Design of a new carrier recovery loop using decision feedback for a 16-state QAM demodulator

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DESIGN OF A NEW CARRIER RECOVERY LOOP
USING DECISION FEEDBACK
FOR A 16-STATE QAM DEMODULATOR

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SUMMARY

This graduation work deals with decision-directed carrier recovery circuits for 16-state QAM signals. At present, these circuits employ baseband remodulation techniques to regenerate a coherent reference carrier signal; during this graduation project the concept of decision-directed IF remodulation has been investigated. In this report several IF remodulation circuits are proposed and compared to the currently used baseband remodulation circuits. The most promising IF remodulation circuit is analyzed in detail; it converts the 16-state QAM signal into a 3-ary ASK signal, which contains a discrete carrier component that can be tracked by a Phase Locked Loop (PLL). Several parts of the remodulation loop are frequency dependent; therefore a heterodyne PLL configuration is chosen to ensure good performance even when the carrier frequency of the received 16-state QAM signal departs from its nominal value. For the symbol timing recovery, an Early-Late tracking loop is proposed, which uses decision feedback as well. Several parts of the decision-directed demodulator have been realized in hardware, for a 16-state QAM signal employing a symbol rate of 1 Msymbols/sec (corresponding to a bit rate of 4 Mbits/sec) and an IF carrier frequency of 70 MHz. Most of the remaining circuits are already designed in detail for hardware realization.
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1. INTRODUCTION

During the last two decades, communication systems have been employing digital modulation techniques more and more frequently. In the 1960s and early 1970s the predominant modulation techniques were binary and four-phase PSK (with a spectral efficiency of 1 b/s/Hz and 2 b/s/Hz respectively), but the crowded conditions prevailing in many regions of the radio spectrum have created a need for improved spectrum utilization techniques.

Speaking about digital communications, one might first think of satellite communications, where bandwidth is very precious indeed, but modulation methods which have a spectral efficiency of more than 2 b/s/Hz require more signal power (a higher carrier-to-noise ratio at the input of the receiver) for a given bit-error-rate. Most operational satellite systems are power limited: the available ratio of energy per bit to noise power density (or the ratio of carrier power to noise power) is insufficient to enable the utilization of spectral efficient (more than 2 b/s/Hz) modulation techniques. In digital terrestrial communications however, the available power is not such a limiting factor as in satellite communications. In the late 1970s and early 1980s digital terrestrial microwave systems with a spectral efficiency between 3 and 6 b/s/Hz have been developed.

One of these spectral efficient modulation methods is known as quadrature-amplitude-modulation (QAM), also known as amplitude-phase-keying (APK) because the information is contained in both the amplitude and the phase of the modulated signal. In the following we will be concentrating on 16-ary QAM signals (or 16-QAM signals for short): two four-level data streams are used to modulate the amplitude of two carrier signals (of the same frequency) shifted by exactly 90°, and the sum of the two resulting AM signals gives the 16-state QAM signal. The four-level data streams result from a binary data stream which first is

* Another frequently encountered term is quadrature amplitude-shift keying (QASK).
commuted into two separate binary data streams (each having half the bit rate of the original data stream); next each of these binary data streams is converted into a four-level data stream having a symbol rate equal to one-fourth of the original bit rate. Thus the 16-QAM signal has a theoretical spectral efficiency of 4 b/s/Hz.

A 16-QAM modulator produces a suppressed-carrier signal, and therefore it is not possible to use a simple carrier-tracking loop (or a narrow bandpass filter) in the demodulator to recover the carrier signal. The carrier recovery circuit must contain a suitable nonlinearity to regenerate a discrete spectral component at the carrier frequency. This nonlinearity can precede the actual tracking loop, but it is also possible to introduce nonlinearities within the tracking loop itself. An example of the former is the squaring loop; examples of the latter are the Costas loop and the remodulator (also called inverse modulator or reverse modulator). To obtain a discrete carrier component from a 16-QAM signal, at least a fourth-order nonlinearity is required in the demodulator. This means that the variance of the noise-caused jitter of the recovered carrier phase will be approximately $4^2 = 16$ times as large as that of an ordinary loop tracking a pure carrier of the same amplitude in the same noise. Where a simple PLL might be able to hold lock down to 0 dB loop signal-to-noise ratio, a tracking loop for a 16-QAM demodulator can be expected to lose lock around +12 dB.

In essence, the demodulators mentioned above remove modulation from the carrier to be tracked by multiplying the demodulated message waveform in analog form. Better noise rejection is possible if the message symbols are optimally detected and the digital message value is used for the modulation-removal multiplication. This type of carrier recovery circuit uses decision feedback; it is said to be decision directed or data aided. It has less noise-caused jitter of the reference carrier because the operation of data detection rejects noise better than the analog-multiplication circuits do.

* The exact number depends on the input signal-to-noise ratio and the nature of the nonlinearity being used.
Several types of decision directed circuits for the demodulation of 16-QAM signals have been developed, all being modifications of an ordinary Costas loop. The principle of operation is the same for all of these circuits: the modulation is removed by multiplying the baseband analog message waveform by the detected digital message value. The objective of this graduation project was to develop a decision directed 16-QAM demodulator, which removes the modulation by modifying the incoming IF signal using the detected data. The bit rate should be 4 Mbits/sec, so the symbol rate is 1 Msymbols/sec. The IF carrier frequency of the incoming 16-QAM signal is 70 MHz.

In Chapter 2 the principle of operation of a 16-QAM modem is explained. The baseband remodulators mentioned above are briefly discussed in Chapter 3; IF remodulation is investigated and compared with the baseband circuits in Chapter 4. In Chapter 5 the chosen method of IF remodulation is further discussed; several problems that might occur in implementing this remodulation method (and the way to avoid them) are discussed as well. The operation of the proposed carrier recovery loop in the presence of noise is examined mathematically in Chapter 6, along with the symbol error probability for the 16-QAM system. Symbol timing is of vital importance in decision-directed circuits; the clock recovery circuits which are necessary to obtain a reliable symbol timing reference are discussed in Chapter 7. In Chapter 8 the actual remodulation control circuits are discussed; in Chapter 9 a closer look is taken at an important subcircuit of the carrier recovery loop (the loop filter). The realized electronic circuits are shown and discussed in Appendix E.
2. THE 16-QAM MODEM: PRINCIPLE OF OPERATION

A block diagram of a 16-QAM suppressed-carrier modulator is shown in Fig. 2.1. The data stream from a binary source, having a bit rate of \( f_b \) bits/sec, is commuted into two binary data streams, each having a rate of \( f_b/2 \). The following two-to-four-level baseband converters convert these \( f_b/2 \) rate data streams into four-level PAM signals having a symbol rate of \( f_s = f_b/4 \) symbols/sec. If premodulation LPFs are used, as shown in Fig. 2.1, then the minimum bandwidth of these filters is \( f_s/2 = f_b/8 \) Hz. The minimum IF bandwidth requirement equals the double-sided minimum baseband bandwidth, that is \( f_s = f_b/4 \) Hz. Thus a (theoretical) spectral efficiency of 4 b/s/Hz has been obtained.

The transmitted signal can be represented as

\[
s(t) = \sqrt{2} x(t) \cos \omega_c t - \sqrt{2} y(t) \sin \omega_c t,
\]

where \( \omega_c \) is the carrier radian frequency, and

\[
x(t) = \sum_{k=-\infty}^{\infty} a_k p(t-kT_s) \quad \text{and} \quad y(t) = \sum_{k=-\infty}^{\infty} b_k p(t-kT_s).
\]

\( T_s = 1/f_s \) is the symbol duration; \( p(t) \) is the baseband pulse shape of each transmitted symbol, often assumed to be confined to the time interval \( 0 \leq t \leq T_s \), although this is not always true (e.g. in the case of Nyquist filtering). The quadrature amplitudes \( a_k \) and \( b_k \) can take on equally likely values from the set \{-3A, -A, A, 3A\}. The signal-state space diagram (also called the phase-amplitude plane) for the 16-QAM signal is shown in Fig. 2.2. It consists of two concentric squares of signal points (or

Fig. 2.1. 16-QAM modulator block diagram [1],[2].
phasors) with each pair of adjacent points separated by 2A.

Fig. 2.2. Signal-state space diagram of a 16-QAM signal.

Fig. 2.3. 16-QAM demodulator block diagram [1],[2].
A generalized block diagram of a coherent 16-QAM demodulator is shown in Fig. 2.3. The blocks CR and STR stand for "carrier recovery" and "symbol timing recovery" respectively; these blocks are of vital importance for the performance of the demodulator. After lowpass filtering the demodulated signals in the I arm and the Q arm, the resulting signals $x(t)$ and $y(t)$ are fed to the four-to-two-level PAM converters, each consisting of a multiple-threshold comparator (implemented here as three single-threshold binary-output comparators) and a logic circuit that converts the output of the comparator into a $f_b/2$ rate binary signal. Finally, the $x2$ data combiner (a parallel-to-serial converter) provides the desired binary signal output at bit rate $f_b$. 
3. DECISION DIRECTED DEMODULATORS BASED ON A COSTAS LOOP

A number of decision directed carrier recovery loops for 16-QAM signals have been developed [8], [9]; all of these are modifications of a four-phase Costas loop. Fig. 3.1a and b show the block diagrams of two of these circuits. The remodulation is a baseband process: the demodulated (analog) quadrature signals and the detected data (the digital output of the decision blocks) are mixed to obtain a control signal for the carrier regenerating VCO. In section 3.1 this control signal (or error signal) will be derived.

**Fig. 3.1a. Block diagram of a decision-directed 16-QAM demodulator using baseband remodulation.**
Fig. 3.1b. Block diagram of another decision-directed 16-QAM demodulator using baseband remodulation; this circuit actually is a modification of the circuit in Fig. 3.1a.

3.1. The error signal and the phase jitter variance in the absence of noise

The circuits in Fig. 3.1a and b cannot remove the modulation completely from the 16-QAM signal: the error signal fed to the loop filter will still contain some sort of modulation. This will show up as phase jitter in the VCO output signal, thus deteriorating the stability of the recovered carrier signal. It is instructive to calculate the phase jitter variance in the absence of noise at the input of the receiver, and use the result as a figure of merit for these carrier recovery circuits.

The input BPF is assumed to be symmetrical about the carrier frequency; its bandwidth is much greater than $1/T_S$ to avoid distortion of the received signal. The output of the BPF can be represented as

$$s_i(t) = \sqrt{2}\cdot x(t)\cos(\omega_1 t + \theta_1) - \sqrt{2}\cdot y(t)\sin(\omega_1 t + \theta_1),$$

where $x(t)$ and $y(t)$ are the four-level data signals as defined in Chapter 2. We will assume that the pulse shape $p(t)$ is rectangular:

$$p(t) = \begin{cases} 
1 & \text{for } 0 < t < T_S \\
0 & \text{for all other } t 
\end{cases}$$
When the loop is locked, the VCO output signal is $\sqrt{2}V_o \cos(\omega_i t + \theta_o)$ and the +90° shifted signal is $-\sqrt{2}V_o \sin(\omega_i t + \theta_o)$. The I-arm multiplier output is (discarding the double-frequency terms):

$$V_o \left[ x(t) \cos \theta_e - y(t) \sin \theta_e \right],$$

where $\theta_e = \theta_i - \theta_o$ (the phase error).

The Q-arm multiplier output is:

$$V_o \left[ x(t) \sin \theta_e + y(t) \cos \theta_e \right].$$

Assuming that $\theta_e$ is small, the output of the I-arm decision circuit is $x(t)$, and the output of the Q-arm decision circuit is $y(t)$. The output of the summator (a differential amplifier) is the error signal $v_d(t)$:

$$v_d(t) = K V_o \left[ x^2(t) + y^2(t) \right] \sin \theta_e,$$

where $K_s$ is the summator gain factor.

The signal $x^2(t) + y^2(t)$ can take on three different values: $2A^2$, $10A^2$ or $18A^2$. The values $2A^2$ and $18A^2$ have a probability of occurrence of $\frac{1}{4}$ each, and the probability for $10A^2$ is $\frac{1}{2}$. The power density spectrum $S(f)$ of $x^2(t) + y^2(t)$ can be found from Appendix A, Equation (A.4):

$$S(f) = (10A^2)^2 \delta(f) + 32A^4 T_s \sin^2(fT_s) \quad (3.1)$$

$v_d(t)$ can be split in a wanted part (the average $\overline{v_d(t)}$ ) and a modulation ripple $n_m(t)$, also called the modulation noise:

$$v_d(t) = \overline{v_d(t)} + n_m(t).$$

$$\overline{v_d(t)} = E[v_d(t)] = K V_o \sin \theta_e \ E[x^2(t) + y^2(t)] = 10A^2 K_s V_o \sin \theta_e.$$  

We define $K_d = 10A^2 K_s V_o$; then $\overline{v_d(t)} = K_d \sin \theta_e$. $K_d$ is analogous to the phase-detector gain factor of an ordinary PLL.

* If $\theta_e$ is not small enough, the decision circuits might make a wrong decision, depending on the level of their input signals. It can be shown that the characteristic of the error signal is an odd and periodic function of $\theta_e$ with period $\pi/2$. This is not surprising when one realizes that the 16-QAM signal set has quadrant symmetry, i.e. a rotation of $\pi/2$ radians produces no change in the constellation of signal points. (See also Fig. 2.2).
The power density spectrum of $n_m(t)$ can be found from (3.1):

$$S_m(f) = 32A_s^4(K_v V_0 \sin \theta_e)^2 T_s \text{sinc}^2(fT_s) = \frac{8}{25} K_d^2 T_s \sin^2 \theta_e \text{sinc}^2(fT_s)$$

We define $N_m = \frac{16}{25} K_d^2 T_s \sin^2 \theta_e$; then $S_m(f) = (N_m/2) \text{sinc}^2(fT_s)$.

The phase of the VCO output signal is $\theta_o(t) = \theta_o(t) + \theta_{no}(t)$; from Appendix C we can find the variance of the phase jitter $\theta_{no}$:

$$\theta_{no}^2 = \frac{N_m B_L}{K_d^2} = \frac{16}{25} T_s B_L \sin^2 \theta_e$$

where $B_L = \int |H(f)|^2 df$ and $H(f) = \frac{K_d K_v F(f)}{j2\pi f + K_d K_v F(f)}$.

