Spectral trajectories and duality

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SPECTRAL TRAJECTORIES AND DUALITY

by

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Summary

Let $E$ be a spectral measure on a ring of sets $\Sigma$, with values in the set of projection operators in a Hilbert space $H$. An $H$-valued set function $\mu$ on $\Sigma$ is called a spectral trajectory controlled by the measure $E$ if for any $\Delta_1, \Delta_2 \in \Sigma$, $E(\Delta_1)\mu(\Delta_2) = \mu(\Delta_1 \cap \Delta_2)$. In other words $\mu$ is a countably additive orthogonally scattered measure on $\Sigma$, controlled by $E$ (cf. [7], [9]).

For a given locally convex space $S_R$ originating from a "generating" family $R$ of operators on the Hilbert space $H$ (cf. the authors' papers [2,3,5]). It is proved that the topological dual $S'_R$ is isomorphic to the space $T_R$ of $R$-bounded spectral trajectories controlled by the joint spectral measure of the family $R$. The present paper contains a study of the duality between the spaces $S_R$ and $T_R$, and of the topological properties of the space $T_R$.

It extends the results of the authors' papers [2,3,4,5], which are the main references to the subject.
Introduction

Our previous paper [3] contains the construction of a locally convex topological vector space $S_R$ (called the initial space), which is associated with a generating family $R$ of operators acting on a Hilbert space $H$. In comparison with our former constructions, which were based on a generating family of functions (cf. [2], [5]), the approach in [3] lacks a proper description of the topological dual of the space $S_R$. Presently we prove that the strong dual of $S_R$ can be represented as a locally convex topological vector space $T_R$ consisting of spectral trajectories (i.e. $R$-bounded countably additive orthogonally scattered measures) over the joint spectrum $\Lambda$ of the family $R$.

The general idea behind these topics originates from our previous papers [2], [3], [5]. However, the use of orthogonally scattered measures, somewhat in the spirit of the Riesz' representation theorem for spaces of continuous functions, seems to be the most complete and elegant one. Moreover, it may provide an interesting global point of view on certain spaces of generalized functions and the generalized eigenvalue problem for self-adjoint operators. The spectral theory connected with our construction will be studied elsewhere.

The main reference is our paper [3], some technicalities come from [2], [4], and [5]. We use freely the general topics on locally convex topological vector spaces presented in the monograph of H.H. Schaefer [10].
1. Preliminaries

Let $L$ be a family of bounded operators in a Hilbert space $H$.

(1.1) Definition

A densely defined linear operator $L$ in a Hilbert space $H$ is called $L$-bounded if for each $A \in L$ the operator $LA$ is everywhere defined and bounded.

The vector space of $L$-bounded operators is denoted by $LB(H)$.

(1.2) Definition

Let $K \subseteq LB(H)$ be a family of operators. The set

$$K^c = \{K \in LB(H) | \forall L \in K \forall A \in L L = LB(H), LK \in LB(H), LKA = KLA\}$$

is called the $L$-commutant of $K$.

The $L$-bicommutant of $K$ is defined by $K^{cc} = (K^c)^c$.

Now let $H$ be a separable Hilbert space. As in our paper [3] we introduce a generating family of operators $R$:

(1.3) Definition

Let $R \subseteq L(H)$ consist of bounded self-adjoint operators in the Hilbert space $H$. $R$ is called a generating family of operators if:

1. $\forall A \in R : 0 \leq A \leq I$ (positivity and boundedness).
2. $\forall A, B \in R : AB = BA$ (commutativity).
3. \( \forall A, B \in R \exists C \in R \\
A \leq C \text{ and } B \leq C \\
(\text{directedness}). \)

4. \( \forall A \in R \exists B \in R \\
A \downarrow \leq B \\
(\text{sub-semigroup property}). \)

5. In the \( W^\ast \)-algebra \( R \), generated by \( R \) and the identity \( I \), there exists a sequence \( \{p_n\}_{n \in \mathbb{N}} \) of mutually orthogonal projections \( P_n \) such that \( \sum_{n=1}^{\infty} P_n = I \), and

a) \( \forall n \in \mathbb{N} \exists A \in R \exists c_1 \in R \uparrow, c_1 > 0 \\
P_n \leq c_1 \cdot A. \)

b) \( \forall A \in R \exists B \in R \exists c_2 \in R \uparrow, c_2 > 0 \\
\forall n \in \mathbb{N} : \\
n^2 \|AP_n\| \leq c_2 \inf \{ \|BP_n\| | \|P_n\| = 1, \gamma \in H \} \)

6. Let \( R^{**} = \{ L \in R'' | \forall L' \in R^C \text{ } L'L \in R'' \} \).

Then \( \forall Q \in R^{**} \exists A \in R \exists c_3 \in R \uparrow, c_3 > 0 \\
Q^*Q \leq c_3 \cdot A^2 \).

Let \( E \) be the joint resolution of the identity for the generating family \( R \) (or simply the joint spectral measure) \( E \) can be defined as a projection valued countably additive measure which extends the product measure \( \chi E_A \) (cf. [1]). Here the measure \( \chi E_A \) is defined on cylinders in the Tichonoff compact space \( \Lambda = \chi \supp E_A \), where each \( E_A, A \in R \), denotes the spectral measure of an element \( A \) of \( R \).
We recall that a cylinder in the product space $\Lambda$ is a set of the form:

$$U(A_1, \ldots, A_k, V_1, \ldots, V_k) =$$

$$= \{ (\lambda_A) \in \Lambda \mid \lambda_{A_i} \in V_i, i=1, \ldots, k \},$$

where each $A_i \in \mathbb{R}$, and where each $V_i$ is a bounded Borel set in the support $\text{supp } E_{A_i}$ of the spectral measure $E_{A_i}$ corresponding to the operator $A_i$.

We introduce the following semi-ring $\Sigma$ of subsets of $\Lambda$:

$$(1.4) \quad \Sigma = \{ \Delta \subset \Lambda \mid \Delta \text{ is Borel and}$$

$$E\text{-measurable, } \exists A \in \mathbb{R}, \exists c \in \mathbb{R}_+, c > 0,$$

$$E(\Delta) \leq c \cdot A \}. $$

It follows from Def. 1.3 3) and 5) that (1.4) defines a ring of subsets, which, however, is not a $\sigma$-ring in general.

$$(1.5) \quad \textbf{Lemma}$$

For each $\Delta \in \Sigma$ there exists a natural number $n_0$ such that

$$E(\Delta) \leq \sum_{n=1}^{n_0} P_n,$$

where the projections $P_n$ are given by Def. 1.3 5).

