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ANALYTICITY SPACES OF SELF-ADJOINT OPERATORS SUBJECTED TO
PERTURBATIONS WITH APPLICATIONS TO HANKEL
INARIANT DISTRIBUTION SPACES

by

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Abstract

A new theory of generalized functions has been developed by one of the authors (De Graaf). In this theory the analyticity domain of each positive self-adjoint unbounded operator \( A \) in a Hilbert space \( X \) is regarded as a test space denoted by \( S_{X,A} \). In the first part of this paper, we consider perturbations \( P \) on \( A \) for which there exists a Hilbert space \( Y \) such that \( A + P \) is a positive self-adjoint operator in \( Y \). In particular, we investigate for which perturbations \( P \) and for which \( \nu > 0 \), \( S_{X,A}^{\nu} \subset S_{Y,(A+P)^{\nu}} \). The second part is devoted to applications. We construct Hankel invariant distribution spaces. The corresponding test spaces are described in terms of the \( S_{\alpha}^{\delta} \)-spaces introduced by Gelfand and Shilov. It turns out that the modified Laguerre polynomials establish an uncountable number of bases for the space of even entire functions in \( S_{\mu}^{\nu} \left( \frac{1}{2} \leq \mu \leq 1 \right) \). For an even entire function we give necessary and sufficient conditions on the coefficients in the Fourier expansion with respect to each basis such that \( \phi \in S_{\nu}^{\mu} \).

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Introduction

Let $X$ be a separable infinitely dimensional Hilbert space and let $L$ be a linear operator in $X$. Then $D^\omega(L)$, the analyticity domain of $L$, consists of all vectors $v \in \bigcap_{n=1}^{\infty} D(L^n)$ satisfying

$$\exists a > 0 \exists b > 0 \forall n \in \mathbb{N} : \|L^n v\| \leq n! a^n b .$$

For a positive self-adjoint operator $A$ in $X$, Nelson ([13]) proved that $D^\omega(A)$ can also be described as

$$D^\omega(A) = \bigcup_{t>0} e^{-t A}(X) = \{e^{-t A} w | w \in X, t > 0\} .$$

Instead of $D^\omega(A)$ we use the notation $S_{X,A}$ introduced by De Graaf. The spaces of type $S_{X,A}$ are called analyticity spaces. They are non strict inductive limits of Hilbert spaces. Together with their strong duals $T_{X,A}$ they establish the functional analytic description of the distribution theory in [G].

For each positive constant $v$ the operator $A^v$ is well-defined, positive and self-adjoint in $X$. So it makes sense to write $S_{X,A}^v$. The question arises for which perturbations $P$ on $A$ there can be found a Hilbert space $Y$ such that $A + P$ is a positive self-adjoint operator in $Y$ and $S_{X,A}^v \subseteq S_{Y,(A+P)^v}$. In the paper ([1]) the case $v = 1$ has been considered. Also some results concerning analytic dominancy can be found there.

In the second part of this paper we study a class of Hankel invariant test- and distribution spaces, and, also their relations to the $S_{\alpha}^\beta$-spaces of Gelfand and Shilov ([9]). With our papers [2] and [4] we have started this study. There we have shown that the space of even functions in $S_{i\frac{1}{2}}^\frac{1}{2}$ remains invariant.
under the modified Hankel transforms $H_\alpha$, $\alpha > -1$, defined by

$$(H_\alpha f)(x) = \int_0^\infty (xy)^{-\alpha} J_\alpha(xy) f(y) y^{2\alpha+1} \, dy.$$  

Moreover, for each $\alpha > -1$ the space of even functions in $S_\frac{1}{4}$ equals the analyticity space $S_{X_\alpha, A_\alpha}$ where $X_\alpha = L_2((0, \infty), x^{2\alpha+1} dx)$ and

$$A_\alpha = -\frac{d^2}{dx^2} + x^2 - (2\alpha + 1)x \frac{d}{dx}.$$  

The operator $A_\alpha$ has an orthonormal basis of eigenvectors $(\ell_n^{(\alpha)})_{n=0}^{\infty}$ with eigenvalues $4n + 2\alpha + 2$. So for each even $f \in S_\frac{1}{4}$ there exists an $\ell_2$-sequence $(\omega_n^{(\alpha)})_{n=0}^{\infty}$ and $t > 0$ such that

$$f = \sum_{n=0}^{\infty} \exp(-(4n + 2\alpha + 2)t) \omega_n^{(\alpha)}.$$  

Here we prove similar results for the spaces $S_{X_\alpha, (A_\alpha)^\nu}$ with $\nu \geq \frac{1}{2}$ and $\alpha > -1$. It will follow that for all $\alpha, \beta > -1$ and all $\nu \geq \frac{1}{2}$

$$S_{X_\alpha, (A_\alpha)^\nu} = S_{X_\beta, (A_\beta)^\nu}.$$  

For $\nu \in [\frac{1}{2}, 1]$ the analyticity space $S_{X_{-\frac{1}{2}}, (A_{-\frac{1}{2}})^\nu}$ contains just the even functions in $S_{\frac{1}{2}, (A_{-\frac{1}{2}})^\nu}$.  

