MASTER

On the determination of electrical epicardial potentials from skin potential and torso geometry data

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ON THE DETERMINATION OF ELECTRICAL EPICARDIAL POTENTIALS FROM SKIN POTENTIAL AND TORSO GEOMETRY DATA

by J.W.M. Vermeulen

Submitted in partial fulfillment of the requirements for the degree of ir. (M.Sc.) at the Eindhoven University of Technology. The work was carried out from September 1976 until January 1978 under the directorship of prof. dr. ir. P. Eykhoff. Advisor: ir. A.A.H. Damen.
Summary.

The main aim of this investigation was the estimation of a number of epicardial potentials simultaneously, exclusively from recorded skin potentials (ECG's) and torso geometry data. Neither a priori knowledge about the electrical properties of (healthy) heart activity, nor any model representing this activity should be used. Several assumptions, however, had to be made, due to imperfect knowledge or estimations of the heart geometry and of inhomogeneities inside the torso.

We started, as many investigators, from Green's second identity to derive integral equations between torso skin and epicardial potentials. These equations were discretized, giving a transfer matrix defining the linear relations between the epicardial and skin potentials. Several ways of calculating the matrix elements were used, and checked for their properties. Finally, the combination of two methods proved to be the best one.

Since the final estimations of the heart potentials were very bad, due to intrinsic amplification of spatially high frequency disturbances, we tried to reduce the sensitivity to noislike signals by limiting locally the second spatial derivative of the heart potential. This was achieved by minimizing the integral of the square value of this derivative over the heart surface.

This way proved to be quite effective, and an encouraging one to proceed by.
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### List of symbols and abbreviations

#### Potentials \( \Phi \) and double layer strengths \( \tau \).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Phi_H(Q) )</td>
<td>Epicardial (heart surface) potential in ( Q ) (analogue, unbounded medium form).</td>
</tr>
<tr>
<td>( \Phi_H )</td>
<td>Epicardial potential vector.</td>
</tr>
<tr>
<td>( \Phi_{Hj} ) or ( \Phi_H(Q_j) )</td>
<td>( j^{\text{th}} ) element of ( \Phi_H ), epicardial potential of ( j^{\text{th}} ) point ( Q_j ).</td>
</tr>
<tr>
<td>( \Phi_{HA} )</td>
<td>Epicardial potential vector (triangle form).</td>
</tr>
<tr>
<td>( \Phi_{HAj} ) or ( \Phi_H(Q^H_{j}) )</td>
<td>( j^{\text{th}} ) element of ( \Phi_{HA} ), average potential of ( j^{\text{th}} ) epicardial triangle.</td>
</tr>
<tr>
<td>( \Phi_s(Q) )</td>
<td>Bounded medium skin potential in ( Q ), measurable (analogue form).</td>
</tr>
<tr>
<td>( \Phi_s )</td>
<td>Bounded medium potential vector.</td>
</tr>
<tr>
<td>( \Phi_{sj} ) or ( \Phi_s(Q_j) )</td>
<td>( j^{\text{th}} ) element of ( \Phi_s ), bounded medium skin potential of ( j^{\text{th}} ) point ( Q_j ).</td>
</tr>
<tr>
<td>( \Phi_{sR} )</td>
<td>( \Phi_s ) reduced, contains all ( \Phi_{sj} ), the ( \Phi_{sj} ) in practice unmeasurable ones excepted.</td>
</tr>
<tr>
<td>( \Phi_{sA} )</td>
<td>Bounded medium potential vector, (triangle form).</td>
</tr>
<tr>
<td>( \Phi_{sA j} ) or ( \Phi_s(Q^A_{j}) )</td>
<td>( j^{\text{th}} ) element of ( \Phi_{sA} ), average, bounded medium potential of ( j^{\text{th}} ) torso triangle.</td>
</tr>
<tr>
<td>( \Phi_\infty(Q) )</td>
<td>Unbounded medium skin potential in ( Q ), not measurable (analogue form).</td>
</tr>
<tr>
<td>( \Phi_\infty )</td>
<td>Unbounded medium potential vector, discretized form</td>
</tr>
<tr>
<td>( \Phi_\infty(P_i) )</td>
<td>( i^{\text{th}} ) element of ( \Phi_\infty ), unbounded medium skin potential of point ( P_i ).</td>
</tr>
<tr>
<td>( \tau_H )</td>
<td>(Equivalent) epicardial double layer strength. Further indices and subscripts: cf. ( \Phi_H ).</td>
</tr>
</tbody>
</table>
### Matrices: Roman Capitals.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>transfer matrix ((\Phi_s = A \Phi_H, A = B^{-1}D))</td>
<td>AGP x AKP</td>
</tr>
<tr>
<td>(A_R)</td>
<td>(A)-reduced ((\Phi_{sR} = A_{R} \Phi_{H}))</td>
<td>RED x AKP</td>
</tr>
<tr>
<td>B</td>
<td>transfer matrix ((\Phi_s = B \Phi_H))</td>
<td>AGP x AGP</td>
</tr>
<tr>
<td>C</td>
<td>transfer matrix ((\Phi_s = C \Phi_H))</td>
<td>AGP x AKP</td>
</tr>
<tr>
<td>(C_A)</td>
<td>triangle form of (C) ((\Phi_s = C_A \Phi_H))</td>
<td>AGP x AKD</td>
</tr>
<tr>
<td>D</td>
<td>triangle form of G ((\Phi_s = D\Phi_H))</td>
<td>AGP x AKP</td>
</tr>
<tr>
<td>(D_A)</td>
<td>triangle form of (D) ((\Phi_s = D_A \Phi_H))</td>
<td>AGP x AKD</td>
</tr>
<tr>
<td>E</td>
<td>transfer matrix ((\Phi_s = E \Phi_H, E = B^{-1}D))</td>
<td>AGP x AKP</td>
</tr>
<tr>
<td>(E_R)</td>
<td>E-reduced ((\Phi_{sR} = E_{R} \Phi_{H}))</td>
<td>RED x AKP</td>
</tr>
<tr>
<td>F, G</td>
<td>coefficient matrices (second derivatives correction, (\Delta Z \Phi_H = F \Phi_H, G = F^T F))</td>
<td>AKP x AKP</td>
</tr>
<tr>
<td>I</td>
<td>identity matrix</td>
<td>square</td>
</tr>
<tr>
<td>T</td>
<td>triangle-to-point transformation matrix ((D = D_A T, C = C_A T))</td>
<td>AKD x AKP</td>
</tr>
<tr>
<td>W</td>
<td>inverse transfer matrix ((\Phi_s = W \Phi_sR, W = (E_{R})^+))</td>
<td>AKP x RED</td>
</tr>
<tr>
<td>X</td>
<td>inverse transfer matrix ((\Phi_s = X \Phi_sR, X = (A_{R})^+))</td>
<td>AKP x RED</td>
</tr>
<tr>
<td>(Z, Z_c)</td>
<td>part of (B) ((B = I - Z_c, Z_c = (1/8 \pi)Z))</td>
<td>AGP x AGP</td>
</tr>
<tr>
<td>(Z_A)</td>
<td>triangle form of (Z)</td>
<td>AGP x AGD</td>
</tr>
<tr>
<td>(M^T)</td>
<td>transpose of matrix (M) ((M^T_{ij} = M_{ji}))</td>
<td>any</td>
</tr>
<tr>
<td>(M^{-1})</td>
<td>inverse of matrix (M)</td>
<td>square</td>
</tr>
<tr>
<td>(M^+)</td>
<td>pseudo-inverse of matrix (M)</td>
<td>any</td>
</tr>
<tr>
<td>(M_{ij})</td>
<td>element of matrix (M): row (i), column (j)</td>
<td>any</td>
</tr>
</tbody>
</table>

### Vectors: underlined variables.

- **\(\vec{x}, \vec{r}\)**: a vector, coordinate vector, \(\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}\)
- **\(\vec{x}^T\)**: transpose of \(\vec{x}\), \(\vec{x}^T = (x_1, x_2, \ldots)\)
- **\(e\)**: \(e^T = (1, 1, \ldots, 1)\)
- **\(e_k\)**: \(e_k^T = (0, 0, \ldots, 1, 0, \ldots, 0) \) \(k\)th element = 1.
- **\(\vec{n}\)**: normal vector, pointing outward. \(\vec{n}_T\): normal to torso surface; \(\vec{n}_H\): normal to heart surface.
- **\(\vec{\varepsilon}_1, \vec{\varepsilon}_2, \vec{\varepsilon}_3\)**: vertex coordinate vectors (seven point method)
\( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_7, \mathbf{x}_i \): interpolation point coordinate vector \((i = 1 \ldots 7)\) 

\((\text{seven point method})\)

\(|\mathbf{x}|\) : length of vector \(\mathbf{x}\)

\(x_i\) : \(i\)th element of vector \(\mathbf{x}\)

**Constants and variables.**

- \(\text{NO (NH)}\) : point grid meshwidth parameter for the torso skin (epicardial) surface.
- \(\text{AGP (AKP)}\) : number of skin (heart) points:
  \[ \text{AGP} = 4 \cdot \text{NO}^2 + 2; \text{AKP} = 4 \cdot \text{NH}^2 + 2. \]
- \(\text{AGD (AKD)}\) : number of skin (epicardial) triangles:
  \[ \text{AGD} = 8 \cdot \text{NO}^2; \text{AKD} = 8 \cdot \text{NH}^2. \]
- \(\text{RED}\) : number of torso skin points, that can be measured at \(\text{AKP} < \text{RED} \leq \text{AGP}\).
- \(a\) : radius of heart surrounding sphere.
- \(c\) : some constant
- \(d (d_i)\) : radius of skin observation point \(P (P_i)\), analogue (discretized) form.
- \(i, j, k, l\) : indices.
- \(r, s, t, u, w, w_1, w_2, \ldots, w_7\) : constants (seven point method).
- \(\text{iref}\) : reference torso point (zero potential).
- \(\text{jref}\) : reference epicardial point (zero double layer strength).
- \(\gamma\) : angle, \(\angle\) QOP.
- \(\delta\) : small quantity, small angle.
- \(\varphi\) : angle, elevation (latitude) angle.
- \(\lambda\) : constant, to be chosen at liberty (second derivatives correction), eigenvalue (deflation).
- \(\delta_{ij}\) : Kronecker delta : \(\delta_{ij} = 1\), if \(i = j\) and zero otherwise.
- \(r, e_{ij}, e_{ijk}\) : distance between : \(P\) and \(Q\), \(P_i\) and \(Q_j\), \(P_i\) and \(Q_{jk}\).
- \(\varphi\) : rotation (longitude) angle.
- \(\Psi\) : Green's function.
- \(\Omega\) : solid angle.

\(\Delta \Omega \mathbf{p}_{ij}, \Delta \Omega \mathbf{r}_{ij}\) : solid angle, subtended at \(P_i, P_i\) by triangle \(j\).
Miscellaneous.

THE : Eindhoven University of Technology.
THE/ER : Group Measurement and Control, Department of Electrical Engineering, Eindhoven University of Technology.
TPT : Triangle-to-point transformation.
SpHF : spatially high frequent.
$A_{\Delta}$, $A_j$ : area of ($j^{th}$) triangle.
0 : Origin, centre of coordinate system (= centre of the heart).
P, $P_i$ : "observation" point on the skin, analogue (discretized) form.
Q, $Q_j$ : "observed" point on the heart or on the skin.
$Q^*_j$ : centre of $j^{th}$ triangle.
R, $R_i$ : "observation" point, image point of P ($P_i$) with respect to the heart surrounding sphere.
$S_H$, $S_T$ : heart surface, torso surface.
$\Delta$ : triangle, or, in combinations: small part of, or:
second spatial derivative operator ($= \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$).
$\Delta_2$, $\Delta_3$ : part of $\Delta$, $\Delta_3 = \Delta$.
$\angle$, $\angle (OQP)$ : angle, angle between OQ andQP.
$\sum_i x_i$ : sum of all $x_i$ existing.
$\sum_i \sum_j A_{ij}$ : sum of all $A_{ij}$ existing, excepted those with $j = i$.
$\nabla$, $\nabla^2$ : 1st, 2nd, spatial derivative, $\nabla$ = gradient operator;
$\nabla^2$ = Laplace operator ($\nabla^2 = \Delta = \nabla \cdot (\nabla \ )$).
(1), (la) : formula no. 1, la.
Introduction.

One of the main aims of biomedical technology research in the group THE/ER is the measurement of a patient's state, without operating on him. One of the problems is the proper estimation of epicardial electrical potentials, from recorded ECG's and torso geometry data only.

An appropriate computer program, as a result of a future solution of this problem, would make diagnosis of heart diseases - possibly in an early stage - easier. The patient's recovery, afterwards, could easily be monitored, too.

Since we didn't want the technique, we are developing, to exclude, intrinsically, any heart disease, we tried to come out with as little a priori information, from healthy hearts, as possible. Therefore, we didn't use e.g. the well known course of the depolarization wave. For the same reason, we didn't use any model, which could impose, implicitly, some restrictions on the source.

For reasons of simplicity and lack of data, however, we had to make some assumptions concerning the torso linearity, homogeneity and isotropy. If the geometry and the electrical properties of inhomogeneities (etc.) inside the torso can be estimated properly, these data may be taken into account in future investigations.

The present report describes the work done as a final year's student in the group THE/ER, the results of this work and its mathematical backgrounds:

Firstly, we had to bring the problem into a manageable shape. The physical side of the problem, and the way in which torso potentials are gathered, are treated in Chapter I.

In Chapter II mathematical integral relations are derived between the skin potentials, the torso geometry data and the epicardial potentials. The discretization of these relations is described in Chapter III, which Chapter also deals with a - necessary - reshaping and inversion of the discretized relations, and with the resulting estimations of the epicardial potentials.

THE/ER will be used as abbreviation of: "Group Measurement and Control, Department of Electrical Engineering, Eindhoven University of Technology".
Another discretization method is described in Chapter IV, as well as the final calculation procedure used for the discretized relations. Since the calculated epicardial potentials pointed towards some instability of the estimation procedure, we tried to improve this procedure as described in Chapter V. Finally, Chapter VI deals with some aspects of the computer and the programs used.
Chapter I. Physical modelling of the problem and data gathering.

Since we aim at estimating the potential distribution on the epicardial surface from skin potential and torso geometry measurements only, without operating upon the patient, we have to make some assumptions concerning:

a. the geometry of the heart,
b. the electrical properties of the medium inside the torso,
c. the number of skin potentials to be measured.

In this investigation we assumed:

a. the epicardial surface to be a sphere, as the heart geometry cannot be measured or estimated well, and for mathematical reasons. This sphere should narrowly enclose the heart, having a radius of about 60 - 80 mm. (We have to estimate this radius, too!) 
b. the medium inside the torso to be linear, homogeneous and isotropic, as far as its electrical properties are concerned. Thus, the specific conductivity $\sigma$ is assumed to be a constant throughout the torso. 
c. the number of measured skin potentials to be limited to about 60 - 100, due to:
   1. the size of the measuring electrodes, 
   2. the degree of interdependence of the skin potentials measured,

The limit of the number of skin potentials limits the number of points on the epicardial surface, the potential of which can be calculated from those skin potentials.

We approximated the torso skin and the epicardial surfaces by a number of plane triangles. The vertices of the torso triangles were to be the points at which the skin potentials would be measured. The vertices of the epicardial triangles were to be the points, of which the epicardial potentials should be calculated. Thus, both on the skin and on the epicardial surface a grid of points will be laid. This grid (or point distribution), however, should be well planned.

Damen [1] and Nicola [13] proposed a point distribution method, the
points of which are about equidistant on a spherical surface (Thus, the resolution is about a constant over a sphere).

We used this method, since the torso geometry and skin potential data available were measured according to this method.

The degrees of subtlety of the torso and the epicardial point distributions, which may be chosen at liberty, are denoted by $NO$ and $NH$ respectively. Fig. 1 gives an example of this point distribution method. (Only the 1st and 5th octants are shown, $NH = 3$).

![Diagram of point distribution on a sphere](image)

Fig. 1. Example of point distribution on a sphere.
The numbers are used to compose vectors from the potentials and coordinates of the points.

The number of torso points $AGP = 4NO^2 + 2$, the number of torso triangles $AGD = 8NO^2$.

Likewise, the number of epicardial points $AKP = 4NH^2 + 2$, the number of epicardial triangles $AKD = 8NH^2$.

Generally, $NO$ is taken 4 or 5, giving an $AGP$ of 66 or 102 (as was stated on page 11), limiting $NH$ to 3 or 4, giving an $AKP$ of 38 or 66.
The radii of the torso skin points are not equal one to another (the origin 0 being in the centre of the heart). The point distribution method, however, prescribes the \( \hat{\theta} \) (elevation angle) and \( \varphi \) (rotation angle) coordinates only, a NO or NH being given.

The influence of extremities (arms, legs, head) has been neglected, thus "truncating" the body mathematically.

In order to obtain a clear view (in two-dimensional drawings) of the events on closed surfaces, we used a projection method, as described by Nicola 13', following this scheme:
- project all points radially upon an octahedron, positioned symmetrically with respect to the origin (centre of the heart),
- open the 4 posterior faces of the octahedron,
- project all points perpendicularly upon a vertical square plane.

Figs. 2 give examples of this projection method, giving a clear view of the numbering of points. These figures also show the situation of the triangles and a "map" of the epicardial surface.

![Diagram](image)

**Fig. 2a.** Example of the projection method, showing the point numbering method and the situation of the triangles. NH (or NO) = 3, number of points = 38, number of triangles = 72.
Fig. 2b. Example of point distribution on and projection of the torso surface. NO = 4, AGP = 66, AGD = 128.