\textbf{Proof:}

Assume the contrary. Let $\Delta \in \Sigma$ be such that there exists a subsequence $\{P_{n_k}\}_{k \in \mathbb{N}}$ of the sequence $\{P_n\}_{n \in \mathbb{N}}$, such that for each $k = 1, 2, \ldots$

$$P_{n_k} E(\Delta) \neq 0,$$

and

$$\sum_{k=1}^{\infty} P_{n_k} E(\Delta) = E(\Delta).$$
Let $A \in \mathbb{R}$ be such that
$$E(\Delta) \leq c \cdot A$$
for some number $c > 0$.
Then by Def. 1.3 5.b) the inequality
$$\|P_{\mathcal{B}} E(\Delta)\| = 1 < c \cdot \|P_{\mathcal{B}} A\| < \frac{1}{2} \|B\|$$
holds for some $B \in \mathbb{R}$. This yields a contradiction.

\[\blacklozenge\]

(1.6) **Definition**

Let $\mathcal{E}$ be a ring of subsets of $\Lambda$ and let $E$ be a projection valued measure on the $\mathcal{V}$-algebra generated by the ring $\mathcal{E}$ in $\Lambda$.
A function $\mu : \mathcal{E} \rightarrow \mathbb{H}$ is called a spectral trajectory controlled (or propagated) by $E$ if

$$\forall \Delta, \Delta' \in \mathcal{E}, \quad E(\Delta) \mu(\Delta') = \mu(\Delta \cap \Delta').$$

($\mu$ is an $E$-trajectory)

Each spectral trajectory $\mu$ satisfies the following conditions

(1.7) Whenever $\Delta_1 \cap \Delta_2 = \phi$, $\Delta_1, \Delta_2 \in \mathcal{E}$
then $\mu(\Delta_1) \perp \mu(\Delta_2)$,
i.e. $\mu$ is a so called orthogonally scattered measure.

(1.8) If $\{\Delta_n\}_{n \in \mathbb{N}} \subseteq \mathcal{E}$, $\Delta_n \cap \Delta_m = \phi$
for \( n \neq m \), and \( \bigcup_{n \in \mathbb{N}} \Delta_n \in \Sigma \), then

\[
\mu\left( \bigcup_{n \in \mathbb{N}} \Delta_n \right) = \sum_{n=1}^{\infty} \mu(\Delta_n),
\]

where the series is convergent in norm in \( H \).

Therefore \( \mu \) is a so called countably additive orthogonally scattered measure (c.a.o.s., cf. P. Masani [7]).

The converse is also true, i.e. each c.a.o.s. measure \( \xi \) on the semi-ring \( \Sigma \) is controlled by some spectral measure \( Q \), the so called spatial measure of \( \xi \) (cf. [9]). However, we maintain our terminology originating from the theory of analyticity and trajectory spaces (cf. [2]).

1.9 Definition

Let \( R \) be a generating family of operators in a Hilbert space \( H \), \( E \) its joint spectral measure and \( \Sigma \) the ring of subsets of the joint spectrum \( \Lambda \) of \( R \), given by (1.4).

An \( E \)-spectral trajectory \( \mu : \Sigma \to H \) is called an \( R \)-bounded spectral trajectory if for every \( A \in R \) the map

\[
\Sigma \ni \Delta \mapsto A \mu(\Delta) \in H
\]

is bounded, i.e.

\[
\sup_{\Delta \in \Sigma} \| A \mu(\Delta) \| < \infty.
\]

The (linear) set of all \( R \)-bounded spectral trajectories is denoted by \( T_R \).
1.10 Lemma

For each $A \in \mathbb{R}$ and each $\mu \in \mathbb{T}_{\mathbb{R}}$ there exists a vector $\mu_A \in \mathcal{H}$ such that for each $\Delta \in \Sigma$

$$E(\Delta) \mu_A = A \mu(\Delta).$$

Proof

In $\Sigma$ we can introduce an order relation $\geq: \Delta \geq \Delta'$ iff $\Delta \geq \Delta'$. Thus $\Sigma$ becomes a directed set.

Consider the net of elements of $\mathcal{H}$: $\{A \mu(\Delta)\}_{\Delta \in \Sigma}$. It is bounded and hence it has weak cluster points. Let $x \in \mathcal{H}$ be one of them. We have

$$\|x\| \leq \sup_{\Delta \in \Sigma} \|A \mu(\Delta)\|.$$ Let $\Delta \in \Sigma$. Then for every $\varepsilon > 0$ and $z \in \mathcal{H}$ there exists $\Delta', \Delta \in \Sigma$ such that $\Delta' \geq \Delta$ and $\|E(\Delta)z - A \mu(\Delta')\| < \varepsilon$.

It follows that

$$\|E(\Delta)x - A \mu(\Delta)\| < \varepsilon.$$

This holds for arbitrary $z \in \mathcal{H}$ and $\varepsilon > 0$, hence $E(\Delta)x = A \mu(\Delta)$. Let $r(A)$ be the right support of the operator $A$ in $\mathbb{R}^n$. Put $\mu_A = r(A)x$.

It is easy to see that $\mu_A$ does not depend on the choice of $x$.

1.11 Corollary

The vector $\mu_A$ is uniquely defined and $r(A)\mu_A = \mu_A$. Moreover the set
$H_A = \{ \mu_A \mid \mu \in T_R \}$

is dense in the Hilbert space $r(A)H$.

**Proof**

Obviously $H_A \subset r(A)H$.

Let $v \in r(A)H$ and $v \perp H_A$.

Then for each $\mu \in T_R$

$$(r(A)v \mid \mu_A) = 0.$$  

In particular it holds for all $\xi \in T_R$ of the form

$$\xi(\Delta) = E(\Delta)z$$

where $z \in H$.

Since $\xi_A = A z$ we have:

$$(Ar(A)v|z) = 0 \text{ for all } z \in H.$$  

It follows that $v = 0$.

$\blacksquare$

We have $\sup_{\Delta \in \Sigma} \| A\mu(\Delta) \| = \| \mu_A \|$ for all $A \in R$, $\mu \in T_R$. So we can introduce the seminorms $\| \cdot \|_A$ on the space $T_R$. 
1.13 Definition

For $A \in \mathbb{R}$ and $\mu \in T_R$ put

$$\|\mu\|_A = \|\mu_A\|.$$  

The locally convex topology on the space $T_R$ generated by the seminorms $\|\cdot\|_A$ will be denoted by $\tau_{\text{proj}}$.

It easily follows from Def. 1.3.5 a) that the topology $\tau_{\text{proj}}$ is Hausdorff and so we shall always refer to the space $T_R$ as to the locally convex topological vector space endowed with the topology $\tau_{\text{proj}}$.

1.14 Lemma

Let $A, B \in \mathbb{R}, \mu \in T_R$. Then

$$A \mu_B = B \mu_A.$$  

Proof

For each $\Delta \in \Sigma$ and $z \in H$ we have

$$(E(\Delta)z \mid A \mu_B - B \mu_A) =$$

$$= (z \mid AB \mu(\Delta) - BA \mu(\Delta)) = 0.$$  

Since $\{E(\Delta)z \mid \Delta \in \Sigma, z \in H\}$ is a dense set in $H$ the result follows.
Now we give a complete characterization of the elements of $T_R$ by means of the strong bicommutant $R^{cc}$ of $R$.

