Dynamics of vortices in shallow fluid layers

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Abstract

The dynamics of monopoles and dipoles in shallow fluid layers are studied numerically as well as by laboratory experiments. Three-dimensional direct numerical simulations were performed to study these flows in detail. In the laboratory, dipoles in thin layers of fluid were created by electromagnetic forcing. Commonly, it is tacitly assumed that flows in thin layers of fluid can be considered as two-dimensional. However, such flows exhibit a strong vertical dependence of the vertical velocity field, which results in the presence of recirculations in the planar vortices. It turned out that monopoles and dipoles in shallow fluid layers can be considered quasi-two-dimensional (Q2D) only under certain conditions, being a sufficiently small Reynolds number and a not-too-large fluid depth. It was found that the geometrical confinement is an equivocal parameter to qualify a flow as quasi-two-dimensional.
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Chapter 1

Introduction

From innocent clouds of milk in coffee to killer tornadoes in the south east of America, vortices of all sorts and sizes can be observed everywhere around us. Apart from the importance to understand the physics behind these vortices, they are an interesting problem as well. Who has not been standing in front of the bath tub looking at the swirling vortex wondering if it is supposed to rotate in that direction\(^1\).

The dynamics of two-dimensional (2D) vortices is relevant in the field of geophysical fluid dynamics. Large-scale vortices with a long life-span appear as high and low pressure cells in the atmosphere and, for instance, as Meddies and Gulf Stream rings in the Atlantic Ocean. Nice examples of large-scale vortices are shown in figure 1.

These large-scale flows tend to behave two-dimensionally due to three different effects: the Coriolis force, as a result of the rotation of the Earth, buoyancy forces due to the density stratifications in the oceans and in the atmosphere and the geometrical confinement of the flow. Vortices in the atmosphere have typical horizontal dimensions of 1000 kilometres, while their vertical dimension is typically 10 km. In order to gain more insight in the dynamics of such flows, small-scale laboratory experiments have been performed in rotating tanks (see e.g. Hopfinger and van Heijst 1993, van Heijst and Kloosterziel 1989), in stratified fluids (see e.g. Beckers 1999, van Heijst and Flór 1989, Maassen et al. 1999) and in rotating-stratified systems (see e.g. Linden et al. 1995).

As mentioned above, a flow could also be considered two-dimensional if a significant geometrical confinement is imposed. It is then assumed that the limited vertical dimension will confine the flow to an almost planar one. It is thus possible to study 2D flows by performing experiments in thin layers of fluid. Several experiments of this type have been performed in soap films (Kellay et al. 1995, Rutgers 1998) and in thin layers of fluid inside a container (see e.g. Antonova et al. 1985, Tabeling et al. 1991). In the experimental studies on thin-layer flows of this type, usually a single layer of fluid is used. Recently, in the experiments of Paret and Tabeling (1997a), in which the flow was forced electromagnetically, a system of salt-stratified fluid layers was used. The stable two-layer stratification provides an additional mechanism for two-dimensionalization by inhibiting vertical motions.

\(^{1}\)In fact, the rotation direction has nothing to do with the rotation of the Earth. On such a small scale, the rotation direction of the vortex is determined by the geometry of the tub.
An important question arises if one thinks about three-dimensional effects in these thin-layer experiments. In general, one could wish to verify purely 2D numerical and theoretical models by means of laboratory experiments. However, three-dimensional effects always play an additional role in laboratory experiments. In confined rotating systems, one encounters the influence of an Ekman boundary layer, which results into a secondary circulation. In stratified fluids, the vertical shear in the velocity results in a secondary circulation as well, which is nicely demonstrated by Beckers (1999). In soap film experiments, the fluid exhibits thickness fluctuations, which results in stretching and compression of vortices. It can thus be expected that three-dimensional effects must play a role in thin layer experiments as well.

In Paret et al. (1997b) it was shown that, for the flow parameters used in their experiments, a stratified thin-layer configuration can be considered as two-dimensional. Supporting evidence was provided by laboratory experiments, although these experiments did not allow full 3D flow measurements. A numerical study concerning 2D issues in thin-layer experiments was performed by Jüttner et al. (1997), but in this case no information was obtained about the full three-dimensional structure of the flow field either.

As yet, three-dimensional effects on flows in thin fluid layers have not been properly investigated. In this thesis, we will try to understand if, and how, three-dimensional effects play a role in the dynamics of monopolar and dipolar vortices in thin fluid layers. Monopolar and dipolar vortices are relatively simple vortex structures, compared to, for example, a field with numerous vortices. If 3D effects significantly alter the 2D character of a flow, the flow cannot be regarded as 2D anymore. In order to develop criteria that will
characterize a flow as 2D or not, the three-dimensionality must be quantified. As a first step, the dynamics of an axisymmetric monopole will be studied in detail. A theoretical model as well as direct 3D numerical simulations will be used to gain more insight in its dynamics. After that, the dipolar vortex will be discussed. A dipole is a more complicated vortex structure due to the non-zero advective term, which is absent for an axisymmetric monopole. Numerical simulations as well as laboratory experiments will be performed to understand the dynamics of dipolar structures in shallow fluid layers.

The remainder of this thesis is organized in the following way. In Chapter 2, some theoretical background as well as theoretical models for monopolar and dipolar vortex structures in thin layers of fluid will be discussed. The experimental and numerical methods will be described in Chapter 3. Then, in Chapter 4, the numerical results of monopoles will be discussed. The numerical simulations will be carried out for a monopole in a single layer of fluid as well as for a monopole in a two-layer stratified fluid. In Chapter 5, the dipolar vortex is analyzed in more detail. The results of numerical simulations will be discussed, which are eventually compared with preliminary laboratory experiments. The general conclusions and the discussion are given in Chapter 6.
Chapter 2

Theory

2.1 Governing Equations

Consider the motion of a homogeneous, incompressible fluid. It is assumed that the fluid behaves like a continuum. Such a fluid is completely governed by conservation of momentum and conservation of mass. The law of conservation of momentum in differential form can be achieved by applying Newton’s law of motion to an infinitesimal fluid element. In combination with the constitutive equation, which relates the stress and deformation in a continuum, we obtain the equation of motion for a Newtonian fluid. This equation is commonly known as the Navier-Stokes equation. For incompressible fluids, conservation of mass implies

\[ \nabla \cdot \mathbf{u} = 0, \quad (2.1) \]

and the Navier-Stokes equation, which expresses conservation of momentum, can be written in vector notation as (see e.g. Kundu 1990)

\[ \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u}, \quad (2.2) \]

where \( \frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \) is the time derivative following a fluid parcel, \( \mathbf{u} \) the fluid velocity, \( p \) the pressure, \( \rho \) the density, \( \mathbf{g} \) the gravitational acceleration and \( \nu \) the kinematic viscosity.

The term on the left hand side of (2.2) describes the rate of change of the velocity as one follows a fluid element. It consists of two parts: \( \partial \mathbf{u} / \partial t \) is the local rate of change of \( \mathbf{u} \) at a given point, whereas \( (\mathbf{u} \cdot \nabla) \mathbf{u} \) describes the change in \( \mathbf{u} \) as a result of advection of the particle from one location to another. The first term on the right hand side is the so-called pressure gradient force (per unit mass). This force is the force that is exerted on a fluid parcel as a result of pressure differences in the fluid. The second term on the r.h.s. is the gravitational force (per unit mass). The last term on the r.h.s. describes dissipative viscous effects.

\[ ^1 \]A Newtonian fluid is a fluid in which the stress is a linear function of the rate of strain.
The vorticity of a fluid element, $\omega$, is related to the velocity by $\omega = \nabla \times \mathbf{u}$. The vorticity is a measure of the rotation of individual fluid elements. By taking the curl of (2.2) the full three-dimensional Navier-Stokes equation can be expressed in terms of the vorticity as (see e.g. Kundu 1990)

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)\mathbf{u} + \nu \nabla^2 \omega + \frac{\nabla p \times \nabla \rho}{\rho^2}.$$  \hspace{1cm} (2.3)

The vorticity of a material fluid element can thus be changed due to three different effects: the first term on the right hand side describes the stretching and tilting of vortices, the term $\nu \nabla^2 \omega$ represents the rate of change of $\omega$ due to diffusion of vorticity in the same way that $\nu \nabla^2 \mathbf{u}$ represents acceleration due to diffusion of momentum and the third term describes baroclinic vorticity production. Note that pressure and gravity terms do not appear in the vorticity equation, since these forces act through the centre of mass of an element and therefore generate no torque. For a derivation of (2.3), the reader is referred to appendix A.

### 2.2 Equations of Motion for Two-dimensional Flows

The Navier-Stokes equation simplifies for a two-dimensional (2D) incompressible flow, in which the flow takes place in the $(x,y)$-plane. In such a 2D flow, $\nabla \rho = 0$, and the first term on the right hand side of (2.3) is also equal to zero. This term represents the rate of change of vorticity due to stretching and tilting of vortex tubes, so it can be understood that it must be zero for a 2D flow. For purely 2D flows, the velocity vector reduces to $\mathbf{u} = (u, v, 0)$, so the vorticity vector becomes $\omega = (0, 0, \omega_z)$. The vorticity equation 2.3 then reduces to

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = \nu \nabla^2 \omega.$$  \hspace{1cm} (2.4)

This equation can be simplified by defining a stream function $\psi(x, y, t)$, such that

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}.$$  \hspace{1cm} (2.5)

This definition is based on the validity of the continuity equation $\nabla \cdot \mathbf{u} = 0$. The vorticity can then be written in terms of $\psi$ as $\omega = -\nabla^2 \psi$. An alternative (non-dimensional) formulation of the vorticity equation (2.4) is given by

$$\frac{\partial \omega}{\partial t} + J(\omega, \psi) = \frac{1}{Re} \nabla^2 \omega,$$  \hspace{1cm} (2.6)

where $J(\omega, \psi)$ is the Jacobian, which is defined as

$$J(\omega, \psi) = \frac{\partial \omega}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \omega}{\partial y} \frac{\partial \psi}{\partial x}.$$  \hspace{1cm} (2.7)
We have written (2.6) in dimensionless form by introducing the Reynolds number, defined as \( \text{Re} = \frac{\omega L^2}{\nu} \). Here, \( \omega \) and \( L \) represent typical values for the vorticity and the length scale of the problem under consideration.

Summarizing, (2.4) is the governing equation for a two-dimensional, viscous flow, in which the density is assumed to be constant. In this type of flow, the vorticity changes due to two different mechanisms: diffusion of vorticity (\( \nu \nabla^2 \omega \)) and advection of vorticity ((\( u \cdot \nabla \))\( \omega \)). In dimensionless form, it can be written as (2.6).

### 2.3 Two-dimensional Turbulence

Vortices in our daily three-dimensional world exhibit the tendency to break up into smaller and smaller structures. A nice example are the smoke rings that experienced smokers can produce. They soon break up into smaller and smaller vortices until no trace is left. A remarkable property of two-dimensional flows is that exactly the opposite happens: an initially disordered flow spontaneously organizes into large and coherent structures. These structures are weakly dissipative and therefore have a relatively long life-span. This process is often referred to as self-organization, in which two mechanisms play an important role.

First of all, it can be understood from the form of the diffusion operator, that small-scale structures dissipate on shorter time scales than the larger scales in the flow.

Besides, a mechanism known as the inverse energy cascade occurs in 2D flows, which can be explained in the following way. The governing equation for a 2D flow is given by (2.4). For the moment, let us assume that the flow is inviscid, so that the last term is zero:

\[
\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = 0.
\] (2.8)

This equation expresses conservation of vorticity of a material fluid element. This can also be understood from the fact that in a 2D flow the effect of vortex stretching is absent, and thus the vorticity is conserved.

For this type of flow, the kinetic energy and the enstrophy are conserved quantities, as we will show. Multiplying (2.8) by \( \psi \) gives an equation for the kinetic energy \( E \) (per unit area) (van Heijst 1992)

\[
\frac{\partial E}{\partial t} + \nabla \cdot S = 0,
\] (2.9)

with \( E \equiv (\nabla \psi)^2/2 \) and \( S = -\psi \nabla \left( \frac{\partial \psi}{\partial t} \right) - u\psi \nabla^2 \psi \) the energy flux vector. An equation for the enstrophy \( V \) can be obtained by multiplying (2.8) by \( \nabla^2 \psi \), where \( \psi \) is the streamfunction

\[
\frac{\partial V}{\partial t} + \nabla \cdot Q = 0.
\] (2.10)

Here, \( V \equiv -(\nabla^2 \psi)^2/2 = \omega^2/2 \) is the enstrophy (per unit area) and \( Q = uV \) is the enstrophy flux vector. If we assume that either the flow domain is finite and the normal
velocity on the boundary is zero, or that the velocities go to zero for infinity, integration of (2.9) and (2.10) yields

$$\frac{\partial}{\partial t} \iint E \, dA = 0, \quad \frac{\partial}{\partial t} \iint V \, dA = 0.$$ (2.11)

These results imply that the kinetic energy and the enstrophy are indeed conserved quantities for inviscid two-dimensional flows.

The implications of these conservation laws become more clear when the enstrophy and energy are written in a spectral form

$$E \sim \int_0^\infty E(k, t) \, dk, \quad V \sim \int_0^\infty k^2 E(k, t) \, dk,$$ (2.12)

where $k$ is the wave number and $E(k, t)$ is the energy density in the wave-number interval between $k$ and $k + dk$ at time $t$. Now suppose that the distribution of energy at $t = 0$ is peaked around a certain wavenumber $k_0$, as shown schematically in figure 2.1. Due to nonlinear interactions (e.g. vortex merging, filamentation), energy will be transferred to larger and smaller wave numbers, so that the peak will be spread out. Conservation of energy implies that the total area of the peak must remain constant in time. Simultaneously, the enstrophy of the flow must be constant in time, which means that the energy peak in the spectrum will shift to smaller wave numbers ($k_1 < k_0$). Conservation of both energy and enstrophy thus causes a spectral energy flux to smaller wave numbers, i.e. to larger scales of motion. This phenomenon has been observed in several laboratory experiments (e.g. Cardoso et al. 1994) and in numerical simulations (e.g. McWilliams 1984). In figure 2.2, a numerical simulation shows the effect of self-organization in a purely 2D viscous flow in a square container with no-slip boundaries (Clercx et al. 1999).

### 2.4 Vortices in Two-dimensional Flows

Practically all studies on 2D flows show the emergence of coherent vortices from initially unstable or random-flow conditions (e.g. McWilliams 1984, van Heijst and Flór 1989).
Figure 2.2: Numerical simulation of a decaying two dimensional flow in a square container with no-slip boundaries (Clercx et al., 1999). The effect of self-organization is clearly visible. An initially random vorticity field organizes into larger, coherent vortex structures.

In all these laboratory experiments and numerical simulations, only three types of stable coherent vortex structures have been observed: the monopole, the dipole and the tripolar vortex. Triangular and square vortices can also be generated in the laboratory, but they are found to be unstable (e.g. Beckers 1999).

The monopolar vortex is the simplest coherent vortex structure. It consists of a single set of closed streamlines around one common centre (see e.g. van Heijst 1992). Monopoles can be observed in the ocean (e.g. Meddies, Gulf Stream Rings) and in the atmosphere (high and low pressure cells, tornadoes). Although most monopolar vortices have an almost axisymmetric shape, they can also be elliptical. Jupiter’s Great Red Spot is a well-known example of an elliptical vortex.

The dipolar vortex (see e.g. van Heijst and Flör 1989) consists of two closely packed patches of opposite vorticity. A dipole translates through the fluid and therefore contains a net linear momentum. If both patches of the dipole are equally strong, the dipole will propagate along a straight line; if the dipole is not symmetric, it has a curved trajectory.

The third type is the tripole: a linear arrangement of three oppositely-signed vorticity patches, in which an elliptical core is accompanied by two satellites of oppositely signed vorticity. The tripole has a net angular momentum and it shows a rotation around the centre of the core vortex (see e.g. van Heijst and Kloosterziel 1989).

A schematic view of these three vortex types is given in figure 2.3. The monopole is only shown in its shielded, or isolated version, where a core of positive vorticity is surrounded by a ring of negative vorticity. A non-isolated monopole consists only of a single-signed patch of vorticity. In the next two parts of this section, we will give purely two-dimensional analytical models for the monopole and the dipole.

2.4.1 Monopolar Vortices

Several models have been formulated in the past to describe two-dimensional monopoles. Some of these are: the potential vortex, the Rankine-, Lamb- and Bessel vortex. Although these models can be used to understand some features of monopoles, they all do have
restrictions. In order to model a ‘real’ monopole, the vorticity distribution, and its deriva­tive, should be continuous and time-dependent. Furthermore, the total kinetic energy and the total angular momentum must be finite. Finally, one usually has to take into account viscous effects.

We will give two examples of 2D monopole models, which fulfill these conditions and are commonly referred to in literature (see e.g. Kloosterziel 1990): the Gaussian non-isolated and isolated vortex model. The time evolution of the vorticity distribution of the isolated vortex is given by

\[
\omega(r, t) = \frac{a_0}{(r_0^2 + 4\nu t)^2} \left(1 - \frac{r^2}{r_0^2 + 4\nu t}\right) \exp \left(-\frac{r^2}{r_0^2 + 4\nu t}\right),
\]

and the vorticity distribution of the non-isolated vortex is

\[
\omega(r, t) = \frac{b_0}{r_0^2 + 4\nu t} \exp \left(-\frac{r^2}{r_0^2 + 4\nu t}\right),
\]

where \(a_0\), \(b_0\) and \(r_0\) determine the initial amplitude and the initial radius of the vortex. The Lamb vortex is a snapshot of the non-isolated vortex. The vorticity distributions are plotted in figure 2.4 for three different times \(t\). These are just two examples of a non-isolated and an isolated monopole, but they share a remarkable property: both functions are self-similar. It means that the function can be written in the following form

\[
\omega(r, t) = \frac{1}{a(t)} \omega \left(\frac{r}{b(t)}, 0\right),
\]

or, to put it another way, the vorticity distributions are at all times scaled versions of themselves. The non-isolated vortex is a scaled version of the function \(f(r) = \exp(-r^2)\), and the isolated vortex is a scaled version of \(f(r) = (1 - r^2) \exp(-r^2)\) for all times \(t > 0\). Note that these models are actually asymptotic solutions to the diffusion equation (see e.g Kloosterziel 1990).
Figure 2.4: Vorticity distributions for (a) the non-isolated Gaussian vortex and (b) the isolated Gaussian vortex at three different times $t, 4\nu t = 0, 4\nu t = 0.5, 4\nu t = 1$. $a_0 = 1, b_0 = 1$ and $r_0 = 1$.