LA = left atrium
RA = right atrium
LV = left ventricle
RV = right ventricle

Fig. 2c. Example of point distribution on and projection of the epicardial surface. NH = 3, AKP = 38, AKD = 72.

Note, that the heart is reclining in the torso, causing small parts of the atrium surfaces to be projected in the lower half of the square.
The torso point geometry data are measured with a special apparatus, designed and developed by the group THE/ER. By sticking an electrode on each torso point, as is designated by the point distribution method, potential time signals (normal ECG's) are measured and recorded, giving the torso potential data set required.
Chapter II. Mathematical description of the problem.

II.1 Introduction and derivation of the integral equations.

Consider a uniform conducting body of volume $V_1$ (cf. fig. 3, [2], [7]) with an outer boundary $S_T$, $V_1$ containing electrical sources, as is indicated.

Fig. 3. Situation of the uniform conducting body, containing electrical sources.

Green's second identity states, that

$$
\iiint_{V_1} (\vec{\Phi} \cdot \nabla^2 \psi - \psi \cdot \nabla^2 \vec{\Phi}) dV_1 = \oint_{S_T} \left( \vec{\Phi} \cdot \frac{\nabla \psi}{n} - \psi \frac{\partial \vec{\Phi}}{\partial n} \right) dS_T
$$

(1)

in which $\psi$ is a Green's function

- $n$ is the normal on $S_T$, pointing outward
- $\vec{\Phi}$ is the local potential, due to the sources inside $V_1$.

For an observation point $P$, just inside $S_T$, we take as a Green's function: $\psi = \psi_1 = \frac{1}{|r - r_p|}$, in which $r_p$ is the coordinate vector of $P$. (This means physically: the potential in some point $r$, due to a point source in $P$)

Using this $\psi_1$, the volume integral can be evaluated, because

$$
\nabla^2 \psi_1 = -4\pi \delta(r - r_p)
$$

$$
\iiint_{V_1} \vec{\Phi} \cdot \nabla^2 \psi_1 dV_1 = -4\pi \vec{\Phi}(P).
$$

$$
\frac{1}{4\pi} \iiint_{V_1} \psi_1 \nabla^2 \vec{\Phi} dV_1
$$

will be called the "unbounded medium potential"
of \( P \), denoted by \( \Phi_\infty(P) \).

This name can be explained in the following way:

Suppose, there are, within \( V_1 \), a number of point sources of strengths \( I_i \). Suppose for a moment also, that the conductive medium is infinitely wide (equal conductivities inside \( V_1 \) and in its environment).

Using the superposition theorem – we assumed linearity – we may state, that the potential in some observation point \( \mathbf{r} \), due to these sources is equal to

\[
\sum_i \frac{I_i}{|\mathbf{r}_i - \mathbf{r}|} = \Phi_\infty(\mathbf{r}).
\]

As a consequence,

\[
\nabla^2 \Phi_\infty = -4\pi \sum_i I_i \delta(\mathbf{r}_i - \mathbf{r}).
\]

Substitution of this result into

\[
\iint_{V_1} \nabla \cdot \nabla \Phi \, dV_1 \quad \text{gives (we integrate over all points } \mathbf{r} \text{)}:
\]

\[
-4\pi \sum_i \frac{I_i \delta(\mathbf{r}_i - \mathbf{r})}{|\mathbf{r}_i - \mathbf{r}|} = -4\pi \sum_i \frac{I_i}{|\mathbf{r}_i - \mathbf{r}_p|} = -4\pi \Phi_\infty(\mathbf{r}).
\]

Note that, if the medium is not unbounded, as it really is, the fields, due to such a number of point sources, will differ from \( \Phi_\infty \), but \( \nabla^2 \Phi = \nabla^2 \Phi_\infty \).

\[ \frac{\partial \psi_1}{\partial \mathbf{n}} \] (normal derivative of \( \psi_1 \)) = \( (\nabla \cdot \psi_1) \cdot \mathbf{n} = -\frac{(\mathbf{r} - \mathbf{r}_p)}{|\mathbf{r} - \mathbf{r}_p|^3} \cdot \mathbf{n} \), leading to

\[
\iint_{S_T} \frac{\partial \Phi}{\partial \mathbf{n}} \, dS_T = -\iint_{S_T} \frac{\psi_1}{|\mathbf{r} - \mathbf{r}_p|^3} \cdot \mathbf{n} \, dS_T.
\]

\[ \frac{\partial \Phi}{\partial \mathbf{n}} \] on the skin is proportional to the current flowing into, or out of the torso. \( \frac{\partial \Phi}{\partial \mathbf{n}} = 0 \) on the skin, because the air around the torso is non-conductive.

Consequently, \( \iint_{S_T} \psi_1 \frac{\partial \Phi}{\partial \mathbf{n}} \, dS_T = 0 \).

If point \( P \), being just inside the torso, is close enough to the skin, \( \Phi(\mathbf{P}) \) will be equal to the potential measured on the skin. We will, therefore, denote \( \Phi(\mathbf{P}) \) by \( \Phi_s(\mathbf{P}) \), measurable skin potential.

Using the relations derived above, we may, for some observation point \( \mathbf{p} \) rewrite (1) as:

\[
4\pi \Phi_s(\mathbf{P}) - 4\pi \Phi_\infty(\mathbf{P}) = \iint_{S_T} \frac{\Phi_s(\mathbf{Q})}{|\mathbf{r}_Q - \mathbf{r}_p|^3} \cdot \mathbf{n} \, dS_T.
\]

in which \( \mathbf{Q} \) is the centre if the infinitesimally small surface element \( dS_T \).
In order to derive a relation between the epicardial potentials and the unbounded medium ones, we will consider a system, as is depicted in fig. 4.

**Fig. 4. Unbounded medium potential concept.**

$S_{\infty}$ is a sphere, the radius $r$ of which is increasing beyond all bounds. $V_2$ is the volume, bounded by $S_H$ and $S_{\infty}$, having a constant specific conductivity $\sigma_1 > 0$.

All sources are contained within $S_H$, making $\nabla^2 \Phi = 0$ throughout $V_2$.

We apply Green's second identity (1) to this system, too:

$$\iiint_{V_2} \left( \Phi \nabla^2 \psi_2 - \psi_2 \nabla^2 \Phi \right) dV_2 = - \oint_{S_H} \left( \Phi \frac{\partial \psi_2}{\partial n_H} - \psi_2 \frac{\partial \Phi}{\partial n_H} \right) dS_H +$$

$$+ \oint_{S_{\infty}} \left( \Phi \frac{\partial \psi_2}{\partial n_{\infty}} - \psi_2 \frac{\partial \Phi}{\partial n_{\infty}} \right) dS_{\infty} \quad (1a)$$

Note, that $n_H$ is pointing inward with respect to $V_2$.

We would like to have a Green's function $\Phi_2$, such that for some point $P$ inside $V_2$ : $\nabla^2 \psi_2 = - 4\pi \cdot \delta(r - r_p)$, and, that $\psi_2 = 0$ on $S_H$.

If $S_H$ is a sphere of radius $a$, and for a point $Q (r = r_Q)$, it turns out, that $\psi_2 = \frac{1}{|r_Q - r|} - \frac{a}{d} \cdot \frac{1}{|r_Q - r_H|}$ is a proper choice

(Cf. fig. 5 and [2]).

**Fig. 5. Situation of parameters around the heart.**

$$\rho_{pq} = r_Q \quad OQ = |r_Q| \quad OP = d$$

$$\rho_{pq} = PQ \quad OR = a^2/d$$
(Physically, this $\psi_2$ describes the potential in some point $Q$, outside $S_H$, due to a unit point source in $P$, opposite to a grounded conducting sphere of radius $a$, inducing in $R$ ($r_R = \frac{a^2}{d}$, $r_P$) an image source of $-\frac{a}{d}$.)

On $S_H$, $\psi_2 = 0$. For any point $P$ inside $V_2$, $R$ is always inside $S_H$.

Thus, $\nabla^2 \psi_2 (P) = -4K \delta (r - r_P)$.

Consider a point $Q$ on $S_\infty$ to evaluate $
\int_{S_\infty} \left( \frac{\partial \psi_2}{\partial n_\infty} - \psi_2 \frac{\partial \Phi}{\partial n_\infty} \right) dS_\infty$

For sufficiently large radius $r$ of $S_\infty$, we may state, that

$\Phi (Q)$ is proportional to $r^{-a}$, $a \geq 2$) and $\psi_2 (Q)$ is proportional to $r^{-1}$.

This causes the integrand to be proportional to $r^{-(a+2)}$, $a \geq 2$.

$S_\infty$ being $4K r^2$, the integral will vanish, if $r$ increases beyond all bounds.

For any point $P$ inside $V_2$, we already saw, that

$\nabla^2 \psi_2 (P) = -4K \delta (r - r_P)$, and $\nabla^2 \Phi (P) = 0$.

Consequently, $\iiint_{V_2} \left( \Phi \nabla^2 \psi_2 - \psi_2 \nabla^2 \Phi \right) dv_2 = -4K \Phi_\infty (P)$.

(unbounded medium potential, because the outer bound $S_\infty$ of $V_2$ is at an infinitely large distance).

For any point $Q$, very close to $S_H$, $\psi_2 (Q) = 0$, which reduces (1a), p.18 to

$\Phi_\infty (P) = \frac{1}{4K} \iiint_{S_H} \Phi_H (Q) \frac{\partial \psi_2}{\partial n_H} dS_H = \text{ (using a similar expression for } \frac{\partial \psi_2}{\partial n_H} \text{ as before) }$

$\Phi_\infty (P) = \frac{1}{4K} \iiint_{S_H} \Phi_H (Q) \left( \frac{r_Q - r_P}{r_Q - r_P} \right) \cdot \frac{n_H - \frac{a}{d} \cdot \left( \frac{r_Q - r_P}{r_Q - r_P} \right)}{r_Q - r_P} dS_H$ (3)

*) This is the mathematical reason, for which we approximated the heart surface by a sphere.

**M** Since monopoles (point current sources) would load the body continuously, causing a disaster, we will not take such sources into account.
If $Q$ approaches $S_H \ (r_Q = a)$, $\Phi_H(Q)$ is the **unbounded medium potential** of epicardial point $Q$, $Q$ being the centre of the (infinitesimally small) surface element $dS_H$. 
II.2 The relation between the unbounded medium and the bounded medium torso skin potentials.

We evaluate (2) for some "observed" point Q on the torso surface:

\[ \phi_{\infty}(P) = \phi_s(P) - \frac{1}{4\pi} \oint_{S_T} \phi_s(Q) \left( \frac{r_Q - r_P}{|r_Q - r_P|^3} \cdot n \right) ds_T \]  

(2a)

cf. fig. 6. \( e_{PQ} = |r_Q - r_P| \), the distance between P and Q. 

|\( n \)| being 1, we may write the dot product as:

\[ \frac{\cos \theta}{e_{PQ}^2} \]

Consider a surface element \( ds_T \) around Q. This \( ds_T \) subtends a solid angle \( d\Omega_P \) at P, 

\[ d\Omega_P = ds_T \cdot \frac{\cos \theta}{e_{PQ}^2} \]

Using these results, the relation between the bounded medium and the unbounded medium skin potentials is:

\[ \phi_{\infty}(P) = \phi_s(P) - \frac{1}{4\pi} \oint_{S_T} \phi_s(Q) \frac{\cos \theta}{e_{PQ}^2} ds_T = \]

\[ = \phi_s(P) - \frac{1}{4\pi} \oint_{S_T} \phi_s(Q) d\Omega_P \]  

(4)
11.3 The relation between the unbounded medium skin and epicardial potentials.

For some point Q, near the epicardial surface \( S_H \) (\( S_H \) being a sphere) we evaluate (3)

\[
\Phi_\infty(P) = \frac{1}{4\pi} \int_{S_H} \Phi_H(q) \left( \frac{(r_Q - r_P) \cdot n_H - \frac{a}{d}}{|r_Q - r_P|^3}, \frac{(r_Q - r_R) \cdot n_H }{|r_Q - r_R|^3} \right) dS_H \quad (3)
\]

Like in Section 11.2, we may rework these dot products.

Denoting \(|r_Q - r_P|\) by \( e_{PQ} \) and \(|r_Q - r_R|\) by \( e_{RQ} \), and using the cosine rule we can calculate these distances from:

\[
\begin{align*}
e_{PQ}^2 &= |r_Q|^2 + d^2 - 2|r_Q|^2 \cos \gamma, \\
e_{RQ}^2 &= |r_Q|^2 + a^2 - 2|r_Q|^2 \frac{a^2}{d^2} \cos \gamma; \quad \gamma = \angle POQ
\end{align*}
\]

Substitution of these results in (3) leads to:

\[
\Phi_\infty(P) = \frac{1}{4\pi} \int_{S_H} \Phi_H(q) \left( \frac{d \cos \gamma - |r_Q|^2 - \frac{a^2}{d^2} \cos \gamma - |r_Q|^2}{e_{PQ}^2}, \frac{a^2}{d^2} \right) dS_H, \quad \text{or}
\]

taking the terms together and substituting \(|r_Q| = a\):

\[
\Phi_\infty(P) = \frac{1}{4\pi} \int_{S_H} \Phi_H(q) \left( \frac{d \cos \gamma - a^2}{e_{PQ}^2} \right) dS_H \quad (6)
\]

We may also write (3) as an integral over solid angles:

The solid angle \( d\Omega_P \), subtended at \( P \) by surface element \( dS_H \) (around \( Q \) on \( S_H \)) is:

\[
d\Omega_P = dS_H \cdot \cos(\angle QOP) \quad \text{and}
\]

Likewise, the solid angle at \( R \) is:

\[
d\Omega_R = dS_H \cdot \cos(\angle QOR) \quad \text{with}
\]

Thus, (3) may also be written as:

\[
\Phi_\infty(P) = \frac{1}{4\pi} \frac{a}{d} \int_{S_H} \Phi_H(q) d\Omega_R - \frac{1}{4\pi} \int_{S_H} \Phi_H(q) d\Omega_P \quad (7)
\]

\( * \) Since \( S_H \) is a sphere, the direction of \( QQ(Q) \) is the same as the direction of \( n_H \) in \( Q \).
II.4 The equivalent double layer model.

Instead of calculating epicardial potentials, often a so-called "equivalent double layer" model is used in electrical heart activity studies. Such a double layer consists of a current dipole density, which may be present on the epicardial surface. The relation between the unbounded medium potential $\Phi_\infty$ and the epicardial double layer strength is to be found.

Consider an (infinitesimally small) area element $dS_H$ (of the epicardial surface), around $Q$, on which the double layer strength is $\tau_H(Q)$. This area element acts as a dipole of amplitude $\tau_H(Q) dS_H$, causing a potential in $P$ (unbounded medium): $d\Phi_\infty(P)$. Cf. fig. 7.

![Diagram](image)

**Fig. 7.** Situation of points and vectors to evaluate a relation between the epicardial double layer strength in $Q$ and the unbounded medium potential in $P$.

$(a = \text{radius of heart surrounding sphere}, d = \text{radius of torso point } P)$. 

$n$ giving the orientation of the dipole, being the normal on $S_H$ in $Q$ itself, $d\Phi_\infty(P) = \tau_H(Q) dS_H \cos \gamma_{PQ}$. 

Integration of this formula over $S_H$ gives the relation required: 

$$\Phi_\infty(P) = \oint_{S_H} \tau_H(Q) \cos \gamma_{PQ} dS_H,$$

which is equal to (Cf. p. 22):
\[ \Phi_\infty (P) = \int_{S_H} \tau_H(q) \, d\Omega_P \]  

(8)

d\Omega_P \text{ is the solid angle subtended at } P \text{ by the surface element } dS_H. \]
Chapter III. Discretization of the integral equations, Solving methods and solutions.

III.1 Discretization, first step.

We would like to have a set of linear relations between the epicardial potentials \( \Phi_B \), and the measurable bounded medium skin potentials \( \Phi_s \). Starting from the integral equations (4), (6), (7) and (8) it looks practical to find relations between

1. the unbounded medium potentials \( \Phi_\infty \) and \( \Phi_s \);
2. \( \Phi_\infty \) and \( \Phi_B \) (the latter being of unbounded medium form, too);
3. \( \Phi_\infty \) and \( \tau_H \) (we are interested in estimations of the latter, too).

Since, however, the point distribution method, as described in Chapter I, discretizes surfaces into triangles, and all integral equations are dependent upon area elements \( dS_B \) or \( dS_T \), we firstly discretize these equations as : (definitions : see below).

\[
\Phi_\infty(P_i) = \frac{d^2 - a^2}{4\pi a} \sum_{j=1}^{AKD} \frac{\Phi_B(Q_j)}{e_{ij}} \cdot \Delta S_{Hj} \quad (6a)
\]

\[
\Phi_\infty(P_i) = \sum_{j=1}^{AKD} \tau_H(Q_j) \cdot \cos \theta_{ij} \cdot \Delta S_{Hj} \quad (8a)
\]

\[
\Phi_\infty(P_i) = \Phi_s(P_i) - \frac{1}{4\pi} \sum_{j=1}^{AGD} \frac{\Phi_s(Q_j) \cdot \cos \gamma_{ij}}{e_{ij}} \cdot \Delta S_{Tj} \quad (4a)
\]

or, in Solid Angle form :

\[
\Phi_\infty(P_i) = \frac{1}{4\pi} \sum_{j=1}^{AKD} \left\{ \frac{\Delta \Omega_{P_i j}}{d_i} - \Delta \Omega_{Q_j} \right\} \Phi_H(Q_j) \quad (7b)
\]

\[
\Phi_\infty(P_i) = \sum_{j=1}^{AKD} \tau_H(Q_j) \cdot \Delta \Omega_{P_i j} \quad (8b)
\]

\[
\Phi_\infty(P_i) = \Phi_s(P_i) - \frac{1}{4\pi} \sum_{j=1}^{AGD} \Phi_s(Q_j) \cdot \Delta \Omega_{P_i j} \quad (4b)
\]
in which: \( P_i \) is the \( i \)th torso point
\( Q_j \) is the centre of the \( j \)th triangle
\( \Phi_H(Q_j) \) is the average potential of the \( j \)th heart triangle
\( \Phi_s(Q_j) \) idem, of the \( j \)th torso triangle
\( \tau_H(Q_j) \) is the average double layer strength of the \( j \)th heart triangle
\( \Delta S_{Hj} (\Delta S_{Tj}) \) is the area of the \( j \)th heart (torso) triangle
\( \Delta \Omega_{R_i} (\Delta \Omega_{P_i j}) \) is the solid angle, subtended at \( R_i \) (\( P_i \) ) by the \( j \)th triangle
\( d_i \) is the radius of \( P_i \)
\( e_{ij} \) is the distance between \( P_i \) and \( Q_j \)
\( \gamma_{ij} \) is the angle between \( P_i Q_j \) and the normal on \( S_T \) at \( Q_j \).