$F(f)$ is the transfer function of the loop filter; $K_v$ is the VCO gain factor. $H(f)$ is called the closed-loop transfer function (by analogy to an ordinary PLL); $B_L$ is the noise bandwidth of the loop.

In the next chapter, several decision-directed configurations using IF remodulation will be proposed; the result obtained here will be compared with similar results derived for these new IF remodulators.
4. MODIFYING THE IF 16-QAM SIGNAL TO REGENERATE A DISCRETE CARRIER COMPONENT

The decision directed circuits of Fig. 3.1a and b use baseband remodulation to obtain a control signal for the carrier-generating VCO. It is also possible to modify the IF 16-QAM signal to regenerate a discrete spectral component at the carrier frequency; this component can be tracked by an ordinary PLL to obtain a "clean" carrier signal. The fundamental block diagram of such a demodulator is shown in Fig. 4.1.

Refering to the signal-state space diagram of Fig. 2.2, we see that the 16-QAM signal (consisting of 16 phasors) contains no discrete carrier component. If the 16 phasors could be reduced to 8 phasors in two adjacent quadrants (by means of "remodulation"), a discrete carrier component would be obtained. This could be done by multiplying the 16-QAM signal by +1 or -1, depending on the quadrant in which the momentary 16-QAM phasor is situated. This results in the signal-state space diagram of Fig. 4.2a.
Fig. 4.2. Signal-state space diagrams for the "remodulated" 16-QAM signal:

a. reduction to 8 phasors in two adjacent quadrants,
b. reduction to 4 phasors in one quadrant,
c. reduction to a 3-ary ASK signal.

If necessary, the resulting 8 phasors can be reduced to 4 phasors in one quadrant, by means of a 90° phase shift if the momentary phasor is situated in the unwanted quadrant. Now the signal-state space diagram of Fig. 4.2b results.

Another way of reducing the 8 phasors in Fig. 4.2a is by shifting the phases of these phasors in such a manner, that all resulting phasors have the same phase. In Fig. 4.2c the resulting signal-state space diagram is shown. The "one-phase signal" can have three different amplitudes: it is a 3-ary ASK signal (or 3-ASK signal for short).

So we have three options to investigate:

option 1: reduction to 8 phasors in two adjacent quadrants, by means of multiplication by +1 or -1. (See Fig. 4.2a).

option 2: reduction to 4 phasors in one quadrant, by means of multiplication by +1 or -1 (yielding the solution of option 1), and a 90° or 0° phase shift. (See Fig. 4.2b).

* ASK = Amplitude Shift Keying.
option 3: reduction to 3 phasors having the same phase, by means of multiplication by +1 or -1 (yielding the solution of option 1), and a phase shift that yields the desired phase. (See Fig. 4.2c).

Whatever solution will be chosen from the options in Fig. 4.2a, b or c, the resulting IF signal will still contain some sort of modulation, just like the "baseband remodulators" in the previous chapter. Again this will show up as phase jitter in the VCO output signal. In Appendix B is derived that in all three cases the output of the phase detector (the error signal) can be written as:

\[ v_d(t) = v'_d(t) + n_m(t) = K_d \sin \theta_e + n_m(t), \]

where \( K_d \) is the phase-detector gain factor, \( \theta_e \) is the phase error and \( n_m(t) \) is the modulation noise. The power density spectrum of \( n_m(t) \) is

\[ S_m(f) = \frac{N_m}{2} \text{sinc}^2(\frac{f_T}{2}). \]

\( N_m \) is a function of \( K_d \), \( T_s \) and \( \theta_e \): \( N_m = K_d^2 T_s f(\theta_e) \), where \( f(\theta_e) \) is a function of \( \theta_e \), dependent on the chosen option. For all three cases, \( f(\theta_e) \) is derived in Appendix B. The results are:

- for option 1: \( f(\theta_e) = \frac{1 + 4 \cos^2 \theta_e}{2} \)
- for option 2: \( f(\theta_e) = \frac{1}{4} \)
- for option 3: \( f(\theta_e) = \frac{22 - 8 \sqrt{5}}{9 + 4 \sqrt{5}} \sin^2 \theta_e \)

The variance of the VCO output phase can be found from Appendix C:

\[ \theta_{no}^2 = \frac{N_B}{L} \]

\( B \) is the noise bandwidth of the PLL as defined in Chapter 3. The ratio \( N_m/K_d^2 \) can be calculated for the three different options:

- option 1: \( \frac{N_m}{K_d^2} = T_s \frac{(1 + 4 \cos^2 \theta_e)}{2} \)
- option 2: \( \frac{N_m}{K_d^2} = T_s / 4 \)
option 3: \[ \frac{N_m}{K_d^2} = T_s \frac{22-8\sqrt{5}}{9+4\sqrt{5}} \sin^2 \theta_e = 0.23T_s \sin^2 \theta_e \]

If \( \theta_e \) is small, then \( \cos^2 \theta_e \approx 1 \) and \( \sin^2 \theta_e \ll 1 \); option 1 then reduces to
\[ \frac{N_m}{K_d^2} \approx 2.5T_s. \]

For the baseband remodulators we found:
\[ \frac{N_m}{K_d^2} = \frac{16}{25} T_s \sin^2 \theta_e = 0.64T_s \sin^2 \theta_e \]

Clearly, option 1 is not interesting anymore, since it will yield a large phase jitter compared to the baseband remodulators. The same is true for option 2. Option 3 however yields a gain of 4.5 dB (a factor 2.8) compared to the baseband remodulators of Chapter 3. This option will be further examined in the next chapters.
5. DATA-AIDED PHASE SHIFTING TO OBTAIN A 3-ARY ASK SIGNAL

In order to convert the 16-QAM signal into a 3-ary ASK signal, a multiplier and several decision-directed phase shifters are needed. This is depicted in Fig. 5.1; the recovered data determine which phase shifter will be chosen. These phase shifters can be implemented as a tapped delay line, a tapped coaxial cable, see Fig. 5.2. Thus the phase shifts are actually realised by making use of the time delay that occurs in a cable.

![Diagram](image)

Fig. 5.1. Converting the 16-QAM signal into a 3-ary ASK signal.

![Diagram](image)

Fig. 5.2. A tapped coaxial cable can be used to implement the phase shifters in Fig. 5.1.
To obtain a phase shift of 180° for an IF signal at 70 MHz, a cable length of about 1.4 m is needed. At radio frequencies (RF) the cable length would have to be several centimeters or even smaller than a centimeter, and then this small cable has to be tapped at several spots. This is quite impractical, and therefore the phase shifting will be implemented in the IF stage of the receiver. One should not choose the IF frequency too low: for a 180° phase shift at 5 MHz, a cable length of 20 m is needed, which again is not very practical. Furthermore a 16-QAM signal at 5 MHz is relatively more broadband than a 16-QAM signal at 70 MHz (with the same symbol rate of the modulating data); the distortion caused by the (non-ideal) cable might be considerably large if the former frequency is used.

Another problem is the timing error that occurs as a result of the fact that the delay time is not equal for all taps. (Of course not, otherwise the different phase shifts would not be implemented!). Suppose that the timing is correct for the signal at the beginning of the cable (the first "tap"), i.e. the (recovered) data-clock is "in phase" with the data (modulated on the IF carrier) at the first tap. At time t=kT switching occurs from one tap to another (possibly the same tap); if the "new" tap is the first one, there is no problem, but if it is not the first one, the "old" (phase-shifted) signal belonging to the time interval <(k-1)T_s,kT_s> will be fed to the following circuits, until the delay time from the beginning of the cable to this tap has passed. The time period during which the "wrong" signal is present at the input of the PLL, is Δt_1 = ℓ_j / ω, where ℓ_j is the length of the cable up to the i-th tap (the tap from which the signal is taken) and ω is the electromagnetical propagation velocity in the cable. The maximum delay time is Δt_max = ℓ / ω, where ℓ is the total length of the cable; this occurs at the last tap (the end of the cable). The longer the cable, the longer the delay time will be. The result is a larger phase jitter in the recovered carrier and a larger timing jitter in the recovered clock. (The clock recovery circuits will be discussed in Chapter 7). So this is another reason why the IF frequency should not be

* During the delay time, the correct IF phase will be present at this tap, but the data signal modulating the IF signal belongs to the previous signaling interval <(k-1)T_s,kT_s>.
too low. The total length of the cable is

\[ \ell = \frac{\phi_{\text{max}}}{360^\circ} \cdot \lambda_c = \frac{\phi_{\text{max}}}{360^\circ} \cdot \frac{45}{f_c}, \]

where \( \phi_{\text{max}} \) is the maximum phase shift (expressed in degrees), and \( f_c \) is the frequency of the carrier signal (\( \lambda_c \) is its wavelength in the cable).

Thus \( \Delta t_{\text{max}} = \frac{\phi_{\text{max}}}{360^\circ} \cdot f_c^{-1} \). Relative to the symbol duration \( T_S \) this becomes

\[ \frac{\Delta t_{\text{max}}}{T_S} = \frac{\phi_{\text{max}}}{360^\circ} \cdot f_S \]

where \( f_S = 1/T_S \).

\[ \phi_{\text{max}} = 180^\circ - 2 \cdot \arctan(1/3) \approx 143^\circ \] (see also Fig. 2.2).

If an IF signal at 5 MHz is being used, modulated by a data signal of 1 Msymbols/sec, the maximum delay time (relative to the symbol duration) will be about 8%, which is considerably large. A signal at 70 MHz however (with the same symbol rate of the modulating data) experiences a maximum (relative) time delay of only 0.6%.

The phase shifts obtained by the tapped cable are frequency dependent; if the carrier frequency of the received 16-QAM signal is not exactly equal to the nominal frequency for which the tap positions are calculated, then the phase shifts will not be correct, resulting in a degradation of the recovered carrier and thus a degradation of the performance of the complete demodulator. Therefore it is better to use a heterodyne PLL configuration, as shown in Fig. 5.3. Now the carrier frequency of the 16-QAM signal will be converted and "locked" to the fixed (and highly stable) frequency of a crystal oscillator. The demodulation and remodulation circuits of the receiver will be calculated for this fixed frequency; no degradation results if the carrier frequency of the received signal departs from its nominal value (within certain limits of course). A frequency of 100 MHz is chosen for the crystal oscillator; then the total length of the cable is about 80 cm, and the timing error will only be 0.4%. A VCO frequency of 30 MHz will be used to convert the input carrier frequency of 70 MHz to 100 MHz (see Fig. 5.3).
Fig. 5.3. Carrier recovery using a heterodyne PLL configuration.
6. DERIVATION OF THE PHASE DETECTOR CHARACTERISTIC IN THE PRESENCE OF NOISE AND THE SYMBOL ERROR PROBABILITY

In this chapter the operation of the loop will be analyzed mathematically, and an expression for the symbol error probability (conditioned on a given phase error in the regenerated coherent carrier signal) will be derived.

6.1. Analysis of the operation of the loop

In this section the stochastic differential equation which describes the operation of the loop in Fig 5.3 will be derived; this derivation will also give us the phase detector characteristic (also known as the loop S-curve). Several assumptions will be made:

- The pulse shape $p(t)$ of the data symbols is rectangular as in Chapter 3.
- The received signal is corrupted by additive white Gaussian noise with a spectral density $N_0/2$.
- The input BPF is symmetrical about the carrier frequency $f_c$; its bandwidth is much greater than the symbol rate $f_s$, but much smaller than $f_c$ (the fixed frequency of the crystal oscillator).
- The loop bandwidth is much smaller than the symbol rate.

The third assumption implies that the input BPF causes no cross talk between the I and Q channels and that the pulse shape of its output signal can be approximated as rectangular. Furthermore it implies that the noise at its output can be characterized as a narrow-band Gaussian process; the noise process correlation time is much less than the length of each signaling interval, which means that the BPF output signal sees the noise as essentially white. The fourth assumption implies that the phase process (the phase of the VCO output signal) varies much more slowly than the signal or noise process.

The output of the BPF can be written as

$$ r_1(t) = s_1(t) + n_1(t) , $$

where

$$ s_1(t) = \sqrt{2} \cdot [ x(t) \cos(\omega_1 t + \theta_1) - y(t) \sin(\omega_1 t + \theta_1) ] $$

as in Chapter 3, and
\( n_1(t) = \sqrt{2} \cdot \left[ n_c(t) \cos(\omega_1 t + \theta_1) - n_s(t) \sin(\omega_1 t + \theta_1) \right], \)

the narrow-band Gaussian noise.

When the loop is locked, the VCO output signal is
\( \sqrt{2} \cdot V_o \cos((\omega_c - \omega_1) t + \theta_0) \),
and the output of the first multiplier is
\( r(t) = s(t) + n(t), \)

where
\[
s(t) = K_m V_o \left[ x(t) \cos(\omega_c t + \theta_c) - y(t) \sin(\omega_c t + \theta_c) \right]
\]
and
\[
n(t) = K_m V_o \left[ n_c(t) \cos(\omega_c t + \theta_c) - n_s(t) \sin(\omega_c t + \theta_c) \right].
\]

\( \theta_c = \theta_1 + \theta_0; K_m \) is the multiplier gain factor.

The signal at the input of the coaxial cable is \( \exp(-pT_d) r(t) \); the Laplace operator \( \exp(-pT_d) \) (where \( p = j\omega \)) is used to denote the delay caused by the fixed-delay circuit prior to the cable.