2.4.2 Dipolar Vortices

Only a few models for 2D dipolar vortices are known, of which the Lamb dipole is probably the most familiar. The difficulty in finding an analytical model for the dipolar vortex lies in the fact that the advective term $(u \cdot \nabla)\omega$ in (2.4) is non-zero (as is the case for axisymmetric monopolar vortices). Two analytical models, solutions of the simplified vorticity equation (2.4), are discussed next: the Lamb dipole and the Stokes dipole.

The Lamb Dipole

An analytical model for a stationary, inviscid, dipolar vortex with a continuous vorticity distribution was found by Lamb (Lamb 1932), and independently by Chaplygin (see Meleshko and van Heijst 1994). For the case of a stationary and inviscid flow, the 2D vorticity equation (2.4) reduces to $(u \cdot \nabla)\omega = 0$, or equivalently $J(\omega, \psi) = 0$. To solve this equation, Lamb assumed a linear relationship between the vorticity and the streamfunction

\[ \omega = k^2 \psi, \tag{2.16} \]

with $k$ a constant. One can easily verify that $J(\omega, \psi) = 0$ is always satisfied then. In general, the relation between the vorticity and the streamfunction can be written as $\omega = -\nabla^2 \psi$, which takes the following form in plane polar coordinates

\[ \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = -k^2 \psi \quad 0 \leq r \leq a, \]

\[ = 0 \quad r > a. \tag{2.17} \]
It is thus assumed that the flow in the exterior region \((r > a)\) is irrotational. The first differential equation can be solved by the method of separation of variables, which leads to a Bessel equation in \(r\). The general interior solution \(\psi_i(r, \theta) = R(r)\Theta(\theta)\) can then be written as (see e.g. van Heijst 1992)

\[
\psi_i(r, \theta) = \sum_{n=0}^{\infty} \{A_n J_n(kr) + B_n Y_n(kr)\} \{C_n \sin(n\theta) + D_n \cos(n\theta)\} ,
\]

with \(n\) an integer, \(A_n, B_n, C_n\) and \(D_n\) constant coefficients, and \(J_n\) and \(Y_n\) the \(n\)-th Bessel functions of the first and second kind, respectively. By taking \(n = 1\) one obtains a dipolar solution. As the solution has to be finite in the domain \((0 \leq r \leq a)\), \(B_n\) must be zero for all \(n\) (\(Y_n\) goes to infinity for \(r \to 0\)).

We assume that the flow in the exterior region is a potential flow around the circular dipole. This problem is equivalent to a 2D potential flow around a circular cylinder with uniform velocity \(U\). The stream function of such a flow is given by (see e.g. Kundu 1990)

\[
\psi_e(r, \theta) = -U \left( r - \frac{a^2}{r} \right) \sin \theta \quad r > a .
\]

One sees that \(\psi_e = 0\) at \(r = a\) for all values of \(\theta\), which shows that the streamline \(\psi = 0\) represents a circle of radius \(a\). The exterior flow is symmetric with respect to the line \(\theta = 0, \pi\). It is thus desirable that \(\psi_i \sim \sin \theta\). This means that we must choose \(D_n = 0\) for all \(n\), leading to the following solution for the interior stream function

\[
\psi_i(r, \theta) = C_1 J_1(kr) \sin \theta , \quad 0 \leq r \leq a , \quad 0 \leq \psi \leq 2\pi .
\]

The two unknowns, the wavenumber \(k\) and the constant \(C_1\), are determined by continuity requirements. Both the stream function and the azimuthal velocity need to be continuous at \(r = a\). The first condition

\[
\psi_i(r = a) = \psi_e(r = a) ,
\]

implies \(J_1(ka) = 0\). The first zero of \(J_1\) is \(ka = 3.83\). The azimuthal velocity is defined as \(u_\theta = \frac{\partial \psi}{\partial r}\), so the second condition can be written as

\[
\left. \frac{\partial \psi_i}{\partial r} \right|_a = \left. \frac{\partial \psi_e}{\partial r} \right|_a ,
\]

and this gives us the following expression for \(C_1\)

\[
C_1 = -\frac{2U}{k J_1'(ka)} = -\frac{2U}{k J_0(ka)} ,
\]

where the equality \(J_1'(x) = J_0(x)\) was used (see e.g. Kreyszig 1993). The interior solution of the Lamb dipole is then given by

\[
\psi_i(r, \theta) = -\frac{2U}{k J_0(ka)} J_1(kr) \sin \theta \quad 0 \leq r \leq a ,
\]
whereas the external flow is governed by (2.19). The corresponding vorticity distribution in the interior region is given by

\[ \omega(r, \theta) = -\frac{2Uk}{J_0(ka)} J_1(kr) \sin \theta \quad 0 \leq r \leq a, \]  

where the relation \( \omega = -\nabla^2 \psi \) has been used. The vorticity in the exterior region is zero. In figure 2.5, the vorticity cross-section along the vorticity extremes is given, as well as a 3D representation of the vorticity field.

![Vorticity Cross-Section](a)  

![3D Representation](b)

Figure 2.5: (a) A 3D representation of the vorticity \( \omega \) of the Lamb dipole. (b) The vorticity distribution of the Lamb dipole along the line \( \theta = 0, \pi \).

The solution that was found by Lamb, (2.19) and (2.24) describes a stationary, inviscid dipole. In several laboratory experiments, dipoles are observed to expand radially in time while at the same time, their vorticity amplitude decreases. The vorticity appears to be 'smeared out' over a larger region. The Stokes model, which we will discuss next, describes a dipole with viscosity, which is non-steady. Furthermore, it shows a radial increase and a decrease of maximum vorticity.

### The Stokes Dipole

In the Stokes limit, advection of vorticity is neglected. This is a plausible assumption, since in a dipolar structure the vorticity gradient is approximately perpendicular to the velocity vector everywhere in the flow field, so that \( \mathbf{u} \cdot \nabla \omega \approx 0 \). We thus assume that the advection term in (2.6) is equal to zero, leaving us with the (dimensionless) diffusion equation in two dimensions

\[ \frac{\partial \omega}{\partial t} = \frac{1}{\text{Re}} \left[ \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right]. \]  

(2.26)
The flow is thus completely governed by diffusion. We will show that similarity solutions, which were also mentioned in section 2.4.1, can be used to model this dipole.

- Solution of the diffusion equation in terms of similarity solutions

Kloosterziel has shown that if one can expand the initial conditions of a diffusion problem in a series of similarity solutions, general asymptotics exist on infinite domains (Kloosterziel 1990). This means that the long-term behaviour of a diffusion problem is directly related to one specific function. Of course, one could write the initial condition in a Fourier-series representation, but the long-term behaviour would not be directly obvious then. We will only use this method to model a dipole, so not all the mathematical details will be discussed here. For these, the reader is referred to Kloosterziel (1990).

Let us begin with a solution to the one-dimensional diffusion equation, in which the diffusion of some scalar distribution $C(x, t)$ is evaluated. We have already used similarity solutions in the models for the monopoles (see section 2.4.1), where the isolated and the non-isolated Gaussian monopole could be written in terms of similarity solutions. In general, they can be written as a scaled version of themselves at all times, that is, for all $t > 0$ one has

$$C(x, t) = \frac{1}{a(t)} \frac{C^0}{b(t)} \left( \frac{x}{b(t)} \right),$$

(2.27)

with $a(0) = b(0) = 1$ and $C^0(x) \equiv C(x, t = 0)$.

These similarity solutions are a valuable tool if one wishes to study the large-time behaviour of solutions of the diffusion equation. Kloosterziel shows that if the initial condition of a diffusion problem is quadratically integrable with respect to the weight function $w(x) = \exp \left( \frac{1}{2} x^2 \right)$, that is

$$\int_{-\infty}^{\infty} |C^0(x)|^2 \exp \left( \frac{1}{2} x^2 \right) dx < \infty,$$

(2.28)

then $C^0$ can be expanded in a sum of similarity solutions

$$C^0(x) = \sum_{n=0}^{\infty} a_n \Omega_n(x),$$

(2.29)

where

$$\Omega_n(x) = \frac{H_n(x/\sqrt{2}) \exp \left( \frac{1}{2} x^2 \right)}{\sqrt{2^n n! \sqrt{2\pi}}},$$

(2.30)

is a normalized set of similarity solutions. $H_n$ are the Hermite polynomials, which are given by

$$H_n(y) \equiv (-1)^n \exp \left( y^2 \right) \frac{d^n \exp \left( -y^2 \right)}{dy^n},$$

(2.31)
so the first three Hermite polynomials are \( H_0 = 1 \), \( H_1 = 2y \), \( H_2 = 4y^2 - 2 \) and so on. The coefficients \( a_n \) in (2.30) can be expressed in terms of \( \Omega_n \)

\[
a_n = \int_{-\infty}^{\infty} C^0(x) \Omega_n(x) w(x) \, dx.
\] (2.32)

The complete time evolution of the field with initial condition \( C^0(x) \) is then given by

\[
C(x, t) = \sum_{n=0}^{\infty} \frac{a_n}{b(t)^{n+1}} \Omega_n \left( \frac{x}{b(t)} \right).
\] (2.33)

Kloosterziel shows that the large-time evolution of the flow field is attached to the smallest, non-zero \( a_n \). Thus, if one wishes to find the large-time behaviour of a solution of the diffusion equation, one has to expand the initial condition in similarity solutions. Then, the coefficients \( a_n \) are easily calculated by (2.32), and the asymptotic behaviour is determined by the solution attached to the lowest non-zero coefficient \( a_n \). For example, if \( a_0 = 0 \) and \( a_1 \neq 0 \), then the slowest-decaying ‘mode’ gives the large-time behaviour, which is given by

\[
\lim_{t \to \infty} C(x, t) = \frac{a_1}{b(t)^2} \Omega_1 \left( \frac{x}{b(t)} \right).
\] (2.34)

The results presented above can straightforwardly be generalized to higher dimensions. For example, if we have a two-dimensional diffusion problem, and the initial condition can be written in terms of similarity solutions, the initial condition \( C^0(x, y) \) takes the following form

\[
C^0(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{nm} \Omega_n(x) \Omega_m(y),
\] (2.35)

where the expansion coefficients are calculated according to

\[
a_{nm} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C^0(x, y) \Omega_n(x) \Omega_m(y) \exp \left( \frac{1}{2} (x^2 + y^2) \right) \, dx \, dy.
\] (2.36)

The time evolution of the problem then is

\[
C(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_{nm}}{b(t)^2+n+m} \Omega_n \left( \frac{x}{b(t)} \right) \Omega_m \left( \frac{y}{b(t)} \right).
\] (2.37)

- **The Stokes dipole**

We will use this method for the following problem in order to get a dipole solution of
the diffusion equation (see Satijn et al. 2000b). Assume the following simple initial condition for a dipolar-like vorticity distribution:

\[ \omega_0 = \begin{cases} 
1 & -1 < x < 1 \quad 0 < y < 1, \\
-1 & -1 < x < 1 \quad -1 < y < 0,
\end{cases} \]  

(2.38)

like is shown in figure 2.6. The coefficients \( a_{nm} \) can be calculated straightforwardly using (2.36) which results into

\[ a_{00} = 0, \quad a_{10} = 0, \quad a_{01} = \frac{2}{\sqrt{2\pi}}. \]  

(2.39)

Thus, starting with the initial condition (2.38), the long-term solution of the diffusion equation becomes

\[ \omega(x, y, t) = \frac{2}{\sqrt{2\pi}} \frac{1}{b(t)^3} \Omega_0 \left( \frac{x}{b(t)} \right) \Omega_1 \left( \frac{y}{b(t)} \right) \]

\[ = \frac{1}{\pi} \frac{1}{(1 + \frac{2t}{Re})^{3/2}} \left( \frac{y}{\sqrt{1 + \frac{2t}{Re}}} \right) \exp \left( -\frac{x^2 + y^2}{2(1 + \frac{2t}{Re})} \right), \]

(2.40)

in which \( b(t) = (1 + 2t/Re)^{1/2} \). In plane polar coordinates \((r, \theta)\), the solution can be written as

\[ \omega(r, \theta, t) = \frac{1}{\pi} \frac{1}{(1 + \frac{2t}{Re})^{3/2}} \left( \frac{r}{\sqrt{1 + \frac{2t}{Re}}} \right) \exp \left( -\frac{r^2}{2(1 + \frac{2t}{Re})} \right) \sin \theta, \]

(2.41)

which describes a decaying dipolar vortex. The vorticity amplitude decays as \((1+2t/Re)^{-3/2}\), and the radius increases as \((1+2t/Re)^{1/2}\). In figure 2.7(a) a 3D representation of the vorticity is shown. Figure 2.7(b) shows the vorticity distribution along the line \( \theta = 0, \pi \) at three different times.
2.5 An Analytical Model of a Monopolar Vortex in a Shallow Fluid Layer


Thus far we have only considered purely 2D models for monopoles and dipolar vortices. However, we are living in a three-dimensional world, and purely two-dimensional vortices do not exist. As was mentioned before, vortices in shallow fluid layers are assumed to behave 2D. However, the velocities in such a system are strongly dependent on the vertical coordinate. Although the vertical velocities may be small, the flow field still has a dependence on \( z \). Therefore we will use the term quasi-two-dimensional for these type of flows.

In this section, a model for an axisymmetric quasi-2D monopole is described.

Consider a single circular vortex in a shallow layer of fluid, i.e. it is assumed that the horizontal dimension \( L \) is much larger than the vertical dimension \( H \). The domain is vertically bounded by a no-slip bottom and a free surface. Further, we assume that the density \( \rho \) and the viscosity \( \nu \) are constant.

In general, it is not possible to solve the Navier-Stokes equation analytically. However, since the problem under consideration is axisymmetric and takes place in a shallow layer of fluid, the following assumptions can be made. Axisymmetry implies that \( \partial/\partial \theta = 0 \). In a shallow layer of fluid, the vertical scale is much smaller than the horizontal scale, and we may assume that the vertical velocity \( u_z \) is much smaller than the azimuthal velocity \( u_\theta \). Mass conservation (\( \nabla \cdot \mathbf{u} = 0 \)) implies that then also \( u_r \ll u_\theta \). Summarizing, we have a
quasi-2D axisymmetric flow with

\[ u_\theta = u_\theta(r, z, t), \quad u_r \ll u_\theta, \quad u_z \ll u_\theta, \quad \text{and} \quad \frac{\partial}{\partial \theta} = 0. \quad (2.42) \]

The symmetry of the problem suggests us to use cylindrical coordinates, \( r, \theta \) and \( z \). With the assumptions (2.42), the equations of motion for this type of flow reduce to (for details see appendix B):

\[
\frac{\partial u_\theta}{\partial t} = \nu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_\theta}{\partial r} \right) - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} \right],
\quad (2.43)
\]

\[
\frac{u_\theta^2}{r} = \frac{1}{\rho} \frac{\partial p}{\partial r},
\quad (2.44)
\]

\[
-\frac{1}{\rho} \frac{\partial p}{\partial z} = g.
\quad (2.45)
\]

The expressions (2.43 - 2.45) represent the azimuthal, radial and vertical components of the reduced form of the Navier-Stokes equations. The radial and vertical components of these equations describe the so-called cyclostrophic and hydrostatic balances. The azimuthal component is a diffusion equation in two dimensions (\( r \) and \( z \)), and it indicates that the flow is entirely governed by diffusion.

### 2.5.1 A Solution to the Diffusion Equation

First, consider the diffusion equation (2.43). A similar diffusion problem for the decay of axisymmetric vortices in a linearly stratified fluid has been solved recently by Beckers et al. (2000), thus some aspects of the flow analysis will not be discussed in full detail here. The diffusion equation can be solved by using the method of separation of variables. Assume a solution \( u_\theta \) that depends on \( r, z \) and \( t \) as

\[ u_\theta(r, z, t) = R(r) Z(z) T(t). \quad (2.46) \]

Substituting this solution into (2.43) and dividing by \( R(r) Z(z) T(t) \) gives three differential equations, one for each component:

\[
\frac{1}{T} \frac{dT}{dt} = -\nu(p^2 + q^2),
\quad (2.47)
\]

\[
\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} - \frac{1}{r^2} = -p^2,
\quad (2.48)
\]

\[
\frac{1}{Z} \frac{d^2 Z}{dz^2} = -q^2.
\quad (2.49)
\]
• **Temporal part**

The time dependent part of the diffusion equation reflects the behaviour of $u_\theta(r, z, t)$ in time. It has the following solution:

$$T(t) = \exp(-\nu (p^2 + q^2)t) = \exp(-\nu p^2 t) \exp(-\nu q^2 t),$$  

(2.50)

where $-p^2$ and $-q^2$ are the separation constants of the radial and the axial part of the diffusion equation, respectively. Both exponential terms lead to a damping of the velocity field, where the first one is related to the radial part of the diffusion equation, and the second one is associated with the axial part.

