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Citation for published version (APA):

Document status and date:
Published: 01/01/1984

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
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Memorandum 1984-03

On the Lebesgue constants
for cardinal L-spline interpolation

by

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May 1984

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1. Introduction and summary

Throughout this paper $p_{2k+1}$ denotes the monic polynomial $p_{2k+1}(x) = x(x^2 - \alpha_1) \cdots (x^2 - \alpha_k)$, where $\alpha_1, \ldots, \alpha_k$ are real numbers such that $0 \leq \alpha_1 \leq \ldots \leq \alpha_k$. The linear differential operator having $p_{2k+1}$ as its characteristic polynomial is denoted by $L_{2k+1}$, i.e., $L_{2k+1}(D) = p_{2k+1}(D)$, where $D$ is the ordinary first order differentiation operator.

A complex-valued function $s$ is called a cardinal L-spline with respect to $L_{2k+1}$ if it satisfies the conditions

$$1.1 \left\{ \begin{array}{l} (i) \quad s \in C^{(2k-1)}(\mathbb{R}) , \\
(ii) \quad L_{2k+1} s(t) = 0 \quad (\nu < t < \nu + 1 , \nu = 0, \pm 1, \pm 2, \ldots ) . \end{array} \right.$$
The set of cardinal $L$-splines with respect to $L_{2k+1}$ is denoted by $S_{2k+1}$.

Obviously, $S_{2k+1}$ depends on $a_1, \ldots, a_k$; this, however, is suppressed in our notation. The following interpolation property holds.

**Lemma 1.1** (Michelli [3])

Let $(y_v)_{v=0}^{\infty}$ be a bounded sequence of complex numbers. Then a unique bounded function $s \in S_{2k+1}$ exists such that

$$1.2 \quad s(v+\frac{1}{2}) = y_v \quad (v = 0, \pm 1, \pm 2, \ldots).$$

The boundedness of the interpolant $s$ in Lemma 1.1 is required to ensure the unicity of $s$.

Let $S_{2k+1}$ be the linear operator mapping the set of bounded sequences $\mathcal{Z} = (y_v)_{v=0}^{\infty}$ onto the set of bounded functions in $S_{2k+1}$ by way of interpolation according to 1.2. The purpose of this paper is to study the asymptotic behaviour of the operator norm

$$1.3 \quad \|S_{2k+1}\| = \sup_{y+0} \frac{\|S_{2k+1} y\|_{\infty}}{\|y\|_{\infty}}$$

as $k \to \infty$.

Taking in particular the sequence $(y_v) = (\delta_{v,0})$ in 1.2 we obtain the so-called fundamental solution $L_{2k+1}$ of the interpolation problem. In Schoenberg [7] it is shown that $|L_{2k+1}(t)| < A e^{-\alpha|t|}$ ($t \in \mathbb{R}$) for appropriate positive constants $A$ and $\alpha$. Hence, for any bounded sequence $y = (y_v)_{v=0}^{\infty}$, the corresponding bounded interpolant $S_{2k+1} y$ may be written in the form
1.4 \[ S_{2k+1} y(t) = \sum_{\nu=-\infty}^{\infty} y_{\nu} L_{2k+1}(t - \nu) \quad (-\infty < t < \infty). \]

It immediately follows from 1.4 that

\[ \| S_{2k+1} \| \leq \sup_{t \in \mathbb{R}} L_{2k+1}(t), \]

where

1.5 \[ L_{2k+1}(t) = \sum_{\nu=-\infty}^{\infty} |L_{2k+1}(t - \nu)| \]

is the Lebesgue function associated with the given cardinal interpolation problem.

In Section 3 it is proved that on \([-\frac{1}{2}, \frac{1}{2}]\) the function \( \tilde{L}_{k+1} \) coincides with the cardinal \( L \)-spline

1.6 \[ \tilde{L}_{2k+1}(t) = \sum_{\nu=-\infty}^{\infty} \tilde{y}_{\nu} L_{2k+1}(t - \nu) \quad (-\infty < t < \infty), \]

where

1.7 \[ \tilde{y}_{\nu} = \begin{cases} (-1)^{\nu} & (\nu = 0, 1, 2, \ldots) , \\ (-1)^{\nu+1} & (\nu = -1, -2, \ldots). \end{cases} \]

We also show that

1.8 \[ \| S_{2k+1} \| = \tilde{L}_{2k+1}(0). \]

In view of this the operator norm \( \| S_{2k+1} \| \) (cf. 1.3) is also called the Lebesgue constant for the interpolation problem. Our study of the asymptotic behaviour
of $\|S_{2k+1}\| (k \to \infty)$ is based on an integral representation of $\|S_{2k+1}\|$; cf. also Section 3. In order to derive this representation, some known results in the theory of cardinal $L$-splines are needed; these are collected in Section 2.

Finally, the asymptotic behaviour of $\|S_{2k+1}\|$ is studied in Section 4. The following result is obtained.