The remodulation control circuits decide which tap will be chosen for each signaling interval. The chosen phase shift (which might be wrong due to the noise) for the signaling interval \( <kT_s,(k+1)T_s> \) is \( \hat{\phi}_k \); for all signaling intervals this can be written as
\[
\hat{\phi}(t) = \sum_{k=-\infty}^{\infty} \hat{\phi}_k u(t-kT_s),
\]
where
\[
u(t) = \begin{cases} 
1 & \text{for } 0 < t < T_s \\
0 & \text{for all other } t 
\end{cases}
\]

* It takes a time \( T_d \) to detect the data and form the remodulation control signals; typically \( T_d \) is in the order of \( T_s \), the symbol duration.
The correct "phase shift signal" is

$$\phi(t) = \sum_{k=-\infty}^{\infty} \phi_k u(t-kT_s).$$

We can define $$\phi_k = \hat{\phi}_k + \Delta \phi_k,$$ where $$\Delta \phi_k$$ is the phase shift error for the signaling interval $$<kT_s,(k+1)T_s>$$; the "phase shift error signal" can be written as

$$\phi_e(t) = \sum_{k=-\infty}^{\infty} \Delta \phi_k u(t-kT_s) = \phi(t) - \hat{\phi}(t).$$

The next step is multiplying the signal by +1 or -1, depending on the quadrant in which the momentary 16-QAM phasor is situated. Mathematically this operation can be included in the phase shifts $$\phi_k$$ and $$\hat{\phi}_k$$. Thus $$\phi_k$$ and $$\hat{\phi}_k$$ can take on 12 different values, corresponding to the 12 different phase angles in the signal-state space diagram of Fig. 2.2.

The easiest way to find a convenient mathematical expression for the output signal $$z(t)$$ of the second multiplier (the one after the cable) is by using complex notation:

$$r(t) = \text{Re}[R(t)];$$

$$R(t) = \mathcal{R}(t) + \mathcal{N}(t),$$

where

$$\mathcal{R}(t) = K_m V_m [x(t) + jy(t)] \exp(j(\omega_c t + \theta_c))$$

and

$$\mathcal{N}(t) = K_m V_s [n_c(t) + jn_s(t)] \exp(j(\omega_c t + \theta_c)).$$

Now the output signal of the second multiplier can be written in complex notation as

$$Z(t) = K_{m1} K_{m2} V \exp(-\rho T_d) \left[Z_1(t) + Z_2(t)\right],$$

where

$$Z_1(t) = [x(t) + jy(t)] \exp(j(\omega_c t + \theta_c)) \exp(-j\hat{\phi}(t))$$

and

$$Z_2(t) = [n_c(t) + jn_s(t)] \exp(j(\omega_c t + \theta_c)) \exp(-j\hat{\phi}(t)).$$

$$K_{m2}$$ is the gain factor of the second multiplier.

$$Z_1(t)$$ can be written as

$$Z_1(t) = [x(t) + jy(t)] \exp(-j\phi(t)) \exp(j\phi_e(t)) \exp(j(\omega_c t + \theta_c)).$$
The factor \( [x(t) + jy(t)] \exp(-j\phi(t)) \) represents the 3-ASK modulation that ideally should be obtained (no decision errors). This can be expressed as
\[ \sqrt{x^2(t) + y^2(t)} \exp(-j\phi(t)) \exp(j(\omega t + \theta - \theta_w)) \]
where
\[ \theta_w = 90^\circ - \arctan(1/3) = 71.6^\circ \] (see also Fig. 4.2c).

Thus \( Z_1(t) \) can be written as
\[ Z_1(t) = \sqrt{x^2(t) + y^2(t)} \exp(j\phi_e(t)) \exp(j(\omega_c t + \theta - \theta_w)). \]

Now we write \( Z_2(t) \) as
\[ Z_2(t) = [n_c(t) + jn_s(t)] \exp(-j(\hat{\phi}(t) - \theta_w)) \exp(j(\omega_c t + \theta - \theta_w)). \]

\( z(t) = \text{Re}[Z(t)] = K_{m1}K_{m2}V_x \exp(-pT_d) \{\text{Re}[Z_1(t)] + \text{Re}[Z_2(t)]\} \]
\[ = K_{m1}K_{m2}V_x \exp(-pT_d) [z_1(t) + z_2(t)]. \]

\( z_1(t) = \sqrt{x^2(t) + y^2(t)} \left[ \cos\phi_e(t) \cos(\omega_c t + \theta - \theta_w) - \sin\phi_e(t) \sin(\omega_c t + \theta - \theta_w) \right] \)
\( z_2(t) = N_1(t, \hat{\phi}(t)) \cos(\omega_c t + \theta - \theta_w) - N_0(t, \hat{\phi}(t)) \sin(\omega_c t + \theta - \theta_w), \)

where
\[ N_1(t, \hat{\phi}(t)) = n_c(t) \cos(\hat{\phi}(t) - \theta_w) + n_s(t) \sin(\hat{\phi}(t) - \theta_w) \] (6.3)
and
\[ N_0(t, \hat{\phi}(t)) = n_s(t) \cos(\hat{\phi}(t) - \theta_w) - n_c(t) \sin(\hat{\phi}(t) - \theta_w) \] (6.4)

The output of the crystal oscillator is \(-V_x \sin(\omega_x t + \theta_x)\). The output \( e(t) \) of the phase detector (the third multiplier) is the error signal; neglecting double frequency terms we find
\[ e(t) = K_{m1}K_{m2}K_{m3}V_x \exp(-pT_d) [e_1(t) + e_2(t)] , \]

where
\[ e_1(t) = \sqrt{x^2(t) + y^2(t)} \left[ \cos\phi_e(t) \sin\theta_e + \sin\phi_e(t) \cos\theta_e \right] \]
and
\[ e_2(t) = N_1(t, \hat{\phi}(t)) \sin\theta_e + N_0(t, \hat{\phi}(t)) \cos\theta_e. \]

\( K_{m3} \) is the gain factor of the third multiplier, and \( \theta_e = \theta - \theta_w - \theta_x \), the loop phase error.
The instantaneous frequency of the VCO output signal is proportional to the error signal via the relation
\[ \theta_0(t) = K_v F(p)e(t) , \]
where \( K_v \) is the VCO gain factor and \( F(p) \) is the transfer function of the loop filter (using Laplace transform notation); the dot denotes differentiation with respect to time.

\[ \theta_0(t) = \theta_c(t) - \theta_i(t) = \theta_e(t) + \theta_w + \theta_x - \theta_i(t) , \]

and thus
\[ \theta_0(t) = \theta_e(t) - \theta_i(t) ; \]

therefore we can write
\[ \theta_e(t) = \theta_1(t) + K_v F(p)e(t) \]
\[ = \theta_1(t) + K K K K V V/V V^2 F'(p)exp(-pT_d) [e_1(t) + e_2(t)] \]

The phase process \( \theta_e(t) \) varies much more slowly than the signal or noise process, and therefore we can take the statistical average of this stochastic equation over the data and the noise:
\[ \theta_e(t) = \theta_1(t) + K K K K V V/V V^2 F'(p)exp(-pT_d) \times \]
\[ \times \{E[e_1(t)|\theta_e] + E[e_2(t)|\theta_e]\} \]

where
\[ E[e_1(t)|\theta_e] = E[\sqrt{\dot{x}^2(t)+\dot{y}^2(t)} \cos \phi_e(t)|\theta_e] \sin \theta_e + \]
\[ + E[\sqrt{\dot{x}^2(t)+\dot{y}^2(t)} \sin \phi_e(t)|\theta_e] \cos \theta_e \]

and
\[ E[e_2(t)|\theta_e] = E[N_\| \| t, \phi(t) \| \| \sin \theta_e + N_\| \| t, \phi(t) \| \| \cos \theta_e |\theta_e] \]

Using (6.3) and (6.4), \( e_2(t) \) can be written as
\[ e_2(t) = [n_c(t) \sin \theta_e + n_s(t) \cos \theta_e] \cos(\phi(t) - \theta_w) + \]
\[ - [n_c(t) \cos \theta_e - n_s(t) \sin \theta_e] \sin(\phi(t) - \theta_w) \]
\[ = N_\|_1(t, \theta_e) \cos(\phi(t) - \theta_w) - N_\|_0(t, \theta_e) \sin(\phi(t) - \theta_w) \]

\( \theta \) is essentially constant for the significant noise process correlation time; therefore \( N_\|_1(t, \theta_e) \) and \( N_\|_0(t, \theta_e) \) can be characterized as zero mean Gaussian random processes \( N_\|_1(t) \) and \( N_\|_0(t) \) respectively, independent of \( \theta_e \).
with a spectral density \( N_0/2 \) and \( E[N'_1(t_1)N'_1(t_2)] = 0 \) for all values of \( t_1 \) and \( t_2 \). Furthermore the noise process correlation time is much less than the length of the signaling interval, and therefore \( E[e_2(t)\theta_e] \) can be written as

\[
E[e_2(t)\theta_e] = \sqrt{E[\cos^2(\phi(t)-\theta_e) + \sin^2(\phi(t)-\theta_e)]} N_\xi(t) = \sqrt{E[1]} N_\xi(t)
\]

where \( N_\xi(t) \) is white Gaussian noise of spectral density \( N_0/2 \). Thus \( E[e_2(t)\theta_e] \) turns out to be independent of \( \theta_e \).

We can define \( G(\theta_e) = E[e_1(t)\theta_e] \) and normalize this function of \( \theta_e \) to unit slope at \( \theta_e = 0 \):

\[
g(\theta_e) = G(\theta_e)/G(0),
\]

where the prime denotes differentiation with respect to \( \theta_e \). (6.5) can now be written as

\[
\dot{\theta}_e(t) = \theta_i(t) + K_dK_v G(0)/\sqrt{2} \cdot F(p) \exp(-pT_d) \{ g(\theta_e) + N_\xi(t)/G(0) \}.
\]

Defining \( K_d = K_v m_1 m_2 m_3 G(0)/\sqrt{2} \) this becomes

\[
\dot{\theta}_e(t) = \theta_i(t) + K_dK_v \exp(-pT_d) \{ g(\theta_e) + N_\xi(t)/G(0) \}.
\]

\( K_d \) is analogous to the phase-detector gain factor of an ordinary PLL.

The exponential factor \( \exp(-pT_d) \) is approximately unity for all \( \omega \) within the loop bandwidth \( \omega_L \), as \( \omega L T_d \ll 1 \). Therefore it can be neglected with respect to the steady-state performance of the loop. Now we arrive at

\[
\dot{\theta}_e(t) = \theta_i(t) + K_dK_v \{ g(\theta_e) + N_\xi(t)/G(0) \},
\]  

(6.6)

the stochastic differential equation which describes the operation of the loop; \( g(\theta_e) \) is the normalized phase detector characteristic or normalized loop S-curve.
In this section an expression for the average probability of symbol error (conditioned on a given loop phase error) will be established.

The signal to be demodulated is given by (6.1):

\[ r(t) = s(t) + n(t) \]

\[ s(t) = K_{m1} V \{ x(t) \cos(\omega t + \theta_c(t)) - y(t) \sin(\omega t + \theta_c(t)) \} \]

and

\[ n(t) = K_{m1} V \{ n_c(t) \cos(\omega t + \theta_c(t)) - n_s(t) \sin(\omega t + \theta_c(t)) \} \]

\[ \theta_c(t) = \theta_e(t) + \theta_w + \theta_x. \]

The output of the crystal oscillator is \(-\sqrt{2} V \sin(\omega t + \theta_w)\); shifting the phase by \(\theta_e + \theta_{es}\), where \(\theta_{es}\) is the static loop phase error, yields

\[ -\sqrt{2} V \sin(\omega t + \theta_e + \theta_{es}) = -\sqrt{2} V \sin(\omega t + \theta_{es}) \]  \hspace{1cm} (6.7)\]

This reference carrier can be shifted by \(-90^\circ\) to yield the quadrature reference carrier

\[ \sqrt{2} V \cos(\omega t + \theta_{es}) \]  \hspace{1cm} (6.8)\]

Demodulating the signal \(r(t)\) using the reference carriers in (6.7) and (6.8), we find the baseband signals \(r_I(t)\) and \(r_Q(t)\), where

\[ r_I(t) = K_{m1} m_4 V_0 \sqrt{2} \{ x(t) \cos(\theta_c(t) - \theta_e) - y(t) \sin(\theta_c(t) - \theta_e) \} + K_{m1} m_4 V_0 \sqrt{2} \{ n_c(t) \cos(\theta_c(t) - \theta_{es}) - n_s(t) \sin(\theta_c(t) - \theta_{es}) \} \]

\[ \text{and} \]

\[ * \text{ This static loop phase error results from the difference between the} \]
\[ \text{incoming signal frequency and the free-running (zero control voltage)} \]
\[ \text{frequency of the VCO: } \theta_{es} = \omega/K_{d} F(0). \text{ If } \omega \text{ is known, then the} \]
\[ \text{influence of } \theta_{es} \text{ on the carrier signal used for demodulation can be} \]
\[ \text{canceled by an appropriate phase shift.} \]
\[ r_q(t) = K \left( K_{m1} x(t) \sin(\theta_c(t) - \theta_{cs}) + y(t) \cos(\theta_c(t) - \theta_{cs}) \right) + K_{m4} \left( n_c(t) \sin(\theta_c(t) - \theta_{cs}) + n_s(t) \cos(\theta_c(t) - \theta_{cs}) \right) \]

\[ r_I(t) = K \left( x(t) \cos(\theta_e(t)) - y(t) \sin(\theta_e(t)) \right) + K \left( n_c(t) \cos(\theta_e(t)) - n_s(t) \sin(\theta_e(t)) \right) \]

\[ r_0(t) = K \left( x(t) \sin(\theta_e(t)) + y(t) \cos(\theta_e(t)) \right) + K \left( n_c(t) \sin(\theta_e(t)) + n_s(t) \cos(\theta_e(t)) \right) \]

\( K_{m1} \) is the gain factor of the demodulation multiplier.

\( \theta_c(t) - \theta_{cs} = \theta_{e} - \theta_{es} \); in the following will be assumed that \( \theta_{es} = 0 \) (i.e. there is no frequency deviation and/or a high-gain loop filter is being used).