(1) **General theory**

Let $A$ be a positive self-adjoint operator in a Hilbert space $X$ and let $\nu > 0$. It makes sense to write $A^\nu$ and the operator $A^\nu$ is positive and self-adjoint in $X$. So the space $S_{X, A^\nu}$ is well-defined. Its elements are characterized by
(1.1) Lemma

For each \( f \in D(A^{\infty}) \subset X \) the following statements are equivalent

(i) \( \exists a > 0 \exists b > 0 \forall k \in \mathbb{N} : \| A^k f \| \leq (k!)^{1/v} a b \);

(ii) \( f \in S_{X, A^v} \).

Proof

(i) \( \Rightarrow \) (ii). Let \( N \in \mathbb{N} \) and let \( \tau > 0 \). Consider the following estimation

\[
(*) \quad \sum_{k=0}^{N} \frac{\tau^k}{k!} \| A^k f \| \leq \sum_{k=0}^{N} \frac{\tau^{k+1}}{k!} \| A^{-1+\nu k-\lfloor \nu k \rfloor} \| \| A^{\lfloor \nu k \rfloor+1} f \| \leq \leq b_1 \sum_{k=0}^{N} \frac{\tau^k}{k!} ([\nu k] + 1)! v^k \]

where \( b_1 = b \sup_{k \in \mathbb{N}_0(0)} (\| A^{-1+\nu k-\lfloor \nu k \rfloor} \|) \). The following inequalities are valid

\[
([\nu k] + 1)! \leq ([\nu k] + 1)([\nu k] + 1)^{1/v} \leq e([\nu k] + 1)(\nu k)^{\nu k}.
\]

So \( ([\nu k] + 1)! \leq (e([\nu k] + 1))^{1/v}(\nu e)^{1/v} \), and for \( \tau < (\nu e)^{-1} \) the series \( (*) \) converges. It implies that \( f \in \exp(-\tau A^v)(X) \).

(ii) \( \Rightarrow \) (i) Suppose \( g \in S_{X, A^v} \). Then there exists \( s > 0 \) and \( w \in X \) such that \( g = \exp(-s A^v)w \). Let \( k \in \mathbb{N} \). Then we estimate as follows

\[
\| A^k f \| \leq \| A^k \exp(-s A^v) \| \| w \| = \| w \| \left( \frac{k}{s} \right)^{k/v} e^{-k/v} \leq \leq \| w \| \left( \frac{1}{s} \right)^{k/v} \cdot (k!)^{1/v}.
\]

With \( a = (s) \) and \( b = \| w \| \) the implication \( (ii) \Rightarrow (i) \) has been proved. □
Let $L$ be an unbounded linear operator in $X$. Then the operators $L^2, L^3, \ldots$ are well-defined. As a corollary of the previous theorem we get

(1.2) **Corollary**

Let $n \in \mathbb{N}$ and let $f \in D^\omega(L)$. The following statements are equivalent.

(i) $\exists a > 0 \exists b > 0 \forall k \in \mathbb{N}$ : $\| L^k f \| \leq (k!)^{1/n} a b$;

(ii) $f \in D^\omega(L^D)$.

As mentioned in the introduction we investigate perturbations $P$ on $A$ such that $D^\omega((A + P)^{\nu}) \supset S_{X, A^{\nu}}$. For $\nu = 1$ the following result has been proved in [1]. Here we consider general $\nu > 0$.

(1.3) **Theorem**

Let $P$ be a linear operator in $X$ with $D(P) \supset S_{X, A^{\nu}}$. Suppose the following conditions are satisfied

(i) There exists a Hilbert space $Y$ such that $\exp(-tA^{\nu})$ maps $X$ into $Y$ for all $t > 0$;

(ii) In addition, $A + P$ defined on $S_{X, A^{\nu}}$ is positive and essentially self-adjoint in $Y$.

(iii) There exists an everywhere defined, monotone non-increasing function $\varphi$ on $(0, 1)$ such that

$$\forall r: 0 < r < 1 : \| \exp(rA^{\nu}) PA^{-1} \exp(-rA^{\nu}) \|_X \leq \varphi(r).$$

Then $S_{X, A^{\nu}} \subset S_{Y, (A + P)^{\nu}}$. 

