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A Strong Relationship Between New and Old Inversion Mechanisms

Application of the theory on cognate six-bars of Watt's type makes it possible to link the well-known inversion mechanisms through mechanical cognation. It is proved that the inversors of Peaucellier, Hart, and Sylvester are cognates, though they do not all have the same number of links and turning-joints. As side-lines, two new inversion linkages and two other straight-line mechanisms are found. All of them are each other's cognate and produce the same straight line. The presented cognates cover a wide range in sizes.

1 Introduction

As early as 1864 the French captain A. Peaucellier invented a nowadays well-known planar inversor [1].[1] The invented linkage was based on the mathematical principle of inversion with respect to some unit circle in the plane. This unit circle enables the mathematician to transform an arbitrarily chosen curve into what may be termed the *inverted* curve. So it is known that a circle transforms itself onto another one, except the circle going through the inversion center O, which will transform itself into a straight line. This line is the radical axis of the two circles. It is the latter property which makes it possible to design straight-line mechanisms. To carry this out, one merely needs an inversion mechanism which possesses two coupler-points P and Q moving so that $\overline{OP} \cdot \overline{OQ}$ = constant. (The inversion center O must then be made a fixed center of pivot on the frame.) The *Peaucellier-cell*, having a rhombus and a kite as kinematic sub-chains, possesses the required condition and may be used for designing a straight-line generating mechanism as described above. This straight-line mechanism of Peaucellier consists of 8 bars and 6 turning-joints (see Fig. 1).

Later, in 1875, H. Hart discovered another well-known inversor, which is called the *contraparallelogram chain* of *Hart* [2]. This chain is merely a four-bar with the points O, P, and Q located on three successive sides of the contraparallelogram. (The points also lie—and remain lying so during motion—on some straight line parallel to the diagonals of the contraparallelogram.) A straight-line mechanism using Hart's cell is shown in Fig. 2. Here the mechanism consists of only 6 bars and 7 turning-joints.

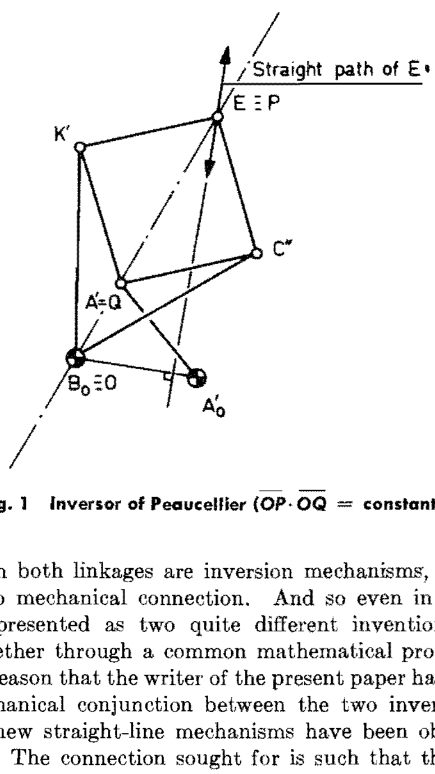

Fig. 1 Inversor of Peaucellier ($\overline{OP} \cdot \overline{OQ}$ = constant)

Although both linkages are inversion mechanisms, they seem to have no mechanical connection. And so even in literature they are presented as two quite different inventions merely linked together through a common mathematical property. It is for this reason that the writer of the present paper has searched for a mechanical conjunction between the two inversors. By doing so, new straight-line mechanisms have been obtained as side-lines. The connection sought for is such that the two inversors are *cognates* of each other. That is to say that the two mechanisms are linked together in such a way that the two corresponding points, generating the same straight line, coincide at any point of time. Besides, they have a common fixed center of pivot on the frame, and corresponding links move at the same angular velocity at any point of time. It will be proved, therefore, that both mechanisms are cognates of another less-known

[1] Numbers in brackets designate References at end of paper.