• **Axial-temporal part of the diffusion equation**

The axial part of the diffusion equation (2.49) describes the vertical dependence of the azimuthal velocity. This is an ordinary differential equation, with the general solution

$$Z(z) = C \sin qz + D \cos qz.$$  

(2.51)

The boundary conditions are the following. At the bottom, the flow has to satisfy the no slip condition, $u_\theta|_{z=0} = 0$, and at the free surface, a stress free condition, $\frac{\partial u_\theta}{\partial z}|_{z=H} = 0$ has to be satisfied. In terms of $Z(z)$ this corresponds to $Z(0) = 0$ at the bottom, and $\frac{\partial Z}{\partial z}|_{z=H} = 0$ at the surface. The first condition implies $D = 0$, and by applying the second condition one obtains an expression for $q$

$$q = \frac{\pi(2n+1)}{2H}.$$  

(2.52)

The solution of the axial-temporal part of the problem, $\hat{Z}(z, t)$, can thus be written as a linear combination of all the solutions $Z(z)$ and the part of the temporal solution which is related to $q$ (see (2.50)):

$$\hat{Z}(z, t) = \sum_{n=0}^{\infty} C_n \sin qz \exp(-\nu q^2 t).$$  

(2.53)

Here, it is used that the overall solution for $u_\theta$ can be separated into a part related to $p$ and a part related to $q$. It is then possible to sum, or to integrate, these parts separately.

The decay times of the different modes are proportional to $1/\nu q^2$ (see (2.53)), which means that higher-order modes decay much faster. For example, the second mode $(n = 1)$ decays nine times faster than the first mode $(n = 0)$. So, we can assume that the higher order modes can be neglected. As a consequence, any appropriate vertical profile of $u_\theta(z)$ will soon evolve towards the following solution

$$Z(z) = \sin \left( \frac{\pi z}{2H} \right).$$  

(2.54)
Without loss of generality, we can take $C_0 = 1$. The damping associated with this shear is thus $\exp(-\lambda t)$, where $\lambda = \frac{\pi^2 \nu}{4H^2}$ (see (2.53)). The vertical velocity profile is shown in figure 2.8.

![Diagram of a shallow fluid layer with a vortex](image)

Figure 2.8: In a shallow layer of fluid, the vertical dependence of the azimuthal velocity $u_\theta$ will soon evolve towards a sine-like profile.

- **Radial-temporal part of the diffusion equation**

If the radial part of the diffusion equation, (2.48), is written in the following way

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(p^2 - \frac{1}{r^2}\right) R = 0,$$  \hspace{1cm} (2.55)

one immediately recognizes a Bessel equation (of zeroth order). The general solution is

$$R(r, p) = A(p)J_1(pr) + B(p)Y_1(pr),$$  \hspace{1cm} (2.56)

where $A$ and $B$ are constants, which are determined by initial and boundary conditions. The solution needs to be finite in the domain, so $B(p) = 0$ ($Y_1 \to \infty$ for $r = 0$). The separation constant $p$ is not restricted to certain discrete values, because the domain in the $r$-direction is not limited. The general solution of the radial-temporal part $\hat{R}(r, t)$ is thus the integral of the product of $\hat{R}(r, p)$ (note the index $p$!) and the temporal part $\exp(-\nu p^2 t)$, where we need to integrate over $p$

$$\hat{R}(r, t) = \int_0^\infty A(p) J_1(pr) \exp(-\nu p^2 t) dp,$$  \hspace{1cm} (2.57)

in which the function $A(p)$ is determined by the initial (radial) velocity distribution.

One particular solution $\hat{R}(r, t)$ that solves (2.55) is the shielded Gaussian vortex (see section 2.4.1). It turned out to be a useful model in several related previous studies (see e.g. Trieling & van Heijst 1998, Vosbeek 1998, Beckers 1999). A shielded vortex is chosen for the following reason: vortex lines$^2$ have to close within the fluid, or have to end at a

$^2$A vortex line is a curve in the fluid such that its tangent at any point gives the direction of the local vorticity vector.
free surface. They cannot end at the bottom (because of the no-slip condition). In our case, they will end at the free surface, which means that a patch of single-signed vorticity is always accompanied by a ring of oppositely signed vorticity. Indeed, in a thin layer experiment performed by Paireau et al. (1997), where a single vortex subjected to a shear flow was studied, this vortex appeared to be shielded. Further, it is reasonable to assume a Gaussian velocity profile, since this is a self-similar solution of the two-dimensional diffusion equation. It was found that any appropriate axisymmetric distribution of vorticity with zero net circulation eventually evolves to this one. In section 2.4.1 this vortex model was given as an example for a 2D vortex. The time-dependent velocity profile of this vortex has the following form:

\[
\dot{R}(r, t) = \frac{a_0 r}{2(r_0^2 + 4\nu t)^2} \exp \left( -\frac{r^2}{r_0^2 + 4\nu t} \right). \tag{2.58}
\]

The quantities \(a_0\) and \(r_0\) determine the initial amplitude and radius of the vortex. For this specific profile, \(A(p)\), as used in (2.57), can be found by solving the following Hankel integral (see e.g. Oberhettinger 1972),

\[
A(p) = a_0 p \int_0^\infty \frac{u}{2p^2} \exp(-u^2) J_1(r_0 pu) \ u \ du = \frac{\pi}{8} a_0 p^2 \exp \left( \frac{1}{4} p^2 r_0^2 \right). \tag{2.59}
\]

The axial vorticity \(\dot{\omega}_z(r, t)\), which is associated with \(\dot{R}(r, t)\) is thus

\[
\dot{\omega}_z(r, t) \equiv \frac{1}{r} \frac{\partial}{\partial r} (r\dot{R}(r, t)) = \frac{a_0}{(r_0^2 + 4\nu t)^2} \left( 1 - \frac{r^2}{r_0^2 + 4\nu t} \right) \exp \left( -\frac{r^2}{r_0^2 + 4\nu t} \right). \tag{2.60}
\]

The peak vorticity, or amplitude, is given by \(a_0/r_0^2\) and the vorticity changes sign at \(r = r_0\) for \(t = 0\). It can be verified that the shielded Gaussian vortex has zero net circulation \((\Gamma = 2\pi \int_0^\infty r \omega_z \ dr = 0)\), which means that it is isolated.

The radial part of the solution, is thus governed by ordinary two-dimensional lateral diffusion. This leads to a decay of the vortex amplitude as \(a_0/(r_0^2 + 4\nu t)^2\) and to an increase of the vortex radius as \((r_0^2 + 4\nu t)^{1/2}\).

- **Complete solution of the diffusion equation**

The complete solution of the azimuthal velocity \(u_\theta\) in an axisymmetric vortex can thus be written as a combination of \(\dot{Z}(z, t)\) and \(\dot{R}(r, t)\):

\[
\begin{align*}
\dot{u}_\theta(r, z, t) &= \frac{a_0 r}{2(r_0^2 + 4\nu t)^2} \exp \left[ -\frac{r^2}{r_0^2 + 4\nu t} \right] \sin \frac{\pi z}{2H} \exp \left[ -\lambda t \right],
\end{align*}
\]

with \(\lambda = \pi^2 \nu/4H^2\), sometimes referred to as the external friction parameter. In our case this is the bottom friction. In figure 2.9, the velocity profile and the associated vorticity
profile are shown for three different times, $t = 0$, $t = 10$ and $t = 20$. The values of $a_0$, $r_0$ and $\nu$ are typical values for a numerical simulation. It can be observed that lateral diffusion plays a minor role.

![Figure 2.9](image)

Figure 2.9: Radial distributions of the azimuthal velocity (a) and vorticity (b) of the monopolar shielded vortex, using the isolated Gaussian vortex model at $t = 0$, $t = 10$ and $t = 20$. Further parameters are $a_0 = 1$, $r_0 = 1$ and $\nu = 0.002$.

It can be shown that the governing equation of a vortex with this quasi-2D velocity profile (2.61) can be formulated as a 2D Navier-Stokes equation with an additional linear term. For this specific velocity profile, the three-dimensional Laplacian $\nabla^2$ can be written as a sum of the Laplacian in plane polar coordinates plus an extra term: $\nu\nabla_{r\theta z}^2 = \nu\nabla_{r\theta}^2 - \lambda$. One can easily verify that the $z$-component of $\nabla_{r\theta z}^2$ results into the extra term $-\lambda$, when applied to (2.61).

The two decay mechanisms, lateral diffusion and additional exponential decay due to vertical diffusion, are now effectively separated, which makes it possible to define two different Reynolds numbers in this problem: the usual Reynolds number $\text{Re} = \frac{\omega L^2}{\nu}$, which is associated with lateral diffusion, and a Reynolds number, $\text{Re}_\lambda$, that is associated with the exponential damping, which is defined as $\text{Re}_\lambda = \frac{\omega}{\lambda}$. The quantity $\omega$ represents a typical value for the vorticity, and for our problem we will take the peak vorticity. The typical horizontal length scale, $L$, is the radius where the vorticity changes sign. The three-dimensional Navier-Stokes equation for this problem, using the assumptions made above, could thus be rewritten in the following quasi-two-dimensional form

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p + \frac{1}{\text{Re}} \nabla^2 \mathbf{u} - \frac{1}{\text{Re}_\lambda} \mathbf{u}. \quad (2.62)$$

The bottom friction has now been parameterized by an additional linear term in the 2D Navier-Stokes equation. Note that for extremely shallow water flows ($H \ll L$) the decay
is mostly governed by vertical diffusion. In that case $\text{Re}_\lambda \ll \text{Re}$, which means that the last term on the right-hand side will dominate the second one.

Summarizing: equation (2.61) describes a solution of a Gaussian shielded vortex with a sine-like axial dependence which is decaying in time. The solution has an essentially three-dimensional (vertical) structure, but can be considered as quasi-two-dimensional since both $u_r \ll u_\theta$ and $u_z \ll u_\theta$.

### 2.5.2 The Hydrostatic and Cyclostrophic Balance

Now that we have found a solution for the $\theta$-component of the Navier-Stokes equation, we are left with the equations for the $r$ and $z$ component. These are much easier to solve, while the results will be quite remarkable.

The assumptions $(u_r, u_z \ll u_\theta)$ lead to the hydrostatic and the cyclostrophic balances

\[
-\frac{1}{\rho} \frac{\partial p}{\partial z} = g,
\]

\[
-\frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r}.
\]

By solving (2.63), one obtains an expression for the pressure in the fluid

\[
p(r, z) = \rho g [h(r) - z],
\]

in which the integration constant $h$ can depend on $r$.

The cyclostrophic balance (2.64) reveals that in a swirling flow, a radial pressure gradient is present. The only mechanism for this pressure gradient in the radial direction is a fluid height that depends on $r$, thus $h = h(r)$. The pressure gradient force $F_p$ simply follows from (2.65)

\[
F_p = -\frac{\partial p}{\partial r} = -\rho g \frac{\partial h(r, t)}{\partial r},
\]

which is constant along the vertical coordinate $z$. In a swirling flow, we also have a centrifugal force $F_c$, which is proportional to the azimuthal velocity. In contrast with the pressure gradient force, it is not constant along $z$. As $u_\theta \sim \sin z$ (see (2.61)), the centrifugal force is proportional to $\sin^2 z$. The flow situation is schematically drawn in figure 2.10(a). The two radial forces are not balanced everywhere. The pressure gradient force exceeds the centrifugal force near the bottom, which results in a flow towards the axis. Near the surface, the centrifugal force is greater than the pressure gradient force, and a flow radially outwards will occur. As a result, a secondary circulation in the $(r, z)$-plane is set up, as is indicated in figure 2.10(b). A similar secondary circulation is known from ‘Einstein’s tea leaves experiment’. In this experiment it can be observed that tea leaves on the bottom gather in the centre of the cup after one has stirred the tea.
2.5 An Analytical Model of a Monopolar Vortex in a Shallow Fluid Layer

Let us pause for a moment here. The boundary conditions of our problem, a no-slip bottom and a free slip surface, resulted in a solution of the diffusion equation (2.61) with a vertical sine-like dependence. However, if we consider the cyclostrophic balance again (2.64), we see that the left hand side depends on $z$, while the right hand side only depends on $r$. Apparently, the solution of $u_\theta$ is not ‘allowed’ to have a vertical dependence! So, by solving the diffusion equation with these specific boundary conditions, (2.65) and (2.64) are no equalities anymore, per definition, which explains that a secondary circulation should occur.

- Dimples in the free surface

Vortices in a fluid cause a deformation of the free surface, as was mentioned before. One can easily derive an expression for the evolution of the free surface associated with this simplified vortex model. A combination of (2.64), the cyclostrophic balance, and (2.65), gives a relation between the deformation of the free surface and the azimuthal velocity

\[ u_\theta = gr \frac{\partial h}{\partial r}. \]  

(2.67)

By substitution the solution for $u_\theta$, (2.61), and integrating (2.67) over the total fluid depth $H$, one obtains an approximation for $h(r,t)$:

\[ h(r,t) = H - \frac{a_0^2 \exp(-2\lambda t)}{32g (r_0^2 + 4\nu t)^3} \exp \left[ -\frac{2r^2}{r_0^2 + 4\nu t} \right]. \]  

(2.68)

The solution describes a dimple which broadens slowly in time, and decays due to the two damping mechanisms. In figure 2.11, the dimple is shown for $H = 0.9$, $a_0^2/(32 g) = 0.2$, $r_0 = 1$ for three different times $t$. For typical flow conditions, the deformation will be very

Figure 2.10: (a) Forces that act on a vertical fluid column in a monopolar vortex. The pressure gradient force, which is directed towards the centre of the vortex, does not depend on the depth. However, the centrifugal force depends on $z$ as $\sin^2(z)$, causing a secondary circulation in the $(r,z)$-plane (b).
weak, since its initial amplitude is given by \(a_6^2/(32 g r_0^6)\). A typical value for the amplitude of the dimple in our laboratory experiment would be 0.04 mm.

![Figure 2.11: Schematic picture of the time evolution of a dimple in a monopolar vortex, according to our model. The dimple is shown for \(t = 0\), \(t = 10\) and \(t = 20\), where the following parameters have been used: \(H = 0.9\), \(a_6^2/32g = 0.2\) and \(r_0 = 1\). The dimple is shown along the line \(\theta = 0, \pi\).](image)

### 2.5.3 Complete Three-dimensional Structure of the Flow Field

The results that were found so far in this section can be summarized as follows. The boundary conditions of our problem result in a vertical shear in the velocity field according to \(\sin(\pi z/2H)\). This vertical shear results in the occurrence of a secondary circulation in the \((r, z)\)-plane of the vortex.

The three-dimensional structure of the flow field of a monopolar vortex in a shallow layer of fluid, according to our model, is drawn schematically in figure 2.12. The secondary circulation can be seen in three dimensions as a doughnut. The fluid moves upwards in the centre of the vortex, and it moves downwards in the outer region. Meanwhile, the doughnut is rotating in the azimuthal direction. The path of an individual fluid parcel lies on a toroid. The vertical dependence of the azimuthal velocity is shown along a vertical line, at an arbitrary position in the vortex. For an axisymmetric flow, the vorticity vector, in cylindrical coordinates, is given by

\[
\omega = (\omega_r, \omega_\theta, \omega_z) = \left( -\frac{\partial u_\theta}{\partial z}, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}, \frac{1}{r} \frac{\partial}{\partial r} (ru_\theta) \right). \tag{2.69}
\]

The three components of the vorticity vector are shown in figure 2.12. The radial component, \(\omega_r\), is related to the vertical shear of \(u_\theta\). The vertical component, \(\omega_z\), is just defined by the vorticity distribution of the isolated Gaussian vortex. The \(\omega_\theta\) component is associated with the secondary flow in the \((r, z)\)-plane.
It is one of the purposes of the numerical simulations that are presented in the next sections to determine whether this secondary circulation is large or not. If it is large, then the Q2D approximation \((u_r, u_z \ll u_\theta)\) no longer holds. In addition, the parameterization of the friction as in (2.62) is no longer valid. If so, the flow is essentially three-dimensional, and quasi-2D theories cannot be used to describe the time evolution of such flows.
2.6 A 3D Model of a Dipolar Vortex in a Shallow Fluid Layer

Based on the findings of the monopolar vortex, a similar three-dimensional structure is proposed for the dipolar vortex in a shallow layer of fluid. It is assumed that the dipole halves can be regarded as two closely packed counter-rotating monopoles. The three-dimensional structure of the dipolar vortex is schematically given in figure 2.13. The dipole consists of two cells of oppositely signed vorticity. The planar vorticity distribution $\omega(r, \theta)$ may be Lamb-like or could be modeled by the Stokes dipole. The vertical dependence of the velocity field is assumed to be sine-like, in analogy with the monopole. Also, it is expected that a secondary circulation is present in both cells. They can be seen as two doughnut-shaped circulation tubes, in such a way that the fluid rises in the centre of a dipole-half and sinks at the separatrix of one cell. Due to the interaction of both dipole halves, the dipole translates through the fluid.

![Figure 2.13: Three-dimensional model of a dipole in a shallow layer of fluid. Indicated are the vertical dependence of the velocity field $u(r, \theta, z, t) \sim \sin(\pi z/2H)$, the secondary circulation in both dipole halves and the peak vorticity vector $\omega_z$ in the positive half of the dipole.](image-url)
Chapter 3

Experimental and Numerical Methods

3.1 Experimental Set-up

To study the dynamics of vortices in shallow fluid layers experimentally, a small square container (length 52 cm x width 52 cm x height 4 cm) was designed and constructed. The container is made out of plexi-glass; the bottom consists of a thin plate of ordinary glass (1 mm thick). The container is filled with a solution of Sodium Chlorine (salt), so that the fluid depth, $H$, in the central part of the container is a few millimeters ($4 - 12$ mm). Permanent magnets are located just below the bottom of the container. These are neodymium flux magnets, with a diameter of 25 mm and a thickness of 5 mm. Each magnet produces a magnetic field which has a maximum value of 1.09 T. Measurements showed that the magnetic field decays over a typical length of 4 mm. A PVC plate, in which a square array of 100 holes is fraised, is used to position the magnets. Several arrangements of magnets can be used, varying from one single magnet up to a $10 \times 10$ chess-board-pattern where the magnets are positioned side by side with alternating poles. Two platinum electrodes are positioned along two facing side walls. In all the experiments, a constant electric current $I$ is driven through the electrolyte from one electrode to the other. A typical value for $I$ is 2 A. The ionized sodium will drift to the negative electrode, while the chlorine ions drift to the positive electrode. This drift velocity is only a few

![Figure 3.1: Schematic cross-section of the experimental set-up. Here, a two-layer stratification is indicated by the dotted line. The poles of the magnets are indicated by N(orth) and S(outh).](image-url)
tens of millimeters per second. The presence of a magnetic field induces a Lorentz force, \( F_L = q v \times B \), which results into a motion perpendicular to the drift velocity of the ions (see figure 3.2).

\[
F_L = q v \times B
\]

Figure 3.2: Moving ions in a magnetic field experience a Lorentz force, which is directed perpendicular to the magnetic field and the drift velocity. In a shallow layer of fluid, this type of forcing will create a dipolar vortex structure, since a net amount of linear momentum is introduced.

