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Numerical-Experimental Confrontation in Non-Invasive Determination of Hemodynamic Parameters

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Research Report

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Abstract

Atherosclerosis and its consequences is one of the biggest unsolved problems in medicine. Cardiovascular diseases are the leading cause of death in western societies. In general it is assumed that wall shear stress plays an important role in the genesis of vascular disorders. To gain insight in the mechanisms of atherogenesis, and to verify the effects of therapeutical interventions wall shear stress has to be assessed. Using ultrasound equipment the non-invasive assessment of wall shear rate is possible on patients in a clinical environment. However, these measurements are not very accurate near the vessel wall. In contradiction, the ultrasound measurements of blood velocity in the center of the blood vessel are more accurate. A mathematical model is described which uses the in-vivo ultrasound measurements of only the more accurate center-line blood velocity to reconstruct wall shear stress. In order to validate this mathematical model it is compared to the in-vitro LDA measurements. The results show that reconstructed wall shear rate and measured wall shear rate coincide closely. Convective terms cannot be neglected in the mathematical model. With this mathematical model it is also possible to calculate compliance of an artery, a hemodynamic parameter which describes the mechanical state of an artery which can change with age and due to cardiovascular diseases like atherosclerosis. The mathematical model and the clinical ultrasound measurements can be combined to reconstruct wall shear stress and to calculate compliance because a good agreement between model results and measurements is obtained. To examine the inaccuracy of ultrasound measurements they are compared to the more accurate Laser Doppler Anemometry (LDA) measurements which can only be conducted in-vitro because a transparent blood vessel is needed. It appears that ultrasound and LDA differ most in the region where wall shear stress has to be known.
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Chapter 1

Introduction

One of the biggest unsolved problems in medicine is atherosclerosis and its consequences. Cardiovascular diseases are the leading cause of death in western societies [Tardy, 1992]. Another phenomenon is arteriosclerosis, which is the structural change in arterial walls due to normal aging processes, mainly stiffening and dilatation of the vessel walls. Methods to investigate the state of the arteries and the effects of therapeutical interventions are therefore highly desirable [Tardy, 1992], [Brands, 1996].

Different studies have suggested that only in-vivo data can provide physiologically relevant information on the mechanical properties of arteries. It is demonstrated that even gentle dissection of the artery environment significantly lowers its compliance. This suggests that invasive measurements violate the anatomical and metabolic environment of the arteries [Tardy, 1992].

The assessment of hemodynamic parameters such as pressure, blood volume flow, vessel wall distension, arterial impedance, compliance, blood viscosity, shear rate and pulse wave velocity is important. In general, it is assumed that wall shear stress plays an important role in the genesis of vascular disorders. Both high as well as low magnitudes of wall shear stress have been cited as factors leading to vessel wall anomalies [Brands, 1996]. Arterial impedance is the mechanical load the heart encounters in pumping the blood into the arterial system. Arterial compliance is a part of arterial impedance and is an important characteristic of an artery considering the ability of an artery to adapt to changing pressure. Because of the elasticity of the aorta the highly pulsating pressure pulse produced by the heart is smoothed.

Methods have been developed to determine wall shear stress, arterial compliance and impedance on the basis of invasive or non-invasive measurements. Ling and Atabek [Ling, Atabek, 1973] invasively measured pressure and wall distension on a living dog to calculate the mentioned hemodynamic parameters. Tardy [Tardy, 1992] used non-invasive measurements of vessel distension in the forearm arteries and finger pressure to solve the mathematical equations describing the blood flow for wall shear rate and compliance of the forearm artery. In this method the measurement of pressure in the finger is a critical step for the determination of pressure in more proximal vessels like the carotic arteries and aorta.

On the basis of in-vivo ultrasound measurements of velocity profiles and wall distension of arteries Brands [Brands, 1996] is able to calculate wall shear rate and dimensionless arterial impedance. Unfortunately ultrasound measurements of flow velocity near the vessel wall are not very accurate due to limited spatial resolution of the ultrasound equipment and difficulty in distinguishing between slowly moving structures like the vessel wall and the slowly moving blood near the vessel wall. However, measurements of centerline blood velocity with ultrasound and wall distension are assumed to be more accurate. Ultrasound measurements have been compared with the more accurate Laser Doppler Anemometry measurements in an in-vitro experiment of Rutten and Brands [Brands, 1996].

On the basis of measurements of only center line blood velocity, radial distension and longitudinal gradient of radial distension Jaubert [Jaubert, 1995] is able to reconstruct velocity profiles and thus calculate wall shear stress solving the mathematical equations governing the blood flow. It is also possible to calculate compliance on the basis of these measurements. This method can be used in clinical practice because ultrasound measurements of blood flow velocity and radial distension can easily be obtained.

We use the results of the in-vitro Laser-Doppler Anemometry measurements of Rutten and Brands to reconstruct velocity profiles and wall shear rate with the method of Jaubert and compare the results to the measured velocity profiles and wall shear rate.
Chapter 2
Mathematical Model

A mathematical model is adapted and solved in an iterative way. This model is developed at the IMF\(^3\) at the University of Paul Sabatier in Toulouse, France. Non-invasive measurements of center-line blood velocity and vessel distension at two adjacent axial locations are used to reconstruct the complete velocity profiles and the compliance of the artery. In this chapter, the derivation of this model and the iterative method are explained.

2.1 Navier-Stokes Equations

The iterative model for determining velocity profiles, pressure gradient and compliance is based on a modified system of the Navier-Stokes equations. We consider the Navier-Stokes equations in a cylindrical coordinate system. We assume a Newtonian fluid and an axisymmetric flow. The momentum equation and all derivatives in \(\theta\)-direction are omitted. The circumferential velocity \(V_\theta = 0\).

\[
\begin{align*}
\frac{\partial V_z}{\partial t} + V_z \frac{\partial V_z}{\partial z} + V_r \frac{\partial V_z}{\partial r} &= -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_z}{\partial r} \right) + \frac{\partial^2 V_z}{\partial z^2} \right) \\
\frac{\partial V_r}{\partial t} + V_z \frac{\partial V_r}{\partial z} + V_r \frac{\partial V_r}{\partial r} &= -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_r}{\partial r} \right) + \frac{\partial^2 V_r}{\partial z^2} \right) \\
\frac{\partial V_z}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r V_r) &= 0
\end{align*}
\]

Due to distensibility of the tube, pressure and flow waves will propagate with a finite wavespeed \(c\). First, a properly scaled dimensionless form of the NS equations will be derived in order to estimate the importance of the different terms. To this end the radial coordinates are made dimensionless using the mean radius of the tube, i.e. \(r' = r/R_0\). The axial coordinates, however, must be scaled with the wavelength \(\lambda\): \(z' = z/\lambda\). The axial velocity is made dimensionless with its characteristic value over a cross-section: \(V'_z = V_z/V\). From the continuity equation (2.1a) it can be derived that the radial velocity must then be made dimensionless as: \(V'_r = (V_r/V)\lambda/R_0\). The characteristic time \(t' = \omega t\) can be written as \(t' = (c/\lambda)t\). Together with a dimensionless pressure \(\rho' = \rho/\rho Vc\) the dimensionless Navier-Stokes equations read ([van de Vosse, 1996]):

\[
\begin{align*}
\frac{\partial V'_z}{\partial t'} + V'_z \frac{\partial V'_z}{\partial z'} + V'_r \frac{\partial V'_z}{\partial r'} &= -\frac{1}{\rho} \frac{\partial P'}{\partial z'} + \frac{1}{\alpha^2} \left( \frac{1}{r'} \frac{\partial}{\partial r'} \left( r' \frac{\partial V'_z}{\partial r'} \right) + \varepsilon^2 \frac{\partial^2 V'_z}{\partial z'^2} \right) \\
\varepsilon^2 \left( \frac{\partial V'_r}{\partial t'} + V'_z \frac{\partial V'_r}{\partial z'} + V'_r \frac{\partial V'_r}{\partial r'} \right) &= -\frac{1}{\rho} \frac{\partial P'}{\partial r'} + \frac{1}{\alpha^2} \left( \frac{1}{r'} \frac{\partial}{\partial r'} \left( r' \frac{\partial V'_r}{\partial r'} \right) + \varepsilon^2 \frac{\partial^2 V'_r}{\partial z'^2} \right) \\
\frac{\partial V'_z}{\partial z'} + \frac{1}{r'} \frac{\partial}{\partial r'} (r' V'_r) &= 0
\end{align*}
\]

