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Analysis of an infinite array of microstrip antennas embedded in a thick substrate

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Analysis of an infinite array of microstrip antennas embedded in a thick substrate

by

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Abstract

Microstrip antennas, especially broadband microstrip antennas, have been one of the most innovative topics in antenna theory and design in recent years, and are increasingly finding application in a wide range of modern microwave systems. Many models have been developed to study these planar antennas, varying from a simple and time efficient cavity model to a sophisticated and time consuming method of moments. The microstrip antenna discussed in this report is fed by a coaxial cable. Since a broadband microstrip antenna is constructed by using an electrically thick substrate, a suitable model to describe the currents on the feeding coaxial probe is necessary. In the world of antenna design, the feeding probe is often modelled as a vertical monopole. A single isolated microstrip antenna has a wide radiation pattern. Therefore, microstrip antennas are often integrated into an array structure in order to match the demands of several applications such as radar and mobile satellite communication.

The overall performance of microstrip antennas is strongly affected by the mutual coupling between the array elements. The calculation of these couplings is very time consuming, especially for large arrays. However, the inner elements of such an array behave almost uniformly and can be approximated by the uniform behaviour of elements in an infinite array. The performance of such an infinite array can be calculated more easily.

The influence of the substrate thickness on the scan performance of an infinite array of probe-fed microstrip antennas has been studied. Initially, various arrays of vertically printed monopoles are considered, because of their relations to the problem of probe-fed microstrip patches.

A full wave analysis is presented for the problem of an infinite array of vertical monopoles and electromagnetically coupled (EMC) microstrip antennas embedded in a grounded dielectric substrate. The dielectric substrate is assumed to be electrically thick, so that the electric field components parallel to the ground plane throughout the substrate can not be neglected. A sophisticated magnetic current frill is used to model the coaxial aperture in the ground plane that excites the probes and the patches. The Green's functions in the spectral domain are involved to find electromagnetic fields inside and outside the substrate. The periodicity of the array makes it possible to write the fields within a periodic cell as an infinite number of Floquet modes. A method-of-moments procedure is then applied to the unit cell. The induced patch and probe currents are expanded into a set of respectively sinusoidal and overlapping rooftop basis functions. Solving the resulting linear equations yields the unknown coefficients of the basis functions. The computed currents are then used to determine the array properties...
such as the input impedance, reflection coefficients and scan blindness phenomenon. A graphical technique, based on the grating lobe diagram is employed to visualize the interaction of various Floquet modes and surface wave modes which leads to a scan blindness.

Special attention is paid to the scan performance of arrays of the EMC microstrip antennas fed by quarter-wavelength monopoles. It is found that it is difficult to match these antennas to an impedance of 50Ω for a grating lobes-free spacing because of high couplings between array elements. However, a low reflection coefficient can be achieved within a quite large frequency band by a proper choice of the distance between array elements. An impedance bandwidth of about 27 percent (VSWR ≤ 2) and a scan range of 20° can be obtained for this type of antenna with a low dielectric constant of the substrate layer. For substrates with high dielectric constants an improvement of the scan range can be obtained.
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Chapter 1

General Introduction

Conventional microstrip antennas, consisting of a single conducting patch on a grounded dielectric substrate, have received much attention in recent years [1]. In comparison with other conventional microwave antennas, microstrip antennas have attractive features such as a low profile, light weight, low production cost, and easy integration with printed circuits. In addition, the benefit of a compact body and low cost feed network may be attained by integrating the microstrip feed structure with the antenna on the same substrate. This is especially useful in arrays. However, due to their resonant behavior, they radiate efficiently only over a narrow band of frequency with a typical bandwidth of a few percent. One of the factors that affects the bandwidth is the substrate thickness. Other factors are substrate permittivity and patch structure. Because a single isolated microstrip antenna has a relatively wide radiation pattern, thus low gain pattern, the microstrip antennas are often integrated in a large array to match the requirements for several applications, such as mobile satellite communication and radar. In an array environment, the mutual coupling between the array elements be-

Figure 1.1: Cross section of a coaxial-fed microstrip antenna.

comes important and can cause degradation of the overall performance of the arrays. The calculation of the mutual coupling is time consuming, especially for large arrays. However, the inner elements of such an array behave almost identically and can be approximated by the uniform behaviour of elements in an infinite array. The analysis of an infinite array is less
complex.

Since the bandwidth improvement is realised using an electrically thick substrate, the influence of the feeding element has to be incorporated. Many different types of feeds are possible: probe feeds, microstrip line feeds, aperture feeds, etc. In this report a short coaxial probe is employed as the feeding element for a microstrip patch. However, the electrically thick substrates give rise to an inductive shift in the input impedance, which means that a good impedance match can only be achieved if a compensating input network is used [7]. The use of such a network would increase the complexity and thus production costs of the total antenna. A possible solution for this problem can be the so-called electromagnetically coupled (EMC) microstrip structure [2]. In this case, the patch is not physically connected to the feeding coaxial cable.

A number of full-wave solutions for the input impedance and radiation properties of probe-fed printed patch antennas and arrays have been proposed in [3, 4, 5, 6, 7]. An idealised feed model was used in which the current distribution along the feeding coaxial probe is assumed to be constant. Thus, the self-impedance of the probe is ignored. This idealised probe feed model can yield a good agreement with measured input impedance data for patches printed on electrically thin substrates. The model, however, fails to accurately predict input impedance when the substrate is thicker than approximately $0.02\lambda$ where $\lambda$ is the electrical wavelength in the substrate [4, 6]. When a thicker substrate is used, the current distribution along the probe feed will not be constant anymore. Therefore, a rigorous source model which includes the current variation on the feed is inevitable. Pozar [8, 9] has developed a source model by using a delta-gap generator. The delta-gap voltage generator is placed at the base of probes. This source model is a somewhat simplistic model for the source region because only the base of probes is excited. This assumption is only valid if the probe length is less than circa $0.25\lambda$ where $\lambda$ is electrical wavelength in a substrate. Practically, the entire probe surface is excited by incident fields coming from the aperture of coaxial cables. A more rigorous and realistic source model is presented in [10, 11] discussing respectively an isolated vertical monopole and a finite array of monopoles. The monopoles are embedded in a dielectric substrate above a highly conducting infinite ground plane. The excitation is provided by a coaxial line from below the ground plane. The solution in [10] is not efficient because the calculations are performed in the space domain. A more appropriate method can be found in [11] where the spectral domain Green's function is involved. Using the same source model, the properties of an infinite array of microstrip antennas in an electrical thick substrate can be investigated accurately. The effect of the thickness of the dielectric slab on the scan performance of a large array of microstrip antennas is our primary investigation.

As a first step toward a rigorous feed model, we will initially consider the simpler geometry of an array of vertical monopoles. This step is intended to obtain a more complete understanding of the scan performance in case of infinite arrays of probe-fed patches. A full-wave analysis of an infinite array of rectangular microstrip antennas will be presented. For that purpose the method of moments is used with the exact expression for the Green's function in the spectral domain. The electric field integral equation, representing the boundary conditions that the total tangential electric field must vanish on both the patch and feed,
is discretised using the Poisson summation formula. The unknown current coefficients are evaluated by solving the resulting matrix equation. Once the current coefficients are known, the input impedance and other array characteristics can be calculated. The results obtained from this analysis are compared with available references, such as [8, 10, 11, 14].

The structure of this report is as follows. Chapter 2 presents a solution for the problem of an infinite array of identical vertical monopoles in a dielectric substrate. The periodic grounded slab Green's functions are used, limiting the analysis to a single unit cell of the array. The source is assumed to be a one-Volt frill of annular magnetic currents located at the base of the monopole. The current on the monopole is expanded into a set of overlapping rooftop basis functions with unknown coefficients. Solving the resulting matrix equation yields the current distribution along the monopole surface. The calculated coefficient of the half-rooftop basis function is used in the calculation of the input impedance. In chapter 3, some results are given and compared to [8, 11, 14]. Using the same strategy as in chapter 2, infinite arrays of the EMC rectangular microstrip antennas are examined in chapter 3. Three unknown currents are involved in the calculation, i.e. the z-directed current on the probe and x- and y-directed currents on the patch. The unknown surface currents on the patch and the probe feed are expanded into a set of entire domain sinusoidal and subdomain rooftop basis functions, respectively. The results from this analysis are compared to [12]. At the end of this report, some topics for further research are given.
Chapter 2

Infinite Array of Vertical Monopoles

2.1 Introduction

As an introduction for the rigorous analysis of an infinite array of the probe-fed thick microstrip antennas, an infinite array of monopoles embedded in a dielectric substrate will be discussed. The monopole length is assumed to be sufficiently long that the current distribution along the monopole surface is no longer constant. A magnetic frill source model is employed in order to account for the feeding coaxial cables. Using the method of moments, the unknown current’s variation on the monopole surface is expanded into a set of elementary basis functions, each of them with a different amplitude.

2.2 Model Description

Figure 2.1 shows the cross section of a coaxial-fed monopole antenna. The perfectly conducting monopole, height $z_f$, is fed by a coaxial cable with inner radius $r_a$ and outer radius $r_b$. The infinite ground plane is also a perfect electric conductor ($\sigma \rightarrow \infty$). The dielectric substrate with thickness $d$ extends to infinity in the $x$- and $y$-directions and is made of an isotropic, homogeneous and lossy material with a loss tangent of $\tan \delta$. The permittivity $\varepsilon$ of the substrate is complex

$$\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon_0 \varepsilon_r (1 - j\tan \delta), \quad (2.1)$$

and the permeability is $\mu_0$. The perfect electric conducting monopole has a diameter of $2r_a$ and is assumed to be small enough that only the axial currents are present. At frequencies for which $kr_b < 0.1$ ($k = \omega\sqrt{\varepsilon\mu_0}$), the coaxial cable supports only the groundmode, i.e. TEM mode [10]. In addition, the monopole current $\tilde{J}_f$ is assumed to be located at $r = r_a$ and has
Figure 2.1: Cross section of a coaxial-fed monopole embedded in a dielectric slab.

no component in the directions parallel to the ground plane,

\[ \vec{J}_f = \frac{I(z)}{2\pi r_a} \hat{e}_z \quad \text{at} \quad r = r_a. \]  

(2.2)

With these assumptions, the electric field in the coaxial aperture is \( \phi \)-independent and has only a radial component described by [16, equation 2.16]

\[ \vec{E}_r(r) \approx \vec{E}_{r,TEM}(r) = \frac{U}{r \ln \left( \frac{r_b}{r_a} \right)} \hat{e}_r, \quad \text{for} \quad z = 0, \quad r_a \leq r \leq r_b, \]  

(2.3)

and the magnetic field in the coaxial aperture is given by

\[ \vec{H}_\phi(r) \approx \vec{H}_{\phi,TEM}(r) = \frac{I(z = 0)}{2\pi r} \hat{e}_\phi, \quad \text{for} \quad r_a \leq r \leq r_b, \]  

(2.4)

where \( U \) is the excitation voltage between the inner and outer conductor of the coaxial cable. \( I(z = 0) \) is the total current at the base of the monopole. The corresponding magnetic current distribution at the coaxial-line opening can be found by the equivalence principle [13, page 106-110]

\[ \dot{M}_{mii} = M_\phi \hat{e}_\phi = \vec{E}_r \times \hat{e}_z = - \frac{U}{r \ln \left( \frac{r_b}{r_a} \right)} \hat{e}_\phi, \quad \text{for} \quad r_a \leq r \leq r_b. \]  

(2.5)
2.3 Electromagnetic Fields in a Grounded Dielectric Slab

The excitation fields that give rise to the induced currents on the monopoles result from an annular frill of magnetic current at the ground plane that is used to model the coaxial aperture [16]. In order to describe these fields the Green’s functions of a magnetic dipole must be known. By virtue of the Lorentz reciprocity theorem [13, equation 3-34] the spectral domain Green’s functions due to an infinitesimal magnetic dipole can be derived from the Green’s functions due to an electric dipole. The details of the derivation of the spectral Green’s functions of an electric dipole are given in [2, 5, 19].

2.3.1 Space and Spectral Domain

Because the Green’s functions can be found in closed form in the spectral domain, the problem and solution presented in this report shall be formulated in the spectral domain. It means that all quantities are Fourier-transformed with respect to $x$- and $y$-coordinates.

A general function $F(x, y)$ and his corresponding Fourier transform $F(k_x, k_y)$ are defined as

$$F(k_x, k_y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k_x, k_y) e^{ik_x x} e^{ik_y y} dk_x dk_y.$$  (2.6)

2.3.2 Green’s Functions of an Electric Dipole

In figure 2.2 an arbitrary directed dipole embedded in a dielectric slab is depicted. The electromagnetic fields in the dielectric slab can be calculated once the Green’s functions of a dipole are known. A Green’s function is a vector potential which results from a unit source.
The unit source could be an electric or a magnetic current. In our model, the fields in the dielectric slab are generated by an electric as well as a magnetic current. The electric currents are located along the surface of monopole \((r = r_a)\) whereas the magnetic currents are found at the base of the monopole \((z = 0)\). In this section the Green's functions of an electric current source will be given. In the next section the Green's function of a magnetic dipole will be derived. The finite electric current source on the monopole surface can be found by subdividing the source into an infinite number of vertically directed electric dipoles. Then the electric vector potential is obtained by integration of the contribution of all the electric dipoles, weighted by the Green's functions. In appendix A the spectral domain Green's function is given of a vertically directed electric dipole embedded in a substrate above an infinite and perfectly conducting ground plane. The resulting electromagnetic fields inside the substrate are presented in appendix B. The electric field created by an infinitesimal electric dipole can be written in terms of the spectral Q-functions\(^1\)

\[
\tilde{E}(x, y, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}(k_x, k_y, z) e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y
\]

\[
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int_{z'} \tilde{Q}^{EE}(k_x, k_y, z, z') \cdot \tilde{J}(k_x, k_y, z') \, dz' \right] e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y,
\]

\(1:\)The \(Q\)-functions are derived from the Green's function. Combining these \(Q\)-functions with the spectral current densities results to the solution of the spectral electromagnetic fields.
2.3. ELECTROMAGNETIC FIELDS IN A GROUNDED DIELECTRIC SLAB

where \( Q^{EE} \) is a 3 \times 3 matrix, formulated by the following equations:

\[
Q^{EE} = \begin{pmatrix}
0 & 0 & Q^{EE}_{zz} \\
0 & 0 & Q^{EE}_{yz} \\
0 & 0 & Q^{EE}_{xz}
\end{pmatrix}.
\]

The corresponding magnetic field at \((x, y, z)\) is given by

\[
\vec{H}(x, y, z) = \frac{1}{4\pi^2} \int \int_{-\infty}^{\infty} \int \int_{-\infty}^{\infty} \vec{H}(k_x, k_y, z) e^{-jk_x x} e^{-jk_y y} dk_x dk_y
\]

\[
= \frac{1}{4\pi^2} \int \int_{-\infty}^{\infty} \int \int_{-\infty}^{\infty} \left[ \int \vec{Q}^{HE}_{(k_z, k_y, z', z')} \cdot \vec{J}(k_x, k_y, z') dz' \right] e^{-jk_x x} e^{-jk_y y} dk_x dk_y,
\]

with

\[
\vec{Q}^{HE} = \begin{pmatrix}
0 & 0 & Q^{HE}_{z'} \\
0 & 0 & Q^{HE}_{y'} \\
0 & 0 & 0
\end{pmatrix},
\]

\(2\) A general function \( Q^{EE}_{\varphi \varphi} \) indicates the \( \varphi \)-component of the spectral electric field \( \vec{E} \) at \( \vec{r} \) due to a unit \( \varphi \)-directed electric current at \( \vec{r} \).