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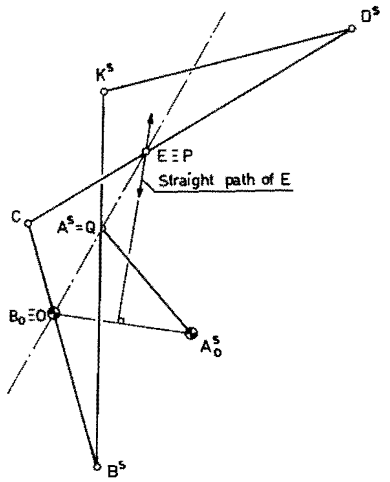

Fig. 2 Inversor of Hart ($\overline{OP} \cdot \overline{OQ}$ = constant)

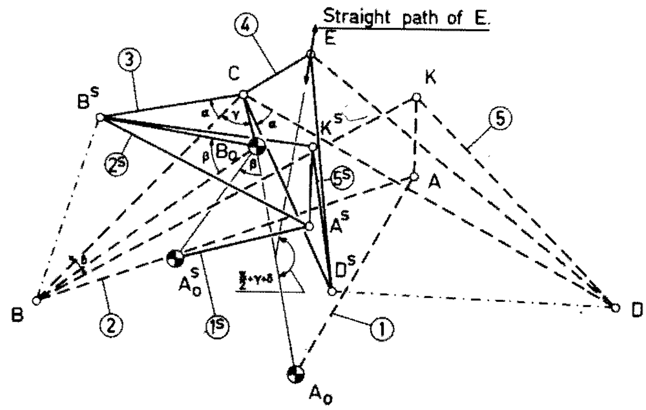

Fig. 4 The quadruplanar inversor of Sylvester and Kempe transformed into another one

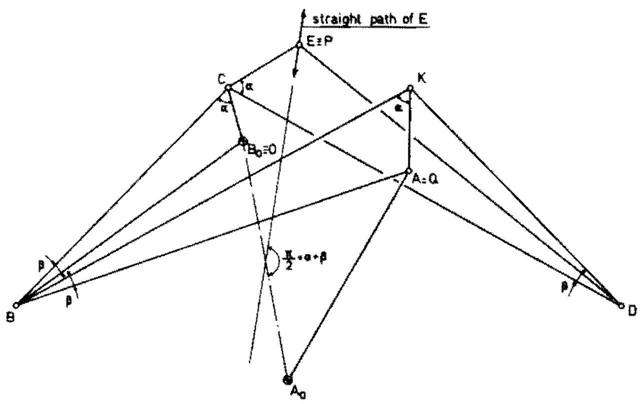

Fig. 3 Quadruplanar inversor of Sylvester and Kempe ($\overline{OP} \cdot \overline{OQ}$ = constant)

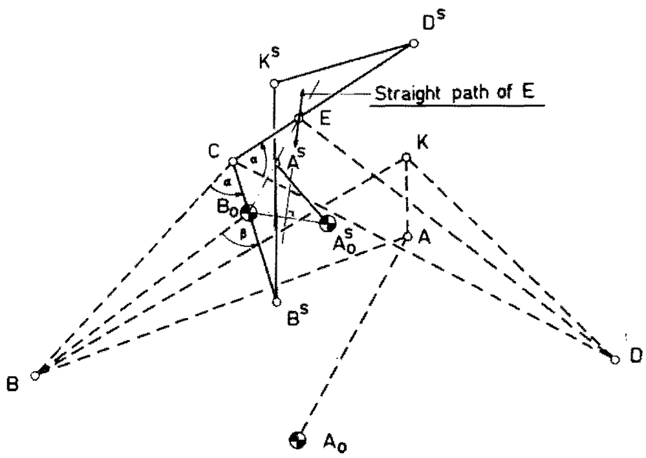

Fig. 5 The quadruplanar inversor of Sylvester and Kempe transformed into the inversor of Hart

inversor, which is called the *quadruplanar inversor of Sylvester and Kempe* [3] (see Fig. 3).