Proof

We note first that $S_x, A^\nu = \bigcup \exp(-t A^\nu)(X)$. So let $0 < t < 1$, and let $0 < \tau < t$. Put $s = t - \tau$. We want to estimate the norm of the operator

$\exp(\tau A^\nu)(A + P)^k \exp(-t A^\nu)$ for each $k \in \mathbb{N}$. Therefore we factor as follows

$$
\exp(\tau A^\nu)(A + P)^k \exp(-t A^\nu) =
$$

$$
= \prod_{j=0}^{k-1} \left\{ \exp\left( (\tau + \frac{j}{k} s)A^\nu \right) \left( I + PA^{-1} \right) \exp\left( -\left( \tau + \frac{j}{k} s \right)A^\nu \right) \right\}.
$$

This factoring yields the estimate

$$
\| \exp(\tau A^\nu)(A + P)^k \exp(-t A^\nu) \| \leq \| A \exp(-s A^\nu) \|^k.
$$

$$
= \prod_{j=0}^{k-1} \| \exp\left( (\tau + \frac{j}{k} s)A^\nu \right) \left( I + PA^{-1} \right) \exp\left( -\left( \tau + \frac{j}{k} s \right)A^\nu \right) \| \leq
$$

$$
\leq (k!)^{1/\nu} \left( \frac{1}{\nu s} \right)^{k/\nu} \prod_{j=0}^{k-1} \left( 1 + \phi(\tau + \frac{j}{k} s) \right).
$$

Since $\phi(\tau + \frac{j}{k} s) \leq \phi(\tau)$ for all $j = 0, 1, \ldots, k-1$, we get

$$
\prod_{j=0}^{k-1} \left( 1 + \phi(\tau + \frac{j}{k} s) \right) \leq (1 + \phi(\tau))^k.
$$

Thus we have proved that

$$
\forall \tau > 0, \forall 0 < \tau < t \exists a > 0 \forall k \in \mathbb{N} : \| \exp(\tau A^\nu)(A + P)^k \exp(-t A^\nu) \| \leq (k!)^{1/\nu} a^k.
$$

Let $t > 0$ and let $w \in X$. Set $f = \exp(-t A^\nu)w$. Then for $0 < \tau < t$ fixed there
exists a $a > 0$ such that
\[
\|(A + P)^k f\|_Y \leq \left\| \exp(-t A^\nu) \right\|_{X \to Y} \| \exp(t A^\nu) (A + P)^k f\|_X \\
\leq \left\| \exp(-t A^\nu) \right\|_{X \to Y} \| w\|_X a^k (k!)^{1/\nu}.
\]

From Lemma (1.1) it follows that $f \in S_{Y,(A+P)^\nu}$.

**Remark:** Suppose there exists $k \in \mathbb{N}$ such that the operator $A^{-k}$ maps $X$ continuously into $Y$. Then Condition (1.3.i) is fulfilled because
\[
\| \exp(-t A^\nu) \|_{X \to Y} \leq \| A^{-k} \|_{X \to Y} \| A^k \exp(-t A^\nu) \|_X.
\]

**Corollary (1.4)**

Let $P$ be an operator in $X$ and let $n \in \mathbb{N}$ with $D(P) \supset S_{X,A^n}$. Suppose there exists an everywhere defined monotone non-increasing function $\phi$ on $(0,1)$ such that
\[
\forall 0 < \tau < 1 : \| \exp(\tau A^n) PA^{-1} \exp(-\tau A^n) \| \leq \phi(\tau).
\]

Then $S_{X,A^n} \subset D^\Omega((A + P)^n)$.

**Proof**

As in the proof of the previous theorem: $\forall t > 0 \forall \tau, 0 < \tau < \exists a > 0 \forall k \in \mathbb{N}$:
\[
\| \exp(\tau A^n) (A + P)^k \exp(-t A^n) \| \leq (k!)^{1/n} a^k.
\]

So for $f = \exp(-t A^n)w, \ t > 0, \ w \in X$, we get
\[
\|(A + P)^k f\|_X \leq \left\| \exp(\tau A^n) (A + P)^k \exp(-t A^n) \right\| \| w\| \leq \\
\leq (k!)^{1/n} a^k \| w\|.
\]
Remark: If $P$ satisfies the conditions in Corollary (1.4), then $A^n$ analytically dominates $(A + P)^n$. (For the terminology, see [6]).

In order to prove the converse statement of Theorem 3, i.e.

$$S_{\gamma, (A+P)^\nu} \subset S_{\nu, A^\nu}$$

we have to interchange the roles of $A$ and $A + P$. Put differently, if we write $B = A + P$ and hence $A = B - P$, then we have to check whether the pair $B, P$ satisfies the conditions required in Theorem (1.3).

2. Hankel invariant distribution spaces

In our papers [2], [4] on Hankel invariant distribution spaces the following results have been proved.

Let $A_y$ denote the differential operator $-\frac{d^2}{dx^2} + x^2 - \frac{2\gamma + 1}{x} \frac{d}{dx}$ and let $X$ denote the Hilbert space $L_2((0,\infty), x^{2\gamma+1} dx)$ where we take $\gamma > -1$. Then for every $\alpha, \beta > -1$ we have shown that

$$S_{x, A_{\alpha}} = S_{x, A_{\beta}}.$$  

Moreover, $f \in S_{x, A_{\gamma}}$ if and only if $f$ is extendable to an even function in $S_{\frac{1}{2}}$. Also, it has been proved that the space $S_{x, A_{\gamma}}$ remains invariant under the modified Hankel transform $H_{\gamma}$ defined by