1.15 Theorem

Let $\mu$ be a c.a.o.s. measure on the ring $\Sigma$. Then $\mu \in T_R$ if and only if there exists $L \in R^{cc}$ and $y \in H$ such that for every $\Delta \in \Sigma$

$$\mu(\Delta) = L E(\Delta) y.$$ 

Proof

$\Rightarrow$ If $\mu(\Delta) = L E(\Delta) y$ then $\mu$ is an $E$-spectral trajectory and it is enough to notice that the measure $A^* \mu$ is bounded on $E$ since $R^{cc} \subseteq RB(H)$. 

$\Leftarrow$ Let $\mu \in T_R$. Let $\{P_n\}_{n \in \mathbb{N}}$ be the sequence of projections given by Definition 1.3.5, and let $A_n \in R$, $n = 1, 2, \ldots$ be such that $r(A_n) P_n \leq c_n \cdot A_n$ for some numbers $c_n > 0$. Thus, for each $n \in \mathbb{N}$ the vector $y_n = A_n^{-1} P_n \mu A_n$ is well defined. Put 

$$r_n = \|y_n\|.$$ 

Since 

$$\sum_{n \in \mathbb{N}} \left( \frac{1}{r_n} \right) r_n y_n \leq \sum_{n \in \mathbb{N}} \frac{1}{n^2} < \infty$$

the series
\[ y = \sum_{n \in N} \frac{1}{\sum_{r \neq 0}^n r} y_n \]  

(1.16)

converges in \( H \) and so \( y \in H \).

Put

\[ L = \sum_{n \in N} \frac{n}{\sum_{r \neq 0}^n r} \cdot P_n. \]  

(1.17)

We shall prove that \( L \in R^{cc} \).

Let \( A \in R \). Then we have

\[
\|L A\| \leq \sup_{n \in N} n \cdot \| A \| \| P_n \| \leq c_2 \sup_{n \in N} \frac{1}{n} \cdot \sum_{r \neq 0}^n \| B^{-1} P_n \|^{-1},
\]

where \( c_2 > 0 \) and \( B \in R \) are chosen as in Def. 1.3.5 b).

Also we have

\[
r_n = \| A^{-1} P_n A_n \| = \| B^{-1} P_n A^{-1} A_n P_n B \| \leq \| B^{-1} P_n \| \| P_n B \| \leq \| B^{-1} P_n \| \| B \| \]

(1.18)

So eventually we get

\[
\|L A\| \leq c_2 \| B \| < \infty.
\]

Thus we have proved that \( L \in RB(H) \).

Obviously \( L \in R^{cc} \) and for each \( \Delta \in E \) \( E(\Delta) y \in D(L) \).
Now we compute $L E(\Delta) y$ for $\Delta \in \Sigma$.

By virtue of Lemma 1.5 we deal with finite sums only:

Let $n_0 \in \mathbb{N}$ be such that

$$\sum_{n=1}^{n_0} p_n E(\Delta) = E(\Delta).$$

Then

$$\sum_{n=1}^{n_0} n \cdot r_n \cdot p_n \cdot E(\Delta) \cdot \frac{1}{n \cdot r_n} \cdot y_n =$$

$$= \sum_{n=1}^{n_0} p_n E(\Delta) \cdot A_n^{-1} \cdot p_n \cdot \mu A_n$$

$$= \left( \sum_{n=1}^{n_0} E(\Delta) r(A_n) p_n \right) \mu(\Delta) = \mu(\Delta)$$

(1.19)

The last equality follows from the identities:

$$r(A_n) p_n = p_n, \text{ and } r(A_n) p_n \mu(\Delta) = 0$$

whenever $r_n = \|A_n^{-1} p_n \mu_0 A_n\| = 0$.

We conclude that $L E(\Delta) y = \mu(\Delta)$.
At the end of this section we recall the definition of the space $S_{R}$ constructed in [3].

Let $AH$ denote the Hilbert space consisting of all elements of $H$ of the form $Ax$, with $A \in \mathbb{R}$ and $x \in H$.

$AH$ is endowed with the scalar product

$$(A x | A y)_A = (r(A)x | r(A)y)_H.$$ 

1.20 Definition

The space $S_{R}$ is the locally convex inductive limit of the family

$\{AH\}_{A \in \mathbb{R}}$ of Hilbert spaces with respect to the family of embeddings

$\{\pi_{AB}\}_{A, B \in \mathbb{R}}$:

$$\pi_{AB} : BH \hookrightarrow AH,$$

such that for $s \in BH$, $A \geq B$, $s = Bx$, $x \in H$

$$\pi_{AB}(s) = A(A^{-1}Bx).$$

1.21 Remark

We can identify $S_{R}$ with the dense linear subspace $\cup_{A \in \mathbb{R}} AH$ of the Hilbert space $H$. Since the embedding $S_{R} \hookrightarrow H$ is continuous $S_{R}$ is a Hausdorff topological vector space. In the construction of the inductive limit $S_{R}$ condition 6 in Def. 1.3 is not involved.

It has been proved in [3] that the locally convex inductive limit topology is given by the family of seminorms:
Moreover (cf. 5.10 [3]), it has been proved that $S_R$ is a bornological, barreled, semireflexive (hence reflexive) locally convex topological vector space. $S_R$ is dense in $H$. If we assume that the family $R$ fulfills also condition 6 of Def. 1.3 then we obtain the completeness of $S_R$. 

\[ S_R \ni s \iff \|Ls\|_H, \ L \in R^{cc} \] (1.22)

(cf. 5.4 [3]).
In this section we prove that the space $T_R$ is endowed with the projective limit topology $\tau_{\text{proj}}$ is an inductive limit of Hilbert spaces. A corresponding result has been proved for the space $T_{\phi}(A)$ introduced in [5], where the generating family of operators was obtained by applying a generating family of functions $\phi$ to an $n$-tuple of commuting operators $A$. Also we present here a nice characterization of the bounded sets in $T_R$.

This characterization is essential for the proof of the main result of this paper—namely, the duality theorem in Section 3.

To represent $T_R$ as an inductive limit we have to assume that the generating family of operators $R$ fulfills also condition 6 of Def. 1.3. Let $R^\text{cc} = \{L \in R^\text{cc} | L \text{ is essentially self-adjoint on } S_R \text{ and } \exists \varepsilon > 0 \text{ s.t. } L \geq \varepsilon I\}$. Let $L \in R^\text{cc}$. We define

$$L \cdot H = \{\mu \in T_R \mid \exists y \in H, \mu(\Delta) = LE(\Delta)y, \Delta \in \Sigma\}.$$

For $\mu_1, \mu_2 \in L \cdot H$ put

$$\langle \mu_1 | \mu_2 \rangle_L = \langle y_1 | y_2 \rangle_H \quad (2.1)$$

where

$$\mu_1(\cdot) = L E(\cdot)y_1$$

$$\mu_2(\cdot) = L E(\cdot)y_2$$

$y_1, y_2 \in H$. 


It is easy to see that (2.1) defines a scalar product on $L\cdot H$ under which $L\cdot H$ becomes a Hilbert space.