In the above equations \(\varepsilon = \frac{\kappa}{\lambda}\) and \(\alpha = R_0 \sqrt{\varepsilon}\) the Womersley number. We introduce certain simplifications into this system. It is remarked by Reuderink [Reuderink, 1991] that the numerical solution of the equations above is insensible to values of \(\varepsilon = \frac{\kappa}{\lambda} = \frac{\kappa_0 \lambda}{\rho \sqrt{\varepsilon} < 0.1}\). The mean radius is near \(1 \cdot 10^{-2} \text{ m}\),

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and the wave velocity is around 6 [m \cdot s^{-1}] (see paragraph 4.1.2). The frequency of the pressure wave is \( f_0 = 1 \) [Hz]. The value of \( \varepsilon \) thus is \( \frac{1 \cdot 10^{-2}}{6} = 1.6 \cdot 10^{-3} \). We neglect all terms with \( \varepsilon \) in the dimensionless equations of Navier-Stokes thus also the corresponding terms in the original equations. However, the convective terms are kept. The simplified system of Navier-Stokes now looks like:

\[
\left( \frac{\partial V_z}{\partial t} + V_z \frac{\partial V_z}{\partial z} + V_r \frac{\partial V_z}{\partial r} \right) = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left( \frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r} \frac{\partial V_z}{\partial r} \right)
\]

\[
\frac{\partial P}{\partial r} = 0
\]

\[
\frac{\partial V_z}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r V_r) = 0
\]

According to Ling and Atabek [Ling, Atabek, 1972] we now introduce a parameter \( \eta = \frac{r}{R(z,t)} \). The derivation of the different terms is presented in Appendix A. The equations now read as follows:

\[
\frac{\partial V_z}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \left( \frac{\eta}{R} \frac{\partial R}{\partial t} - \frac{V_z}{R} \right) \frac{\partial V_z}{\partial \eta} + \frac{V_z}{R} \left( \frac{\partial V_r}{\partial \eta} + \frac{V_r}{\eta} \right) + \frac{\nu}{R^2} \left( \frac{\partial^2 V_z}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial V_z}{\partial \eta} \right)
\]

\[
\frac{\partial}{\partial \eta} (\eta V_r) = -R \eta \frac{\partial V_z}{\partial z} + \eta^2 \frac{\partial R}{\partial \eta} \frac{\partial V_z}{\partial z}
\]

After partially integrating the second expression of this system and using the boundary condition \( V_z(\eta, z, t)_{\eta=1}=0 \) the system looks as follows:

\[
\frac{\partial V_z}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \left( \frac{\eta}{R} \frac{\partial R}{\partial t} - \frac{V_z}{R} \right) \frac{\partial V_z}{\partial \eta} + \frac{V_z}{R} \left( \frac{\partial V_r}{\partial \eta} + \frac{V_r}{\eta} \right) + \frac{\nu}{R^2} \left( \frac{\partial^2 V_z}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial V_z}{\partial \eta} \right)
\]

\[
V_r = \frac{\partial R}{\partial z} \eta V_z - \frac{2}{\eta} \frac{\partial R}{\partial z} \int_0^\eta \xi V_z d\xi - R \int_0^\eta \frac{\xi}{\eta} \frac{\partial V_z}{\partial z} d\xi
\]

with boundary conditions \( (V_r)_{\eta=0} = 0, (V_r)_{\eta=1} = \frac{\partial R}{\partial t} \) and \( (V_z)_{\eta=1} = 0 \)

This system will have to be solved for \( V_z(\eta, t) \). The main problem is that the term \( \frac{\partial V_z}{\partial \eta} \) is unknown. So the next step is to rewrite this derivative in terms of known quantities.

### 2.2 Elimination of \( \frac{\partial V_z}{\partial \eta} \)

Here, two methods are presented to eliminate \( \frac{\partial V_z}{\partial \eta} \) from the second equation of system (2.5). The first one is from Ling and Atabek [Ling, Atabek, 1972], the second one is more recent and is from Cavalcanti.

#### 2.2.1 Ling and Atabek's Method

Ling and Atabek [Ling, Atabek, 1972] introduce a hypothesis for the development of the longitudinal velocity. This hypothesis is based on experimental data for small \( \Delta z \) and a rigid tube.

\[
V_z(\eta, z + \Delta z, t) \approx K(z, t) V_z(\eta, z, t)
\]

\[
\frac{\partial V_z}{\partial z} \approx \frac{1}{\Delta z} \text{sgn} \left( \frac{\partial V_z}{\partial z} \right) \cdot |(K(z, t) - 1) \cdot V_z(\eta, z, t)|
\]

To eliminate the term \( \frac{\partial V_z}{\partial \eta} \) from equation (2.5b) we first project this equation on the vesselwall which gives equation (2.8). Next we substitute the expression for \( \frac{\partial V_z}{\partial \eta} \), equation (2.6b) into equation (2.8). Now we can isolate the term \( K(z, t) \) from this result and substitute this into equation (2.6b). Now the term \( \frac{\partial V_z}{\partial \eta} \) no longer depends explicitly on \( K(z, t) \) and we substitute this one into equation (2.5b). The resulting expression is as follows:
The disadvantage of this expression is that the complete axial velocity field has to be known.
In their article ([Ling, Atabek, 1973]), Ling and Atabek present a method for determining velocity profiles and wall shear stress starting from the invasive measurements of pressure, pressure gradient, diameter-pressure relation and tapering angle of the artery using the above equations. They conducted these experiments on a living dog. The solution is reached by solving the above equations in an iterative way. Their results indicate a large difference between the solution with the equations in their full non-linear form (equations (2.5a) and (2.7)) and the linearized equations, in which the convective terms are discarded. This is probably due to the importance of the convective terms in their experiments, because a natural tapering angle is present which causes the convective terms for a major part. The other part of the convective terms is caused by the distensibility of the tube.

2.2.2 Cavalcanti's Method

Cavalcanti presents another method to eliminate the term $\frac{\partial V_z}{\partial x}$. Equation (2.5b) is projected on the vessel wall. This means that $V_z=0$, $\eta=1$ and $V_z(\eta,z,t)_{\eta=1} = \frac{\partial R}{\partial t}$. This gives:

$$\frac{\partial R}{\partial t} = -2 \frac{\partial R}{\partial z} \int_0^1 \xi V_z d\xi - R \int_0^1 \xi \frac{\partial V_z}{\partial z} d\xi$$

(2.8)

Because $\frac{\partial R}{\partial t}$ is independent of $\eta$ we use $\frac{\partial R}{\partial t} = \int_0^1 2 \xi \frac{\partial R}{\partial t} d\xi$.

$$\int_0^1 \frac{2 \xi}{R} (\frac{\partial R}{\partial t} + \frac{\partial R}{\partial z} V_z) d\xi = - \int_0^1 \xi \frac{\partial V_z}{\partial z} d\xi$$

(2.9)

Considering this integral as a local form we obtain:

$$\frac{\partial V_z}{\partial z} = \frac{2}{R} (\frac{\partial R}{\partial t} + \frac{\partial R}{\partial z} V_z)$$

(2.10)

After substituting this last result into the original equation (2.5b) for $V_z$ the following expression for $V_z$ is obtained:

$$V_z = \eta \left( \frac{\partial R}{\partial t} + \frac{\partial R}{\partial z} V_z \right)$$