The latter may be seen as a generalized inversor of Hart. The basic cell in this mechanism also consists of a contraparallelogram. The points O, P, and Q, however, no longer lie on the sides themselves, but coincide with the respective apices of the similar triangles attached to three successive sides of the contraparallelogram. As before, the inversion center O is made a fixed center of pivot on the frame. The points P and Q, however, are no longer on a straight line with O. (As has been shown by Burmester [4], the points O, P, R, and Q always form a parallelogram as long as R is attached to the fourth remaining side of the contraparallelogram similarly to the points O, P, and Q.) Notwithstanding that, it is possible to prove that $\overline{OP} \cdot \overline{OQ} = \overline{OB} \cdot \overline{OC} \cdot \dfrac{\overline{CD}^2 - \overline{BC}^2}{\overline{BC}^2}$ = constant, and the basic cell may therefore be used as an inversor.

2 Two Cognate Inversors of Sylvester and Kempe

The quadruplanar inversor of Sylvester and Kempe may be seen as a six-bar linkage of Watt. As a consequence, one may obtain ∞^2 cognates [5] using the inversor as the initial mechanism from which to start the design. The cognation is shown by the following sequence of design instructions (see Fig. 4):

(*a*) The initial mechanism consists of the contraparallelogram $BCDK$, the four-bar A_0ABB_0 (with $A_0A = A_0B_0$), and the similar and rigid triangles ΔBB_0C, ΔBAK, and ΔDEC.

(*b*) Turn the contraparallelogram $BCDK$ about C over an arbitrarily chosen angle $\alpha = \sphericalangle BCB^s = \sphericalangle DCD^s$ and multiply the contraparallelogram geometrically from C by the arbitrarily chosen factor $f_\alpha = B^sC/BC$.

(*c*) One thus obtains $\square\, B^sCD^sK^s \sim \square\, BCDK$.

(*d*) Form the rigid and similar triangles ΔB^sCB_0 and ΔD^sCE.

(*e*) Turn the four-bar A_0ABB_0 about B_0 over the angle $\beta = \sphericalangle BB_0B^s = \sphericalangle A_0B_0A_0^s$ and multiply simultaneously from B_0 by the factor $f_\beta = B^sB_0/BB_0 = A_0^sB_0/A_0B_0$.

(*f*) One thus obtains the four-bar $\square\, A_0^sA^sB^sB_0 \sim \square\, A_0ABB_0$ with $A_0^sA^s = A_0^sB_0$.

(*g*) Form the rigid and similar triangles $\Delta A^sB^sK^s$ and ΔB_0B^sC.

The cognate quadruplanar inversor so obtained consists of the contraparallelogram $B^sCD^sK^s$, the four-bar $A_0^sA^sB^sB_0$ (with $A_0^sA^s = A_0^sB_0$), and the similar triangles B^sB_0C, $B^sA^sK^s$, and D^sEC. The initial and the obtained cognates both have the fixed center B_0, the turning-joint C, and the generating point E in common. Both mechanisms are inversors and generate the same straight line, produced by the coupler-point E.

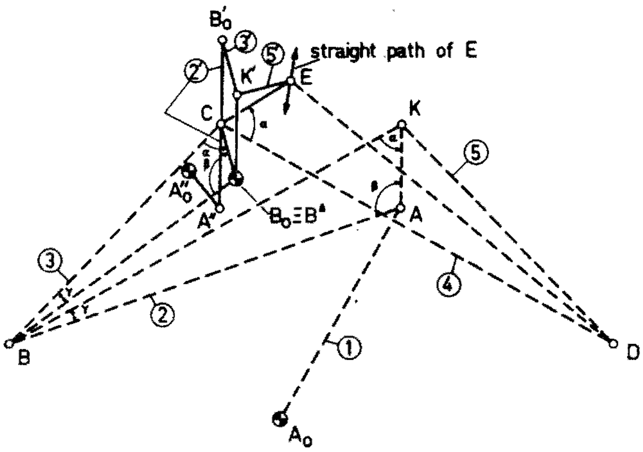

Fig. 6 The quadruplanar inversor of Sylvester and Kempe transformed into a new inversor with 8 links and 7 turning-joints