\(3\) A general function \( Q^{HE}_{\varphi \varphi} \) indicates the \( \varphi \)-component of the spectral magnetic field \( \vec{H} \) at \( \vec{r} \) due to a unit \( \varphi \)-directed electric current at \( \vec{r} \).
In the above relations \( k_0 = \omega \sqrt{\varepsilon_0 \mu_0} \) is the free space wavenumber and \( \omega = 2\pi f \) is the radial frequency. The restriction that \( \text{Im}(k_2) < 0 \) follows from the radiation condition that the fields are outward propagating waves, decaying with distance from the source. The zeros of the functions \( T_m \) and \( T_e \) correspond to solutions of the characteristic equation for transverse magnetic (TM) respectively transverse (TE) surface waves in a grounded dielectric slab. These zeros correspond to the first order poles in the Green’s function. In case of infinite array of microstrip antennas, these zeros can cause a blindness phenomenon for a certain scan angle \( \theta \).

2.3.3 Green’s Functions of a Magnetic Dipole

In this section, the exact Green’s function of a magnetic dipole embedded in a dielectric substrate will be given. Using the Lorentz reciprocity theorem [13], the spectral Green’s functions can be constructed from the spectral Green’s functions of an electric dipole. The fields produced by the magnetic currents at the base of monopole act as sources. Because of the \( \phi \)-directed current source in the coaxial aperture, only two components of the dyadic \( Q \)-functions are necessary to describe the incident electric fields on the vertical monopole, namely \( Q_{xx}^{EM} \) and \( Q_{zy}^{EM} \).

The reciprocity theorem states that the response of a system to a source is unchanged when source and observer are interchanged. In a more general sense, the reciprocity theorem

\[
Q_{xx}^{HE} = -\frac{j k_y}{k_1 T_m} \begin{cases} 
\cos k_1 z' \left[ \varepsilon_r k_2 \sin k_1 (d - z) - j k_1 \cos k_1 (d - z) \right], & z' \leq z, \\
\cos k_1 z \left[ \varepsilon_r k_2 \sin k_1 (d - z') - j k_1 \cos k_1 (d - z') \right], & z \leq z',
\end{cases}
\]

\[
Q_{zy}^{HE} = \frac{j k_x}{k_1 T_m} \begin{cases} 
\cos k_1 z' \left[ \varepsilon_r k_2 \sin k_1 (d - z) - j k_1 \cos k_1 (d - z) \right], & z' \leq z, \\
\cos k_1 z \left[ \varepsilon_r k_2 \sin k_1 (d - z') - j k_1 \cos k_1 (d - z') \right], & z \leq z',
\end{cases}
\]

where

\[
T_m = \varepsilon_r k_2 \cos k_1 d + j k_1 \sin k_1 d,
\]

\[
T_e = k_1 \cos k_1 d + j k_2 \sin k_1 d,
\]

\[
k_1^2 = \varepsilon_r k_0^2 - k_x^2 - k_y^2,
\]

\[
k_2^2 = k_0^2 - k_x^2 - k_y^2, \quad \text{Im}(k_2) < 0
\]

\( Q_{yy}^{EM} \) indicates that we deal with the problem where magnetic currents act as sources. A general function \( Q_{\psi \psi}^{EM} \) indicates the \( \psi \)-component of the spectral electric field \( \vec{E} \) at \( \vec{r} \) due to a unit \( \phi \)-directed magnetic current at \( \vec{r} \).
relates a response at one source due to a second source to the response at the second source due to the first source. Consider two sets of source currents, \((\mathbf{J}_a, \mathbf{M}_a)\) and \((\mathbf{J}_b, \mathbf{M}_b)\), existing in the same linear, homogeneous and isotropic medium, see figure 2.4. Denote the field produced by the \(a\) sources by \((\mathbf{E}_a, \mathbf{H}_a)\), and by \((\mathbf{E}_b, \mathbf{H}_b)\) for the \(b\) sources. The reciprocity theorem states that

\[
\int \int \int \left( \mathbf{E}_b \cdot \mathbf{J}_a - \mathbf{H}_b \cdot \mathbf{M}_a \right) dV_a = \int \int \int \left( \mathbf{E}_a \cdot \mathbf{J}_b - \mathbf{H}_a \cdot \mathbf{M}_b \right) dV_b. \tag{2.17}
\]

The range of the integration in (2.17) can be limited to the area that contains only the sources. If the source \(a\) only consists of a magnetic current and \(b\) of an electric current, then equation (2.17) can be replaced by

\[
- \int \int \int_{V_a} \left( \mathbf{H}_b \cdot \mathbf{M}_a \right) dV_a = \int \int \int_{V_b} \left( \mathbf{E}_a \cdot \mathbf{J}_b \right) dV_b. \tag{2.18}
\]

First \(Q_{zx}\) will be calculated. For that purpose it will be assumed that the current source in \(V_a\) is an \(x\)-directed magnetic current located at \(\mathbf{r}_a = (x'_a, y'_a, z'_a)\) and in \(V_b\) a \(z\)-directed electric current at \(\mathbf{r}_b = (x'_b, y'_b, z'_b)\). The calculation of \(Q_{xy}^{EM}\) can be done in a similar way.

\[
\mathbf{M}_a(\mathbf{r}) = \delta(x - x'_a) \delta(y - y'_a) \delta(z - z'_a) \hat{e}_x,
\]

\[
\mathbf{J}_b(\mathbf{r}) = \delta(x - x'_b) \delta(y - y'_b) \delta(z - z'_b) \hat{e}_z. \tag{2.19}
\]
Using equation (2.6) shows that the spectral currents $\tilde{M}_a$ and $\tilde{J}_b$ are

$$
\tilde{M}_a(k_x, k_y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - x'_a) \delta(y - y'_a) \delta(z - z'_a) \hat{e}_x e^{jk_x x} e^{jk_y y} dx dy
$$

$$
= \delta(z - z'_a) e^{jk_x x'_a} e^{jk_y y'_a} \hat{e}_x
$$

$$
= M_{ax} \hat{e}_x, \tag{2.20}
$$

$$
\tilde{J}_b(k_x, k_y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - x'_b) \delta(y - y'_b) \delta(z - z'_b) \hat{e}_x e^{jk_x x} e^{jk_y y} dx dy
$$

$$
= \delta(z - z'_b) e^{jk_x x'_b} e^{jk_y y'_b} \hat{e}_x
$$

$$
= J_{bz} \hat{e}_x \tag{2.21}
$$

Similar to equations (2.7) and (2.12) the electric field $\tilde{E}_a$ and the magnetic field $\tilde{H}_b$ can be written in terms of the spectral domain $Q$-functions

$$
\tilde{E}_a(\vec{r}) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}_a(k_x, k_y, z) e^{-jk_x x} e^{-jk_y y} dk_x dk_y
$$

$$
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int_{z'} Q^{EM}(k_x, k_y, z, z') \cdot \tilde{M}_a(k_x, k_y, z') dz' \right] e^{-jk_x x} e^{-jk_y y} dk_x dk_y \tag{2.22}
$$

$$
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \tilde{e}_x Q^{EM}_{xx} \tilde{e}_x + \tilde{e}_y Q^{EM}_{yx} \tilde{e}_x + \tilde{e}_z Q^{EM}_{xz} \tilde{e}_x \right] e^{-jk_x (x - x'_a)} e^{-jk_y (y - y'_a)} \hat{e}_z dk_x dk_y,
$$

where $Q^{EM}_{\nu z} = Q^{EM}_{\nu z}(k_x, k_y, z, z'_a)$ and

$$
\tilde{H}_b(\vec{r}) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{H}_b(k_x, k_y, z) e^{-jk_x x} e^{-jk_y y} dk_x dk_y
$$

$$
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int_{z'} Q^{HE}(k_x, k_y, z, z') \cdot \tilde{J}_b(k_x, k_y, z') dz' \right] e^{-jk_x x} e^{-jk_y y} dk_x dk_y \tag{2.23}
$$

$$
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \tilde{e}_x Q^{HE}_{xx} \tilde{e}_x + \tilde{e}_y Q^{HE}_{yx} \tilde{e}_x + \tilde{e}_z Q^{HE}_{xz} \tilde{e}_x \right] e^{-jk_x (x - x'_b)} e^{-jk_y (y - y'_b)} \hat{e}_z dk_x dk_y
$$

where $Q^{EM}_{\nu z} = Q^{EM}_{\nu z}(k_x, k_y, z, z'_b), \nu \in \{x, y, z\}$. Substitution of equations (2.19), (2.22) and
(2.23) in (2.18) yields
\[
- \iiint_{V_a} \left[ \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \hat{e}_x Q_{zz}^{HE} + \hat{e}_y Q_{yy}^{HE} + \hat{e}_z Q_{zz}^{HE} \right) e^{-jk_x(x-x'_a)} e^{-jk_y(y-y'_a)} dk_x dk_y \right] \cdot \\
\delta(x-x'_a) \delta(y-y'_a) \delta(z-z'_a) \hat{e}_x \, dx \, dy \, dz = \tag{2.24}
\]
\[
- \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \hat{e}_x Q_{zz}^{HE} + \hat{e}_y Q_{yy}^{HE} + \hat{e}_z Q_{zz}^{HE} \right) e^{-jk_x(x-x'_a)} e^{-jk_y(y-y'_a)} dk_x dk_y \cdot \hat{e}_x = \tag{2.25}
\]
\[
\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \hat{e}_x Q_{zz}^{EM} + \hat{e}_y Q_{yy}^{EM} + \hat{e}_z Q_{zz}^{EM} \right) e^{-jk_x(x-x'_a)} e^{-jk_y(y-y'_a)} dk_x dk_y \cdot \hat{e}_x = \tag{2.26}
\]

Changing \( k_x = -k_x \) and \( k_y = -k_y \) in the left side of equation (2.26) gives
\[
- Q_{zz}^{HE}(-k_x, -k_y, z'_a, z'_a) = Q_{zz}^{EM}(k_x, k_y, z'_a, z'_a). \tag{2.27}
\]

Using expression for \( Q_{zz}^{HE} \) as given previously in equation (2.14) one gets,
\[
- Q_{zz}^{HE}(-k_x, -k_y, z'_a, z'_a) = Q_{zz}^{HE}(k_x, k_y, z'_a, z'_a). \tag{2.28}
\]

Substituting equation (2.28) in equation (2.27) yields
\[
Q_{zz}^{HE}(k_x, k_y, z'_a, z'_b) = Q_{zz}^{EM}(k_x, k_y, z'_b, z'_a) \tag{2.29}
\]

Evidently,
\[
Q_{yz}^{HE}(k_x, k_y, z'_a, z'_b) = Q_{yz}^{EM}(k_x, k_y, z'_b, z'_a) \tag{2.30}
\]

The general form of the Green's functions of an \( x \)- and \( y \)-directed magnetic dipole is
\[
Q_{zz}^{EM} = \begin{cases} 
    \frac{jk_y}{k_1 T_m} \left( \cos k_1 z \left[ \varepsilon_r k_2 \sin k_1 (d - z') - j k_1 \cos k_1 (d - z') \right], \ z \leq z', \\
    \cos k_1 z' \left[ \varepsilon_r k_2 \sin k_1 (d - z) - j k_1 \cos k_1 (d - z) \right], \ z' \leq z, 
\end{cases} \tag{2.31}
\]
The derivation of the other components of the dyadic Green’s functions can be carried out in a similar way. The results will be given in the next chapter when an infinite array of electromagnetically coupled microstrip antennas is analyzed.

2.4 Fields from an Infinite Array of Identical Monopoles

In this section the electric field from an infinite array of monopoles will be derived. The calculation of the magnetic field can be done in an analogous manner. The geometry of an infinite array of identical coaxial-fed monopoles is depicted in figure 2.5. The distance between elements in the x-direction and in the y-direction is \( a \) and \( b \), respectively. The center of the array is represented by the \((m = 0, n = 0)\)-th element. If the center of the array is located at the origin of the Cartesian co-ordinate system, then the location of the \((m,n)\)-th element is
2.4. FIELDS FROM AN INFINITE ARRAY OF IDENTICAL MONOPOLES

given by \((x = ma, y = nb)\) where \(m\) and \(n\) are integer indices with \(-\infty < (m, n) < \infty\). For scanning at the angle \(\theta, \phi\) the currents of \((m, n)\)-th source must be phased as

\[ e^{-jk_0(mau+nbv)}, \quad \text{with } u = \sin \theta \cos \phi \text{ and } v = \sin \theta \sin \phi. \]  

(2.33)

The total electric field generated by an infinite array can be found via a superposition of the contribution of each element [18]. Hence

\[
\mathbf{E}(x, y, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{4\pi^2} \int \int \int \int_{V'} \bar{Q}^E(k_x, k_y, z, z') \cdot \bar{J}(k_x, k_y, z') \, dz' \, e^{-jk_0x} e^{-jk_0y} dk_x dk_y
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{4\pi^2} \int \int \int \int_{V'} \bar{Q}^E(k_x, k_y, z, z') \cdot \bar{J}(x', y', z') \, e^{-jk_0(mau+nbv)}
\]

\[
\cdot e^{jk_0(x'+ma)} e^{jk_0(y'+nb)} \, dV' \, e^{-jk_0x} e^{-jk_0y} \, dk_x \, dk_y,
\]

(2.34)

where \(\zeta\) indicates the source that is used to generate \(E\)-field, i.e. \(\zeta \in \{\text{Electric, Magnetic}\}\) source. \(V'\) denotes the volume in which the current density \(\bar{J}\) is present. Rewriting equation (2.34) in a somewhat compact form,

\[
\mathbf{E}(x, y, z) = \frac{1}{4\pi^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int \int \int_{V'} \bar{J}(x', y', z') \, e^{-jk_0(mau+nbv)}.
\]

\[
\left[ \int \int_{-\infty}^{\infty} \bar{Q}^E(k_x, k_y, z, z') \, e^{-jk_0(x-x'-ma)} \, e^{-jk_0(y-y'-nb)} \, dk_x dk_y \right] \, dV'
\]

\[
= \frac{1}{4\pi^2} \int \int \int_{V'} \bar{J}(x', y', z') \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-jk_0(mau+nbv)}.
\]

\[
\left[ \int \int_{-\infty}^{\infty} \bar{Q}^E(k_x, k_y, z, z') \, e^{-jk_0(x-x'-ma)} \, e^{-jk_0(y-y'-nb)} \, dk_x dk_y \right] \, dV'
\]

\[
= \int \int \int_{V'} \bar{E} \cdot \bar{J}(x', y', z') \, dV',
\]

(2.35)

with

\[
\bar{E} = \frac{1}{4\pi^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-jk_0(mau+nbv)}.
\]

(2.36)

The dyadic \(\bar{E}\) gives the electric field due to an infinite periodic array of infinitesimal dipoles. Equation (2.36) is a rigorous expression but clearly not in a very usable form for a direct
numerical evaluation. By twice applying the Poisson summation formula \([3, 15]\), equation (2.36) may be rewritten in the following form

\[
\vec{E} = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \vec{Q}(k_x, k_y, z, z') e^{-jk_x(x-x')} e^{-jk_y(y-y')},
\]

where the wavenumbers \(k_x\) and \(k_y\) are the Floquet mode phase constants defined as

\[
k_x = k_0 u + \frac{2\pi m}{a}, \quad \text{and} \quad k_y = k_0 v + \frac{2\pi n}{b}
\]

for a rectangular array grid. For a triangular grid, \(k_y\) is replaced by

\[
k_y = k_0 v + \frac{2\pi n}{b} - \frac{2\pi m}{a \tan \alpha}
\]

where \(\alpha\) is the skew angle of the array grid relative to the \(x\)-axis (\(\alpha = 90^\circ\) for a rectangular grid). Equation (2.37) is also known as the standard expansion in terms of the doubly infinite set of Floquet modes with propagation constants \(k_x\) and \(k_y\) in the transverse directions. The expressions for the total magnetic field are of a similar form as compared to the previous equations, therefore they will not be given. The method of moments can now be applied to a monopole in a single unit cell. Due to the periodicity, all the monopoles in the infinite array are then accounted for.