(2.11)

2.3 Iterative Method

In the iterative method of Jaubert [Jaubert, 1995], we will start from the measurements of center-line blood velocity and diameter working towards the solution of velocity profiles, wall shear rate and longitudinal gradient of pressure. This is in fact the other way around from Ling and Atabek because they started from the measurements of longitudinal gradient of pressure working towards the solution of velocity profiles and wall shear rate. The governing set of equations for the iterative model thus is:

$$\frac{\partial V_z}{\partial t} = \frac{1}{\rho} \frac{\partial P}{\partial z} + \left( \frac{\eta}{R} \frac{\partial R}{\partial t} - V_z \right) \frac{\partial V_z}{\partial \eta} + \frac{V_z}{R} \left( \frac{\partial V_z}{\partial \eta} + \frac{V_z}{\eta} \right) + \frac{\nu}{R^2} \left( \frac{\partial^2 V_z}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial V_z}{\partial \eta} \right)$$

with $V_r = \frac{\partial R}{\partial z} \left( \eta V_z - \left\{ U_1(\eta,t) - \frac{U_1(1,t)}{U_2(1,t)} U_2(\eta,t) \right\} \right) + \frac{1}{\eta} \frac{\partial R}{\partial t} \frac{U_2(\eta,t)}{U_2(1,t)}$

(2.12)

or $V_r = \eta \left( \frac{\partial R}{\partial t} + \frac{\partial R}{\partial z} V_z \right)$

This iterative method consists of solving this set of equations starting from the non-invasive measurements of longitudinal center line blood velocity on the axis and the wall distension at two adjacent axial positions.
Because the radial variation of pressure is negligible we can project equation (2.12a) on the axis in order to determine $\frac{\partial P}{\partial z}$. This gives:

$$\frac{\partial P}{\partial z} = -\rho \left( \frac{\partial V_z}{\partial t} \right)_{\eta=0} + \frac{2 \rho V_z}{R} \left( \frac{\partial V_r}{\partial \eta} \right)_{\eta=0} + \frac{2 \rho V_z}{R^2} \left( \frac{\partial^2 V_r}{\partial \eta^2} \right)_{\eta=0}$$ (2.13)

For this purpose the following Taylor expansion is used:

$$\lim_{\eta \to 0} \frac{V_r(\eta)}{\eta} = \lim_{\eta \to 0} \frac{1}{\eta} \left( V_r |_{\eta=0} + \left( \frac{\partial V_r}{\partial \eta} \right)_{\eta=0} \cdot \eta + o(\eta^2) \right) = \left( \frac{\partial V_r}{\partial \eta} \right)_{\eta=0}$$ (2.14)

$(V_r)_{\eta=0} = 0$ because of the assumption of axisymmetry. Equation (2.13) is an equation of volume forces and expresses that the gradient of pressure on the axis is equilibrated by in-stationary forces, convective forces mostly due to natural tapering, and viscous forces. The iterative method consists of initializing a pressure gradient with a modification of equation (2.13), and from the assumption that $\frac{\partial P}{\partial z}$ does not change much in the radial direction we can calculate the radial and axial velocity fields from equations (2.12a with b or c). After having calculated the velocity fields, a new $\frac{\partial P}{\partial z}$ can be calculated using equation (2.13). This process is continued until the difference between the measured velocity on the axis and the recalculated velocity on the axis is beneath a certain, small value.

For initialization of $\frac{\partial P}{\partial z}$, equation 2.13 is modified. Because the initial velocity profile is not known, a quadratic velocity profile is assumed which fits the measured velocity on the axis.

$$V_z(z_0, \eta, t) = V_{z,measured}(z_0, t) \cdot (1 - \eta^2)$$ (2.15)

$$\frac{\partial^2 V_z}{\partial \eta^2} = -2V_{z,measured}(z_0, t)$$ (2.16)

The initial radial velocity field is not known neither. In the case of Cavalcanti, we can approximate the initial radial velocity field with equation (2.11) which we substitute into equation (2.13). Jaubert [Jaubert, 1995] showed that the iterative method when initializing according to equation (2.18) converges faster. In the case of Ling-Atabek, the derivative of $V_r$ is not calculated to initialize $\frac{\partial P}{\partial z}$.

$$\text{Ling-Atabek} \quad \frac{\partial P}{\partial z} = -\rho \left( \frac{\partial V_z}{\partial t} \right)_{\eta=0} - \frac{4 \rho V_z}{R^2} V_{z,measured}$$ (2.17)

$$\text{Cavalcanti} \quad \frac{\partial P}{\partial z} = -\rho \left( \frac{\partial V_z}{\partial t} \right)_{\eta=0} - \frac{4 \rho V_z}{R^2} V_{z,measured} + \frac{2 \rho V_z}{R} \left( \frac{\partial R}{\partial t} + \frac{\partial R}{\partial z} V_{z,measured} \right)$$ (2.18)

### 2.3.1 Compliance Evaluation

Compliance can be calculated from this measurements as follows:

$$C = \frac{\partial A}{\partial P} = 2\pi R \frac{\partial R}{\partial P} = 2\pi R \frac{\partial R}{\partial \frac{\partial P}{\partial z}}$$ (2.19)

Because $\frac{\partial R}{\partial \frac{\partial P}{\partial z}}$ is measured and $\frac{\partial P}{\partial z}$ is reconstructed with the model the compliance of the artery can be calculated.

### 2.4 Analytical Model

An analytical solution is available for the case that the convective terms are negligible. Considering the Navier-Stokes equations in a dimensionless form (equation (2.2a) without the term with $\varepsilon$):

$$\alpha^2 \frac{\partial V'}{\partial t} + \frac{\rho V'}{c} \left( V'_z \frac{\partial V'}{\partial z} + V'_r \frac{\partial V'}{\partial r} \right) = -\alpha^2 \frac{\partial P'}{\partial z} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial P'}{\partial r} \right)$$ (2.20)

The ratio of convective terms to in-stationary term and pressure gradient is on the order of $\frac{V'}{V}$. The maximal velocity $V$ is 0.6 [m·s⁻¹] and the wave velocity if 6.0 [m·s⁻¹], so the convective terms constitute
Mathematical Model

about 10% of the in-stationary and pressure forces. When neglecting the convective terms, the momentum
equation of Navier-Stokes reduces to:

\[
\frac{\partial V_z}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial Z} + \nu \left( \frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r} \frac{\partial V_z}{\partial r} \right)
\]  

(2.21)

For this last equation an analytical solution is available for velocity profiles, wall shear rate, flow and
longitudinal gradient of pressure. This equation can be solved for each harmonic in the frequency domain.
We use real Fourier coefficients and the signal can be written in terms of this real coefficients as follows:

\[
v_z(r, t) = \sum_{n=-N}^{N} a_n \cos(\omega_n t) + b_n \sin(\omega_n t)
\]

\[
= \sum_{n=-N}^{N} \frac{1}{2} \left( e^{i\omega_n t} + e^{-i\omega_n t} \right) + \frac{1}{2i} \left( e^{i\omega_n t} - e^{-i\omega_n t} \right)
\]

\[
= \sum_{n=-N}^{N} \frac{1}{2} (a_n - ib_n) e^{i\omega_n t} + \frac{1}{2} (a_n + ib_n) e^{-i\omega_n t}
\]

\[
= \sum_{n=-N}^{N} a_n e^{i\omega_n t}
\]

We substitute the following harmonic solutions into equation (2.21):

\[
v_z = v_z^+ e^{i\omega_n t} + v_z^- e^{-i\omega_n t}
\]

\[
\frac{\partial P}{\partial Z} = \frac{\partial P^+}{\partial Z} e^{i\omega_n t} + \frac{\partial P^-}{\partial Z} e^{-i\omega_n t}
\]