We define: "point potential" = potential, attributed to or measured at a vertex point;
"triangle potential" = average potential of a triangle, attributed to its centre.

The above equations (4a, b, 6a, 7b, and 8a, b) will be called to have the "triangle form", i.e. they describe the transfer from triangles to points.

We vectorize the \( \Phi_\infty(P_i) \) into \( \Phi_\infty \), the \( \Phi_s(Q_j) \) into \( \Phi_s \), the \( \Phi_H(Q_j) \) into \( \Phi_H \) and the \( \tau_H(Q_j) \) into \( \tau_H \).

Using these vectors, the above equations may be rewritten as a matrix vector multiplication:
\[
\Phi_\infty = D_\Delta \Phi_{H\Delta} \quad (6a, 7b), \text{ in which } D_{\Delta ij} = \frac{1}{4\pi} \left\{ \frac{a_i}{d_i} \cdot \Delta \Omega_{R_i j} - \Delta \Omega_{P_i j} \right\}.
\]
\[
\Phi_s = C_\Delta \Phi_{H\Delta} \quad (8a, 8b), \text{ in which } C_{\Delta ij} = \Delta \Omega_{P_i j}.
\]
\[
\Phi_\infty = B_\Delta \Phi_s \quad (4a, 4b).
\]

An explicit formula for \( B_{\Delta ij} \) will be derived in Section III.3, due to the impossibility of calculating \( \Delta \Omega_{P_i j} \), if \( P_i \) is a vertex of triangle \( j \).

\[\text{Note, that } Q_j \text{ may be the centre of a heart triangle (6a, 7b, 8a, 8b) or of a torso triangle (4a, 4b).}\]
\[\text{HM) } \Phi_\infty(P_i) \text{ is a point potential, the other ones being of "triangle" form.} \]
We would like to have relations between the measured - bounded medium - torso potentials $\Phi_s$ and the unbounded medium ones $\Phi_\infty$, instead of relations between bounded medium torso triangle potentials $\Phi_{s\Delta}$ and $\Phi_{\infty\Delta}$. We therefore have to transform the $B_\Delta$ matrix. Furthermore, we will transform the $C_\Delta$ and $D_\Delta$ matrix, too, since the number of epicardial triangles is about double the number of epicardial points.
III.2 The triangle-to-point transformation.

We have to find a relation between the potentials attributed to the centres of the triangles ("triangle potentials") and the potentials to be attributed to its vertices ("point potentials").

As a consequence, we have to make a retribution of the elements of the $C_\Delta$, $D_\Delta$, and $Z_\Delta$ matrices ($Z_\Delta$ is part of $B$).

Consider, therefore, triangle $ABC$, as shown in fig. 8.

![Fig. 8](image)

Let the local potential in the triangle be:

$$\Phi = \frac{p}{a} y,$$

being a linear function of $y$.

The triangle's mean potential is:

$$\frac{1}{A_\Delta} \int \int_{\Delta_{ABC}} \frac{p}{a} y \, dx \, dy = \frac{2p}{a} \int_{a}^{b} y \, dy \int_{0}^{c-a} dx = \frac{1}{3} p = \frac{1}{3} \Phi(A).$$

$A_\Delta$ = area of triangle $ABC$.

By applying the superposition theorem (because this "system" is supposed to be linear), the average potential of the triangle is:

$$\Phi_\Delta = \frac{1}{3} \left( \Phi(A) + \Phi(B) + \Phi(C) \right),$$

which will be defined as the potential of the triangle's centre.

Note, that the potential has been linearly interpolated between the potentials of the vertices, or, in other words, we assumed the potential gradient to be a constant over the triangle.

We postulate, that we may derive the triangle potentials from vertex potentials by taking as the potential of a triangle the average potentials of its vertices.
We may write this procedure as a matrix-vector product:
\[ \Phi_A = T \Phi \] (\( T \) is transformation matrix, 
\( \Phi_A \) is triangle potential vector 
\( \Phi \) is vertex potential vector).

\( T \) consists of elements \( T_{ij} \), being \( 1/3 \), if point \( j \) is vertex of triangle \( i \) and zero otherwise.

Take e.g. the vectorized form of (8b):
\[ \Phi_\infty = C_A T \vec{r}_H = C \vec{r}_H \]
(\( C = C_A T \), \( C \) and \( \vec{r}_H \) are of "point form").

Thus we haven't repartitioned the double layer strengths, but the
\[ C_{ij} = \Delta Q P_{ij} \]

Or, in integral form: we approximated \( \int \vec{r}_H dQ_p \) by
\[ \vec{r}_{HA} \Delta j \int_{A} dQ_p = \Delta Q P_{ij} \cdot \frac{1}{3} \cdot \left\{ \vec{r}_H(A) + \vec{r}_H(B) + \vec{r}_H(C) \right\} \]
taking the constant of \( 1/3 \) to \( \Delta Q P_{ij} \), instead of to the sum of \( r_H \)'s.

In the paragraphs and chapters to follow, this method will be referred to as the Triangle-to-Point Transformation (TPT).

Using the TPT, we transform the \( D_A \) and \( C_A \) matrices into "point-to-point matrices:
\[ \Phi_\infty = D_A \Phi_H = D_A T \Phi_H = D \Phi_H \] and
\[ \Phi_\infty = C_A \vec{r}_H = C_A T \vec{r}_H = C \vec{r}_H \], in which \( \Phi_H \) and \( \vec{r}_H \) are epicardial point potential and double layer strength vectors respectively.
III.3 The relation between unbounded and bounded medium torso skin potentials.

Generally, we are not able to measure unbounded medium skin potentials $\Phi_\infty$. We, therefore, have to process the measured skin potentials $\Phi_s$ (bounded medium potentials) into the unbounded medium form $\Phi_\infty$.

The discretized relation between some $\Phi_\infty(P_i)$ and all $\Phi_s(Q^x_j)$, $P_i$ being situated just inside the torso, and $Q^x_j$ being the centre of torso skin triangle $j$ is ("triangle form"):

$$\Phi_\infty(P_i) = \Phi_s(P_i) - \frac{1}{4\pi} \sum_{j=1}^{AGD} \Phi_s(Q^x_j) \cdot \Delta Q_{P_i j}$$

(4b)

Here, another inconvenience appears:

Since we assumed the $P_i$ to be situated just inside the torso, the sum of all solid angles $\Delta Q_{P_i j}$ subtended at any $P_i$ by the torso skin facets, should be $4\pi$. However, if we calculate these $\Delta Q_{P_i j}$ using as $P_i$'s the torso measurement points, it is impossible to compute those $\Delta Q_{P_i j}$, if $P_i$ is a vertex of triangle $j$, by which this solid angle at $P_i$ is subtended.

We, therefore, have followed this scheme:

After the computation of all $\Delta Q_{P_i j}$ calculable, giving a $Z_\Delta$ matrix (triangle form, $Z_{\Delta ij} = \Delta Q_{P_i j}$), we apply the TPT to this $Z_\Delta$, giving a $Z$ matrix (point form).

No solid angles at $P_i$ by triangles $j$, having $P_i$ as a vertex, being taken into account in this calculation, all $Z_{ii}$ terms will be zero. The sum of all solid angles at some $P_i$ being $4\pi$, we take as $Z_{ii}$ terms:

$$Z_{ii} = 4\pi - \sum'_{j=1}^{AGP} Z_{ij}$$

(5)

The $\Phi_s(P_i)$ term in (4b) resulting in an identity matrix, the $B$ matrix, giving the relation between the $\Phi_\infty$ and the $\Phi_s$ (point form) is to be calculated from:

$$B = I - \frac{1}{4\pi} Z, \text{ or: } \Phi_\infty = (I - \frac{1}{4\pi} Z) \Phi_s = B \Phi_s.$$
The sum of the elements of a row of \( \frac{1}{4K} Z \) being equal to 1, implicitly, we see, that the row sums of 
\[ B = (I - \frac{1}{4K} Z) \] 
are zero. Consequently, B is a singular matrix, and cannot be inverted conventionally.

(zero row sums means : B has a zero eigenvalue, the corresponding eigenvector is \( e; e^T = (1, 1, ..., 1) \)).
III.4 The "deflation" technique.

The inconvenience of matrices being singular, i.e. matrices having a zero eigenvalue, may be overcome by applying the "deflation" technique [5].

We know, that with our present ECG registration method a reference point, henceforth called iref, on the torso is defined to be at zero potential. (\( \Phi_{iref} = 0 \)).

Thus, all terms (\( B_{i,iref} \times \Phi_{s,iref} \)) being zero, we may freely add something to all \( B_{i,iref} \).

Defining : \( Z_c = \frac{1}{n} Z \), we may write (cf. III.3, p. 30):
\[ B = I - Z_c, \text{ and } Z_c e = e \quad (e^T = 1, 1, \ldots, 1). \]

If we change the unit eigenvalue, (e.g. into a zero one), without affecting the other eigenvalues (all eigenvectors remain unchanged), \( B \) would be invertible.

To achieve this, we take a vector \( p \), such that \( e^T \cdot e = 1 \), and define a matrix \( Y(p) = Z_c - (e \cdot p^T) \).

Using this \( Y(p) \), we see, that \( Y(p) \cdot e = Z_c \cdot e - (e^T \cdot e) \cdot e = e - e = 0 \), and, that, if \( v = Z_c \cdot w, Y(p) \cdot w = v - (e^T \cdot w) \cdot e \).

We, therefore, would like \( (p^T \cdot w) \) to be zero. \(^{*}\)

The consequences of this choice are, replacing \( w \) by \( \Phi_s(p) \):

\( e.g. \) - Make \( p = e_k \) (the \( k \)th unit vector). Then \( p^T \cdot \Phi_s(p) \) implies, that \( \Phi_s(p) \) of the \( k \)th point be zero, or,
- If \( \Phi_s,iref \) is defined to be zero, \( p = e_{iref} \) is a good choice.

- Make \( p = \frac{1}{n} \cdot e, n \) being the number of elements of \( p \).

Then \( p^T \cdot \Phi_s(p) \) should be zero, or, the sum of all elements of \( \Phi_s(p) \) should be zero, meaning in practice, that if the average value of the potential is defined to be zero, \( p = \frac{1}{n} \cdot e \) is a proper choice.

If we make \( B \) such, that \( B = I - Z_c + (e \cdot p^T) \), in which \( p = e_{iref} \), \( B \) will be non-singular, and may be inverted.

\(^{*}\) \( w \) is to be compared with \( \Phi_s, v \) with a part of \( \Phi_{\infty} \).
The row sums \( K \) of the \( C \) matrix are zero, too, due to:

the sum of the solid angles \( \sum_{i} A_{ij} \) is zero, since the \( P_i \) are situated outside the closed heart surface; which means physically, that a uniform double layer on a closed surface is not detectable outside that surface.

Thus, \( \Phi_{\infty 1} = C \tau_{H1} \) and \( \Phi_{\infty 2} = C \tau_{H2} \) are equal, if \( (\tau_{H1} - \tau_{H2}) \) is proportional to \( e \).

Consequently, inversion \(^{\text{MN}}\) would lead to solutions \( \tau_{H} \) containing unknown "constant" parts (i.e. proportional to \( e \)).

Since we want to know \( \tau_{H} \) on a number of instants, the changes of the unknown constant vectors could completely blur the solution wanted.

For this reason we applied the deflation technique to \( C \), too.

We may e.g. assume, that the top point of the heart (top of septum between the atria) be relatively inactive during the QRS complex, which means, that its \( \tau_{H} \) may be defined as zero. Accordingly, we take for a \( p \) vector the corresponding unit vector, and add \( (e \cdot p^T) \) to \( C \), or, in practice, calling the point concerned : jref, we add 1 to all \( C_i,j \).

A different assumption may be, that the average double layer strength be zero. We, here, take as a \( p \) vector : \( p = \frac{1}{n} \cdot e \), \( n \) being the number of elements of \( P \), for \( C : n = APK \), and add \( (e \cdot p^T) \) to \( C \).

In practice, \( \frac{1}{n} = \frac{1}{APK} \) is added to all elements of \( C \).

Note : Henceforth, the \( B \) and \( C \) matrices will supposed to be deflated!

\(^{\text{M}}\) row sum = sum of all elements of a row.

\(^{\text{MN}}\) Note, that, \( C \) being rectangular, normal inversion is not possible.

Pseudo-inversion, however, is. Cf. section III.6, pp. 36-38
III.5 The problem of the non-measurable extremities.

We now have a complete system of equations between the unbounded medium skin potentials $\Phi_\infty$ and:
- the bounded medium skin potentials $\Phi_s$;
- the (unbounded medium) epicardial potentials $\Phi_H$;
- the (unbounded medium) epicardial equivalent double layer strengths $L_H$.

Generally, $AGP$ (the number of skin points) is unequal to (and always greater than) $AKP$ (the number of heart points) due to:
1. some torso points cannot be measured at (extremities);
2. from the point of view of information transfer, $AKP$ should be less than, or equal to, the number of torso points.

Firstly, the problem of the non-measurable extremities will be treated:
Interpolation of the unmeasurable point potentials from neighbouring point potentials would introduce dependencies, and would not add any information.

Since we are not able to get the information required, we proceed otherwise:
Suppose, we have got the right $\Phi_H$, inducing on the skin a $\Phi_s = B^{-1}D \Phi_H$, whether it be measurable or not.
If we "scratch" from the $\Phi_s$ vector the unmeasurable elements, leaving $\Phi_{sR}$ (reduced $\Phi_s$, number of elements : RED), we obtain new sets of linear equations:

$$\Phi_{sR} = A_R \Phi_H \quad \text{and} \quad \Phi_{sR} = E_R L_H \quad (9a)$$

In this way, we have "scratched" from $A$ and $E$ the rows, corresponding to the $\Phi_s$ eliminated, the $\Phi_H$ being unaffected, still.

The fact, that RED will not be equal to AKP, generally, causes another problem:
Neither $A_R = (B^{-1}D)_R$ nor $E_R = (B^{-1}C)_R$ can be inverted ordinarily. This problem will be dealt with in the next section.

Note: $A_R$ and $E_R$ will henceforth be denoted by $A$ and $E$, as well as $\Phi_{sR}$ by $\Phi_s$.

If no reduction was required, RED equals AGP.
III.6 Matrix Pseudo-Inversion.

Since, for reasons already described, A (or \( A_R \)) isn't invertible ordinarily, we would like to find a \( \Phi_H \) vector such, that:

\[
\left| A \Phi_H - \Phi_s \right|^2 \text{ be a minimum (least squares criterion as a function of the } \Phi_H \).
\]

Thus, \[
\frac{\partial}{\partial \Phi_H} \left[ (A \Phi_H - \Phi_s)^T (A \Phi_H - \Phi_s) \right] \text{ should be equal to zero,}
\]

or: \[
0 = \frac{\partial}{\partial \Phi_H} \left[ \Phi_H^T A^T A \Phi_H - 2 \Phi_H^T A^T \Phi_s + \Phi_s^T \Phi_s \right],
\]

leading to: \[
0 = 2A^T A \Phi_H - 2A^T \Phi_s.
\]

We would calculate the heart potentials from \( \Phi_H = \Phi_s \), \( X \) being the so-called "pseudo-inverse" (denoted by \( X = A^+ \)).

We therefore would like to find an \( X \), such that:

\[
A^T \Phi_X - A^T \Phi_s = 0.
\]

If \( A \), being a \((m \times n)\) matrix, is of rank \( r \) ( \( \min (m,n) \geq r > 0 \)),

\( A \) may be factorized as:

\[
A = BC,
\]

\( B \) and \( C \) being of rank \( r \); \( B \) is a \((m \times r)\), \( C \) a \((r \times n)\)-matrix.

The "Penrose-Moore pseudo-inverse" \( A^+ \) of \( A = BC \) is defined by:

\[
A^+ = C^T (C C^T)^{-1} (B^T B)^{-1} B^T.
\]

Two ways of calculating \( A^+ \) have been used:

1. The procedure "pseudo", programmed by ir. J.A. Blom, using the above expression.

2. The "singular value decomposition" \([10]\).

As some details of the calculation of matrix pseudo-inverses may be clearly seen from the latter method, it will be treated in short.

The "Singular Value Decomposition" factorizes a matrix \( A \) into:

\[
A = U \Lambda V^T,
\]

in which

\[\text{\( \times \)} \] If \( A^T A \) is non-singular, it easily may be seen, that \( X \) is equal to: \( X = (A^T A)^{-1} A^T \)

\[\text{\( \times \times \times \)} \] Since \( C C^T \) and \( B^T B \) are both \((r \times r)\) matrices of rank \( r \), they may be inverted ordinarily.

\[\text{\( \times \times \times \times \times \times \)} \] ir. J.A. Blom is with the group THE/ER.
A is a diagonal matrix, whose elements \( \Lambda_{ii} \) are the square roots of the eigenvalues of \((A^T A)\) or \((AA^T)\), occasional zero ones included. 