Let

$$\beta_k = \frac{2}{\pi} + 4\pi \sum_{j=1}^{k} \frac{1}{\alpha_j + \pi^2},$$

and let $\gamma$ denote Euler's constant. It is shown that

$$\|S_{2k+1}\| = \frac{2}{\pi} (\ln \beta_k + 3 \ln 2 - \ln \pi + \gamma) + O(\beta_k^{-2}) (k \to \infty),$$

as $\beta_k \to \infty (k \to \infty)$. If the sequence $(\beta_k)$ converges then it is proved that $\|S_{2k+1}\|$ converges as well.

2. Preliminaries

Let the polynomial $\tilde{p}_{2k+1}$ be defined by

$$\tilde{p}_{2k+1}(z) = (z-1)(z-e^{-\sqrt{\alpha_1}})(z-e^{\sqrt{\alpha_1}}) \cdots (z-e^{-\sqrt{\alpha_k}})(z-e^{\sqrt{\alpha_k}}),$$

where $z \in \mathbb{C}$.

For all $z \in \mathbb{C}$ with $\tilde{p}_{2k+1}(z) \neq 0$ and for all $t \in \mathbb{R}$ the function $\psi(z,t)$ is then defined by

$$\psi(z,t) = \frac{\tilde{p}_{k+1}(z)}{2\pi i} \oint_{C} \frac{e^{t\xi}}{(z-e^{\xi}) \tilde{p}_{2k+1}(\xi)} d\xi.$$
where $p_{2k+1}$ is given in Section 1, and where $C$ is any contour in the complex plane surrounding the zeros of $p_{2k+1}$ but excluding the zeros of $\zeta \mapsto z - e^\zeta$.

In the sequel the following properties of $\psi(z,t)$ are needed; they are contained in Ter Morsche [5] as well as in Michelli [3], where, apart from a normalisation factor, $\psi(z,t)$ is also used.

One has

2.3 $t \mapsto \psi(z,t) \in \text{Ker}(L_{2k+1})$, the kernel of $L_{2k+1}$,

2.4 $\left. \left( \frac{\partial}{\partial t} \right)^j \psi(z,t) \right|_{t=1} = z \left. \left( \frac{\partial}{\partial t} \right)^j \psi(z,t) \right|_{t=0}$ (j = 0, 1, ..., 2k - 1),

2.5 $\psi(z,1-t) = z^{2k} \psi(z^{-1},t)$,

2.6 $\psi(z,t) = \sum_{j=0}^{2k} A_j(t) z^j$,

with $A_j \in \text{Ker}(L_{2k+1})$, $A_{2k}(t) > 0$ (t ≠ 0), $A_{2k}(0) = 0$.

Apart from these relations the following property of $\psi(z,t)$ is of interest.

**Lemma 1.2** (Michelli [3])

If $z < 0$ the function $t \mapsto \psi(z,t)$ has precisely one zero in $(0,1]$.

Furthermore, if $t \in [0,1)$ then the polynomial $z \mapsto \psi(z,t)$ has only real zeros; these zeros are negative and simple.
The polynomial \( z \mapsto \psi(z,t) \) is usually called the exponential \( L \)-polynomial, and in case \( a_1 = a_2 = \ldots = a_k = 0 \) it is the well-known Euler-Frobenius polynomial of degree at most \( 2k \) (cf. Ter Morsche [5, p.62]). From 2.4 it follows that \( \psi(z,1) = z \psi(z,0) \). Therefore, by Lemma 2.1, \( \psi(z,1) \) has \( 2k - 1 \) negative simple zeros and, in addition, \( z = 0 \) is also a zero.

Let the zeros of \( z \mapsto \psi(z,t) \) \( (t \in (0,1]) \) be denoted by \( \lambda_1(t), \ldots, \lambda_{2k}(t) \) with

\[
-\infty < \lambda_1(t) < \lambda_2(t) < \ldots < \lambda_{2k}(t) \leq 0 .
\]

In Schoenberg [7] it is shown that the functions \( t \mapsto \lambda_i(t) \) \( (i = 1, \ldots, 2k) \) are increasing on \( (0,1] \), satisfying the inequalities

\[
2.7 \quad \lambda_{i-1}(1) < \lambda_{i}(t_1) < \lambda_{i}(t_2) < \lambda_{i}(1) \leq 0 ,
\]

where \( 0 < t_1 < t_2 < 1 \) and, by definition, \( \lambda_0(1) = -\infty \).

In the polynomial case, i.e., the case \( a_1 = a_2 = \ldots = a_k = 0 \), the inequalities 2.7 are already contained in Ter Morsche [4].

In view of 2.5 the zeros of \( \psi(z,1) \) are ordered as follows

\[
2.8 \quad \left\{ \begin{array}{l}
\lambda_1(\frac{1}{i}) < \ldots < \lambda_k(\frac{1}{i}) < -1 < \lambda_{k+1}(\frac{1}{i}) < \ldots < \lambda_{2k}(\frac{1}{i}) < 0 , \\
\lambda_{k+i}(\frac{1}{i}) \lambda_{k-i+1}(\frac{1}{i}) = 1 \quad (i = 0, 1, \ldots, k) .
\end{array} \right.
\]

According to Ter Morsche [4, p. 68] the relation

\[
2.9 \quad \sum_{j=0}^{2k} A_j(\frac{1}{i}) s(\mu + j + t) = \sum_{j=0}^{2k} A_j(t) y_{\mu+j} \quad (0 \leq t < 1, \mu = 0, \pm 1, \ldots)
\]
holds for all functions \( s \in S_{2k+1} \) satisfying 1.2; here the functions \( A_j \) are given by 2.6.