$$(H_{\gamma} f)(x) = \int_0^\infty f(y)(xy)^{-\gamma} J_{\gamma}(xy)y^{2\gamma+1} dy.$$
Here $J_\gamma$ denotes the Bessel function of the first kind and of order $\gamma$. The Hankel transform $H_\gamma$ extends to a unitary operator on $X_\gamma$ and $H_\gamma A_\gamma = A_\gamma H_\gamma$. It follows that for all $\alpha, \beta > -1$, $H_\alpha$ maps the space $S_{X_\beta, A_\beta}$ onto itself. By duality, each $H_\alpha$ leaves invariant each space of generalized functions $T_{X_\beta, A_\beta}$ corresponding to $S_{X_\beta, A_\beta}$. The functions $L_n^{(\gamma)}$ defined by

$$L_n^{(\gamma)}(x) = \left(\frac{2^{\gamma}(n + 1)}{\pi(n + \gamma + 1)}\right)^{\frac{1}{2}} e^{-\frac{1}{4}x^2} L_n^{(\gamma)}(x^2), \quad n \in \mathbb{N} \cup \{0\}, \quad x > 0$$

establish an orthonormal basis in $X_\gamma$ and they are the eigenfunctions of the self-adjoint operator $A_\gamma$ with respective eigenvalues $4n + 2\gamma + 2$. Here $L_n^{(\gamma)}$ denotes the $n$-th generalized Laguerre polynomial of order $\gamma$. We note that

$$H_\gamma L_n^{(\gamma)} = (-1)^n L_n^{(\gamma)}.$$

We recall that for each $\alpha, \beta > -1$ the functions $f \in S_{X_\alpha, A_\alpha}$ can be written as $f = \sum_{n=0}^{\infty} \omega_n L_n^{(\beta)}$ where $\omega_n = O(e^{-nt})$ for some $t > 0$.

With the aid of the theory presented in the first part of this paper we extend the mentioned results and prove that

$$S_{X_\alpha, (A_\alpha)^\nu} = S_{X_\beta, (A_\beta)^\nu}$$

for all $\nu \geq \frac{1}{2}$ and all $\alpha, \beta > -1$. In addition, we show that for each $\nu \in \left[\frac{1}{2}, 1\right]$ and all $\alpha > -1$ the space $S_{X_\alpha, (A_\alpha)^\nu}$ contains just the even functions of the Gelfand-Shilov space $S_{1/2\nu}^{1/2\nu}$. So each even function $f \in S_{1/2\nu}^{1/2\nu}$ admits Fourier expansions $f = \sum_{n=0}^{\infty} \rho_n L_n^{(\alpha)}$ with $\rho_n^{(\alpha)} = O(\exp(-n^\nu t))$. 
Let $\alpha, \beta > -1$. Then $A_\alpha$ can be written as

$$A_\alpha = A_\beta + 2(\alpha - \beta)R$$

where we put $R = \frac{1}{x} \frac{d}{dx}$. Obviously, $A_\alpha$ can be obtained from $A_\beta$ by means of the 'perturbation' $2(\alpha - \beta)R$, and $A_\beta$ from $A_\alpha$ by means of $2(\beta - \alpha)R$. In order to show that $R$ and hence $cR$, $c \in \mathbb{C}$, is a perturbation in the sense of Theorem (1.3) we compute the matrix of $R$ with respect to the orthonormal basis $(L^{(\gamma)}_n)_{n=0}^\infty$. To this end, we mention that

$$RL^{(\gamma)}_n = -L^{(\gamma)}_{n-1} - 2L^{(\gamma)}_n$$

where the relation $\frac{d}{dx} L^{(\gamma)}_n = -L^{(\gamma+1)}_{n-1}$ is used.

Now $L^{(\gamma+1)}_k = \sum_{j=0}^k L^{(\gamma)}_j$ and hence

$$RL^{(\gamma)}_n = -\left(\frac{2\Gamma(n+1)}{\Gamma(n+\gamma+1)}\right)^{\frac{1}{2}} \left[\left(\frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)}\right)^{\frac{1}{2}} L^{(\gamma)}_n + 2 \sum_{m=0}^{n-1} \left(\frac{\Gamma(m+\gamma+1)}{2\Gamma(m+1)}\right)^{\frac{1}{2}} L^{(\gamma)}_m\right].$$

Thus we obtain the matrix of $R$ with respect to the basis $(L^{(\gamma)}_n)_{n=0}^\infty$

$$(R L^{(\gamma)}_k, L^{(\gamma)}_\ell) = \begin{cases} 0 & \text{if } \ell > k, \ k \in \mathbb{N} \\ -2 \left(\frac{\Gamma(k+1)}{\Gamma(k+\gamma+1)}\right)^{\frac{1}{2}} \left(\frac{\Gamma(\ell+1)}{\Gamma(\ell+\gamma+1)}\right)^{\frac{1}{2}} & \text{if } 0 \leq \ell < k, \ k \in \mathbb{N} \end{cases}$$
The inequality (cf. [11])

\[ n^{1-s} \leq \frac{\Gamma(n+1)}{\Gamma(n+s)} \leq (n+1)^{1-s}, \quad 0 \leq s \leq 1, \ n \in \mathbb{N} \]

yields

\[
| (R_L^{(\gamma)} \bar{L}_\ell^{(\gamma)}) | \leq \begin{cases} 2 & \text{if } \gamma \geq 0, \ 0 \leq \ell < k, \ k \in \mathbb{N} \cup \{0\} \\
2k^{-\gamma/2} & \text{if } -1 < \gamma < 0, \ 0 \leq \ell < k, \ k \in \mathbb{N} \cup \{0\}.
\end{cases}
\]