The drift velocity of sodium is opposite to that of chlorine and because of their opposite charges, the direction of the Lorentz force will be the same. Both will thus be accelerated in the same direction, resulting in a transfer of momentum into the fluid. As the ions are moving in the same direction now, i.e. perpendicular to the drift velocity, there will no longer be any influence of the magnetic field on the flow. Of course, there is still a Lorentz force, but as it is directed oppositely for sodium and chlorine, the net influence on the flow will be zero. The result of the flow forcing using one single magnet is a dipolar vortex structure, because a linear momentum is introduced (see figure 3.2).

The flow is visualized by floating particles on the surface of the fluid, 250 \( \mu \text{m} \) in size. The particles are visualized by a light sheet, which is produced by a 1200 W Xenon lamp.

The experimental procedure is as follows. The electrical current \( I \) is imposed at time \( t = -\tau \) and switched off at \( t = 0 \), then leaving the system relaxing. The flow is recorded from above using a Kodak Megaplus digital video camera, type ES1.0. The resolution of the CCD camera is 1008 \( \times \) 1019 pixels. Following completion of the experiment, the data is analyzed by using particle velocimetry methods. The principal data obtained were the location of particles, which can be used to calculate the velocity field as well as the vorticity field.

### 3.2 Experimental Methods

As a first step, experiments have been performed with a single magnet configuration to study the dynamics of dipolar vortices. Two fluid configurations were used: a single layer of salt water, and a two-layer stratification, in which the fluid was composed of a layer of salt water beneath a layer of fresh water. The (stable) two-layer stratification was produced as follows: the set-up was filled with the necessary amount of fresh water. Then, the set-up
was filled from below with salt water up to the desired fluid depth. The experiments were
carried out for three different fluid depths, being 4, 8 and 12 mm. The electrical current
and the forcing time were kept roughly the same in all experiments. Note that the forcing
time was regulated manually, so the initial Reynolds number of the dipoles could not be
controlled accurately.

A configuration of $10 \times 10$ magnets, positioned in a chess-board-like pattern, has been
used next for a preliminary study of the behaviour of decaying quasi-two-dimensional
turbulent flows.

### 3.2.1 Particle Velocimetry

Particle Velocimetry is a well-known method to acquire quantitative information about
velocity fields in flows. Here, the flow is visualized with small particles that float on the
free surface of the fluid. The flow is then recorded with a digital camera. In general,
the velocities are obtained by comparing two sequential video frames. Two methods that
will be described here are PIV (Particle Image Velocimetry) and PTV (Particle Tracking
Velocimetry). The first method determines the average displacement of a cluster of particles
in corresponding image segments between two sequential images. The average displacement
is found by correlating the image segments. This method can only be used when the
velocity gradients in the image segments are small. PTV tracks individual particles in
two sequential images. By using PIV as a preprocessing for PTV, one obtains a higher
accuracy. This combination of PIV and PTV is called High resolution Particle Velocimetry
(HPV) (see van der Plas and Bastiaans 1998).

The velocity of each particle can now easily be determined as one knows the translation
of the particle from frame to frame and the frame-rate of the video camera. A calibration
needs to be performed to map pixel-coordinates of the camera and ‘world’-coordinates.
The unstructured velocity field found by the two-dimensional particle tracking procedure,
is then interpolated on a regular grid by a linear Delaunay interpolation method. The
vertical vorticity component $\omega_z$ can now easily be calculated numerically at each grid
point. The velocities can be determined with an accuracy of approximately 8 %, resulting
in an error in the vorticity of roughly 20 %. An example of a particle tracking experiment
dipole collision) is shown in figure 3.3. The velocities on an irregular grid are shown in
figure 3.3(a). Figure 3.3(b) shows the interpolated velocities on a regular grid, and the
the corresponding vorticity field is given in figure 3.3(c).

### 3.3 Numerical Methods

The numerical method used for the simulations of an axisymmetric monopole is based
on a finite difference code in cylindrical coordinates. Because of the axisymmetry of the
problem, the numerical problem is reduced to a two-dimensional problem. The flow is
independent of $\theta$, so the equations only need to be solved in the $(r, z)$-plane. The viscous
term in the equations of motion is calculated implicitly with the Crank-Nicholson method.
The non-linear and buoyancy terms are calculated explicitly by using a third-order Runge-Kutta scheme, leading to a second-order accuracy in time. For details, the reader is referred to Verzicco and Orlandi (1996).

The simulations presented here were performed using 128 grid points in both the $r$ and $z$ direction. The grid convergence has been checked by performing simulations with double resolution ($256^2$) and half resolution ($64^2$). It was found that the flow is well resolved when a resolution of $128^2$ is used. The results of the computation seem indistinguishable from a run with $256^2$ grid points. Further, a fully three-dimensional simulation (with $128^3$ grid points) yielded the same results as the corresponding axisymmetric computation and revealed that the flow indeed remained axisymmetric. The cylindrical computation domain is bounded by a no-slip bottom and stress-free upper and lateral walls. A last check with a larger domain in radial sense showed that the finiteness of the domain in the radial direction did not affect the results.

For the simulations in the two-layer system, the equations of motion are solved in the Boussinesq approximation. In the Boussinesq approximation, the density changes can be neglected in the fluid, except in the gravity term where $\rho$ is multiplied by $g$. The pressure $p$ and the fluid density $\rho$ can be written as

$$p = p_0 + p', \quad \rho = \rho_0 + \rho',$$

where $p'$ and $\rho'$ are (small) perturbations to the basic state $p$ and $\rho$. The equations of motion which have to be solved take the following form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p' - \frac{1}{Fr^2} \mathbf{e}_z + \frac{1}{Re} \nabla^2 \mathbf{u}, \quad (3.2)$$

and

$$\frac{DS}{Dt} = \frac{1}{Sc Re} \nabla^2 S, \quad (3.3)$$

Figure 3.3: Data processing method demonstrated for a ‘dipole collision’. (a) The result of the HPV particle tracking method is an unstructured velocity field. (b) The data is then interpolated on a regular grid by using standard interpolation methods. (c) The final vorticity field of the colliding dipoles.
with $S$ the salinity and $\mathbf{e}_z$ the unit vector in the $z$-direction. It will be assumed here that the density and the salinity are linearly related: $\rho = \alpha S$. The Froude number $Fr$, the Reynolds number $Re$ and the Schmidt number $Sc$ are defined as

$$Fr = \sqrt{\frac{g \rho_0}{\omega^2 L \rho'}}, \quad Re = \frac{L^2 \omega}{\nu}, \quad Sc = \frac{\nu}{\kappa}. \quad (3.4)$$

The quantities $\omega$ and $L$ are typical values for the vorticity and the horizontal length scale in the flow; $\kappa$ is the diffusivity of the stratifying agent.

In our simulations, the density deviation $\rho'$ will not exceed 10%. It is assumed that for this value the Boussinesq approximation can be applied successfully (Kundu, 1990).

Numerical simulations have also been performed for dipolar vortices in shallow fluid layers. The numerical scheme used for the dipole is in fact the same as the code described above. The equations are now solved in a Cartesian coordinate frame, so the code has to be solved in three dimensions ($x$, $y$ and $z$) instead of two. One simulation of a dipolar vortex lasts 64 times longer than a monopole simulation, since the resolution in the $x$, $y$ and $z$ direction is taken 128 by 128 by 64. A dipole simulation typically takes 12 hours of CPU time.
Chapter 4

The Monopolar Vortex in a Shallow Fluid Layer


4.1 Introduction

Laboratory experiments on flows in shallow fluid layers cannot provide enough information about the complete three-dimensional structure of the flow inside a vortex. Only measurements of the flow at the free surface can be made. By performing additional 3D numerical simulations, one can obtain quantitative data of the entire flow field. In the numerical simulations presented here, two parameters are varied: the Reynolds number associated with vertical diffusion, $Re_x$, and the ordinary Reynolds number associated with lateral diffusion, $Re$. We will show that the three-dimensional character of the flow is entirely determined by these two variables. The height-to-width aspect ratio $\mu = H/2L$ of the vortex is mentioned as well, to picture the shape of the vortex. Note that it can be expressed in $Re$ and $Re_x$ as $\mu \sim (Re_x/Re)^{1/2}$.

The effect of $Re$ and $Re_x$ on the quasi-two-dimensionality of the flow has been studied for two cases: a monopole in a single homogeneous layer of fluid and a monopole in a two-layer stratified system. The simulations for a stratified fluid have been performed, since experiments in thin stably stratified fluid layers have been reported in literature (see e.g. Marteau et al. 1994).
**4.2 Numerical Simulations of Monopoles in a Homogeneous Layer of Fluid**

**4.2.1 Initial Conditions**

For the initial condition, a purely azimuthal flow is taken, with a shielded Gaussian vorticity distribution and with a sine-like vertical dependence, as is described by the diffusion model in section 2.5. The initial velocity profile is thus given by equation (2.61) for \( t = 0 \), where we take \( a_0 = 1 \) and \( r_0 = 1 \)

\[
u_0(r, z) = \frac{r}{2} \exp(-r^2) \sin\left(\frac{\pi z}{2H}\right).
\]

In the first simulation presented below, the (ordinary) Reynolds number is taken \( Re = 500 \). The Reynolds number is defined earlier as \( Re = L^2 \omega_p/\nu \) (see page 21). Here, the peak vorticity at the axis \( \omega_p \) is taken as a typical value for the vorticity. The radius \( L \), where the vorticity profile changes sign, is taken as the typical length scale. The quantities \( a_0 \) and \( r_0 \), which determine the initial amplitude and radius of the vortex, are thus used to non-dimensionalize the problem. In most of the simulations, \( \omega_p = 1 \) and \( L = 1 \), so that the Reynolds number is inversely proportional to the viscosity, \( Re = 1/\nu \). For the simulations where the Reynolds number is varied, it is changed by changing the horizontal length scale \( L \).

In the diffusion model, we assumed that the higher order modes of the axial-temporal solution (2.54) can be neglected. Simulations for low values of the fluid depth \( H \), in which a uniform vertical velocity profile was taken, indeed showed a rapid relaxation towards a sine-like vertical velocity profile. For the sake of simplicity, we take this initial profile for all of the numerical simulations.

**4.2.2 Quantitative Characterization of the Quasi-two-dimensionality**

Basically, we will compare the evolution of the ‘real’ flow field with the Q2D solution of the diffusion model, as discussed in the previous chapter. However, it has been explained in section 2.5 that a secondary circulation should always be present in a monopolar vortex. In order to verify these findings, a numerical simulation of a monopole has been performed with \( Re = 500 \) and height \( H = 0.4 \) (\( Re_\lambda = 32 \)). Note that one can either use the height \( H \) or the associated Reynolds number \( Re_\lambda \). The aspect ratio was \( \mu = H/2L = 0.20 \).

In figure 4.1(a), the velocities in the \((r, z)\)-plane are shown at \( t = 5 \) for the first simulation. Indeed, the existence of a recirculation inside the vortex is confirmed. Note that the horizontal and vertical scales are not equal.

The central question that we would like to discuss is the following: under what conditions can a flow situation be qualified as quasi-two-dimensional? If the secondary circulation in the vortex is large, one could say that the flow is three-dimensional. But, what can be considered ‘large’, or what is ‘small’ here? This secondary flow thus needs to be quantified in some way. One quantity that determines the strength of the secondary flow is its...
The Manapolar Vortex in a Shallow Fluid Layer

Figure 4.1: Numerical simulation of a monopolar vortex in a shallow fluid layer with $Re = 500$ and $Re_\lambda = 32$. The three-dimensional recirculation in the $(r, z)$-plane of the monopole is shown at $t = 5$.

kinetic energy, compared to the kinetic energy of the azimuthal flow. For each component of the velocity $u_i$, where $i = (r, \theta, z)$ denotes the specific component, the kinetic energy is defined as

$$E_{k,i} = 2\pi \int_0^H \int_0^R \frac{1}{2} \rho(z) u_i^2(r, z) r \, dr \, dz,$$

where $H$ represents the total fluid depth and $R$ is the radius of the computational domain. In the first set of simulations, $\rho(z)$ is constant. Later on, numerical simulations with a two-layer stratified fluid will be shown, where the density is not constant along $z$.

In figure 4.2(a), the time evolution of the kinetic energies $E_{k,\theta}, E_{k,r}$ and $E_{k,z}$ is shown. The kinetic energies related to $r$ and $z$ appear to be small. In figure 4.2(b) these energies are plotted relative to the kinetic energy of the azimuthal flow. At $t = 0$, $E_{k,r}$ and $E_{k,z}$ are zero, according to the assigned initial condition. Then, they both increase in time, indicating that a secondary circulation is set up inside the vortex.

One effect of the secondary circulation can be seen in figure 4.3(a). The radial profile of the axial vorticity at the free surface, $\omega_z(r)$, is shown for three different times ($t = 0$, $t = 5$ and $t = 10$). Remember that in the diffusion model the vorticity profile could be rescaled at all times. Here, the vorticity profile clearly deforms as time proceeds: the core of the vortex seems to relax towards a state of solid-body rotation, since $\omega_z$ is almost constant in the core of the vortex. This deformation is a consequence of the secondary circulation: the fluid that is located near the bottom and close to the axis of the vortex is transported upwards due to the secondary circulation (see figure 4.1). These fluid parcels have a low amplitude of axial vorticity $\omega_z$. High-amplitude vorticity located near the surface and
4.2 Numerical Simulations of Monopoles in a Homogeneous Layer of Fluid

Figure 4.2: Numerical simulation of a monopolar vortex in a shallow fluid layer with \( Re = 500 \) and \( Re_\lambda = 32 \). Shown are (a) the time evolutions of the kinetic energies \( E_{k,r}, E_{k,\theta}, \) and \( E_{k,z} \), and (b) the energies \( E_{k,i} \) relative to \( E_{k,\theta} \), where \( i = r, z \).

Figure 4.3: Numerical simulation of a monopolar vortex in a shallow fluid layer with \( Re = 500 \) and \( Re_\lambda = 32 \). Shown are (a) the time evolution of the radial vorticity profile \( \omega_z(r) \) for three different times \( (t = 0, t = 5 \text{ and } t = 10) \) and (b) the time evolution of the peak vorticity at three different levels in the fluid, and the time evolution of the peak vorticity according to the model (solid line).
near the axis is transported outwards, which results in a smoothened vorticity distribution in the core of the vortex.

Another method to examine 3D effects on the evolution of the vortex is to monitor the peak vorticity \( \omega_p(t) \), which is located at the axis \( (r = 0) \). According to the Q2D diffusion model, the time evolution of the peak vorticity would be (see also Eq. (2.61))

\[
\omega_p(t) \sim \exp(-\lambda t) \frac{1}{(1 + 4\nu t)^2}.
\]  

(4.3)

In figure 4.3(b), the result of the numerical simulation of \( \omega_p(t) \) is shown at three different depths in the fluid, being \( h = H \) (free surface), \( h = H/2 \) and \( h = H/4 \). Note that the initial peak vorticity at \( r = 0 \) decreases with height as \( \sin(\pi z/2H) \). The solid line represents the time evolution of the peak vorticity at the surface according to (4.3).

It can be seen that for this vortex (\( \text{Re} = 500, \ H = 0.4 \)), the actual decay of \( \omega_p \) at the surface differs significantly from the prediction based on the model (solid line), and besides that, it is observed that the decay is not uniform in \( z \): the peak vorticity at the surface decreases even below the peak vorticity at \( h = H/2 \) for \( 7 < t < 12 \). This enhanced decay can also be understood by considering the secondary circulation, which transports low-amplitude axial vorticity located near the bottom upwards.

A third quantity that can be considered to obtain information about the three-dimensional structure of the monopole, is the divergence field at the surface. Here, the horizontal divergence, which is defined as

\[
\nabla_h \cdot \mathbf{u} = \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} = -\frac{\partial u_z}{\partial z},
\]  

(4.4)

has been considered, since the total divergence obviously is equal to zero. If we consider the effect of the recirculation pattern on the flow field at the surface, there must be a positive divergence near the axis, accompanied by a ring of negative divergence, where the fluid moves downwards. The radial divergence profile has been calculated for this simulation with \( \text{Re} = 500 \) and \( H = 0.4 \). It is shown in figure 4.4. The divergence has been rescaled with the maximum vorticity of the flow field. For this specific simulation, the horizontal divergence reaches a maximum value of roughly 20 % of the maximum vorticity at \( t = 5 \).

The results of the first simulation indicated that a vortex with \( \text{Re} = 500 \) and \( H = 0.4 \) cannot be described very well by the Q2D model, and it cannot be considered as perfectly quasi-two-dimensional. However, we need criteria to quantify the quasi-two-dimensionality in some way. The first criterion that we will introduce is related to the kinetic energy of the secondary circulation; the second criterion is related to the deformation of the vorticity profile.