Relation 2.9 may be considered as a linear difference equation for the unknown sequence \( (s(\mu + t))^\infty_{\mu=-\infty} \) having \( \psi(z, \frac{t}{2}) \) as its characteristic polynomial.

We know, however, that \( \psi(z, \frac{t}{2}) \) is a polynomial of degree \( 2k \) with \( 2k \) distinct negative zeros. Since, in view of 2.8, \( \psi(-1, \frac{t}{2}) \neq 0 \), the polynomial \( \psi(z, \frac{t}{2}) \) has no zeros on the unit circle in the complex plane, and therefore Lemma 3.4.1 of Ter Morsche [5, p.74] may be applied to 2.9. This yields the following result.

**Lemma 2.1** Let \( (y_\nu)^\infty_\nu \) be a bounded sequence of complex numbers. Then a unique bounded function \( s \in S_{2k+1} \) exists satisfying 1.2. Moreover, this interpolating function \( s \) can be written in the form

\[
2.10 \quad s(\mu + t) = \sum_{j=-\infty}^{\infty} \omega_j(t)y_{\mu+j} \quad (0 \leq t < 1, \mu = 0, \pm 1, \ldots)
\]

where \( \omega_j(t) \) is given by the contour integral

\[
2.11 \quad \omega_j(t) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{\psi(z, t)}{z^{j+1}} \psi(z, \frac{t}{2}) \, dz \quad (j = 0, \pm 1, \pm 2, \ldots).
\]

3. The Lebesgue function and an integral representation of \( \|S_{2k+1}\| \)

An application of Formula 2.10 to the particular sequence \( (y_\nu) = (\delta_\nu, 0) \) yields the fundamental solution \( L_{2k+1} \) as introduced in Section 1. In view of
Lemma 2.1 one has

3.1 \[ L_{2k+1}(t-\mu) = \frac{1}{2\pi i} \int_{|z|=1} \frac{\psi(z,t)}{z^{\mu+1}} \frac{dz}{\psi(z,\frac{1}{2})} \quad (0 \leq t < 1, \mu = 0, \pm 1, \pm 2, \ldots). \]

Using the residue theorem and 2.8, we obtain the representation

3.2 \[ L_{2k+1}(t-\mu) = \sum_{\xi=k+1}^{2k} \psi(\lambda_{\xi}(\frac{1}{2}),t) \frac{\psi(\lambda_{\xi}(\frac{1}{2}),\xi)}{\psi_{z}(\lambda_{\xi}(\frac{1}{2}),\frac{1}{2})} \quad (0 \leq t < 1, \mu = -1,-2,\ldots), \]

here \( \psi_z \) denotes the partial derivative of \( \psi(z,t) \) with respect to \( z \). It follows from 2.7 that

3.3 \[ \text{sgn} \left( \frac{\psi(\lambda_{\xi}(\frac{1}{2}),t)}{\psi_{z}(\lambda_{\xi}(\frac{1}{2}),\frac{1}{2})} \right) = \begin{cases} -1 & (\frac{1}{2} < t \leq 1), \\ 0 & (t = \frac{1}{2}), \\ 1 & (0 \leq t < \frac{1}{2}). \end{cases} \]

Consequently,

3.4 \[ \text{sgn} L_{2k+1}(t-\mu) = (-1)^{\mu} \text{sgn}(t-\frac{1}{2}) \quad (0 \leq t < 1, \mu = -1,-2,\ldots). \]

Since, by Lemma 2.1, the function \( L_{2k+1} \) is uniquely determined, one has

3.5 \[ L_{2k+1}(\frac{1}{2}+t) = L_{2k+1}(\frac{1}{2}-t) \quad (\infty < t < \infty). \]

Therefore

3.6 \[ \text{sgn} L_{2k+1}(t-\mu) = (-1)^{\mu} \text{sgn}(\frac{1}{2}-t) \quad (0 < t \leq 1, \mu = 1,2,\ldots). \]
Taking $\mu = 0$ and applying the residue theorem, we obtain

\[ L_{2k+1}(t) = \frac{\psi(0,t)}{\psi(0,\frac{1}{2})} + \sum_{l=k+1}^{2k} \frac{\psi(\lambda_2(\frac{1}{2}), t)}{\lambda_2(\frac{1}{2}) \psi_z(\lambda_2(\frac{1}{2}), \frac{1}{2})} \quad (0 \leq t < 1). \]

From 2.6 it follows that $\psi(0,t)\psi^{-1}(0,\frac{1}{2}) > 0$ ($t \in [0,1]$).

Using this and Formulae 2.8, 3.3, we conclude that $L_{2k+1}(t) > 0$ ($t \in [\frac{1}{2},1]$).

Hence, in view of 3.5,

\[ \text{sgn}(L_{2k+1}(t)) = 1 \quad (0 < t < 1). \]

The fundamental solution $L_{2k+1}$ thus changes sign at the points $v + \frac{1}{2}$ ($v = \pm 1, \pm 2, \ldots$), and these points are the only zeros of $L_{2k+1}$.