\( V \) consists of the corresponding orthonormalized eigenvectors of \( A^T A \).

\( U \) idem, those of \( AA^T \).

Consequently, \( U^T U = I \) and \( V^T V = I \).

A's pseudo-inverse \( X \) is found by "inverting" \( \Lambda \) as:

\[
\Lambda^+_ii = (\Lambda_{ii})^{-1} \text{ if } \Lambda_{ii} \text{ is significantly unequal to zero,} \\
= 0, \text{ otherwise.} \\
\Lambda^+_ij (i \neq j) = 0.
\]

\( X \) is to be calculated from: \( X = V \Lambda^+ U^T \).

Both pseudo-inverting procedures have been tested, and found equal (the numerical precision of the computer being regarded).

Note, that it is important to take into account the numerical precision of the computer used.

For example:

A's pseudo-inverse is found by inversion of all (diagonal) elements of \( \Lambda \). Suppose the numerical precision of the computer to be \( 10^{-11} \).

All \( \Lambda_{ii} \) being smaller than about \( 10^{-10} \) times the largest one, should then be considered insignificantly small. Inversion of these \( \Lambda_{ii} \) would result into very large numbers having no significance!

We would like \( A \) to be of maximum rank (i.e. the rank be equal to the number of columns).

Otherwise, \( A^T A \) may have some zero eigenvalues.

Be \( \mathbf{v} \) an eigenvector of \( A^T A \), corresponding to such a zero eigenvalue.

We, then, may write: (cf. p.36)

\[
A^T A \mathbf{\Phi}_H + A^T A \mathbf{v} - A^T \mathbf{\Phi}_S = 0.
\]

As a consequence, we may add \( \mathbf{v} \) to the solution \( \mathbf{\Phi}_H \), without affecting the result of this equation.

We, thus, are afraid, that, if \( A \) is not of maximum rank, the resulting \( \mathbf{\Phi}_H \) will contain unknown multiples of these \( \mathbf{v} \) vectors.

In this section we have silently supposed \( A \) to be reduced already.

Its pseudo-inverse \( X \) gives, consequently, the relations between the
\( \Phi_H \) wanted and the measurable skin potentials. (\( \Phi_{SR} \)).

\( E, \) from \( \Phi_s = E \frac{1}{\Phi_H} \), being of the reduced form, too, will be pseudo-inverted similarly, giving \( W (\frac{1}{\Phi_H} = E^+ \Phi_s = W \Phi_s) \).
III.7 Computations: Checks on matrices calculated.

Resultant estimations of $\tilde{\Phi}_H$ and $\tau_H$.

Starting from formulae (6a) and (8a), p. 25, using the TPT and the deflation technique, we calculated the C and D matrices ("direct formulae").

We applied the "solid angle formulae" (4b), (7b) and (8b), too, for comparison reasons.

Note, that the extra work to calculate the elements of C, when calculating the elements of D, is almost nihil, since:

either $\Delta \Omega_{p_{ij}}$ or $\Delta S_{Hj}$ and $Q_{1j}$ are readily calculated.

We have, therefore, calculated D and C in parallel.

We have to find some criteria to check the matrices calculated.

1. Concerning the solid angles only:

   a. The $P_i$ being outside the closed heart surface, the sum of all solid angles $\Delta \Omega_{p_{ij}}$, subtended at some $P_i$ by the heart facets $\Delta S_{Hj}$ should be zero ($\sum_{j} \Delta \Omega_{p_{ij}} = 0$), the sign of the $\Delta \Omega_{p_{ij}}$ depending upon the direction of the outward normal on $\Delta S_{Hj}$ and the $P_i Q_i$ vector.

   b. The $R_i$ being inside the heart surface, the sum of the $\Delta \Omega_{R_{ij}}$ should be $4 \pi$ ($\sum_{j} \Delta \Omega_{R_{ij}} = 4 \pi$).

Implementation of this check revealed, that the procedure used for the calculation of the solid angles was very accurate, the deviations slightly increasing with increasing number of facets (AKD), probably due to machine-made round-off errors.

The relative deviations were:

NH being 3 (AKD = 72): about $10^{-10}$, and
NH being 4 (AKD = 128): about $10^{-9}$.

2. Concerning both calculation methods:

   If we assume a uniform heart potential (or double layer) distribution, e.g. $\tilde{\Phi}_H = 1$ (or $\tau_H = 1$) on all heart points, then:

   $\tilde{\Phi}_\alpha(p) = \frac{1}{4 \pi} \left( \frac{a}{d} \cdot \oint \frac{d \Omega}{R} - \oint \frac{d \Omega}{p} \right) = \frac{1}{4 \pi} \left( \frac{a}{d} \cdot 4 \pi - 0 \right) = \frac{a}{d}$,  

   (7b)
or, \( \Phi_{\infty}(P) = \oint 1.d\Omega_P = 0 \quad \Rightarrow (8b) \)

In discretized form: \( \Phi_{\infty} = D \Phi_H \) or \( \Phi_{\infty} = C \mathbf{t}_H \), we choose as \( \Phi_H \) and \( \mathbf{t}_H \) vectors: \( \mathbf{a}, \mathbf{e}_T = (1, 1, \ldots, 1) \).

If the elements of D and C are calculated exactly,

\[
\sum_j D_{ij} \cdot l \quad \text{should be} \quad \frac{a}{d_i} \quad (d_i = \text{radius of torso point } P_i)
\]

\[
\sum_j C_{ij} \cdot l \quad \text{should be zero} \quad \Rightarrow (\text{II})\).
\]

Implementation of this row sum check revealed, that the "direct" method was very inaccurate. The "solid angle" method proved to be superior in this aspect.

The deviations of the "direct" method were always negative, and decreasing with increasing number of facets (AKD).

Typical deviations were:
- NH being 3 (AKD = 72) : about -8%,
- NH being 4 (AKD = 128) : about -5%.

The fact, that the direct method gives too small matrix elements, is probably caused by:

(cf. fig. 9 for a two-dimensional example)

![Fig. 9. Two-dimensional view of the situation of the spherical triangle with area \( S_\Theta \) and centre \( Q^\Theta \) and the plane triangle, with area \( S \), and centre \( Q^\pi \).]

\( PQ^\Theta < PQ^\pi \)

\( S_\Theta > S \)

\( ^{\text{II}} \) A uniform double layer strength on a closed surface inside a conducting body, is not detectable on the body's outer boundary, the potential inside that closed surface being a constant as well.

\( ^{\text{II}} \) Note, that these sums represent the sums of all elements of the \( i^{\text{th}} \) row of D and C: These will be referred to as "row sums".
1. the area of a plane triangle (as this method is calculating it) is smaller than the area of a spherical one, having the same vertices.

2. the distances $e_{ij}$ between the observation points $P_i$ and the centres $Q_j$ of the observed triangles are calculated too large, since the $Q_j$ are calculated as the centres of the above mentioned plane triangles, thus being positioned inside the heart surface. A finer discretization improves the approximation of the heart surrounding sphere, thus reducing deviations.

These results made us proceed by the solid angle method only, following this scheme:

- Calculate the $Z_\Delta$, $C_\Delta$, and $D_\Delta$ matrices (triangle form),
- Apply the triangle-to-point transformation (TPT), as described above,
- Apply the deflation technique to $B (= I - Z_C)$ and $C$,
- Invert $B$, and calculate $A (= B^{-1}D)$ and $E (= B^{-1}C)$,
- Reduce $A$ and $E$, giving $A_R$ and $E_R$,
- Pseudo-invert $A_R$, giving $X$, and $E_R$, giving $W$,
- Calculate $\Phi_H = X \Phi_{SR}$ and $I_H = W I_{SR}$.

As $\Phi_{SR}$, we used two sets of torso point potentials:

1. A set of recorded real ECG's from a normal healthy man (Cf. Annex 1).
2. A synthetic set, originating from the "string model", a stylized model of the electrical heart activity (Cf. e.g. [1] and Annex 5).

The field of this simulated source is evaluated by a time-varying series of multipoles. Generally, we have taken multipoles up to the $10^{th}$ order into account.

---

* We supposed the electrical occurrences inside the torso to be quasi-stationary, i.e. we neglected occasional inductive or capacitive effects on potential transfer.

Thus, the $\Phi_{SR}$ vectors, measured at, say, 100 instants, e.g. during the QRS complex, are supposed to be independent one from another. The calculated $\Phi_H$ and $I_H$ vectors on those instants then give an estimation i.a. of the course of the depolarization wave over the heart surface.
For both sets we took the same torso geometry data, to be able to distinguish method faults from noise-originated faults, because the synthetic set is noise-free, while the real set is more or less affected by noise or noiselike disturbances.

The resultant $\Phi_H$ and $\Upsilon_H$ had no resemblance, neither to the expected electrical activity of a real healthy heart, nor to the known epicardial potentials of the string model.

The disturbances $\Phi$) in the $\Phi_H$ and $\Upsilon_H$ looked stronger with real (noise-corrupted) signals than with synthetic ones.

Several calculations of $\Phi_H$ and $\Upsilon_H$, using the same set of skin potentials, but different values of $NH$ (the heart point grid parameter), revealed, that
- the double layer strength in its reference point ($j_{ref}$) was, correctly, about zero,
- the calculation method looked instable for the heart top and bottom points; especially the potentials of the right back side of the heart bottom showed up this instability,
- in case of a very coarse heart point distribution ($NH = 1, AKP = 6$) these disturbances were reduced to a large extent,
- the ranks of the A and E matrices were always two less than AKP ($A$ and $E$ are (RED x AKP) matrices, RED > AKP).

See for an example of the $\Phi_H$ calculated: Annex 2.

*) The deviations of the estimations of $\Phi_H$ (or $\Upsilon_H$) from the expected (real $\Phi_s$) or calculated (synthetic $\Phi_s$) values will be called disturbances, too.
III.8 Discussion.

Concerning the apparent instability of the calculation method:

Consider fig. 10, a schematic representation of the torso (T) and the heart (H) therein:

- The torso bottom points (B) are at a considerably larger distance from the heart than the precordial points (P).
- Moreover, the distance between the back side of the heart, foremost its right bottom, and the closest torso observation points is larger than the distance between the heart front side and the precordial (breast) area.

Suppose the heart potential distribution to be a complex one (as is indicated by the + and - signs).
Torso point potentials of the precordial area P will show up many details from the heart front potential distribution, mainly.
At greater distances, however, interference of opposed potentials on heart points close to one another, will cause:
- torso bottom potentials to have only little detail, and
- complex potential distributions on the heart back side to be badly observable on the torso (cf. Van der Kaa [10]).

Analogously to signals varying in time, using e.g. Fourier expansions,
we have quasi-stationary signals, varying in space, on subsequent instants - we may develop the potentials into a series of "spatial frequencies". Thus, complex or finely detailed potential distributions have large spatially high frequency*) components, while "smooth" distributions mainly consist of spatially low frequent components. Analogous to functions of time, we develop the potential transfer function (which is a function of place) into a series of spatial frequencies, too. Its characteristics are listed below, using well-known filter denominations, here, however, concerning spatial frequencies.
- transfer from heart front to precordial area : all pass filter;
- transfer from any heart point to torso bottom : low pass filter;
- transfer from heart (right) back to any torso point : low pass filter.

The inverse transfer function has following properties, because the product of the forward and the inverse transfer functions should be equal to 1.
- from precordial torso area to heart front : all pass filter;
- from torso bottom to any heart point : high pass filter;
- from any torso point to heart back : high pass filter.

The inverse transfer function is, thus, rather sensitive to SpHF disturbances, especially, in the case that these disturbances exceed the level of SpHF signals on the torso, due to the heart potential distribution.

We have to deal with two kinds of disturbances:
- noise and other disturbances, caused by electrical apparatus,
- "disturbances", caused by the transfer function : due to occasional electrode displacements, torso inhomogeneities etc., the transfer function is not calculated exactly. We may take deviations of the $A_{ij}$ from their right values into account as noiselike components of the $\Phi_s$, because $\Phi_s = A \Phi_H$.
- The potentials of the heart front side are mainly determined by the precordial torso potentials, the latter containing large SpHF components.

Since no extra amplification of SpHF signals occurs in this case,

*) Spatially high frequency will be denoted by SpHF.
and, if the SpHF components of noise are small, the $\Phi_H$ calculated of the front side of the heart are tolerably well estimated. This can be seen from numerical output data, only, because the scale of the images (Annex 2) is very large, due to the large "noise" signals on other points.

- The estimated potentials of the heart back side are determined by many torso potentials at rather large distances. The latter contain only small SpHF components, possibly being submerged in noiselike signals. Estimations of heart posterior potentials are, thus, intrinsically, poor due to the "amplification" of SpHF signals, whether noise or not.

The fact, that a coarse heart point distribution "reduces" the effects of noiselike signals, though a 6 point discretization is too coarse to be of practical use, proves, that the average of the evident disturbances is zero, because the highest possible spatial frequency is very low in such a case.\footnote{If we attribute all equator points (4.NH points) alternating a positive and a negative potential, the spatial frequency of this potential distribution is 2.NH. NH = 1 thus gives a very low maximal spatial frequency, omitting all SpHF effects.}

Concerning the rank of our transfer matrices always being 2 less than AKP, we found out, that this was caused by the TPT:

We already saw, that we may write the TPT as a matrix-matrix product:

$D = D_\Delta T$ and $C = C_\Delta T$, in which:

- $D$ and $C$ are (AGP x AKP) matrices (point form), $D_\Delta$ and $C_\Delta$ are (AGP x AKD) matrices (triangle form).
- T, the transformation matrix is a (AKD x AKP) matrix, and consists of elements $T_{ij}$, being 1/3 if point j is vertex of triangle i and 0 otherwise.

We are afraid of introducing two independent "solutions", being absent in the $\Phi_0$, since $D_\Delta T \Phi_0$, if $\Phi_0$ is such a solution, is equal to zero.

We will explain this, using NH = 1 (8 triangles, 6 points), cf. fig.11 (p. 46).
Fig. 11. Stereometric and projected image of octahedron, NH = 1.

We choose three potentials (□, ○ and Δ), such that □ + ○ + Δ = 0. Thus, two potentials may be chosen freely (e.g. □ and ○). We are able, now, to attribute all 6 points such potentials, that all average triangle potentials (1/3 of sum of vertex potentials) are zero.

We, thus, have created (for any pair of potentials □, ○) "solutions" $\Phi_H'$ independent of the skin potentials.

We may prove this once more by reducing T explicitly.

The T matrix is in this case:

(multiplied by 3, for practical reasons)

<table>
<thead>
<tr>
<th>△</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>II</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>III</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>IV</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>V</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>VI</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>VII</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>VIII</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Making T's column vectors independent from one another, two zero column vectors will appear.

Thus, T's rank is 4.
We reduced several T matrices, with different values of NH, showing, that T always had two dependent column vectors. This may also be seen from the fact, that, for any NH, it is possible to attribute potentials □, ○ or Δ to all points, such that the sum of the vertex potentials of any triangle = 0 (□ + ○ + Δ = 0). This may be seen on one face of the octahedron.

Suppose, this property to be proved for some value of NH, say n. From fig. 12 may now easily be proved, that it is possible to attribute the points on line (n + 1) such potentials, that all new triangles have average zero potentials, the fact, that □, ○ and Δ appear always cyclically, playing an important role.

![Fig. 12. Part of face of octahedron, showing, that it is possible to attribute to the vertices such potentials, that all new triangles have zero average potentials like the "old" ones.](image)

As was stated in Section III.6, p. 37, it is important to take into account the numerical precision of the computer used in pseudo-inversion procedures. No column vector of the matrix to be inverted will be completely dependent on other vectors due to round-off errors. If this precision is not regarded, maximal rank will be found.

The resultant $\Phi_H$, however, will contain solutions of $A \Phi_H = 0$, as indicated above. We found these "solutions" to dominate over the solutions of $\Phi_S = A \Phi_H$, giving very "regular" potential patterns, because the sum of the vertex potentials of any triangle was about zero.
Chapter IV. The "seven point approximation" and its results.

IV.1 Introduction to the seven point approximation.

We have seen in Chapter III, that the triangle-to-point transformation (TPT) may cause some troubles, since

1. it causes the ranks of A and E to drop by 2 (introduction of interdependence);
2. it attributes $1/3$ of each $D_{\Delta ij}$ (or $C_{\Delta ij}$) to 3 $D_{ik}$'s (or $C_{ik}$'s), the points $k$ being vertices of triangle $j$.

It may be doubted, whether this fixed retribution constant of $1/3$ be a correct one.

Note, that the sum of the three retribution constants for any $D_{\Delta ij}$ should always be equal to 1.

The constant of $1/3$ has been derived by supposing a constant potential over the triangle, with a level, equal to the averaged potential of the three vertices, cf. fig. 13.

![Diagram of potentials and gradient planes](image)

Fig. 13. Potentials of vertices a, b and c of a triangle, and the difference between an average potential plane and a constant gradient plane.

Albrecht and Collatz [8] proposed a method to evaluate numerically two- and multi-dimensional integrals.
Integrals over an equilateral triangle, like \( \iint f(\mathbf{x}) \, dA \) are to be approximated by \( A \cdot \sum_{i=1}^{3} w_i \cdot f(\mathbf{x}_i) \), in which \( A \) is the area of equilateral triangle \( \Delta \), \( w_i \) are weighing factors, \( f(\mathbf{x}) \) is a function of the coordinate vector \( \mathbf{x} \), \( \mathbf{x}_i \) is the \( i \)th point, as will be indicated below. Following constants \( w_i \) and vectors \( \mathbf{x}_i \) should be chosen, the coordinates of the vertices being \( \mathbf{x}_1 \), \( \mathbf{x}_2 \) and \( \mathbf{x}_3 \) respectively. Cf. fig. 14.

