}$ is given by:

$$\sin \phi \mathbf{n} = \frac{1}{2} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix} \quad (31)$$

With $\mathbf{n}^T \mathbf{n} = 1$ and $\sin \phi \geq 0$ this equation can be solved for $\sin \phi$, which results in:

$$\sin \phi = \frac{1}{2} \sqrt{(R_{32} - R_{23})^2 + (R_{13} - R_{31})^2 + (R_{21} - R_{12})^2}. \quad (32)$$

Apart from this $\cos \phi$ can be calculated from (30) by adding up the components on the principal diagonal of the matrices. Then:

$$3 \cos \phi + (1 - \cos \phi) \text{trace}(\mathbf{n}\mathbf{n}^T) = \text{trace}(\frac{1}{2}(\mathbf{R} + \mathbf{R}^T)), \quad (33)$$

and because of $\text{trace}(\mathbf{n}\mathbf{n}^T) = \mathbf{n}^T \mathbf{n} = 1$ it follows:

$$\cos \phi = \frac{1}{2}(R_{11} + R_{22} + R_{33} - 1). \quad (34)$$

Both equations (32) and (34) can be used to calculate ϕ . For numerical reasons it is preferred to use (32) if $\sin \phi \leq \frac{1}{2}\sqrt{2}$ and (34) if $\sin \phi > \frac{1}{2}\sqrt{2}$.

As soon as $\sin \phi$ is known $\mathbf{n}$ can be determined from (31) if $\sin \phi \neq 0$. From a numerical point of view this is not recommendable if ϕ approaches π . With known $\cos \phi$ it is preferred to use (30) if $\phi > \frac{3}{4}\pi$. While:

$$(1 - \cos \phi)\mathbf{n}\mathbf{n}^T = \frac{1}{2}(\mathbf{R} + \mathbf{R}^T) - \cos \phi \mathbf{I} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3], \quad (35)$$

it is seen that each of the columns $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ of the matrix $\frac{1}{2}(\mathbf{R} + \mathbf{R}^T) - \cos \phi \mathbf{I}$ is a vector in the same direction as $\mathbf{n}$. So, apart from a factor, $\mathbf{n}$ is equal to $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$. Let $\mathbf{b}_i$ ($i = 1, 2$ or 3) be the column with the greatest length $\sqrt{\mathbf{b}_i^T \mathbf{b}_i}$. Then $\mathbf{n}$ is determined by:

$$\mathbf{b}_i^T \mathbf{b}_i = \max(\mathbf{b}_1^T \mathbf{b}_1, \mathbf{b}_2^T \mathbf{b}_2, \mathbf{b}_3^T \mathbf{b}_3); \quad \mathbf{n} = \pm \frac{\mathbf{b}_i}{\sqrt{\mathbf{b}_i^T \mathbf{b}_i}}. \quad (36)$$

The sign of $\mathbf{n}$ must be chosen such that $\sin \phi$ in (31) is positive.

The translation t along the helical axis and the radius vector $\mathbf{s}$ of a point on this axis follow from (25) and (27):

$$t = \mathbf{n}^T \mathbf{v} \quad (37)$$

$$\mathbf{s} = -\frac{1}{2}\mathbf{n} * (\mathbf{n} * \mathbf{v}) + \frac{\sin \phi}{2(1 - \cos \phi)} \mathbf{n} * \mathbf{v}. \quad (38)$$

The relations (36), (37) and (38) hold if $\phi \neq 0$. If $\phi = 0$ there is no rotation at all. In that case the helical axis is not defined and therefore $\mathbf{n}$, $\mathbf{s}$ and t are not unique. If $\phi = 0$ and $\mathbf{v} \neq 0$ then one can put:

$$t = \sqrt{\mathbf{v}^T \mathbf{v}}; \quad \mathbf{n} = \frac{1}{t} \mathbf{v}; \quad \mathbf{s} = 0. \quad (39)$$

If $\phi = 0$ and $\mathbf{v} = 0$ there is no movement at all. This case is of no interest.

Sufficient for the computation of $\mathbf{n}$, ϕ , $\mathbf{s}$ and t are (32) and (34) for ϕ , (31) and (35) until (38) for $\mathbf{n}$, t and $\mathbf{s}$ if $\phi \neq 0$ and (39) if $\phi = 0$.

DISCUSSION

Accurate results are quickly achieved by the described method. Although alternative methods exist, the described method seems more elegant and needs no initial approximation. The analysis is rather long but only few of its equations are required to achieve numerical results.

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NOMENCLATURE

- $\mathbf{a}$ average of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$
- $\mathbf{a}_i$ ($i = 1, 2, \dots, n$): radius vector of marker i before the movement
- $\mathbf{b}_i$ column i of $\frac{1}{2}(\mathbf{R} + \mathbf{R}^T) - \cos \phi \mathbf{I}$
- $\mathbf{D}$ diagonal matrix of square roots of eigenvalues of $\mathbf{M}^T \mathbf{M}$
- $\mathbf{D}_0$ diagonal matrix of eigenvalues of $\mathbf{M}_0$
- $\mathbf{D}_{11}, \mathbf{D}_{22}, \mathbf{D}_{33}$: square roots of eigenvalues of $\mathbf{M}^T \mathbf{M}$
- $\mathbf{D}_{01}, \mathbf{D}_{02}, \mathbf{D}_{03}$: eigenvalues of $\mathbf{M}_0$
- $\mathbf{E}$ error matrix, see (20)
- f to be minimized sum of squares
- F like f , but with Lagrangian multipliers
- $\mathbf{I}$ unit matrix of order 3×3
- $\mathbf{M}$ matrix of order 3×3 , see (5)
- $\mathbf{M}_0$ matrix of order 3×3 , see (20)
- $\mathbf{m}_i$ column i of $\mathbf{M}\mathbf{V}$
- n number of markers
- $\mathbf{n}$ unit vector along helical axis
- $\mathbf{p}$ average of $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$
- $\mathbf{p}_i$ ($i = 1, 2, \dots, n$): measured radius vector of marker i after the movement
- $\mathbf{q}_i$ ($i = 1, 2, \dots, n$): exact radius vector of marker i after the movement
- $\mathbf{R}$ rotation matrix
- $\mathbf{s}$ radius vector of a point on the helical axis
- $\mathbf{S}$ matrix of Lagrangian multipliers
- $\bar{\mathbf{S}}$ symmetric part of $\mathbf{S}$
- t translation along helical axis
- $\mathbf{v}$ translation vector
- $\mathbf{V}$ orthogonal matrix of eigenvectors of $\mathbf{M}^T \mathbf{M}$
- $\mathbf{V}_0$ orthogonal matrix of eigenvectors of $\mathbf{M}_0$
- $\mathbf{w}$ arbitrary vector
- ϕ angle of rotation about the helical axis, $0 \leq \phi \leq \pi$
- δ_i ($i = 1, 2, \dots, n$): error vector, equal to $\mathbf{p}_i - \mathbf{q}_i$

Superscript T denotes transposition.

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