Therefore, on the interval $[-\frac{1}{2}, \frac{1}{2}]$ the Lebesgue function $L_{2k+1}$ as given by 1.5 coincides with the function $\tilde{L}_{2k+1}$ defined by 1.6.

Having established this, our next goal is to show that $\|L_{2k+1}\| = \tilde{L}_{2k+1}(0)$ holds. To this end we introduce the function $L^{[n]}_{2k+1}$ ($n \in \mathbb{N}$), being the unique bounded cardinal $L$-spline in $S_{2k+1}$ interpolating the periodic sequence

\[ y_v^{[n]} = (-1)^v \quad (v = 0, 1, \ldots, 2n), \]

\[ y_{v+2n+1}^{[n]} = y_v^{[n]} \quad (v = 0, \pm 1, \pm 2, \ldots). \]

We emphasize that $y_v^{[n]} = y_{-v}^{[n]}$ ($v \in \mathbb{Z}$). Consequently, the unicity of $L^{[n]}_{2k+1}$ implies that $L^{[n]}_{2k+1}$ is an even and periodic function with period $2n+1$.

Since (cf. 1.7)

\[ y_v^{[n]} = \sim_v \quad (v = -2n, -2n+1, \ldots, 2n) \]
one has
\[ \lim_{n \to \infty} L_{2k+1}^n(t) = \sim_{2k+1}(t), \]
uniformly on every compact interval of \( \mathbb{R} \).

Therefore 1.8 will be established if it is shown that
\[ 3.10 \quad L_{2k+1}^n(0) = \max_{0 \leq t \leq \frac{1}{2}} L_{2k+1}^n(t). \]

This assertion may be proved as follows. Since \( L_{2k+1}^n \) is an even function having at least \( 2n \) zeros in \( (\frac{1}{2}, 2n+\frac{1}{2}) \), its derivative \( L_{2k+1}^n \) has at least \( 2n-1 \) zeros in \( (\frac{1}{2}, 2n+\frac{1}{2}) \), where, in addition,
\[ L_{2k+1}^n(0) = L_{2k+1}^n(2n+1) = 0. \]

In order to prove that these zeros are the only zeros of \( L_{2k+1}^n \) on \( [0, 2n+1] \), we use a generalized version of Rolle's theorem (cf. Ter Morsche [5, Lemma 1.4.11]). Also taking into account that the functions involved, together with their \((2k-1)\)st derivatives, are periodic with period \( 2n+1 \), and the fact that
\[ (D - \sqrt{a_k}I)(D^2 - a_{k+1}I) \cdots (D^2 - a_1I)L_{2k+1}^n \]
has at most \( 2n \) sign changes in \((0, 2n+1)\), imply that \( L_{2k+1}^n \) has at most \( 2n-1 \) zeros in \((0, 2n+1)\). It follows that \( L_{2k+1}^n \) has precisely \( 2n-1 \) zeros in \((0, 2n+1)\), all of which are contained in the subinterval \((\frac{1}{2}, 2n+\frac{1}{2})\).

In view of \( L_{2k+1}^n(v + \frac{1}{2}) = (-1)^v \) \((v = 0, 1, 2, \ldots, 2n)\) we obtain that \( L_{2k+1}^n(t) \leq 0 \) in \((0, \frac{1}{2})\). Hence 3.10 holds, which implies that \( \|S_{2k+1}\| = \sim_{2k+1}(0) \).
An integral representation of $\| S_{2k+1} \|$ is now obtained as follows. We recall (cf. 1.6, 1.7) that $\tilde{L}_{2k+1}$ is the unique bounded cardinal $L$-spline interpolating the sequence $(\tilde{y}_v)$. Formula 2.10 combined with 2.11 yields

$$\tilde{L}_{2k+1}(0) = \frac{1}{2\pi i} \left( \sum_{j=0}^{\infty} \int_{|z|=1-\epsilon} \frac{(-1)^j \psi(z,0)}{z^{j+1} \psi(z,\frac{1}{z})} \, dz \right),$$

where $\epsilon$ is chosen so small that $\psi(z,\frac{1}{z})$ has no zeros in the ring $1 - 2\epsilon < |z| < 1 + 2\epsilon$.

Consequently,

$$\tilde{L}_{2k+1}(0) = \frac{1}{2\pi i} \left( \int_{|z|=1-\epsilon} \frac{\psi(z,0)}{(1+z) \psi(z,\frac{1}{z})} \, dz + \int_{|z|=1+\epsilon} \frac{\psi(z,0)}{(1+z) \psi(z,\frac{1}{z})} \, dz \right).$$

It easily follows from 2.4 and 2.5 that $\psi(-1,0) = 0$.

Hence, by 1.8, we obtain an integral representation of the form

$$3.11 \quad \| S_{2k+1} \| = \frac{1}{\pi i} \int_{|z|=1} \frac{\psi(z,0)}{(1+z) \psi(z,\frac{1}{z})} \, dz.$$ 

This formula will now be used to study the asymptotic behaviour of $\| S_{2k+1} \|$.