Our proof of the representation of $T_R$ as an inductive limit space of the family of Hilbert spaces \{L\cdot H \mid L \in R_+^{cc}\} is a reformulation of the proof of Theorem 1.7 in [2]. It is based on the following facts:

By virtue of Theorem 1.15 the space $T_R$ can be identified with the linear manifold $\bigcup_{L \in R_+^{cc}} L\cdot H$.

Using the usual ordering of essentially self-adjoint operators in $R_+^{cc}$ we can introduce the locally convex inductive limit topology $\tau_{\text{ind}}$ on $T_R$, with respect to the canonical embeddings of the Hilbert spaces $L\cdot H$ into $T_R$.

Recall that a set $O \subset T_R$ is open in the topology $\tau_{\text{ind}}$ if and only if for each $L \in R_+^{cc}$ the set $O \cap L\cdot H$ is open in the Hilbert space $L\cdot H$.

Since for each $A \in R$ the operator $L_A$ is bounded for every $L \in R^{cc}$ the seminorms on $T_R$, $\mu \mapsto \|A\mu\|_A$, are continuous in the topology $\tau_{\text{ind}}$. Thus $\tau_{\text{ind}} > \tau_{\text{proj}}$.

To prove that $\tau_{\text{proj}} > \tau_{\text{ind}}$ we shall use the following result.

2.2 Proposition

Let $O \subset T_R$ be a convex null-neighborhood in the topology $\tau_{\text{ind}}$.

Then there exists $A \in R$ and a number $\delta > 0$, such that the set

$$V_\delta, A = \{\mu \in T_R \mid \|A\mu\| < \delta\}$$

is contained in $O$. 

Proof

For each $L \in \mathbb{R}_+^{cc}$ the set $0 \cap L \cdot H$ is open and convex in $L \cdot H$.

Let $\text{emb}$ denote the canonical embedding of $H$ into $T_R$ defined by

$$\exists \Delta \mapsto \text{emb}(x)(\Delta) \in H, \text{ where } x \in H.$$  \hfill (2.3)

Since $L \in \mathbb{R}_+^{cc}$ and for each $n \in \mathbb{N}$ the Hilbert space $\text{emb}(P_n H)$ is a Hilbert subspace of $L \cdot H$ the set $0 \cap \text{emb}(P_n H)$ is open in $\text{emb}(P_n H)$.

For every $n \in \mathbb{N}$ we define:

$$r_n = \sup \{ p > 0 | \text{emb}(P_n K(0, p)) \subset 0 \cap \text{emb}(P_n H) \} = \sup \{ p > 0 | (u \in \text{emb}(P_n H), \sup_{\Delta \in \Sigma} \| P_n u(\Delta) \| < p) \Rightarrow \mu \in \mathcal{O} \}$$

\hfill (2.4)

Now let us define the following operator

$$K = \sum_{n=1}^{\infty} 2 \cdot \frac{n^2}{r_n} \cdot P_n$$

\hfill (2.5)

We are going to show that $K \in \mathbb{R}^{**}$.

We need the following result:

(2.6) Lemma

For every $L \in \mathbb{R}^{cc}$ there exists $L' \in \mathbb{R}^{cc}$ such that for all $n \in \mathbb{N}$

$$n^2 \| P_n \| \leq c \inf_{\| P_n y \| = 1} \| L' P_n y \|$$
for some number $c > 0$.

**Proof**

Define the following unbounded operator

$$L' = \sum_{n=1}^{\infty} n^2 \|L_P\| P_n.$$ 

Let $A \in \mathbb{R}$. Then there exists $B \in \mathbb{R}$ such that for $N_o = \{n \in \mathbb{N} \mid \|A_P\| \neq 0\}$:

$$\|L'A\| \leq \sup_{n \in N_o} n^2 \|A_P\| \|L_P\| \leq c \cdot \sup_{n \in N_o} \inf_{\|P_n\| = 1} \|B_P y\| \|B^{-1}_P L_P\| \leq$$

$$\leq c \cdot \sup_{n \in N_o} (\|B^{-1}_P n\|^{-1})^{-1} \|B^{-1}_P L_P\| \leq c \cdot \|B\| < \infty.$$ 

Hence $L' \in \mathbb{R}^{cc}$.

Obviously

$$n^2 \|L_P\| = n^2 \|L_P\| \inf_{\|P_n\| = 1} \|P_n y\| \leq \inf_{\|P_n y\| = 1} \|L' P_n y\|.$$ 

This proves Lemma 2.6.

Now let $\mu \in T_R$. Then $\mu \in L \cdot H$ for some $L \in \mathbb{R}_+^{cc}$.

Let the operator $L' \in \mathbb{R}^{cc}$ be defined for $L$ as in Lemma 2.6, i.e. for each $n \in \mathbb{N}$

$$n^2 \|L_P\| \leq c \inf_{\|P_n y\| = 1} \|L' P_n y\|.$$ 

Since the set $0 \cap L' \cdot H$ is a neighborhood of $0$ in $L' \cdot H$ there exists $\epsilon > 0$ such that
For each $n \in \mathbb{N}$ we have:

$$\|P_n\|_{L^1} = \sup_{\Delta \in \Sigma} \|L^{-1}P_n(\Delta)\| \leq \left( \inf_{\|P_n\_{y\|}=1} \|L'P_n\_{y\|}\right)^{-1} \sup_{\Delta \in \Sigma} \|P_n(\Delta)\|.$$ 

So if $\sup_{\Delta \in \Sigma} \|P_n(\Delta)\| < \varepsilon \cdot \inf_{\|P_n\_{y\|}=1} \|L'P_n\_{y\|}$ then $P_n \not\in 0 \cap L' \cdot H \subset 0$.

It follows that for each $L \in \mathbb{R}^{cc}$, $r_n \geq \varepsilon \cdot \inf_{\|P_n\_{y\|}=1} \|L'P_n\_{y\|}$ with $L'$ as in 2.6.

By virtue of Lemma 2.6 we have for all $L \in \mathbb{R}^{cc}$

$$\|KL\| \leq 2 \sup_{n \in \mathbb{N}} \frac{n^2}{r_n} \|LP_n\| \leq 2 \sup_{n \in \mathbb{N}} \frac{n^2}{r_n} \left( \inf_{\|P_n\_{y\|}=1} \|L'P_n\_{y\|}\right)^{-1} \|LP_n\| \leq 2 \cdot \frac{1}{\varepsilon} \cdot c < \infty.$$ 

Thus $K \in \mathbb{R}^{**}$.

By condition 6) of Def. 1.3 there exists $A \in \mathbb{R}$ and a number $c_3 > 0$ such that: $K \leq c_3 \cdot A$. (2.7)

Let $\delta = \frac{1}{c_3}$ and

$$V = \{x_{1/A} = \{x \in \mathbb{R} \mid \|\|_A < \frac{1}{c_3}\} \}.$$ 

We will show that $V \subset 0$.