The above two harmonic solutions imply the absence of reflections. The results after substituting are
as follows, according to Womersley (1957) who initiated these solutions, with the Fourier coefficients of
the flow signal as calculated from the integrated velocity profiles. Also, the boundary conditions will be
linearized when using this description in the frequency domain. So the superposition of harmonic signals
might not be valid with a moving vessel wall.

\[
\frac{\partial P}{\partial Z} = -\frac{8\eta}{\pi R^3} Q_0 + \sum_{n=1}^{N} \frac{\rho \eta}{2\pi R^2} \left\{ \left( -i\omega_n + \frac{\nu R^2}{m} \right) Q_n^+ e^{i\omega_n t} + \left( i\omega_n + \frac{\nu R^2}{m'} \right) Q_n^- e^{-i\omega_n t} \right\}
\]

\[
\tau = -\frac{4\eta}{\pi R^3} Q_0 + \sum_{n=1}^{N} \frac{\eta}{2\pi R^2} \left\{ m^2 \frac{F_{10}(m)}{1 - F_{10}(m)} Q_n^+ e^{i\omega_n t} + m'^2 \frac{F_{10}(m')}{1 - F_{10}(m')} Q_n^- e^{-i\omega_n t} \right\}
\]

\[
v_z = v_{z,0} + \sum_{n=1}^{N} \frac{1}{\pi R^2} \left( 1 - \frac{1}{1 - F_{10}(m)} \left( 1 - \frac{J_0(\alpha i \frac{\omega_n}{R})}{J_0(\alpha i \frac{\omega_n}{R})} \right) Q_n^+ e^{i\omega_n t} + \frac{1}{1 - F_{10}(m')} \left( 1 - \frac{J_0(\alpha i \frac{\omega_n}{R})}{J_0(\alpha i \frac{\omega_n}{R})} \right) Q_n^- e^{-i\omega_n t} \right)
\]

Where

\[
Q_n^+ = \frac{1}{2} (a_n - ib_n) \quad \text{and} \quad Q_n^- = \frac{1}{2} (a_n + ib_n)
\]

\[
v_{z,0} = -\frac{2}{\pi R^2} \left( \frac{1}{\alpha^2} - 1 \right) Q_0
\]

\[
m = \alpha i \frac{\omega_n}{R} \quad \text{and} \quad m' = \alpha i \frac{\omega_n}{R}
\]

\[
\alpha = R\sqrt{\frac{\omega_n}{\nu}}
\]

\[
F_{10}(m) = \frac{2J_1(m)}{mJ_0(m)}
\]

This method is implemented and the results are presented in chapter 5.
Chapter 3

Measurements

The validation of the iterative method for solving the reduced Navier-Stokes equations is performed by comparing in-vitro measurements of velocity profiles with the results of this method. The in-vitro experiments were done at the department of Mechanical Engineering (WFW) of the Eindhoven University of Technology and are available for comparison with the numerical method.

3.1 Experimental Setup

The in-vitro measurements used here were intended to compare the Ultrasound measurements to the more accurate Laser Doppler Anemometry measurements. The former can be used in vivo for measuring blood velocity profiles and wall distension, the latter can only be used in-vitro because a transparent medium is necessary. From the comparison it became clear that the coefficient of variation for the difference in wall shear stress determination with both methods was 9.4% (see figure 3.6).

The distensible tube in the in vitro setup was composed of two component silicon rubber (Sylgard 184 from Dow Corning) and had a length of 545 [mm], an inner radius of 9 [mm], a wall thickness of 1 [mm] and a speed of sound of 1010 [m \cdot s^{-1}]. The tube was installed in a closed flow circuit with a stationary and a non-stationary pump. The stationary pump led the fluid from a reservoir to the distensible tube via a rigid inlet tube with a length of 1.5 meter (75 times the tube diameter). The length of this rigid inlet tube should be long enough to obtain a fully developed velocity profile at the entrance of the distensible tube. A non-stationary pump was installed between the stationary pump and the rigid inlet tube. This pulsatile pump generated a volume flow pulse \( f_0 = 1 \) [Hz] based on the first ten harmonics of an aorta volume flow pulse presented by Milnor [Milnor, 1989]. To create physiological pressure-flow relations in the distensible tube, the tube was terminated with a hydraulic load characteristic for the thoracic aorta.

The fluid in the flow circuit was a water-glycerol mixture (60% water and 40% glycerol) containing small air bubbles (10-30 \( \mu \)m diameter) for ultrasound scattering. The fluid was Newtonian with a density of 1096 [kg \cdot m^3] and a dynamic viscosity of 3.74 [mPa \cdot s].

Distension of the tube and flow velocity distribution inside the tube were measured at three positions (1=106 [mm], 2=231 [mm] and 3=356 [mm] downstream of the entrance of the tube) with pulsed ultrasound and LDA. The pressure in the tube was measured using a catheter-tip pressure transducer (Millar) at the three given positions simultaneously with distension [Brands, 1996], [Slegers, 1995].

Because of interference of the ultrasound equipment when measuring distension and flow velocity distribution simultaneously, it was not possible to measure these quantities at the same time. A synchronous pulse is generated at every second by a computer. The generated input pressure wave form is synchronized to this pulse. Another computer is synchronized to this pulse as well for measuring distension, pressure and flow velocity distribution sequentially. The measurements for the three positions are taken sequentially as well in the same way.
3.2 Ultrasound Measurements

Velocity profiles have been measured in horizontal direction for four periods. The ultrasound probe measured at an angle of $66^\circ$ with flow direction. The velocity profiles are measured in a horizontal plane.

3.2.1 Ultrasound Velocity Profiles

The measurements of velocity profiles are corrected to a normalized radius. The flow velocity distributions for the three different positions are displayed in the following figure.

![Flow Velocity Distribution](image)

**Figure 3.1:** The flow velocity distribution at the three positions as measured with ultrasound.

Position 1 is the most downstream position. The velocity profiles are not perfectly axisymmetric. This is unfortunate for the model input because the mathematical model assumes an axisymmetric velocity profile. The rigid inlet tube had to ensure a fully developed axisymmetric velocity profile, and it is long enough for it. Only the first half of the velocity profiles is measured correctly, the second half couldn't be measured correctly probably due to the scattering of micro air bubbles ([Slegers, 1995]). The velocity profiles at positions 3 and 2 are discarded because they are the least axisymmetric.

3.3 LDA Measurements

In the Laser Doppler Anemometry measurements a measurement grid is placed over an axial position of the tube. The measurement volume of the LDA is displaced in radial direction 16 steps of 200 $\mu$m near the front vessel wall, followed by 38 steps of 400 $\mu$m crossing the central region of the velocity field and again 15 steps of 200 $\mu$m near the back vessel wall. At every step, velocity and pressure have been measured simultaneously during 16 cycles with a sample frequency of 128 [Hz] ([Slegers, 1995]). The measurement points are "attached to the real world", so the distensible tube will move through this measurement grid. Near the vessel wall, the measurement points are closer together in order to be able to assess the high velocity gradient near the vessel wall more accurately. In representing the velocity profiles in terms of dimensionless coordinates it is corrected for this, as well for the vessel wall which changes position in the measurement grid.
3.3.1 LDA Velocity Profiles

![Flow Velocity Distribution](image)

Figure 3.2: The flow velocity distribution at the three positions as measured with LDA.

3.4 Other Measurements

Pressure, flow and distension measurements are conducted in the same way for both ultrasound as well as LDA measurements.

3.4.1 Pressure

Pressure is measured at each position. Because of reflections due to the distal impedance the pressure wave changes with the longitudinal position.

![Pressure waves](image)

Figure 3.3: Pressure waves at the three locations.
3.4.2 Distension

Distension is the change in diameter relative to the diameter in rest, which is $18.43 \cdot 10^{-2}$\text{m}. Distension is measured at each position. The shape and amplitude changes due to reflections. In figure (3.4) the distension for all three positions is displayed.