Fig. 14. Situation of points \( \mathbf{x}_i \) in the triangle.

\[
\begin{align*}
W_1 &= W_2 = W_3 = (155 - \sqrt{15})/1200, \text{ denoted by } t, \\
W_4 &= W_5 = W_6 = (155 + \sqrt{15})/1200, \text{ denoted by } u, \\
W_7 &= 9/40, \text{ denoted by } w.
\end{align*}
\]

With \( r = (1 + \sqrt{15})/7 \) and \( s = (1 - \sqrt{15})/7 \), the \( \mathbf{x}_i \) are:

\[
\begin{align*}
\mathbf{x}_1 &= r \cdot \mathbf{e}_1 + (1-r) \cdot \mathbf{e}_0 \\
\mathbf{x}_2 &= r \cdot \mathbf{e}_2 + (1-r) \cdot \mathbf{e}_0 \\
\mathbf{x}_3 &= r \cdot \mathbf{e}_3 + (1-r) \cdot \mathbf{e}_0 \\
\mathbf{x}_4 &= -s \cdot \mathbf{e}_1 + (1+s) \cdot \mathbf{e}_0 \\
\mathbf{x}_5 &= -s \cdot \mathbf{e}_2 + (1+s) \cdot \mathbf{e}_0 \\
\mathbf{x}_6 &= -s \cdot \mathbf{e}_3 + (1+s) \cdot \mathbf{e}_0 \\
\mathbf{x}_7 &= \frac{1}{3}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3), \text{ being the triangles centre.}
\end{align*}
\]

Using this approximation method, we rewrite (6) as:

\[
\Phi_{\infty}(P) = \frac{d^2 - a^2}{4\pi a} \oint_{S_H} \oint_{\mathcal{E}_{PQ}} \frac{\Phi_H(Q)}{3} \, dS_H = \frac{d^2 - a^2}{4\pi a} \sum_{j=1}^{AKD} \oint_{\mathcal{E}_{PQ}} \frac{\Phi_H(Q_j)}{3} \, dS_{Hj},
\]
approximated by:
\[ \Phi_{\infty}(P_1) = \sum_{j=1}^{\infty} A_j \left\{ \sum_{k=1}^{3} \frac{\Phi_H(Q_{jk})}{e_{ijk}} \right\} \]

in which:
- \( A_j \) is the area of triangle \( j \),
- \( Q_{jk} \) is point \( k \) of triangle \( j \),
- \( e_{ijk} \) is the distance between \( P_i \) and \( Q_{jk} \),
- \( \frac{\Phi_H(Q_{jk})}{e_{ijk}} \) is the \( f(\lambda_k) \).

In order to evaluate a relation between the potentials in the 7 points and the vertex potentials, we have to make an assumption about the potential gradient.

We assumed this gradient to be a constant over the triangle, thus
\[ \Phi(r) = \Phi_0 + a^T \cdot (r - r_0), \quad \Phi_0 = \frac{1}{3} \left\{ \Phi(r_1) + \Phi(r_2) + \Phi(r_3) \right\} \]

being the centre potential \( \Phi_0 \),

\( a \), a constant vector, is to be calculated from:
\[ \begin{align*}
\Phi(r_1) &= \Phi_1 = a^T \cdot (r_1 - r_0) + \Phi_0 \\
\Phi(r_2) &= \Phi_2 = a^T \cdot (r_2 - r_0) + \Phi_0 \\
\Phi(r_3) &= \Phi_3 = a^T \cdot (r_3 - r_0) + \Phi_0
\end{align*} \]

Compare this method to the "direct method", described in Chapter III.

The contribution of heart triangle \( j \) to the potential in torso point \( P_i \) is derived to be (Cf. Chapter II, (6), p. 72):
\[ \Delta \Phi_{\infty}(P_i) = \left( \frac{d^2 - a^2}{4 \pi \alpha} \right) \sum_{j=1}^{\infty} \int_{\Delta j} \frac{\Phi_H(Q_j)}{e_{ij}} \ dS_{Hj}, \quad Q_j \text{ being the centre of the}
\]
surface element \( dS_{Hj} \).

The "direct method" approximates this integral by:
\[ \Delta \Phi_{\infty}(P_i) = \left( \frac{d^2 - a^2}{4 \pi \alpha} \right) \sum_{j=1}^{\infty} \int_{\Delta j} dS_{Hj} = \left( \frac{d^2 - a^2}{4 \pi \alpha} \right) \cdot \left( \frac{\Phi_H(Q_j)}{e_{ij}} \right) \cdot A_j, \]
in which:
- \( Q_j \) is the centre of triangle \( j \);
- \( \Phi_H(Q_j) \) is the average potential of triangle \( j \) (\( \Delta j \));
- \( e_{ij} \) is the distance between \( Q_j \) and \( P_i \);
- \( A_j \) is the area of triangle \( j \).
For the integration, \( \Phi_H(Q_j) \) is thus taken a constant over the triangle (Cf. fig. 13, p. 48).

The seven point method, however, approximates the integral above using the constant gradient assumption, or any other assumption, such that \( \Phi_H(Q_j) \) is not a constant ever triangle j - as:

\[
\Delta \Phi_\infty(P_1) = \left( \frac{d_1^2}{4K a} \right) A_j \cdot \sum_{k=1}^{7} \frac{\Phi_H(Q_{jk})}{\rho_{ijk}} \cdot w_k,
\]

being a closer approximation to the original integral expression.

We may reduce the three-dimensional coordinate vectors to two-dimensional ones, because only vectors in the (plane) triangle are concerned. Thus, for any point \( r \) in the triangle plane, we may write:

\[
r = r_o + \alpha \cdot v_1 + \beta \cdot v_2,
\]

\( \alpha \) and \( \beta \) being the "new coordinates", \( v_1 \) and \( v_2 \) being vectors in the triangle plane.

We chose:

\[
v_1 = \frac{1}{3} \cdot (r_1 - 2 \cdot r_2 + r_3) \quad \text{and} \quad v_2 = \frac{1}{3} \cdot (r_1 + r_2 - 2 \cdot r_3).
\]

Now, the \( (\alpha, \beta) \) coordinates of the \( r_1 \) are:

\[
r_1 = (1, 1), \quad r_2 = (-1, 0), \quad r_3 = (0, -1), \quad r_o = x_7 = (0, 0).
\]

Those of the \( x_1 \) are:

\[
x_1 = (r, r), \quad x_2 = (-r, 0), \quad x_3 = (0, -r), \quad x_4 = (-s, -s), \quad x_5 = (s, 0), \quad x_6 = (0, s).
\]

Likewise, we rewrite the potential equation:

\[
\Phi(r) = \Phi_o + \alpha \cdot \Psi_1 + \beta \cdot \Psi_2,
\]

eliminating simultaneously the problem, that, due to the \( x_1 \) being in one plane, only 2 of the equations (10) as to calculate a three-dimensional \( x_1 \), are independent one from another.

Substitution of two \( x_1 \) and their potentials gives the constants \( \Psi_1 \) and \( \Psi_2 \): \( \Psi_1 = \Phi_o - \Phi_2 \); \( \Psi_2 = \Phi_o - \Phi_3 \), leading to:

\[
\Phi(x_1) = r \cdot \Phi_1 + (1 - r) \cdot \Phi_0; \quad \Phi(x_4) = -s \cdot \Phi_1 + (1 + s) \cdot \Phi_0; \\
\Phi(x_2) = r \cdot \Phi_2 + (1 - r) \cdot \Phi_0; \quad \Phi(x_3) = -s \cdot \Phi_2 + (1 + s) \cdot \Phi_0; \\
\Phi(x_3) = r \cdot \Phi_3 + (1 - r) \cdot \Phi_0; \quad \Phi(x_6) = -s \cdot \Phi_3 + (1 + s) \cdot \Phi_0; \\
\Phi(x_7) = \Phi_o = (\Phi_1 + \Phi_2 + \Phi_3)/3,
\]

which gives the relations required between the \( \Phi(x_k) \) or \( \Phi(Q_{jk}) \), and the vertex potentials.
Note the exact resemblance to the formulae for the $x_i$ vectors, caused by the assumption of a constant potential gradient.

If these equations are written as a matrix equation, exactly the same matrix is used for the calculation of the $x_i$ from the $r_i$ and the $\Phi(x_i)$ from the $\Phi(r_i)$.

These $\Phi(x_i)$ are substituted into (6c), giving,

$$\Phi_{cm}(P_1) = \sum_{j=1}^{AKD} \frac{M_{ij}}{3} \left[ \begin{array}{c} \Phi_{hj1} \left\{ t \left( \frac{1}{3} + \frac{1}{3} - \frac{1}{3} \right) + u \left( \frac{1}{3} + \frac{1}{3} - \frac{1}{3} \right) \right\} \\
\Phi_{hj2} \left\{ t \left( \frac{1}{3} + \frac{1}{3} - \frac{1}{3} \right) + u \left( \frac{1}{3} + \frac{1}{3} - \frac{1}{3} \right) \right\} \\
\Phi_{hj3} \left\{ t \left( \frac{1}{3} + \frac{1}{3} - \frac{1}{3} \right) + u \left( \frac{1}{3} + \frac{1}{3} - \frac{1}{3} \right) \right\} \end{array} \right] \ (11)$$

This equation gives us the $D_{ij}$ wanted:

For instance,

$$D_{ij1} = \frac{M_{ij}}{3} \left\{ t \left( \frac{1}{3} + \frac{1}{3} - \frac{1}{3} \right) + u \left( \frac{1}{3} + \frac{1}{3} - \frac{1}{3} \right) \right\} + \frac{w}{\varepsilon i j 7} \ (11a)$$

$D_{ij}$, now, is the sum of all $D_{ijk}$, where $i$ is the $k$th vertex of triangle $j$ (there are 4 or 6 such $D_{ijk}$, giving together $D_{ij}$).

The $D$ matrix was calculated again, using this method.

Of course, we implemented the aforementioned checks (Cf. Chapter III, p. 39) into the computer program.

The "row sum check" revealed, however, these sums to be much too small, mainly in the precordial (breast area, where heart and torso skin are very close (AKP being 38, deviations were minus 10 - 20%).
These deviations most probably from the same causes as mentioned for the "direct method" (Cf. Chapter III, p. 41), i.e.
- \( E_{ijk} \) is calculated too large,
- \( A_j \) is calculated too small.

Moreover, the triangles are not equilateral, as was stipulated by Albrecht and Collatz ([8], cf. p. 49).

Several calculations showed, that, if the distance between the observation point \( P_i \) and a triangle is larger than any edge of that triangle, the deviations are small. Thus, the effects of non-equilaterality are probably small.

If a triangle is closer to the observation point, the seven point approximation is a bad one.

The C matrix was calculated, too. Its row sums proved to be rather unequal to zero (AKP being 38 : ranging up to about \( 10^{-2} \)).

The resultant \( \Phi_H \) and \( \Sigma_H \) showed some likeliness with the ones, calculated before (Cf. Annex 3): "instabilities" in the heart back and bottom points.

The heart points, being situated near the torso skin (precordial area) were calculated to have, relatively, small, but physiologically probable potentials/ double layer strengths.

The ranks of the \( A_R \) and \( E_R \) matrices proved to be maximal (AKP), as was expected.

Appendix 1 deals with a method, we tried out, to improve the seven point method by forcing the row sums of D and C to the respective correct values.
IV.2. Combination of the solid angle method and the seven point method.

We have seen, that the solid angle method has good properties with respect to accuracy. The seven point method, however, has better properties concerning the triangle-to-point transformation. Combination would therefore probably improve the estimation procedure.

We recalculated the B matrix ($\Phi_{\text{obs}} = B \Phi_{s}$), using the combined method, too. This time, however, we situated the observation points $P_i$ really just inside the torso by reducing the "observing" $P_i$’s radius by one thousandth. Thus, we eliminated the problems arising, if a solid angle, subtended at some $P_i$ by a triangle, of which that $P_i$ is a vertex, is to be calculated. Cf. fig. 15

![Diagram](image.png)

**Fig. 15.** Calculation of solid angle at $P_i$ by triangle of which $P_i$ is a vertex. Two-dimensional example. $\angle Y$ is not calculable, if $P_i$ and $P_i$-obs are identical points.

We started the matrix calculation of $C$, $D$ and $B$ by the solid angle method, using (4b), (7b) and (8b):

$$B_{\text{ij}} = \frac{1}{4\pi} \cdot \Delta \Omega \_{P_i^j}$$  
(4b)

$$C_{\text{ij}} = \frac{1}{4\pi} \cdot \Delta \Omega \_{P_i^j}$$  
(7b)

$$D_{\text{ij}} = \frac{1}{4\pi} \cdot \left\{ \frac{\mathbf{a} \cdot \Delta \Omega \_{P_i^j}}{d_i} - \Delta \Omega \_{P_i^j} \right\}$$  
(8b)

*) This $\Delta \Omega \_{P_i^j}$ is not equal to the ones for $C_{\text{ij}}$ and $D_{\text{ij}}$, since the "observed" triangle $j$ is situated in this case on the torso surface, and not on the epicardial surface.
We calculated these $C_{Aij}$, $D_{Aij}$ and $B_{Aij}$ by the seven point method, too, giving, for each $i$ and $j$; three pseudo-solid angles $\Delta \Psi_{ijk}$ ($k = 1, 2$ or $3$, denoting the vertex number of triangle $j$).

For $C_{Aij}$ and $D_{Aij}$: the "seven point equivalent" of $\Delta \Omega_{P_{ij}}$ is:

$$\sum_{k=1}^{3} \Delta \Psi_{ijk} = \sum_{k=1}^{3} \left( \sum_{k=1}^{3} \cos \angle_{P_{ijh}}. \Delta S_{Hj} \right) \frac{3}{\theta_{P_{ijh}}}$$

$\theta_{P_{ijh}}$ is the angle between $OQ_{j}h$ and $P_{ij}Q_{j}h$; and

$\theta_{P_{ijh}}$ is the length of $P_{ij}Q_{j}h$.

We are able to calculate "seven point equivalents" of $\Delta \Omega_{R_{ij}}$ in a similar way, and, for $B_{Aij}$: those of $\Delta \Omega_{P_{ij}}$ (Cf. note on p. 54).

The solid angle $\Delta \Omega_{P_{ij}}$ (or $\Delta \Omega_{R_{ij}}$), as it was calculated by the solid angle method, is assumed to be exact. We distribute the solid angle to the vertices $k$ of triangle $j$, according to the appropriate scaling factors as follows:

$$\Delta \Psi_{ijk} \cdot \Delta \Omega_{ij} \left( \sum_{k=1}^{3} \Delta \Psi_{ijk} \right), \text{ in which}$$

$\Delta \Omega_{ij}$ = either $\Delta \Omega_{P_{ij}}$ or $\Delta \Omega_{R_{ij}}$, as in (4b), (7b) and (8b).

The resultant C and D matrices were multiplied by $B^{-1}$, giving ordinarily the A and E matrices. After reduction and pseudo-inversion, the $\Phi_{H}$ and $\Upsilon_{H}$ were calculated.

These, however, were not very encouraging (Cf. Annex 4).

We still had to deal with the same instability for high spatial frequencies of the heart back potentials.

The instabilities looked somewhat less seriously than before, probably because of:

\[\text{B is completed in the following way: After the solid angle distribution we have a Z matrix, } B = I - Z/4\Upsilon. (\text{Cf. III.3 and III.4}).\]

B, and C too, are deflated.
- the $\Delta \Omega$'s being calculated exactly, giving appropriate row sums;
- the ranks of $A$ and $E$ being maximal;
- the repartition of the $\Delta \Omega$'s being improved.

Yet, we would like to have a procedure to reduce the sensitivity for spatially high frequency disturbances on "remote" torso points.
Chapter V. Reduction of the effects of spatially high frequency disturbances.

V.1 Extension of the optimization criterion.

Reduction of spatially high frequency signals means, that the spatial derivatives of $\Phi_H$ should be limited. Guided by an idea, launched at the 4th Congress on Electrocardiology, by Taccardi [11], we extended our optimization criterion (cf. p 36) with a term containing the second spatial derivative on a sphere, denoted by

$$\Delta_2 \Phi_H = \frac{1}{a^2} \left\{ \frac{\partial^2 \Phi_H}{\partial y^2} + \frac{1}{\tan \gamma} \frac{\partial \Phi_H}{\partial y} + \frac{1}{\sin^2 \gamma} \frac{\partial^2 \Phi_H}{\partial \gamma^2} \right\}$$

(12)

This is, in fact, a part of the three-dimensional second derivative:

$$\Delta_3 \Phi_H = \frac{\partial^2 \Phi_H}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi_H}{\partial r} + \Delta_2 \Phi_H$$

being zero on the heart surrounding sphere, since all sources are supposed to be contained within that sphere.

We made, at first, our new optimization criterion function:

$$\int_{S_T} \left| \Phi_s - \hat{\Phi}_s(\Phi_H) \right|^2 dS_T + \lambda \int_{S_H} \Delta_2 \Phi_H dS_H$$

(13a)

in which $\Phi_s$ is the real, measurable, skin potential on the torso surface $S_T$;

$\hat{\Phi}_s(\Phi_H)$ is the estimation of $\Phi_s$ from the $\Phi_H$ calculated, $\lambda$ is to be chosen appropriately.