Defining \( K_q = K_{m1} K_{m4} \), we find

\[ r_I(t) = K_q \left( x(t) \cos(\theta_e(t)) - y(t) \sin(\theta_e(t)) \right) + K \left( n_c(t) \cos(\theta_e(t)) - n_s(t) \sin(\theta_e(t)) \right) \]

\[ r_0(t) = K_q \left( x(t) \sin(\theta_e(t)) + y(t) \cos(\theta_e(t)) \right) + K \left( n_c(t) \sin(\theta_e(t)) + n_s(t) \cos(\theta_e(t)) \right) \]

The phase process varies much more slowly than the signal or noise process, so the noise components of \( r_I(t) \) and \( r_0(t) \) can be written as quadrature Gaussian noise processes \( n_I(t) \) and \( n_q(t) \) respectively, independent of \( \theta_e \):

\[ r_I(t) = K_q \left( x(t) \cos(\theta_e) - y(t) \sin(\theta_e) \right) + n_I(t) \]

\[ r_0(t) = K_q \left( x(t) \sin(\theta_e) + y(t) \cos(\theta_e) \right) + n_q(t) \]

Each of these signals passes through a receive filter (e.g. an integrate-and-dump filter or a Nyquist filter), yielding \( r_I'(t) \) and \( r_0'(t) \). The output of each receive filter is sampled every \( T_s \) seconds; this sample is fed to a four-level quantizer for estimation of the received data. The quantizer characteristic is given by

\[ quant(z) = \begin{cases} 
-3A & \text{for } -\infty < z < -2A \\
-A & \text{for } -2A < z < 0 \\
A & \text{for } 0 < z < 2A \\
3A & \text{for } 2A < z < \infty 
\end{cases} \]

where \( z \) is the value of the sample.

In the expression for \( quant(z) \) we have assumed that \( K_q = 1 \); this will not have any influence on the symbol error probability, as \( K_q \) appears in both the signal and the noise.
The data estimates for the k-th signaling interval are \( \hat{a}_k \) and \( \hat{b}_k \); a correct decision is made whenever \( \hat{a}_k = a_k \) and \( \hat{b}_k = b_k \), where \( a_k \) and \( b_k \) are the transmitted data symbols (see also Chapter 2).

Denoting the variance of the Gaussian noise process at the output of each receive filter as \( \sigma^2 \), and defining \( \mu = A/\sigma \), we can find the average probability of symbol error (conditioned on a given loop phase error) as

\[
P_e(\theta_e) = \frac{1}{4} \sum_{l=-2}^{2} \sum_{j=-3}^{3} Q(\mu(1-(l-1)\cos \theta_e + j\sin \theta_e)) + \]
\[
- \frac{1}{4} \sum_{l=-2}^{2} \sum_{m=-2}^{2} Q(\mu(1-(l-1)\cos \theta_e + (m-1)\sin \theta_e)) \times \]
\[
x Q(\mu((m-1)\cos \theta_e - (l-1)\sin \theta_e))
\]

where \( Q(z) = \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} \exp(-y^2/2)dy \).

The summations over \( l \) and \( m \) are for even values only; the summation over \( j \) is for odd values only. The derivation of this expression is given in Appendix D.

For a perfectly synchronized 16-QAM system (\( \theta_e = 0 \)), this reduces to

\[
P_e(0) = \frac{1}{4} \sum_{l=-2}^{2} \sum_{j=-3}^{3} Q(\mu) - \frac{1}{4} \sum_{l=-2}^{2} \sum_{m=-2}^{2} Q(\mu)Q(\mu)
\]
\[
= 3Q(\mu)[1 - \frac{3}{4} Q(\mu)]
\]

which agrees with [10].

In order to find the "overall" average symbol error probability \( P_e \), the probability density function \( p(\theta_e) \) of the (modulo-2\( \pi \) reduced) loop phase error must be known:

\[
P_e = \int_{-\pi}^{\pi} P_e(\theta_e)p(\theta_e)d\theta_e
\]

This involves solving the stochastic differential equation (6.6), an operation which has not been executed yet.
A problem not discussed so far is the symbol synchronization: the receiver clock must be synchronized with the demodulated baseband symbol stream. This is especially important in decision-directed demodulators, because correct symbol timing information must be present in order to use the data for remodulation purposes. If the symbol timing is not correct, the remodulation process might fail to yield a suitable carrier signal, which means that correct demodulation is not possible.

The recovery of the symbol timing information (also called clock recovery) can be done by means of remodulation, just like the carrier recovery in this demodulator. A suitable configuration is the coherent Early-Late tracking loop, which has proven to operate successfully in combination with decision-directed carrier recovery in a BPSK demodulator [11]. Fig. 7.1 shows the block diagram of the Early-Late circuit.

Fig. 7.1. Block diagram of a coherent Early-Late tracking loop.

The input signal for the Early-Late circuit is taken from the carrier recovery circuit: it is the "multiplexed" signal that comes out of the tapped delay line, i.e. the remodulated 16-QAM signal just before it enters the second multiplier in Fig. 5.3. This is a zero-mean 6-ary ASK signal, containing no discrete carrier component. In the second multiplier of Fig. 5.3, this signal is turned into a 3-ary ASK signal (containing a carrier component) using the binary signal
\[ \text{\( d(t) = \sum_{k=-\infty}^{\infty} d_k u(t-kT_s) \),} \]

where \( d_k \) can be +1 or -1 and \( u(t) \) is a rectangular pulse as defined in (6.2). In the Early-Late circuit however, the 6-ary ASK signal \( m(t)\cos(\omega_c t+\theta) \) is being remodulated by two shifted versions of the signal \( d(t) \): the "early" signal \( d(t+\tau+T_s/4) \) and the "late" signal \( d(t+\tau-T_s/4) \). This operation yields two signals containing a discrete carrier component; for \( \tau=0 \) these carrier components are of equal amplitude. After substraction, filtering and coherent detection, the difference in amplitude of the two carrier components is used as an error signal for the tracking loop. The output of the VCXO (which generates the receiver clock) will be adjusted in such a way that the error signal becomes zero (i.e \( \tau=0 \)); when this condition is reached, the loop is locked.

7.1. Derivation of the error signal (or S-curve)

The input 6-ASK signal can be expressed as
\[ \text{\( s(t) = \sqrt{2P} m(t-T)\cos(\omega_c t+\theta) \),} \]
where
\[ \text{\( m(t) = \sum_{k=-\infty}^{\infty} m_k u(t-kT_s) \).} \]

\( u(t) \) is as defined in (6.2). \( m_k \) can take on six different values:
- \( m_k = +1 \) or -1, each having a probability of occurrence of \( \frac{1}{6} \).
- \( m_k = +\sqrt{5} \) or -\( \sqrt{5} \), each having a probability of occurrence of \( \frac{1}{4} \).
- \( m_k = +3 \) or -3, each having a probability of occurrence of \( \frac{1}{6} \).

\( T \) is the unknown time delay that should be estimated by the tracking loop: the estimate of \( T \) determines the clock phase and thus the correct symbol timing.

The early signal is \( d(t+\hat{\tau}+T_s/4) \) and the late signal is \( d(t+\hat{\tau}-T_s/4) \), where \( \hat{\tau} = T - \tau \), the estimate of \( T \); \( \tau \) is the timing error.
Perfect demodulation will be assumed (no phase error); the coherent reference carrier is $\sqrt{2} V_o \cos(\omega_c t + \theta)$. The input signal for the loop filter can easily be verified to be (neglecting double frequency terms):

$$e(t) = K_m V_o \mathbb{V}_p [m(t-T)d(t-T - T_s/4) - m(t-T)d(t-T + T_s/4)],$$

where $K_m$ is the total gain factor of the two successive multiplications.

Assuming that the loop bandwidth is much smaller than the symbol rate, we can take the statistical average over the data:

$$e(t) = K_m V_o \mathbb{V}_p \{E[m(t-T)d(t-T - T_s/4)] - E[m(t-T)d(t-T + T_s/4)]\},$$

where $R_{md}(\alpha)$ is the cross-correlation function of $m(t)$ and $d(t)$:

$$R_{md}(\alpha) = E[m(t)d(t+\alpha)]$$

$$= E[m(t)d(t+\alpha)|A] P(A) + E[m(t)d(t+\alpha)|\bar{A}] P(\bar{A}),$$

where $A$ is the random event that $t$ and $t+\alpha$ occur during the same symbol interval; $\bar{A}$ is the complementary event that $t$ and $t+\alpha$ do not occur during the same interval. $P(A)$ and $P(\bar{A})$ are the respective probabilities of these events; $P(\bar{A}) = 1 - P(A)$. $P(A)$ can be shown to be [3]:

$$P(A) = \begin{cases} 1 - |\alpha|/T_s & \text{for } |\alpha| \leq T_s \\ 0 & \text{for } |\alpha| > T_s \end{cases}$$

$$E[m(t)d(t+\alpha)|\bar{A}] = E[m(t)]E[d(t+\alpha)] = 0$$

$d_k = +1$ if $m_k > 0$ and $d_k = -1$ if $m_k < 0$; therefore we find

$$E[m(t)d(t+\alpha)|A] = \frac{1}{8}(1)(1) + \frac{1}{8}(-1)(-1) + \frac{1}{4}(1)(\sqrt{5}) + \frac{1}{4}(-1)(-\sqrt{5}) +$$

$$+ \frac{1}{8}(1)(3) + \frac{1}{8}(-1)(-3)$$

$$= 1 + \frac{1}{\sqrt{5}}$$

$$R_{md}(\alpha) = \begin{cases} (1 + \frac{1}{\sqrt{5}})(1 - |\alpha|/T) & \text{for } |\alpha| \leq T_s \\ 0 & \text{for } |\alpha| > T_s \end{cases},$$

where

$$A(x) = \begin{cases} 1 - |x| & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$
Thus the error signal is
\[ e(t) = K V_m V P \left( 1 + \frac{1}{2} \sqrt{5} \right) \left[ A \left( \frac{\tau}{T_s} - \frac{1}{4} \right) - A \left( \frac{\tau}{T_s} + \frac{1}{4} \right) \right]. \]

This is depicted in Fig. 7.2.

![Fig. 7.2. S-curve (error signal) of the Early-Late tracking loop; \( a = K V_m V P (1 + \frac{1}{2} \sqrt{5})/2 \).]

The dashed part of the curve in Fig. 7.2 is not really important, as the timing error never exceeds \( \pm T_s/2 \): an error \( \tau = T_s/2 + x \) (where \( 0 \leq x \leq T_s/2 \)) is the same as a negative error \( \tau = -(T_s/2 - x) \). Thus the tracking loop will always be in the full-drawn part of the curve in Fig. 7.2; this means that the clock phase will always be regulated to the stable null at \( \tau = 0 \) and then the loop will be locked.

### 7.2. Symbol timing for the various data and remodulation-control signals

In Fig. 7.3 the block diagram of the whole demodulator is shown, including the clock recovery circuits. Several clock signals are needed:

- a clock signal for sampling the demodulated signal
- a clock signal for the remodulation-control data which regulates the IF remodulation (i.e. the decision which tap of the delay line will be chosen and the succeeding multiplication by \( d(t) \))
- a clock signal for the "early" data signal
- a clock signal for the "late" data signal
If we denote the clock signal for the data signal prior to the fixed-delay circuit "Delay $T_d$" in Fig. 7.3 by $c(t)$, then the clock signals mentioned above can be expressed as

- $c_o(t) = c(t-T_o)$, the "sampling" clock signal
- $c_R(t) = c(t-T_d)$, the remodulation-control clock signal
- $c_E(t) = c(t-T_d+T_s/4)$, the early clock signal
- $c_L(t) = c(t-T_d-T_s/4)$, the late clock signal

$T_o$ denotes the delay time caused by the receive filter, typically in the order of $T_s$. (For an integrate-and-dump filter, $T_o$ will be equal to $T_s$). It will be clear that $T_d$ should be at least $T_o+T_s/4$, in order to be able to form the early data signal!
In Fig. 7.4 the four clock signals are shown; $T_v = T_d - (T_o + T_s/4)$, the "surplus" of delay time caused by the fixed-delay circuit "Delay $T_d$". In Fig. 7.5 is shown how the correct timing for the various signals can be realized. When the Early-Late loop is locked, the timing for the early and late data signals is correct; now "Delay $T_v$" can be adjusted to equal the surplus of delay time mentioned above.

Fig. 7.4. Clock signals for the various data signals.

Fig. 7.5. Realization of the symbol timing for the various data signals. The VCXO is controlled by the Early-Late tracking loop, as shown in Fig. 7.1 and Fig. 7.3.
In Fig. 7.3 is shown how the demodulated signals (which will be called the I signal and the Q signal) pass the receive filter and enter the remodulation control circuit. The block "receive filter" actually consists of two separate, identical circuits: one for the I signal and one for the Q signal. Ideally, when there is no noise and no phase error in the regenerated coherent carrier signal, the demodulated signals are $x(t)$ and $y(t)$, the quadrature data signals as defined in Chapter 2. After lowpass filtering (e.g. by an integrate-and-dump filter or a Nyquist filter) the signals are fed to threshold comparators. This could be implemented using a configuration as shown in Fig. 2.3 (three single-threshold binary-output comparators for each quadrature signal), but this yields redundant information: the four possible states (levels) of each quadrature signal are represented by a three-bits code, while two bits would be sufficient. When the output of the "threshold 0" comparator in Fig. 2.3 is high, the output of the "threshold -2A" comparator is also high; when the output of the "threshold 0" comparator is low, the output of the "threshold +2A" comparator is also low.

![Diagram](image)

*Fig. 8.1. Threshold comparators yielding a two-bits code which represents the four-level PAM signal.*
Fig. 8.1 shows a configuration which yields a two-bits code: the output of the "threshold 0" comparator determines whether the threshold level of the other comparator is +2A or -2A. The two-bit code is then read into two D-flipflops; the timing for this sampling operation is controlled by the clock recovery circuits (see also Fig. 7.5). Two identical circuits are used for the demodulated signals; together they yield a four-bit code representing the 16 different states of the 16-QAM signal. Calling the output bits \( x_1 \) and \( x_2 \) for the signal \( x(t) \), and \( y_1 \) and \( y_2 \) for the signal \( y(t) \), we find the "state-to-code" conversion table in Table 8.1. The left part of the table gives the output code for \( a_k < 0 \), and the right part for \( a_k > 0 \). In each row the codes for two "opposite" states are given, i.e. the codes for the states \( (a_k, b_k) \) and \( (-a_k, -b_k) \).

