The prediction of tool pressures in coining

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THE PREDICTION OF TOOL Pressures IN COINING

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S. Hoogenboom

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THE PREDICTION OF TOOL PRESSURES IN COINING

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SUMMARY.

The coining process in its ultimate phase is considered. Theoretical models are developed which enable the prediction of the tool loads. The theoretical formulas are derived with the help of well known methods from the plasticity theory, upper bound- and slab method.

Two series of experiments are carried out: compression of a blank between flat punches to study the pressing of sharp edges at the rim of the coin, and compression of a blank with a punch provided with conical engravings.

1. INTRODUCTION.

Coining is, as well known, one of the first forming processes in the history of mankind. Yet only a few theoretical and experimental explorations have been made. A major contribution was made by Bocharov, Thomsen, Yang an Kobayashi (1), (2), several approximate solutions were given using the Coulomb friction model. Especially at the end of the coining process when the tool pressure rises quickly this friction model is not valid. An extension of the aforementioned work was given by Ogha and Kondo (3) with the application of extra holes to improve the filling of the corner. Furthermore an analysis was carried out by Kiran and Shaw (4) concerning a saw tooth pattern in the die and using the hodograph method.

The present work concerns two kinds of experiments. In the first series of measurements a flat punch is used and special attention is drawn to the filling of the edges in the corner zone (Fig. 1.1a).

![Image 1.1a](image)

a. Flat die and a circular edging of the blank.

b. A die with conical engravings.

Fig. 1.1.

In the second series the punch was provided with conical engravings (Fig. 1.1b). The relation between the mean tool pressure and the required local pressure to fill the engraving is studied.

The applied methods of analysis are the upper bound method, the slab method and a combination of both. In the analysis the constant-friction model is used, however, comparison with the measurements leads to the conclusion that the influence of the friction on the load is negligible; this is in accordance with findings of others (4, 5). There are some indications that friction still has an important influence on the sliding (displacements) of the material.

2. ANALYSIS AND EXPERIMENTS

2.1. The average punch load \( \sigma_p \) (flat punch).

In appendix A three expressions describing the theoretical relation between the mean punch load \( \sigma_p \) and the "radius" \( r_c \) of the corner zone are derived. In Fig. 2.1 the result of model A2 (eq. A20) is compared with empirical values, and with some calculated points derived from the finite element method (MARC). (see Table 2.1)

![Image 2.1](image)

Fig. 2.1. The mean punch load \( \sigma_p \) \( (= \sigma_p/\sigma_0) \) as a function of the relative radius of the corner zone \( \varrho \) \( (= r_c/s) \).
The experimental mean dimensionless punchload \( p_{\text{exp}} \) is determined through division of the measured force \( F \) by the area of the blank and the actual value of the yield stress, thus:

\[
(2.1) \quad p_{\text{exp}} = \frac{F}{R} \frac{E}{\sigma_y}
\]

using the exponential strain hardening rule:

\[
(2.2) \quad \sigma_y = C (\ln \frac{E}{\sigma_y})^n
\]

This simplified way to deal with the strain hardening can cause some inaccuracy as is demonstrated by the results of the finite element method. The difference between the results for lead and aluminium can be explained by the deformation in the corner zone which is two to three times higher than in the centre region of the blank (Fig. 2.2). Depending on the value of the strain hardening coefficient this can cause a deviation of 20% and even more.

The difference between theory, including F.E.M., and experimental values for very small radii \( \varphi < 0.04 \) can be explained by the elastic behaviour of the coining ring.

![Fig. 2.2. A distorted grid with strain concentration in the corner zone.](image)

The relevant data of the materials used in the experiments are given in Table 2.1.

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Table 2.1. The material data.

2.2. The average load on the coining ring \( \sigma_0 \).

In appendix B a formula (B3) for the load on the coining ring is given. In figure 2.3 theory and experiments are compared.

![Fig. 2.3. The load on the coining ring \( \sigma_0 \) as a function of the relative radius \( \varphi = (r_c/s) \).](image)

The experimental method applied for measuring the toolload \( \sigma_0 \) is probably the cause of the high inaccuracy. The load \( \sigma_0 \) was in fact measured in an indirect way by registering the alteration of the outer diameter of the ring under pressure. From this the inner average pressure was calculated. From a theoretical point of view equation (B3) represents an absolute minimum.