This extension proved to be ineffective, because the latter term is always equal to zero! (Cf. Appendix 2)

Therefore, we changed it into:

$$\int_{S_T} \left| \Phi_s - \hat{\Phi}_s(\Phi_H) \right|^2 dS_T + \lambda \int_{S_H} \left| \Delta_2 \Phi_H \right|^2 dS_H$$

(13b).

In the poles ($\gamma = 0, \pi$ respectively), both $\tan \gamma$ and $\sin \gamma$ are zero.

To make an estimation of $\Delta_2 \Phi_H$ in the poles, we consider a function $f (\varphi, \gamma)$ in points very close to the poles.

- $\gamma$ being $0$ or $\pi$, the value of $f$ cannot depend upon $\gamma$.

- Consider a point $P$, very close to the north pole, its $\gamma$ being $\delta$, $\delta$ being that small, that $f$ doesn't depend upon $\varphi$. 


We may approximate \( f(\varphi, \delta) \) by:

\[
f(\varphi, 0) + \delta \frac{\partial f}{\partial \delta} \bigg|_{\delta = 0}
\]

Thus:

\[
\frac{\partial f}{\partial \varphi} \bigg|_{\varphi = 0} = 0 \quad \text{and, likewise} \quad \frac{\partial f}{\partial \delta} \bigg|_{\delta = \pi} = 0
\]

Consequently, in both poles, the above expression for \( A \Phi_H \) reduces to:

\[
A \Phi_H \text{ pole } = \frac{1}{r^2} \cdot \frac{\partial^2 \Phi_H}{\partial \varphi^2}
\]

We approximate the derivatives to be found in a simple way:

\[
\frac{\partial^2 \Phi_H}{\partial \varphi^2} \bigg|_{ij} \text{ is approximated by } (\Phi_H i-1,j - 2\Phi_H i,j + \Phi_H i+1,j)
\]

in which the \( i,j \) indices refer to "longitude" and "latitude" coordinates respectively. Cf. fig. 16.

\[
(i,j) \quad \text{is the point on which we would like to know the potential derivatives.}
\]

As to the first derivative with respect to \( \varphi \), there are two possibilities of approximation:

\[
\left. \frac{\partial \Phi_H}{\partial \varphi} \right|_{ij} \text{ is either } (\Phi_H i,j - \Phi_H i,j-1) \text{ or } (\Phi_H i,j+1 - \Phi_H i,j)
\]

We took therefore the average value:

\[
\left. \frac{\partial \Phi_H}{\partial \varphi} \right|_{ij} = \frac{1}{2} \cdot (\Phi_H i,j+1 - \Phi_H i,j-1)
\]

\[
\left. \frac{\partial^2 \Phi_H}{\partial \varphi^2} \right|_{ij} \text{ is approximated by } (\Phi_H i,j-1 - 2\Phi_H i,j + \Phi_H i,j+1)
\]

In order to make correct approximations of the derivatives with
respect to \( \phi \) and \( \varphi \), the points \((i,j-1), (i,j)\) and \((i,j+1)\) should have equal longitudes \( \varphi \) (derivatives with respect to \( \phi \)) or the points \((i-1,j), (i,j)\) and \((i+1,j)\) should have equal latitudes \( \phi \) (derivatives with respect to \( \varphi \)).

The point distribution used acts such, that for the derivatives with respect to \( \varphi \) any point has always "neighbours" at equal latitude \( \varphi \), for the derivatives with respect to \( \phi \), however, only points with \( \varphi = 0, \pi/2, \pi \) or \( 3\pi/2 \) have the required property. Cf. fig. 17.

Fig. 17. Top view of the sphere. Clearly is to be seen, that only points on the axes have upper and lower neighbours at equal latitude.

Concerning the other points, we have a point distribution like fig 18.

Fig. 18. Schematic view of the situation of points, not being on the axes for the approximation of the derivatives with respect to \( \varphi \).
We, therefore, take as $\Phi_{H,i,j-1}$ (or $\Phi_{H,i,j+1}$) the average of two potentials:

$\Phi_{H,i,j-1}$ is defined to be $\frac{1}{2} \cdot (\Phi_{H,i,j-1,1} + \Phi_{H,i,j-1,2})$, and

$\Phi_{H,i,j+1}$ is defined to be $\frac{1}{2} \cdot (\Phi_{H,i,j+1,1} + \Phi_{H,i,j+1,2})$.

We have to determine, yet, $\frac{\partial^2 \Phi_{H}}{\partial \nu^2}$ in the poles.

There are, closest to each pole, two pairs of points coming into consideration, cf. fig. 19.

Fig. 19. Top (or bottom) view of the sphere, showing the "north" (or the "south") pole and the 4 closest points.

We, again, take the average value:

$$\frac{\partial^2 \Phi_{H}}{\partial \nu^2} = \frac{1}{2} \cdot \left\{ (\Phi_{H1} + \Phi_{H3} - 2 \Phi_{H\text{pole}}) + (\Phi_{H2} + \Phi_{H4} - 2 \Phi_{H\text{pole}}) \right\}$$

By substituting all approximated derivatives in the $\Delta_2 \Phi_{H}$ formula (12), we approximated, in fact, $\Delta_2 \Phi_{H}$ by $F \Phi_{H}$, $F$ being a square matrix containing many zeroes, because the approximated $\Delta_2 \Phi_{H}$ in some point only depends upon the potential of 4 or 6 other points, and its own potential.

Discretization of the criterion function gives:

$$\left( \Phi_{s} - A \Phi_{H} \right)^T \left( \Phi_{s} - A \Phi_{H} \right) + \lambda \Phi_{H}^T F^T F \Phi_{H}$$

We denote $F^T F$ by $G$, and differentiate with respect to the $\Phi_{H}$, giving:

$$A^T A \Phi_{H} - A^T \Phi_{s} + \lambda G \Phi_{H},$$

which expression should be equal to zero, to attain minimum error.

Thus:

$$\Phi_{H} = \left[ A^T A + \lambda G \right]^{-1} A^T \Phi_{s} \quad (14)$$

Reduced form:

$$\Phi_{H} = \left[ A_R^T A_R + \lambda G \right]^{-1} (A_R)^T \Phi_{s_R} \quad (14a)$$
We haven't met any difficulty in the (normal) inversion of several \((A^T A + \lambda G)\) matrices. In case of singularity, the matrix should be pseudo-inverted.

\(\lambda\) should be given an appropriate value.

We attained the best results (best reduction of noise, relative to the probability of the calculated \(\Phi_H\) being good estimations (real potentials), or best resemblance of the calculated \(\Phi_H\) to their known values (synthetic potentials)) with such values of \(\lambda\), that neither \(\lambda G\) dominates over \(A^T A\), nor \(A^T A\) over \(\lambda G\).
V.2 Results of the extended criterion.

We adapted our computer program to the new formula for $\Phi_H$ (14 / 14a). With this program we calculated the $\Phi_H$, both from real torso potentials and from synthetic ones, using the same torso geometry set. The resulting $\Phi_H$ were encouraging (Cf. Annexes 5, 6 and 7). Especially $\lambda$ values of $10^{-2} - 10^{-3}$ reduced the "noiselike disturbances" to a great extent.

This is to be seen by comparing the $\Phi_H$, calculated from synthetic torso potentials, being not affected by noise (etc.), to those calculated from real ones.

Concerning $G$, we noted, that
- many $G_{ij}$'s were zero, due to the fact, that in the approximation of $\Delta_2 \Phi_H$, on some point, only 4 or 6 other point potentials were taken into account;
- the largest $G_{ij}$'s were found on the main diagonal.

The effects of this correction may be explained as follows:
As was stated before (III.8, pp. 43 - 45) there are, mainly spatially high frequency potential distributions possible on the heart back side, that are not detectable from torso skin potentials.
By limiting the (second) derivatives of $\Phi_H$, we reduce the number of possible and undetectable potential distributions. (We abandon the spatially high frequency solutions).
By the nature of the transfer function, low frequent disturbances have only little effect, since their level is much lower than the level of the spatially low frequent signals.
Note, that we probably loose some information, being rather strongly present in the precordial torso potentials, about spatially high frequency potential distributions on the heart front side.
By a proper choice of $\lambda$ we will probably reduce the disturbances in the heart back potentials to a great extent, without affecting the finely detailed "patterns" on the heart front side very much.
V.3 Another improvement method.

From the above explanations, and from the results of Van der Kam [10] we deduced, that we may divide our transfer matrix $A$ into two parts:
- $A_H$ being the transfer matrix for spatially high frequency heart potential distributions, and having small elements, very small for the transfer from heart back and bottom points and to the torso bottom.
- $A_L$ being the transfer matrix for spatially low frequency distributions, and having relatively large elements.

If $A_H$ could be eliminated, many problems of "disturbance-amplification" would be solved.

We, therefore, calculated $\Phi_H$ once more, ordinarily ($\Phi_H = (A_R)^* \Phi_{SR}$), using the pseudo-inverting procedure, now with a different "significance criterion" value $\varepsilon^H$.

The results, with $\varepsilon = 0.05$, causing the rank to be only 21 (instead of 38), looked much like those, described in V.2 (Cf. Annex 8).

If no noise is present, only machine-originated errors are to be taken into account. In the presence of noise, $\varepsilon$ should be given a value, according to the signal to noise ratio of the skin potentials.

---

$^H$) This procedure firstly converts the matrix to be inverted into one, having independent column vectors. If the ratio of the length of some column vector to the largest one is less than $\varepsilon$, the elements of the former are made zero. The number of independent column vectors is the rank of the matrix.

Formerly, $\varepsilon$ was taken about 100 times the machine precision (being $10^{-11}$). Now, $\varepsilon$ was taken 0.05, causing the rank to drop.
Chapter VI. Computer programs.

A number of computer programs has been written to calculate the matrices required and the $\Phi_H$ (and $\sum_H$). Use has been made of existing procedures (with the group THE/ER, or with the Computing Centre of the THE) for the point distribution, pseudo-inversion and for the plotting of results.

These programs have been written in a dialect of Algol 60, called "BEATHE" (Burroughs Extended Algol - THE), designed for use on the Burroughs' B7700 System, as it is installed in the Computing Centre of the THE.

The programs are available at the group THE/ER [14]. Further information required, concerning BEATHE and intrinsic procedures of this computer system is available at the above mentioned Computing Centre [12].

Calculation costs:
The complete, final, program needs about 5 minutes CPU to be executed by this fast and big computer system, the number of torso points being 66, the number of heart points: 38. These costs increase very sharply with increasing NO and NH, as does the core usage.

The programs are universally usable for any skin potential and torso geometry data set, if its torso point distribution corresponds to the one used, with an integer value for NO and NH.

---

$\Phi_H = \left( A^T A + \lambda I \right)^{-1} A^T \Phi_R$ was implemented in the program, instead of (14a) - with $A_R^T A_R$ - ; the effects of this error, however, are probably small.
Conclusions.

Unrestricted estimation of epicardial potentials, using a discretized integral equation, from skin potential recordings and torso geometry data only - not using any model, nor physiological knowledge of (healthy) hearts - gives 'bad results' (e.g. the course of the depolarization wave over the heart is not discernable), even if the potential transfer function, giving the relation between heart and torso potentials, is discretized and calculated as accurately as possible.

These bad results are due to the following facts:

1. The transfer function is such, that the more detailed the heart potential distribution is, the less it is observable on the skin. Especially fine details of the potential distribution on the heart back and bottom side are attenuated to a great extent. Estimation of such details from skin potentials therefore leads to an unstable algorithm.

2. Noise and noiselike signals, caused by electrode displacements, torso inhomogeneities etc. are added (being a finely detailed pattern!) to the real torso potential distribution. These noiselike signals may be stronger than the torso potentials from subtle heart potential details. This causes these details to be estimated completely wrong, even blurring coarser patterns.

The effects of these disturbances (being present in real torso potential recordings) may be reduced by:

a. putting a restraint upon the heart potentials to be calculated, i.e. only the less complex solutions should come out.

b. minimizing these noiselike signals themselves, e.g.
   - measurements or estimations of the geometry and the electrical properties of inhomogeneities (such as lungs, ribs etc.) inside the torso, being implemented into the calculation of the transfer matrix, to improve its accuracy.
   - a more exact estimation of the heart geometry.

Only a. was implemented, resulting in heart potentials being much less distorted by noiselike signals in the skin potentials. By limiting the "degree of complexity" we get estimations of heart potentials being "incomplete", since - with the noise-originated patterns - really existent detailed potential distributions are kept from the
solutions, too.

Estimations of "heart potentials" of a synthetic (known) source showed this method to be a good one, and method b. to correct to a relatively small extent. Only the transfer matrix will be more exact, the problem of the bad observability of the heart back and bottom sides will remain.

A further improvement would be to apply this restraint to those heart areas only (such as its back and bottom), in which disturbances are dominant over the really existent potentials.
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The numbers in the little squares are the numbers of the skin measurement points, of which the potential is given in that square, as a function of time. Note, that NO = 4.

We used the same projection method, as described on pp. 13-14.

Points 1 and 66 couldn't be measured at (thus RED = 64).

Point 13 was the reference point for the B matrix (cf. p. 32, III.4), thus having zero potential.
The signal on point 1 is just a reference signal for the ECG recorder, and is, of course, not used in the calculation.

Due to the thickness of the drawing pen, fine details (such as noise) are somewhat blurred in this picture, and in the other ones, too.
Annex 2. *Estimations of heart potentials from real skin potentials,*
    using the solid angle method, the TPT and pseudo-inversion.

The skin potentials used were as showed in Annex 1 (real potentials).
NH was taken 3, giving 38 heart points.
Cf. fig. 2c, p. 14 to locate these potentials in terms of atria and
ventricles. We see from this picture, that especially the heart bottom
potentials suffer from severe disturbances (points 23, 32, 37, 38 : right
de bottom side). If anything could be concluded from this picture,
noise and noiselike signals may said to be amplified.
Annex 3. *Estimations of heart potentials from real skin potentials*,

*using the seven point method and pseudo-inversion.*

NH was taken 3. We see, that the disturbances in the heart back and bottom points remained.

The heart front points, however, have - relative to the level of the disturbances - small, but "smoothly" elapsing potentials.
Annex 4. Estimations of heart potentials from real skin potentials, using the combined solid angle - seven point method and pseudo-inversion.

This picture is much like the one of Annex 3. Thus, though the theoretical properties of the combined method are better than those of the single ones, this improvement doesn't show up in practice.
Annex 5. Calculation of heart potentials directly from a stylised model, NH = 3.

Using this model, torso skin potentials were calculated, too, using the same set of torso geometry data as before (NH = 4, AGP = 66). From these skin potentials again heart potentials were calculated (Annex 6). Comparison of Annexes 5 and 6 will result into conclusions for the noisefree case, since no noise is added, and other error causes, such as inhomogeneities and electrode displacements, are absent.
Annex 6. **Estimation of heart potentials from synthetic torso potentials**, using the combined solid angle - seven point method and the 2\textsuperscript{nd} derivatives correction, with $\lambda = 10^{-3}$.

Comparison of this picture to Annex 5 gives, that instabilities have been overcome (noise-free case) and, that fine details have disappeared. Note: one shouldn't compare time functions, but neighbouring point potentials at equal instants!

The attenuation of subtle patterns was to be expected, since we limited the spatial derivatives.
Annex 7. Estimations of heart potentials from real skin potentials, using the combined solid angle - seven point method and the 2nd derivatives correction, with $\lambda = 10^{-3}$.

Though some noise is still present, the resultant potentials are physiologically probable: small potentials on the atria and great activity, above all, on the left ventricle.

Note, that, due to the limitation of spatial frequencies, no subtle patterns are calculated.
Annex 8. Estimations of heart potentials from real skin potentials, using the combined solid angle – seven point method and pseudo-inversion, with $\xi = 0.05$.

This picture looks much like the one of Annex 7, indicating, that both correction methods (the 2nd derivatives correction and pseudo-inversion with an $\xi$, adapted to the noise level) are equally good.
Appendix 1.

A failing attempt to improve the seven point method.

We know from the solid angle method, that each $D_{ij}$ consists principally of two solid angle terms (cf. (7b), p. 25):
- one, proportional to $\left( \frac{a}{d_i} \cdot \Delta \Omega_{R_i k} \right)$, and
- the other, proportional to $-\Delta \Omega_{P_j k}$ (point $j$ is vertex of triangle $k$).

Each $C_{ij}$ is, in principle, proportional to one solid angle $\Delta \Omega_{P_i k}$ (cf. (8b), p. 25).

Moreover, we know, that $\sum_k \Delta \Omega_{P_j k}$ should be equal to zero, and that $\sum_k \Delta \Omega_{R_i k}$ should be equal to $4 \pi$.

This way, the row sums of $C$ give the deviations in the $\Delta \Omega_{P_i k}$ - terms.

We make these row sum deviations equal to zero by subtracting from each of the $C_{ij}$, and adding to the $D_{ij}$:
\[ \frac{1}{AKF} \sum_{j=1}^{AKP} C_{ij} , \text{ giving new matrices } C_1 \text{ and } D_1. \]

The row sums of $D_1$ give the deviations of the $\Delta \Omega_{R_i k}$ - terms.

We make a new matrix $D_2$, with elements:
\[ D_{2ij} = D_{1ij} - \left( \frac{1}{AKF} \sum_{j=1}^{AKP} D_{1ij} \right) + \frac{a}{d_i} , \text{ thus forcing the row sums of } D_2 \]
to be $\frac{a}{d_i}$.

The resultant $\Phi_H$ and $\Theta_H$ were hardly better than the former ones.

Moreover, it may be doubted, whether this correction be, theoretically, a good one, since no distinction is made between the heart point close to the skin and the other ones, the former causing the largest deviations.

This correction was therefore abandoned.
Appendix 2. The fault of the extended criterion function.