![Figure 3.4: Distension waves at the three locations.](image)

3.4.3 Distension-Pressure Relation

The quasi-static diameter-pressure relationship is measured (\cite{Slegers, 1995}). The linear coefficient for the diameter-pressure relation is $0.226 \text{[mm}\cdot \text{kPa}^{-1}]$. This static diameter-pressure relation is measured at position 2. For the dynamic diameter-pressure relation it is $0.2 \text{[mm}\cdot \text{kPa}^{-1}]$ for position 1, $0.222 \text{[mm}\cdot \text{kPa}^{-1}]$ for position 2 and $0.25 \text{[mm}\cdot \text{kPa}^{-1}]$ for position 3. Change in diameter is proportional to and in phase with change in pressure. The assumption of a purely elastic tube thus is correct. Because the coefficients are not equal to each other the tube might not be uniform in its properties.

![Figure 3.5: Dynamic distension-pressure relations at the three locations. Distension here is change in diameter relative to static diameter.](image)
3.5 Measurements as Model Input

From the above mentioned measurements axial velocity and diameter will be used as input for the iterative method solving the NS equations. This iterative method however also needs the measurement of the gradient of radius with respect to the longitudinal axis. This gradient can be measured by means of a second distension measurement close to the first distension measurement. This gradient of radius is however not available, because the three positions are not close enough to each other for calculating the gradient of radius. In order to examine the influence of the measurement of gradient of radius, we will approximate the gradient of radius by means of a model.

One of the assumptions in the model is that the axial movement of the tube is restrained. This is thought to be relevant in vivo due to tethering of the artery in the surrounding tissue. However, the experiments are done in vitro on a tube and the assumption of axial restraint might not be valid.

Another assumption in the iterative method is that the velocity profiles are axisymmetric. As can be seen, the velocity profiles are not perfectly axisymmetric. This will cause differences in the reconstructed velocity profiles. However, the important characteristics to be calculated are wall shear stress and longitudinal gradient of pressure. We will examine the reconstruction of these quantities.

3.6 Calculation of Shear Rate

The shear rate is calculated as the difference between the longitudinal velocity at two adjacent discretized radial positions, divided by the radial distance between those positions. For the LDA velocity profiles, the mean of the two sides of the velocity profiles is taken. For the ultrasound measurements, only the first half is used to calculate the wall shear rate because the other half is not measured accurately enough (see paragraph 3.2.1).

\[
\gamma(r, t) = \frac{\partial v(r, t)}{\partial r} \Rightarrow \gamma(\eta, t) = \frac{1}{R} \left( \frac{\partial v(\eta, t)}{\partial \eta} \right)_{\eta=1} \tag{3.1}
\]

Wall shear rate as calculated from the measurements will have two main influences of errors; the first is the error in the measurement itself. The standard deviation in the velocity profile measurements is about 0.03 with peaks of 0.08. The other source of errors in deriving wall shear rate from the velocity profiles results from the fact that the measurement points are discrete points. By taking the linear slope between the two measurement points near the vessel wall an error is introduced. Therefore, it is better to take a polynomial fit for the first measurement points near the vessel wall and calculate the wall shear rate from this polynomial fit. With this method, both influences of error are diminished.

\[
V_s(\eta, t) = \sum_{i=0}^{N} \alpha_i(t) \eta^i \tag{3.2}
\]

\[
\gamma(r, t) = \frac{1}{R} \sum_{i=1}^{N} i \alpha_i(t) \eta^{i-1}
\]

The comparison between shear rate as calculated from the LDA measurements and from the ultrasound measurements is presented in figure (3.6).
3.7 Validity of Ling-Atabek Hypothesis

Ling and Atabek introduced a hypothesis for the development of $\frac{\partial V_z}{\partial t}$ (see equation (2.6) in paragraph (2.2.1)). We check the validity of this hypothesis for the experimental data we have available.

3.7.1 Calculation of $K(z,t)$ from the Measurements

The quotient of longitudinal velocity is calculated for all steps in time and for all radial positions for the LDA velocity profiles.

\[ K_1(\eta, t) = \frac{V_z(\eta, t)_{pos1}}{V_z(\eta, t)_{pos2}} \]
\[ K_2(\eta, t) = \frac{V_z(\eta, t)_{pos2}}{V_z(\eta, t)_{pos3}} \]

Ling and Atabek state that the coefficients $K_1$ and $K_2$ should be independent of $\eta$ for a small $\Delta z$. To verify this, both coefficients are displayed in figure 3.7. The left column represents coefficient $K_1$, the right column represents coefficient $K_2$. The above two pictures represent the coefficients for $-0.25 < \eta < 0.25$ and so on. In fact, these plots are each 3D-plots as seen from one side so they seem to be 2D-plots. The $\eta$-direction is perpendicular to the paper. As can be seen, for $-0.25 < \eta < 0.25$, the coefficients are fairly constant for all moments in time. For values of $\eta$ above 0.5 the coefficients can no longer be considered as constant, except for the systolic peak between $t=0.15$ and $t=0.30$. In the boundary layer where $\eta > 0.75$ the coefficients are not at all constant any more. The $\Delta z$ used here from the experiments is small compared to the wavelength, $\Delta z = 0.125$ [m] and $\lambda = 6.0$ [m]. For improving the resulting plots in figure (3.7) it should be better to filter the results so that the high-frequency noise is reduced.
Figure 3.7: Coefficients $K_1$ and $K_2$
Chapter 4

Calculation of Longitudinal Gradient

The longitudinal gradient of radius \( \frac{\partial R}{\partial z} \) is not measured. However, this is an input parameter for the iterative method. One of the output quantities of the iterative method is the longitudinal gradient of pressure \( \frac{\partial P}{\partial z} \). In this chapter the gradient of radius \( \frac{\partial R}{\partial z} \) will be calculated in order to obtain the proper input for the iterative method. Also, the \( \frac{\partial P}{\partial z} \) will be calculated from the measurements in order to be able to compare with the \( \frac{\partial P}{\partial z} \) as reconstituted with the iterative method.

4.1 Calculation of Longitudinal Gradient

To examine the importance of \( \frac{\partial R}{\partial z} \), we will check the results obtained from the iterative method with a \( \frac{\partial R}{\partial z} \) subsequently set equal to zero, a linear fit and a quadratic fit between the three positions. Also, a method is tried to calculate \( \frac{\partial R}{\partial z} \) with another method which needs the value of the wave velocity. The same method is used to calculate \( \frac{\partial P}{\partial z} \) at one of the axial positions in order to compare with the \( \frac{\partial P}{\partial z} \) as calculated from the iterative method. These methods might not be valid because reflections are present in the system, and therefore another method is tried based on an analytical solution of the linearized equations of Navier-Stokes and the static pressure-diameter relation.

4.1.1 Quadratic Fit

Because the distension is measured at three axial positions, it is possible to fit a quadratic curve through the measurements of distension at these three positions.

\[
R(z, t) = a_1(t)z^2 + \beta_1(t)z + \gamma_1(t)
\]

\[
\frac{\partial R}{\partial z}(z_i, t) = 2a_1(t)z_i + \beta_1(t) \tag{4.1}
\]

The same can be done for the pressure:

\[
P(z, t) = a_2(t)z^2 + \beta_2(t)z + \gamma_2(t)
\]

\[
\frac{\partial P}{\partial z}(z_i, t) = 2a_2(t)z_i + \beta_2(t) \tag{4.3}
\]

The parameters \( a_i, \beta_i \) and \( \gamma_i \) are calculated for each step in time. The fitted distension and pressure wave forms as function of \( z \) can be visualized as in figure (4.1). The calculated signal of \( \frac{\partial R}{\partial z} \) is presented in figure (4.2), the calculated signal of \( \frac{\partial P}{\partial z} \) is presented in figure (4.2). As can be seen, the shapes of the left and right curves are not the same and this stems from the fact that although the relation for pressure and radius is linear in a fit, there is some deviation as can be seen in figure (3.5). The way of plotting the curves in figure (4.1) is very sensitive to little deviations from this linear fit as in figure (3.5).
Chapter 4

Polyfit of Pressure as function of Z

14.5
14
13.5
13
12.5
12
11.5
11

0 100 200 300
Z-axis [mm]

Polyfit of Diameter as function of Z

0.7
0.6
0.5
0.4
0.3
0.2
0.1

0 100 200 300
Z-axis [mm]

Figure 4.1: Quadratic fit of pressure and distension measurements. The numbers next to the curves indicate the moment in time, where 128 equals 1 second.