4. The asymptotic behaviour of $\| S_{2k+1} \|$.

We first observe that the sum of the residues of the function

$$\zeta \mapsto \frac{e^{\zeta t}}{(z - e^{\zeta}) p_{2k+1}(z)}$$

is zero in case $0 \leq t \leq 1$ as can be shown rather easily.
Consequently, if $\psi \neq 0 \pmod{2\pi}$, (2.2) yields

$$
\psi(e^{i\varphi}, t) = p_{2k+1}(e^{i\varphi}) \sum_{m=-\infty}^{\infty} \frac{e^{i(t-1)(2m\pi + \varphi)}}{p_{2k+1}(2m i \pi + i \varphi)}.
$$

Recalling that $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k$ (cf. 2.1), we define the polynomial $q_{2k+1}$ by

$$
q_{2k+1}(z) = z(z^2 + \alpha_1) \ldots (z^2 + \alpha_k).
$$

Since

$$
p_{2k+1}(iz) = (-1)^k i q_{2k+1}(z)
$$

one has

$$
\frac{\psi(e^{i\varphi}, 0)}{\psi(e^{i\varphi}, 1)} = e^{-i\varphi} \frac{\sum_{m=-\infty}^{\infty} (-1)^m q_{2k+1}(\varphi + 2m\pi)}{\sum_{m=-\infty}^{\infty} (-1)^m q_{2k+1}(\varphi + 2m\pi)}.
$$

Substituting $z = e^{i(\pi - \tau)}$ in 3.11, we then obtain

$$
4.1 \quad \|S_{2k+1}\| = \frac{1}{\pi} \int_{0}^{\pi} \frac{\sum_{m=-\infty}^{\infty} (-1)^m q_{2k+1}((2m+1)\pi - \tau)}{\sum_{m=-\infty}^{\infty} (-1)^m q_{2k+1}((2m+1)\pi + \tau)} \frac{d\tau}{\sin(\tau/2)}.
$$

Now let $u_{m,k}^\pm (m = 0, 1, \ldots)$ be defined by

$$
u_{m,k}^\pm = q_{2k+1}^\pm((2m+1)\pi - \tau) \pm q_{2k+1}^\pm((2m+1)\pi + \tau) \quad (0 \leq \tau < \pi).
$$
One easily verifies that
\[
\sum_{m=-\infty}^{\infty} q_{2k+1}^{-1}((2m+1)\pi-\tau) = \sum_{m=0}^{\infty} u^{-}_{m,k} (\tau),
\]
and
\[
\sum_{m=-\infty}^{\infty} (-1)^m q_{2k+1}^{-1}((2m+1)\pi-\tau) = \sum_{m=0}^{\infty} (-1)^m u^{+}_{m,k} (\tau).
\]

Define the functions \( R^+_k, R^-_k, \) and \( v_k \) on \([0,\pi] \) by
\[
\begin{align*}
R^+_k (\tau) &= q_{2k+1}(\pi-\tau) \sum_{m=1}^{\infty} (-1)^m u^+_{m,k} (\tau), \\
R^-_k (\tau) &= q_{2k+1}(\pi-\tau) \sum_{m=1}^{\infty} u^-_{m,k} (\tau), \\
v_k (\tau) &= q_{2k+1}(\pi-\tau) q_{2k+1}^{-1}(\pi+\tau).
\end{align*}
\]

In view of 4.1 we then have
\[
\|S_{2k+1}\| = \frac{1}{\pi} \int_{0}^{\pi} \frac{1 - v_k(\tau) + R^-_k (\tau)}{1 + v_k(\tau) + R^+_k (\tau)} \frac{d\tau}{\sin(\pi/2)}.
\]

Let the increasing sequence \( (\omega_k)_{k=1}^{\infty} \) be defined by
\[
\omega_k = \sum_{j=1}^{k} (\alpha_j + 1)^{-1}.
\]

From now on we distinguish between two cases, i.e.,
\[
\lim_{k \to \infty} \omega_k = \infty \quad \text{and} \quad \lim_{k \to \infty} \omega_k < \infty.
\]
I. \[ \lim_{k \to \infty} \omega_k = \infty. \]

We first give a couple of assertions concerning the behaviour of the functions \( u_{m,k}^+ \) and \( u_{m,k}^- \) as \( k \to \infty \). Their verification involves straightforward, but rather tedious computations, which are omitted here. The two relations are: a positive constant \( c \) exists such that for all \( m \in \mathbb{N} \) and all \( \tau \in [0, \pi] \)

\[
\begin{align*}
q_{k+1}(\pi - \tau) u_{m,k}^-(\tau) &= \frac{1}{\pi} \left( 1 - v_k^\prime(\tau) \right) \sin(\tau/2) \left( 1 + O(e^{-\omega_k^m}) \right) + O(e^{-\omega_k^m}), \\
q_{k+1}(\pi - \tau) u_{m,k}^+(\tau) &= \frac{1}{\pi} \left( 1 + v_k^\prime(\tau) \right) \sin(\tau/2) \left( 1 + O(e^{-\omega_k^m}) \right) + O(e^{-\omega_k^m}),
\end{align*}
\]

uniformly in \( m \) and \( \tau \).

From 4.2 and 4.5 it immediately follows that

\[
\begin{align*}
R_k^-(\tau) &= \tau O(e^{-\omega_k^m}), \\
R_k^+(\tau) &= 0(e^{-\omega_k^m}),
\end{align*}
\]

uniformly in \( \tau \).