For each \( \nu \geq \frac{1}{2} \), the operator \( \exp(\nu(A_\gamma) \nu) R(A_\gamma)^{-1} \exp(-\nu(A_\gamma) \nu) \) has to satisfy Condition (iii) of Theorem (1.3). We define the weighted shift operators \( \bar{W}_{\gamma, \nu}(r), n \in \mathbb{N} \cup \{0\}, \)

\[
(\bar{w}_{\gamma, \nu}(n) L_L^{(\gamma)} \bar{L}_\ell^{(\gamma)})_\gamma = \begin{cases} 0 & \text{if } k \neq \ell + n \\
(R_L^{(\gamma)} \bar{L}_\ell^{(\gamma)}) \frac{\exp(-\nu((n+1)\gamma - \ell \gamma))}{4(n+1) + 2\gamma + 2} & \text{elsewhere}
\end{cases}
\]

with norms

\[
\| \bar{w}_{\gamma, \nu}(n) \|_{X_\gamma} = \sup_{\ell \in \mathbb{N} \cup \{0\}} | (R_L^{(\gamma)} \bar{L}_\ell^{(\gamma)}) | \frac{\exp(-\nu((n+1)\gamma - \ell \gamma))}{4(n+1) + 2\gamma + 2}.
\]

So \( \| \bar{w}_{\gamma, \nu}(0) \| \leq \frac{1}{2\gamma + 2} \). Now let \( n \in \mathbb{N} \). The inequality

\[
(n+1)\nu - \ell \nu \geq (n+1)^{\frac{1}{2}} - \ell^{\frac{1}{2}}
\]

is valid for all \( \ell \in \mathbb{N} \cup \{0\} \) and all \( \nu \geq \frac{1}{2} \). In addition, the matrix elements \( | (R_L^{(\gamma)} \bar{L}_\ell^{(\gamma)}) | \) are smaller than \( 2(n+1)^{-\gamma/2} \) for \(-1 < \gamma < 0\) and smaller than 2 for \( \gamma \geq 0 \). If \(-1 < \gamma \leq 0\) we therefore get

\[
\| \bar{w}_{\gamma, \nu}(n) \| \leq \sup_{\ell \in \mathbb{N} \cup \{0\}} \frac{2(n+1)^{-\gamma/2}}{4(n+1) + 2\gamma + 2} \exp(-\nu((n+1)^{\frac{1}{2}} - \ell^{\frac{1}{2}})) \leq \sup_{\ell \in \mathbb{N} \cup \{0\}} (1(n+1)^{-\gamma} \exp(-\nu \ln(n+1)^{-\frac{1}{2}})) \leq \frac{1}{2}(1 + \frac{1}{n})^{2+\gamma} \left( \frac{1}{r} \right)^{2+\gamma} \left( \frac{1}{n} \right)^{2+\gamma} \exp(2 + \gamma) =: d_1 \left( \frac{1}{r} \right)^{2+\gamma} \left( \frac{1}{n} \right)^{2+\gamma}.
\]
Since
\[ \exp(r(A_\gamma)^v) R(A_\gamma)^{-1} \exp(-r(A_\gamma)^v) = \sum_{n=0}^{\infty} \omega(n)^v(r) \]
we can use the following straightforward estimate for all \( r > 0 \)

\[ \| \exp(r(A_\gamma)^v) R(A_\gamma)^{-1} \exp(-r(A_\gamma)^v) \| \leq \sum_{n=0}^{\infty} \| \omega(n)^v(r) \| \leq \frac{1}{2\gamma + 2} + d_1 \left( \frac{1}{r} \right)^{2+\gamma} \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^{2+\gamma} \leq \]

\[ \leq d_1 \left( \frac{1}{r} \right)^{2+\gamma} + \frac{1}{2\gamma + 2} \]

where \( d_1 = d_1 \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^{2+\gamma} \). Summarized

(2.1) Lemma

Let \( \gamma > -1 \). Then there exist constants \( d_1 > 0 \) and \( p_1 > 0 \) such that

\( \forall r > 0 : \| \exp(r(A_\gamma)^v) R(A_\gamma)^{-1} \exp(-r(A_\gamma)^v) \| \leq d_1 \left( \frac{1}{r} \right)^{p_1} + \frac{1}{2\gamma + 2} \).