Table 8.1. state-to-code conversion accomplished by the threshold detectors. The 16 different states are written as \( a_k, b_k \).

<table>
<thead>
<tr>
<th>state</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>state</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3A, -3A</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3A, 3A</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-A, -A</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>A, A</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>-3A, -A</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3A, A</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>-A, -3A</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>A, 3A</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-3A, 3A</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3A, -3A</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-A, A</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>A, -A</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>-3A, A</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3A, -A</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>-A, 3A</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>A, -3A</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Refering to Fig. 7.5, Table 8.1 shows that \( x_1 \) can be used as the remodulation signal \( d(t) \). Furthermore \( x_2, y_1 \) and \( y_2 \) can be used as a three-bits code for controlling the tap-selection switches of the delay line in the carrier recovery circuit. In Table 8.1 we see that this three-bits code is not the same for the states \( (a_k, b_k) \) and \( (-a_k, -b_k) \) (what it actually should be), but they are each others inverse. This problem can be solved by using \( x_1 \) as an "inverting signal": if \( x_1 = 1 \) then the three-bits code must be inverted, and if \( x_1 = 0 \) this code must remain unchanged. This can be done using exclusive-or gates. Fig. 8.2 shows this solution, together with the circuits of Fig. 8.1.

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Fig. 8.2. Remodulation control circuit, yielding the remodulation signal $d(t)$ and a three-bits code for controlling the tap-selection switches of the delay line.

The delay line has only six taps, while the three-bits code yields eight different control codes. For the states in the first and the second row of Table 8.1, the same tap must be chosen; the same is true for the states in the fifth and the sixth row. If there is a switch for each of the eight possible codes, then this simply means that the two switches which are selected by the codes in row one and two must be connected to the same tap; the same must be done for the two switches which are selected by the codes in row five and six. A suitable switching device is the integrated CMOS circuit 4051, which contains eight switches, capable of switching analog signals; selection of a switch is accomplished by a three-bits code as discussed above.
9. IMPLEMENTATION OF THE LOOP FILTER IN THE CARRIER RECOVERY LOOP

In Appendix E the hardware realization of the demodulator as shown in Fig. 7.3 will be discussed. It is good however, to pay some extra attention to the loop filter in the carrier recovery loop, as it is mainly this circuit that determines parameters like the loop (noise) bandwidth, the lock-in time, the hold-in range, the static loop phase error, et cetera [4].

Usually, two conflicting demands are made on the PLL (and thus on the loop filter):
- a fast lock-in time, implying a large loop (noise) bandwidth
- insensitivity to disturbances, implying a small loop (noise) bandwidth

In order to meet both demands, the bandwidth of the PLL will be "switched": when the system has to acquire lock the bandwidth will be large (20 kHz), and when lock is acquired the bandwidth will be changed to 1 kHz. This is done by switching resistors "in and out" of the loop filter [12], as shown in Fig. 9.1.

The first operational amplifier (OA1) forms together with the resistors R₁ to R₅ and the capacitors C₁ and C₂ the actual high-gain loop filter. The state of the switches S₁ and S₂ determines whether the loop bandwidth is large or small: when the switches are open, the loop bandwidth is 1 kHz, and when they are closed, the bandwidth is 20 kHz. The circuitry around comparators C₀₁ and C₀₂ forms a "window detector" to check whether the output voltage \( v_f \) of the loop filter lies within a certain "voltage window". If it does, the loop is considered to be locked; the detector output is high and all switches are open. If \( v_f \) exceeds the window limits, the loop has lost lock; the detector output becomes low, causing all switches to close immediately. Now \( C₂ \) is (dis)charged by OA2 until \( v_f \) lies within the window again; the detector output becomes high again, causing S₃ to open immediately. After a certain delay time (to let the loop acquire lock again), S₁ and S₂ are opened as well, thereby reducing the loop bandwidth to 1 kHz.
The advantage of using OA2 to bring $v_f$ rapidly within the window limits, is that thus the frequency error will always be within the lock-in range (which is about equal to the larger bandwidth, i.e. 20 kHz), so fast lock will be acquired. If the frequency error would not be kept within the lock-in range, considerable time might elapse before lock is acquired [4].

9.1. Dimensioning the loop filter

The switches are implemented by the integrated CMOS circuit 4053, which contains three separate switches, each controlled by a binary "enable" signal. The on-resistance is about 120 $\Omega$, and the off-resistance can be assumed infinitely high for our purposes.

When the PLL is locked, switches S1 and S2 are open. OA1 is assumed to be a broad-band amplifier with a very large gain. The transfer function of the loop filter can easily be found to be:
\[
F(p) = \frac{1 + pR_1C_2}{p(R_1 + R_2 + R_3)C_2 \left[ 1 + \frac{pR_1(R_1 + R_2)C_1}{R_1 + R_2 + R_3} \right]}
\]

\[
= \frac{1 + p\tau_1}{p\tau_2(1 + p\tau_3)}
\]

\[
\tau_1 = R_1 C_2
\]

\[
\tau_2 = (R_1 + R_2 + R_3)C_2
\]

\[
\tau_3 = \frac{R_1(R_1 + R_2)C_1}{R_1 + R_2 + R_3}
\]

For \( \tau_3 \ll \tau_2 \) this can be approximated by

\[
F(p) = \frac{1 + p\tau_1}{p\tau_2}
\]

The closed-loop transfer function of the PLL-system is

\[
H(p) = \frac{K_d K_v F(p)}{p + K_d K_v F(p)}
\]

where \( K_d \) is the phase-detector gain factor and \( K_v \) is the VCO gain factor.

Substituting (9.1) into (9.2), we find

\[
H(p) = \frac{K_d K_v (p\tau_1 + 1)/\tau_2}{p^2 + p(K_d K_v \tau_1 / \tau_2) + K_d K_v / \tau_2}
\]

This can be written as

\[
H(p) = \frac{2\zeta \omega_n p + \omega_n^2}{p^2 + 2\zeta \omega_n p + \omega_n^2}
\]

\( \zeta = \omega_n \tau_1 / 2 \) and \( \omega_n^2 = K_d K_v / \tau_2 \).

The 3 dB loop bandwidth is given by [5]

\[
B_{3dB} = \frac{\omega_n}{2\pi} \left[ 2\zeta^2 + 1 + \sqrt{(2\zeta^2 + 1)^2 + 1} \right]^{1/2}
\]

\[
= 0.328 \cdot \omega_n \text{ for } \zeta = \frac{1}{2}
\]

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N.b.: The loop noise bandwidth, defined by \( B_L = \int_{0}^{\infty} |H(j2\pi f)|^2 df \),
can be found to be [5]
\[
B_L = \frac{\omega_n}{4} \left( \frac{1}{2\zeta} + 2\zeta \right) = 0.530 \cdot \omega_n = 1.62 \cdot B_{3dB} \quad \text{for} \ \zeta = \frac{1}{\sqrt{2}}.
\]
In the in-lock situation, \( B_{3dB} = 1 \text{ kHz} \), and thus \( \omega_n = 3.05 \cdot 10^3 \text{ rad/s} \).

\( K_d \) is measured to be 0.244 V/rad, and \( K_v \) is 2.41 \cdot 10^5 rad/V·s. For \( \tau_1 \) and \( \tau_2 \) we find
\[
\tau_1 = 4.64 \cdot 10^{-4} \text{ s}
\]
\[
\tau_2 = 6.33 \cdot 10^{-3} \text{ s}
\]

\( R_1 = 47 \Omega \) and \( C_1 = 33 \text{ nF} \); together they form a lowpass filter to reject the double-frequency components in the output signal of the phase detector. The cut-off frequency of this filter is \( f_1 = 103 \text{ kHz} \). For \( f \gg f_1 \) the input impedance as seen by the phase detector is about 50 \( \Omega \).

Choosing \( C_2 = 100 \text{ nF} \), we find
\[
R_4 = 4.64 \text{ k\Omega}
\]
\[
R_2 + R_3 = 63.3 \text{ k\Omega}
\]

Now \( \tau_3 \) is found to be \( 1.55 \cdot 10^{-6} \); this indeed is much smaller than \( \tau_2 \).

When the PLL is not locked, switches S1 and S2 are closed. Assuming that \( R_2 \gg R_{\text{on}} \) and \( R_4 \gg R_{\text{on}}+R_{\text{on}} \), where \( R_{\text{on}} \) is the on-resistance of the switches, \( F(p) \) becomes
\[
F(p) = \frac{1 + p(R_{\text{on}}+R_5)C_2}{p(R_1+R_2)C_2 + \frac{pR(R_{\text{on}}+R_5)C_1}{R_1+R_2} \left[ 1 + \frac{pR(R_{\text{on}}+R_5)C_1}{R_1+R_2} \right]}
\]
\[
= \frac{1+p\tau_4}{p\tau_5(1+p\tau_6)}
\]
\[ \tau_4 = (R_1 + R_2)C \]
\[ \tau_5 = (R_2 + R_3 + R_4)C \]
\[ \tau_6 = \frac{R_1 (R_1 + R_3)C}{R_1 + R_3 + R_2} \]

For \( \tau_5 \ll \tau_6 \), \( F(p) \) can be approximated by

\[ F(p) = \frac{1+pt_4}{pt_5} \]

(9.3) and (9.4) are still valid; now \( \tau_4 = 2\xi/\omega_n \) and \( \tau_5 = K_d K_v/\omega_n^2 \) (again \( \xi = 1/\sqrt{2} \)).

\( B_{3dB} = 20 \, \text{kHz} \), and thus \( \omega_n = 6.10 \times 10^4 \, \text{rad/s} \). Now we find

\[ \tau_4 = 2.32 \times 10^{-5} \, \text{s} \]
\[ \tau_5 = 1.58 \times 10^{-5} \, \text{s} \]

This results in

\[ R_1 + R_2 = 232 \, \Omega \]

and

\[ R_2 + R_3 = 111 \, \Omega \]

This would mean that \( R_1 = 112 \, \Omega \) and \( R_3 = -9 \, \Omega \). \( R_1 \) is chosen to be 0 \( \Omega \) (i.e., a short-circuit); this hardly changes anything to the other parameters. \( R_2 \) is now found to be 63.3 \( \text{k}\Omega \).

\[ \tau_6 = 1.11 \times 10^{-6} \]; this indeed is much smaller than \( \tau_5 \).

9.2. Dimensioning the window detector

The voltage window is chosen from 6.20 V to 6.60 V; this is a voltage swing of ±0.20 V around 6.40 V, which is chosen to be the nominal VCO control voltage for an input carrier frequency of 70 MHz. A voltage swing of ±0.20 V implies a frequency swing of \( t(K_v/2\pi) \cdot 0.20 \, V = ±7.7 \, \text{kHz} \). This is well within the loop bandwidth of 20 kHz, while it gives enough "headroom" for a possible frequency deviation in the received 16-QAM signal.
The IC LM319 is used for implementing the comparators. It contains two integrated comparators with open-collector output circuits; therefore the joined output in Fig. 9.1 will be low whenever any of the two comparator outputs is low. The threshold levels are set by $R_6$, $R_7$ and $R_8$; assuming that the input impedance of each comparator is much larger than these resistances, we find

$$\frac{R_8}{R_6 + R_7 + R_8} \cdot 12V = 6.20 \, V$$

and

$$\frac{R_7 + R_8}{R_6 + R_7 + R_8} \cdot 12V = 6.60 \, V$$

Expressing $R_6$ and $R_8$ in terms of $R_7$ yields

$$R_6 = 13.5 \cdot R_7 \quad \text{and} \quad R_8 = 15.5 \cdot R_7.$$ 

Choosing $R_7 = 940 \, \Omega$, we find $R_6 = 12.7 \, k\Omega$ and $R_8 = 14.6 \, k\Omega$. 

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10. CONCLUSIONS AND RECOMMENDATIONS

From the comparisons made in Chapter 4 it is clear that decision-directed IF remodulation techniques for 16-QAM carrier recovery can yield a better performance than the existing baseband remodulation circuits. The proposed IF remodulation circuit yields a better reduction of the modulation noise, thereby reducing the phase jitter in the regenerated reference carrier signal. The problem that the proposed circuit uses frequency-dependent phase shifts is solved by using a heterodyne PLL configuration; thus no degradation results if the carrier frequency of the received signal departs from its nominal value of 70 MHz. The implementation of the actual remodulation control circuit does not require excessive electronic circuitry, but it can be realized with relatively few components.

Several parts of the demodulator have been realized in hardware (see Chapter 9 and Appendix E), but the project turned out to be too extensive to finish both the theoretical and the practical aspects in a graduation period of 9 months. However, most subcircuits are already designed in detail for hardware realization.

Although it is not possible (yet) to test the proposed demodulator, the theoretical analysis presented in this report shows that the concept of IF remodulation is a promising one. Therefore it is worthwhile to continue the practical realization of the demodulator, and investigate further possibilities of improving the stability of the regenerated carrier signal. There are several aspects one can think of:

- Reducing the amplitude modulation of the 3-ASK signal to obtain a "cleaner" carrier signal at the PLL input; perhaps this can be done by remodulation as well.

- Optimizing the phasor distance in the signal-state space diagram of the 16-QAM signal, to obtain a "cleaner" carrier signal at the PLL input: choosing other PAM-levels than ±A and ±3A might yield a better regenerated carrier signal. However, the probability of symbol error will not be distributed evenly among the 16 different states anymore, so this requires a thorough analysis to predict the "overall" effect.
- Design of an adaptive equalizer to reduce the degrading effects of fading and multipath distortion that will inevitably occur in "real-world" applications.
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APPENDIX A: DERIVATION OF THE POWER DENSITY SPECTRA
OF SEVERAL RELEVANT DATA SIGNALS

In the calculations in Chapters 3 and 4 and Appendix B, the power density spectra of some specific data signals have to be known; these spectra will be derived in this appendix. Four different data signals are considered; they can all be expressed as

\[ z(t) = \sum_{k=-\infty}^{\infty} z_k p(t-kT_s) \]

where \( p(t) \) is a rectangular pulse:

\[ p(t) = \begin{cases} 1 & \text{for } 0 < t < T_s \\ 0 & \text{for all other } t \end{cases} \]

The first signal is a 2-ary ASK signal, where \( z_k \in \{A, 3A\} \); both values are equally likely.