Since in view of 4.2 one has \( v_k(\tau) \geq 0 \) on \([0, \pi]\), it follows from 4.3 and 4.6 that

\[
\begin{align*}
\|S_{2k+1}\| &= \frac{1}{\pi} \int_0^\pi \frac{1 - v_k^\prime(\tau)}{1 + v_k(\tau)} \sin(\tau/2) \left( 1 + O(e^{-\omega_k^m}) \right) + O(e^{-\omega_k^m}),
\end{align*}
\]

as \( k \to \infty \).

In order to analyze 4.7, it is convenient to write \( v_k \) in the form

\[
v_k(\tau) = \exp \left[ \ln \left( \frac{\pi - \tau}{\pi + \tau} \right) + \sum_{j=1}^k \ln \left( \frac{a_j + (\pi - \tau)^2}{a_j + (\pi + \tau)^2} \right) \right].
\]
Hence,
\[
\ln v_k(\tau) = \ln \left( \frac{1 - \tau/\pi}{1 + \tau/\pi} \right) + \sum_{j=1}^{k} \ln \left( \frac{1 - 2\pi \tau (a_j + \pi^2 + \tau^2)^{-1}}{1 + 2\pi \tau (a_j + \pi^2 + \tau^2)^{-1}} \right).
\]

We observe that \(0 < \tau < \pi\) implies
\[
0 \leq 2\pi \tau (a_j + \pi^2 + \tau^2)^{-1} \leq 2\pi \tau (\pi^2 + \tau^2)^{-1} < 1.
\]

An application of the Taylor expansion
\[
\ln \left( \frac{1 - t}{1 + t} \right) = -2 \sum_{\ell=0}^{\infty} \frac{t^{2\ell+1}}{2\ell + 1} \quad (-1 \leq t < 1)
\]
now yields
\[
4.8 \quad v_k(\tau) = \exp(-\tau g_k(\tau) - \tau^3 h_k(\tau)) \quad (0 \leq \tau < \pi),
\]
where
\[
4.9 \quad \begin{cases}
    g_k(\tau) = \frac{2}{\pi} + \sum_{j=1}^{k} \frac{4\pi}{a_j + \pi^2 + \tau^2}, \\
    h_k(\tau) = 2 \left( \sum_{\ell=1}^{\infty} \pi^{-2\ell-1} \frac{2\ell-2}{\tau} + \sum_{j=1}^{k} \sum_{\ell=1}^{\infty} \left( \frac{2\pi}{a_j + \pi^2 + \tau^2} \right)^{2\ell+1} \frac{2\ell-2}{\tau} \right).
\end{cases}
\]

Apparently, the function \(g_k\) satisfies on \([0,\pi]\) the inequalities
\[
4.10 \quad g_k(\tau) > \sum_{j=1}^{k} \frac{4\pi}{a_j + \pi^2 + \tau^2} \geq \sum_{j=1}^{k} \frac{4\pi}{a_j + 2\pi^2} \geq \frac{2}{\pi} \omega_k.
\]
Since
\[
\sum_{j=1}^{k} \sum_{\ell=1}^{\infty} \left( \frac{2\pi}{\alpha_j + \pi + \frac{2}{2}} \right)^{2\ell+1} \frac{2\ell-2}{2\ell+1} = \sum_{j=1}^{k} \sum_{\ell=1}^{\infty} \left( \frac{2\pi}{\alpha_j + \pi + \frac{2}{2}} \right)^{2\ell+1} \frac{1}{2\ell+1} \left( \frac{2\pi}{\alpha_j + \pi + \frac{2}{2}} \right)^{2\ell-2}
\]

\[
\leq \sum_{j=1}^{k} \frac{2\pi}{\alpha_j + \frac{2}{2}} \sum_{\ell=1}^{\infty} \frac{1}{2\ell+1} \left( \frac{2\pi}{\alpha_j + \pi + \frac{2}{2}} \right)^{2\ell-2} \leq \omega_k \sum_{\ell=1}^{\infty} \frac{2\pi}{2\ell+1} \left( \frac{2\pi}{\pi + \frac{2}{2}} \right)^{2\ell-2},
\]

one has (cf. 4.9)

\[4.11 \quad 0 \leq h_k(\tau) \leq g(\tau) + \omega_k h(\tau) \quad (0 \leq \tau < \pi),\]

where the functions \( g \) and \( h \) are given by

\[
\begin{align*}
g(\tau) &= 2 \sum_{\ell=1}^{\infty} \left( \frac{1}{\pi} \right)^{2\ell+1} \frac{2\ell-2}{2\ell+1}, \\
h(\tau) &= 4\pi \sum_{\ell=1}^{\infty} \frac{1}{2\ell+1} \left( \frac{2\pi}{\pi + \frac{2}{2}} \right)^{2\ell-2}.
\end{align*}
\]

Obviously, \( g \) and \( h \) are positive on \([0,\pi]\) and, moreover, \( g(\tau) \to \infty \) and \( h(\tau) \to \infty \) as \( \tau \to \pi \).