Proof

For \( -1 \leq \gamma \leq 0 \) the assertion has already been proved. For \( \gamma > 0 \) it follows from the matrix expressions for \( R \) that

\[ \| \exp(r(A_\gamma)^v) R(A_\gamma)^{-1} \exp(-r(A_\gamma)^v) \| \leq d_1 \left( \frac{1}{r} \right)^{p_1} + \frac{1}{2\gamma + 2} . \]

In addition, we show that given \( r > 0, \gamma, \delta > -1 \), the operator \( \exp(-r(A_\gamma)^v) \) maps \( X_\gamma \) into \( X_\delta \). In [2], p. 17, the following result has been proved

\( \forall s \in \mathbb{N} \exists \ell \in \mathbb{N} : \| Q^{2s}(A_\gamma)^{-\ell} \| \gamma < \infty . \)
Here $Q$ denotes the multiplication operator in $X\gamma$ given by

$$(Qf)(x) = xf(x).$$

Now let $\delta > -1$ and let $f \in X\gamma$. Put $s := \lceil \max\{0, \frac{\delta - \gamma}{2}\} \rceil + 1$. Then there exists $\ell_0 \in \mathbb{N}$ such that $\|Q^{2s} A^{-\ell}\|_\gamma < \infty$ for all $\ell \geq \ell_0$. So we derive

$$(*) \quad \int_1^\infty \left| \left( (A_\gamma)^{-\ell} f \right)(x) \right|^2 x^{2s+1} \, dx = \int_1^\infty x^{2(\delta - \gamma)} \left| \left( (A_\gamma)^{-\ell} f \right)(x) \right|^2 x^{2\gamma+1} \, dx \leq \int_1^\infty x^{4s} \left| \left( (A_\gamma)^{-\ell} f \right)(x) \right|^2 x^{2\gamma+1} \, dx \leq \|Q^{2s} (A_\gamma)^{-\ell}\|_\gamma^2 \|f\|_\gamma^2.$$

Following [12], p. 248, there exists $\ell_1 \in \mathbb{N}$ and $d > 0$ such that

$$\max_{x \in [0,1]} |f_k^{(\gamma)}(x)| \leq d(k+1)^{\ell_1}.$$

For $\ell > \ell_1$ it yields

$$(**) \quad \int_0^1 \left| \left( (A_\gamma)^{-\ell} f \right)(x) \right|^2 x^{2s+1} \, dx \leq \left( \max_{x \in [0,1]} \left| \left( (A_\gamma)^{-\ell} f \right)(x) \right| \right)^2 \int_0^1 x^{2s+1} \, dx \leq \frac{1}{25 + 2} \left( \sum_{k=0}^\infty \left( \frac{d^2}{4k + 2\gamma + 2} \right)^{2\ell} \max_{x \in [0,1]} |f_k^{(\gamma)}(x)| \right)^2 \leq \frac{1}{25 + 2} \left( \sum_{k=0}^\infty \frac{(k+1)^{2\ell}}{(4k + 2\gamma + 2)^{2\ell}} \right) \|f\|_\gamma^2.$$
From (*) and (**) we get

\[ \forall \gamma > -1 \forall \delta > -1 \exists \ell \in \mathbb{N} \exists c > 0 \forall f \in X_\gamma : \]

\[ \| (A_\gamma)^{-\ell} f \|_\delta^2 = \int_0^\infty \left( |(A_\gamma)^{-\ell} f(x)|^2 x^{2\delta + 1} \right) dx \leq c \| f \|_\gamma^2 \]

i.e. \((A_\gamma)^{-\ell}\) is a continuous linear operator from \(X_\gamma\) into \(X_\delta\).

(2.2) **Lemma**

Let \(\gamma > -1\). Then for every \(r > 0\), \(v > 0\) and \(\delta > -1\) the operator \(\exp(-r(A_\gamma)^v)\) is a continuous linear operator from \(X_\gamma\) into \(X_\delta\).

**Proof**

Let \(r > 0\), \(v > 0\) and let \(\delta > -1\). Then there exists \(\ell \in \mathbb{N}\) such that \((A_\gamma)^{-\ell}\) is a continuous linear mapping from \(X_\gamma\) into \(X_\delta\). Hence \(\exp(-r(A_\gamma)^v) = (A_\gamma)^{-\ell} \{(A_\gamma)^\ell \exp(-r(A_\gamma)^v)\}\) is also a continuous linear mapping from \(X_\gamma\) into \(X_\delta\). \(\square\)

Lemmas (2.1) and (2.2) yield the following important result.