Let $\mu \in V$. Assume that $\mu \in L \cdot H$ for some $L \in \mathbb{R}^{cc}$. Then

$$\|\mu\|_{L^2} = \sum_{n=1}^{\infty} \|P_n\mu\|_{L^2} < \infty.$$ 

By virtue of (2.7) for each $n \in \mathbb{N}$ we have

$$n^2 \sup_{\Delta \in \Sigma} \|P_n(\Delta)\| = \frac{1}{2} r_n \sup_{\Delta \in \Sigma} \|AP_n(\Delta)\| \leq \frac{1}{2} \cdot c_3 \cdot r_n \|\mu\| A < r_n.$$ 

Thus, by (2.4) we have
\( n^2 \mu \in 0 \quad (2.8) \)

Consider the following decomposition:

\[
\mu = \sum_{n=1}^{n_o} \frac{1}{2n} 2n^2 \mu + \left( \sum_{n=n_o+1}^{\infty} \frac{1}{2n} \right) \mu_{n_o}, \quad (2.9)
\]

where \( \mu_{n_o} = \left( \sum_{n=n_o+1}^{\infty} \frac{1}{2n} \right)^{-1} \sum_{n=n_o+1}^{\infty} \mu_n \).

Using again Lemma 2.6 we can find \( L' \in \mathbb{R}^{cc} \) such that

\[
\| \mu_{n_o} \|_{L'}^2 = \left( \sum_{n=n_o+1}^{\infty} \frac{1}{2n} \right)^{-2} \sum_{n=n_o+1}^{\infty} \| \mu_n \|_{L'}^2 \leq 4 \cdot c \cdot \sum_{n=n_o+1}^{\infty} \frac{1}{n} \| \mu_n \|_{L'}^2 \leq 4 \cdot \mu_{n_o} \mu_{n_o} \| \mu_n \|_{L'}^2 \leq \frac{\epsilon}{\mu_{n_o}} \quad \text{(2.8)}
\]

Hence \( \| \mu \|_{L'} \rightarrow 0 \) as \( n_o \rightarrow \infty \).

The first term in the decomposition (2.9) belongs to 0. The second term for sufficiently large \( n_o \) also belongs to 0. Thus \( \mu \) is a convex combination of elements of the convex set 0, and therefore \( \mu \in 0 \).

We proved that \( V \subset 0 \).

This ends the proof of Proposition 2.2.

\[ \Box \]

Taking into account Proposition 2.2 and the previously stated fact that \( \tau_{\text{ind}} > \tau_{\text{proj}} \) we can formulate the following result.
2.10 Theorem

The locally convex topological vector space \( T_R \) with topology \( \tau_{\text{proj}} \) can be identified with the locally convex inductive limit space

\[ U \bigoplus_{L \in R^+_c} L \cdot H \text{ with topology } \tau_{\text{ind}}. \]

Now we prove another useful result:

2.11 Theorem

Let \( R \) be a generating family of operators fulfilling the conditions 1-6 of Def. 1.3. Let \( T_R \) be the space of spectral trajectories. Then a set \( B \subseteq T_R \) is bounded in the topology \( \tau_{\text{proj}} \) if and only if there exists \( L \in R^+_c \) such that \( B \subseteq L \cdot H \) and \( B \) is bounded in the Hilbert space \( L \cdot H \).

Proof

The idea of the proof is based on the proof of Theorem 2.3 in [5] and the above formulated result representing \( T_R \) as the inductive limit

\[ U \bigoplus_{L \in R^+_c} L \cdot H. \]

Therefore we present here only an outline of it.

If a set \( B \subseteq T_R \) is contained in a space \( L \cdot H \) for some \( L \in R^+_c \) then it is bounded in \( T_R \) whenever it is bounded in \( L \cdot H \) because the embedding \( L \cdot H \subseteq T_R \) is continuous.

On the other hand assume now that \( B \) is bounded in \( T_R \).

Then for each \( n \in \mathbb{N} \) the number
is well defined if $A$ is chosen as in Def. 1.3.5 a). We define in $H$ an unbounded operator

$$L = \sum_{n=1}^{\infty} n \cdot s_n \cdot P_n .$$

Obviously $S_R \subset D(L)$.

Using Def. 1.3.5 b) we can prove that $L \in RB(H)$, and hence $L \in R^c$.

For $\mu \in B$ put

$$x_\mu = \sum_{n=1}^{\infty} \frac{1}{n \cdot s_n} \cdot A^{-1} P_n \mu_A$$

with $A$ as in 2.12 and $\mu_A$ given by Lemma 1.10.

We have the easy estimation

$$\|x_\mu\|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Therefore $x_\mu \in H$ and the set $B_0 = \{x_\mu \mid \mu \in B\}$ is uniformly bounded in $H$. One can see that $B = L \cdot B_0$. 

$\square$
3. **Duality**

In this section we discuss the duality between the spaces $S_R$ and $T_R$. We shall do it on two levels. Under the assumption of conditions 1-5 of Definition 1.3 we prove an algebraic identification of the space $T_R$ with the (strong) dual $S_R'$ of the space $S_R$, and similarly an algebraic identification of the space $S_R$ with the (strong) dual $T_R'$ of the space $T_R$. However, to prove the topological identification of the spaces $S_R$ and $T_R$ with the strong duals of $T_R$ and $S_R$, respectively, we have to assume also condition 6 of Def. 1.3.

We define the following pairing between spaces $S_R$ and $T_R$.

**3.1 Definition**

Let $\mu \in T_R$ and $s \in S_R$, with $s = A \times$ for some $A \in R$ and $x \in H$. We define the number

$$< \mu, s > = (\mu_A | x)_H,$$

where $\mu_A$ is given by Lemma 1.10.

**3.2 Remark**

The numbers $< \mu, s >$ are well defined, i.e. they do not depend on the decomposition $s = A \times$, $A \in R$, $x \in H$. 
Proof

Let \( s = A x = A'x' \), \( A, A' \in \mathbb{R}; x, x' \in H \).

By the definition the vectors \( \mu_A \) and \( \mu_{A'} \) are weak cluster points of the nets \( \{E(\Delta)\mu_A\}_{\Delta \in \Sigma} \) and \( \{E(\Delta)\mu_{A'}\}_{\Delta \in \Sigma} \), respectively.

For each \( \Delta \in \Sigma \) we have

\[
(E(\Delta)\mu_A \mid x)_H = (E(\Delta)\mu_{A'} \mid x')_H.
\]

Thus it follows that

\[
(\mu_A \mid x)_H = (\mu_{A'} \mid x')_H = \langle \mu, s \rangle.
\]

\[\square\]

3.3 Proposition

The function

\[
T_R \times S_R \ni \langle \mu, s \rangle \mapsto \langle \mu, s \rangle \in \mathcal{C}^1
\]

is a non-degenerate sesquilinear form.

Proof

The antilinearity of the form \( \langle \mu, s \rangle \) in the first argument \( \mu \in T_R \) follows directly from the definition of the vectors \( \mu_A \), \( A \in \mathbb{R} \), Lemma 1.10.