In the first criterion, the kinetic energy of the secondary flow is compared with the kinetic energy of the azimuthal flow. It is assumed that the flow behaves Q2D if

\[
q_r(t) = \frac{E_{k,r}(t)}{E_{k,\theta}(t)} \leq 0.01, \quad q_z(t) = \frac{E_{k,z}(t)}{E_{k,\theta}(t)} \leq 0.01,
\]  

(4.5)
4.2 Numerical Simulations of Monopoles in a Homogeneous Layer of Fluid

![Graph showing the radial profile of the horizontal divergence for three different times (t = 0, t = 5 and t = 10). The divergence has been rescaled with the maximum vorticity at all times.](image)

Figure 4.4: Numerical simulation of a monopolar vortex in a shallow fluid layer with Re = 500 and Re<sub>λ</sub> = 32. Shown is the radial profile of the horizontal divergence for three different times (t = 0, t = 5 and t = 10). The divergence has been rescaled with the maximum vorticity at all times.

stating that the values of E<sub>k,r</sub> and E<sub>k,z</sub> should not exceed 1% of the value of E<sub>k,θ</sub>. This seems rather strict; however, keep in mind that if the criterion is satisfied, the maximum values of the velocities u<sub>r</sub> and u<sub>z</sub> are roughly only one order smaller than the maximum value of the azimuthal velocity u<sub>θ</sub>. The first simulation that has been discussed clearly does not satisfy the criterion, since the maximum value of q<sub>r</sub> almost measures 0.05 (see figure 4.2(b)).

Condition (4.5) is associated with the complete 3D flow field. The laboratory experiments, however, only provide velocities and vorticities evaluated at the free surface. The second criterion is therefore involved with the vorticity profile at the free surface. We will quantify its deformation by calculating the enstrophy Z at the surface, which is defined as

\[ Z = \iint_A \omega_z^2 \, da = 2\pi \int_0^R \omega_z^2 r \, dr, \quad (4.6) \]

where A represents the area of the free surface and R the radius of the computational domain. The second criterion states that the flow can be considered Q2D if the ratio Q, which is defined as

\[ Q = \frac{\int_0^R (\omega'_z(r,0) - \omega'_z(r,t))^2 r \, dr}{\int_0^R \omega'^2_z(r,0) r \, dr}, \quad (4.7) \]

does not exceed 0.10 at t = 5. The quantity \( \omega'_z(r,t) \) represents the rescaled vorticity profile with respect to its amplitude and radius. The deformation of the vorticity profile is thus
characterized quantitatively by comparing the shape of a rescaled profile at time $t$ with the initial condition at $t = 0$, which also serves as a weight function. For the case with $\text{Re} = 500$ and $H = 0.4$, as discussed above, this criterion is not satisfied either, since $Q = 0.97$.

In terms of the divergence, a flow will be considered to behave Q2D if the horizontal divergence does not exceed 15% of the maximum value of the vorticity field at the free surface.

Of course, these criteria are in some sense arbitrary. One could argue that a flow can still be considered Q2D if $Q \leq 0.20$ at $t = 5$. We also want to point out that the second criterion is based on the first one. No additional criteria will be formulated, since this would not reveal new information that is essentially different from the criteria cited above.

### 4.2.3 Variation of the Fluid Depth

In the simulations that will be presented below, the fluid depth was varied to determine its effect on the 3D structure of the vortex and the influence on the decay properties of the flow. While $L$ was kept constant, five different fluid depths $H$ were taken, $H = 0.1, 0.2, 0.4, 0.5$ and 1.0. The corresponding values for $\text{Re}_\lambda$ are 2.0, 8.1, 32, 51 and 203. The height-to-width aspect ratios are given by $\mu = 0.05, 0.1, 0.2, 0.25$ and 0.5, respectively. The value of $\text{Re}_\lambda$, and thus the height $H$, is thus systematically decreased for fixed Reynolds number $\text{Re}$. The influence of the ordinary Reynolds number $\text{Re}$ will be discussed in the next part of this section. We have seen that a monopole with $\text{Re} = 500$ and $H = 0.4$ ($\text{Re}_\lambda = 32$) can

not be regarded Q2D, according to our criteria. Halving the fluid depth to $H = 0.2$ (thus $\mu = 0.1$ and $\text{Re}_\lambda = 8.1$) shows that the vortex becomes 'more 2D', as can be concluded from figure 4.5. The maximum value of the ratio $q_r$ is only 0.007 and $q_z$ is even smaller. Considering the shape of the vorticity profile, we observe that its deformation is less severe.

![Figure 4.5: Results of a numerical simulation with $H = 0.2$ ($\text{Re}_\lambda = 8.1$) and $\text{Re} = 500$. Shown are (a) the time evolution of the kinetic energy of the secondary flow, (b) time evolution of the vorticity profile $\omega_z(r)$ at the free surface, (c) the decay of the peak vorticity $\omega_p$ at three different levels in the fluid; the solid line represents the decay according to the analytical model.](image)
4.2 Numerical Simulations of Monopoles in a Homogeneous Layer of Fluid

than for $H = 0.4$. In quantitative terms, the value of $Q$ is 0.093 at $t = 5$. Also, the decay of the peak vorticities is not significantly different from the analytical model (compare with 4.3(b)!”. This vortex can thus be considered Q2D, since $q_{r,z} \leq 0.01$ and $Q \leq 0.1$.

As one would expect, the vortex becomes even ‘more 2D’ if the fluid depth is decreased to $H = 0.1$ ($\mu = 0.05$, $Re_\lambda = 2.0$). The results for this simulation are given in figure 4.6(a). The ratio $q_r$ reaches a maximum value of 0.0005; $q_z$ is almost equal to zero. The vorticity profile does not deform significantly, according to the low value of $Q$ at $t = 5$, being $Q = 0.0043$. The decay scenario fits almost perfectly with the analytical model now, as can be concluded from the decay of the peak vorticities.

However, if we increase the fluid depth to $H = 0.5$ ($\mu = 0.25$, $Re_\lambda = 51$), the consequences of the secondary circulation are more pronounced and are clearly visible in figure 4.6(b). Not only the energy of the secondary circulation increases, $q_r$ and $q_z$ rise to 0.07 and 0.01, respectively, but also the radial vorticity profile deforms considerably, with a $Q$ of 1.5 at $t = 5$. The vortex can therefore not be considered Q2D anymore. The dynamics become more 3D when the vortex height is doubled to $H = 1.0$ ($\mu = 0.5$, $Re_\lambda = 203$), see figure 4.6(c). The maximum values of $q_r$ and $q_z$ are 0.17 and 0.05, respectively, and $Q = 1.64$ at $t = 5$. The secondary circulation results in a large deformation of the vorticity profile, where the location of the maximum vorticity is even shifted outwards. The peak vorticities do not show a uniform decay at all. At $t = 5$, the peak vorticities at $H = H/2$ and $H = H/4$ have become larger than the vorticity at the free surface.

One may conclude that the flow loses its Q2D character for larger fluid depths. Based on the simulations performed with $Re = 500$, the analytical model can only be applied to a vortex in a very shallow fluid layer ($H \leq 0.2$). The results of the numerical simulations in this section are summarized in appendix C.

4.2.4 Variation of the Reynolds Number

In this section, we will analyze the influence of the Reynolds number $Re$ on the dynamics of the vortex. In all cases, the fluid depth $H$ (and thus $Re_\lambda$) remains constant. Changing the Reynolds number influences the action of lateral diffusion. Let us start with a simulation with $H = 0.4$ and $Re = 500$, as discussed in the beginning of this section. Two simulations will be compared with this case: one with a lower Reynolds number ($Re = 125$, so that $\mu = 0.40$) and one with a higher Reynolds number ($Re = 2000$, so that $\mu = 0.10$). The aspect ratio of the vortex changes, because the Reynolds number is changed by varying the radius $L$ of the vortex. The value of $Re_\lambda$ is clearly not affected by changing the horizontal scale of the flow. The results of the three simulations are shown in figure 4.7.

It can be observed that the kinetic energy of the secondary flow decreases as the Reynolds number decreases. For $Re = 125$ (figure 4.7(a)), $q_r$ has a maximum value of about 0.03. Also, $Q$ decreases to a value of $Q = 0.44$. The simulation with $Re = 2000$ (figure 4.7(c)) shows an increasing secondary circulation, according to a maximum value for $q_r$ of 0.05 and a $Q$-value of $Q = 1.19$ at $t = 5$. From the decay properties of the flow, it can also be observed that the case with $Re = 125$ is ‘more 2D’ than the case in which $Re = 2000$. Upon first sight, this may seem rather strange. A vortex with a height-to-width
aspect ratio of $\mu = 0.4$ appears to be more 2D than one for which $\mu = 0.1$? Apparently, the aspect ratio itself is not an essential parameter for the evolution of the flow, and what
4.2 Numerical Simulations of Monopoles in a Homogeneous Layer of Fluid

Figure 4.7: Results of numerical simulations with varying Reynolds numbers. Shown are the time evolution of the kinetic energy of the secondary flow, the time evolution of the vorticity profile $\omega_z(r)$ at the free surface and the decay of the peak vorticity $\omega_p$ at three different levels for (a) $Re = 125$ ($\mu = 0.40$), (b) $Re = 500$ ($\mu = 0.20$) and (c) $Re = 2000$ ($\mu = 0.10$). In all cases, the fluid depth was $H = 0.4$ ($Re = 32$).

is even more important: the geometrical confinement, which is a commonly used argument for two-dimensionality, does apparently not determine whether a flow is Q2D! The flow is
merely governed by Re and Re_\alpha. It is thus possible to create two vortices with the same aspect ratio where one can be considered Q2D and the other one is essentially 3D. By considering the 3D Navier Stokes equation (2.2), one can see that the non-linear advective term becomes more important for higher Reynolds numbers. In our case, the advective term is related to the secondary circulation in the (r, z)-plane. Hence, the secondary circulation becomes relatively more important for higher Reynolds numbers. In other words, the dynamical aspect of changing the Reynolds number is apparently stronger than the geometrical effect.

It is yet not completely understood whether very small-scale vortices can be considered Q2D, as their aspect ratio is relatively high. However, a small-scale vortex has a small Reynolds number and if one takes into account the results that were just presented, one might expect that these vortices can be considered Q2D after all. In order to study this in more detail, a simulation has been performed with aspect ratio \mu = 5 (H = 0.4, Re_\alpha = 32 and Re = 0.80. The vortex radius is thus five times smaller than its height. The results are given in figure 4.8. Indeed, the kinetic energy of the secondary circulation is extremely small, \qr has a maximum value of \qz = 2 \cdot 10^{-5}. A clear lateral expansion is also observed since lateral diffusion now plays an important role in the evolution of the flow field (figure 4.8(b)).

Several additional simulations have been performed to construct a regime diagram in which the quasi-two-dimensional character of the flow is indicated as a function of Re and Re_\alpha. The diagram is shown in figure 4.9. According to our criteria, the flow can be considered Q2D if it is located below the shaded line. Above the line, it should be considered 3D.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.8}
\caption{Evolution of a small-scale vortex structure, where Re = 0.80 and Re_\alpha = 32, \mu = 5. Shown are (a) the time evolution of the kinetic energy of the secondary flow and (b) the evolution of the vorticity profile \omega_z(r) (rescaled with the peak vorticity at the free surface for all times.)}
\end{figure}
4.2 Numerical Simulations of Manapoles in a Homogeneous Layer of Fluid

4.2.5 The Effect of the Initial Condition

It was mentioned previously that the Gaussian isolated vortex is a self-similar solution to the diffusion equation. Eventually, isolated vortices in a purely 2D flow will evolve towards this Gaussian vorticity profile. A useful analytical expression for the vorticity distribution of a family of isolated monopolar vortices was given by Carton et al. (1989)

\[ \omega(r, \alpha) = (1 - \frac{1}{2}sr\alpha) \exp(-r\alpha), \]  

(4.8)

where \(\alpha\) is a parameter that controls the steepness of the vorticity profile. The vorticity profile for the isolated Gaussian vortex is a special case of (4.8) with \(\alpha = 2\).

To study the effect of slightly different initial conditions on the dynamics of monopolar vortices in shallow fluid layers, and to test the robustness of our criteria, two additional simulations have been performed: one in which the vorticity profile was an \(\alpha\)-profile with \(\alpha = 3\), and one was performed with an initial profile for which the steepness parameter was \(\alpha = 4\). These simulations have been compared with the simulation where the initial condition was an isolated Gaussian vortex (\(\alpha = 2\)). The results are shown in figure 4.10. The Reynolds number and the height of the vortex remained constant (\(Re = 500\) and \(H = 0.4\) (\(Re_\lambda = 32\))). Small differences can be observed between the different simulations. The general behaviour, though, is similar. The kinetic energy of the secondary flow slightly increases for steeper \(\alpha\)-profiles. The deformation of the vorticity profile at \(t = 5\) is \(Q = 0.97, 0.98\) and \(0.99\) for \(\alpha = 2, 3\) and 4, respectively. Also, the decay of the peak vorticities along
Figure 4.10: Results of numerical simulations with different initial conditions. Shown are the time evolution of the kinetic energy of the secondary flow, the time evolution of the vorticity profile $\omega_z(r)$ at the free surface and the decay of the peak vorticity $\omega_p$ at three different levels for an $\alpha$-profile with (a) $\alpha = 2$, (b) $\alpha = 3$ and (c) $\alpha = 4$. In all cases, the Reynolds number is 500, and the fluid depth is $H = 0.4$ ($Re_\lambda = 32$).
4.3 Numerical Simulations of Monopoles in a Stratified Fluid

In order to construct a similar regime diagram for monopolar vortices in a two-layer stratified fluid, the same numerical simulations as described in the previous section have been performed. Instead of taking a homogeneous layer of fluid, now a two-layer stratified fluid has been used. The density difference in the two layers is 10%, corresponding to a Froude number of Fr = 1.0. The density changes within a region which has a thickness of 10% of the total fluid depth. The Schmidt number in all cases is Sc = 700, which is representative for the case that the stratifying agent is Sodium Chloride (salt).

We may expect that the model, as presented in section 2.5, describing vortices in a homogeneous fluid layer, can also be used to describe the vertical structure and the decay properties of vortices in a two-layer stratified system at least to some extent. One problem arises in the different values of $v_1$ in the upper and lower layer, which depends on the density $\rho$. As a consequence, the boundary condition at the interface yields $\nu_1 \rho_1 \frac{\partial u_1}{\partial z} \big|_1 = \nu_2 \rho_2 \frac{\partial u_2}{\partial z} \big|_2$, so that a slight kink is expected in the vertical velocity profile at the density interface. Another difference arising in a two-layer stratified fluid is the deformation of the density interface. This will be discussed later in this section.

The remaining part of this section is organized in a similar way as the previous section about the monopoles in a single layer of fluid. First, the dynamics and 3D structure of a vortex in a two-layer stratified fluid is analyzed. Then, the variation of $Re_A$ and $Re$ will be discussed.

The parameters of the first simulation that will be presented here are, in analogy with the single-layer case, $Re = 500$ and $H = 0.4$ (so that $Re_A = 32$ and $\mu = 0.20$). The results are presented in figure 4.11. Compared to the corresponding simulation in a homogeneous layer of fluid, we observe that the energy of the secondary circulation is approximately four times smaller now; the maximum value of $q_r$ is 0.012 here. Also, the deformation of the radial vorticity profile is much weaker; the value of $Q$ is only 0.0044, which is small compared to a value of $Q = 0.97$ for the homogeneous case. The pattern of the secondary circulation in the $(r,z)$-plane is remarkably different. The recirculation pattern is shown in figure 4.12 by velocity vectors in the $(r,z)$-plane for $t = 5$ and $t = 10$. Instead of one large recirculation cell, as observed in the case of the single layer, now a multiple-cell structure of counter-rotating circulations is observed. This rather complex cell pattern is not stationary. This is probably due to the action of internal waves, that are excited by the upwards motion of the fluid near the axis. This mechanism will be discussed in more detail later this section. Although the circulation pattern is more complex than for the homogeneous case, it is also much weaker and, as a consequence, its influence on the
behaviour of the flow is less pronounced.

We would like to discuss the horizontal (relative) divergence of the flow field at the surface, in analogy with the case of the homogeneous fluid layer. In figure 4.13, the horizontal divergence at the surface is shown for two different times. Note that the divergence relative to the maximum vorticity $\omega_z$ at the free surface has been calculated. The time evolution of the secondary circulation pattern is clearly visible. The recirculation in the $(r, z)$-plane here results in a sink at the axis at $t = 5$, while at $t = 10$ the divergence at the axis is positive, corresponding to a source. It can also be observed that the secondary circulation is relatively less pronounced for $t = 10$. By comparing figures 4.13 and 4.12 it can be observed that the divergence pattern at the free surface is not representative for the recirculation pattern inside the dipole, which was the case for the monopole in a homogeneous layer. Besides, the maximum value for the horizontal divergence is almost 10 times lower in the case of a stratified fluid for this specific simulation.

When considering figure 4.11(c) again, one notices that the decay in the upper layer of the vortex is much weaker than in the case of a homogeneous layer. At least three mechanisms are believed to play a role in this. The main reason is the density difference of the fluid in the upper and the lower layer. Due to buoyancy forces, the advection of low-amplitude axial vorticity from the lower to the upper layer is prevented. Secondly, the use of different densities implies different kinematic viscosities. The kinematic viscosity is related to the density as $\nu = \mu/\rho$. As a consequence, the diffusion in lateral and vertical sense act at different time scales. However, the density difference is only 10%, so one may assume that this effect is very weak.