The second signal is a 4-ary ASK signal, where \( z_k \in \{-3A, -A, A, 3A\} \); all four values are equally likely.

The third signal is a 3-ary ASK signal, where \( z_k \in \{A, \sqrt{3}A, 3A\} \); the values \( A \) and \( 3A \) have a probability of occurrence of \( \frac{1}{4} \) each, and the probability for \( \sqrt{3}A \) is \( \frac{1}{2} \).

The fourth signal is a 3-ary ASK signal, where \( z_k \in \{2A^2, 10A^2, 18A^2\} \); the values \( 2A^2 \) and \( 18A^2 \) have a probability of occurrence of \( \frac{1}{4} \) each, and the probability for \( 10A^2 \) is \( \frac{1}{2} \).

The autocorrelation function for all four signals can be written as

\[ R_{zz}(\tau) = E[z(t)z(t+\tau)] = E[z(t)z(t+\tau)|B] P(B) + E[z(t)z(t+\tau)|\bar{B}] P(\bar{B}) \]

where \( B \) is the random event that \( t \) and \( t+\tau \) occur during the same symbol interval; \( \bar{B} \) is the complementary event that \( t \) and \( t+\tau \) do not occur during the same interval. \( P(B) \) and \( P(\bar{B}) \) are the respective probabilities of these events; \( P(\bar{B}) = 1 - P(B) \). \( P(B) \) can be shown to be [3]:

\[ P(B) = \begin{cases} 1 - |\tau|/T_s & \text{for } |\tau| \leq T_s \\ 0 & \text{for } |\tau| > T_s \end{cases} \]
This can be written as
\[ P(B) = A(\tau/T_s) \]

where
\[
A(x) = \begin{cases} 
1 - |x| & \text{for } |x| \leq 1 \\
0 & \text{for } |x| > 1
\end{cases}
\]

The power density spectrum \( S_z(f) \) of \( z(t) \) is the Fourier transform of \( R_{zz}(\tau) \).

Now \( R_{zz}(\tau) \) will be calculated for the four different ASK signals.

### 2-ary ASK signal

\[
E[z(t)z(t+\tau)|B] = \frac{1}{2}(A)(A) + \frac{1}{2}(3A)(3A) = 5A^2 \\
E[z(t)z(t+\tau)|\bar{B}] = E^2[z(t)] = (2A)^2 = 4A^2
\]

\[
R_{zz}(\tau) = 5A^2A(\tau/T_s) + 4A^2[1-A(\tau/T_s)] \\
= A^2[4 + A(\tau/T_s)]
\]

\[
S_z(f) = A^2[4\delta(f) + T_s\text{sinc}^2(fT_s)] \quad (A.1)
\]

### 4-ary ASK signal

\[
E[z(t)z(t+\tau)|B] = \frac{1}{4}(-3A)(-3A) + \frac{1}{4}(-A)(-A) + \frac{1}{4}(A)(A) + \frac{1}{4}(3A)(3A) = 5A^2 \\
E[z(t)z(t+\tau)|\bar{B}] = E^2[z(t)] = 0
\]

\[
R_{zz}(\tau) = 5A^2A(\tau/T_s) \\
S_z(f) = 5A^2T_s\text{sinc}^2(fT_s) \quad (A.2)
\]

### First 3-ary ASK signal

\[
E[z(t)z(t+\tau)|B] = \frac{1}{4}(A)(A) + \frac{1}{2}(AV5)(AV5) + \frac{1}{4}(3A)(3A) = 5A^2 \\
E[z(t)z(t+\tau)|\bar{B}] = E^2[z(t)] = [(1+\frac{1}{2}V5)A]^2 \\
R_{zz}(\tau) = 5A^2A(\tau/T_s) + [(1+\frac{1}{2}V5)A]^2[1-A(\tau/T_s)] \\
= A^2[(1+\frac{1}{2}V5)^2 + \frac{11-4V5}{4}A(\tau/T_s)]
\]
Second 3-ary ASK signal

\[ E[z(t)z(t+\tau)|B] = \frac{1}{4}(2A^2)^2 + \frac{1}{2}(10A^2)(10A^2) + \frac{1}{4}(18A^2)(18A^2) = 132A^4 \]

\[ E[z(t)z(t+\tau)|B] = E^2[z(t)] = (10A^2)^2 \]

\[ R_{zz}(\tau) = 132A^4 A(\tau/T_s) + (10A^2)^2[1-A(\tau/T_s)] \]

\[ = (10A^2)^2 + 32A^4 A(\tau/T_s) \]

\[ S_z(f) = (10A^2)^2 \delta(f) + 32A^4 T_s \text{sinc}^2(fT_s) \] (A.4)
APPENDIX B: DERIVATION OF THE VARIOUS ERROR SIGNALS
INCLUDING MODULATION NOISE

In Chapter 4, three different options to obtain an IF signal containing a discrete carrier component are described. This signal enters a PLL, which has to track the carrier component, see Fig. B.1. All three options yield an error signal (the phase detector output) which can be expressed as

\[ v_d(t) = v_d(t) + n_m(t) = K_d \sin \theta + n_m(t), \]

where \( n_m(t) \) is the modulation noise with a power density spectrum

\[ S_m(f) = \left(\frac{N_m}{2}\right) \text{sinc}^2(fT_s). \]

\[ N_m = K_d^2 T_s f(\theta_e), \]

where \( f(\theta_e) \) is a function of \( \theta_e \), dependent on the chosen option.

In this appendix, the above will be derived for the three different options.

![PLL block diagram](image)

**Fig. B.1.** PLL tracking the carrier component of the modulated IF signal.

**Option 1: reducing the 16 phasors to 8 phasors**

The input signal for the PLL can be expressed as

\[ \sqrt{2} z_1(t) \cos(\omega_1 t + \theta_1) - \sqrt{2} z_2(t) \sin(\omega_1 t + \theta_1) \]

where \( z_1(t) = \sum_{k=-\infty}^{\infty} z_{1k} p(t-kT_s) \) and \( z_2(t) = \sum_{k=-\infty}^{\infty} z_{2k} p(t-kT_s) \).
The random variables $z_{1k}$ and $z_{2k}$ are independent. $z_{1k}$ can take on equally likely values from the set \{A, 3A\}; $z_{2k}$ can take on equally likely values from the set \{-3A, -A, A, 3A\} (see also the signal-state space diagram in Fig. 4.2a). $p(t)$ is a rectangular pulse:

$$p(t) = \begin{cases} 1 & \text{for } 0 < t < T_s \\ 0 & \text{for all other } t \end{cases}$$

The VCO output signal is $-\sqrt{2} V_o \sin(\omega_1 t + \theta_o)$; the phase detector output is (discarding double frequency terms):

$$v_d(t) = K V_m \left[ z_1(t) \sin \theta_e + z_2(t) \cos \theta_e \right]$$

where $\theta_e = \theta_1 - \theta_o$, the phase error, and $K_m$ is the multiplier gain factor. Averaging over the data we find:

$$\overline{v_d(t)} = K V_m \left[ \overline{z_1(t)} \sin \theta_e + \overline{z_2(t)} \cos \theta_e \right]$$

$$= 2AK V_m \sin \theta_e$$

$$= K_d \sin \theta_e$$

$K_d$ is called the phase-detector gain factor.

Now we can write

$$v_d(t) = \overline{v_d(t)} + n_m(t) \quad (\text{with } \overline{n_m(t)} = 0)$$

where

$$n_m(t) = K V_m [z_1(t) - z_{2A}] \sin \theta_e + K V_m z_2(t) \cos \theta_e$$

$$= n_s(t) \sin \theta_e + n_c(t) \cos \theta_e$$

The autocorrelation function of $n_m(t)$ is

$$R_{n_m n_m}(\tau) = R_{n_m n_m}(\tau) \sin^2 \theta_e + R_{n_c n_c}(\tau) \cos^2 \theta_e + \frac{1}{2} [R_{n_m n_c}(\tau) + R_{n_c n_m}(\tau)] \sin \theta_e \cos \theta_e$$

$$= K^2 V^2_m [R_{z_1 z_1}(\tau) - 4A^2]$$

$$= K^2 V^2_m [R_{z_2 z_2}(\tau) - 4A^2]$$

$$= K^2 V^2_m R_{z_2 z_2}(\tau)$$

$$= 0$$

because $R_{z_1 z_2}(\tau) = R_{z_2 z_1}(\tau) = \overline{z_1(t) \cdot z_2(t)}$ and $\overline{z_2(t)} = 0$.  

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\[ R_{mn}(\tau) = K_m^2 v^2 \left[ R_{z_1 z_1}(\tau) - 2A z_2(t) \right] \]

\[ = 0 \]

Thus we find

\[ R_{mn}(\tau) = K_m^2 v^2 \left[ 4A z_2(t) - 4A^2 \sin^2 \theta_e + R_{z_2 z_2}(\tau) \cos \theta_e \right] \]

The power density spectrum of \( n(t) \) is the Fourier transform of \( R_{mn}(\tau) \):

\[ S_m(f) = K_m^2 v^2 \left[ 4A^2 \delta(f) \sin^2 \theta_e + S_{z_2}(f) \cos \theta_e \right] \]

where \( S_{z_1}(f) \) and \( S_{z_2}(f) \) are the power density spectra of \( z_1(t) \) and \( z_2(t) \) respectively.

From Appendix A, Equations (A.1) and (A.2), we find

\[ S_{z_1}(f) = 4A^2 \delta(f) + A^2 T_s \sin^2(\frac{fT_s}{2}) \]

and

\[ S_{z_2}(f) = 5A^2 T_s \sin^2(\frac{fT_s}{2}) \]

This yields

\[ S_m(f) = K_m^2 v^2 A^2 T_s \left( \sin^2 \theta_e + 5 \cos^2 \theta_e \right) \sin^2(\frac{fT_s}{2}) \]

\[ = K_m^2 v^2 A^2 T_s \left( 1 + 4 \cos^2 \theta_e \right) \sin^2(\frac{fT_s}{2}) \]

\[ = K_d^2 T_s / \left( N_m / 2 \right) \sin^2(\frac{fT_s}{2}) \]

where \( N_m = K_d^2 T_s / \left( 1 + 4 \cos^2 \theta_e \right) \)

Option 2: reducing the 16 phasors to 4 phasors

In this case, the input signal for the PLL can be written as

\[ v_2 \cdot z_1(t) \cos(\omega_1 t + \theta_1) - v_2 \cdot z_2(t) \sin(\omega_1 t + \theta_1) \]

where \( z_1(t) = \sum_{k=-\infty}^{\infty} z_{1k} p(t-kT_s) \) and \( z_2(t) = \sum_{k=-\infty}^{\infty} z_{2k} p(t-kT_s) \).
The independent random variables $z_{1k}$ and $z_{2k}$ can take on equally likely values from the set \{A,3A\} (see also Fig. 4.2b); $p(t)$ is assumed to be rectangular again.

The VCO output signal is $\sqrt{2}V_o \cos(\omega_o t + \frac{3}{4} \pi + \theta_o)$; the phase detector output is (discarding double frequency terms):

$$v_d(t) = K_m V_m [z_1(t) \cos(\theta - \frac{3}{4} \pi) - z_2(t) \sin(\theta - \frac{3}{4} \pi)]$$

where $\theta = \theta_e - \theta_o$, and $K_m$ is the multiplier gain factor as before.

Averaging over the data we find:

$$\overline{v_d(t)} = K_m V_m [z_1(t) \cos(\theta - \frac{3}{4} \pi) - z_2(t) \sin(\theta - \frac{3}{4} \pi)]$$

$$= 2AK_m V_m \cos(\theta - \frac{3}{4} \pi) - \sin(\theta - \frac{3}{4} \pi)$$

$$= 2\sqrt{2}AK_m V_m \sin(\theta_e)$$

$K_d$ is called the phase-detector gain factor again.

Now we can write

$$v_d(t) = \overline{v_d(t)} + n_m(t) \quad (\text{with} \quad n_m(t) = 0)$$

where

$$n_m(t) = K_m V_m [z_1(t) \cos(\theta - \frac{3}{4} \pi) - z_2(t) \sin(\theta - \frac{3}{4} \pi) - 2\sqrt{2}A \sin(\theta_e)]$$

$$= K_v \left[\frac{\sqrt{2}}{2}z_1(t) \{\sin(\theta - \cos(\theta_e)\right) + \frac{\sqrt{2}}{2}z_2(t) \{\sin(\theta + \cos(\theta_e)\right) - 2\sqrt{2}A \sin(\theta_e)\right]$$

$$= \frac{\sqrt{2}}{2}K_m V_m [z_1(t) + z_1(t) - 4A] \sin(\theta_e) + \frac{\sqrt{2}}{2}K_m V_m [z_2(t) - z_1(t)] \cos(\theta_e)$$

$$= n_s(t) \sin(\theta_e) + n_c(t) \cos(\theta_e)$$

Again the autocorrelation function of $n_m(t)$ is

$$R_{n_m n_m}(\tau) = \frac{1}{2} R_{n_s n_s}(\tau) \sin^2(\theta_e) + \frac{1}{2} R_{n_c n_c}(\tau) \cos^2(\theta_e) + \frac{1}{2} \left[R_{n_s n_c}(\tau) + R_{n_c n_s}(\tau)\right] \sin(2\theta_e)$$

* The reason for using the phase term $3\pi/4$ is to avoid constant phase terms in the expression for $\overline{v_d(t)}$, as will become clear later on in this text.
\[
R_{nn}(\tau) = K_0^2 V_0^2 [R_{zz}(\tau) + R_{zz}(\tau) + R_{zz}(\tau) + R_{zz}(\tau) + R_{zz}(\tau) + R_{zz}(\tau) + R_{zz}(\tau) + R_{zz}(\tau) - 8A(z_1(t) + z_2(t)) + 16A^2]
\]

because \(R_{zz}^2(\tau) = R_{zz}^2(\tau)\), \(R_{zz}^2(\tau) = R_{zz}^2(\tau) = \frac{z_1(t)}{z_2(t)}\)

and \(\frac{z_1(t)}{z_2(t)} = 2A\).