Let
\[
\frac{\pi}{0} \frac{1 - v_k(\tau)}{1 + v_k(\tau)} \frac{dt}{\sin(\tau/2)} = I_1 + I_2,
\]

where
\[
I_1 = \int_{0}^{\pi} \frac{e^{-\tau g_k(\tau)}}{1 + v_k(\tau)} \left( 1 - \frac{e^{-\tau h_k(\tau)}}{\tau} \right) \frac{\tau dt}{\sin(\tau/2)},
\]
\[
I_2 = \int_{0}^{\pi} \frac{1 - e^{-\tau g_k(\tau)}}{1 + v_k(\tau)} \frac{dt}{\sin(\tau/2)}.
\]
Using 4.10, the inequality

\[ 1 - e^{-t} \leq 2t(t+1)^{-1} \quad (t \geq 0) \]

and the observation that

\[ \frac{h_k(t)}{1 + \frac{3}{2} h_k(t)} = O(\omega_k) \quad (k \to \infty), \]

uniformly on \([0,\pi)\), we may conclude that

\[ I_1 = O\left( \sqrt{\int_0^\pi \omega_k^2 e^{-\frac{2}{\pi} \omega_k^2} \, d\tau} \right). \]

Hence

\[ I_1 = O\left( \omega_{k}^{-2} \right) \quad (k \to \infty). \]

In a similar way one can prove that

\[ I_2 = \int_0^\pi \frac{1 - e^{-\frac{1}{2} g_k(\tau)}}{1 + e^{-\frac{1}{2} g_k(\tau)}} \frac{d\tau}{\sin(\tau/2)} + O\left( \omega_{k}^{-2} \right). \]

In view of 4.7 this leads to

\[ \| S_{2k+1} \| = \frac{1}{\pi} \int_0^\pi \frac{1 - e^{-\frac{1}{2} g_k(\tau)}}{1 + e^{-\frac{1}{2} g_k(\tau)}} \frac{d\tau}{\sin(\tau/2)} \left( 1 + O(e^{-\frac{c}{\omega_k}}) \right) + O(\omega_{k}^{-2}). \]

On account of 4.9 the function \( g_k \) may be written in the form

\[ g_k(\tau) = \beta_k - \tau^2 r_k(\tau) \quad (0 \leq \tau < \pi), \]

where

\[ \beta_k = \frac{2}{\pi} + 4\pi \sum_{j=1}^{k} \frac{1}{\alpha_j + \pi^2}, \]
and

$$4.18 \quad r_k(\tau) = 4\pi \sum_{j=1}^{k} \frac{1}{(a_j + \pi^2 + \tau^2)(a_j + \pi^2)}.$$  

We observe that positive constants $c_1$ and $c_2$ exist such that $c_1 \omega_k \leq \beta_k \leq c_2 \omega_k$ ($k \in \mathbb{N}$). Therefore $O(\omega_k^{-2})$ may be replaced by $O(\beta_k^{-2})$, and vice versa.

From $4.18$ it easily follows that

$$4.19 \quad 0 < r_k(\tau) < \frac{\beta_k}{\pi^2 + \tau^2} \quad (k = 1, 2, \ldots; 0 \leq \tau < \pi).$$

Now, let

$$\int_{0}^{\pi} \frac{1 - e^{-\tau g_k(\tau)}}{1 + e^{-\tau g_k(\tau)}} \frac{d\tau}{\sin(\tau/2)} = J_1 + J_2,$$

where

$$J_1 = \int_{0}^{\pi} \frac{e^{-\beta_k \tau} (1 - e^{-\tau r_k(\tau)})}{l + e^{-\tau g_k(\tau)}} \frac{d\tau}{\sin(\tau/2)},$$

$$J_2 = \int_{0}^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\tau g_k(\tau)}} \frac{d\tau}{\sin(\tau/2)}.$$

Using $4.19$ together with the inequality $e^t - 1 \leq te^t$ ($t \geq 0$), we conclude that

$$J_1 = 0 \left( \int_{0}^{\pi} \frac{\tau^2 e^{-\beta_k \tau} e^{-\tau r_k(\tau)}}{r_k(\tau) d\tau} \right).$$
Similarly one has

\begin{align*}
J_2 &= \int_0^\pi \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{d\tau}{\sin(\tau/2)} + O\left(\beta_k^{-2}\right) \quad (k \to \infty).
\end{align*}

These relations for \(J_1\) and \(J_2\) yield (cf. 4.15)

\begin{align*}
4.20 \quad \|S_{2k+1}\| &= \frac{1}{\pi} \int_0^\pi \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{d\tau}{\sin(\tau/2)} \left(1 + O\left(e^{-\omega_k}\right)\right) + O\left(\beta_k^{-2}\right).
\end{align*}

The integral in the right-hand side of 4.20 can be written as follows.

\begin{align*}
\int_0^\pi \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{d\tau}{\sin(\tau/2)} &= \int_0^\pi \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{2}{\tau} d\tau + \int_0^\pi \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \left(\frac{1}{\sin(\tau/2)} - \frac{2}{\tau}\right) d\tau =
\end{align*}

\begin{align*}
&= \int_0^\pi \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{2}{\tau} d\tau + \int_0^\pi \left(\frac{1}{\sin(\tau/2)} - \frac{2}{\tau}\right) d\tau + O\left(\beta_k^{-2}\right).
\end{align*}
The second integral can be evaluated quite easily; in fact
\[ \int_{0}^{\pi} \left( \frac{1}{\sin(\tau/2)} - \frac{2}{\tau} \right) d\tau = 4 \ln 2 - 2 \ln \pi. \]