(2.3) **Theorem**

Let \(\alpha, \beta > -1\). Then for every \(v \geq \frac{1}{2}\)

\[ S_{X_{\alpha},(A_\alpha)^v} = S_{X_{\beta},(A_\beta)^v} \]
Proof

Let \( v \geq \frac{1}{2} \). We have shown that

- \( \exp(-t(A_\alpha)^v), t > 0 \), maps \( X_\alpha \) continuously into \( X_\beta \);
- \( D(R) \supset S_{X_\alpha}, (A_\alpha)^v \), and \( A_\beta = A_\alpha + 2(\alpha - \beta)R \) is positive and self-adjoint in \( X_\beta \);
- There exist constants \( d_\alpha, p_\alpha > 0 \) such that for all \( r > 0 \)

\[
\|\exp(r(A_\alpha)^v) R(A_\alpha)^{-1} \exp(-r(A_\alpha)^v)\|_\alpha \leq d_\alpha \left( \frac{1}{r} \right)^{p_\alpha} + \frac{1}{2a+2}.
\]

So by Theorem (1.3), \( S_{X_\alpha}, (A_\alpha)^v \subset S_{X_\beta}, (A_\beta)^v \). Interchanging \( \alpha \) and \( \beta \) we get the wanted result.

Let \( \alpha > -1 \). Since \( H_\alpha A_\alpha = A_\alpha H_\alpha \), also \( H_\alpha (A_\alpha)^v = (A_\alpha)^v H_\alpha \). So the Hankel transform \( H_\alpha \) is a continuous bijection on the space \( S_{X_\alpha}, (A_\alpha)^v, \nu \geq \frac{1}{2} \), and hence on the spaces \( S_{X_\beta}, (A_\beta)^v, \nu \geq \frac{1}{2}, \beta > -1 \). By duality each transform \( H_\alpha \)

leaves invariant the spaces of generalized functions \( T_{X_\beta}, (A_\beta)^v \). For \( \alpha = -\frac{1}{2} \)

we get \( X_{-\frac{1}{2}} = L_2((0,\infty)) \) and \( A_{-\frac{1}{2}} = -\frac{d^2}{dx^2} + x^2 \). The functions \( L_k^{(-\frac{1}{2})} \) are the even

Hermite functions. With the aid of the papers [8] and [10] the following

categorization of the spaces \( S_{X_{-\frac{1}{2}}}, (A_{-\frac{1}{2}})^v, \nu \in [\frac{1}{2}, 1] \), can be obtained,

\[
f \in S_{X_{-\frac{1}{2}}}, (A_{-\frac{1}{2}})^v \iff f \text{ is extendable to an even function in the space } S_{1/2v}.
\]

The spaces \( S^p_q, p + q \geq 1, p, q \geq 0 \), are introduced by Gelfand and Shilov in [9]. In this connection we note that in our paper [5] we have proved that the

spaces \( S^{k/k+1}_{1/k+1} \) are analyticity spaces; explicitly

\[
S^{k/k+1}_{1/k+1} = L_2(\mathbb{R}), B_k^{(k+1)/2k}
\]

with \( B_k = \left(-\frac{d^2}{dx^2} + x^{2k}\right)^{(k+1)/2k} \).
Relevant for the present paper are the spaces $S^\mu_\mu$, $\frac{1}{2} \leq \mu \leq 1$. We have

$$\varphi \in S^\mu_\mu, \quad \frac{1}{2} \leq \mu \leq 1 \quad \text{if and only if} \quad \varphi \text{ is an entire function satisfying } \exists A, B, C > 0 : |\varphi(x + iy)| \leq C \exp(-A|x|^{1/\mu} + B|y|^{1/1-\mu})$$

and

$$\varphi \in S^1_1 \quad \text{if and only if} \quad \varphi \text{ is analytic on a strip about the real axis say of width } r > 0 \text{ and satisfying }$$

$$\exists A, C > 0 : \sup_{|y| < r} |\varphi(x + iy)| \leq C \exp(-A|x|).$$

Now Theorem (2.3) leads to the following important results.

(2.4) Corollary
Let $\alpha > -1$ and let $\nu \in [\frac{1}{2}, 1]$. Then $f \in S_{\alpha}^{1/2\nu}(A_{\alpha})^{\nu}$ if and only if $f$ is extendable to an even function in the space $S_{1/2\nu}^{1/2\nu}$.

(2.5) Corollary
Let $f \in S_{1/2\nu}^{1/2\nu}$ be even, with $\nu \in [\frac{1}{2}, 1]$. Then for each $\gamma > -1$, there exists an $\ell_2$-sequence $(\omega_n^{(\gamma)})_{n=0}^\infty$ and $t > 0$ such that $f = \sum_{n=0}^\infty \exp(-n^\nu t) \omega_n^{(\gamma)} L_n^{(\gamma)}$ where the series converges pointwise.
Appendix

The set of so-called entire vectors for a positive self-adjoint operator $A$ in a Hilbert space $X$ is equal to

$$D^\infty(A) = \bigcap_{t>0} e^{-tA}(X).$$

In [3], Van Eijndhoven has used the Fréchet space $D^\infty(A)$ as the test space in a theory of generalized functions which is a kind of reverse of the theory in [7]. The space $D^\infty(A)$ is denoted by $\tau(X,A)$ and it may be called the entire-ness space. To our opinion the well-known theory of tempered distributions is considerably generalized in [3]. (Put $A = \log(-\frac{d^2}{dx^2} + x^2 + 1)$. Then $\tau(L^2(\mathbb{R}),A)$ is the space $S(\mathbb{R})$ of functions of rapid decrease.)