A third mechanism is related to the deformation of the density interface. In general, the pressure in a moving fluid is lower than in a still fluid, according to Bernoulli’s law (inviscid, stationary flow). If we consider the pressure in our two-layer system, a pressure drop occurs in the upper layer, where the fluid rotates faster, which is bigger than the
pressure drop in the lower layer. The density interface will thus be lifted during the first time units, storing an amount of potential energy. As the vortex decays in time, the density interface slowly relaxes to its original position releasing the potential energy. The decay of the peak vorticity in a two-layer system can thus be explained in the following way (see figure 4.11(c)): due to the lift of the interface, the vortex tubes above the interface are compressed leading to a substantial decrease of axial vorticity at the free surface. Beneath the interface, the vortex tubes are stretched leading to increasing vorticity. At \( t \approx 3 \), the interface starts to relax to its original position, so that vortex tubes above the interface are being stretched: the vorticity increases here. At the same time, the vortex tubes beneath the interface are being compressed, so here the vorticity decreases. One can observe in figure 4.11(c) that from \( t = 7 \), the decay of axial vorticity in the upper layer, i.e. at \( h = H \), is in agreement with the diffusion model. The time evolution of the density interface for this simulation is shown in figure 4.14 for three different times. Shown are isopycnals, i.e. isolines of the density.

Note that in the case of a homogeneous layer a similar argumentation can be made. An imaginary interface within the fluid will be lifted due to the same pressure drop in the upper part of the fluid, where the velocities are higher. One can indeed observe in figure 4.3(b) that there is also an enhanced decay for \( 0 < t < 3 \), similar to the case of a stratified fluid. The difference lies in the fact that there is no potential energy stored by an interface, and there will not be an equilibrium between pressure and buoyancy forces.
Figure 4.13: Numerical simulation of a monopolar vortex in a two-layer stratified fluid with $Re = 500$ and $Re\alpha = 32$. Shown is the radial profile of the horizontal divergence for three different times ($t = 0$, $t = 5$ and $t = 10$). The divergence has been rescaled with the maximum vorticity.

- Variation of the fluid depth and the Reynolds number

In order to construct a similar regime diagram for a monopolar vortex in a two-layer stratified fluid, additional numerical simulations have been performed. In the first series of simulations, the fluid depth has been varied. It can be concluded that the flow loses its two-dimensional character for larger fluid depths, similar to monopoles in a homogeneous fluid layer. Only for small fluid depths the decay can be described with the diffusion model very well.

The Reynolds number has been varied in the second series of numerical simulations, ranging from $125 < Re < 2000$. As in the case of a homogeneous fluid layer, decreasing the Reynolds number two-dimensionalizes the flow, whereas the flow loses its Q2D character for higher Reynolds numbers. The different runs will not be discussed here, but some results are given in appendix D. The regime diagram is shown in figure 4.15. Below the shaded line, the flow can be considered Q2D, above the line, the vortex should be considered essentially three-dimensional. Comparing the two regime diagrams, one can see that by using a two-layer stratified fluid the regime in which the flow can be considered Q2D increases considerably, especially for low Reynolds numbers. If one wishes to verify 2D theoretical and/or numerical models in laboratory experiments, it may be useful to use a two-layer stratified system.
4.3 Numerical Simulations of Monopoles in a Stratified Fluid

Figure 4.14: Numerical simulation of a monopolar vortex in a shallow layer of fluid, with $Re = 500$ and $H = 0.4$ ($Re_\lambda = 32$). Shown are isolines of density in the $(r,z)$-plane for (a) $t = 0$, (b) $t = 5$ and (c) $t = 10$.

Figure 4.15: Regime diagram for the quasi-two-dimensionality of flows in a two-layer stratified fluid as a function of $Re$ and $Re_\lambda$. Below the shaded line, the flow can be considered as Q2D; above the line it should be considered 3D.
Chapter 5

Dipolar Vortices in Shallow Fluid Layers

The numerical simulations performed for the monopolar vortex have provided more insight in its three-dimensional structure. We would like to verify these results by experiments, but it is difficult to create monopoles with our experimental set-up. The simplest coherent vortex structure that can easily be produced, is the dipolar vortex. It can be expected that the dynamics and the 3D structure of a dipole are more complicated than for the monopole due to the presence of non-linearity in the horizontal flow field (which corresponds to a non-zero advective term in the Navier-Stokes equation). Hence, the next logical step is to perform numerical simulations for dipolar vortices in a thin layer of fluid. The results of these simulations will be compared with laboratory experiments.

5.1 Numerical Simulations

5.1.1 Initial Conditions

The numerical simulations of the dipolar vortex were performed in a square domain with periodic boundaries in the $x$ and $y$ direction. The domain ranges from -5 to 5 in both directions. The domain is vertically bounded by a no-slip bottom and a stress-free surface. For the initial condition, a Lamb dipole (see section 2.4) was taken, with a sine-like vertical dependence of the vorticity distribution

$$\omega(r, \theta) = -\frac{1}{b} J_1(kr) \sin \theta \sin \left(\frac{\pi r}{2H}\right) \quad 0 \leq r \leq a .$$

The vorticity distribution in the exterior region ($r > a$) is equal to zero. The quantity $b$ represents the value of the first maximum of $J_1$, which is $b = 0.853$, which is located at $r = 1$. The first zero of $J_1$ is given by $r = a = 2.07$, so that $k = 1.85$. Here, the dipole has been scaled, so that $L$, being the distance of the centre of the dipole to the vorticity maximum, and the peak vorticity $\omega_p$ are unity. The Reynolds number has been defined as
5.1 Numerical Simulations

5.1.1 Results of a numerical simulation with $H = 0.4$ ($Re_\lambda = 32$) and $Re = 500$. Shown are isolines of vertical vorticity $\omega_z$ at the free surface for (a) $t = 0$, (b) $t = 5$ and (c) $t = 10$. The increments in the contour values are $\Delta \omega_z = 0.1$. Dashed lines indicate positive vorticity; solid lines represent negative vorticity.

Re = $L^2 \omega_p / \nu$, thus $Re = 1/\nu$. The Reynolds number associated with the vertical diffusion has been defined as $Re_\lambda = \omega_p / \lambda$ (see page 21). In the first set of simulations, the density was uniform along $z$. In addition, one simulation has been performed in which the fluid had a two-layer stratification, in analogy with the simulations for the monopoles. Also, one simulation has been performed in which the initial condition was a Stokes dipole (see section 2.4) with a sine-like vertical dependence.

5.1.2 Visualization of the Three-dimensional Flow Structure

In analogy with the monopole, we will first discuss a particular numerical simulation of a dipolar vortex in order to understand its three-dimensional flow properties. The first simulation is performed with $Re = 500$ and $H = 0.4$ (corresponding to $Re_\lambda = 32$), similar to the first simulation of the monopole. In figure 5.1, the vorticity distribution of the dipole at the free surface is shown for three different times ($t = 0$, $t = 5$ and $t = 10$). The contours represent isolines of vertical vorticity ($\omega_z$). The dipole translates through the fluid from right to left, while its maximum vorticity decreases, and its radius increases slowly.

The vertical structure of the dipole is visualized in figure 5.2(a), where the dipole is shown at three different levels in the fluid for $t = 5$. Shown are isolines of vertical vorticity ($\omega_z$) which are rescaled according to $\sin(\pi z / 2H)$. It can be observed that the vorticity distribution at $h = H$ is already showing a deformation for $t = 5$. Besides, the location of the extreme vorticities at $h = H$ have been shifted towards the front of the dipole. In figure 5.2(b), the corresponding velocity vector field at each level is shown. Note that the velocity vectors are also scaled with $\sin(\pi z / 2H)$. The translation of the dipole is an interesting problem itself. One would expect that the vertical dependence of the dipole structure is sine-like, just like the components of the velocity and the vertical vorticity. By
Figure 5.2: Results of a numerical simulation of a dipole with $H = 0.4$ ($Re_\lambda = 32$) and $Re = 500$. Shown are (a) isolines of vertical vorticity at three different levels for $t = 5$. The increments in the contour values are $\Delta \omega_z = 0.1, 0.07$ and $0.012$, respectively; (b) velocity vectors in the $(x, y)$-plane at three different levels, which are rescaled according to $\sin(\pi z/2H)$. 
5.1 Numerical Simulations

Figure 5.3: Numerical simulation of a dipolar vortex in a shallow fluid layer with \( Re = 500 \) and \( H = 0.4 \) (\( Re_\lambda = 32 \)). Shown are contours of the vertical velocity \( u_z \) at \( h = H/2 \) for (a) \( t = 5 \) and (b) \( t = 10 \). Solid lines represent positive \( u_z \); dashed lines negative \( u_z \). The increments in the contour values are \( \Delta u_z = 0.003 \).

considering the contours of the vertical vorticity in figure 5.2(a), it appears that the dipole translates more or less like a columnar structure through the fluid, and only a small vertical dependency, which is not sine-like, can be observed. As can be observed in figure 5.2(b), the velocities in the \((x, y)\)-plane are clearly sine-like dependent with \( z \), since the velocities have been scaled with \( \sin(\pi z/2H) \) in these figures. Although the horizontal velocities at each level vary in magnitude, there is somehow a mechanism involved which translates the dipole more or less like a columnar structure through the fluid. We still do not exactly understand this phenomenon, but it is very likely that it is an effect of the secondary circulation.

In section 2.5, we have tried to model the structure of the secondary flow inside a dipole, by extending the results that were found for the monopolar vortex. Here, it was assumed that each dipole half can be considered as an individual monopole. We pictured the secondary flow in the dipole as two doughnut-shaped rings, one in each dipole half. The secondary flow was supposed to be directed upwards in the centres of the dipole halves and it had a motion downwards near the separatrix. Well, that is what we expected.

The secondary circulation of the dipole, according to our simulation, is visualized in figure 5.3. Shown are isolines of vertical velocity \( u_z \) in a horizontal plane. Solid contours represent positive \( u_z \) (out of the plane), dashed contours represent negative \( u_z \). The contours are shown at level \( h = H/2 \) for \( t = 5 \) and \( t = 10 \). In accordance with our expectations, the flow is directed upwards in the centres of both dipole halves. There is a motion downwards at the front of the dipole, and behind the centre of the dipole, near the \( y = 0 \) axis. At the dipole sides, the vertical velocities are small, but negative. The structure of the secondary motion does not change significantly in time, as can be observed in figure 5.3. Also, it has been checked that the pattern does not change considerably with the vertical coordinate \( z \).

The velocity vectors in the two vertical symmetry planes of the dipole are shown in
Figure 5.4. Figure 5.4(a) shows the velocity vector field in the \((x, z)\)-plane at \(t = 5\). The view is thus in a direction, perpendicular to the translation direction of the dipole. The dipole translates from the right to the left. The downward motions at the front of the dipole and behind the centre of the dipole are clearly visible. Right in the centre of the dipole, the motion is directed upwards. In figure 5.4(b) the velocity vectors in the \((y, z)\)-plane are shown for \(t = 5\). The view is thus in the translation direction of the dipole, and in the vertical cross-section through the vorticity extremes. The flow is directed upwards in the centres of the dipole halves and there is a very small downward motion near the separatrix of the dipole. According to our simple model, we would expect four cells of counter-rotating vortices in this symmetry plane (see figure ??). Clearly, a 'real' dipole is much more complicated than our simplified model, which was based on the results obtained for the monopolar vortex.

![Diagram of velocity vectors in the \((x, z)\)-plane and \((y, z)\)-plane at \(t = 5\).](image)

To conclude this section, we would like to discuss the horizontal divergence of the flow field at the free surface. Here, the (horizontal) divergence is defined in Cartesian coordinates as

\[
\nabla_h \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \left( = - \frac{\partial w}{\partial z} \right). \tag{5.2}
\]

As was mentioned previously, we can only measure velocities at the free surface in the laboratory experiments. By considering the horizontal divergence of the flow at the free
surface, information about the interior flow can be obtained. In figure 5.5, a contour plot of the divergence field at the free surface has been plotted for the simulation with Re = 500 and $H = 0.4$ at $t = 5$ and $t = 10$. The divergence field has the same structure as the contour plots of $u_z$. The divergence has been scaled with the maximum vorticity of the flow field at the surface. It appears that the (relative) divergence is stronger at $t = 10$, which means that the secondary circulation has become more important at $t = 10$. The increments in the contour values are $\Delta (\nabla_h \cdot u) = 0.005$, so the maximum divergence is roughly 4% of the maximum vorticity. We have seen that a monopole with the same dynamical properties (i.e. $Re = 500$ and $H = 0.4$) showed a maximum (relative) divergence of 20% at $t = 5$, which is roughly five times higher.

![Figure 5.5](image)

**Figure 5.5:** Numerical simulation of a dipolar vortex in a shallow fluid layer with $Re = 500$ and $H = 0.4$ ($Re_\lambda = 32$). Shown are contours of the divergence at the free surface for (a) $t = 5$ and (b) $t = 10$. Solid lines represent positive divergence; dashed lines negative. The divergence has been scaled with the maximum vorticity. The increments in the contour values are $\Delta (\nabla_h \cdot u) = 0.005$ (non-dimensional).

### 5.1.3 Quantitative Characterization of the Quasi-two-dimensionality

The criteria that were used to characterize the three-dimensional structure of the monopole will also be used for the dipolar vortex. However, the coordinate system that is used forces us to formulate the criteria in a slightly different way.

The first criterion that is used for the monopole states that the radial and vertical component of the kinetic energy should not exceed 1% of the kinetic energy of the azimuthal (main) flow. Here, we use a Cartesian coordinate system, so this criterion has to be reformulated. For each component of the velocity $u_i$, where $i = (x, y, z)$ denotes the specific component, the kinetic energy is defined as

$$E_{k,i} = \int_0^H \int_{-\frac{1}{2}y}^{\frac{1}{2}y} \int_{-\frac{1}{2}x}^{\frac{1}{2}x} \rho(z) u_i^2 \, dx \, dy \, dz ,$$

(5.3)
where $H$ represents the total fluid depth and $X$ and $Y$ are the horizontal dimensions of the computational domain.

In figure 5.6(a), the time evolution of the kinetic energies $E_{k,x}$, $E_{k,y}$ and $E_{k,z}$ is shown, scaled with $E_{k,x}$. The value of $E_{k,z}$ is too small to be visible in this figure. The value of $E_{k,x}$ is the largest, since the dipole propagates in the $x$-direction. In the case of the monopole, the choice of the coordinate system allowed to us to calculate the energy of the secondary flow directly, by using the $r$ and $z$ component of the velocity. For the dipole, we use a Cartesian coordinate system, and the $x$ and $y$ components of the secondary flow cannot be distinguished from the $x$ and $y$ components of the main flow. For sure, the $z$-component of the velocity can be assigned to the secondary flow. The first criterion will be formulated as follows. The flow in a dipole will be considered Q2D if

$$q_z(t) = \frac{3E_{k,z}(t)}{E_{k,x}(t) + E_{k,y}(t) - 2E_{k,z}(t)} \leq 0.01. \quad (5.4)$$

Here, it is assumed that the kinetic energies of the secondary flow in the $x$ and $y$ direction are almost the same as $E_{k,z}$. In figure 5.6(b), the time evolution of $q_z$ is shown. It reaches a maximum value of 0.0045 around $t = 10$.

![Graphs](a) and (b) showing the time evolution of kinetic energies and the ratio $q_z$.

Figure 5.6: Numerical simulation of a dipolar vortex in a shallow fluid layer, with $Re = 500$ and $H = 0.4$ ($Re_\lambda = 32$). Shown are (a) the time evolutions of the kinetic energies $E_{k,x}$, $E_{k,y}$ and $E_{k,z}$, scaled with $E_{k,x}$, (b) the ratio $q_z$, which is an estimate of the energy of the secondary flow relative to the energy of the main flow.
5.1 Numerical Simulations

Figure 5.7: Numerical simulation of a dipolar vortex in a shallow fluid layer with \( Re = 500 \) and \( H = 0.4 \) (\( Re_\lambda = 32 \)). Shown are (a) the vorticity cross-section through the vorticity extremes of the dipole, for three different times, \( t = 0 \), \( t = 5 \) and \( t = 10 \), (b) the decay of the (rescaled) kinetic energy \( E_{k,x} + E_{k,y} \) compared with the model (solid line).

In analogy with the monopole, the second criterion, which quantifies the deformation of the vorticity profile at the free surface can now be formulated as

\[
Q = \frac{\int_{-\frac{1}{4}y}^{\frac{1}{4}y} \int_{-\frac{1}{4}x}^{\frac{1}{4}x} \left( \omega'_z(x,y,H,0) - \omega''_z(x,y,H,t) \right)^2 \, dx \, dy}{\int_{-\frac{1}{4}y}^{\frac{1}{4}y} \int_{-\frac{1}{4}x}^{\frac{1}{4}x} \omega''_z(x,y,H,0) \, dx \, dy}, \quad (5.5)
\]

where \( \omega'_z(x,y,H,0) \) and \( \omega''_z(x,y,H,t) \) represent the rescaled vorticity profile at the surface with respect to its amplitude and radius. The double prime indicates that the vorticity profile is shifted towards the origin, in order to correct for the translation of the dipole. In analogy with the monopole, we characterize the flow in the dipole to behave Q2D if

\[
Q \leq 0.10, \quad (5.6)
\]

where \( Q \) is evaluated at \( t = 5 \). For the simulation with \( Re = 500 \) and \( H = 0.4 \) (\( Re_\lambda = 32 \)), the value for \( Q \) is 0.22 at \( t = 5 \), so this flow can not be considered Q2D, according to our criteria. The simulation of the monopole with \( Re = 500 \) and \( H = 0.4 \) showed a value of \( Q = 0.97 \), indicating that three-dimensional effects (recirculation) play a more important role in the case of a monopole.
In figure 5.7(a), the vorticity profile through the vorticity extremes of the dipole is shown for three different times, \( t = 0, t = 5 \) and \( t = 10 \). The deformation of the vorticity profile is clearly visible, especially if one takes into consideration that the vorticity amplitude should be roughly 0.75 at \( t = 10 \), according to the combination of the Stokes model with vertical diffusion. The model predicts a decay of the vorticity amplitude proportional to \((1 + 2\nu t)^{-3/2} \exp(-\lambda t)\). One effect of lateral diffusion is nicely demonstrated in the same figure. The 'kink' in the initial vorticity distribution is soon smoothened due to diffusion of vorticity through the separatrix.