To prove the linearity of \( \langle \mu, s \rangle \) in the second argument \( s \in S_R \), let us take \( s = A x = A_1x_1 + A_2x_2 \) with \( A, A_1, A_2 \in \mathbb{R}, x, x_1, x_2 \in H \).
and \( \mu \in T_R \). Then for every \( \Delta \in \Gamma \),
\[
(\mu(\Delta) \mid A x)_H = (\mu(\Delta) \mid A_1 x_1)_H + (\mu(\Delta) \mid A_2 x_2)_H = (E(\Delta)\mu_{A_1} \mid x)_H =
\]
\[
= (E(\Delta)\mu_{A_2} \mid x_2)
\]
Again remembering that the vectors \( \mu_A, \mu_{A_1}, \) and \( \mu_{A_2} \) are weak cluster points of the corresponding nets we obtain the result by "2\( \varepsilon \)-argument".

To prove that the form \( \langle , \rangle \) is non-degenerate assume the contrary: suppose there exists \( \nu \in T_R, \nu \neq 0 \) such that \( \langle \nu, s \rangle = 0 \) for all \( s \in S_R \). Then we have for each \( x \in H \), each \( A \in R \) and each \( \Delta \in \Gamma \):
\[
(\nu_A \mid E(\Delta) x)_H = 0
\]
It follows that \( \nu_A = 0 \) so \( \nu = 0 \). A contradiction.

Now let \( s_0 \in S_R \) be such that for all \( \mu \in T_R, \langle \mu, s_0 \rangle = 0 \).

Considering the trajectories of the form
\[
\xi(\Delta) = E(\Delta)s_0
\]
we get:
\[
\|E(\Delta)s_0\|^2 = 0 \quad \text{for all} \Delta \in \Gamma.
\]
Thus \( s_0 = 0 \).

Now let \( S_R' \) denote the (strong) topological dual of the inductive limit space \( S_R \), and let \( T_R' \) be the (strong) topological dual of the space \( T_R \) endowed with the topology \( \tau_{\text{proj}} \).
3.4 Theorem

Let $R$ be a generating family of operators fulfilling conditions 1-5 of Def. 1.3. Then the following algebraic dualities hold:

$$S_R = T'_R \quad \text{and} \quad T_R = S'_R.$$ 

Proof

We shall prove the theorem in four steps showing the existence of four anti-linear injections:

I. $\alpha_1 : S_R \to T'_R$

II. $\alpha_2 : T'_R \to S_R$

III. $\beta_1 : T_R \to S'_R$

IV. $\beta_2 : S'_R \to T_R$

and the identities:

$$\alpha_1 \circ \alpha_2 = \text{id}_{T'_R}, \quad \alpha_2 \circ \alpha_1 = \text{id}_{S_R}$$

$$\beta_1 \circ \beta_2 = \text{id}_{S'_R}, \quad \beta_2 \circ \beta_1 = \text{id}_{T_R}.$$ 

I. The injection $\alpha_1$ is defined by

$$S_R \ni s \mapsto \alpha_1(s) \in T'_R$$

where for each $\mu \in T_R$

$$\alpha_1(s)(\mu) = \langle \mu, s \rangle,$$  \hspace{1cm} (3.5)
and $T_R^*$ is the algebraic dual of the space $T_R$.

By proposition 3.3 the map $\alpha_1$ is well defined and anti-linear.

It is easy to see that for each $s \in S_R$, $\alpha_1(s)(\mu)$ is $\tau_{proj}$-continuous on $T_R$. Hence $\alpha_1(s) \in T_R'$.

II. The injection $\alpha_2$ is constructed as follows:

Let $\varphi \in T_R'$. Then there exists $A \in \mathbb{R}$ and a constant $c > 0$ such that

$$|\varphi(\mu)| \leq c \cdot \|\mu\|_A.$$

On the pre-Hilbert space with the norm $\|\cdot\|_H$

$$H_A = \{\mu_A \in H \mid \mu \in T_R\}$$

a functional $\tilde{\varphi}(\mu_A) = \varphi(\mu)$.

$\varphi$ is linear and bounded hence it can be extended onto the Hilbert space $r(A)H$ which contains $H_A$ as a dense subset. So there exists $x \in r(A)H$ such that for each $\mu \in T_R$

$$\varphi(\mu) = \tilde{\varphi}(\mu_A) = (x | \mu_A)_H = \langle \mu, s \rangle$$

where $s = A x \in S_R$. It is easy to show that the thus defined $s$ is unique. Hence the map $T_R' \ni \varphi \rightarrow \alpha_2(\varphi) = s \in S_R$ is a well defined anti-linear injection.

One can easily check that the identities

$$\alpha_1 \circ \alpha_2 = \text{id}_{T_R},$$

and $\alpha_2 \circ \alpha_1 = \text{id}_{S_R}$ hold.
It means that the vector spaces $S_R$ and $T_R'$ can be identified by means of the anti-linear invertible one-to-one mappings $\alpha_1$ and $\alpha_2$.

III. Now we shall construct the injection

$$\beta_1 : T_R \rightarrow S_R'. $$

Let $\mu \in T_R$ and $s \in S_R$. Define

$$\beta_1(\mu)(s) = <\mu, s>. $$

It follows from Proposition 3.3 that $\beta_1$ is a well defined anti-linear map from $T_R$ into the algebraic dual $S_R^*$ of the space $S_R$.

Let us consider the linear functional

$$r(A) \exists x \mapsto \beta_1(\mu)(Ax) \in C^1, \quad (3.6)$$

with $A \in R$. We have

$$|\beta_1(\mu)(Ax)| = |(\mu_A \downarrow x)_H| \leq \|r(A)\|\|\mu_A\|$$

Hence the functional (3.6) is continuous. It is sufficient for the continuity of the functional $\beta_1(\mu)(\cdot)$ on the inductive limit space $S_R$ (cf. [3]). Hence $\beta_1(\mu) \in S_R'$. 

IV. Now we construct the injection

$$\beta_2 : S_R' \rightarrow T_R. $$
By virtue of (1.4) for each $\Delta \in \Sigma$ the Hilbert space $E(\Delta)H$ can be isometrically embedded into the Hilbert space $AH$ for some $A \in \mathbb{R}$.

Let $\ell \in \mathcal{S}_{\mathbb{R}}'$. We can restrict $\ell$ to the space $E(\Delta)H$, for every $\Delta \in \Sigma$.