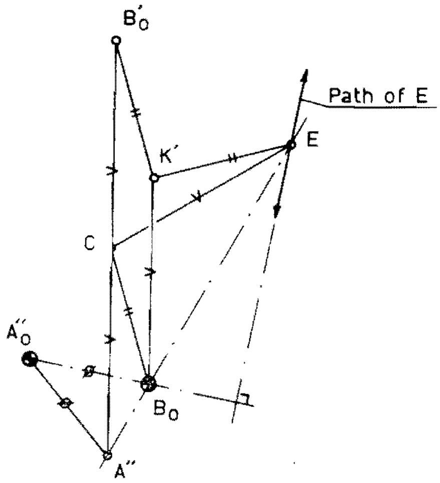

Fig. 7 Inversor of the first kind ($\overline{B_0E} \cdot \overline{B_0A''}$ = constant)

3 The Inversor of Hart Obtained as a Cognate From the Quadruplanar Inversor of Sylvester and Kempe

By a special choice of α in the design presented in the preceding section, one may obtain the well-known inversor of Hart. This may be done by making $\alpha = \sphericalangle BCB_0$ and by choosing f_α arbitrarily, but $\neq B_0C/BC$.

The obtained cognate is shown in the Figs. 2 and 5. The straight line generated by point E is perpendicular to $B_0A_0^*$. (In this special case the points B_0, A^*, and E remain on the same straight line, moving parallel to the diagonals of the contraparallelogram $B^*CD^*K^*$ and they are points on the sides of this contraparallelogram.)

4 The Design of a New Inversor Through Cognation With the Inversor of Sylvester and Kempe

Taking the quadruplanar inversor of Sylvester and Kempe as the initial mechanism, one may obtain a quite new inversor through the following pattern of instructions in design (see Fig. 6):

(*a*) Turn the contraparallelogram $BCDK$ about C over $\alpha = \sphericalangle BCB_0 = \sphericalangle DCE$ and multiply simultaneously from C by the factor $f_\alpha = CB_0/CB = CE/CD$.

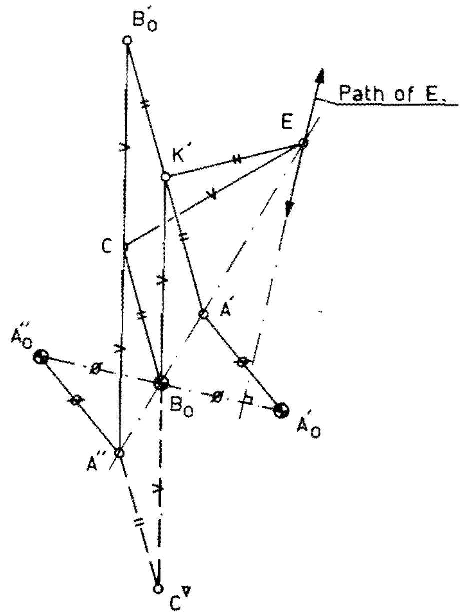

Fig. 8 Two cognate inversion mechanisms of the first and second kind

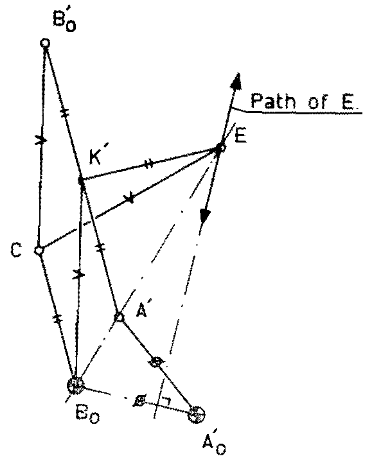

Fig. 9 Inversor of the second kind ($\overline{B_0A'} \cdot \overline{B_0E}$ = constant)

(*b*) One thus obtains the contraparallelogram $B_0CEK' \sim \square BCDK$.

(*c*) Frame the linkage parallelogram $CB_0K'B_0'$.

(*d*) Next, turn the four-bar A_0ABB_0 about B_0 over $\beta = \sphericalangle BB_0C$ and multiply the four-bar geometrically from B_0 by the factor $f_\beta = CB_0/BB_0$.

(*e*) One obtains the four-bar $A_0''A''CB_0 \sim \square A_0ABB_0$ with $A_0''A'' = A_0''B_0$.