With respect to the first integral one has
\[ \int_{0}^{\beta_k \pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} d\tau = \int_{0}^{\beta_k \pi} \frac{1 - e^{-\tau}}{1 + e^{-\tau}} d\tau = \]
\[ = \int_{0}^{\beta_k \pi} \tanh(\tau/2) d\ln \tau = 2 \ln(\beta_k \pi) \ln(\beta_k) - \frac{1}{2} \int_{0}^{\beta_k \pi} \frac{\ln \tau}{\cosh^2(\tau/2)} d\tau = \]
\[ = \ln(\beta_k) - \frac{1}{2} \int_{0}^{\infty} \frac{\ln \tau}{\cosh^2(\tau/2)} d\tau + O(e^{-d \omega_k}), \]
where \( d \) is a positive constant.

Using Formula 4.371(3) in Gradshteyn and Ryzhik [1, p. 580], we obtain
\[ \int_{0}^{\infty} \frac{\ln \tau}{\cosh^2(\tau/2)} d\tau = 2(\ln \pi - \ln 2 - \gamma), \]
where \( \gamma \) denotes Euler's constant.

We thus arrive at the following theorem.

**Theorem 4.1** If \( \beta_k = 2/\pi + 4\pi \sum_{j=1}^{k} (a_j + \pi^2)^{-1} \to \infty \) as \( k \to \infty \), then
\[ \| S_{2k+1} \| = \frac{2}{\pi} (\ln \beta_k + 3 \ln 2 - \ln \pi + \gamma) + O(\beta_k^{-2}) \quad (k \to \infty). \]

Having dealt with \( \omega_k \to \infty \) as \( k \to \infty \), we now consider the case that the sequence \( (\omega_k) \) is convergent.
II. $\lim_{k \to \infty} \omega_k < \infty$.

The convergence of the sequence $(\omega_k)$ implies that $\lim_{j \to \infty} \alpha_j = \infty$, so a positive integer $k_0$ exists such that $\alpha_j > 0$ for $j \geq k_0$. The polynomial $q_{2k+1}$, introduced at the beginning of this section, may therefore be written in the form

$$q_{2k+1}(z) = \gamma_k z^{2k_0-1} \left(1 + \frac{z^2}{\alpha_{k_0}}\right) \ldots \left(1 + \frac{z^2}{\alpha_k}\right) \quad (k \geq k_0),$$

where

$$\gamma_k = \alpha_k^{k_0} \alpha_{k_0+1} \ldots \alpha_k.$$

Since $\sum_{j=k_0}^{\infty} \alpha_j^{-1}$ is finite, the product $\prod_{j=k_0}^{k} (1 + z^2 \alpha_j^{-1})$ converges uniformly in $z$ on every bounded set of $\mathbb{C}$. As a consequence its limit function, which we denote by $q$, is a holomorphic function.

Taking into account 4.1 we obtain

**Theorem 4.2** If $\sum_{j=1}^{\infty} (\alpha_j + 1)^{-1} < \infty$ then

$$4.22 \quad \lim_{k \to \infty} \|S_{2k+1}\| = \frac{1}{\pi} \int_{0}^{\pi} \frac{\sum_{m=-\infty}^{\infty} q_{m+1}^{1}(2m+1)\pi - \tau)}{\sin(\tau/2)} \, d\tau,$$

where

$$q(z) = \prod_{j=k_0}^{\infty} (1 + z^2 \alpha_j^{-1}).$$
Finally, we examine a few particular cases.

(a) The polynomial case: \( \alpha_j = 0 \) (\( j = 1, 2, \ldots \)).

Since \( \beta_k = \pi^{-1}(2 + 4k) \to \infty \), Theorem 4.1 may be applied. A simple computation yields

\[
\|S_{2k+1}\| = \frac{2}{\pi} \left( \ln k + \ln \frac{32}{\pi^2} \right) + O(k^{-1}) \quad (k \to \infty),
\]

which is in agreement with results obtained by Meinardus [2], and Richards [6].

(b) The hyperbolic case: \( \alpha_j = j^2 \) (\( j = 1, 2, \ldots \)).

Obviously \( \sum_{j=1}^{\infty} (\alpha_j + 1)^{-1} < \infty \), and thus Theorem 4.2 may be applied. Using the well-known relation

\[
\frac{\sinh(\pi z)}{\pi} = z \prod_{j=1}^{\infty} \left( 1 + \frac{z^2}{j^2} \right),
\]

we conclude from 4.22 that

\[
\lim_{k \to \infty} \|S_{2k+1}\| = \frac{1}{\pi} \int_{0}^{\infty} \frac{\sum_{m=-\infty}^{\infty} (-1)^m \sinh^{-1}(\pi((2m+1)\pi - \tau))}{\sin(\tau/2)} \ d\tau.
\]

A numerical computation of the integral yields

\[
\lim_{k \to \infty} \|S_{2k+1}\| \approx 2.1314.
\]
REFERENCES