2.5 Method of Moments Formulation

The intention of this chapter is to present an accurate analysis of an infinite array of vertical monopoles. For that purpose, the method of moments \([17]\) is used to calculate the various quantities. By means of the method of moments, the unknown current density along the monopoles is expanded into a set of elementary basis functions. Next, the resulting integral equations are transformed to a matrix equation. Solving this matrix equation yields the unknown coefficients of the basis functions.

We start this section with the formulation of the boundary condition at the surface of a perfectly conducting monopole \(S_f\). The total tangential electric field on the surface of a monopole must vanish, i.e.

\[
\hat{n} \times (\vec{E}_f + \vec{E}_{\text{full}}) = \vec{0}, \quad \text{for} \quad r = r_a,
\]

where the normal vector \(\hat{n}\) points outward from the surface of the perfectly conducting monopole. \(\vec{E}_f\) and \(\vec{E}_{\text{full}}\) are the total electric fields due to the electric currents on the monopole and due to the magnetic currents in the coaxial aperture, respectively. Because an infinite array of identical monopoles is investigated, this boundary condition must hold for all other monopoles in the array. A finite conductivity monopole could be dealt with by assuming a total tangential electric field proportional to the surface current \([13, 16]\). The next step is the expansion of the unknown current distribution \(\vec{J}_f\) on the monopoles into a set of \(N_{\text{max}}\) basis functions,

\[
\vec{J}_f(x', y', z') = \sum_{j=1}^{N_{\text{max}}} I_j \vec{J}_{fj}(x', y', z'),
\]
in which \( I_j \), with \( j = 1, 2, 3, \ldots, N_{\text{max}} \), are the unknown complex mode coefficients. \( \mathcal{J}_{fj} \) are the basis functions. A proper choice of the basis functions is discussed in section 2.6. Since the method of moments requires large amounts of numerical computations, a proper choice of the basis functions is of great importance. Inserting equation (2.41) in equation (2.35), the electric field \( \vec{E}_f \) may be written as

\[
\vec{E}_f(x, y, z) = \iiint_{V'} \vec{E} \cdot \mathcal{J}_{fj}(x', y', z') \, dV' = \iiint_{V'} \vec{E} \cdot \sum_{j=1}^{N_{\text{max}}} I_j \mathcal{J}_{fj}(x', y', z') \, dV' = \sum_{j=1}^{N_{\text{max}}} I_j \vec{E}_{fj}(x, y, z),
\]

where

\[
\vec{E}_{fj}(x, y, z) = \iiint_{V'} \vec{E} \cdot \mathcal{J}_{fj}(x', y', z') \, dV' = \iiint_{S_f} \vec{E} \cdot \mathcal{J}_{fj}(\tau') \, dS',
\]

is the total electric field generated by the \( j \)-th basis function. \( S_f \) is a closed surface which bounds the interior volume \( V' \). Substitution of equation (2.42) into equation (2.40) yields

\[
\hat{n} \times \left( \sum_{j=1}^{N_{\text{max}}} I_j \vec{E}_{fj} + \vec{E}_{\text{frill}} \right) = \vec{0}, \quad \text{for } r = r_a.
\]

Introduce a residual \( \vec{R} \), which is defined as

\[
\vec{R} = \hat{n} \times \left( \sum_{j=1}^{N_{\text{max}}} I_j \vec{E}_{fj} + \vec{E}_{\text{frill}} \right) = \vec{0}, \quad \text{for } r = r_a.
\]

If \( \vec{R} = \vec{0} \) for every point on \( r = r_a \), the boundary condition in equation (2.40) is satisfied. However, it is impossible to fulfill the boundary condition for the infinite number of points on the monopole surface. Therefore the condition (2.40) has to be relaxed somewhat. We will apply the following condition

\[
\langle \vec{R} ; \mathcal{J}_{fi} \rangle = \int_{S_f} \mathcal{J}_{fi} \cdot \vec{R} \, dS = 0, \quad \text{for } i = 1, 2, 3, \ldots, N_{\text{max}}.
\]

\( \mathcal{J}_{fi} \) is called a weighting function or test function. Substitution of equation (2.42) into (2.46) gives

\[
\sum_{j=1}^{N_{\text{max}}} I_j \sum_{i=1}^{N_{\text{max}}} Z_{ij} = V_i^f, \quad \text{for } i = 1, 2, 3, \ldots, N_{\text{max}},
\]

\[
\sum_{j=1}^{N_{\text{max}}} I_j Z_{ij} = V_i^f, \quad \text{for } i = 1, 2, 3, \ldots, N_{\text{max}},
\]
with

\[
Z_{ij} = -\iiint_{\tilde{S}_j} \tilde{J}_{fi} \cdot \tilde{E} \cdot \tilde{J}_{fj} \, dS' \, dS,
\]

\[
V^t_i = \iiint_{\tilde{S}_j} \tilde{J}_{fi} \cdot \tilde{E}_{\text{null}} \, dS.
\]  

(2.48)

The superscript \( t \) in the term \( V^t_i \) indicates that this term is based on the test mode. In matrix notation equation (2.47) becomes

\[
[Z] \begin{bmatrix} I \end{bmatrix} = [V^t].
\]  

(2.49)

The matrix \([Z]\) contains \( N_{\text{max}} \times N_{\text{max}} \) elements. \([I]\) is a vector containing the \( N_{\text{max}} \) unknown mode coefficients of the basis functions, and \([V^t]\) is the excitation vector with \( N_{\text{max}} \) elements.

### 2.6 Expansion Functions on the Monopole

It has been noticed in section 2.5 that the kind of basis functions used in a moment method solution is of great importance. The intention is to choose suitable basis functions so that the computation time can be limited without losing significant accuracy. In the method of moments formulation, the unknown currents on the monopoles are expanded into a set of \( N_{\text{max}} \) basis functions. In order to find the real current, the number of basis functions must be set to infinity. It is obvious that this number of basis functions from a practical point of view must be limited to a finite number \( N_{\text{max}} \). In this section we will take a closer look to some type of basis functions that are commonly used in field computations with the method of moments.

In theory, there is an infinite number of basis functions. From a computational point of view, however, only a few of them can be used. Our aim is to describe those which have the ability to represent the real current \( \tilde{J}_f \) fairly accurately, even if the number of basis functions is limited.

Generally, two classes of basis functions can be distinguished:

- subdomain, and
- entire domain basis functions.

In this section the subdomain basis functions will be discussed. The second class will be described in chapter 4 when an infinite array of EMC microstrip antennas is discussed. The subdomain basis functions are defined only over a small part of the domain of the unknown function. These basis functions can be used without prior knowledge of the function they represent. So the induced current distribution on an arbitrarily shaped microstrip antenna can be described accurately by these basis functions [7]. By choosing a small distance between the center of subdomains, results with a good accuracy can be obtained. The price that has to be paid, however, is very long computation since the entire domain of the unknown function
must be written in a summation of overlapping subdomain basis functions. Examples of this class of basis functions which are often used for the determination of the unknown current density $\mathcal{J}_f$ are:

- piece-wise constant (PWC),
- piece-wise sinusoidal (PWS) and
- piece wise linear (PWL or rooftop).

In this report, only PWL or rooftop basis functions will be considered. The description of other basis functions can be found in [5, 7]. An equidistance $z$-dependent rooftop basis function has the following form

$$g_j(z) = \begin{cases} 
\frac{2}{h}(z - \frac{h}{2}) & \text{for } j = 1, 0 \leq z \leq \frac{h}{2}, \\
\frac{2}{h}(z - z_{j-1}) & \text{for } j \geq 2, z_{j-1} \leq z \leq z_j, \\
\frac{2}{h}(z_{j+1} - z) & \text{for } j \geq 2, z_j \leq z \leq z_{j+1}.
\end{cases}$$ (2.50)

Notice that the first basis function is a half-rooftop. In the previous section, it was assumed that the induced current distribution on the monopole has no $\phi$-component and that the currents depend only on their location along the monopole. According to equation (2.41), this unknown current distribution can thus be written as a superposition of functions with the form

$$\mathcal{J}_{f_j}(x, y, z) = \frac{1}{2\pi r_a}\delta\left(\sqrt{x^2 + y^2} - r_a\right) g_j(z) \hat{e}_z.$$ (2.51)

The Fourier transform of $\mathcal{J}_{f_j}$ can be found by using equation (2.6)

$$\mathcal{J}_{f_j}(k_x, k_y, z) = \hat{e}_z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi r_a}\delta(\sqrt{x^2 + y^2} - r_a) g_j(z) e^{ik_x x} e^{ik_y y} dx dy$$

$$= J_0(\sqrt{k_x^2 + k_y^2} r_a) g_j(z) \hat{e}_z$$

$$= J_0(k_0 r_a) g_j(z) \hat{e}_z,$$ (2.52)
with
\[
k_0 \beta = \sqrt{k_x^2 + k_y^2},
\]
(2.53)

\(J_0\) in equation (2.52) is the Bessel function of the first kind of order 0.

### 2.7 Calculation of the Elements of the Matrix \([Z]\)

An element of the matrix \([Z]\) can be calculated by using expression (2.48). Because the Green’s function is known in closed form in the spectral domain, we shall rewrite this expression in terms of spectral domain quantities. Substituting equation (2.37) in (2.48) gives

\[
Z_{ij} = -\int_{S_f} \int_{S_f} \mathcal{J}_{fi}(x, y, z) \cdot \mathcal{J}_{fj}(x', y', z') dS' dS
\]
(2.54)

\[
= -\int_{S_f} \int_{S_f} \mathcal{J}_{fi}(x, y, z) \cdot \left[ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \tilde{Q}_{EE}^{ij}(k_x, k_y, z, z') e^{-jk_x(x-x')} e^{-jk_y(y-y')} \right] \cdot \mathcal{J}_{fj}(x', y', z') dS' dS
\]

\[
= -\frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int \mathcal{J}_{fi}(k_x, k_y, z) \cdot \left[ \int \tilde{Q}_{EE}^{ij}(k_x, k_y, z, z') \cdot \mathcal{J}_{fj}(k_x, k_y, z') dz' \right] dz,
\]

with

\[
\mathcal{J}_{fi}(k_x, k_y, z) = \int_{S_f} \mathcal{J}_{fi}(x', y', z) e^{jk_x x'} e^{jk_y y'} dx' dy',
\]
(2.55)

the Fourier transform of the basis function \(\mathcal{J}_{fi}\) and with

\[
\mathcal{J}_{fj}^*(k_x, k_y, z) = \int_{S_f} \mathcal{J}_{fj}(x, y, z) e^{-jk_x x} e^{-jk_y y} dx dy
\]
(2.56)

the Fourier transform of the complex conjugate of the test function \(\mathcal{J}_{fj}\). The next step is to write the spectral current densities \(\mathcal{J}_{fj}, \mathcal{J}_{fi}\) in terms of basis functions. Inserting equation (2.52) in (2.55) and using the Galerkin solution \([5, 7]\), i.e. test- and expansion functions are of the same form, \(Z_{ij}\) can now be written as

\[
Z_{ij} = -\frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( \int g_i(z) \left[ \int Q_{zz}^{EE}(k_x, k_y, z, z') g_j(z') dz' \right] dz \right) \cdot J_0^2(k_0 \beta r_a)
\]
(2.57)
2.7. **CALCULATION OF THE ELEMENTS OF THE MATRIX \([Z]\)**

The two \(z\)-integrations can be calculated analytically in the case of rooftop basis functions. Because the Galerkin solution has been used in expression (2.57), the matrix \([Z]\) is symmetric. To find an analytical expression for the double \(z\)-integrations in (2.58) three situations must be examined, i.e.

- \(i = j\), self term:

\[
I_{jj}^{zz}(\beta) = \int_z z g_i(z) \left[ \int_z g_j(z') Q_{zz}^{EE}(k, k_y, z, z') g_j(z') \, dz' \right] \, dz. \tag{2.58}
\]

The two \(z\)-integrations can be calculated analytically in the case of rooftop basis functions.

![Figure 2.7: Subdomain rooftop basis functions with \(i = j\) (selfterm).](image)

The element of the matrix \([Z]\) can now be written as

\[
Z_{jj} = -\frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} I_{jj}^{zz}(\beta) J_0^2(k_0 \beta r_n), \tag{2.59}
\]

with

\[
I_{jj}^{zz}(\beta) = \int_{z_{j-1}}^{z_j} g_j(z) \left[ \int_{z_{j-1}}^{z} g_j(z') Q_{zz}^{EE}(z' < z) \, dz' \right. \\
+ \int_z^{z_{j+1}} g_j(z') Q_{zz}^{EE}(z' > z) \, dz' + \int_{z_{j-1}}^{z_{j+1}} g_j(z') Q_{zz}^{EE}(z' > z) \, dz' \\
+ \int_{z_{j-1}}^{z_{j+1}} g_j(z) \left[ \int_{z_{j-1}}^{z_{j+1}} g_j(z') Q_{zz}^{EE}(z' < z) \, dz' \right. \\
+ \int_{z_{j-1}}^{z} g_j(z') Q_{zz}^{EE}(z' > z) \, dz' + \int_{z_{j-1}}^{z_{j+1}} g_j(z') Q_{zz}^{EE}(z' > z) \, dz' \\
+ \frac{j \omega \mu_0}{\varepsilon r_0 k_0^2} \int_{z_{j-1}}^{z_j} g_j(z) g_j(z_{j-1} \leq z \leq z_j) \, dz
\]
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\[
- \frac{j \omega \mu_0}{\varepsilon_r k_0^2} \int_{z_j}^{z_{j+1}} g_j(z) g_j(z \leq z \leq z_{j+1}) \, dz. \tag{2.60}
\]

The last two terms in expression (2.60) are due to the \( \delta \)-function in \( Q^E_{zz} \). According to expression (2.11), \( Q^E_{zz}(z' < z) \) and \( Q^E_{zz}(z' > z) \) can be written as

\[
Q^E_{zz}(z' < z) = - \frac{j \omega \mu_0 k_0^2 \beta^2 \cos k_1 z'}{k_0^2 k_1 T_m} \left[ \varepsilon_r k_2 \sin k_1 (d - z) - j k_1 \cos k_1 (d - z) \right],
\tag{2.61}
\]

\[
Q^E_{zz}(z' > z) = - \frac{j \omega \mu_0 k_0^2 \beta^2 \cos k_1 z}{k_0^2 k_1 T_m} \left[ \varepsilon_r k_2 \sin k_1 (d - z') - j k_1 \cos k_1 (d - z') \right].
\]

Substituting expression (2.61) in (2.60) and using \( g_j \) according to (2.50), the final expression for \( I_{ij}^E(\beta) \) takes the following form

\[
I_{ij}^E(\beta) = \frac{j \omega \mu_0}{\varepsilon_r} \left\{ \left( - \frac{h \varepsilon_r}{3 k_0^2 (\beta^2 - \varepsilon_r)} + \frac{4 \beta^2}{h k_1^4} \right) \varepsilon_j - \frac{4 \beta^2}{h^2 k_1^4 T_m} \times \right.