(Since $A''C$ and CB_0' move with identical angular velocity and both have a common turning-joint C, they form one rigid bar. This bar is a stretched one since $\sphericalangle (A''C, B_0'C) = \beta + \gamma + \alpha = \pi$ rad. Moreover,

$$\frac{A''C}{B_0'C} = \frac{A''C}{AB} \cdot \frac{BK}{B_0'C} \cdot \frac{AB}{BK} = \frac{B_0C}{B_0B} \cdot \frac{BC}{B_0C} \cdot \frac{B_0B}{BC} = 1$$

Thus $A''C = B_0'C$.)

The obtained cognate inversor consists of the contraparallelogram B_0CEK', the parallelogram $CB_0K'B_0'$, and the four-bar

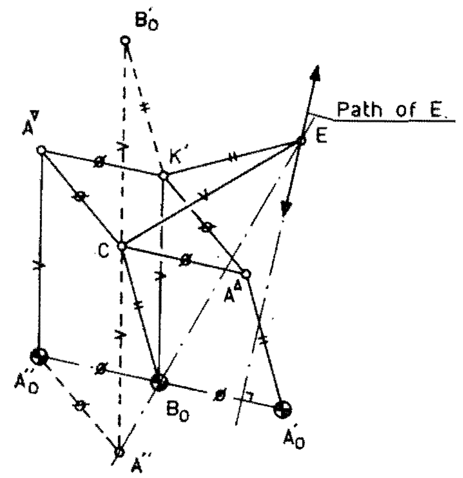

Fig. 10 Transformation of an inversor into two straight-line mechanisms

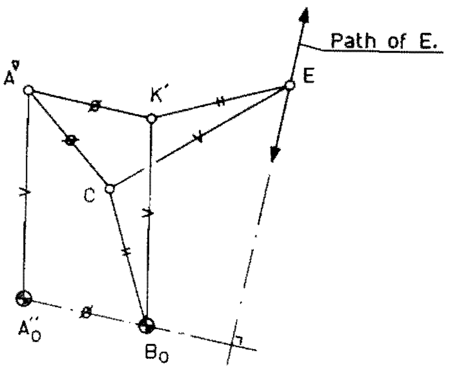

Fig. 11 Straight-line mechanism with 8 bars and 6 turning-joints

$A_0''A''CB_0$. The points A_0'' and B_0 are the fixed centers of pivot on the frame; the link $A''CB_0'$ forms one stretched bar. Moreover, $A''C = CB_0' = B_0K' = CE$ and $K'B_0' = B_0C = EK'$ and $A_0''A'' = A_0''B_0$ (see Fig. 7).

The turning-joint E of the cognate generates the same straight line as point E of the initial mechanism.

In the contraparallelogram B_0CEK' the product $\overline{A''B_0} \cdot \overline{B_0E} = \overline{CK'} \cdot \overline{B_0E} = \overline{CE}^2 - \overline{B_0C}^2$ = constant with respect to time. Since also $A''B_0//CK'//B_0E$ and A'', B_0, and E always remain in line, the obtained linkage is an inversion mechanism. That is to say, as long as $A_0''A'' = A_0''B_0$, point E will describe a straight line perpendicular to $A_0''B_0$. (Otherwise, E will generate a circle with its center on the fixed link $A_0''B_0$.)

Finally, it may be clear that the straight-line mechanism, shown in Fig. 7, allows the designer to choose the fixed center A_0'' somewhere on the perpendicular bisector to the distance $A''B_0$. In any case, however, the straight path of E remains perpendicular to the fixed link $A_0''B_0$.

The mechanism, shown in Fig. 7, will be called an inversion mechanism of the *first* kind. It possesses 8 links and 7 turning-joints.

5 Two Cognate Inversion Mechanisms of the First and the Second Kind

Taking the mechanism of Fig. 7 as the initial one, one may obtain another inversor by following the instructions in design below (see Fig. 8):

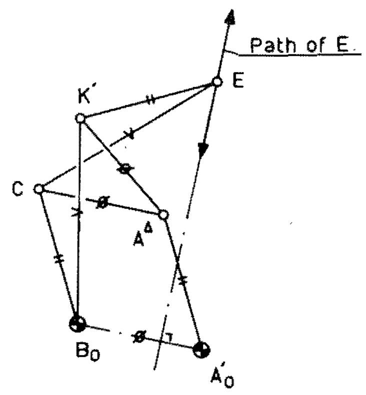

Fig. 12 Straight-line mechanism with 8 bars and 6 turning-joints

(*a*) Frame the linkage parallelogram $A''CB_0C^\nabla$.