2.3. The filling of the engraving.

An expression for the stress \( \sigma_{ze} \) required to fill the engraving is given in appendix C (equation C12). For the experimental points in the figure 2.4 the diameters \( d \) and \( d' \) were measured. The local required stress \( \sigma_{ze} \) was derived from the average punchload \( p_{\text{exp}} \).

There is a rather large spreading of the experimental points. Although, until now, a satisfactory explanation has not been found, we assume that local disturbances in the lubrication are responsible for this behaviour. On the one hand a high friction in the conical engraving can lead to a shortage of material with inward flow \( (D_e = d) \) and so filling hardly occurs. On the other hand a hindrance in the radial flow of the material along the punch surface to the rim of the coin, will cause a great width \( (D_e > d) \) of the influenced zone. Therefore causing a well filled engraving but requiring a high pressure \( \sigma_{ze} \). This could also be one of the reasons for the rather unpredictability of the life endurance of coining-tools, especially when using mechanical presses that exert the material flow.
CONCLUSIONS.

- In coining toolloads can reach high values of several times (5 to 10) the yield stress of the product material, therefore a correct tool design is required.

- According to the theoretical results the toolload at the very end of the process will tend to infinity. In practice the tools and press will deform elastically. Therefore a complete theoretical description of the process should take this elastic behaviour into account.

- The influence of the friction is not unequivocal. Because of the relative small slip the friction shearstress will be small, but minor local disturbances of the lubrication can cause high local pressures or poor filling of the engraving.

- Investigation of the reasons behind the spread in the required stresses (σₜₑ; Fig. 2.4) would probably enlighten some of the backgrounds behind the unpredictability of tool life-endurance.

ACKNOWLEDGEMENTS.

The authors wish to thank Mrs. Borg and Mr. Van der Meulen for typing of the manuscript and making the drawings. Also we like to thank Mr. Saeets and Mr. Aarts for the experimental part of this investigation.

REFERENCES.


APPENDIX A.

Application of the upperbound theorem to the corner zone.

Introduction.

The upperbound theorem states that among all kinematically admissible velocity fields the one that occurs minimizes the expression (2):

\[ P = P_D + P_T + P_F \]

where

\[ P_D = \oint \sigma_{ij} \frac{\partial u}{\partial x_j} \, dv, \] the internal power dissipation, summation takes place over the subdomain.

\[ P_T = \oint \frac{\partial u}{\partial x_j} \, dS_T, \] the power dissipation on the discontinuity surfaces.

\[ P_F = \oint \sigma_{ij} \left( \frac{\partial u}{\partial x_j} \right)^2 \, dS_T, \] the power dissipation on the contactsurfaces between tool and blank.
Dividing the total power $P$ by $\pi R^2 a_U \bar{u}$ gives the dimensionless mean tool pressure.

\[(A5) \quad \bar{P} = \frac{P}{\pi R^2 a_U \bar{u}}\]

Three different velocity fields will be examined, with the following assumptions in common (Fig. A1):
- axial symmetry,
- plane strain ($i_w = 0$) in the corner zone because of $r_c < R$.

In all three models the workpiece is divided into three subdomains.

**Model A1.**

For the regions I, II and III it is assumed that:
- region I $\bar{u}_x \neq \bar{u}_x(z)$
- region II $\bar{u}_x = 0$
- region III $u_z = 0$

Because of invariancy of volume, this leads to:

\[(A6) \quad \bar{u}_z = \frac{1}{2} \bar{u}_x \quad \text{and} \quad \bar{u}_z = -\bar{z} \bar{z}
\]

\[(A7) \quad \bar{u}_z = \frac{R}{R} \bar{u}_x\]

\[(A8) \quad \bar{u}_z = \frac{1}{2} \bar{u}_x\]

Thus with $\bar{u} = r_c/s$ it follows for the whole blank:

\[(A9) \quad P_{D1} = s R^2 a_U \dot{\bar{u}}\]

\[(A10) \quad P_{R1} = \pi R^2 a_U \bar{u} \frac{\sqrt{3}}{2} R\]

\[(A11) \quad P_{R2} = \pi R^2 a_U \bar{u} \frac{\sqrt{3}}{2} (\frac{1}{2} + 2\bar{u})\]

Because the friction losses can be neglected ($4, \bar{u}$), the dimensionless mean tool pressure is calculated as:

\[(A12) \quad \bar{p}_{D1} = 1 + \frac{\sqrt{3}}{2} \frac{R}{2} + \frac{1}{2} (\frac{1}{2} + 2\bar{u})\]