\left[ \varepsilon_r k_2 \left( \cos k_1 z_{j-1} \sin k_1 (d - z_{j-1}) - 4 \cos k_1 z_{j-1} \sin k_1 (d - z_j) \\
+ 2 \cos k_1 z_{j-1} \sin k_1 (d - z_{j+1}) + 4 \cos k_1 z_j \sin k_1 (d - z_j) \\
- 4 \cos k_1 z_j \sin k_1 (d - z_{j+1}) + \cos k_1 z_{j+1} \sin k_1 (d - z_{j+1}) \right) \\
- j k_1 \left( \cos k_1 z_{j-1} \cos k_1 (d - z_{j-1}) - 4 \cos k_1 z_{j-1} \cos k_1 (d - z_j) \\
+ 2 \cos k_1 z_{j-1} \cos k_1 (d - z_{j+1}) + 4 \cos k_1 z_j \cos k_1 (d - z_j) \\
- 4 \cos k_1 z_j \cos k_1 (d - z_{j+1}) + \cos k_1 z_{j+1} \cos k_1 (d - z_{j+1}) \right) \right\}, \tag{2.62}
\]

with \( \varepsilon_j = 1 \) for \( j \geq 2 \). For \( j = 1 \), i.e. half rooftop basis function, \( \varepsilon_j = \frac{1}{2} \) and \( z_j, z_{j-1} \) should be set to zero. For large \( \beta \)-values numerical difficulties may occur due to a limitation of the computing capacity of available compilers. Inclusion of the asymptotic values for calculating \( Z_{ij} \) becomes necessary to avoid these numerical problems. The
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Following approximations can be made for large \(\beta\)

\[
\begin{align*}
  k_1 & \approx -j k_0 \beta, \\
  k_2 & \approx -j k_0 \beta, \\
  \cos k_1 d & \approx \frac{1}{2} e^{jk_0 d}, \\
  \sin k_1 d & \approx -\frac{j}{2} e^{jk_0 d}, \\
  T_z & \approx -j k_0 \beta e^{jk_0 d}, \\
  T_m & \approx -j k_0 \beta \left(\frac{\varepsilon_r + 1}{2}\right) e^{jk_0 d}.
\end{align*}
\] (2.63)

By substituting the above approximations into equation (2.62), it can easily be shown that the asymptotic value of \(I_{jj}^z(\beta)\) is given by

\[
I_{jj}^z(\beta) = \frac{\mu_0}{\varepsilon_r} \times \left\{
\begin{array}{ll}
-\frac{1}{h^2 k_0 \beta^2} \left(\frac{4\varepsilon_r + 8}{k_0 \beta(\varepsilon_r + 1)} - 2h\right) - \frac{h\varepsilon_r}{6\beta^2 k_0^2} & j = 1 \land z_{j+1} = d \\
-\frac{1}{h^2 k_0 \beta^2} \left(\frac{6}{k_0 \beta} - 2h\right) - \frac{h\varepsilon_r}{6\beta^2 k_0^2} & j = 1 \land z_{j+1} < d \\
-\frac{1}{h^2 k_0 \beta^2} \left(\frac{12\varepsilon_r + 16}{k_0 \beta(\varepsilon_r + 1)} - 4h\right) - \frac{h\varepsilon_r}{3\beta^2 k_0^2} & j = 2 \land z_{j+1} = d \\
-\frac{1}{h^2 k_0 \beta^2} \left(\frac{14}{k_0 \beta} - 4h\right) - \frac{h\varepsilon_r}{3\beta^2 k_0^2} & j = 2 \land z_{j+1} < d \\
-\frac{1}{h^2 k_0 \beta^2} \left(\frac{10\varepsilon_r + 14}{k_0 \beta(\varepsilon_r + 1)} - 4h\right) - \frac{h\varepsilon_r}{3\beta^2 k_0^2} & j > 2 \land z_{j+1} = d \\
-\frac{1}{h^2 k_0 \beta^2} \left(\frac{12}{k_0 \beta} - 4h\right) - \frac{h\varepsilon_r}{3\beta^2 k_0^2} & j > 2 \land z_{j+1} < d
\end{array}\right.
\] (2.64)

- \(i = j - 1\):

The element of the matrix \([Z]\) has now the following form

\[
Z_{jj-1} = -\frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} I_{jj-1}^z(\beta) J_0(k_0 \beta r_a),
\] (2.65)
with

\[
I_{j_{j-1}}^{zz}(\beta) = \int_{z_{j-1}}^{z_j} g_j(z) \left[ \int_{z_{j-2}}^{z_j} g_{j-1}(z') Q_{zz}^{EE}(z' < z) \, dz' \right. \\
+ \int_{z_{j-1}}^{z_j} g_{j-1}(z') Q_{zz}^{EE}(z' < z) \, dz' + \int_{z}^{z_{j-1}} g_{j-1}(z') Q_{zz}^{EE}(z' > z) \, dz' \right] \, dz \\
+ \int_{z_j}^{z_{j+1}} g_j(z) \left[ \int_{z_{j-2}}^{z_j} g_{j-1}(z') Q_{zz}^{EE}(z' < z) \, dz' \\
+ \int_{z_{j-1}}^{z_j} g_{j-1}(z') Q_{zz}^{EE}(z' > z) \, dz' \right] \, dz \\
+ \frac{j \omega \mu_0}{\varepsilon_r k_0^2} \int_{z_{j-1}}^{z_j} g_j(z) g_{j-1}(z_{j-1} \leq z \leq z_j) \, dz.
\]

(2.66)

The last term of the above expression is caused by the \(\delta\)-function in \(Q_{zz}^{EE}\). After performing the double \(z\) integrations analytically, one may have the following expression for \(j \geq 3\)

\[
I_{j_{j-1}}^{zz}(\beta) = \frac{j \omega \mu_0}{\varepsilon_r} \left\{ \left( -\frac{h \varepsilon_r}{12 k_0^2 (\beta^2 - \varepsilon_r)} - \frac{2 \beta^2}{h k_1^4} \right) - \frac{4 \beta^2}{h^2 k_1^3 T_m} \times \\
\varepsilon_r k_2 \left( \cos k_1 z_{j-2} \sin k_1 (d - z_{j-1}) - 2 \cos k_1 z_{j-2} \sin k_1 (d - z_j) \\
+ \cos k_1 z_{j-2} \sin k_1 (d - z_{j+1}) + 5 \cos k_1 z_{j-1} \sin k_1 (d - z_j) \\
- 2 \cos k_1 z_{j-1} \sin k_1 (d - z_{j-1}) - 2 \cos k_1 z_{j-1} \sin k_1 (d - z_{j+1}) \\
- 2 \cos k_1 z_{j-1} \sin k_1 (d - z_{j-1}) + \cos k_1 z_{j-1} \sin k_1 (d - z_{j+1}) \right) \\
\right. \\
- j k_1 \left( \cos k_1 z_{j-2} \cos k_1 (d - z_{j-1}) - 2 \cos k_1 z_{j-2} \cos k_1 (d - z_j) \\
+ \cos k_1 z_{j-2} \cos k_1 (d - z_{j+1}) + 5 \cos k_1 z_{j-1} \cos k_1 (d - z_j) \\
\right) \right\}.
\]
2.7. Calculation of the Elements of the Matrix \([Z]\)

\[
Z_{ij} = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} I_{ij}^{zz}(\beta) J_0^2(k_0 r a),
\]

with

\[
I_{ij}^{zz}(\beta) = \int_{z_i}^{z_{i+1}} g_j(z) \left[ \int_{z_{j-1}}^{z_j} g_i(z') Q_{zz}^{EE}(z' < z) \, dz' \right. + \int_{z_j}^{z_{j+1}} g_i(z') Q_{zz}^{EE}(z' > z) \, dz' \left. \right] \, dz
\]

\[
+ \int_{z_{j-1}}^{z_i} g_j(z) \left[ \int_{z_{i-1}}^{z_i} g_i(z') Q_{zz}^{EE}(z' < z) \, dz' \right. + \int_{z_i}^{z_{i+1}} g_i(z') Q_{zz}^{EE}(z' > z) \, dz' \left. \right] \, dz.
\]

Performing the double \(z\) integrations analytically one may obtain

\[
I_{ij}^{zz}(\beta) = \frac{j \omega \mu_0}{\varepsilon_r} \left[ 2 \cos k_1 z_i - \cos k_1 z_{i-1} - \cos k_1 z_{i+1} \right] \times
\]

\[
\left[ \varepsilon_r k_2 \left( 2 \sin k_1 (d - z_j) - \sin k_1 (d - z_{j-1}) - \sin k_1 (d - z_{j+1}) \right) \right. \]

\[
- j k_1 \left( 2 \cos k_1 (d - z_j) - \cos k_1 (d - z_{j-1}) - \cos k_1 (d - z_{j+1}) \right) \right].
\]

For \(j = 2\), \(z_{j-2}\) and \(z_{j-1}\) should be made zero. The asymptotic value of \(I_{jj-1}^{zz}\) is given by

\[
I_{jj-1}^{zz}(\beta) = \frac{j \omega \mu_0}{\varepsilon_r} \left[ -\frac{1}{h^2 k_0^2 \beta^2} \left( 2h - \frac{8}{k_0 \beta} \right) \right. \]

\[
- \frac{h \varepsilon_r}{12 \beta^2 k_0^2} \right].
\]
2.8 Calculation of the Elements of the Excitation Vector \([V^t]\)

In this section we will derive an expression for an element of the excitation vector \([V^t]\). Contrary to the theory for a finite array [11, 12] the reaction concept can not be used. This problem is caused by the infinite extension of the array dimension. Therefore we have to calculate the electric field produced by magnetic currents located at the base of monopoles. As mentioned in section 2.2, the induced electric current distribution on the monopoles is assumed to be concentrated at the outer surface of the monopoles and has no \(\phi\)-component. The consequence is that only two dyadic Green’s functions are sufficient to describe the incident fields on the monopole surface. These Green’s functions are \(Q_{xz}^{EM}\) and \(Q_{zy}^{EM}\). The derivation of these Green’s functions has been performed in section 2.3.3. We start our calculation with expression (2.48) which represents an element of \([V^t]\):

\[
V^t_i = \int_{S_f} \vec{J}_{fi}(\vec{r}) \cdot \vec{E}_{\text{trill}}(\vec{r}) \, dS,
\]

where \(\vec{E}_{\text{trill}}\) is the electric field caused by the magnetic frill at the coaxial aperture. Each monopole is excited by a coaxial cable from below the ground plane. A frill of magnetic currents is used to model the coaxial aperture, depicted in figure 2.10. The voltage \(U\) between the inner and outer radius of the coaxial cable is used to generate the magnetic currents \(M_{\text{trill}}\), according to equation (2.5). Using equation (2.37), the electric field due to a magnetic current at the coaxial aperture \(S_{\text{trill}}\) can be written as

\[
\vec{E}_{\text{trill}}(\vec{r}) = \int_{S_{\text{trill}}} \vec{E} \cdot \hat{\mu}_{\text{trill}}(\vec{r}') \, dV' = \int_{S_{\text{trill}}} \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} Q_{x\text{M}}^{EM}(k_x, k_y, z, 0) e^{-jk_y(x-x')} \, e^{-jk_y(y-y')} \cdot \hat{\mu}_{\text{trill}}(x', y', 0) \, dS'.
\]

\(Q_{x\text{M}}^{EM}(k_x, k_y, z, 0)\) is given by expressions (2.31) and (2.32) by assigning \(z' = 0\):

\[
Q_{x\text{M}}^{EM}(k_x, k_y, z \geq 0, 0) = -\frac{j k_y}{k_1 T_m} \left[ \varepsilon_r k_2 \sin k_1 (d - z) - j k_1 \cos k_1 (d - z) \right],
\]

\[
Q_{y\text{M}}^{EM}(k_x, k_y, z \geq 0, 0) = \frac{j k_x}{k_1 T_m} \left[ \varepsilon_r k_2 \sin k_1 (d - z) - j k_1 \cos k_1 (d - z) \right].
\]
2.8. CALCULATION OF THE ELEMENTS OF THE EXCITATION VECTOR $[V^T]$ 27

Figure 2.10: Vertical monopole and magnetic current frill.
Substituting (2.75) in (2.73) gives

\[ V^t_i = \int_{S_{\text{trial}}} \int \mathcal{J}_{fi} \cdot \left[ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{Q}^{EM}(k_x, k_y, z, 0) e^{jk_x(x' - x)} e^{jk_y(y' - y)} \cdot \mathcal{M}_{\text{trial}} dS' dS \right] \]

\[ = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{S_{\text{trial}}} \mathcal{J}_{fi}(x, y, z) e^{-jk_x x} e^{-jk_y y} \cdot \left[ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{Q}^{EM}(k_x, k_y, z, 0) \cdot \mathcal{M}_{\text{trial}}(x', y', 0) e^{jk_x x'} e^{jk_y y'} dS' \right] dS. \]  

(2.76)

After performing \(x\)- and \(y\)-integrations, the above expression can be written as

\[ V^t_i = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[ \int_{S_{\text{trial}}} \mathcal{J}_{fi}(k_x, k_y, z) \cdot \mathcal{Q}^{EM}(k_x, k_y, z, 0) dz \right] \cdot \mathcal{M}_{\text{trial}}(k_x, k_y, 0). \]  

(2.77)

The Fourier transform of \( \mathcal{M}_{\text{trial}} \), denoted by \( \tilde{\mathcal{M}}_{\text{trial}} \), is calculated by using equations (2.5) and (2.6), giving

\[ \tilde{\mathcal{M}}_{\text{trial}}(k_x, k_y, 0) = \int_{S_{\text{trial}}} \mathcal{M}_{\text{trial}}(x', y', 0) e^{jk_x x'} e^{jk_y y'} dS' \]

\[ = \frac{U}{\ln \left( \frac{T_{k}}{r_{a}} \right)} \int_{S_{\text{trial}}} \frac{1}{r} \tilde{\mathcal{E}} \cdot e^{jk_x x'} e^{jk_y y'} dS'. \]  

(2.78)

\( \mathcal{J}_{fi}^*(k_x, k_y, z) \) is the conjugate of the Fourier transform of \( \mathcal{J}_{fi}(x, y, z) \) with

\[ \mathcal{J}_{fi}^*(k_x, k_y, z) = \left[ J_0(k_0 \beta r_a) g_i(z) \hat{e}_z \right]^* = \tilde{\mathcal{J}}_{fi}(k_x, k_y, z). \]  