(*b*) Turn the four-bar $A_0''A''C^\nabla B_0$ about B_0 over π rad.

(*c*) One thus obtains the four-bar $A_0'A'K'B_0 \cong \square A_0''A''C^\nabla B_0$.

(*d*) Frame the stretched bars $A_0''B_0A_0'$ and $B_0'K'A'$.

The obtained cognate inversor is shown in Fig. 9. The linkage will be called an inversion mechanism of the *second* kind since it contains the same properties as the inversor of the first kind. Here also $\overline{B_0A'} \cdot \overline{B_0E}$ = constant and both cognates have 8 bars and 7 turning-joints. The straight line, generated by E, here, too, is perpendicular to $A_0'B_0$.

6 A New Straight-Line Mechanism as a Cognate of the Inversor of the First Kind

As before, taking the inversor of Fig. 7 as the initial mechanism, one may obtain a straight-line mechanism having 8 bars and only 6 turning-joints. This may be done using the following instructions (see Fig. 10):

(*a*) Make the four-bar linkage $\square A^\nabla CB_0'K' \cong \square A_0''A''CB_0$.

(*b*) Frame the linkage parallelogram $A_0''B_0K'A^\nabla$.

One thus obtains the cognate straight-line mechanism shown in Fig. 11. The mechanism consists of the contraparallelogram B_0CEK', the parallelogram $A_0''B_0K'A^\nabla$, and the four-bar $A_0''A^\nabla CB_0$ with $A^\nabla C = A_0''B_0$. The designer, using such a mechanism, may freely choose the lengths $A^\nabla K'$, $A^\nabla C$, and $A_0''B_0$, but they have to remain equal to each other.

Although the mechanism bears some resemblance to the *planar Kempe linkage* of the *first* kind [3, 6], they are no cognates of each other, since the four-bar $A^\nabla K'EC$ (or another one, derived from this one, through some change in the sequence of the links) is not materialized in any sub-chain of the linkage of Kempe (see Fig. 17).

7 Cognate Straight-Line Mechanisms

Taking the mechanism of Fig. 11 as the initial one, another straight-line mechanism may be obtained through cognation. The design instructions are (see Fig. 10):

(*a*) Frame the kite $CA^\nabla K'A^\Delta$ and the linkage parallelogram $B_0CA^\Delta A_0'$.

(*b*) Take point A_0' as a fixed center of pivot on the frame.

The obtained cognate is the straight-line linkage of Fig. 12. As before, the straight path of E is perpendicular to frame link B_0A_0'.

Starting with the inversor of Fig. 7 as the initial mechanism, one seemingly obtains a *third* straight-line mechanism bearing some resemblance to the mechanisms of the Figs. 11 and 12. The design instructions are (see Fig. 13):

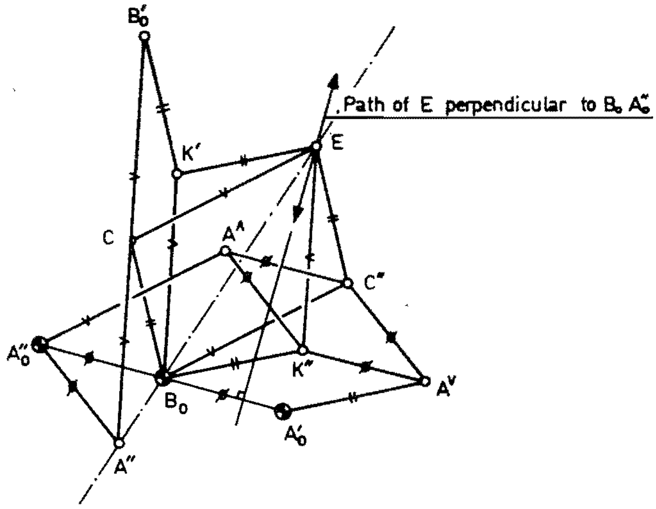

Fig. 13 Second transformation of an inversor into two straight-line mechanisms, both having 8 bars and 6 turning-joints