The energy of the main flow will be estimated by the sum of \( E_{k,x} \) and \( E_{k,y} \). A fraction of this energy will be related to the secondary circulation within the dipole, but this is assumed to be small. The kinetic energy is proportional to \(|u|^2\), so

\[
E \sim |u|^2 \sim E_0 \exp(-2\lambda t). \tag{5.7}
\]

For our simulation with \( H = 0.4 \), the energy decay in the simulation has been compared with the energy decay according to (5.7) which is shown in figure 5.7(b). This again confirms that the dipole can not be considered Q2D for the parameters used here. Note that by using equation (5.7) we do not take into account the decay of the flow due to lateral diffusion. For the range of Re and \( \text{Re}_\lambda \) in our numerical simulations presented here, we may assume that the effect of lateral diffusion on the scale of the dipole structure is negligible.

### 5.1.4 Variation of the Fluid Depth

In analogy with the numerical simulations of the monopole, the fluid depth has been varied to determine its influence on the Q2D character of the flow. While \( L \) remained constant (so that \( \text{Re} \) remained constant), five different fluid depths \( H \) were taken, being \( H = 0.1, 0.2, 0.4, 0.5 \) and \( H = 1.0 \), corresponding to values of \( \text{Re}_\lambda = 2.0, 8.1, 32, 51 \) and 203, respectively. For each fluid depth, we will consider three aspects of the flow: the kinetic energy of the secondary flow, which is characterized by \( q_z \), the deformation of the vorticity distribution at the free surface (characterized by \( Q \)) and the decay of the kinetic energy of the main flow.

The results of the simulation for half the fluid depth \( H = 0.2 \) (\( \text{Re}_\lambda = 8.1 \)) are shown in figure 5.8(a). The decay of the kinetic energy of the main flow is almost similar to that of the model, so that we expect a deformation of the vorticity profiles, which is not severe. This is indeed confirmed by the second figure, with a corresponding value of \( Q \) of 0.085 at \( t = 5 \). The maximum value of \( q_z \) is only \( 2.4 \cdot 10^{-4} \) (third figure). This dipole can thus be considered as Q2D, since \( q_z \leq 0.01 \) and \( Q < 0.10 \). Decreasing the fluid depth to \( H = 0.1 \) (\( \text{Re}_\lambda = 2.0 \)), reveals that the flow is 'more' 2D, since the maximum value of \( q_z \) is only \( 5 \cdot 10^{-6} \). The simulation is shown in figure 5.8(b). The decay of the kinetic energy \( E_{k,x} + E_{k,y} \) is in excellent agreement with the model. The vorticity profiles can be scaled almost perfectly, which is indicated by the very low \( Q \)-value of 0.007. Clearly, this dipole can also be considered as Q2D.
5.1 Numerical Simulations

Figure 5.8: Results of numerical simulations for a dipole with \( \text{Re} = 500 \). Shown are the decay of the energy \( E_{k,x} + E_{k,y} \); the solid line represents the decay according to the model, the time evolution of the cross-section of the vorticity profile \( \omega_z(y) \) through the vorticity extremes at the free surface and the time evolution of the kinetic energy of the secondary flow \( q_z \) for (a) \( H = 0.2 \) (\( \text{Re}_\lambda = 8.1 \)), (b) \( H = 0.1 \) (\( \text{Re}_\lambda = 2.0 \)).

If we make the fluid deeper than \( H = 0.4 \), it is expected that the flow loses its Q2D character. A simulation is performed for \( H = 0.5 \) (\( \text{Re}_\lambda = 51 \)), which is shown in figure 5.9(a). The kinetic energy of the secondary flow \( q_z \) reaches a maximum of almost 0.01, which is roughly twice as much as in the case of \( H = 0.4 \). The value of \( Q \) measures 0.19 here, which is smaller than for \( H = 0.4 \), where \( Q = 0.22 \) at \( t = 5 \). To evaluate the quantity \( Q \), a correction must be made for the translation of the dipole. The dipole at \( t = 5 \) must be shifted to the origin of the domain in order to subtract the profiles for \( t = 5 \) and \( t = 0 \). The error in determining this distance is 1 grid point, which results in a typical error in \( Q \) of 0.01. Although the value of \( Q \) is lower, the dipole with \( H = 0.5 \) is very likely to behave ‘more’ 3D than the dipole with \( H = 0.4 \), since the kinetic energy of the secondary circulation is twice as large. Note that the two criteria contradict each other at this point. Based on the value of \( q_z \), both vortices can be regarded Q2D, since for both cases \( q_z < 0.1 \), but if the value of \( Q \) is taken into account, both dipoles are to be considered 3D, since \( Q > 0.10 \).

Doubling the fluid depth to \( H = 1.0 \) (\( \text{Re}_\lambda = 203 \)) leads to a value for \( Q \) of 0.23 at \( t = 5 \).
Figure 5.9: Results of numerical simulations for a dipole with \( Re = 500 \). Shown are the time evolution of the kinetic energy \( E_{k,x} + E_{k,y} \); the solid line represents the decay according to the model, the time evolution of the cross-section of the vertical vorticity \( \omega_z(x) \) through the vorticity extremes at the free surface and the time evolution of the kinetic energy of the secondary flow \( q_z \) for (a) \( H = 0.5 \) (\( Re = 51 \)) and (b) \( H = 1.0 \) (\( Re = 203 \)).

The maximum value for \( q_z \) cannot be determined from figure 5.9(b), but it is larger than 0.01. This dipole cannot be considered as Q2D anymore, according to our criteria.

We can conclude that for the dipolar vortex three-dimensional effects become more important for larger fluid depths, similar to the monopolar vortex. The dipole is relatively ‘more’ 2D than the monopolar vortex, based on the values of \( q_z \) and \( Q \). While the kinetic energy of the secondary flow increases for increasing fluid depths, the value of \( Q \) seems to converge to a value of about 0.2. An overview of the simulations presented in this section, is given in appendix C.
5.1 Numerical Simulations

5.1.5 Variation of the Reynolds number

In analogy with the monopole, the Reynolds number has been varied to study its effect on the quasi-two-dimensionality of the dipolar vortex. The simulation with \( \text{Re} = 500 \) and \( H = 0.4 \) (\( \text{Re}_\lambda = 32 \)) will be compared to a simulation with a lower Reynolds number, \( \text{Re} = 125 \), and to a simulation with a higher Reynolds number, \( \text{Re} = 2000 \). In both cases, the value of \( H \) will remain constant. The value of \( \text{Re}_\lambda \) also remains constant, since the Reynolds number has been varied by varying \( L \), which is the typical length scale of the flow. Based on the results for the monopole, we expect that the dipole becomes 'more' 2D for lower Reynolds numbers and vice versa. In figure 5.10, the vorticity cross-sections through the vorticity extremes are shown for the simulations with \( \text{Re} = 125 \), \( \text{Re} = 500 \) and \( \text{Re} = 2000 \), respectively. For each simulation, the cross-section is given for \( t = 0 \), \( t = 5 \) and \( t = 10 \). The values for \( Q \) increases for higher Reynolds numbers, \( Q = 0.20, 0.22 \) and \( 0.24 \) for \( \text{Re} = 125, 500 \) and \( 2000 \), respectively, which indicates that the dipole indeed gets 'more 2D' for lower Reynolds numbers! We may conclude that both the dipolar vortex and the monopole get 'less 2D' for higher Reynolds numbers and for larger fluid depths.

![Figure 5.10: Results of numerical simulations with different Reynolds numbers. Shown are the time evolutions of the vorticity cross-section through the vorticity extremes of the dipole at \( t = 0 \), \( t = 5 \) and \( t = 10 \) for (a) \( \text{Re} = 125 \), (b) \( \text{Re} = 500 \) and (c) \( \text{Re} = 2000 \). In all cases, the fluid depth was equal to \( H = 0.4 \) (\( \text{Re}_\lambda = 32 \)).](image)

5.1.6 Simulations with Different Initial Conditions

We have performed two numerical simulations to study the effects of different initial conditions on the evolution of the flow field. In both simulations, the Reynolds number was \( \text{Re} = 500 \), and the fluid depth was \( H = 0.4 \) (\( \text{Re}_\lambda = 32 \)). Both runs can thus be compared with the first simulation that was presented (see page 56).

The initial vorticity distribution has been varied in the first simulation. Whereas a Lamb dipole was chosen for all the simulations presented earlier, now a Stokes dipole is taken. A 'real' dipole has not the vorticity distribution of a Stokes dipole and for sure,
the vorticity distribution of the Lamb dipole is not realistic either, considering the ‘kink’ in the vorticity distribution. A simulation with a different initial vorticity distribution has been performed to see whether the qualitative behaviour changes considerably. If the initial condition does not alter the flow properties substantially, the results of the numerical simulations can be used for laboratory experiments, in which the initial vorticity profile resembles a Lamb dipole or a Stokes dipole. Some results of the simulation with the Stokes dipole are shown in figure 5.11(a). The kinetic energy of the main flow decays similar to the case of the Lamb dipole. Also, the time evolution of the vorticity profile is similar, although the value of $Q$ is 0.32, which is much larger. The Stokes dipole has a non-zero vorticity distribution outside the separatrix at $t = 0$. This vorticity is advected towards the wake of the dipole for $t > 0$. It can indeed be observed in figure 5.11(a) that the radius of the dipole becomes smaller for larger time, although the effect is fairly small. Eventually, this results in the formation of ‘tails’ behind the dipole. This is probably a reason why $Q$ has a higher value for the Stokes dipole. The value of $q_z$ is smaller than for the Lamb dipole, $q_z \approx 0.0008$ which is almost twice as small. In the second simulation, a

![Diagram](a)

![Diagram](b)

Figure 5.11: Results of numerical simulations for a dipole with $Re = 500$ and $H = 0.4$ ($Re_\theta = 32$). Shown are the decay of the kinetic energy $E_{k,x} + E_{k,y}$, the solid line represents the decay according to the model, the time evolution of the vorticity cross-section through the vorticity extremes of the dipole $\omega_z(x)$ at the free surface and the time evolution of the kinetic energy of the secondary flow for (a) a single-layer system where a Stokes dipole was taken as the initial condition, (b) a two-layer stratified system with a Lamb dipole as initial condition.
two-layer stratification has been used, instead of a single layer of fluid. Note that the initial vorticity distribution is again a Lamb dipole here. The results are shown in figure 5.11(b). The decay of the kinetic energy of the main flow is similar to the single layer case. The difference in time evolution of the vorticity profile is striking. The value of $Q$ is only 0.055, whereas $Q = 0.22$ for the single layer. The kinetic energy of the secondary flow is also much smaller. The value of $q_z$ is roughly four times smaller for the stratified case. A two-layer stratified fluid is much more 'Q2D', which is very likely caused by the suppression of the secondary circulation due to the density interface (this was also observed for the monopole).

To study the 3D structure in this case in more detail, vector plots of the velocity fields in the $(x,z)$-plane and in the $(y,z)$-plane are given in figure 5.12. There are some remarkable differences with the single-layer case, see figure 5.4 on page 54. In figure 5.4(a), the motion in the $(x,z)$-plane is downward almost everywhere, which is clearly different from figure 5.12(a). Considering figure 5.12(b), it can be observed that three regions with significant upward velocities exist, whereas in the case of a single layer, two regions with considerable upward velocities could be observed. The difference between a dipole in a single-layer fluid and a two-layer stratified fluid is probably better demonstrated by the pattern of the secondary circulation in the $(x,y)$-plane. This pattern has been visualized in figure 5.13 by contour plots of the vertical velocity $u_z$ in the $(x,y)$-plane at half the fluid depth ($H/2$). The increments in the contour values are $\Delta u_z = 0.02$. The recirculation pattern is shown for $t = 5$ and for $t = 10$. It has been observed that the pattern of $u_z$ is roughly similar to that of $t = 10$ for greater times ($t = 15$). There is a region in the middle of the dipole, in

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.12.png}
\caption{Visualization of the flow in the interior of the dipole. A two-layer stratification has been used. Shown are (a) the velocity vectors in the symmetry plane of the dipolar vortex ($x = 0$), (b) the velocity vectors in the plane through the vorticity extremes of the dipole at $t = 5$.}
\end{figure}
which the fluid rises, which is embedded in a complicated structure where the fluid moves downwards.

With respect to laboratory experiments, where only velocities at the free surface can be obtained, the horizontal divergence field at the free surface is shown for $t = 5$ and $t = 10$ in figure 5.14. The horizontal divergence has been scaled with the maximum vorticity at $t = 5$ and $t = 10$, respectively. A comparison with figure 5.5 is worthwhile. Note that the increments in the contour values are 0.002 here, indicating that the maximum relative divergence is approximately 2 %, whereas the dipole in the homogeneous layer showed a maximum divergence of about 4 %. Note that the pattern of the horizontal divergence field at the free surface is almost the same to the contour plots of the vertical velocity inside the dipole, and can therefore be used in laboratory experiments to ‘measure’ the recirculation pattern within the dipole.

Although the interior structure of a dipole in a two-layer stratified fluid has been visualized in this section, it is yet not completely understood what the exact nature is of the complicated secondary circulation pattern within the dipole.

Figure 5.13: Numerical simulation of a dipolar vortex in a two-layer stratified fluid, with $Re = 500$ and $H = 0.4$ ($Re_a = 32$). Shown are contours of the vertical velocity $u_z$ at $H/2$ for (a) $t = 5$ and (b) $t = 10$. Solid lines represent positive velocities; dashed lines negative. The increments in the contour values are $\Delta u_z = 0.02$. 
5.2 Laboratory Experiments

Preliminary laboratory experiments have been performed in order to verify some aspects that were found in the previous section. In the laboratory, dipoles were created by using a single magnet, as is described in chapter 3. In analogy with the numerical simulations, experiments were performed for different fluid depths, ranging from $H = 4$ mm to $H = 12$ mm. For each fluid depth, one experiment was carried out by using a single layer of fluid and another was performed with a two-layer stratification. A stratified layer of 8 mm is thus composed of two layers of 4 mm, with a density difference of 10%. A typical laboratory experiment is shown in figure 5.15. This experiment was done in a single layer of fluid, with a fluid depth of $H = 8$ mm. The flow was forced for 2 seconds with a current of $I = 2$ A. The corresponding initial maximum vorticity is $2.5 \text{ s}^{-1}$ at $t = 0$. Here, $t = 0$ has been determined as the time where the vorticity reaches a maximum, and thus at the moment that the vortex starts to decay. The (initial) Reynolds number is approximately 1000, and the Reynolds number associated with vertical diffusion is $Re_v = 65$. The upper pictures in figure 5.15 show the vorticity field at the free surface at $t = 0$ s, $t = 1.7$ s and $t = 3.4$ s. It can be observed that the dipole, which has a circular shape, travels along a straight line. The velocity field, and thus the vorticity field, decreases in time, while the radius of the vortex slowly increases in time. The maximum values of the vorticity are $2.5 \text{ s}^{-1}$, $2.0 \text{ s}^{-1}$ and $1.4 \text{ s}^{-1}$ for $t = 0$ s, $t = 1.7$ s and $t = 3.4$ s, respectively. In figure 5.15(b), the corresponding measured horizontal divergence field $(\nabla_h \cdot u)$ at the free surface is shown. Apart from some local patches of non-zero divergence, due to measurement errors, the divergence field is close to zero. The maximum values of the divergence are $0.4 \text{ s}^{-1}$, $0.4 \text{ s}^{-1}$ and $0.3 \text{ s}^{-1}$ for $t = 0$ s, $t = 1.7$ s and $t = 3.4$ s, respectively. According to the numerical simulations, we would expect a divergence in the order of magnitude of
Figure 5.15: Results of a laboratory experiment with $H = 8$ mm ($Re_\alpha = 65$) and $Re \approx 1000$. The horizontal and vertical scales are given in centimeters. Shown are (a) the vorticity distribution $\omega_z$ at the free surface for three different times, $t = 0$ s, $t = 1.7$ s and $t = 3.4$ s and (b) the corresponding measured divergence $\nabla_h \cdot u$ of the flow field at the free surface. The increments in the contour values are $\Delta \omega_z = 0.25$ and $\Delta (\nabla_h \cdot u) = 0.1$, respectively.
5% in the centres of both dipole halves. If we take into account that the accuracy of the vorticity and the divergence measurements is roughly 15 - 20%, it can be explained that the actual divergence cannot be resolved in the measurements.

To make a proper comparison between the experiments and the simulations, the experiments need to be made dimensionless. In that case, a quantitative comparison can be made if both the Reynolds number and the parameter $Re_x$ are the same. In our experimental set-up, it was difficult to create a dipole with a well-defined Reynolds number. The Reynolds numbers of our experiments range between 300 and 3500. It is thus very difficult to make a quantitative comparison between the experiments and the simulations. In this section, we will compare the experiments and the numerical simulations qualitatively by considering the time evolution of the vorticity cross-sections through the vorticity extremes of the dipole at the free surface. If this deformation is weak, the effects of the secondary circulation, and thus its strength, is relatively small. If the deformation of the vorticity distribution is large, the flow can not be qualified as Q2D anymore. Besides, the value of $Q$, which is a measure for the deformation of the total vorticity field, will be evaluated for each experiment, and a mutual comparison is made.

First, we need to determine the form of the initial vorticity profile of our laboratory dipole. As an example, the experiment which is shown above will be taken. In figure 5.16, the rescaled initial cross-section of the vorticity distribution of the experiment (crosses) is compared to that of the Lamb dipole (solid line) and the Stokes dipole (dashed line). Apparently, the initial vorticity distribution of a real dipole is somewhere in between that of a Lamb or a Stokes dipole. Fortunately, this is what we could have expected! In the Stokes limit, advection of vorticity is neglected, so it represents the asymptotic behaviour of a dipole for very small Reynolds numbers. On the other hand, the Lamb dipole is inviscid, which correspond to very high Reynolds numbers. Since the Reynolds number of
Table 5.1: Values of $Q$ for five different experiments. Three different fluid depths were taken. A two-layer stratification and a single layer of fluid has been used. The value of $Q$ has a relative error of 50%.