(2.79)

Equation (2.77) shows that the solution for each element of the excitation vector \([V^t] \) can be presented in the spectral quantities. An easy way to find the spectral magnetic current \( \tilde{\mathcal{M}}_{\text{trial}} \) is by using the polar coordinate system \((\rho, \phi, z)\). Equation (2.78) in polar notation takes the form

\[ \tilde{\mathcal{M}}_{\text{trial}}(k_x, k_y, 0) = -\frac{U}{\ln \left( \frac{T_{k}}{r_{a}} \right)} \int_{r_{a}}^{r_{b}} 2\pi \rho d\rho \int_{0}^{2\pi} e^{j(k_0 \beta \cos \alpha)(\rho \cos \phi)} e^{j(k_0 \beta \sin \alpha)(\rho \sin \phi)} d\rho d\phi \]

\[ = -\frac{U}{\ln \left( \frac{T_{k}}{r_{a}} \right)} \int_{r_{a}}^{r_{b}} 2\pi \rho d\rho \int_{0}^{2\pi} e^{j(k_0 \beta \rho \cos \phi \cos \alpha + \sin \phi \sin \alpha)} d\rho d\phi \]

\[ = -\frac{U}{\ln \left( \frac{T_{k}}{r_{a}} \right)} \int_{r_{a}}^{r_{b}} 2\pi \rho d\rho \int_{0}^{2\pi} e^{j(k_0 \beta \rho \cos \phi \cos \alpha)} d\rho d\phi \]

\[ = \frac{U}{\ln \left( \frac{T_{k}}{r_{a}} \right)} \left[ I_{\text{trial}}^{\tilde{e}_x} + I_{\text{trial}}^{\tilde{e}_y} \right], \]  

(2.80)
2.8. CALCULATION OF THE ELEMENTS OF THE EXCITATION VECTOR \([V^T]\)

Figure 2.11: Magnetic current at \(z = 0\) is decomposed into two perpendicular components.

where

\[ I_{\text{trill}}^x = \int_{r_a}^{r_b} \int_0^{2\pi} \sin \phi \, e^{j k_0 \beta \rho \cos(\phi - \alpha)} \, d\rho d\phi, \]

\[ I_{\text{trill}}^y = -\int_{r_a}^{r_b} \int_0^{2\pi} \cos \phi \, e^{j k_0 \beta \rho \cos(\phi - \alpha)} \, d\rho d\phi. \]  

(2.81)

In (2.80) the following expressions have been used

\[ k_x = k_0 \beta \cos \alpha, \quad x' = \rho \cos \phi, \]

\[ k_y = k_0 \beta \sin \alpha, \quad y' = \rho \sin \phi. \]  

(2.82)

Substitution of \(\phi' = \phi - \alpha\) in (2.81) gives

\[ I_{\text{trill}}^x = \int_{r_a}^{r_b} \int_0^{2\pi} \sin(\phi' + \alpha) \, e^{j k_0 \beta \rho \cos \phi'} \, d\phi' d\rho \]

\[ = \int_{r_a}^{r_b} \left[ \int_0^{2\pi} \sin \phi' \cos \alpha \, e^{j k_0 \beta \rho \cos \phi'} d\phi' \right] d\rho + \int_{r_a}^{r_b} \left[ \int_0^{2\pi} \cos \phi' \sin \alpha \, e^{j k_0 \beta \rho \cos \phi'} d\phi' \right] d\rho \]

\[ = 2 \sin \alpha \int_{r_a}^{r_b} \int_0^{\pi} \cos \phi' \, e^{j k_0 \beta \rho \cos \phi'} d\phi' d\rho. \]
\[ \beta = 2\pi_J \sin \alpha \int_{r_a}^{r_b} J_1(k_0 \beta \rho) \, d\rho \]

\[ = -\frac{2\pi_J \sin \alpha}{k_0 \beta} \left[ J_0(k_0 \beta r_b) - J_0(k_0 \beta r_a) \right], \tag{2.83} \]

and

\[ I^y_{\text{full}} = \frac{2\pi_J \cos \alpha}{k_0 \beta} \left[ J_0(k_0 \beta r_b) - J_0(k_0 \beta r_a) \right]. \tag{2.84} \]

Inserting expressions (2.75), (2.79), (2.80), (2.83) and (2.84) into (2.77), we now find the final expression for an element of the excitation vector \([V^t]\)

\[ V^t_i = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} -\frac{2\pi U}{\ln \left( \frac{T_k}{r_a} \right)} \frac{J_0(k_0 \beta r_b) - J_0(k_0 \beta r_a)}{k_1 T_m} J_0(k_0 \beta r_a) I^y_i(\beta), \tag{2.85} \]

with

\[ I^y_i(\beta) = \int g_i(z) \left[ \varepsilon_r k_2 \sin k_1(d - z) - jk_1 \cos k_1(d - z) \right] \, dz. \tag{2.86} \]

Using rooftop subdomain basis functions for \(g_i(z), I^y_i(\beta)\) can be evaluated analytically. Two cases are distinguished:

- \(i = 1\) (half-rooftop basis function)

\[ I^y_{1\text{-h}}(\beta) = \int_0^{\frac{h}{2}} \frac{2}{h} \left( \frac{h}{2} - z \right) \left( \varepsilon_r k_2 \sin k_1(d - z) - jk_1 \cos k_1(d - z) \right) \, dz \]

\[ = -\frac{T_m}{k_1} + \frac{2}{hk_1^2} \left[ \varepsilon_r k_2 \left( \sin k_1 d - \sin k_1 \left( d - \frac{h}{2} \right) \right) \right. \]

\[ \left. -jk_1 \left( \cos k_1 d - \cos k_1 \left( d - \frac{h}{2} \right) \right) \right]. \]

- \(i \geq 2\)

\[ I^y_{i\geq2}(\beta) = \int_{z_{i-1}}^{z_i} \frac{2}{h} \left( z - z_{i-1} \right) \left( \varepsilon_r k_2 \sin k_1(d - z) - jk_1 \cos k_1(d - z) \right) \, dz \]

\[ + \int_{z_i}^{z_{i+1}} \frac{2}{h} \left( z_{i+1} - z \right) \left( \varepsilon_r k_2 \sin k_1(d - z) - jk_1 \cos k_1(d - z) \right) \, dz \]

\[ = \frac{2}{hk_1^2} \left[ \varepsilon_r k_2 \left( 2 \sin k_1(d - z_i) - \sin k_1(d - z_{i-1}) - \sin k_1(d - z_{i+1}) \right) \right. \]

\[ -jk_1 \left( 2 \cos k_1(d - z_i) - \cos k_1(d - z_{i-1}) - \cos k_1(d - z_{i+1}) \right) \].
2.9. CALCULATION OF THE INPUT IMPEDANCE

For large values of $\beta$, $I_i^*(\beta)$ can be reduced to its asymptotic form

$$
\frac{1}{k_1 T_m} I_i^*(\beta) = \begin{cases} 
\frac{1}{k_0^2 \beta^2} - \frac{2}{h k_0^3 \beta^3}, & \text{for } i = 1, \\
\frac{2}{h k_0^3 \beta^3}, & \text{for } i = 2, \\
0, & \text{for } i \geq 3.
\end{cases}
$$

(2.87)

2.9 Calculation of the Input Impedance

In the previous section, it has been explained how the method-of-moments matrix equation is constructed in terms of integrals that are evaluated numerically. Solving the matrix equation yields the basis function amplitude coefficients which, when combined with the basis functions, approximate the true current distribution on the array structure. The input admittance of an antenna excited by a magnetic frill is given in [16]:

$$
Y_{in} = -\frac{1}{|U|^2} \oint \oint \mathcal{H}_f \cdot \mathcal{M}_{frill}^* dV_{frill} = -\frac{1}{|U|^2} \oint \oint \mathcal{M}_{frill}^* \cdot \oint \oint \mathcal{H} \cdot \mathcal{J}_f dV_f dV_{frill},
$$

(2.88)

in which $\mathcal{H}_f$ is the total magnetic field generated by the monopole current $\mathcal{J}_f$. $\mathcal{M}_{frill}^*$ is the complex conjugate of the magnetic frill current in the aperture of the coaxial cable and $U$ is the voltage of the frill. The dyadic $\mathcal{H}$ represents the total magnetic field produced by an infinite array of electric dipoles and can be written as (see also equation (2.37))

$$
\mathcal{H} = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \tilde{Q}^{HE}(k_x, k_y, z, z') e^{-j k_x (x-x')} e^{-j k_y (y-y')}.
$$

(2.89)

The $Q$-function in the above equation has already been derived in section 2.3.2, see (2.14) and (2.15). The other components of the $Q$-function can be found in [12, page 8]. Substitution of equation (2.89) in (2.88) results in the following equation

$$
Y_{in} = -\frac{1}{U^2} \sum_{i=1}^{N_{max}} \int \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{M}_{frill}^*(k_x, k_y, 0) \cdot \tilde{Q}^{HE}(k_x, k_y, 0, z') \cdot \mathcal{J}_f(k_x, k_y, z') dz'
$$

$$
= -\frac{1}{U^2} \sum_{i=1}^{N_{max}} \int \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{M}_{frill}^* \cdot \int_{z'} \left( \tilde{\hat{e}}_x Q_{xx}^{HE} \hat{e}_z + \tilde{\hat{e}}_y Q_{yy}^{HE} \hat{e}_z \right) \cdot \mathcal{J}_f dz'
$$

$$
= \frac{1}{U^2} \sum_{i=1}^{N_{max}} I_i V_i^t.
$$

(2.90)

Note that in the above equations we have used a real excitation voltage.
In [10, 11, 16] another approach is suggested to calculate the input admittance. If one uses the approximation given by equation (2.4), a simple relation can be obtained:

\[
Y_{\text{in}} = -\frac{1}{U^2} \int \int \int_{V} \vec{R}_f \cdot \vec{M}^*_{\text{null}} \, dV = \frac{1}{U^2} \int_{0}^{r_b} I(z = 0) \frac{U}{2\pi \rho} \ln \left( \frac{r_a}{r_s} \right) \rho \, d\rho \, d\phi
\]

where \( I(z = 0) \) is the current at the base of the monopole found via the method-of-moments procedure. In the case of electrically thick substrates relation (2.91) gives quite accurate results. However, if the substrate layer is thin, results obtained from (2.90) and (2.91) differ significantly because the assumption in equation (2.4) is no longer valid. Inverting \( Y_{\text{in}} \) yields the input impedance \( Z_{\text{in}} \). When the input impedance is known, the active reflection coefficient \( R(\theta, \phi) \) of a planar array can be calculated as

\[
R(\theta, \phi) = \frac{Z_{\text{in}}(\theta, \phi) - Z_0}{Z_{\text{in}}(\theta, \phi) + Z_0},
\]

with \( Z_0 \) denoting the impedance of the feed network.
Chapter 3

Properties of Infinite Arrays of Monopoles

3.1 Introduction

In chapter 2 formulae were derived in order to analyze the properties of an infinite array of monopoles embedded in a dielectric substrate. These formulae were implemented in a software package. The software package is written using the Microsoft (MS) Fortran-77 Power Station compiler. In this chapter properties of infinite arrays, such as input impedance and scan blindness, will be examined. The results will be compared with data from the measurements using waveguide simulators as given in [8, 14]. Therefore the same arrays will be investigated to verify the developed model. Comparison with results from the finite array theory, presented in [11], will be given as well. First, analysis of the convergence of infinite number of Floquet modes is performed. Next, a second convergence check is applied to the expansion and test modes. Both convergence checks will be applied to an array configuration with an air-filled substrate. Much attention is paid to the characteristics of arrays consisting of quarter-wavelength monopoles.

Finally, we discuss another array property, namely the occurrence of surface waves. Because of the fact that thicker substrates are used, the effect of surface waves to the overall performance of large arrays needs to be discussed. Destructive interference of surface wave power can occur, raising the radiation efficiency, although at certain scan angles a constructive interference may be possible, leading to a scan blindness effect. At the end of this chapter, this blindness phenomenon will be investigated by using a surface wave diagram.

3.2 Convergence Consideration

Saving computation time is a very important factor in analyzing an antenna array, especially if the array's dimension is very large in terms of the number of elements in the array. In the
case of infinite arrays, we deal with the summation over an infinite number of Floquet modes according to equations (2.57) and (2.85). Moreover, the number of basis functions included in the analysis must also be limited to a finite number, see equation (2.41). In the following subsections, the convergence properties of an infinite array will be examined.

3.2.1 Number of Floquet Modes

The properties of equations (2.57) and (2.85) are analysed. Each of these equations contains a double infinite sum of Floquet modes. Evidently, the number of Floquet modes taken into account, must be kept as small as possible since the computation time increases quadratically with the number of Floquet modes. Pozar [8] has employed about 161 Floquet modes in each direction, i.e. \(-80 \leq m \leq 80, -80 \leq n \leq 80\). This number is greater than 121 which is needed to analyze an infinite array of microstrip antennas [6]. In [8] some results are given of measurements of the input impedance of vertical monopoles using a waveguide simulator. In order to analyse the convergence of the input impedance of an infinite array of monopoles, the same configuration is taken and the input impedance is computed using the developed software.

antenna 1

- monopole height \(z_f = 14.7\) mm
- permittivity \(\varepsilon_r = 1\) (air)
- inner radius 50 \(\Omega\) coax \(r_a = 0.565\) mm
- outer radius 50 \(\Omega\) coax \(r_b = 1.853\) mm
- element spacing \(a = 60.6\) mm, \(b = 60.6\) mm
- number of basis functions \(N_z = 5\)

The antenna given above is steered to a scan angle with \(\phi = 45^\circ\), and \(\theta = \arcsin[\lambda_0/(a\sqrt{2})]\). Thus, the values of \(\theta's\) depend also on the evaluated frequencies. The frequency range is chosen to be at 4 - 6 GHz. Notice that \(z_f = \lambda_0/4\) at \(f = 5.1\) GHz, where \(\lambda_0\) is the freespase wavelength. In figure 3.1 some results are shown for various numbers of Floquet modes. From this figure it is clear that the input impedance is not strongly sensitive to the number of Floquet modes. If this number is equal or greater than 161 modes in each direction, a stable result can be achieved. It is therefore that 161 Floquet modes will be used in investigations of the array properties.

3.2.2 Number of Basis Functions

Using subdomain basis functions, the number of matrix elements of \([Z]\) and \([V^t]\) that have to be calculated, can become very large. Generally, the more basis functions taken into calculation, the more accurate results might be achieved. From reference [10] it can be
3.2. CONVERGENCE CONSIDERATION

Figure 3.1: Input impedance for various number of Floquet modes NMax, (a) Re(Zin), (b) Im(Zin), (c) Smith chart with Z₀ = 50 Ω.
CHAPTER 3. PROPERTIES OF INFINITE ARRAYS OF MONOPOLES

concluded that about twenty basis functions per wavelength have to be used in order to obtain acceptable results. For the same array as specified in section 3.2.1 the second convergence investigation with respect to the expansion modes will be applied. The results are given in figure 3.2. This figure suggests that using more than five basis functions for quarter-wavelength monopoles does not change the results significantly. Therefore, the best choice to obtain accurate results within acceptable computation time is five PWL subdomain basis functions for an infinite array of quarter wavelength monopoles. In addition, Pozar has performed some measurements with a waveguide simulator [8, figure 2]. It can be observed that a little discrepancy between our results and [8] can be seen especially at lower frequencies, see figure 3.1. In the region above 5 GHz the resemblance of both curves is quite good. It is suggested in [8] that the discrepancy is probably due to reflections from the absorber in the waveguide simulator. The induced current on the monopole surface is depicted in figure 3.3 for various number of the basis functions. The form of this current converges to a sinusoidal form as $N_z$ increases.