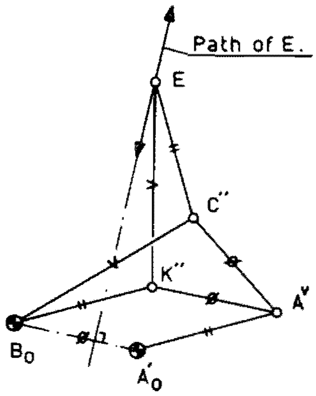

Fig. 15 Straight-line mechanism with 8 bars and 6 turning-joints

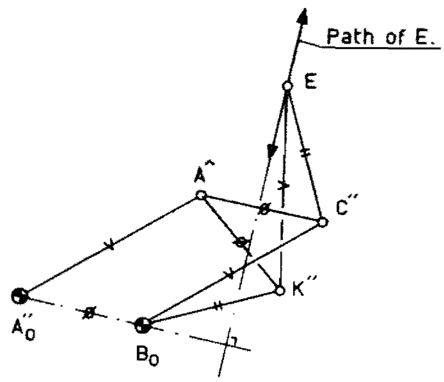

Fig. 14 Straight-line mechanism with 8 bars and 6 turning-joints

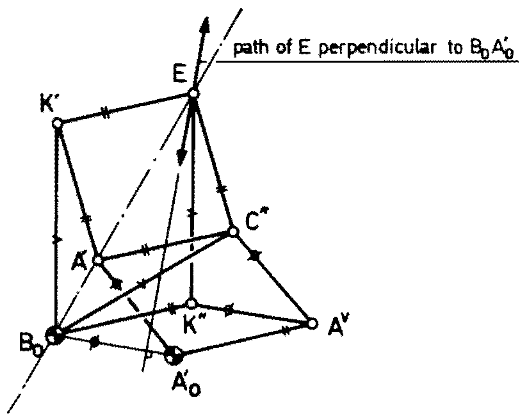

Fig. 16 The inversor of Peaucellier as a cognate of a straight-line linkage

(*a*) Frame the parallelograms B_0CEC'' and $B_0K'EK''$.
(*b*) Make the four-bar $\square A^{\wedge}K''EC'' \cong \square A_0''A''CB_0$.
(*c*) Frame the parallelogram $B_0C''A^{\wedge}A_0''$.

The obtained cognate is shown in Fig. 14. (One may observe however, that the mechanism of Fig. 14 is identical to that of Fig. 11: They are merely drawn in different positions.)

Apparently a *fourth* straight-line mechanism of this kind may be obtained, using the mechanism of Fig. 14 as the initial mechanism. The design instructions are (see Fig. 13):

(*a*) Frame the kite $K''A^{\wedge}C''A^{v}$ and the linkage parallelogram $B_0K''A^{v}A_0'$.

(*b*) Turn the joint A_0' into a fixed center of pivot on the frame. One thus obtains the cognate straight-line mechanism shown in Fig. 15. This is identical to the straight-line mechanism shown in Fig. 12: They merely differ in position.

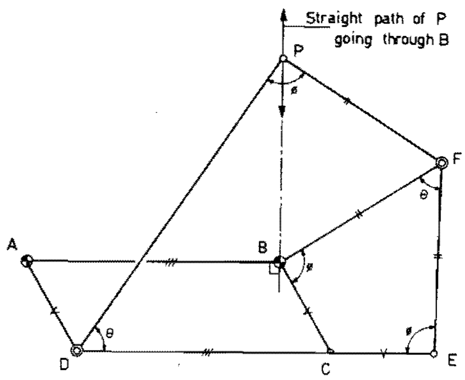

Fig. 17 Planar Kempe linkage of the first kind

8 The Inversor of Peaucellier as a Cognate of the Fourth Straight-Line Linkage

The well-known inversor of Peaucellier may be obtained from the straight-line mechanism shown in Fig. 15, through cognation. Taking the mechanism of Fig. 15 as the initial one, one may design the inversor with the following instruction (see Fig. 16):

(*a*) Frame the parallelograms $A_0'A^{v}C''A'$ and $B_0K''EK'$ and the kite $A'C''EK'$.