4.1.2 Phase Shift Method

In the next method, $\frac{\partial R}{\partial z}$ and $\frac{\partial P}{\partial x}$ are calculated from the assumption that the shape and amplitude of the distension and pressure waves do not change much for small variations in the axial position. In fact, the shape and amplitude of the pressure and distension waves do change due to reflections and damping (see figures (3.3) and (3.4)). Nevertheless we try this method. The following expression is valid for small $\Delta z$:

$$\frac{\partial R}{\partial z}(z, t) = \frac{R(z + \Delta z, t) - R(z, t)}{\Delta z}$$

(4.5)

The above mentioned assumption reads like:

$$R(z + \Delta z, t) = R(z, t - \Delta t)$$

(4.6)

The value of $\Delta t$ corresponds to the time the pressure or distension wave takes for traveling a distance $\Delta z$. Thus, the wave velocity $c$ is needed and the $\Delta t$ can be substituted by $\frac{\Delta z}{c}$. The equation for $\frac{\partial R}{\partial z}$ reads as follows:

$$\frac{\partial R}{\partial z}(z_i, t) = \frac{R(z_i, t - \frac{\Delta t}{c}) - R(z_i, t)}{\Delta z}$$

(4.7)

For large Womersley numbers the wave velocity $c$ can be approximated by the following equation. The Womersley Number $\alpha$ is in this case equal to 12.23.

$$c = \sqrt{\frac{A_0}{\rho G}}$$

(4.8)

with $C(P) = \frac{\partial A}{\partial P} = 2\pi R \frac{\partial R}{\partial P}$. This can be calculated from the static diameter and distension and pressure measurements. Calculated compliance and wave velocity are displayed in figure (4.2). In this figure, the whole static range for pressure is indicated. During the experiments the dynamic pressure range is from 11 to 14.5 [kPa] (see figure (3.3)). In this dynamic pressure range the wave velocity is about 5.9 [m·s$^{-1}$].
4.2 Results

The calculated $\frac{\partial R}{\partial z}$s for the linear method, the quadratic method and the phase shift method are presented in figure (4.2). The results for $\frac{\partial P}{\partial z}$ are presented in figure (4.2). They are different from each other but the amplitude is of the same order. These signals can be used as input for the iterative method and be used for comparison with the output of this iterative method. As will appear later, the $\frac{\partial R}{\partial z}$ of this magnitude will have a negligible influence on the results. Notice that there is just a constant difference in amplitude between the linear and quadratic method and the analytical solution. The shape of the phase shift result is the same as for the other three methods but not for the second peak between $t=0.2$ and $t=0.3$. In conclusion we can say that the longitudinal gradient of pressure and radius are of the order as indicated in the figures and have the shape as indicated.

![Figure 4.2: Compliance and Wave Velocity for the tube according to quasi static diameter-pressure relationship.](image)

![Figure 4.3: The evolution of $\frac{\partial R}{\partial z}$ and $\frac{\partial P}{\partial z}$ with time for position 1 as calculated with the linear, the quadratic and the phase shift method and the analytical solution for a rigid straight tube without convective terms.](image)
Chapter 5

Results

Results of the iterative method for different model parameters are presented. The results comprise wall shear rate, velocity profiles, longitudinal pressure gradient and flow. Parameters are axial velocity, radius, gradient of radius, number of positions in radial mesh and method of calculation of radial velocity in the iterative method. A comparison is made with an analytical model described in chapter 2.

5.1 Temporal Resolution

The number of steps in time in the measurements for one period is 128. However, no convergence is reached with the iterative method. The Δt is too large to reach converge because we use an explicit numerical scheme to solve the Navier-Stokes equations. Therefore, the number of steps in time is increased to 4096 so the Δt is smaller. This is reached by calculating the Fourier coefficients from the original measurements with 128 steps in time and reconstructing the signals but now with 4096 steps in time. The program is changed in such a way that only the output data for 128 steps in time equally spread over one period is written to output files for visualization.

5.2 Ultrasound

Influence of Gradient Measurement

In this section, the iterative method using the Cavalcanti equations is used. Wall shear rate is calculated according to section (3.6) as the maximum of the linear slope between two measured points within 0.9 mm from the vessel wall according to [Brands, 1996]. We can question this method for its validity and as can be seen in figure (5.1), the maximum within 0.9 mm is not even enough to coincides with the reconstructed wall shear rate. For the ultrasound measurements, only the first half of the velocity profiles is used to calculate wall shear rate because the second half is not measured correctly (section (3.2)). The different $g_s$ have no influence on the reconstructed wall shear rate. However, quite a large difference is present between the reconstructed and measured wall shear rate. The iterative method calculates the velocity profiles and thus the flow as function of time. This flow is compared with the integrated measured velocity profiles in figure (5.2). Reconstructed flow coincides well with measured flow (integrated measured velocity profiles). Because the LDA measurements are more reliable and more axisymmetric, and the reconstructed wall shear rate is closer to the LDA wall shear rate the ultrasound measurements are further discarded. This coincides with the previous statement that velocity measurements near the vessel wall done with ultrasound are not precise.

5.3 Laser Doppler Anemometry

5.3.1 Influence of Method

Three different versions of the iterative method are tested. The first version calculates $V_r$ according to Cavalcanti (equation (2.12c)). The second version according to Ling-Atabek (equation (2.12b)). The third version discards the convective terms so there is no need to calculate the $V_r$ because $V_z$ does not
Figure 5.1: Influence of $dR/dZ$. Comparison reconstructed wall shear rate for different $\frac{dR}{dz}$ with ultrasound wall shear rate.

Figure 5.2: Influence of $dR/dZ$. Comparison reconstructed flow (solid lines) for different $\frac{dR}{dz}$ with integrated ultrasound velocity profiles (dashed line).

depend on $V_r$ in this case (see equation (2.21)). The longitudinal gradient of radius is set to 0. The dynamic radius is calculated from the static pressure-diameter relationship measured at position 2. Flow is presented in figure (5.3). Longitudinal gradient of pressure in figure (5.4), shear rate in figures (5.5) and (5.6). Notice that the wall shear rate peak at $t=0.35$ for the Cavalcanti and Ling-Atabek method coincides with the peak according to the polynomial fit (see figure (3.6)). We can conclude that the influence of the method has a very small effect on reconstructed flow and $\frac{dR}{dz}$. However, the influence of
the method on the reconstructed wall shear rate is of importance. For the case that the convective terms are neglected, the wall shear rate differs about 15% for the systolic and diastolic peaks.

Figure 5.3: Influence of Method. Comparison reconstructed flow for different methods with integrated LDA velocity profiles with $R=R(t)$ and $\frac{\partial R}{\partial z}=0$. **Left**: reconstructed Flow signals solid lines, Integrated Flow signal dashed line. **Right**: Difference between reconstructed and integrated flow signal. Cavalcanti (solid line), Ling-Atabek (..) and Linear Method (-.)

Figure 5.4: Influence of Method. Comparison reconstructed $\frac{\partial P}{\partial z}$ for different methods with shifted linear fit for LDA measurements with $R=R(t)$ and $\frac{\partial R}{\partial z}=0$. 
5.3.2 Influence of $\frac{\partial R}{\partial z}$

In Figure (5.7) the difference in flow calculation from the iterative model is displayed for the three methods with $\frac{\partial R}{\partial z} = 0$ respectively equal to a model value. The model value is equal to the output of the iterative method for the $\frac{\partial R}{\partial z}$ multiplied by the static pressure-diameter relation $\frac{\partial R}{\partial p}$ (see section (3.4.3)). The difference in Figure (5.7) between the flow as calculated with $\frac{\partial R}{\partial z} = 0$ and $\frac{\partial R}{\partial z}=$ model value is less than 1.
% for all three methods. We can conclude that the influence of longitudinal gradient of radius on the reconstructed flow signal is negligible.