Similar to Theorem (1.3) we prove.

(a.1) **Theorem**

Let $P$ be a linear operator in $X$ with $D(P) \supset \exp(-\sigma A^\nu)(X)$ for some $\sigma > 0$ sufficiently large. Suppose the following conditions are satisfied.

(i) There exists a Hilbert space $Y$ such that $\exp(-tA^\nu)$ maps $X$ into $Y$ for all $t > 0$.

(ii) Also, $A + P$ defined on $\exp(-\sigma A^\nu)(X)$ is a positive essentially self-adjoint operator in $Y$.

(iii) There exist positive constants $r_0 \geq 1, d > 0$ and $0 < q < 1/\nu$ such that for all $r > r_0$

$$\|\exp(rA^\nu)PA^{-1}\exp(-rA^\nu)\|_X < dr^q.$$

Then $\tau(X,A^\nu) \subset \tau(Y,(A+P)^\nu)$.
Proof

Since \( \tau(X, A^\nu) = \bigcap_{t > r_0} \exp(-t A^\nu)(X) \), we consider \( t > r_0 \) only. Let \( 0 < \tau < 1 \) with \( s = t - \tau > 1 \). The factoring used in Theorem (1.3) yields the following estimate

\[
\| \exp(\tau A^\nu)(A + P)^k \exp(-t A^\nu) \| \leq k! \left( \frac{1}{\nu s} \right)^{k/v} \prod_{j=0}^{k-1} (1 + d(\tau + js/k)^q).
\]

Put \( b_\tau = 1 + dt^q \). Then

\[
\prod_{j=0}^{k-1} (1 + d(\tau + js/k)^q) \leq b_\tau \prod_{j=1}^{k-1} \left( \frac{k + js}{k} \right)^q \leq b_\tau (1 + d)^{2qk} s^q.
\]

Set \( a = (1 + d)^{2q} \left( \frac{1}{\nu s} \right)^{1/v} \). Then

\[
\| \exp(\tau A^\nu)(A + P)^k \exp(-t A^\nu) \| \leq (k!)^{1/v} \left( \frac{1}{s} \right)^{q+1/v} a^k b_\tau.
\]

For \( f \in \exp(-t A^\nu)(X) \) it yields

\[
\| (A + P)^k f \|_X \leq \| \exp(-\tau A^\nu) \|_{X \to Y} \| \exp(\tau A^\nu)(A + P)^k \exp(-t A^\nu) \|_X \| \exp(t A^\nu) f \|_X \leq (k!)^{1/v} \left( \frac{1}{s} \right)^{q+1/v} a^k b_\tau \| \exp(-\tau A^\nu) \|_{X \to Y} \| \exp(t A^\nu) f \|_X.
\]

Thus we find that \( f \in \exp(-r(A + P)^\nu)(Y) \) for all \( r < \frac{1}{\nu a e} s^{-q+1/v} \). Now put

\[
r(t) = \frac{1}{\nu a e + 1} s^{-q+1/v}
\]

with \( s = t + \frac{1}{t} - 1 \) for instance. Then we get

\[
\tau(X, A^\nu) = \bigcap_{t > r_0} \exp(-t A^\nu)(X) = \bigcap_{t > r_0} \exp(-r(A + P)^\nu)(Y) = \bigcap_{r > 0} \exp(-r(A + P)^\nu)(Y) = \tau(Y, (A + P)^\nu).
\]
It is not hard to see that the spaces $\tau(X_\alpha, (A_\alpha)^\nu)$, $\alpha > -1$, are Hankel invariant, and hence their strong duals $\sigma(X_\alpha, (A_\alpha)^\nu)$. The previous theorem and the Lemmas (2.1) and (2.2) lead to the following classification.

(a.2) Theorem

Let $\alpha, \beta > -1$ and let $\nu \geq \frac{1}{2}$. Then

$$\tau(X_\alpha, (A_\alpha)^\nu) = \tau(X_\beta, (A_\beta)^\nu).$$

By [2] and [8] we obtain the following characterizations

$f \in \tau(X_{-\frac{1}{2}}, (A_{-\frac{1}{2}})^\nu)$ iff $f$ is extendable to an even entire function for which

$$\forall 0 < a < 1 \exists C > 0 \forall x + iy \in \mathbb{C} : |f(x + iy)| \leq C \exp(-\frac{1}{a}x^2 + \frac{1}{2a}y^2)$$

and

$f \in \tau(X_{-\frac{1}{2}}, (A_{-\frac{1}{2}})^{\frac{1}{2}})$ iff $f$ is extendable to an even entire function for which

$$\forall r > 0 : \sup_{|y| < r, -\infty < x < \infty} e^r |x| |f(x + iy)| < \infty.$$ 

Finally, Theorem (a.2) gives the characterization in classical analytic terms of the elements in each $\tau(X_\alpha, A_\alpha)$, respectively $\tau(X_\alpha, (A_\alpha)^{\frac{1}{2}})$, $\alpha > -1$. 
References


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