<table>
<thead>
<tr>
<th>$H$ (mm)</th>
<th>$Re$</th>
<th>Layers</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3500</td>
<td>1</td>
<td>0.59</td>
</tr>
<tr>
<td>8</td>
<td>1000</td>
<td>1</td>
<td>0.11</td>
</tr>
<tr>
<td>12</td>
<td>1000</td>
<td>1</td>
<td>0.45</td>
</tr>
<tr>
<td>8</td>
<td>750</td>
<td>2</td>
<td>0.17</td>
</tr>
<tr>
<td>12</td>
<td>300</td>
<td>2</td>
<td>0.42</td>
</tr>
</tbody>
</table>

our laboratory experiment is neither very small nor very large, it can be expected that we find a vorticity distribution that lies in between. It is thus plausible to make a qualitative comparison between the experiments and the numerical simulations.

As mentioned before, the value of $Q$ has been calculated for each experiment by comparing the initial vorticity distribution with the distribution at $t = 5$ (dimensionless time). Here, the time is made dimensionless by the maximum vorticity at $t = 0$. Take into account that this value for $Q$ has quite a large error. Apart from the error in the vorticities ($\approx 20\%$), an additional error is introduced by translating the vortex to its initial position. This error is less than 5%. The total error in $Q$ is thus roughly 50%, since two vorticity profiles are being subtracted. In table 5.1, the $Q$-values for five different experiments are given. The experiments for $H = 12$ mm both show a higher $Q$ value than the case of 8 mm. The $Q$-value for $H = 4$ mm is larger than in the case of $H = 12$ mm, but this might have something to do with the high value of the Reynolds number (3500). With respect to the high error in $Q$, the only conclusion that can be made at the moment is that the experimental values for $Q$ are in the same order of magnitude as the values obtained by the numerical simulations.

As mentioned before, the amplitude of the vorticity distribution decays as

$$(1 + 2\nu t)^{-3/2} \exp(-\nu t).$$

We will compare this with the decay of the vertical vorticity $\omega_z$ measured in two experiments. In the first experiment, the fluid was fairly shallow $H = 4$ mm, corresponding to $Re_\lambda = 25$. The maximum vorticity was $\omega_p = 8 \text{s}^{-1}$, corresponding to a Reynolds number $Re = 3500$. The second experiment was carried out in a 'deep' fluid layer. Here, $H = 12$ mm (corresponding to $Re_\lambda = 70$), and the initial peak vorticity was $2.4 \text{s}^{-1}$, corresponding to $Re = 1000$. In both cases, the fluid consisted of a homogeneous layer. The results are shown in figure 5.17. In both cases, the decay in the experiment has a deviant behaviour from the decay predicted by the Q2D model. Very likely, both dipoles cannot be considered as Q2D. This is also indicated by the values of $Q$. For $H = 4$ mm, $Q$ is equal to 0.59 and for $H = 12$ mm, $Q = 0.45$ (see table 5.1). It seems that the effect of the fluid depth on the three-dimensional structure of the dipole cannot be observed accurately in these two experiments. It is expected that the dipole in the shallow fluid behaves 'more' 2D, than the dipole in the deeper layer. However, the values of $Re_\lambda$ are quite large in both experiments, $Re_\lambda = 25$ and $Re_\lambda = 70$ for $H = 4$ mm and $H = 12$ mm. Based on the Reynolds number of both experiments, and based on the regime diagram for
the monopolar vortex, which is in fact not completely legitimate, we can suspect that the flows are in the 3D regime.

The effect of the two-layer stratification on the dynamics of the dipole is illustrated in figure 5.18. Two experiments were performed in a fluid layer of 8 mm. The first experiment was done in a single layer, the second was performed with a two-layer configuration. The initial Reynolds numbers were equal to $\text{Re} = 1000$ and $\text{Re} = 750$, respectively. The value of $\text{Re}_\lambda$ was 30 for the single-layer, and it was equal to 25 for the stratified layer. These figures show that the stratified case is ‘more’ 2D than the single layer experiment, which was also found in the numerical results.

It can be concluded that some qualitative aspects found in the numerical simulations can also be observed in the laboratory experiments. A deformation of the vorticity profile at the free surface can be observed in the cases that the flow is expected to behave 3D. For a better verification of the numerical results, experiments need to be conducted in which the initial conditions can be defined with higher accuracy.
5 Dipolar Vortices in Shallow Fluid Layers

5.2.1 Decaying Quasi-two-dimensional Turbulence

As was mentioned before, a lot of experimental and numerical work has been done to study quasi-two-dimensional turbulence. Sophisticated numerical models have been developed, which have been compared with quasi-2D laboratory experiments (see e.g. Jüttner et al.). In this thesis, only the quasi-two-dimensionality of monopoles and dipoles has been studied. Although two-dimensional turbulence is far more intricate than single coherent vortex structures, it can be expected that turbulent flows are quasi-two-dimensional too only under certain conditions. In figure 5.19, a nice (preliminary) laboratory experiment of Q2D turbulence is shown. The magnet configuration was a $10 \times 10$ chessboard-like pattern. The flow has been forced for 2 s, with a current of $I = 2$ A. The effect of self-organization is visible. At $t = 0$ (end of forcing), one hundred dipoles are created. Note that figure 5.19 does not show the entire (experimental) domain. The individual dipoles soon organize into an array of one hundred single monopoles. These monopoles start to interact with...
5.2 Laboratory Experiments

Figure 5.19: Results of a laboratory experiment of decaying turbulence. The fluid depth is $H = 8\text{mm}$ ($Re_\lambda = 35$) and $Re \approx 750$. The fluid is a two-layer stratified system. The scales are in cm.
each other by clustering and merging, resulting in larger vortex structures. The quasi-
two-dimensionality of such a flow is also determined by the strength of the secondary
flow. This could perhaps be measured by considering the relative horizontal divergence of
the flow field. Future studies will hopefully reveal more information about the quasi-two-
dimensionality of such turbulent flows.
Chapter 6

Conclusions and Recommendations

6.1 Conclusions

The dynamics of monopolar and dipolar vortices in thin fluid layers were studied numerically and by laboratory experiments. Commonly, it is assumed that flows in thin layers of fluid can be considered as quasi-two-dimensional. We derived a diffusion model which showed that the vertical vorticity distribution of the vortex can be scaled at all times, in the case that one can speak of a quasi-two-dimensional flow. The flow can then be parameterized by the 2D Navier-Stokes equation with an additional linear term, which describes vertical diffusion.

However, vortices in thin layers of fluid do exhibit three-dimensional effects due to the vertical shear of the velocity field. This vertical shear causes a secondary flow within the planar vortex, which may result in a deformation of the vorticity distribution at the free surface, so that the vorticity profiles cannot be scaled anymore. Depending on the strength of this secondary circulation, a flow can be considered Q2D or not.

Two criteria were formulated to quantify the quasi-two-dimensionality of the flow. The first criterion is related to the kinetic energy of the secondary flow and the second criterion quantifies the deformation of the vorticity distribution of the vortex at the free surface.

Three-dimensional direct numerical simulations were performed for monopolar and dipolar vortex structures. It turned out that the dynamics of the flows, which are determined by the ordinary Reynolds number Re, and by the Reynolds number associated with the vertical diffusion, Re\(\lambda\), determine whether a monopole or dipole can be considered Q2D. In general, a vortex can only be considered Q2D for small enough Re and Re\(\lambda\). Moreover, it was found that the geometrical confinement of a flow is not a good qualification to determine its Q2D properties, because the dynamical effect of changing the Reynolds number is stronger than the geometrical effect. A comparison between monopoles and dipolar vortices reveals that dipoles are ‘more’ 2D than monopoles under similar conditions, according to our criteria.

Numerical simulations with a two-layer stratification, which has been used in several laboratory experiments in literature, showed that the flow can be regarded Q2D for larger
fluid depths compared to a single layer configuration. It is thus desirable to perform Q2D experiments in two-layer stratified fluids, if one wishes to compare experiments with 2D theoretical and/or numerical methods.

Preliminary laboratory experiments on dipoles have been performed, to verify some aspects of the numerical results. The dipoles were created by means of electromagnetic forcing. The initial vorticity distribution of such a dipole appears to be in between the vorticity distribution of a Stokes and Lamb dipole. The experiments confirmed the deformation of the vorticity distribution at the free surface in cases that the flow was expected to be three-dimensional.

At last, we would like to analyze the Q2D character of a few experiments in thin layers of fluid that are described in literature. The comparison is based on the regime diagrams of the monopolar vortex. We will make a comparison between our numerical simulations and the experiments performed by Antonova et al. (1991) and by Tabeling et al. (1991). To make a proper comparison, the absolute values of the fluid depth and a typical value for the peak vorticity, or some typical value for the velocity and length scale, needs to be known for proper scaling of the flows. None of the authors mentioned above provide all these values, so it is necessary to make some estimation concerning the typical peak vorticity. It is reasonable to assume a typical value for the peak vorticity in these laboratory experiments of $\omega_p \sim 4 \text{s}^{-1}$ (taken from Paireau et al. (1997)). The experiments performed by Antonova et al. were carried out by using a single layer of fluid (water) with a depth of 2.0 cm. In combination with $\omega_p$, this corresponds to a value of $Re_\lambda \approx 648$. These experiments cannot be considered as Q2D, according to the criteria that were found in this thesis. Note that for this value of $Re_\lambda$, the result is independent of $Re$ and it is thus possible to draw some conclusions about the quasi-two-dimensionality. The experiments of Tabeling et al. were conducted in a single layer of fluid for depths of 2.5 mm and 4.0 mm. The case of 2.5 mm can be considered as Q2D, but the case of 4.0 mm should be considered 3D, since the values of $Re_\lambda$ are estimated $Re_\lambda \approx 10$ and $Re_\lambda \approx 26$, respectively. We may conclude that thin layer experiments on 2D flows reported in literature can not be considered (quasi)-2D in several cases, according to our criteria.

6.2 Recommendations

There are still some unanswered questions that need to be solved. The complex secondary circulation pattern in the dipolar vortex, which is much more complicated than in the case of the monopole, is not yet understood. Furthermore, it is not clear why the dipole moves more or less like a columnar structure through the fluid, as we expected that the displacement of the dipole would be vertically dependent as $\sin(\pi z/2H)$. Eventually, the results presented in this thesis could be used if one wishes to understand the effects of the three-dimensional structure on decaying turbulent flows, or, to make it more complicated, on forced turbulent flows. Of course, these flows are far more intricate, due to interactions of vortices, which are unequal in strength and in size. It is recommended to study a simple vortex interaction, for example a colliding dipole, first. It is not well understood in what
way the additional linear term in the Q2D Navier-Stokes equation affects the dynamics of 2D turbulence.

It is worthwhile to apply and test our method for characterizing quasi-two-dimensionality to vortices in rotating and/or stratified fluids as well.
References


Appendix A

The Vorticity Equation

The following derivation is taken from Kundu (1990). For incompressible fluids, the Navier-Stokes equation takes the following form

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u}.
\]  

(A.1)

An equation for rate of change of the vorticity \( \omega \) is obtained by taking the curl of (A.1). We shall see that pressure and gravity are eliminated during this operation. The gravity force is a conservative force, and can be expressed as the gradient of a potential function, \( \mathbf{g} = -\nabla \phi \). Using the vector identity

\[
(\mathbf{u} \cdot \nabla) \mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla (\mathbf{u} \cdot \mathbf{u}) = \omega \times \mathbf{u} + \nabla \rho^2,
\]

and noting that the curl of a gradient vanishes, the curl of (A.1) gives

\[
\frac{\partial \omega}{\partial t} + \nabla \times (\omega \times \mathbf{u}) = \nu \nabla^2 \omega + \frac{\nabla p \times \nabla \rho}{\rho^2}.
\]  

(A.3)

where we have also used the identity \( \nabla \times \nabla^2 \mathbf{u} = \nabla^2 (\nabla \times \mathbf{u}) \) in rewriting the viscous term. The second term in (A.3) can be written as

\[
\nabla \times (\omega \times \mathbf{u}) = (\mathbf{u} \cdot \nabla) \omega - (\omega \cdot \nabla) \mathbf{u},
\]

(A.4)

where we have used the vector identity

\[
\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \nabla \cdot \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} - \mathbf{B} \nabla \cdot \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B},
\]  

(A.5)

and that \( \nabla \cdot \mathbf{u} = 0 \) and \( \nabla \cdot \omega = 0 \) (the divergence of a curl vanishes). Equation (A.3) then becomes

\[
\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla) \omega = (\omega \cdot \nabla) \mathbf{u} + \frac{\nabla p \times \nabla \rho}{\rho^2} + \nu \nabla^2 \omega.
\]  

(A.6)

For purely 2D flows, \( \omega = (0, 0, \omega_z) \) and \( \mathbf{u} = (u, v, 0) \), thus the first term on the right hand side vanishes. As this term represents rate of change of vorticity due to stretching and tilting of vortex lines, it can be understood that it must be zero in a purely 2D flow.
Appendix B

The Reduced Form of the Navier-Stokes Equation

The three components of the Navier-Stokes equation in cylindrical coordinates are given by

\[
\frac{\partial u_r}{\partial t} + (\mathbf{u} \cdot \nabla) u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right), \tag{B.1}
\]

\[
\frac{\partial u_\theta}{\partial t} + (\mathbf{u} \cdot \nabla) u_\theta + \frac{u_ru_\theta}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \nu \left( \nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right), \tag{B.2}
\]

\[
\frac{\partial u_z}{\partial t} + (\mathbf{u} \cdot \nabla) u_z = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 u_z + g. \tag{B.3}
\]

In cylindrical coordinates the operator \((\mathbf{u} \cdot \nabla)\) and the Laplace operator take the following form:

\[
(\mathbf{u} \cdot \nabla) = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}, \tag{B.4}
\]

\[
\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{B.5}
\]

We have assumed that \(u_r \ll u_\theta\), so the first two terms on the left hand side of Eq. (B.1) are negligible. The second and third term on the right hand side vanish for the same reason. Axisymmetry of the vortex implies \(\partial/\partial \theta = 0\), which makes the last term on the r.h.s. vanish. This yields

\[
\frac{u_\theta^2}{r} = \frac{1}{\rho} \frac{\partial p}{\partial r}. \tag{B.6}
\]
This is known as the cyclostrophic balance. Equation (B.2) can be reduced in a similar way, which yields

\[
\frac{\partial u_\theta}{\partial t} = \nu \left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} \right). \tag{B.7}
\]

This equation describes the diffusion of momentum in the \( r \) and \( z \) direction. Equation (B.3) can be written as

\[
-\frac{1}{\rho} \frac{\partial p}{\partial z} = g, \tag{B.8}
\]

which is recognized as the hydrostatic balance.
Appendix C

Overview of Numerical Simulations

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<th>Re</th>
<th>$H$</th>
<th>$Q$</th>
<th>$q_{r,z,\text{max}}$</th>
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<td>0.0043</td>
<td>0.0005</td>
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<tr>
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<td>0.2</td>
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<td>0.007</td>
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<tr>
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<tr>
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<td>0.5</td>
<td>1.50</td>
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<td>1.0</td>
<td>1.64</td>
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</tr>
<tr>
<td>125</td>
<td>0.4</td>
<td>0.44</td>
<td>0.03</td>
</tr>
<tr>
<td>2000</td>
<td>0.4</td>
<td>1.19</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table C.1: Overview of the numerical monopole simulations in a homogeneous layer of fluid.

<table>
<thead>
<tr>
<th>Re</th>
<th>$H$</th>
<th>$Q$</th>
<th>$q_{z,\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>0.1</td>
<td>0.007</td>
<td>$5 \cdot 10^{-6}$</td>
</tr>
<tr>
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<tr>
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<td>$4.5 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>500</td>
<td>0.5</td>
<td>0.19</td>
<td>0.01</td>
</tr>
<tr>
<td>500</td>
<td>1.0</td>
<td>0.23</td>
<td>$&gt; 0.05$</td>
</tr>
<tr>
<td>125</td>
<td>0.4</td>
<td>0.20</td>
<td>-</td>
</tr>
<tr>
<td>2000</td>
<td>0.4</td>
<td>0.24</td>
<td>-</td>
</tr>
</tbody>
</table>

Table C.2: Overview of the numerical dipole simulations in a homogeneous layer of fluid.
Appendix D

Numerical Simulations of a Monopole in a Stratified Fluid

Figures D.1(a) and D.1(b) show two simulations for a monopolar vortex in a shallow, two-layer stratified fluid.

Figure D.1: Results of numerical simulations in a two-layer stratified fluid with (a) Re = 500 and \( H = 0.2 \) \((Re_x = 8)\) and (b) Re = 500 and \( H = 1.0 \) \((Re_x = 203)\). Shown are the time evolution of the kinetic energy of the secondary flow, the time evolution of the vorticity profile \( \omega_z(r) \) at the free surface, the decay of the peak vorticity \( \omega_p \) at three different levels.
Appendix E

Technology Assessment

The local climate is for the greater part determined by geophysical flows. For example, Western Europe is relatively warm compared to the West coast of America, due to the presence of the Gulf Stream, which transports relatively warm water towards Europe. Geophysical flows in the atmosphere determine the path of high and low pressure areas, and thus the local amount of rainfall.

With respect to the present global climate change, the interest in Geophysical Fluid Dynamics has grown. One can imagine that a global warming could affect the circulation patterns in the oceans and in the atmosphere, resulting in a change of (locale) climate\(^1\).

Vortices in the atmosphere and in the ocean are relatively thin compared to their size. While their horizontal dimensions are typically 1000 km, they are only 1-10 km thick. On small scales, vortices are studied in rotating fluids, in stratified fluids as well as in shallow fluid layers to try to understand some aspects that can be related to these huge vortices in geophysical flows.

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\(^1\)Since 1979, global temperatures have been monitored by satellite with the Microwave Sounding Units (MSU), which revealed no significant temperature change of the lower troposphere.