In Chapter V, p.57, we stated an extended criterion function to reduce the effects of spatially high frequency disturbances (13a). We may write this formula, in discretized form, as:
\[ |\Phi_s - A \Phi_H| \leq + \lambda \sum_j (\Delta_2 \Phi_H)_j \], in which \( \sum_j \) denotes a sum over all heart points \( j \).

To get an impression of the effects of this extended criterion function, we approximated \( (\Delta_2 \Phi_H)_k \) by:
\[ \{(\Phi_{i-1,j} - 2\Phi_{i,j} + \Phi_{i+1,j}) + (\Phi_{i,j-1} - 2\Phi_{i,j} + \Phi_{i,j+1})\} \], in which the \( i,j \) indices refer to longitude and latitude coordinates respectively. \( (i,j) \) itself refers to point \( k \). Cf. fig. 20.

![Diagram](image)

\( \Phi \)

\( \bullet i,j-l \)

\( i-l,j \)

\( \bullet i,j \)

\( \bullet i+1,j \)

\( \bullet i,j+1 \)

Fig. 20. Situation of points to approximate the second derivatives of the heart potential.

All points are, thus, in turn, left-hand, right-hand, upper and lower neighbour to another point, or
\[ \lambda \sum_j (\Delta_2 \Phi_H)_j \] is approximated by \[ \lambda \sum_j \Phi_{Hj} \cdot (1 - 2 + 1 + 1 - 2 + 1) \]
which is always equal to zero!

Thus, this "correction method" will not correct, implicitly.

This can be explained, using a "smooth" function \( f(x) \).

Then, \[ \int f'(x) \, dx = 0 \] and \[ \int f''(x) \, dx = 0 \], too, since, at the "beginning" \( b \) and at the "end" \( e \) of the integration line, the \( f(b) \) equals \( f(e) \), and \( f'(b) \) equals \( f'(e) \).
Therefore, \( \iint_{S_H} (\Delta_2 \Phi_H) \, dS_H = 0 \), making this extra term of the error function useless.

This may be proved in the following way:

\[ \Delta_3 \Phi_H = 0 \] on the heart surrounding sphere, therefore \( \iint_{S_H} \Delta_3 \Phi_H \, dS_H = 0 \).

\( \iint_{S_H} \frac{\partial \Phi_H}{\partial r} \, dS_H = 0 \), since \( \frac{\partial \Phi_H}{\partial r} \) is proportional to the current density, normal to the (spherical) surface, and no net current should flow into, or out of, the heart, the average \( \frac{\partial \Phi_H}{\partial r} \bigg|_{S_H} \) should be zero.

\( \iint_{S_H} \frac{\partial^2 \Phi_H}{\partial r^2} \, dS_H \) is equal to zero, too, as can be shown by developing \( \Phi_H \) into a series of multipoles, the elements of which are dependent upon \( \cos(m \varphi) \), \( \sin(m \varphi) \), \( \cos(n \varphi) \) and \( \sin(n \varphi) \) and upon \( 1/r^{n+1} \), \( n \) being the order of the multipole, (unbounded medium potential).

After differentiating this series twice with respect to \( r \) and integration of the result over a closed surface, only the monopole term (point current source) remains, because its potential field is not dependent upon \( \varphi \) and \( \varphi \) (\( n = 0 \)).

This monopole term, however, should be zero, since otherwise the heart, or the body, would be loaded continuously, causing a disastre.

Consequently,

\[
\begin{align*}
\iint_{S_H} (\Delta_2 \Phi_H) \, dS_H &= \iint_{S_H} (\Delta_3 \Phi_H) \, dS_H - \iint_{S_H} \frac{\partial^2 \Phi_H}{\partial r^2} \, dS_H - \frac{2}{r} \iint_{S_H} \frac{\partial \Phi_H}{\partial r} \, dS_H \\
&= 0 - 0 - 0 = 0.
\end{align*}
\]
Vakgroep Meten en Regelen
Afdeling der Elektrotechniek
Technische Hogeschool Eindhoven

BESCHRIJVING VAN DE COMPUTER-
PROGRAMMA'S, behorende bij het
afstudeerverslag

door J.W.M. Vermeulen
Inhoud.

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Toelichting.


Om het eigenlijke verslag geen al te grote omvang te laten aannemen zijn de programma's en de resultaten daarvan separaat beschreven.

De titel van het afstudeerverslag luidt:

"On the determination of electrical potentials from skin potential and torso geometry data", by J.W.M. Vermeulen.

Het verslag is gedateerd januari 1978.

In het hierna volgende wordt een aantal malen verwezen naar publicatie's van het Rekencentrum THE. Met uitzondering van het werken met files op pack wordt de lezer met deze publicatie's bekend verondersteld.

Alle programma's zijn geschreven in BEATHE, een variant op ALGOL 60, zoals dat tijdens de afstudeerperiode op het B 6700 (later B 7700) computersysteem van het Rekencentrum THE geïmplementeerd was. De lezer wordt verondersteld met deze "taal" vertrouwd te zijn.
I. Inleiding

De programma's zijn, voor zover mogelijk, bloksgewijs opgebouwd, d.w.z. een programma is te veranderen door het eruitnemen van een blok en/of het erin plaatsen van een ander blok, zonder dat aan de rest van het programma iets veranderd moet worden.

De grote procedures en enkele grote matrices zijn op pack *) opgeslagen, om de omvang van het kaarten-pakket te beperken, daar deze procedures en data toch steeds in ongewijzigde vorm gebruikt werden.

Een listing van deze procedures is ook opgenomen.

Ook zijn stuurkaartconfiguraties opgenomen om procedures en data van/naar pack te copiëren, of van pack te verwijderen.

*) Voor het gebruik van ruimte op pack is toestemming van het RC nodig!
II. De programma's.

II.1 Algemeen opzet.

In het algemeen beginnen alle programma's vrijwel identiek:
- Eerst worden de constantes gedeclareerd en ingelezen (resp. berekend), die de puntenverdeling bepalen. Vaak worden ook nog het aantal tijdstippen, de referentiepunten en de orde van de multipolen gedeclareerd en ingelezen.
- Het tweede blok koopt ook tot aan het einde van het programma.
  Hierin worden de overal in het programma gebruikte arrays (coördinaten, projecties, X, W etc.) gedeclareerd, evenals vaak de proceduregroep "matrixprocs" (zie III.3).
- Daar "coördinaten" slechts op één plaats in het programma gebruikt wordt: locale declaratie (zie III.1).
- Vanaf hier komen er een aantal varianten voor:
  w A is al berekend, en staat op kaart of pack:
    A wordt ingelezen.
  w B, D (en C) moeten nog berekend worden:
    1. "directe methode"
    2. "ruimtehoek-methode"
    3. "zevenpunten-methode"
    4. gecombineerde methode. 
    declaratie benodigde procedures, berekening B, D (en C)
    Let op: bij de zevenpunts- en de gecombineerde methode: inlezen van de waarneemdiepte,
    b.v. 1^15/00 verkorte straal : 0,999 inlezen.
- Eerst hier worden de bemeten punten ingelezen !
  Let op ! VUE [i] = 'FALSE' indien het punt i wel bemeten is !
- Op B (en C) wordt deflatie toegepast, waarna B geïnverteerd wordt.
  Let op ! Na de inversie bevat het array B : B^{-1}.
- B^{-1} wordt vervolgens met D (en C) vermenigvuldigd.
  Let op ! Hierna bevat het array D (resp. C) : B^{-1}D = A (resp. B^{-1}C = E).
- Na de reductie van A (en E) wordt pseudo-geïnverteerd (behalve bij de zgn. \(\lambda\)-correctie). Locale declaratie van "PSEUDO" (of PSEUDOSVD met singular value decomposition).
  Hiermee wordt dit blok (nesting 3) gesloten, omdat de erin voorko-
mende procedures en variabelen niet langer nodig zijn.
- Vervolgens worden de huidpotentialen ingelezen of berekend.
  - echte huidpotentialen: inlezen van een card-deck;
  - synthetische huidpotentialen:
    - inlezen van de multipool-coefficienten (van card of pack),
    - berekenen van de overdrachtsmatrix (of het inlezen daarvan van pack);
      de overdrachtsmatrix "BIF" (tussen mpc's en huidpotentialen) bestaat uit het product van \(B^{-1}\) en \(F\).
      \(F\) wordt met behulp van de procedures "LlPl" en "OVERDRACHT" berekend. Let OP! De procedure "OVERDRACHT" is geschreven voor radii in meters. Daar in deze programma's mm. gebruikt worden dienen alle \(B^{-1}F\) [i, j] met \(10^{3(n+1)}\) vermenigvuldigd te worden (n = orde van de betreffende multipool).
    - De synthetische huidpotentialen als functie van de tijd worden nu gevonden uit:
      \[\Phi_s = B^{-1}F M,\]
      waarbij M een matrix is, waarin het element \(M_{ij}\) de waarde is van de i\(^e\) multipool op tijdstip j.

Door de overdrachtsmatrices X en W met \(\Phi_s\) te vermenigvuldigen worden de hartpotentialen en dubbellaagsterktes verkregen. Deze worden uitgeprint en geplot.
Om een indruk te krijgen van o.a. het verloop van de depolarisatiegolf over het hart worden voor een twaalftal tijdstippen hoogtekaartjes getekend van de momentane epicardiale activiteit.
Deze 12 tijdstippen worden ingelezen in "TIJDPLOT".
II.2 De $\lambda$-correctie.

Bij de programma's, waarin de "$\lambda$-correctie" de gevoeligheid voor hoge ruimtelijke frequenties vermindert, zijn een aantal procedures nodig om 1. de linker-, rechter-, onder- en boven-"buren" van een punt te vinden, teneinde de afgeleides van $\Phi$ in dat punt te kunnen bepalen;
2. de matrix $X$ te berekenen.

Deze procedures zijn:

'PROCEDURE' ZOEK (P, G, B, PE);
'VALUE' PE; 'INTEGER' 'ARRAY' P[x]; 'INTEGER' B, PE; 'BOOLEAN' G;

'PROCEDURE' KOEZ (P, I, R, L);
'VALUE' I; 'INTEGER' 'ARRAY' P[x]; 'INTEGER' I, R, L;
- KOEZ bepaalt het linker- (L) en rechter- (R) buur-punt van I.

Aan de hand van een schets van de puntenverdeling (b.v. met NH = 2 of 3) wordt de werking van KOEZ duidelijk. Vaak zal immers gelden, dat $L = I - 1$ en $R = I + 1$, echter niet indien het punt I of R in het x-z-vlak liggen ($\Phi = 0$).

'PROCEDURE' ZOEKPLATH (I, O, B, L, R, NH, PLATH);
'VALUE' I, NH; 'INTEGER' I, O, B, L, R, NH; 'INTEGER' 'ARRAY' PLATH[x, x];
met daarin: 'BOOLEAN' 'PROCEDURE' LIJN (PLJ, J, NH, I);
'VALUE' NH, I; 'INTEGER' PLJ, J, NH, I;
LIJN onderzoekt een der aangeduide lijnstukken van PLATH op de aanwezigheid van I daarop.
- indien de positie van I gewond is, kunnen de onder- (O), boven- (B),
linker- (L) en rechter- (R) bepaald worden uit PLATH (alleen voor
punten I met 4 "buren" !)
Voor punten I met 6 "buren" is de bepaling van de "onder- en boven-
buren" iets afwijkend (OL = onder-links, boven-rechts = BR enz.).
Voor de procedurevorm is gekozen om de berekening van G overzichtelijker
te maken.

'PROCEDURE' CORR (LA, ATA, G, AR, X, AKP, RED, OUT, SINGULIER);
'VALUE' LA, AKP, RED; 'REAL' LA; 'INTEGER' AKP, RED; 'BOOLEAN'
SINGULIER; 'FILE' OUT; 'REAL' 'ARRAY' ATA, G, AR, X [\mathbb{R}, \mathbb{R}];
- CORR bepaalt \( X = (A^T A + \lambda G)^{-1} A^T (LA = \lambda) \).
Indien \((A^T A + \lambda G)\) singulier mocht zijn, wordt niet geïnverteerd.
Eventueel kan - indien dit daadwerkelijk zou optreden - pseudo -
geïnverteerd worden.
Let op! A^T A moet actueel gelijk zijn aan \( A_R^T A_R \) (de gereduceerde vorm).

Dit in tegenstelling tot het programma.
III. De procedures op pack.

III.1 'PROCEDURE' COORDINATEN (NO, AGP, AGD, COHP, HOOG, PLAT, DHP, STRAAL, I, OUT);

'VALUE' NO, AGP, AGD; 'INTEGER' NO, AGP, AGD, I; 'REAL' 'ARRAY' COHP [x, x]; 'INTEGER' 'ARRAY' HOOG, PLAT, DHP [x, x]; 'REAL' STRAAL; 'FILE' OUT;

- COORDINATEN "vult" de arrays HOOG, PLAT en DHP (de eerste 2 bevatten de projecties van de punten op een vierkant, DHP[i,j] is het nummer van het j° hoekpunt (rechtsomgaande) van driehoek i). Tevens worden de x, y, z, φ en ϑ coordinaten van de punten berekend, uitgaande van een gegeven NO en de "straal" van de punten. I wordt als Jensen-parameter gebruikt.

Voor de punten op de huid zijn de actuele parameters resp. :

NO, AGP, AGD, COHP, HOOG, PLAT, DHP, COHP[I,6], I, OUT.

Voor hartpunten op een bol met straal "STRAAL" resp. :

NH, AKP, AKD, COHPH, HOOGH, PLATH, DHPH, STRAAL, I, OUT.

(Indien de hartpunten niet op een bol liggen moet COHPH met 1 kolom uitgebreid worden en STRAAL vervangen worden door COHPH[I,6]). Let op ! De formules in het verslag zijn afgeleid, gebruikmakend van de bolvorm van het hart.

COHP[I,6] en STRAAL (resp. COHPH[I,6]) evenals NO, AGP, AGD en NH, AKP, AKD moeten voor de aanroep van COORDINATEN een waarde hebben.

Let op ! kaart CO OOOOB : pi hoeft globaal niet meer bekend te zijn (pi wordt locaal in de procedure gedeclareerd, alwaar pi ook zijn waarde krijgt : pi = π).

- interne, locaal gedeclareerde, procedures :

'REAL' 'PROCEDURE' SID (X);

'VALUE' X; 'REAL' X;

SID genereert, als functie van X, een oneven driehoeksfunctie met amplitude 1 ; zie figuur.
'REAL' 'PROCEDURE' COD (X);
'VALUE' X; 'REAL' X;
COD genereert, als functie van X, een even driehoeksfunctie
met amplitude 1, zie figuur.

\[ \text{COD}(x) \]

\[ \begin{align*}
-2 & \quad -1 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad x \\
-1 & \quad & \quad 1 & \quad & \quad & \quad & \\
\end{align*} \]

'PROCEDURE' VULDHP (K, L, DHP);
'VALUE' K; 'INTEGER' K, L; 'INTEGER' 'ARRAY' DHP [X, X];
Deze procedure is een verkorting van de 4 erin voorkomende
assignment statements, voortkomend uit de systematische
nummering van punten en driehoeken.

COORDINATEN bevat ook een aantal kaarten (HOPLA 00 - HOPLA 99),
waarin HOOG en PLAT uitgeprint worden. Voor NO>6 moeten deze kaarten
gewijzigd worden, omdat dan de papierbreedte overschreden wordt.
III.2 De "meetkundeprocs".

Dit is een verzameling korte procedures om bewerkingen met driesimensionale vectoren uit te voeren.

De procedures en het erbij behorende commentaar spreken voor zich, behalve wellicht bij RHB.

Gebruik wordt n.l. gemaakt van de formule, dat de ruimtehoek tussen 3 vectoren \( \vec{a} \), \( \vec{b} \) en \( \vec{c} \) gelijk is aan:

\[
\pm \left\{ \arctg \left( \frac{|(\vec{c} \times \vec{a}) \cdot (\vec{a} \times \vec{b})|}{|\vec{c} \times \vec{a}| \cdot |\vec{a} \times \vec{b}|} \right) + \arctg \left( \frac{|(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})|}{|(\vec{a} \times \vec{b}) \cdot (\vec{b} \times \vec{c})|} \right) + \arctg \left( \frac{|(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})|}{|(\vec{b} \times \vec{c}) \cdot (\vec{c} \times \vec{a})|} \right) \right\}
\]

Het teken hangt af van de richting van de normaal op het oppervlak, opgespannen door \( \vec{a} \), \( \vec{b} \) en \( \vec{c} \).
III.3 De "matrixprocs".

Met deze procedures worden matrices ingelezen en uitgevoerd.
(Inlezen van card of pack, uitvoeren naar printer, ponsers of pack)

'PROCEDURE' PRINTKOLOMMEN (A, AGP, SEL, O, B, OUT);
'VALUE' AGP, O, B; 'REAL' 'ARRAY' A [JE, JE]; 'INTEGER' AGP, O, B;
'INTEGER' 'ARRAY' SEL [JE]; 'FILE' OUT;
- A is een \[1 : AGP, 0 : B\] -matrix, waarvan 8 kolommen afgedrukt worden (format: E12.5), vergezeld van (links) het rijnummer en (erboven) het kolomnummer.
Het array SEL[1 : 8] wordt extern gevuld met de nummers van de af te drukken kolommen.
Indien een SEL[1] buiten de grenzen (0 of B) van het array A valt, wordt "0" afgedrukt.
Oudere versies van PRINTKOLOMMEN hebben deze beveiliging niet.

'PROCEDURE' SCHRIJFMATRIX (A, M, N, s, OUT);
'VALUE' M, N; 'REAL' 'ARRAY' A [JE, JE]; 'INTEGER' M, N; 'STRING' S;
'FILE' OUT;
- A is een \[1 : M, 1 : N\] - matrix, waarvan de naam in de string S staat (max. 10 karakters). Via de file OUT wordt deze matrix uitgeprint, telkens 8 kolommen naast elkaar, met PRINTKOLOMMEN.