Next, a comparison between our results and results obtained from the finite array theory is performed. It is expected that the finite array results would converge to the infinite array solution as the array size increases. For that purpose, the same array dimension as above is investigated using the theory developed in [11, 12]. In figure 3.2 the input impedance of the center element of an air-filled array of $15 \times 15$ monopoles is plotted. The input impedance curve of the center element of this array shows a less smooth course. This whimsical behaviour is usually found with a finite array. Since the mutual coupling in an infinite array theory is assumed to be uniform for all elements, the input impedance of an infinite array can be regarded as an average of the input impedance of a large finite array. The in-phase coupling by surface and space waves from infinite number of elements can not cause sudden variations in the input impedance.

At last, the scan performance of monopoles embedded in an air-filled substrate is considered. Figure 3.4 shows the calculated reflection coefficients against scan angle $\theta$ for $a = b = \lambda_0/2$. It can be seen that the reflection coefficient minimum is seen to occur in the vicinity of $\theta = 60^\circ$. Based on figure 3.4, scanning with a low reflected radiation power is possible from approximately $45^\circ$ to $70^\circ$ from broadside.

3.3 Scan Blindness and Reflection Coefficient

Scan blindness refers to a resonance condition whereby, for certain scan angles, no real power is coupled to or from a phased array. This condition is caused by the forced resonance of a surface wave of the loaded dielectric substrate. The surface wave is bound to the infinite array surface, so that no real power enters or leaves the array; the surface waves stores energy. A surface wave will be excited whenever a Floquet mode propagation constant $\beta = \sqrt{k_z^2 + k_y^2/k_0}$
3.3. SCAN BLINDNESS AND REFLECTION COEFFICIENT

Figure 3.2: Input impedance for various number of PWL basis functions $N_z$, (a) $\text{Re}(Z_{in})$, (b) $\text{Im}(Z_{in})$, (c) Smith chart with $Z_0 = 50$ $\Omega$. 
Figure 3.3: Current distribution along the monopole surface, \( f = 5 \text{ GHz} \),
(a) Real part, (b) Imaginary part.
3.3. SCAN BLINDNESS AND REFLECTION COEFFICIENT

equals the surface wave propagation constant $\beta_{sw}$ of the loaded dielectric substrate

\[ \beta_{sw} = \sqrt{\left(\frac{k_x}{k_0}\right)^2 + \left(\frac{k_y}{k_0}\right)^2} = \sqrt{\left(\frac{m\lambda_0}{a} + \sin \theta \cos \phi\right)^2 + \left(\frac{n\lambda_0}{b} + \sin \theta \sin \phi\right)^2}, \]

where $\lambda_0$ is the free-space wavelength. Strictly speaking, $\beta_{sw}$ must be the propagation constant of the surface wave in presence of the monopoles, and so should be found from the moment method solution. For most problems of practical interest, however, this loading effect will be very small, and $\beta_{sw}$ can be closely approximated by using the propagation constant of the unloaded dielectric slab, as determined by the zeros of $T_m$ and $T_e$ in equation (2.16). It has to be noted that the propagation constant of the surface wave is a complex number and that the propagation constant of the Floquet mode is a real number, so the $T_m$ or $T_e$ function will never be exactly zero and the reflection coefficient does not have to be one in a blind spot of an infinite array. For substrates with

\[ k_0d\sqrt{\varepsilon_r - 1} < \frac{\pi}{2} \]

only the lowest order TM surface wave mode can exist. All the substrates considered in this report satisfy this condition. In figure 3.5 the surface wave propagation constant of a grounded dielectric substrate is plotted against substrate thickness for various dielectric

Figure 3.4: Calculated reflection coefficient magnitude against scan angle for an infinite array of quarter wave-length monopoles with a $\lambda_0/2$ element spacing.
constants. From this figure it can be seen that the values of $\beta_{sw}$ start at the value 1 and end at the point where the second surface wave mode begins to propagate. For rectangular grids,

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3_5}
\caption{Surface wave propagation constants $\beta_{sw}$ for a grounded dielectric substrate.}
\end{figure}

it is the $(m = -1, n = 0)$ or the $(m = 0, n = -1)$-th Floquet mode which couples to the TM surface wave. Figure 3.6 shows the maximum scan range in a principal plane to avoid a blind spot. This set of curves is based on equation (3.1), with the element spacing in the plane of scan. It can be observed from figure 3.6 that the scan blindness always occurs before the onset of the grating lobes.

3.3.1 Scan Blindness Observed from the Input Impedance

In order to detect the occurrence of scan blindness, we consider an infinite array of quarterwave monopoles on a square grid with the following sizes:

antenna 2

- monopole length $z_f = 0.15 \lambda_0$
- substrate thickness $d = 0.15 \lambda_0$
- permittivity $\varepsilon_r = 2.55$, $\tan \delta = 0.0005$
- element spacing $a = b = 0.5 \lambda_0$
- inner radius coax $r_a = 0.001 \lambda_0$
3.3. SCAN BLINDNESS AND REFLECTION COEFFICIENT

Figure 3.6: Maximum scan range before scan blindness, for various surface wave propagation constants, for scanning in the principal plane (E- or H-plane) of an array with a rectangular grid. The element spacing is in the plane of scan.

Because of the presence of the dielectric material, the monopoles are approximately \( \lambda/4 \) long in the substrate in the frequency band under consideration. \( \lambda \) is the electrical wavelength in the substrate and is defined as \( \lambda = \lambda_0/\sqrt{\varepsilon_r} \). The input impedance versus scan angle is shown in figure 3.7 for \( \phi = 0^\circ \) (E-plane) and \( \phi = 45^\circ \) (D-plane). Note that the characteristic impedance for the Smith chart is \( 2S_1 \) and the number of Floquet modes taken into account is 201 in each direction. Figure 3.8 shows the reflection coefficient magnitude for E- and D-scan plane. Two reflection coefficients near unity are found for both scan planes, i.e. for \( \theta = 0^\circ \) and \( \theta = 90^\circ \). The total reflection at \( \theta = 0^\circ \) is due to the fact that vertical monopoles have a zero in their radiation pattern at broadside, whereas at endfire (\( \theta = 90^\circ \)) no real power can leave the plane of the infinite array. It can be seen from the input impedance curve that the input resistance \( R_{in} \) is roughly zero at both angles. In the vicinity of \( \theta = 54^\circ, \phi = 0^\circ \), a scan blindness can be expected. As was mentioned above, a scan blindness can occur at a certain scan angle such that the surface wave propagation constant \( \beta_{sw} \) matches the transverse propagation constant of one of the Floquet modes. From the data of figure 3.5, the substrate is seen to support a TM surface wave with \( \beta_{sw} = 1.19259 \). The condition for surface wave propagation (3.1) is satisfied for the \( E \)-plane for the \( (m = -1, n = 0) \)-th Floquet mode resulting in a blind angle \( \theta = 53.8^\circ \), see figure 3.6. In the same way, an identical scan blindness can be observed for the \( H \)-plane (\( \phi = 90^\circ \)) at \( \theta = 53.8^\circ \). Along the \( D \)-plane no blindness angle is detected. Comparing our results and the delta-gap source method of Pozar [8, figure 3], one finds a slight discrepancy between the two figures at the scan angles near
Figure 3.7: Active input impedance of an infinite array of quarter wavelength monopoles
(a) E-scan plane ($\phi = 0^\circ$). (b) D-scan plane ($\phi = 45^\circ$). $\theta_{\text{step}} = 2.5^\circ$
3.3. SCAN BLINDNESS AND REFLECTION COEFFICIENT

Figure 3.8: Active reflection coefficient with square grid. The antenna is matched to 50 Ω.

θ = 60° for the D-plane.

Another array geometry which is often investigated, is an array with a triangular grid. This triangular grid has certain advantages over a rectangular and square grid. The number of array elements required for a grating lobe free scan region is less than that for the case of rectangular grids. For a equilateral triangular grid, the maximum distance between the elements in the x-direction is about $\frac{\lambda_0}{\sqrt{3}}$ while $\frac{\lambda_0}{2}$ for a rectangular grid. Since a greater spacing is allowed, the coupling values between the elements can be decreased. The relation between these couplings and the element spacing has been analysed in [11]. Figure 3.9 shows the reflection coefficient of the above array configuration if a triangular grid is used with $a = 0.5774\lambda_0$, $b = 0.5\lambda_0$. This figure also shows that there is no advantage, in general, in using triangular grids to avoid a blindness angle.

Another infinite array that has been analyzed in [8] by means of a waveguide simulator, has the dimensions:

**antenna 3**

- monopole length $z_f = 10$ mm
- substrate thickness $d = 10$ mm
- permittivity $\varepsilon_r = 2.5$ (acrylic plastic), tan δ = 0.0005
- element spacing $a = b = 60.6$ mm
• inner radius coax $r_a = 0.565$ mm

The input impedance is calculated over a frequency range of 4.0 to 6.1 GHz. The results are plotted in figure 3.10. This figure shows a very good agreement with the measured data presented in [8, figure 4]. As can be seen, this antenna is badly matched to a 50 $\Omega$ characteristic impedance. The reflection coefficient is very large (but not unity) at a frequency of 6 GHz, corresponding to a scan angle of $\theta = 35.7^\circ$. For comparison, the calculated center element reflection coefficients are also given for two array sizes. These arrays have already been investigated in [11]. Note that the reflection coefficient of the center element, in general, can exceed unity for a finite array. A significant difference can be seen between the two approaches. This discrepancy can be explained if we look at the role of surface waves. Generally, the surface wave power generated by a single element increases with substrate thickness and dielectric constant. The surface waves radiated by one element will interfere with those radiated by other elements. At a certain phase difference, this interference will be either constructive or destructive. For a constant element spacing, a phase difference can be caused by changing the frequency or the scan angles. The partly quenching and strengthening the surface wave lead to up- and downward peaks of the reflection coefficient curve in figure 3.10. Pozar in [8] has observed that the coupling coefficients for an air-filled substrate will decay with a factor $\frac{1}{s}$, while a slower decay is detected in a dielectric case. This slow decay of the mutual coupling may be the reason for the discrepancy in figure 3.10.
3.3. SCAN BLINDNESS AND REFLECTION COEFFICIENT

Figure 3.10: Input impedance (a) and reflection coefficient (b) against frequency with \( \phi = 45^\circ \), \( Z_0 = 50\,\Omega \) and \( \sin \theta = \lambda_0 / (a\sqrt{2}) \).
3.3.2 Scan Blindness Predicted by a Surface Wave Circle Diagram

The previously stated condition for scan blindness and the effects of element spacing and surface wave propagation constants given by equation (3.1) can be graphically portrayed by means of a modified grating lobe diagram, referred as a surface wave circle diagram. Figure 3.11 shows the surface wave circle diagram of antenna 3.

The solid circles are the usual grating lobe circles, with centers at
\[ u = -\frac{m}{a/\lambda_0}, \quad v = -\frac{n}{b/\lambda_0}, \] (3.3)

and a radius of unity. The dotted circles are the surface wave circles, with the same centers as the grating lobe circles but with a radius of \( \beta_{sw} \). Whenever a surface wave circle intersects the visible region, i.e. the region for which \( |u|^2 + |v|^2 < 1 \), the condition in equation (3.1) is satisfied, and scan blindness will occur along this section of the surface wave circle. In the present array configuration, the grid spacing \( a, b = 1, 2 \lambda_0 \) is greater than \( \lambda_0/2 \) at \( f = 6 \text{ GHz} \) so the grating lobe circles overlap. The substrate thickness \( d/\lambda_0 \) is about 0.2. Through the developed programs the value of \( \beta_{sw} \) is found to be 1.29024. Since the evaluation is made in the \( D \)-scan plane, we look for intersections of the surface wave circles with the 45° line in the visible space. There are six surface wave circles that intersect this line near \( \theta = 35^\circ \). They are from the \(( -2, 0), (0, -2), ( -2, -1), ( -1, -2), ( -1, 1) \) and \(( 1, -1)\)-th Floquet mode. Half of these Floquet modes are due to symmetry with respect to the \( D \)-plane. As comparison, in most printed antenna arrays a scan blindness can occur through the \(( -1, 0) \) and \((0, -1)\)-th Floquet mode for \( E \)- and \( H \)-plane scan, respectively. In the present case, higher order Floquet modes are important because of the diagonal scan plane and the large element spacing.
3.3. SCAN BLINDNESS AND REFLECTION COEFFICIENT

Figure 3.11: Grating lobe and surface wave circle diagram with a element spacing of, (a) $\lambda_0/2$, (b) $1.21\lambda_0$
Chapter 4

Infinite Array of EMC Microstrip Antennas

4.1 Introduction

In many cases the narrow bandwidth of the traditional microstrip antenna element is its most serious disadvantage, preventing its use in many practical microwave applications. Thus a large amount of effort has been put into the development of creative design techniques for improving the bandwidth.

The most direct method of increasing the bandwidth of a microstrip element is to use a thick substrate with a low permittivity. However, an electrically thick substrate gives rise to an inductive shift in the input impedance and therefore the use of a compensating network would be inevitable [7, 9]. The use of such a network would increase the complexity and production costs of the total antenna. A possible solution for this problem can be the so-called electromagnetically coupled (EMC) microstrip structure [2, 12], which means that the microstrip patches are not physically connected to the feeding coaxial cables. In this chapter a theoretical study of an infinite array of microstrip antennas with EMC elements is presented. A sophisticated magnetic frill excitation is included in the model. In the next chapter, the calculated input reflection coefficients will be given and compared with the measured reflection coefficients of the center element of a finite array [12].

4.2 Model Description

The geometry of a unit cell in an infinite array of EMC microstrip antennas is depicted in figure 4.2. It is assumed that each element fills one unit cell, and that all elements are identical in structure and excitation, except for a uniform phase shift given by equation (2.33). Only rectangular microstrip antennas will be considered in this report. However, extension of this model to other patch geometries is straightforward. Each patch is placed at \( z = z_p \) and can
have a dielectric cover. The feeding coaxial cables with inner and outer radius \( r_a \) and \( r_b \), respectively, are located at a distance \((x_s, y_s)\) from the center of each patch. The substrate with a complex permittivity \( \varepsilon_r \) and thickness \( d \) is situated on a thin and perfectly conducting infinite ground plane. The probes are represented by a perfectly conducting cylinder with radius \( r_a \) and height \( z_f \). These probes are not physically connected to the patches, \( z_f < z_p \). It is assumed that the z-directed surface current on this probe only depends on the z-coordinate. Furthermore, the array has a rectangular grid with a element spacing of \( a \) in the x-direction and \( b \) in the y-direction. The electromagnetic fields corresponding to the TEM-mode in the coaxial apertures, at \((z=0)\), of each array element are used as sources. These sources can be represented by an annular magnetic current distribution at the aperture of each array element. Because the coaxial cable is now placed at \((x_s, y_s)\), the magnetic current distribution is written as

\[
\vec{M}_{\text{full}}(k_x, k_y, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{M}_{\text{full}}(x, y, 0) e^{jk_x x} e^{jk_y y} dx dy
\]

Figure 4.1: Geometry of an infinite array of microstrip antennas.
4.3. **Method of Moments Formulation**

The method of moments formulation for an infinite array of EMC microstrip antennas can be performed in the same manner as in the case of an infinite array of vertical monopoles. The only difference is that apart from \( z \)-directed currents on the probes, we now also have to

\[
\hat{e}_y \left[ \frac{2\pi y \cos \alpha}{k_0} \right] \left[ J_0(k_0 \beta r_b) - J_0(k_0 \beta r_a) \right] + \hat{e}_x \left[ \frac{2\pi y \sin \alpha}{k_0} \right] \left[ J_0(k_0 \beta r_b) - J_0(k_0 \beta r_a) \right]
\]

Note that the above equation has a similar form to equation (2.83) and (2.84). The only difference is the presence of \( e^{jk_x x_s} e^{jk_y y_s} \) due to the translation \( (x_s, y_s) \).