\[
R_{nn}(\tau) = \frac{1}{2} K_0^2 V_0^2 [R_{zz}(\tau) - R_{zz}^2(\tau) - R_{zz}^2(\tau) - R_{zz}^2(\tau)]
\]

\[
= K_0^2 V_0^2 [R_{zz}^2(\tau) - 4A^2]
\]

\[
= R_{nn}(\tau)
\]

\[
R_{nn}(\tau) = \frac{1}{2} K_0^2 V_0^2 [R_{zz}^2(\tau) - R_{zz}^2(\tau) - R_{zz}^2(\tau) - R_{zz}^2(\tau) + 4A(z_1(t) - z_2(t))]\]

\[
= 0
\]

Thus we find

\[
R_{nn}(\tau) = R_{nn}(\tau) [\sin^2 \theta_e + \cos^2 \theta_e]
\]

\[
= K_0^2 V_0^2 [R_{zz}^2(\tau) - 4A^2]
\]

The power density spectrum of \(n_m(t)\) is

\[
S_m(f) = K_0^2 V_0^2 [S_z(f) - 4A^2 \delta(f)]
\]

\(S_z(f)\) is given by Equation (A.1) in Appendix A:

\[
S_z(f) = 4A^2 \delta(f) + A^2 T_s \text{sinc}^2(f T_s)
\]
and thus
\[
S_\text{m}(f) = K_m \sqrt{2} V^2 \frac{A^2 T_s}{D} \text{sinc}^2(f T_s) \\
= K_d \frac{T_s}{s} \text{sinc}^2(f T_s) \\
= (N_m/2) \text{sinc}^2(f T_s)
\]
where \(N_m = K_d T_s/4\)

**Option 3: reducing the 16-QAM signal to a 3-ASK signal**

For this option, the input signal for the PLL can be expressed as

\[
V_2 \cdot z(t) \cos(\omega t + \theta_i), \text{ where } z(t) = \sum_{k=-\infty}^{\infty} z_k p(t-kT_s).
\]

\(z_k\) can take on three different values (see also Fig. 4.2c):

- \(z_k = A\) with a probability of occurrence of \(\frac{1}{4}\),
- \(z_k = AV_S\) with a probability of occurrence of \(\frac{1}{2}\),
- \(z_k = 3A\) with a probability of occurrence of \(\frac{1}{4}\).

Again \(p(t)\) is assumed to be rectangular.

The VCO output signal is \(-V_2 \cdot V_o \sin(\omega t + \theta_o)\); the phase detector output is (discarding double frequency terms):

\[
v_d(t) = K \sqrt{2} V_o z(t) \sin \theta_m e
\]

where \(\theta_m = \theta_I - \theta_o\), and \(K_m\) is the multiplier gain factor as before.

The power density spectrum of \(z(t)\) can be found from Appendix A, Equation (A.3):

\[
S_z(f) = [(1+\frac{1}{2})V_S]\delta(f) + \frac{11-4V_S}{4} \frac{A^2 T_s}{D} \text{sinc}^2(f T_s)
\]

Averaging (B.1) over the data, we find:

\[
\overline{v_d(t)} = K \sqrt{2} V_o z(t) \sin \theta_m e \\
= (1+\frac{1}{2})K \sqrt{2} V_o A \sin \theta_m e \\
= K_d \sin \theta_m e
\]

\(K_d\) is the phase-detector gain factor again.
We can write

\[ v_d(t) = v_d(t) + n_m(t) \quad (\text{with } n_m(t) = 0) \]

where

\[ n_m(t) = K_m \overline{v}/z(t) - z(t) \mid \sin \theta_e \]

The power density spectrum of \( n_m(t) \) is

\[
S_m(f) = \frac{11-4\sqrt{5}}{4} K_m^2 \overline{V_o^2} T_s^2 \sin^2 \theta_e \sin^2(fT_s)
\]

\[
= \frac{11-4\sqrt{5}}{9+4\sqrt{5}} K_d^2 T_s \sin^2 \theta_e \sin^2(fT_s)
\]

\[
= \left( \frac{N_m}{2} \right) \sin^2(fT_s)
\]

where \( N_m = \frac{22-8\sqrt{5}}{9+4\sqrt{5}} K_d^2 T_s \sin^2 \theta_e \)

Summarizing the results for the different options, we have found:

\[ v_d(t) = K_d \sin \theta_e + n_m(t) \]

with \( S_m(f) = \left( \frac{N_m}{2} \right) \sin^2(fT_s) \) and \( N_m = K_d^2 T_s \sin^2 \theta_e \):

for option 1: \( K_d = 2AK_m \), \( f(\theta_e) = (1+4\cos^2 \theta_e)/2 \)

for option 2: \( K_d = 2\sqrt{2}AK_m \), \( f(\theta_e) = 1/4 \)

for option 3: \( K_d = (1+\sqrt{5})AK_m \), \( f(\theta_e) = \frac{22-8\sqrt{5}}{9+4\sqrt{5}} \sin^2 \theta_e \)

N.b.: any extra gain factors in the circuit can be included in \( K_d \).
APPENDIX C: CALCULATION OF THE VCO OUTPUT PHASE JITTER
AS A RESULT OF THE MODULATION NOISE

In Chapter 3 the error signal for the baseband remodulators has been
derived including the modulation noise. In Chapter 4 and Appendix B the
same has been done for the three different IF remodulators. All error
signals can be expressed as

\[ v_d(t) = K_d \sin \theta_e + n_m(t) \]  \hspace{1cm} (C.1)

where \( n_m(t) \) is the modulation noise with a power density spectrum

\[ S_m(f) = \frac{(N/2) \text{sinc}^2(f T_s)}{m} \]

the values of \( K_d \) and \( N_m \) are dependent on the chosen configuration. In this
appendix the variance of the phase jitter in the VCO output signal as a
result of the modulation noise will be calculated. The result of the
calculations presented here can be applied to all configurations discussed
in Chapters 3 and 4.

The analysis presented here will be in terms of transfer functions and
spectral densities; therefore it is necessary to make a linearizing
approximation with respect to the signal part \( K_d \sin \theta_e \) in (C.1). Assuming
that \( \theta_e \) will be small (which can safely be expected when the modulation
noise is not too strong in proportion to the carrier component and when a
high-gain loop filter is being used), then the error signal can be written
as

\[ v_d(t) = K_d \theta_e + n_m(t). \]

A block diagram of the linearized loop is shown in Fig. C.1.

The transfer function relating \( \theta_o \) to \( \theta_1 \) is

\[ H(p) = \frac{K_d K_F(p)}{p + K_d K_F(p)} \hspace{1cm} \text{where} \hspace{0.5cm} p = j \omega = j 2 \pi f \]

From Fig. C.1 it is clear that the same transfer function divided by \( K_d \)
relates \( \theta_o \) to \( n_m(t) \).
Fig. C.1. Block diagram of the linearized tracking loop.

\[ \theta_o \text{ can now be written as } \theta_o = \theta_o + \theta_{\text{no}}, \text{ where } \theta_{\text{no}} \text{ is the phase jitter as a result of the modulation noise:} \]

\[ \theta_{\text{no}} = \{H(p)/K_d\}n_m(t). \]

The variance of \( \theta_{\text{no}} \) can be expressed as

\[ \overline{\theta_{\text{no}}^2} = \frac{1}{K_d^2} \int_{-\infty}^{\infty} S_m(f)|H(j2\pi f)|^2 df. \]

Assuming that the loop bandwidth is small compared to the symbol rate, this can be approximated by

\[ \overline{\theta_{\text{no}}^2} = \frac{S_m(0)}{K_d^2} \int_{-\infty}^{\infty} |H(j2\pi f)|^2 df \]

\[ = \frac{2S_m(0)}{K_d^2} B_L, \]

where \( B_L = \int_{-\infty}^{\infty} |H(j2\pi f)|^2 df \), the noise bandwidth of the loop.

* In Chapter 3, \( H(j2\pi f) \) and \( F(j2\pi f) \) are written as \( H(f) \) and \( F(f) \).
Substituting $S_m(0) = \frac{N_m}{2}$, we find:

$$\theta_{no}^2 = \frac{N_mB}{K_d^2}.$$

Clearly the ratio $\frac{N}{K_d^2}$ should be kept small (or equivalently $\frac{K_d^2}{N_m}$ should be large); $\frac{K_d^2}{N_m}$ can be interpreted as a kind of signal-to-noise ratio. Furthermore the noise bandwidth $B_L$ should be small to reduce the effect of $N_m$. 
APPENDIX D: DERIVATION OF THE AVERAGE SYMBOL ERROR PROBABILITY
CONDITIONED ON A GIVEN LOOP PHASE ERROR

In Chapter 6 has been described how the demodulated quadrature signals are sampled and quantized to obtain the estimates \( \hat{a}_k \) and \( \hat{b}_k \) of the received data.

\( \hat{a}_k \) is determined by quantizing \( a_k \cos \theta e - b_k \sin \theta e + n_1(t) \); \( \hat{b}_k \) is determined by quantizing \( b_k \cos \theta e + a_k \sin \theta e + n_2(t) \). where \( i,j \in \{-3,-1,1,3\} \); \( n_1(t) \) and \( n_2(t) \) are zero-mean Gaussian noise processes with variance \( \sigma^2 \).

In this appendix the average probability of symbol error, conditioned on a given loop phase error, will be derived. This will be done for a \( K^2 \)-state QAM signal, a QAM signal where \( i \) and \( j \) can take on \( K \) different values (\( K \) is an even number): \( i,j \in \{-(K-1),-(K-3),\ldots,-3,-1,1,3,\ldots, K-3, K-1\} \). For a 16-QAM signal, \( K=4 \). Generalizing to a \( K^2 \)-QAM signal does not introduce extra difficulties to the derivation; therefore this approach has been chosen.

The average symbol error probability is

\[
P_{se} (\theta) = 1 - P_{ne} (\theta),
\]

where \( P_{ne} (\theta) \) is the probability that no error will be made in determining \( \hat{a}_k \) and \( \hat{b}_k \):

\[
P_{ne} (\theta) = \frac{1}{K^2} \sum_{i=-\frac{K-1}{2}}^{\frac{K-1}{2}} \sum_{j=-\frac{K-1}{2}}^{\frac{K-1}{2}} P(\hat{a}_k = iA | a_k = iA) P(\hat{b}_k = jA | b_k = jA)
\]

In this expression, \( P(\hat{a}_k = iA | a_k = iA) \) and \( P(\hat{b}_k = jA | b_k = jA) \) are for odd values only.
\[ P(\hat{a}_k=1|\lambda, \theta, j) = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} \exp[-(y-1\cos \theta + j\sin \theta)^2/2\sigma^2] dy \]

\[ = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} \exp[-(y+(1-1)\cos \theta + j\sin \theta)^2/2\sigma^2] dy \]

\[ = Q[\mu(1-1)\cos \theta + j\sin \theta] \]

The expression above is valid for all odd \( i \in \{-3, ..., 3\} \).

For \( i = -(K-1) \) we find

\[ P(\hat{a}_k=-(K-1)|\lambda, -(K-1), j) = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} \exp[-(y+(1+1)\cos \theta + j\sin \theta)^2/2\sigma^2] dy \]

\[ = Q[\mu(1+1)\cos \theta + j\sin \theta] \]

Similarly, \( P(\hat{b}_k=j|\lambda, i, j) \) can be found to be

\[ P(\hat{b}_k=j|\lambda, i, j) = Q[\mu(j-1)\cos \theta - j\sin \theta] \]

for all odd \( j \in \{-3, ..., 3\} \).

For \( j = -(K-1) \):

\[ P(\hat{b}_k=-(K-1)|\lambda, i, -(K-1)) = 1 - Q[\mu(-1)\cos \theta - j\sin \theta] \]

and for \( j = K-1 \):

\[ P(\hat{b}_k=(K-1)|\lambda, i, K-1) = Q[\mu(1)\cos \theta - j\sin \theta] \]
In the following, the shortened notation $Q_{x,y}(z)$ will be used:

$$Q_{x,y}(z) = Q[\mu(z-x\cos\theta + y\sin\theta)]$$

An important property of this function is that

$$Q_{x,y}(z) = 1 - Q_{-x,-y}(-z) \quad (D.1)$$

Now we have

$$P(\hat{a}_k = i|A| J, J) = Q_{i,j}(1-1) - Q_{i,j}(1+1) \text{ for } i \in \{-K-3, \ldots, K-3\}$$

$$P(\hat{a}_k = -(K-1)|A| -(K-1), J] = 1 - Q_{-(K-1), J}(-(K-2))$$

$$P(\hat{a}_k = (K-1)|A| K-1, J] = Q_{(K-1), J}(K-2)$$

$$P(\hat{b}_k = j|A| 1, J) = Q_{j,-1}(j-1) - Q_{j,-1}(j+1) \text{ for } J \in \{-K-3, \ldots, K-3\}$$

$$P(\hat{b}_k = -(K-1)|A| i,-(K-1)] = 1 - Q_{-(K-1), -1}(-(K-2))$$

$$P(\hat{b}_k = (K-1)|A| 1, K-1] = Q_{(K-1), -1}(K-2)$$

For $i, j \in \{-K-3, \ldots, K-3\}$ the contribution to $P_{ne}(\theta)$ is

$$P_{ne1}(\theta) = \frac{1}{K^2} \sum_{i=-(K-3)}^{K-3} \sum_{j=-(K-3)}^{K-3} [Q_{i,j}(1-1) - Q_{i,j}(1+1)][Q_{j,-1}(j-1) - Q_{j,-1}(j+1)]$$

The product under the summations can be split into four terms:

$$Q_{i,j}(1+1)Q_{j,-1}(j+1) + Q_{i,j}(1-1)Q_{j,-1}(j-1) - Q_{i,j}(1-1)Q_{j,-1}(j+1) +$$

$$- Q_{i,j}(1+1)Q_{j,-1}(j-1)$$

Now choose $i=x$ and $j=y$ for the first term, $i=-x$ and $j=-y$ for the second term, $i=-y$ and $j=x$ for the third term, and $i=y$ and $j=-x$ for the fourth term. Summing these four terms and making use of the property in (D.1), we find

$$4Q_{x,y}(x+1)Q_{y,-x}(y+1) - 2[Q_{x,y}(x+1) + Q_{y,-x}(y+1)] + 1$$

and thus

$$P_{ne1}(\theta) = \frac{1}{K^2} \sum_{i=-(K-3)}^{K-3} \sum_{j=-(K-3)}^{K-3} \left[4Q_{i,j}(1+1)Q_{j,-1}(j+1) +
- 2[Q_{i,j}(1+1) + Q_{j,-1}(j+1)] + 1 \right]$$

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The sum $2[Q_i,j(i+1) + Q_{j,-1}(j+1)]$ can be reduced to $4Q_{x,y}(i+1)$: choose $i=x$ and $j=y$ for the first term and $i=-y$ and $j=x$ for the second term, and add the results to obtain $4Q_{x,y}(x+1)$.