**Model A2.**

It is assumed that:
- region I $\bar{u}_x \neq \bar{u}_x(z)$
- region II $\bar{u}_x \neq \bar{u}_x(r)$
- region III $u_z = 0$

Because of continuity of volume it follows that:

\[(A13) \quad \bar{u}_z = \frac{R}{R} \bar{u}_x \quad \text{and} \quad \bar{u}_z = -\bar{z} \bar{z}
\]

\[(A14) \quad \bar{u}_z = \frac{1}{2} \bar{u}_x \quad \text{(and) } \bar{u}_z = \frac{R}{R} \bar{u}_x
\]

\[(A15) \quad \bar{u}_z = \frac{1}{2} \bar{u}_x \quad \text{(and) } \bar{u}_z = \frac{R}{R} \bar{u}_x
\]

With $\bar{u}_z = 0$ this leads to the following power terms:

\[(A16) \quad P_{D1} = \pi R^2 a_U \bar{u}\]

\[(A17) \quad P_{DII} = \pi R^2 a_U \bar{u} \frac{\sqrt{3}}{2} (1 - 2\bar{u})\]

\[(A18) \quad P_{R1,3} = \pi R^2 a_U \bar{u} \frac{1}{2} (\frac{1}{2} - 2\bar{u}) + 1 - 2\bar{u} + \frac{3}{2} \frac{R}{R}\]

\[(A19) \quad P_{R2} = \pi R^2 a_U \bar{u} \frac{\sqrt{3}}{2} \bar{u}\]

So, for the mean tool pressure it follows:

\[(A20) \quad \bar{p}_{D1} = 1 + \bar{u} \frac{1}{2} (2 - 4\bar{u} + \frac{1}{2} \frac{R}{R})\]

**Model A3.**

For the regions I, II and III it is assumed that:
- region I $\bar{u}_x \neq \bar{u}_x(z)$
- region II $\bar{u}_x \neq \bar{u}_x(r)$
- region III $u_z = 0$

Because of continuity of volume it follows that:

\[(A13) \quad \bar{u}_z = \frac{R}{R} \bar{u}_x \quad \text{and} \quad \bar{u}_z = -\bar{z} \bar{z}
\]

\[(A14) \quad \bar{u}_z = \frac{1}{2} \bar{u}_x \quad \text{(and) } \bar{u}_z = \frac{R}{R} \bar{u}_x
\]

\[(A15) \quad \bar{u}_z = \frac{1}{2} \bar{u}_x \quad \text{(and) } \bar{u}_z = \frac{R}{R} \bar{u}_x
\]

With $\bar{u}_z = 0$ this leads to the following power terms:

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\[(A17) \quad P_{DII} = \pi R^2 a_U \bar{u} \frac{\sqrt{3}}{2} (1 - 2\bar{u})\]

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\[(A19) \quad P_{R2} = \pi R^2 a_U \bar{u} \frac{\sqrt{3}}{2} \bar{u}\]

So, for the mean tool pressure it follows:

\[(A20) \quad \bar{p}_{D1} = 1 + \bar{u} \frac{1}{2} (2 - 4\bar{u} + \frac{1}{2} \frac{R}{R})\]
THE PREDICTION OF TOOL PRESSURES IN COINING

It is assumed that:

region I \( \dot{u}_x = \dot{u}_z = 0 \)

region II \( \dot{u}_x = \frac{2a}{s-2t} \) (\( \dot{u}_{II} \) parallel to \( r_2 \))

region III \( \dot{u}_z = 0 \) (dead zone)

The parameter \( a \) is a pseudo-independent parameter \((\frac{a}{s})\) which minimizes the total power of deformation. With \( a \ll R \) it holds:

(A21) \( P_{DI} = \pi R^2 \sigma_y \dot{u} \)

(A22) \( P_{II} = \pi R^2 \sigma_y \dot{u} \left( \frac{1}{2} \right) \left( \frac{1}{2} s^2 - 2 \varphi \right) \)

(A23) \( P_{III} = \pi R^2 \sigma_y \dot{u} \frac{1}{2} \left( \frac{1}{2} s^2 - 2 \varphi \right) \)

where \( \delta = \frac{a}{s} \)

Summation over the whole blank yields to:

(A24) \( P_0 = \sigma_y \int_0^R \left( \frac{1}{2} \right) \left( \frac{1}{2} s^2 - 2 \varphi \right) \)

Differentiating with respect to \( \delta \) and setting the derivative equal to zero gives the optimum value of \( \delta \):

(A25) \( \delta_{opt} = \frac{1}{2} \left( \frac{1}{2} s^2 - 2 \varphi \right) \)

and

(A26) \( \sigma_{opt} = \pi R^2 \sigma_y \dot{u} \left( \frac{1}{2} \right) \left( \frac{1}{2} s^2 - 2 \varphi \right) + \left( 2 - 3 \varphi \right) \left( \frac{1}{2} s^2 - 2 \varphi \right) \)

APPENDIX C.