4.3 **Method of Moments Formulation**

The method of moments formulation for an infinite array of EMC microstrip antennas can be performed in the same manner as in the case of an infinite array of vertical monopoles. The only difference is that apart from \( z \)-directed currents on the probes, we now also have to

![Cross section of a probe-fed electromagnetically coupled microstrip antenna.](image-url)
calculate the $x$- and $y$-directed current distribution on the patches.

### 4.3.1 Green’s Function

In order to satisfy the boundary conditions at a perfectly conducting material, the electric field produced by a unit source has to be known. The electric field in the substrate is caused by two kind of sources. The incident field on the patches and the probes is generated by the annular frill of magnetic current source at the ground plane. This incident field gives rise to the second current, i.e. the induced electric currents on the surface of the probes and patches that in turn re-radiate scattered or diffracted fields. In section 2.3.3 the Lorentz reciprocity theorem was introduced that can be used to derive the spectral Green’s function due to a magnetic current source in a substrate. We have constructed $Q_{xx}^{EM}$ and $Q_{xy}^{EM}$ from $Q_{zz}^{HE}$ and $Q_{yz}^{HE}$, respectively. Through a similar way the other components of $Q$-functions can be found.

The incident electric field due to a magnetic current distribution in the substrate is given by

\[
\tilde{E}^{\text{inc}}(x,y,z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}(k_x,k_y,z)e^{-jk_x x}e^{-jk_y y} dk_x dk_y \\
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left[ \int_{z'} \tilde{Q}^{EM}(k_x,k_y,z,z') \cdot \tilde{M}(k_x,k_y,z')dz' \right] e^{-jk_x x}e^{-jk_y y} dk_x dk_y,
\]

with

\[
\tilde{Q}^{EM}(k_x,k_y,z,z') = \begin{pmatrix} Q_{xx}^{EM} & Q_{xy}^{EM} & Q_{xz}^{EM} \\ Q_{yx}^{EM} & Q_{yy}^{EM} & Q_{yz}^{EM} \\ Q_{zx}^{EM} & Q_{zy}^{EM} & Q_{zz}^{EM} \end{pmatrix}
\]

for $z \leq d$ and $z' \leq d$, \hspace{1cm} (4.4)

\[
Q_{xx}^{EM} = -Q_{yy}^{EM} = -\frac{j k_x k_y (\varepsilon_r - 1) \sin k_1 z \cos k_1 z'}{T_e T_m},
\]

\[
Q_{yx}^{EM} = \frac{j k_y^2 (\varepsilon_r - 1) \sin k_1 z \cos k_1 z'}{T_e T_m} + \begin{cases} \cos k_1 z' \left[ \varepsilon_r k_2 \cos k_1 (d - z) + j k_1 \sin k_1 (d - z) \right], & z' \leq z, \\
\sin k_1 z T_m N_m(z'), & z' > z \end{cases},
\]

\[
Q_{xy}^{EM} = -\frac{j k_x^2 (\varepsilon_r - 1) \sin k_1 z \cos k_1 z'}{T_e T_m} - \begin{cases} \cos k_1 z' \left[ \varepsilon_r k_2 \cos k_1 (d - z) + j k_1 \sin k_1 (d - z) \right], & z' \leq z, \\
\sin k_1 z T_m N_m(z'), & z' > z \end{cases}.
\]
4.3. METHOD OF MOMENTS FORMULATION

\[ Q_{zz}^{EM} = -\frac{j k_y}{T_m k_1} \begin{cases} 
\cos k_1 z' N_m(z'), z' \leq z, \\
\cos k_1 z N_m(z'), z \leq z', 
\end{cases} \]

\[ Q_{zy}^{EM} = \frac{j k_x}{T_m k_1} \begin{cases} 
\cos k_1 z' N_m(z), z' \leq z, \\
\cos k_1 z N_m(z'), z \leq z', 
\end{cases} \]

\[ Q_{zx}^{EM} = \frac{j k_y}{T_e k_1} \begin{cases} 
\sin k_1 z' N_e(z), z' \leq z, \\
\sin k_1 z N_e(z'), z \leq z', 
\end{cases} \]

\[ Q_{yz}^{EM} = -\frac{j k_x}{T_e k_1} \begin{cases} 
\sin k_1 z' N_e(z), z' \leq z, \\
\sin k_1 z N_e(z'), z \leq z', 
\end{cases} \]

\[ Q_{zz}^{EM} = 0, \]

and

\[ N_e(z) = k_1 \cos k_1 (d - z) + j k_2 \sin k_1 (d - z), \]

\[ N_m(z) = \varepsilon_r k_2 \sin k_1 (d - z) - j k_1 \cos k_1 (d - z), \]

\[ T_m = \varepsilon_r k_2 \cos k_1 d + j k_1 \sin k_1 d, \]

\[ T_e = k_1 \cos k_1 d + j k_2 \sin k_1 d, \]

\[ k_1^2 = \varepsilon_r k_0^2 - k_x^2 - k_y^2, \]

\[ k_2^2 = k_0^2 - k_x^2 - k_y^2, \quad \text{Im}(k_2) < 0. \]

The scattered electric field due to an electric current distribution in the substrate is given by

\[
\mathcal{E}^{scat}(x, y, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{E}(k_x, k_y, z)e^{-j k_x x}e^{-j k_y y} dk_x dk_y
\]

\[
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int_{L_1} Q_{EE}^{EM} (k_x, k_y, z, z') dz' \right] e^{-j k_x x}e^{-j k_y y} dk_x dk_y, \]

(4.5)
with

\[
Q^{EE}_{xz} (k_x, k_y, z, z') = \begin{pmatrix}
Q^{EE}_{xx} & Q^{EE}_{xy} & Q^{EE}_{xz} \\
Q^{EE}_{yx} & Q^{EE}_{yy} & Q^{EE}_{yz} \\
Q^{EE}_{zx} & Q^{EE}_{zy} & Q^{EE}_{zz}
\end{pmatrix}
\]

for \( z \leq d \) and \( z' \leq d \), \( (4.7) \)

\[
Q^{EE}_{xx} = \frac{\omega \mu_0}{\epsilon_r k_0^2 T_e T_m} \left\{ \begin{array}{l}
\sin k_1 z' \left[ j(k_x^2 - k_0^2 \epsilon_r) N_e(z) T_m - k_x^2 k_1^2 (\epsilon_r - 1) \sin k_1 z \right], \quad z' \leq z, \\
\sin k_1 z \left[ j(k_x^2 - k_0^2 \epsilon_r) N_e(z') T_m - k_x^2 k_1^2 (\epsilon_r - 1) \sin k_1 z' \right], \quad z \leq z',
\end{array} \right.
\]

\[
Q^{EE}_{xy} = Q^{EE}_{yx} = \frac{\omega \mu_0 k_x k_y}{\epsilon_r k_0^2 T_e T_m} \left\{ \begin{array}{l}
\sin k_1 z' \left[ jN_e(z) T_m - k_1^2 (\epsilon_r - 1) \sin k_1 z \right], \quad z' \leq z, \\
\sin k_1 z \left[ jN_e(z') T_m - k_1^2 (\epsilon_r - 1) \sin k_1 z' \right], \quad z \leq z',
\end{array} \right.
\]

\[
Q^{EE}_{yy} = \frac{\omega \mu_0 k_y}{\epsilon_r k_0^2 T_e T_m} \left\{ \begin{array}{l}
\sin k_1 z' \left[ j(k_y^2 - k_0^2 \epsilon_r) N_e(z) T_m - k_y^2 k_1^2 (\epsilon_r - 1) \sin k_1 z \right], \quad z' \leq z, \\
\sin k_1 z \left[ j(k_y^2 - k_0^2 \epsilon_r) N_e(z') T_m - k_y^2 k_1^2 (\epsilon_r - 1) \sin k_1 z' \right], \quad z \leq z',
\end{array} \right.
\]

\[
Q^{EE}_{zz} = \frac{\omega \mu_0 k_x}{\epsilon_r k_0^2 T_e T_m} \left\{ \begin{array}{l}
\cos k_1 z' \left[ \epsilon_r k_2 \cos k_1 (d - z) + jk_1 \sin k_1 (d - z) \right], \quad z' \leq z, \\
\sin k_1 z \ N_m(z'), \quad z \leq z',
\end{array} \right.
\]

\[
Q^{EE}_{xx} = -\frac{\omega \mu_0 k_x}{\epsilon_r k_0^2 T_e T_m} \left\{ \begin{array}{l}
\sin k_1 z' \ N_m(z), \quad z' \leq z, \\
\cos k_1 z \left[ \epsilon_r k_2 \cos k_1 (d - z') + jk_1 \sin k_1 (d - z') \right], \quad z \leq z',
\end{array} \right.
\]

\[
Q^{EE}_{yy} = \frac{\omega \mu_0 k_y}{\epsilon_r k_0^2 T_e T_m} \left\{ \begin{array}{l}
\cos k_1 z' \left[ \epsilon_r k_2 \cos k_1 (d - z) + jk_1 \sin k_1 (d - z) \right], \quad z' \leq z, \\
\sin k_1 z \ N_m(z'), \quad z \leq z',
\end{array} \right.
\]

\[
Q^{EE}_{xx} = -\frac{\omega \mu_0 k_y}{\epsilon_r k_0^2 T_e T_m} \left\{ \begin{array}{l}
\sin k_1 z' \ N_m(z), \quad z' \leq z, \\
\cos k_1 z \left[ \epsilon_r k_2 \cos k_1 (d - z') + jk_1 \sin k_1 (d - z') \right], \quad z \leq z',
\end{array} \right.
\]
4.3. Method of Moments Formulation

4.3.2 Electric Field from an Infinite Array

The next step in analysing an infinite array of microstrip antennas is the formulation of the total electric field $\tilde{E}$ created by an infinite array of elementary dipoles. The discretised form of $\tilde{E}$ can be obtained after using the Poisson sum formula twice:

$$
Q_{z\varepsilon}^{EE} = \frac{2\omega \mu_0}{\varepsilon_r k_0^2} \delta(z - z') - \frac{2\omega \mu_0 (k_m^2 + k_z^2)}{\varepsilon_r k_0^2 k_1 T_m} \left\{ \begin{align*}
\cos k_1 z' N_m(z), & z' \leq z, \\
\cos k_1 z N_m(z'), & z \leq z'.
\end{align*} \right.
$$

(4.8)

4.3.3 Matrix Equation

A linear matrix equation can be derived by applying the method of moments on the boundary conditions on all metallic structures. The total tangential electric field has to be zero on all patches and probes

$$
\tilde{E}_{\text{tan}} + \tilde{E}_{\text{scat}} = 0 \quad \text{on patches and probes,}
$$

(4.9)

where $\tilde{E}_{\text{tan}}$ is the tangential component of the incident field which is excited by the magnetic current distribution at the coaxial aperture,

$$
\tilde{E}_{\text{inc}} = \iiint_{V_{\text{fri}}} \tilde{E} \cdot \mathbf{n}_{\text{fri}} dV_{\text{fri}}.
$$

(4.10)

The term $\tilde{E}_{\text{scat}}$ in equation (4.9) represents the scattered field that results from the induced currents on the patch $\mathcal{J}_p$ and probe $\mathcal{J}_p'$

$$
\tilde{E}_{\text{scat}} = \iint_{V'} \tilde{E} : \mathcal{J} dV',
$$

(4.11)

with

$$
\mathcal{J} = \mathcal{J}_p + \mathcal{J}_p'.
$$

(4.12)

The boundary condition in (4.9) must hold for all other patches and probes. Because we deal with an infinite array of identical microstrip antennas, our analysis can be restricted to a unit cell. The next step is the expansion of the unknown current distribution on the patch and the coaxial probe. The unknown patch current $\mathcal{J}_p$ will be expanded into a set of entire-domain sinusoidal basis functions whereas a set of subdomain rooftop basis functions will be used to
CHAPTER 4. INFINITE ARRAY OF EMC MICROSTRIP ANTENNAS

describe the unknown probe current \( \vec{J}^f \). These basis functions will be discussed in the next sections. We shall use a Galerkin solution, i.e. test- and expansion functions are identical. Using the strategy of section 2.5, a set of the moment method linear equations can be derived which is in matrix notation:

\[
\begin{bmatrix}
    Z
\end{bmatrix}
\begin{bmatrix}
    I
\end{bmatrix} = \begin{bmatrix}
    V^t
\end{bmatrix},
\]

(4.13)

The vector \( [I] \) contains the unknown mode coefficients of the basis functions. The only difference with section 2.5 is that \( [I] \) not only has \( z \)-directed basis function on the probe, but also \( x \)- and \( y \)-directed basis functions on the patch. The matrix \( [Z] \), the test vector \( [V] \) and the current vector \( [I] \) can therefore be written in the following form

\[
\begin{bmatrix}
    Z
\end{bmatrix} = \begin{bmatrix}
    [Z^{ff}] & [Z^{fp}]
    
    [Z^{pf}] & [Z^{pp}]
\end{bmatrix},
\]

(4.14)

\[
\begin{bmatrix}
    V^t
\end{bmatrix} = \begin{bmatrix}
    [V^{tf}]
    
    [V^{tp}]
\end{bmatrix},
\]

(4.15)

\[
\begin{bmatrix}
    I
\end{bmatrix} = \begin{bmatrix}
    [I^f]
    
    [I^p]
\end{bmatrix},
\]

(4.16)

with

\[
Z_{ij}^{\alpha\beta} = -\iiint_V \vec{J}^\alpha_i \cdot \vec{E}^\beta_j \, dV
\]

(4.17)

\[
V_i^{\alpha} = \iiint_V \vec{F}_i^\alpha \cdot \vec{E}_{\text{full}} \, dV
\]

(4.18)

In the above expressions the superscript \( f \) represents the modes on the coaxial feed probe and \( p \) the modes on the patch. If there are \( N_z \) basis functions consisting on the feeding probe, \( N_x \) \( x \)-directed and \( N_y \) \( y \)-directed basis functions on the patch, then \( [Z^{ff}] \) is a \( N_z \times N_z \) matrix, \( [Z^{fp}] \) a \( N_z \times (N_x + N_y) \) matrix, \( [Z^{pf}] \) a \( (N_x + N_y) \times N_z \) matrix and \( [Z^{pp}] \) a \( (N_x + N_y) \times (N_x + N_y) \) matrix. Notice that \( [Z^{pf}] \neq [Z^{fp}]^T \) due to the infinite extension of the array. Solving equation (4.13), yields the currents on the patch and probe. The input impedance can then be determined by using formula (2.90) or its approximation (2.91).
4.3. Method of Moments Formulation