One thus obtains the inversor of Peaucellier, as shown in Fig. 1.

All the foregoing shows that the mechanisms presented in the illustrations 1—16 are related through cognation. It is thus shown that both inversors, viz., the one of Hart and the one of Peaucellier, are cognates of each other.

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DISCUSSION

Fan Y. Chen[2]

This is a remarkable discovery. A century has passed since the invention of the first mechanical inversor, and we are now aware of the two historical inversive mechanisms: The Peaucellier cell and contraparallelogram of the Hart cell are cognates of each other. As a consequence of this, a new class of straight-line mechanisms is being introduced. Aside from theoretical interest, cognates provide alternative linkages with different link sizes and configurations, force transmission characteristics, crank rotations, and fixed pivot locations to provide a variety of design choices.

The writer would like to point out that the crossed parallelogram of Hart can directly replace the Peaucellier cell without using the quadruplanar inversor of Sylvester and Kempe. In Fig. 18, the Peaucellier cell of the first kind[3] and the crossed parallelogram of Hart are presented superimposed upon each other. By using the same mechanism configurations and notations as those of the authors, the rhombus $QK'PC''$ and the kite $B_oK'QC''$ of the Peaucellier cell are shown in solid lines, the crossed parallelogram of Hart $B^sCD^sK^s$ is shown in dashed lines, and B_oA_o and $A_o'Q$ are the auxiliary links. The steps outlined below show how to obtain one from the other:

[2] Associate Professor of Mechanical Engineering, Ohio University, Athens, Ohio.

[3] There are two versions of Peaucellier cell. They differ only by the relative proportions of the link length.

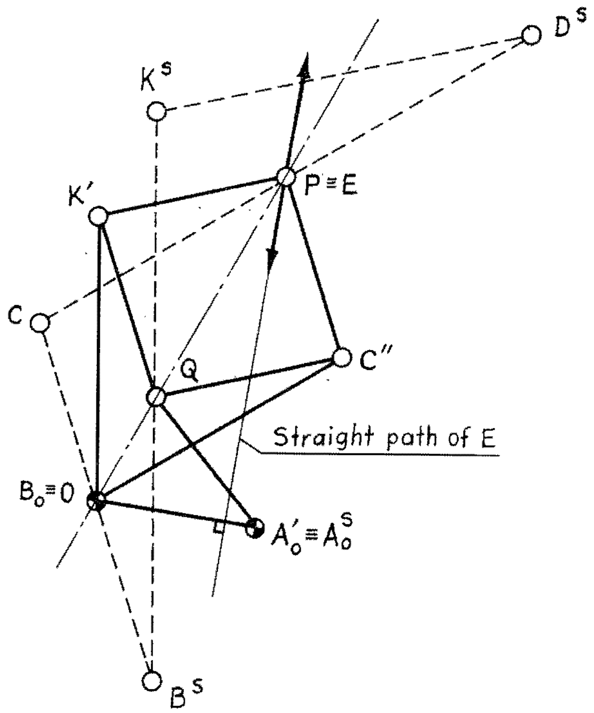

Fig. 18

1 Starting from the given Peaucellier cell, draw CP in parallel to OC'' and extend CP to D^s such that $PD^s = CP = OC''$.

2 Draw CO in parallel to PC'' and extend CO to B^s such that $OB^s = CO = PC''$.

3 From point B^s draw a straight line B^s in parallel to OK' and extend it to point K^s such that $QK^s = B^sQ = OK'$.

4 Join K^sD^s to complete the construction of the crossed parallelogram of Hart.

Note that both mechanisms have the fixed centers B_o and A_o', points P, Q, and B_o are collinear, and the generating point E in common. Without difficulty, we can also show that the Peaucellier cell of the second kind is cognate to the inversor of Hart.

Furthermore, it is conceivable that all cognate mechanisms presented in the paper are extensible to become spherical mechanisms by means of stereographic projection and that some of the cognate mechanisms may be used to generate inverse-square law force (a property which the Peaucellier cells can be used to simulate, as has been shown by the discusser).[4]

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