$\ell|_{E(\Delta)H}$ is continuous on $E(\Delta)H$, so there exists a unique vector $\phi(\Delta) \in E(\Delta)H$, such that for every $x \in H$

$$
\ell( E(\Delta)x ) = (\phi(\Delta) \mid E(\Delta)x)_{H}.
$$

Thus we obtain an $H$-valued set function:

$$
E \ni \Delta \mapsto \phi(\Delta) \in H,
$$

which is obviously an $E$-spectral trajectory. We will show that

$\phi \in T_{\mathbb{R}}$, i.e. $\phi$ is $R$-bounded. Let $A \in \mathbb{R}$. Then

$$
\sup_{\Delta \in \Sigma} \| A \phi(\Delta) \| \leq \sup_{\Delta \in \Sigma} \sup_{\|x\| \leq 1} \| \phi(\Delta) | E(\Delta)Ax \| = \sup_{\Delta \in \Sigma} \sup_{\|x\| \leq 1} \| E(\Delta)Ax \| = \sup_{\Delta \in \Sigma} \| E(\Delta)AH \| \leq \| \ell \|_{AH}.
$$

Hence $\phi \in T_{\mathbb{R}}$. Now we put

$$
\beta_{2}(\ell) = \phi.
$$

(3.7)

Since for each $\Delta \in \Sigma$, $A \in \mathbb{R}$, and $x \in H$ $(\beta_{2}(\ell)(\Delta)|Ax\rangle_{H} = \ell(E(\Delta)Ax)$ we have $\langle \beta_{2}(\ell), Ax \rangle = \ell(Ax)$.

Thus we have defined an anti-linear $\beta_{2}$ map from $\mathcal{S}_{\mathbb{R}}'$ into $T_{\mathbb{R}}$.

A straightforward computation yields: $\beta_{1} \circ \beta_{2} = \text{id}_{\mathcal{S}_{\mathbb{R}}'}$, and $\beta_{2} \circ \beta_{1} = \text{id}_{T_{\mathbb{R}}}$. Thus we have established the one-to-one antilinear correspondence between the vector spaces $T_{\mathbb{R}}$ and $\mathcal{S}_{\mathbb{R}}'$. 

$\blacksquare$
3.8 Remark

To simplify the notation we shall denote \( \alpha = \alpha_1, \alpha^{-1} = \alpha_2, \)
and \( \beta = \beta_1, \beta^{-1} = \beta_2. \)

In the following part of this section we shall consider the duality
between the spaces \( S_R \) and \( T_R \) from the topological point of view.
Having in mind the characterization of bounded sets in \( T_R \) given in
Section 2 we recall also the following result:

3.9 Proposition (cf. [3], Lemma 5.8)

Let the family \( R \) fulfill the conditions 1-6 of Def. 1.3. Then a set
\( B \subset S_R \) is bounded if and only if there exists \( A \in R \) such that \( B \)
is a bounded subset of the Hilbert space \( AH. \)

3.10 Theorem

Let the family \( R \) fulfill the conditions 1-6 of Def. 1.3. Then the
maps \( \alpha \) and \( \beta \) are antiisomorphisms between the space \( S_R \) and the strong
dual \( T'_R \) of the space \( T_R \), and, respectively, between the space \( T_R \)
and the strong dual \( S'_R \) of the space \( S_R. \)
Thus \( S_R \) is homeomorphic with \( T'_R \)
and \( T_R \) is homeomorphic with \( S'_R. \)

Proof

Consider at first the map \( \alpha: S_R \rightarrow T'_R. \)

Let \( B_1 \subset T_R \) be a convex bounded set. Then \( B_1 \subset L^1_H \) for some
\( L_1 \in R^{cc}. \) We recall that the \( l.c. \) topology in the space \( S_R \) is given
by the seminorms $S_R \ni s \mapsto \|Ls\|_H$, where $L \in R^{cc}$ (cf. [3], Theorem 5.4).

Consider the inequality:

$$\sup \langle \alpha(s), \mu >_{TR} = \sup \langle \mu, s > = \sup \|\mu_A x\|_H \leq \sup \sup \|A(L_E(\Delta)y \mid x)\|_H \leq r \cdot \|L_A x\| = r \|L_A s\|$$

where $B_1 \subset L_1 \cdot B(0,r)$, and where $B(0,r)$ is a ball in $H$ of radius $r$.

Thus the map $\alpha : S_R \rightarrow T_R'$ is continuous in the strong dual topology in $T_R'$.

To prove the continuity of the map $\alpha^{-1} : T_R' \rightarrow S_R$, let us consider the following relation:

For any $L \in R^{cc}, \varphi \in T_R'$, we have:

$$\|L \alpha^{-1}(\varphi)\| = \|L Ax\| = \sup \|y \mid L A x\|_H = \sup \|\varphi(\mu)\|_{\|y\| \leq 1}$$

where $\varphi(\mu) = (x \mid \mu_A)H$ with $x \in H$, $A \in R$ (cf. Theorem 3.4).

Since the set $L^* \cdot B(0,1)$ is bounded in $T_R$, the above relation proves the continuity of $\alpha^{-1}$ with respect to the strong dual topology in $T_R'$ and the inductive limit topology in $S_R$.

Thus we proved that the spaces $S_R$ and $T_R'$ are topologically isomorphic by means of the map $\alpha$.

Now let us consider the map $\beta : T_R \rightarrow S_R'$.

Let $B_2 \subset S_R$ be a bounded set in the inductive limit topology in $S_R$.

Thus $B_2 \subset A H$ for some $A \in R$ and

$$\sup \|s\|_A \leq c < \infty$$

for some $c > 0$.

We have for any $\mu \in T_R$:
Hence the map $\beta$ is continuous with respect to the strong dual topology in $S_R'$ and the topology $\tau_{\text{proj}}$ in $T_R$.

The inverse map $\beta^{-1} : S_R' \to T_R$ is also continuous thanks to the following relations:

For any $\ell \in S_R'$ and $A \in R$ we have

$$
\| \beta^{-1}(\ell) \|_A = \sup_{\Delta \in \Sigma} \| A \beta^{-1}(\ell)(\Delta) \| = \sup_{\Delta \in \Sigma} \sup_{x \in H} \| (A\beta^{-1}(\ell)(\Delta)x) \|_H = \sup_{\Delta \in \Sigma} \sup_{\| x \| \leq 1} \| \ell(E(\Delta)Ax) \| \leq \| \ell \|_{AH} = \sup_{\| x \| \leq 1, s=Ax} \| \ell(s) \|
$$

Since the set $\{ s \in S_R' \mid s=Ax, \| x \| \leq 1 \}$ is bounded in the inductive limit topology in $S_R'$ the map $\beta^{-1}$ is continuous with respect to the strong dual topology in $S_R'$ and the topology $\tau_{\text{proj}}$ in $T_R$.

This proves that the spaces $S_R'$ and $T_R$ are topologically isomorphic.

Considering all results of the sections 2 and 3 we have the following final statement:

3.11 Theorem (cf. [3], [5], [6], [10])

Let a generating family of operators $R$ fulfill the conditions 1-6 of Def. 1.3 and let $S_R$ and $T_R$ be the initial space and spectral trajectories space endowed with the inductive limit topology and
the topology \( \tau_{\text{proj}} \), respectively. Then both spaces are the topological representations of the strong dual of each other.