The filling of a conical engraving.

In Fig. A4 the three theoretical models are compared. Theoretically the best solutions are (A12) for \( \varphi < 0.17 \) and (A26) for \( \varphi > 0.17 \). Because the formation of a dead zone for \( \varphi > 0.17 \) is quite doubtful and the interesting area is for \( \varphi < 0.2 \) we consider (A12) as the best solution.

APPENDIX B.

The radial stress on the coining ring.

Equilibrium of forces on the slab gives:

(B1) \( d_0^z = \frac{1}{2} \frac{1}{2} \sigma_y \frac{da}{c} \)

with \( \sigma_y(z = s/2 - r_2) = 0 \) it follows:

(B2) \( \sigma_y = \frac{1}{2} \frac{1}{2} \frac{z - s/2 + r_2}{r_2} \)

And for the average pressure \( \sigma_m \) on the coining ring it holds:

(B3) \( \sigma_{m} = \frac{1}{2} \frac{1}{2} \left( 1 + \frac{1}{2} \sigma_y (1 - 2 \varphi) \right) \)

APPENDIX C.

The filling of a conical engraving.

In Fig. C1 the filling of a conical engraving.
In this analysis a combination of the upperbound (power balance) and the slabmethod is applied. The zone providing material for the filling of the engraving is characterized by its width \(D_e\) and its depth \(s_e\). Three regions are distinguished:

- Subdomain I, the conical engraving itself. The required stress \(\sigma_c\) is calculated with the help of the slabmethod. The dimensionless stress \(\sigma_c/\sigma_v\) becomes:

\[
(C1) \quad \sigma_c/\sigma_v = 2(1 + D/D_e) \sin \alpha / \ln \left( \frac{D}{d} \right)
\]

- Subdomain II and III with inward materialflow. The concerning velocity fields are:

\[
(C2) \quad \dot{u}_{III} = -2 \dot{u}_e \frac{D}{d} \quad \text{and} \quad \dot{u}_{III} = 4 \dot{u}_e \frac{D}{d}
\]

\[
(C3) \quad \dot{u}_{III} = \frac{1}{\frac{D}{d}} \dot{u}_e \left( \frac{\pi}{2} \frac{D}{d} - \sqrt{\frac{D}{d}} \right) \quad \text{and} \quad \dot{u}_{III} = - \dot{u}_e \frac{D}{d}
\]

where \(\dot{u}_e = \dot{u}_{III} (r = d/2)\).

From this the deformation power in region II and the shearpowers on the discontinuity surfaces \(\Gamma_1\) and \(\Gamma_2\) can be calculated:

\[
(C4) \quad P_{II} = \sigma_e \frac{d}{s_e} \quad \sigma_v \dot{u}_e
\]

\[
(C5) \quad P_{\Gamma_1} = \sigma_e \frac{d}{s_e} \quad \sigma_v \dot{u}_e \frac{D}{d}
\]

\[
(C6) \quad P_{\Gamma_2} = \sigma_e \frac{d}{s_e} \quad \sigma_v \dot{u}_e \frac{1}{\frac{D}{d}} \frac{d}{s_e}
\]

Considering the radial stress \(\sigma_e\) as a load compressing the material in region II, then for \(\sigma_e/\sigma_v\) it follows:

\[
(C7) \quad \sigma_e/\sigma_v = 1 + \frac{2}{\frac{D}{d}} \frac{\dot{u}_e}{\dot{u}_v} + \frac{2}{\frac{D}{d}} \left( 1 + \frac{2}{\frac{D}{d}} \right) \sin \alpha \ln \left( \frac{D}{d} \right)
\]

The stress distribution in region III is calculated with the help of the slab method. Equilibrium of forces on the slab yields to:

\[
(C8) \quad \sigma_e + (\sigma_e - \sigma_v) \frac{\dot{u}_v}{\dot{u}_e} + \frac{1}{\frac{D}{d}} \frac{\sigma_v}{s_e} \frac{d}{s_e} = 0
\]