'PROCEDURE' LOCK DISKFILE (F); 'FILE' F;
- Files op disk-pack moeten gesloten worden om tot "permanent file" te worden. Crènchen, d.w.z. het "teruggeven" aan het systeem van de ongebruikte ruimte op het pack is aanbevelenswaardig, omdat veelal teveel ruimte gereserveerd wordt voordat de file gevuld wordt (zie b.v. HS 4.2.3, defaultwaarden).
- Deze procedure cruncht eerst de file F en sluit hem daarna.

'PROCEDURE' LEEMSMATRIX (X, A, B, F, WPR);
'VALUE' A, B, WPR; 'INTEGER' A, B, WPR; 'FILE' F; 'REAL' 'ARRAY' X [JE, JE];
- X is een \[1 : A, 1 : B\] - matrix, die in binaire vorm in F opgeslagen is. Let op ! LEEMSMATRIX kan alleen gebruikt worden, indien X met
BINMATRIX naar F geschreven is. Bovendien moet WPR dan actueel dezelfde waarde hebben, zie voorbeeld na de toelichting bij BINMATRIX.

'PROCEDURE' BINMATRIX (X, A, B, F, WPR);
'VALUE' A, B, WPR; 'INTEGER' A, B, WPR; 'FILE' F; 'REAL' 'ARRAY' X [H" . ];


Indien men de matrix A [1 : C, 1 : D] van carddeck naar pack wil copiëren, met resp. files KAART en SCHIJF, roept men bovenstaande procedures als volgt aan:
LEESMATRIX (A, C, D, KAART, 20);
BINMATRIX (A, C, D, SCHIJF, 30);
LOCK DISKFILE (SCHIJF);

In volgende programma's wordt A van pack gelezen met:
LEESMATRIX (A, C, D, SCHIJF, 30);
III.4 'PROCEDURE' PSEUDO (A, M, N, X, EPS, T, OUT);
   'VALUE' M, N; 'INTEGER' M, N; 'REAL' EPS; 'REAL' 'ARRAY' A, X
   [x, x]; 'STRING' T; 'FILE' OUT;
   - deze procedure berekent de pseudo-inverse X van de matrix A [1 : M,
     1 : N]. De naam van A staat in de string T (max. 15 karakters).
     Via de file OUT worden afgedrukt:
     - de lengtes van de geveegde kolommen van A (format E9.2),
     - de waarde van EPS (format E9.2),
     - de rang van A.
     Deze procedure PSEUDO berekent X op exact dezelfde wijze als de pro-
     cedure PSEUDO, zoals die door ir. J.A. Blom geschreven is.
     Om geheugengebruik te verminderen, en omdat zo'n uitgebreide heading
     veelal niet nodig is, zijn vele arrays locaal in de procedure gede-
     clareerd, en niet in de heading opgenomen.
     Om de tijd, die nodig is om een matrixelement op te zoeken, te verkor-
     ten, worden in de hier gebruikte versie vaak getransponeerden van
     matrices gebruikt.
III.5 De "tekenprocedures".

Met deze procedures worden de tijdsignalen (potentialen en dubbellaagsterktes) en de hoogtekaartjes daarvan geplot.

Let op! 1. Steeds moet vóór de declaratie van deze procedures de kaart $\text{INCLUDE} "\text{DRAWPROCEDURES}" voorkomen.

2. Binnenkort zullen de oude "DRAWPROCEDURES" uit de bibliothek verwijderd worden. In de bijlage bij het afstudeerverslag van J.J. van der Kam staan programma's, die de nieuwe plotprocedures gebruiken.

'PROCEDURE' LOCK PLOT (F); 'FILE' F;

- Nadat alle tekeningen voor een plot-file gemaakt zijn, dient deze gesloten te worden. Deze procedure sluit de file F en geeft tevens een melding via de regeldrukker.

'PROCEDURE' MAMI (I, AFM, AI, MAX, MIN);
'VALUE' AFM; 'INTEGER' I, AFM; 'REAL' AI, MAX, MIN;

'PROCEDURE' BLOK (M, RIJ, KOL, H, N, XO, PL);
'VALUE' RIJ, KOL, N; 'REAL' ARRAY M[$\text{k}, \text{m}$]; 'INTEGER' ARRAY H[$\text{k}, \text{m}$];
'INTEGER' RIJ, KOL, N; 'REAL' XO; 'FILE' PL;
- Via de plotfile PL wordt een tekening gemaakt van de (tijd-) signalen, zoals die in M[$1 : \text{RIJ}, 1 : \text{KOL}$] rijgewijs staan.

H is het array HOOG of PLAT resp. BOOGH of PLATH. Het tijdsignaal op een punt wordt op de plaats van dat punt in H getekend, alsmede het nummer van dat punt.

Let op! MAMI moet gedeclareerd zijn.

N = NO of NH, al naar gelang M huid- of hart-potentialen (dubbellaagsterktes) bevat.

Dit "blok" van tijdsignalen is 25 x 25 cm groot.

XO bepaalt de plaats waar dit blok op de lengte van het papier komt.

In BLOK wordt XO met 30 opgehoogd (en moet dus actueel een variabele
zijn), zodat tweemaal achtereenvan EK geen overschrijven van de eerste tekening veroorzaakt.

'PROCEDURE' PRENTJES (M, RIJ, KOL, H, TIJDPLT, N, XO, ELF, PHI, PL, OUT);
'VALUE' RIJ, KOL, N; 'REAL' 'ARRAY' M [x, x]; 'INTEGER' RIJ, KOL, N;
'FILE' PL, OUT; 'BOOLEAN' ELF, PHI; 'INTEGER' 'ARRAY' TIJDPLT [x];

met intern de

'PROCEDURE' HOOGTEKAART (PL, JTP, MAX);
'VALUE' MAX; 'INTEGER' MAX; 'FILE' PL; 'REAL' 'ARRAY' JTP [x, x];


Na lineaire interpolatie worden rechten getekend van gelijke potentiaal.

Indien de potentiaal over een vierkant een zadelvlak is, worden beide paren mogelijke hoogtekaarten getekend, zie voorbeeld hieronder.

![Diagram](attachment:image.png)

'PROCEDURE' UNI (PL, IZ, X, Y);
'VALUE' IZ, X, Y; 'FILE' PL; 'INTEGER' IZ; 'REAL' X, Y;

- UNI is een hulpprocedure om een stuk programma verkort in te voeren; o.a. wordt het tijdstip (IZ) geplot en worden de grootste en de kleinste potentiaal op dat tijdstip afgedrukt.

- In PRENTJES bevat M de te plotten tijdsignalen (plaats = rijnummer, tijdstip = kolomnummer), RIJ is het totaal aantal plaatsen, KOL het aantal tijdstippen. H is actueel HOOG of PLAT (huidpotentialen) resp. HOOGH of PLATH (hart). TIJDPLT [1 : 12] bevat de tijdstippen, waarvan we de hoogtekaartjes getekend willen hebben. XO geeft weer de (lengte-...
positie van de tekeningen op het papier, en wordt -automatisch zoveel
opgehoogd, dat geen tekening overschreven wordt.

ELF = 'TRUE', indien de resultaten via plotter 11 uitgevoerd worden.
Let op! Beperk de uitvoer via plotter 11 tot ca. 1 m (4 "blokken"
of 1 "blok" met hoogtekaartjes). Deze plotter werkt n.l. vrij
langzaam en on-line, zodat een grote hoeveelheid tekeningen tot
"verstopping" kan leiden.

ELF = 'FALSE', indien via plotter 30 getekend wordt (off-line,
vaak 1 à 2 dagen wachttijd), vooral voor grote en/of veel tekeningen.
(Via plotter 11 is de uitvoer als volgt: eerst een "blok", dan de
6 paar hoogtekaartjes -2 aan 2 boven elkaar- daarna. Via plotter
30 worden de paren hoogtekaartjes in de breedte van het papier
getakend -telkens 2 naast elkaar-).

PHI = 'TRUE' indien het potentia len betreft en 'FALSE' voor dubbel-
lagsterktes.

PL is de plot-file, OUT de regeldrukker-file.
IV. Het werken met packfiles.

IV.1 Inleiding.

Na het verkrijgen van toestemming om een bepaalde hoeveelheid ruimte op pack (b.v. 1000 segmenten = 30 k words, 1 getal = 1 word) te gebruiken, dienen voor de gegevensoverdracht tussen kerngeheugen en pack steeds stuurkaarten (WF1-opdrachten) gebruikt te worden, vgl. HS 4.2.3.

In de hierna volgende voorbeelden zal steeds USER3 als packname gebruikt worden.

Het ? (<I> of invalid character) aan het begin van de extra stuurkaarten is facultatief, hier zal het steeds aan het begin van alle stuurkaarten vermeld worden, om het verschil met programma-kaarten aan te duiden.
IV.2 Programma-files.

Een procedure (of een groep procedures) kan onvertaald op pack opge­
slagen worden, b.v. de procedure PROC.
Om PROC naar USER3 te schrijven kan het volgende programma gebruikt
worden:

? JOB PACK; QUEUE = 2; USER = U....S..../....;
? BEGIN
? COMPILe PROC WITH BEATHE FOR SYNTAX;
? COMPILER FILE NEWTAPE (KIND = PACK; PACKNAME = USER3, TITLE = PROC,
...............)
? DATA CARD
$ 'SET' NEW 'BEGIN'
$ 'RESET' LIST
 'PROCEDURE' PROC (...........);
 .
.
$ 'POP' LIST
 'END'.
$ 'RESET' NEW
? END JOB

*) eventueel zijn nog andere attributen te geven, b.v. securitytype,
securityuse, zie de programma's en HS 4.2.3.
- de kaarten $ 'RESET' LIST en $ 'POP' LIST zorgen ervoor, dat PROC
niet gelist wordt in het programma, waarin PROC gebruikt wordt.
Door de kaart $ 'POP' LIST keert de "waarde" van LIST na de compilatie
van PROC terug naar zijn oude waarde.

Om de procedure in een programma te kunnen gebruiken moet PROC naar
het kerngeheugen gecopieerd worden (WFL-COPY-statement, voor de
compilatie). Tevens moet PROC op een bepaalde plaats in het programma
ingevoegd (= gedeclareerd, met de $ 'INCLUDE'-statement).
? JOB PACK; QUEUE = 2; USER = U....S....;/.....;
? BEGIN
? COPY PROC FROM USER3 (KIND = PACK);
? COMPILe PROGRAM WITH BEATHE;
? FILE ...... ;
? DATA CARD
   'BEGIN' ....
   :
   'BEGIN'
$ 'INCLUDE' "PROC"
   :
   'END';
   :
   'END'.
etc.

Om procedures uit de bibliotheek van een ander te kunnen gebruiken, moet die ander aan het attribuut SECURITYUSE de waarde CLASSA toegekend hebben.
In plaats van ? COPY PROC FROM USER3 (KIND = PACK);
moet de gebruiker de volgende stuurkaart gebruiken:
? COPY (U....S....)PROC AS <name> FROM USER3 KIND = PACK);
De files staan n.l. op pack geadministreerd met het registratienummer van degene, die de file creëerde.
(U....S....) : registratienummer van degene, die de file creëerde.
<name> : mag iedere naam zijn, zelfs PROC.
IV.3 Datafiles.

Een groep getallen (b.v. een matrix) kan binair op pack gezet worden. De matrix dient daartoe in stukken van 30 getallen opgedeeld te worden. (1 record = 30 words = 30 getallen).

Indien de matrix M (100 x 50) in de file MFILE opgeslagen moet worden (op pack), gaan we als volgt te werk:

? JOB MATRIX; QUEUE = 2; USER = U....S..../.....;  
? BEGIN  
? COPY MATRIXPROCS FROM USER3 (KIND = PACK);  
? COMPILe MAT~IXN~ARPACK WITH BEATHE;  
? FILE MFILE (KIND = PACK, PACKNAME = USER3, TITLE = M, UNITS = WORDS,  
 AREAS = 1, AREASIZE = 167);  
'BEGIN!'  
$ 'INCLUDE' "MATRIXPROCS"  
...  
'BEGIN''REAL''ARRAY' M [1 : 100, 1 : 50] ; 'FILE' MFILE;  
...  
BINMATRIX (M, 100, 50, MFILE, 30);  
LOCK DISKFILE (MFILE);  
'END';

enz.

- Gebruik wordt gemaakt van de matrixprocedures, zoals beschreven in III.3, dus eerst copiëren van pack naar kerngeheugen, en invoegen in het programma.
- Zo nodig kunnen nog andere files (input- of output-) gedeclareerd worden. Ter wille van de duidelijkheid zijn de betreffende stuurkaarten hier niet weergegeven.
- Aan de file-attributes van MFILE kunnen b.v. nog worden toegevoegd: SECURITYUSE = SECURED (voor niemand anders toegankelijk dan de eigenaar) of SECURITYUSE = CLAS3A, SECURITYTYPE = IN (toegankelijk als inputfile voor degenen, die het registratienummer van de eigenaar weten).
- 1 area van 167 records = 167 x 30 words = 5010 words, dus juist voldoende.
M kan als volgt weer worden ingelezen:

```
? JOB MATRIX2; QUEUE = 2; USER = USER3;...

? BEGIN

? COPY MATRIXPROCS FROM USER3 (KIND = PACK);
? COMPILE MATRIXVANPACK WITH BEATHE;

? FILE MFILE (KIND = PACK, PACKNAME = USER3, TITLE = M, UNITS = WORDS);

'BEGIN'

$ INCLUDE "MATRIXPROCS"

'BEGIN' FILE MFILE; 'REAL' 'ARRAY' M [1 : 100, 1 : 50] ;

LEESMATRIX (M, 100, 50, MFILE, 30);

'END';
```

- Ook hier wordt weer gebruik gemaakt van de MATRIXPROCS, die onder het eigen registratienummer op USER3 opgeslagen gedacht waren.
IV.4 Diversen.

- De compiler Control cards ($) kaarten) worden niet gelist, als $ in de eerste kolom geponst is. Indien $ in de tweede kolom staat wordt de gehele kaart gelist.

- Files op pack worden verwijderd met de WFL REMOVE-statement:

  b.v.

  ? REMOVE M, MATRIXPROCS FROM USER3 (KIND = PACK);

  Alleen de eigenaar van de betreffende file kan deze verwijderen, ongeacht de waarde van SECURITYUSE of SECURITYTYPE.

- Indien van b.v. een matrix een binair carddeck gemaakt is (door de file, waarin de matrix geschreven wordt aan PUNCH te koppelen) kan deze matrix veelal niet via een RJE (Remote Job Entry) ingelezen worden. Raadpleeg in voorkomend geval steeds de balie. Het inlezen van een binair carddeck via een RJE, die daar niet voor geschikt is heeft veel narigheid tot gevolg!

  Nummer ook steeds de kaarten van zo'n binair deck, anders is het deck onmiddellijk onbruikbaar als er een kaart uitvalt.

- Overzicht van de eigen bibliotheek:

  Een volledig overzicht van de eigen files op een bepaald pack (b.v. USER3), met alle file-attributen wordt verkregen met de stuurkaart:

  ? RUN SYSTEM/FILEDATA ("A: ALL TITLE = USERCODE/<reg.nr> ON USER3);

  <reg.nr.> het volledige U....S....-registratienummer.

  Het is ook mogelijk om een beperkt overzicht van alle files (met weinig of geen attributen) te laten afdrukken. Zie hiervoor HS4.2.3. De uitvoer van deze statement komt op de regeldrukker.

- Listing van een op pack staande procedure (of groep procedures, zoals MATRIXPROCS): met de volgende stuurkaarten:

  ? RUN SYSTEM CARDLINE

  ? FILE CARD (KIND = PACK, PACKNAME = USER3, TITLE = MATRIXPROCS, UNITS = WORDS);

  ? FILE LINE (KIND = PUINTER);

- Copiëren van een file op pack (het maken van een copie-carddeck):

  als boven, nu echter in plaats van:

  ? FILE LINE (KIND = PUINTER); de kaart : ? FILE LINE (KIND = PUNCH);
Op dezelfde wijze kan van datafiles een (copie-) carddeck gemaakt worden.

- Het combineren van verscheidene, niet "samenhangende" WFL-statements tot één JOB is in principe altijd mogelijk:

Na de JOB identificatie en evt. het aangeven van limits voor processtijd, printlimit e.d. kunnen tussen ? BEGIN en ?END JOB een (groot) aantal WFL-statements voorkomen, zoals in het voorbeeld hieronder:

? BEGIN
? REMOVE M FROM USER3 (KIND = PACK);

? COPY MATRIXPROCS FROM USER3 (KIND = PACK);
? COMPILE PROGRAM WITH BEATHE;
? FILE IN (KIND = READER), OUT (KIND = PRINTER);
? FILE MFILE (KIND = PACK, PACKNAME = USER3, TITLE = M, UNITS = WORDS, AREAS = 1, AREASIZE = 167, SECURITYUSE = CLASSA, SECURITYTYPE = IN);
? DATA CARD
'BEGIN'
? 'INCLUDE' "MATRIXPROCS"
programma
'END'.
? DATA IN
data

?RUN SYSTEM/ FILEDATA ("A: ALL TITLE = USERCODE/....S.... ON USER3");

? RUN SYSTEM/CARDLINE;
? FILE CARD (KIND = PACK, PACKNAME = USER3, TITLE = M, UNITS = WORDS);
? FILE LINE (KIND = PUNCH);

?END JOB
(de open regels in het bovenstaande hebben slechts tot doel de bij elkaar horende WFL-statements typografisch van de andere te scheiden).