Thus we have

$$P_{\text{ne1}}(\theta_e) = \frac{4}{K^2} \sum_{i=-(K-3)}^{K-3} \sum_{j=-(K-3)}^{K-3} Q_i,j(i+1)Q_{j,-1}(j+1) +$$

$$- \frac{4}{K^2} \sum_{i=-(K-3)}^{K-3} \sum_{j=-(K-3)}^{K-3} Q_i,j(i+1) +$$

$$+ \frac{(K-2)^2}{K^2} \quad \text{(D.2)}$$

For $i=-(K-1)$ and $i=K-1$, and $j \in \{-K-3, \ldots, K-3\}$, the contribution to $P_{\text{ne1}}(\theta_e)$ is

$$P_{\text{ne2}}(\theta_e) = \frac{1}{K^2} \sum_{j=-(K-3)}^{K-3} \left[ \{1-Q_{-(K-1),j}(-(K-2))\} \{Q_{j,-1}(j-1) - Q_{j,-1}(j+1)\} + \right.$$

$$\left. + Q_{K-1,j}(-(K-2)) \{Q_{j,-1}(j-1) - Q_{j,-1}(j+1)\} \right]$$

Performing the same kind of "tricks" as in the calculation of $P_{\text{ne1}}(\theta_e)$, this can be written as

$$P_{\text{ne2}}(\theta_e) = \frac{2}{K^2} \sum_{j=-(K-3)}^{K-3} Q_{-(K-1),j}(-(K-2))Q_{j,-1}(j+1) +$$

$$+ \frac{2}{K^2} \sum_{i=-(K-3)}^{K-3} Q_{i,-1}(i+1)Q_{-(K-1),,-1}(-(K-2)) +$$

$$- \frac{2}{K^2} \sum_{i=-(K-3)}^{K-3} [Q_{i,-1}(i+1) + Q_{i,-1}(i+1)] +$$

$$- \frac{2}{K^2} \sum_{j=-(K-3)}^{K-3} Q_{-(K-1),j}(-(K-2)) +$$

$$+ \frac{2(K-2)}{K^2} \quad \text{(D.3)}$$
The contribution to \( P_{ne}(\theta_e) \) for \( j=-(K-1) \) and \( j=K-1 \), and \( i \in \{-(K-3), \ldots, K-3\} \) is

\[
P_{ne3}(\theta_e) = \frac{1}{K^2} \sum_{i=-(K-3)}^{K-3} \left\{ (1 - Q_{-(K-1), -(K-2)}) (1 - Q_{-(K-1), -(K-1)}) + Q_{K-1, -(K-2)} Q_{-(K-1), -(K-1)} \right\}
\]

Substituting \( j \) for \(-i\) and using (D.1), this can be written as

\[
P_{ne3}(\theta_e) = \frac{1}{K^2} \sum_{j=-(K-3)}^{K-3} \left\{ (1 - Q_{j, -(K-2)}) (1 - Q_{j, -(K-1)}) + Q_{K-1, j} Q_{j, -(K-1)} \right\}
\]

The last contribution to \( P_{ne}(\theta_e) \) is made by the four terms resulting from \( i=\pm(K-1) \) and \( j=\pm(K-1) \) (adding all four possible combinations):

\[
P_{ne4}(\theta_e) = (1 - Q_{-(K-1), -(K-1)} [-(-K-2)]) (1 - Q_{-(K-1), -(K-1)} [-(-K-2)]) +
\]

After some calculations this is found to be

\[
P_{ne4}(\theta_e) = \frac{4}{K^2} Q_{-(K-1), -(K-1)} Q_{-(K-1), -(K-1)} [-(-K-2)] +
\]

\[
P_{ne}(\theta_e) = P_{ne1}(\theta_e) + P_{ne2}(\theta_e) + P_{ne3}(\theta_e) + P_{ne4}(\theta_e)
\]

\[
P_{ne}(\theta_e) = P_{ne1}(\theta_e) + 2P_{ne2}(\theta_e) + P_{ne3}(\theta_e)
\]

D-5
Adding the first summation in (D.2), the first and the second summation (multiplied by two) in (D.3) and the first summation in (D.4), we find

\[ \frac{4}{k^2} \sum_{i=-(K-1)}^{K-3} \sum_{j=-(K-1)}^{K-3} Q_{i,j}(i+1)Q_{j,-1}(j+1) \]  
(D.5)

Adding the second summation in (D.2), the third and the fourth summation (multiplied by two) in (D.3) and the second summation in (D.4), we find

\[ -\frac{4}{k^2} \sum_{i=-(K-1)}^{K-3} \sum_{j=-(K-1)}^{K-1} Q_{i,j}(i+1) \]  
(D.6)

Adding the last term in (D.2), the last term (multiplied by two) in (D.3) and the last term in (D.4), we find

\[ \frac{(K-2)^2}{k^2} + \frac{4(K-2)}{k^2} + \frac{4}{k^2} = 1 \]  
(D.7)

Now \( P_{\text{ne}}(\theta_e) \) is the sum of (D.5), (D.6) and (D.7):

\[ P_{\text{ne}}(\theta_e) = 1 + \frac{4}{k^2} \sum_{i=-(K-1)}^{K-3} \sum_{j=-(K-1)}^{K-3} Q_{i,j}(i+1)Q_{j,-1}(j+1) + \]

\[ -\frac{4}{k^2} \sum_{i=-(K-1)}^{K-3} \sum_{j=-(K-1)}^{K-1} Q_{i,j}(i+1) \]

The average probability of symbol error \( P_e(\theta_e) \) is \( 1 - P_{\text{ne}}(\theta_e) \); substituting \( i+1=1 \) and \( j+1=m \), the final result is

\[ P_e(\theta_e) = \frac{4}{k^2} \sum_{l=-(K-2)}^{K-2} \sum_{j=-(K-1)}^{K-1} Q[\mu(1-(1-1)\cos\theta_e+j\sin\theta_e)] + \]

\[ -\frac{4}{k^2} \sum_{l=-(K-2)}^{K-2} \sum_{m=-(K-2)}^{K-2} \left[ Q[\mu(1-(1-1)\cos\theta_e+(m-1)\sin\theta_e)] \right] \times \]

\[ Q[\mu((m-1)\cos\theta_e-(1-1)\sin\theta_e)] \]

The summations over \( l \) and \( m \) are for even values only; the summation over \( j \) is for odd values only.
For a 16-QAM signal, $K=4$; $P_e(\theta_e)$ is then found to be

\[
P_e(\theta_e) = \frac{1}{4} \sum_{l=-2}^{2} \sum_{j=-3}^{3} Q[\mu(1 - (l-1)\cos \theta_e + j\sin \theta_e)] + \]
\[- \frac{1}{4} \sum_{l=-2}^{2} \sum_{m=-2}^{2} Q[\mu(1 - (l-1)\cos \theta_e + (m-1)\sin \theta_e)] \times \]
\[Q[\mu(m - (m-1)\cos \theta_e - (l-1)\sin \theta_e)] \]

with $m, l = -2, 0, +2$ and $j = -3, -1, +1, +3$. 
APPENDIX E: REALIZATION OF THE DEMODULATOR

In this appendix the realization of the demodulator is discussed; unfortunately not all parts of the demodulator have been realized in hardware yet. The phase detector, the loop filter and the VCO for the carrier recovery loop have been realized, just like the 100 MHz crystal oscillator. The remodulation control circuits and the timing control circuits are discussed in Chapters 8 and 7 respectively; in Fig. 8.2 and Fig. 7.5 is shown how they can be implemented. An electronic circuit for the Early-Late tracking loop is proposed in this appendix.

No print lay-outs have been made for the circuits that are actually realized; instead they are soldered directly onto special prefabricated printed circuit boards designed at the Twente University of Technology. This had the advantage that changes could be made relatively easy.

The supply voltages for the whole system are +15V and -15V; these are converted to +12V, -12V and +5V wherever needed by the various circuits. The power supply terminals of the subcircuits are decoupled separately by LC-filters; for "critical" subcircuits like oscillators, extra voltage stabilizers are used. The power supply circuits are mounted on one side of the pc-board, the other circuits on the other side. Small tin plates are placed between the subcircuits to minimize their mutual electromagnetical influence; each pc-board is also shielded from external electromagnetical radiation by such tin plates.
The 100 MHz crystal oscillator

The circuits for the 100 MHz crystal oscillator are shown in Fig. E.1. A fifth-overtone crystal is used in the circuit around T₁, the actual oscillator. The inductance L₁ must be large enough to permit oscillation at 100 MHz (the parallel combination of L₁ and C₁ must be capacitive at the oscillation frequency), but small enough to prevent oscillation at the fundamental frequency of 20 MHz or the third overtone at 60 MHz. With C₁ = 39 pF, L₁ should have a value between 0.06 μH and 0.18 μH; an inductance of 0.1 μH proved to be fine. (For L₁ < 0.06 μH no oscillation occurs, and for L₁ > 0.18 μH the circuit oscillates at 60 MHz). The circuit around T₂ is a filter/amplifier; the circuit around T₃ forms a 50 Ω output buffer.

The oscillation frequency is measured to be 100.0037 MHz. This means that the VCO in the carrier recovery loop must be set to 30.0037 MHz for an input carrier frequency (of the 16-QAM signal) of 70 MHz.

The phase detector and the loop filter

In Fig. E.2 the circuits for the phase detector and the loop filter (as discussed in Chapter 9) are shown. The phase detector is implemented by mixer SBL-1; input A is the 3-ASK signal and input B is the 100 MHz signal from the crystal oscillator. The high-gain loop filter is built around OA1 (operational amplifier LF356). The window detector is built around the IC LM319, containing two comparators with open-collector output circuits; P₁ is adjusted to yield a voltage window from 6.20 V to 6.60 V. OA2 is used as a comparator; P₂ is adjusted to a voltage of 6.40 V at the inverting input. The switches are implemented by the CMOS IC 4053. For a more detailed discussion of the loop filter and the window detector, see Chapter 9.

The time delay between the closing of S3 and the other two switches (to let the loop acquire lock) is realized by the monostable multivibrator 74LS123; the delay time is set to about 30 ms. A LED is used to indicate whether S1 and S2 are closed.

E-2
All transistors: BFY90
The 30 MHz VCO

The circuits for the VCO are shown in Fig. E.3. The actual VCO is formed by the circuits around the dual-gate MOS-fet BF961, forming a 10 MHz VCO. The oscillation frequency is controlled by the voltage over the varactors BB109 (the output voltage of the loop filter). $P_1$ is adjusted for minimum harmonic distortion (its setting also determines whether any oscillation will occur at all); with $C_1$ the frequency can be adjusted. The circuits around $T_1$ and $T_2$ form an amplifier/buffer with an output impedance of about 50 Ω.

The circuit around $T_3$ is a 20 MHz crystal oscillator. The circuit around $T_4$ forms an amplifier/filter for the 20 MHz signal; with a transformer (10 turns on the primary side, 1 turn on the secondary side) +7 dBm is coupled into the mixer SBL-1 (which has an input impedance of 50 Ω). The transformer is necessary to keep the load resistance high (and thus realize a high-Q filter); the primary coil serves as the filter inductance.

In the mixer the 10 MHz signal and the 20 MHz signal are multiplied; the output signal is fed to a filter with an input impedance of about 50 Ω (the common-base circuit around $T_5$). This filter rejects the 10 MHz component (the difference frequency) and passes the 30 MHz signal (the sum frequency). The circuit around $T_6$ forms a 50 Ω output buffer.

$C_1$ is adjusted to an output frequency of 30.0037 MHz for a control voltage of 6.40 V over the varactors. In Fig. E.4 the output frequency as a function of the control voltage is shown; from this curve $K_v$ can be determined to be $2.41 \cdot 10^5$ rad/V·s.
Figure F.3

Title: 80 MHz VCO

Bipolar transistors: 2N4184
The Early-Late tracking loop

In Fig. E.5 and Fig. E.6 is shown how the Early-Late tracking loop can be implemented. In Fig. E.5, the 6-ary ASK signal from the carrier recovery loop is split into two identical signals using buffers as shown for the "early" part of the loop (the circuit around $T_1$); then these signals are multiplied by the early signal $d(t+T_5/4)$ and the late signal $d(t-T_6/4)$ respectively, using the mixers SBL-1. The IC TBA120 is used to amplify each output signal; after passing through a tuned transformer circuit, both signals are added and fed to the amplifier built around $T_2$. The circuit around $T_3$ is a buffer with an output impedance of 50 $\Omega$. Now the signal is demodulated using another mixer SBL-1; with $L_1$ the phase of the signal can be adjusted to equal the phase of the reference carrier. A high-gain loop filter is used (like the one in the carrier recovery loop); its output is fed to the VCXO in Fig. E.6.

The VCXO uses a varactor to control the output frequency; with $C_1$ the frequency can be adjusted to the right value for the nominal control voltage. $P_1$ is adjusted to yield a symmetrical clock signal at the output of the Schmitt-trigger 74LS132.

Figure E.4. Output frequency of the VCO as a function of the control voltage.
Figure E.6

Title: 4.096 MHz VCO

Control voltage from loop filter