With the Levy-von Mises stress-strain relations the following is obtained:

\[
(C9) \quad \sigma_e = \frac{1}{3} \left( e - \dot{e}_e \right) \dot{e}_e
\]

With eq. \((C3)\) the strain rates \(\dot{e}_e, \dot{e}_v\) and \(\dot{e}\) can be calculated, substituting in \((C9)\) and rearranging eq. \((C8)\) leads to:

\[
(C10) \quad \sigma_e = \sigma_e + \frac{1}{\frac{D}{d}} \frac{\sigma_v}{s_e} \left( \frac{D}{d} \right)^2 \dot{u}_v - \frac{1}{\frac{D}{d}} \frac{\sigma_v}{s_e} \dot{e}_e - \frac{1}{\frac{D}{d}} \frac{\sigma_v}{s_e} \dot{e}_e - \frac{1}{\frac{D}{d}} \frac{\sigma_v}{s_e} \dot{e}_e
\]

For \(r = D_e/2\) the axial stress \(\sigma_{III}\) has to be equal to the local punch pressure \(\sigma_{pII}\). With \(\sigma_{III} (r = D_e/2) = 0\) and the Von Mises yield criterium the axial pressure \(\sigma_{III}\) required for the filling of the engraving can be derived:

\[
(C11) \quad \sigma_{III} = \frac{D_e}{D_e/2} + \frac{\sigma_v}{\sigma_v} + \frac{1}{\frac{D}{d}} \frac{D_e}{\sigma_v} + \frac{1}{\frac{D}{d}} \ln \left( \frac{D}{d} \right) - \frac{1}{\frac{D}{d}} \frac{D_e/2}{\sigma_v} + \frac{1}{\frac{D}{d}} \ln \left( \frac{D}{d} \right) - \frac{1}{\frac{D}{d}} \frac{D_e/2}{\sigma_v} + \frac{1}{\frac{D}{d}} \ln \left( \frac{D}{d} \right) + \frac{1}{\frac{D}{d}} \frac{D_e/2}{\sigma_v} - \frac{1}{\frac{D}{d}} \ln \left( \frac{D}{d} \right) - \frac{1}{\frac{D}{d}} \frac{D_e/2}{\sigma_v}
\]

\[
\text{Equation (C11) can be simplified to:}
\]

\[
(C12) \quad \sigma_{III} = A + B \ln \left( \frac{D}{d} \right)
\]

\[
\text{Where:}
\]

\[
(C13) \quad A = 1 + \frac{1}{\frac{D}{d}} \left[ \frac{2}{\frac{D}{d}} \frac{1}{\frac{D}{d}} \frac{1}{\frac{D}{d}} - 1 \right] \ln \left( \frac{D}{d} \right)
\]

\[
(C14) \quad B = 2 \left( 1 + \frac{1}{\frac{D}{d}} \right) \left( a = 60° \right)
\]

The width \(D_e\) of the deformation zone is related to the axial geometry of the blank. An optimisation of the total power with respect to this parameter is not carried out. However, with a given width \(D_e\) the parameter \(A\) can be written as a function of the depth \(s_e\) of the zone. This function has a minimum, and in accordance with the upperbound theorie we consider this minimum as the best solution. Table C1 gives a number of calculated values of the depth \(s_e/\sigma_v\) and the parameter \(A\) in the optimum situation.

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Table C1. The calculated values of the optimum \(s_e/\sigma_v\) and \(A\).

When the depth of the deformation zone equals the thickness \(s\) of the blank the parameter \(A\) becomes:

\[
(C15) \quad A(s_e = s) = 1 + \frac{1}{\frac{D}{d}} \left[ \frac{2}{\frac{D}{d}} \frac{1}{\frac{D}{d}} \frac{1}{\frac{D}{d}} - 1 \right] \ln \left( \frac{D}{d} \right) + \frac{1}{\frac{D}{d}} \frac{D_e/2}{\sigma_v} + \frac{1}{\frac{D}{d}} \ln \left( \frac{D}{d} \right) + \frac{1}{\frac{D}{d}} \frac{D_e/2}{\sigma_v} - \frac{1}{\frac{D}{d}} \ln \left( \frac{D}{d} \right) - \frac{1}{\frac{D}{d}} \frac{D_e/2}{\sigma_v}
\]