Hence, the spaces \( S_R \) and \( T_R \) are complete, barreled, bornological, Mackey, reflexive locally convex topological vector spaces.
4. An example

In our previous papers we considered a particular generating family of operators, namely the family \( \Phi(A) \) constructed by means of a generating family \( \Phi \) of Borel functions on the real line and a self-adjoint operator \( A \) in a Hilbert space \( H \). (cf. [2], [5]). Let us recall shortly this construction:

4.1 Definition

A family \( \Phi \) of real valued functions on \( \mathbb{R}^1 \) is called a generating family of functions if it satisfies the following conditions:

A I. Each \( \varphi \in \Phi \) is a nonnegative Borel function bounded by \( 1 \) with inverse \( \varphi^{-1} \) bounded on bounded Borel subsets of the carrier of \( \varphi \):
\[
\varphi = \{ \lambda \in \mathbb{R}^1 \mid \varphi(\lambda) \neq 0 \}.
\]
The family \( \Phi \) is directed with respect to the usual ordering of real functions.

A II. For each \( \varphi \in \Phi \) there exists \( \psi \in \Phi \) such that \( \varphi^\dagger \leq \psi \).

A III. For each \( \varphi \in \Phi \) and each number \( \delta > 0 \) there exists \( \chi_\delta \in \Phi \) and a number \( c > 0 \) such that
\[
\varphi(\lambda + \delta) \leq c \cdot \chi_\delta(\lambda).
\]

A IV. For each \( \varphi \in \Phi \) there exists \( \psi \in \Phi \) and a positive number \( c \) such that
\[
(1 + \lambda^2) \varphi(\lambda) < c \cdot \psi(\mu)
\]
whenever \( |\lambda| - |\mu| < 1 \), \( \lambda, \mu \in \mathbb{R}^1 \).
A V. Let \( \phi^* = \{ f | f \) is a Borel function on \( \mathbb{R}^1 \) and \( \forall \phi \in \phi^* \sup_{\lambda \in \mathbb{R}} |f(\lambda)\varphi(\lambda)| < \infty \} \).

The set \( \phi^{**} = \{ \chi | \chi \) is a Borel function on \( \mathbb{R}^1 \) and \( \forall f \in \phi^* \sup_{\lambda \in \mathbb{R}} |\chi(\lambda)f(\lambda)| < \infty \} \)

has the property:

For every \( \chi \in \phi^{**} \) there exists \( \varphi \in \phi \) and a number \( c > 0 \) such that

\[ |\chi(\lambda)| < c \cdot \varphi(\lambda) , \quad \lambda \in \mathbb{R}^1. \]

There is no difficulty in defining a generating family of functions on \( \mathbb{R}^n \) \( n \geq 1 \). However, the one-dimensional case is illuminating to a satisfactory extent.

Now, let \( A \) be a self-adjoint operator in a separable Hilbert space \( H \), with the spectral measure \( E \). Then

\[ A = \int_{\mathbb{R}} \lambda \ dE(\lambda), \]

and for every finite Borel function \( f \) the operator

\[ f(A) = \int_{\mathbb{R}} f(\lambda) \ dE(\lambda) \]

is a well defined normal operator in \( H \).

In particular the family of bounded self-adjoint operators

\[ \Phi(A) = \{ \varphi(A) | \varphi \in \phi \} \]

is well defined.
It is very easy to check that the family $\phi(A)$ fulfills the conditions imposed on generating families of operators (with a slight modification) given in Def. 1.3.

Thus let $R = \phi(A)$.

For this particular family of operators the spaces $S_R$ and $T_R$ are constructed in our previous papers [2] and [5]. However, it has been done without an explicit application of spectral trajectories. Hence, we give here an alternative description of these spaces.

The definition of the inductive limit space $S_R$ remains unchanged. Instead, the space of spectral trajectories needs a more specific description.

Since the joint spectral measure for the family $R=\phi(A)$ is simply $E$, we have:

4.2 Proposition

Each element of the topological dual space $S'_R$ of the space $S_R$ can be represented as an $R$-bounded spectral trajectory $\mu \in T_R$ in the following form:

$$\mu(\Delta) = \int_{\Delta} f(\lambda)dE(\lambda)y,$$

where $\Delta$ is a bounded Borel set, $y \in H$, and $f$ is a Borel function in $\varphi^*$. The set $\varphi^*(A)$ plays the role of $R^\circ$ (cf. Section 2).

Proof

By Theorem 1.15 we can express every element of $T_R$ in the form
\[ \mu (\Lambda) = \int_{\Lambda} \lambda \, d\nu(\lambda) y , \]

with \( \Lambda \in \mathbb{R}^{cc} \). By Theorem 4.1 [3] we can write:

\[ L = \int_{\mathbb{R}} f(\lambda) \, d\nu(\lambda) , \]

with \( f \in \phi^* (\mathbb{R}) \).

The result easily follows from Theorem 3.4.

The duality between the spaces \( S_\phi(\Lambda) = S_\mathbb{R} \) and \( T_\phi(\Lambda) = T_\mathbb{R} \), defined in Def. 3.1 is given now as follows:

\[ \langle \mu, s \rangle = \int_{\mathbb{R}} \overline{f}(\lambda) \phi(\lambda)(y) \, d\nu(\lambda)x , \quad (4.3) \]

where \( \mu \in T_\mathbb{R} \), \( \mu(\Lambda) = \int_{\Lambda} f(\lambda) \, d\nu(\lambda)y \), with \( y \in \mathcal{H} \), \( f \in \phi^* \)

and \( s \in S_\mathbb{R} \), \( s = \int_{\mathbb{R}} \phi(\lambda) \, d\nu(\lambda)x \), with \( x \in \mathcal{H} \), \( \phi \in \phi \).

4.4 Remark

Every (not necessarily R-bounded) spectral trajectory can be represented in the form:

\[ \mu (\Lambda) = \int_{\Lambda} g(\lambda) \, d\nu(\lambda) y , \]

with \( y \in \mathcal{H} \), and where \( g \) is a Borel function, bounded on bounded Borel sets.
Let $\Lambda$ be the spectrum of the abelian $C^*$-algebra $C^*(\phi(A))$
-generated by the family $\phi(A)$.

Let for a suitable Radon measure $\nu$

$$H = \int_\Lambda H(\lambda) \, d\nu(\lambda) ,$$

be the decomposition of the Hilbert space $H$ into a direct integral of
Hilbert spaces which diagonalizes all elements of the $C^*$-algebra
$C^*(\phi(A))$.

Thus we can represent every $\Delta \in \Sigma$ (cf. 1.4) by an open-closed subset
of $\Lambda$, denoted for simplicity also by $\Delta$.

We have then:

$$E(\Delta) = \int_\Lambda \chi_\Delta(\lambda)I_\lambda \, d\nu(\lambda) ,$$

where $\chi_\Delta$ is the characteristic function of the set $\Delta$ and $I_\lambda$ is the
identity operator in the Hilbert space $H(\lambda)$.

Now every $E$-spectral trajectory can be represented in the form:

$$\mu(\Lambda) = \int_\Lambda \chi_\Lambda(\lambda) f(\lambda)y(\lambda) \, d\nu(\lambda) ,$$

where $f \in \Phi^*$ and $y \in \int_\Lambda H(\lambda) \, d\nu(\lambda)$.

The connection with the operator $A$ is preserved by the choice of the
measure $\nu$. 
References