![Figure 5.7: Influence of dR/dZ. Difference reconstructed flow for different methods with integrated LDA velocity profiles with R=R(t) between $\frac{dR}{dz}$ =0 respectively set to model value, thus $Q\left(\frac{dR}{dz} = 0\right)$ - $Q\left(\frac{dR}{dz} \neq 0\right)$. Cavalcanti(solid line), Ling-Atabek(---), Linear Method (-.)](image)

In figure (5.8) the difference for the calculated $\frac{dP}{dz}$ is displayed for $\frac{dR}{dz} =0$ and for $\frac{dR}{dz}$ equal to the model value. The difference is on the order of 1%. For the linear method the difference found is equal to zero. According to equation (2.17), $\frac{dP}{dz}$ does not depend on $V_z$ when neglecting the convective terms so this is logical. So for the reconstructed longitudinal gradient of pressure the influence of the longitudinal gradient of radius is negligible as well. In figure 5.9 the effect of $\frac{dR}{dz}$ on the reconstructed wall shear rate is expressed for the three methods. For the linear method the difference is the largest, for the other two methods the difference is less than 1 %. This can be explained by the different axial velocity profiles which are calculated with the linear method, because $V_z$ no longer depends on $V_r$ when neglecting the convective terms.

### 5.3.3 Influence of Axial Velocity

To verify the influence of the measurement of axial velocity, the axial velocity which serves as input for the mathematical model is disturbed by an error of + 10 % and - 10 %. The effects on flow, $\frac{dP}{dz}$, and wall shear rate are presented in figure (5.10) for flow, figure (5.11) for pressure gradient and figure (5.12) for wall shear rate. It seems that the differences in results are proportional to the variation in axial velocity for the two large peaks. Jaubert [Jaubert, 1995] remarks that a difference in positioning the ultrasound receiver/transmitter unit of 1° causes a difference in measured axial velocity of 10 %.

### 5.3.4 Influence of Radius

The radius and $\frac{dR}{dz}$ are both set to zero and the results are presented in figure (5.13) t/m (5.15). A comparison is made between the three methods where R is moving and where R is equal to the mean value. In both cases, $\frac{dR}{dz} = 0$. For the flow signal, there is a difference of about 3% for all three methods. The pressure gradient is not much affected because the maximal difference is about 1%. For the linear method, no pressure gradient difference at all is found for distensible or rigid tube. This is because the pressure gradient does not depend anymore on the convective term. For the wall shear rate, the difference for the linear model is the largest, and the difference is not negligible for neither one of the three methods.
5.3.5 Influence of Radial Mesh

For a refined radial mesh, the results for wall shear rate are displayed in figure (5.16). A phaseshift and an amplitudechange for the systolic and diastolic peaks occur when refining the mesh. The amplitudechange could be explained by the discretization of the points and the way in which the shear rate is calculated as the slope between two discretized points.
Figure 5.10: Influence of Axial Velocity. Comparison reconstructed flow signal with Cavalcanti and $\frac{9R}{\partial z}$ equal to modelvalue for axial velocity normal (-), axial velocity-10% (..), axial velocity+10% (-.).

Figure 5.11: Influence of Axial Velocity. Comparison reconstructed $\frac{\partial P}{\partial z}$ signal with Cavalcanti and $\frac{9R}{\partial z}$ equal to modelvalue for axial velocity normal (-), axial velocity-10% (..), axial velocity+10% (-.).

5.4 Results of Analytical Solution for Distensible Tube

In paragraph (2.4), an analytical model is derived to calculate velocity profiles, wall shear rate and longitudinal gradient of pressure $\frac{\partial P}{\partial z}$ starting from the Fouriercoefficients of the flowsignal. This model is implemented and the results are displayed in figures (5.17) and (5.18). Velocityprofiles are compared in chapter 6. Analytical wall shear stress and analytical $\frac{\partial P}{\partial z}$ are almost exactly the same as the ones calculated from the iterative method with $R=R(t)$ and $\frac{9R}{\partial z} \neq 0$ and 210 radial positions.
Figure 5.12: Influence of Axial Velocity. Comparison reconstructed shear rate with Cavalcanti and $\frac{\partial R}{\partial z}$ equal to model value for axial velocity normal (-), axial velocity-10% (..), axial velocity+10% (-.).

Figure 5.13: Influence of Radius. Comparison reconstructed flow signal with the three methods, $\frac{\partial R}{\partial z} = 0$ and $R$ equal to $R_0$, thus $Q(R = R(t)) - Q(R = R_0)$. Cavalcanti(solid line), Ling-Atabek(..), Linear Method (-.)
Figure 5.14: Influence of Radius. Comparison reconstructed flow signal with the three methods, $\frac{\partial R}{\partial t} = 0$ and $R$ equal to $R_0$, thus $Q(R = R(t)) - Q(R = R_0)$. Cavalcanti(solid line), Ling-Atabek(••), Linear Method (-.)

Figure 5.15: Influence of Radius. Comparison reconstructed flow signal with the three methods, $\frac{\partial R}{\partial t} = 0$ and $R$ equal to $R_0$, thus $Q(R = R(t)) - Q(R = R_0)$. Cavalcanti(solid line), Ling-Atabek(••), Linear Method (-.)
Figure 5.16: Influence of Radial Mesh. Comparison reconstructed shear rate with Cavalcanti, $R=R(t)$, $\frac{\partial R}{\partial z}$ equal to modelvalue between 21 and 210 radial positions.

Figure 5.17: Analytical Solution. Comparison of analytical solution with $R = R_0$ with solution of iterative method with $R = R(t)$ and a moving $\frac{\partial R}{\partial z}$. Analytical solution (solid line), Iterative method (dashed line).
Figure 5.18: Analytical Solution. Comparison of analytical solution with \( R = R_0 \) with solution of iterative method with \( R = R(t) \) and a moving \( \partial R / \partial t \). Analytical solution (solid line), Iterative method (dashed line).
Chapter 6

Velocity Profiles

Velocity profiles as calculated from the iterative method and as measured with LDA are compared to each other as function of time and radial position. The relative difference between these velocity profiles indicates that a large difference is present during the period of inverse flow.

6.1 Method of Comparison

In order to compare the velocity profiles as measured with LDA and the velocity profiles as calculated from the iterative method, we calculate the difference between these velocity profiles. The difficulty here is that the calculated velocity profiles are in 42 linearly distributed points in radial direction, and the calculated velocity profiles are in 69 non-linearly distributed discrete measurement points in radial direction (see paragraph (3.3)). So in order to subtract the velocity profiles from each other, we have to rewrite the measured velocity profiles. A polynomial fit is used to describe the measured velocity profiles. It appears that a polynomial fit with time dependent coefficients of degree 20 is enough to describe the measured velocity profiles (see equation (3.2)).

6.2 Velocity profile comparison

First, an overview of the measured velocity profiles is given in figure (6.1) in order to get an impression of what is happening. The relative difference between these velocity profiles as obtained with LDA and as calculated with the Cavalcanti version of the iterative method with $R=R(t)$ and $\frac{\partial R}{\partial t}$ moving is presented in figure (6.2). A contour plot of the same relative difference is presented in figure (6.3). The spatial and temporal location of the largest difference coincides with the location where $K_1$ diverges as first (see figure (3.7)). This largest difference occurs in the boundary layer for $0.5 < |\eta| < 1.0$ and around $t=0.375$. This time coincides with the largest inverse flow (figures (6.4 and 6.5)). In figure (6.4) a velocity profile during the period of inverse flow is presented. The reconstructed velocity profiles during the period of backflow are taller than the measured ones but only during the period of inverse flow. This velocity profile is taken from a specially developed program with which the velocity profiles can interactively be displayed on the screen.
Figure 6.1: Overview of the LDA Measurements.

Figure 6.2: Relative difference between velocity profiles as measured with LDA and as calculated from iterative method with distensible tube and $\frac{\partial R}{\partial z}$ equal to model value and $R=R(t)$. The largest difference occurs in the boundary layer during the period of inverse flow.
Figure 6.3: Contour plot of relative difference as in figure 6.2, between velocity profiles as measured with LDA and as calculated from iterative method with distensible tube and $\frac{dR}{dt}$ equal to model value and $R=R(t)$. The largest difference occurs in the boundary layer during the period of inverse flow and is about 100% for an extreme value.

Figure 6.4: Velocity profile as measured and as reconstructed as function of $\eta$ for $t=0.375$ [s] in the period of the largest difference between calculated and measured velocity profiles.
Figure 6.5: Measured Longitudinal Velocity on the tube axis as function of time. The velocity profile of figure (6.4) is taken at the moment where the axial velocity is smallest.
Chapter 7

Conclusion

Measurements

The polynomial fit of the velocity profiles to calculate wall shear rate coincides better with the results of the iterative method because errors due to discretization and due to measurement inaccuracies are diminished. The way of calculating wall shear rate as the maximum within 0.9 [mm] from the vessel wall according to Brands [Brands, 1996] is not accurate because in this way different errors are introduced into the result.

Results

During the period of inverse flow, the velocity profiles are not reconstructed well. The influence of the method (Cavalcanti, Ling-Atabek and linear method) is only of importance for the reconstructed wall shear rate. In case of neglecting the convective terms, reconstructed wall shear rate differs about 15 % from the wall shear rate as calculated with Cavalcanti or Ling-Atabek. Therefore, convective terms cannot be neglected in the iterative method. The difference between the method of Cavalcanti and Ling-Atabek is negligible, so there is no need to calculate the radial velocity field with Ling-Atabek so the process converges faster.

The influence of longitudinal gradient of radius is negligible in all cases. This is can be explained because long waves are present so on the scale of 12.5 [cm] the radius does not change a lot. The longitudinal gradient of radius as calculated with the different methods as described in chapter 4 is of the order of $1 \cdot 10^{-4}$, and this has a negligible effect on the results from the iterative method. It depends on the longitudinal gradient of radius in vivo whether it can be neglected or not in the calculations.

A difference in measured axial velocity causes a proportional difference in the results of the iterative model, but only for the systolic and diastolic peak. The ultrasound probe should be placed as accurately as possible to measure center-line blood velocity.

The method of Ling-Atabek for the radial position independent development of axial velocity might not be true during a period of inverse flow when the velocity gradients are very large and change rapidly. Because centerline velocity of ultrasound and LDA measurements are the same we can use ultrasound in-vivo to measure centerline blood velocity and calculate wall shear rate with the iterative method as described in this report. Convective terms cannot be neglected but the longitudinal gradient of radius can be, so there is no need to measure radius at two adjacent positions.

Discussion

The analysis of the flow during the period of inverse flow should be considered in more detail. During this period the difference is the largest. For clinical use of this method it is necessary to measure blood velocity and wall distension at the same time with ultrasound. In the in-vitro experiments this was not possible because of interference of the ultrasound probe which measured fluid velocity and the ultrasound probe which measured radial distension. Velocity profiles as measured with LDA are more axisymmetric than the velocity profiles as measured with ultrasound. Therefore ultrasound velocity profiles cannot be used to compare with the output of the iterative method.
Appendix A

We consider the following equations:

\[
\left( \frac{\partial V_z}{\partial t} + V_z \frac{\partial V_z}{\partial z} + V_r \frac{\partial V_z}{\partial r} \right) = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left( \frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r} \frac{\partial V_z}{\partial r} \right)
\]

\[
\frac{\partial P}{\partial r} = 0
\]

\[
\frac{\partial V_z}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r V_r) = 0
\]

Ling and Attabek introduced a parameter \( \eta = \frac{r}{R^2 \eta^2} \) to rewrite these equations in order to obtain a form which can be numerically integrated more easily. The different terms in the above equations have to be rewritten considering that the \( \eta \) itself is a function of \( r \).

\[
\left( \frac{\partial V_z}{\partial t} \right)_r = \left( \frac{\partial V_z}{\partial \eta} \right)_{z,t} \left( \frac{\partial \eta}{\partial t} \right)_r + \left( \frac{\partial V_z}{\partial t} \right)_{\eta,s}
\]

\[
\left( \frac{\partial V_z}{\partial \eta} \right)_{z,t} = \left( \frac{\partial V_z}{\partial \eta} \right)_{z,t} \left( \frac{\partial \eta}{\partial \eta} \right)_r + \left( \frac{\partial V_z}{\partial \eta} \right)_{\eta,s}
\]

\[
\left( \frac{\partial V_z}{\partial \eta} \right)_{z,t} = -\frac{\eta}{R} \left( \frac{\partial R}{\partial t} \right)_z \left( \frac{\partial V_z}{\partial \eta} \right)_{z,t} + \left( \frac{\partial V_z}{\partial t} \right)_{\eta,s}
\]

For the other derivatives, we will skip the intermediate steps of the derivation and give the final results:

\[
\left( \frac{\partial V_z}{\partial z} \right)_r = -\frac{\eta}{R} \frac{\partial R}{\partial \eta} \left( \frac{\partial V_z}{\partial \eta} \right)_{z,t} + \left( \frac{\partial V_z}{\partial \eta} \right)_{\eta,s}
\]

\[
\left( \frac{\partial^2 V_z}{\partial r^2} \right)_z = \frac{1}{R^2} \left( \frac{\partial^2 V_z}{\partial \eta^2} \right)_{z,t}
\]

\[
\left( \frac{\partial V_r}{\partial \eta} \right)_{z,t} = \frac{1}{\eta} \left( \frac{\partial V_r}{\partial \eta} \right)_{z,t}
\]

We substitute the above terms into equation (.1a).

\[
\left( -\frac{\eta}{R} \frac{\partial R}{\partial \eta} \frac{\partial V_z}{\partial t} \right) + V_z \left( -\frac{\eta}{R} \frac{\partial R}{\partial \eta} \frac{\partial V_z}{\partial z} + \frac{\partial V_z}{\partial \eta} \right) + V_r \frac{\partial V_z}{\partial \eta} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left( \frac{\partial^2 V_z}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial V_z}{\partial \eta} \right)
\]

\[
\frac{\partial V_z}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \left( \frac{\eta}{R} \frac{\partial R}{\partial \eta} - \frac{V_r}{R} \right) \frac{\partial V_z}{\partial \eta} + V_z \left( \frac{\eta}{R} \frac{\partial R}{\partial \eta} - \frac{\partial V_z}{\partial \eta} \right) + \nu \left( \frac{\partial^2 V_z}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial V_z}{\partial \eta} \right)
\]

The underlined term in equation (.7) can be rewritten using the continuity equation (.1c):

\[
\frac{\partial V_z}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r V_r) = 0
\]

\[
-\frac{\eta}{R} \frac{\partial R}{\partial \eta} \frac{\partial V_z}{\partial z} + \frac{\partial V_z}{\partial \eta} + \frac{1}{R \eta} \frac{\partial}{\partial \eta} \left( \eta V_r \right) = 0
\]
Equation (7) is now written as equation (2.4a). In the case that the convective terms are neglected, the momentum equation (1a) is written as:

\[
\frac{\partial V_z}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{\eta}{R} \frac{\partial R}{\partial t} \frac{\partial V_z}{\partial \eta} + \frac{\nu}{R^2} \left( \frac{\partial^2 V_z}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial V_z}{\partial r} \right)
\]
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