4.3.4 Basis Function on the Coaxial Probe

Piecewise-linear subdomain basis functions (also called rooftop functions) will be used to describe the induced current on the probe. The i-th basis function on the feeding probe located at \((x_s, y_s)\) has the form

\[
\mathcal{J}_i^f(x, y, z) = \delta_z \frac{1}{2\pi r_a} \delta \left( \sqrt{(x - x_s)^2 + (y - y_s)^2} - r_a \right) g_i(z), \text{ for } i = 1, 2, \ldots, N_z, (4.19)
\]

with \(g_i(z)\) is given by equation (2.50). The Fourier transform of this basis function is

\[
\tilde{\mathcal{J}}_i^f(k_x, k_y, z) = \delta_z J_0(k_0 r_a) g_i(z) e^{jk_x x_s} e^{jk_y y_s},
\]

with \(k_0^2 \beta^2 = k_x^2 + k_y^2\), (4.20)

4.3.5 Basis Function on the Rectangular Patch

The unknown current distribution on the patch \(\mathcal{J}^p\) is expanded into a set of entire-domain basis functions. Generally, as the number of expansion functions increases, the solution should converge and accuracy should increase. In contrast to the subdomain basis functions that are always suited to analyse an arbitrarily microstrip geometry, the entire-domain basis functions are only capable to describe the unknown currents on a patch with a simple geometry, for instance rectangular or circular microstrip antennas [5, 7]. In addition, for the case of a rectangular microstrip antenna sinusoidal basis functions are often employed, see figure 4.3. The basis functions on the patch are chosen in correspondence with solutions obtained from the cavity model. It is assumed that \(x\)-directed basis functions are \(y\)-independent and that \(y\)-directed basis functions are \(x\)-independent. Abusing somewhat the notation, one can write the induced currents on the patch surface located at \(z = z_p\)

\[
\mathcal{J}^p_i(x, y, z) = \mathcal{J}^p_i(x, y) \delta(z - z_p), \quad (4.21)
\]

Figure 4.3: Entire domain sinusoidal basis functions.
with
\[ J_i(x, y) = J_{ix}(x) + J_{iy}(y). \] (4.22)

The \( i \)-th sinusoidal basis function has the following form:

- **x-directed basis function**
  \[ J_{ix}(x) = \hat{\epsilon}_x \frac{1}{W_x} \sin \frac{i\pi}{W_x} \left( x + \frac{W_x}{2} \right), \text{ for } |x| \leq W_x/2, \ |y| \leq W_y/2, \] (4.23)

- **y-directed basis function**
  \[ J_{iy}(y) = \hat{\epsilon}_y \frac{1}{W_y} \sin \frac{i\pi}{W_y} \left( y + \frac{W_y}{2} \right), \text{ for } |x| \leq W_x/2, \ |y| \leq W_y/2. \] (4.24)

The Fourier transforms of these basis functions are given by

\[ J_{ix}^p = \hat{\epsilon}_x F_s(i, k_x, W_x) F_c(k_y, W_y), \]
\[ J_{iy}^p = \hat{\epsilon}_y F_s(i, k_y, W_y) F_c(k_x, W_x), \] (4.25)

with

\[ F_s(i, k_x, W_x) = \begin{cases} 
    \frac{2i\pi W_x \cos \frac{k_x W_x}{2}}{(i\pi)^2 - (k_x W_x)^2} & \text{for } i \text{ odd}, \\
    \frac{-2i\pi W_x \sin \frac{k_x W_x}{2}}{(i\pi)^2 - (k_x W_x)^2} & \text{for } i \text{ even},
\end{cases} \] (4.26)

\[ F_c(k_y, W_y) = \frac{2 \sin \frac{k_y W_y}{2}}{k_y W_y}. \] (4.27)

### 4.4 Calculation of the Elements of the Matrix \([Z]\)

The general form of the matrix \([Z]\) obtained from the method of moments is given by (4.14). The elements of the submatrix \([Z^f]\) and \([Z^{pf}]\) have already been given in section 2.7 and in [3], respectively. Only the submatrices \([Z^p]\) and \([Z^{fp}]\) will be derived in this section.

#### 4.4.1 \([Z^p]: \text{patch modes } \leftrightarrow \text{feed modes}\)

According to (4.8) and (4.17), an element of the submatrix \([Z^p]\) can be written as

\[ Z_{ij}^p = -\iiint_V \bar{J}_i^p \cdot \left[ \iiint_{V'} \tilde{\epsilon} \cdot \tilde{J}_j \ dx' dy' dz' \right] dx dy dz \]
\[ = -\iiint_V J_i^p(x, y, z) \cdot \left[ \iiint_{V'} \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \tilde{Q}^{EE}(k_x, k_y, z', z') \cdot \tilde{J}_j(x', y', z') \right] dx dy dz \]
4.4. CALCULATION OF THE ELEMENTS OF THE MATRIX \([Z]\]

\[
e^{jk_x(x' - x)}e^{jk_y(y' - y)}\int dx' dy' dz' \\
=- \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_\gamma \bar{J}_m(x, y, z)e^{-jk_x x}e^{-jk_y y}.
\]

\[
\left[ \int_\gamma \bar{Q}_{EE}^{\gamma}(k_x, k_y, z, z') \cdot \bar{J}_m^*(x, y, z') e^{jk_x x'}e^{jk_y y'} dx'y' dz' \right] dx dy dz
\]

\[
= - \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \bar{J}_m^*(x, k_y, z_p) \cdot \int_{z'} \bar{Q}_{EE}^{\gamma}(k_x, k_y, z_p, z') \cdot \bar{J}_m^*(k_x, k_y, z') dz',
\] (4.28)

where \(\bar{J}_m^*\) is the complex conjugate form of (4.25). Using (4.20) in the above expression gives

\[
Z_{ij}^{pf} = - \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \bar{J}_m^*(k_x, k_y, z_p) \cdot \bar{I}_j(z') J_0(k_0 \beta r_a) e^{j k_x x} e^{j k_y y},
\] (4.29)

with

\[
\bar{I}_j(z') = \int_{z'} [\hat{e}_x Q_{zz}^{\gamma}(z' < z_p) \hat{e}_z + \hat{e}_y Q_{yz}^{\gamma}(z' < z_p) \hat{e}_z] \cdot g_j(z') \hat{e}_z dz'.
\] (4.30)

According to (4.7) the \(Q\)-functions in the above equation can be written as

\[
Q_{zz}^{\gamma}(z' < z_p) = \frac{\omega \mu_0 k_x \cos k_1 z'}{\varepsilon_r k_0^2 T_m} \left[ \varepsilon_r k_2 \cos k_1 (d - z_p) + j k_1 \sin k_1 (d - z_p) \right],
\] (4.31)

\[
Q_{yz}^{\gamma}(z' < z_p) = \frac{\omega \mu_0 k_y \cos k_1 z'}{\varepsilon_r k_0^2 T_m} \left[ \varepsilon_r k_2 \cos k_1 (d - z_p) + j k_1 \sin k_1 (d - z_p) \right].
\]

First we will find an analytical expression for an \(x\)-directed patch basis function. Therefore the \(x\)-component of \(\bar{I}_j\) will be considered. For a \(y\)-directed patch basis function a similar way can be followed. Inserting \(Q_{zz}^{\gamma}\) of (4.31) in (4.30) yields

\[
I_j^x = \frac{\omega \mu_0 k_x}{\varepsilon_r k_0^2 T_m} \left[ \varepsilon_r k_2 \cos k_1 (d - z_p) + j k_1 \sin k_1 (d - z_p) \right] \int_{z'} \cos k_1 z' g_j(z') dz'.
\] (4.32)

The \(z\)-integration above can be performed analytically for rooftop basis functions. After performing the \(z\)-integration and substituting the result in equation (4.29), one can write an expression for an \(x\)-directed patch basis function in a somewhat more compact form as

\[
Z_{ij}^{pf} = \frac{\omega \mu_0 k_x}{\varepsilon_r k_0^2 T_m} \left[ \varepsilon_r k_2 \cos k_1 (d - z_p) + j k_1 \sin k_1 (d - z_p) \right] \times
\]

\[
J_0(k_0 \beta r_a) e^{j k_x x} e^{j k_y y},
\]

\[
\left\{ \begin{array}{ll}
1 - \cos k_1 \frac{h}{2} & \text{for } j = 1, \\
2 \cos k_1 z_j - \cos k_1 z_{j-1} - \cos k_1 z_{j+1} & \text{for } j \geq 2.
\end{array} \right.
\]

(4.33)

Since the probes never touch the patches in the EMC structure, the asymptotic value of \(Z_{ij}^{pf}\) for large \(\beta\) is equal to zero.
4.4.2 \([Z_{ij}^p]\): feed modes \(\rightarrow\) patch modes

Due to the infinite extension of the microstrip array, the reaction concept cannot be applied. The consequence is that \(Z_{ij}^p \neq Z_{ji}^p\). This is in contrast to the finite-array theory [2, 11]. For an arbitrarily chosen basis function \(\mathcal{J}_i^f\) on the feeding probe and \(\mathcal{J}_j^p\) on the patch, an element of the submatrix \([Z_{ij}^p]\) has been given previously by equation (4.17). Substituting equation (4.8) in (4.17) gives

\[
Z_{ij}^p = -\int \int \int_{V} \mathcal{J}_i^f \cdot \left[ \int \int \int_{V'} \tilde{\mathcal{E}} \cdot \mathcal{J}_j^p \, dx' \, dy' \, dz' \right] \, dx \, dy \, dz
\]

\[
= -\frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{z} J_{i}^{*}(k_x, k_y, z) \cdot Q_{zz}^{EE}(k_x, k_y, z, z_p) \, dz \cdot \mathcal{J}_{j}^{p}(k_x, k_y, z_p), \tag{4.34}
\]

where \(J_{i}^{*}\) is the complex conjugate of the \(i\)-th probe current and the \(Q\)-functions are given by

\[
Q_{zz}^{EE}(z < z_p) = -\frac{\omega \mu_0 k_x \cos k_1 z}{\varepsilon_r k_0^2 T_m} \left[ \varepsilon_r k_2 \cos k_1 (d - z_p) + j k_1 \sin k_1 (d - z_p) \right],
\]

\[
Q_{zz}^{EE}(z < z_p) = -\frac{\omega \mu_0 k_y \cos k_1 z}{\varepsilon_r k_0^2 T_m} \left[ \varepsilon_r k_2 \cos k_1 (d - z_p) + j k_1 \sin k_1 (d - z_p) \right].
\tag{4.35}

Applying the same strategy as used in the previous section, the final expression of \(Z_{ij}^p\) in (4.34) can be found. Only the result for the \(j\)-th \(x\)-directed patch basis function will be given.

\[
Z_{ijx}^p = -\frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_{jx}^{p} \cdot \varepsilon_r k_0^2 T_m \left[ \varepsilon_r k_2 \cos k_1 (d - z_p) + j k_1 \sin k_1 (d - z_p) \right] \times
\]

\[
J_0(k_0 \beta r_a) e^{-j k_x x} e^{-j k_y y} \begin{cases} 1 - \cos k_1 z_i^{1/2} & \text{for } i = 1, \\ 2 \cos k_1 z_i - \cos k_1 z_{i-1} - \cos k_1 z_{i+1} & \text{for } i \geq 2. \end{cases}
\tag{4.36}
\]

In the case of \(y\)-directed patch basis functions a similar expression can be derived. Because the probes are not physically connected to the patches, \(Z_{ij}^p\) can be set to zero for large \(\beta\)-values.

4.5 Calculation of the Elements of the Excitation Vector \([V_{i}^{t}]\)

The excitation vector \([V_{i}^{t}]\) can be divided in two subvectors according to expression (4.15). In this section we will take a closer look to the elements of the subvector \([V_{ij}^{tp}]\). The derivation of each element of the subvector \([V_{ij}^{tp}]\) has already been carried out in section 2.8. According to equation (4.18) an element of the subvector \([V_{ij}^{tp}]\) can be written in combination with the discretised form of the electric field defined by equation (4.8):

\[
V_{ij}^{tp} = \int \int \int_{V} \mathcal{J}_{i}^{p} \cdot \left[ \int \int \int_{V_{\text{full}}} \tilde{\mathcal{E}} \cdot \mathcal{N}_{\text{full}} \, dV_{\text{full}} \right] \, dV
\]
4.5. **CALCULATION OF THE ELEMENTS OF THE EXCITATION VECTOR** \([V^T]\)

\[
\begin{align*}
\mathcal{N}_\text{frill}(x', y', z') \, dx' \, dy' \, dz' \, dx \, dy \, dz &= \\
\frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{J}_i(x, y, z) \cdot \tilde{Q}_\text{frill}^E(k_x, k_y, z, z') e^{jk_x(x'-x)} e^{jk_y(y'-y)}. \\
\mathcal{N}_\text{frill}(x', y', z') \, dx' \, dy' \, dz' \, dx \, dy \, dz &= \\
\frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{J}_i(x, y, z) \cdot \tilde{Q}_\text{frill}^E(k_x, k_y, z_p, 0) \cdot \tilde{M}_\text{frill}(k_x, k_y, 0),
\end{align*}
\]

(4.37)

where the dyadic \(Q\)-functions and the spectral magnetic current \(\tilde{M}_\text{frill}\) have been given previously in equation (4.4) and (4.2), respectively:

\[
\begin{align*}
Q_{xx}^E(z = z_p, z' = 0) &= -Q_{yy}^E(z = z_p, z' = 0) = \frac{-jk_x k_y (\varepsilon_r - 1)}{T_m T_e} \sin k_1 z_p, \\
Q_{xy}^E(z = z_p, z' = 0) &= \frac{1}{T_m T_e} \left[ jk_x^2 (\varepsilon_r - 1) \sin k_1 z_p + T_e N'_m(z_p) \right], \\
Q_{yx}^E(z = z_p, z' = 0) &= -\frac{1}{T_m T_e} \left[ jk_y^2 (\varepsilon_r - 1) \sin k_1 z_p + T_e N'_m(z_p) \right],
\end{align*}
\]

(4.38)

with

\[
N'_m(z_p) = \varepsilon_r k_2 \cos k_1 (d - z_p) + j k_1 \sin k_1 (d - z_p).
\]

(4.39)

Substitution of equations (4.2), (4.38) and (4.39) in (4.37) and application of some elementary algebra leads to an expression for the \(i\)-th \(x\)-directed patch basis function

\[
V_{ix}^p = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{J}_i^* \left[ Q_{xx}^E M_{\text{frill}}^x + Q_{xy}^E M_{\text{frill}}^y \right],
\]

(4.40)

with

\[
\begin{align*}
Q_{xx}^E M_{\text{frill}}^x + Q_{xy}^E M_{\text{frill}}^y &= \\
-\frac{N'_m(z_p)}{T_m} \frac{2\pi j k_x U}{k_1^2 \beta_1 \ln \frac{\tau_k}{\tau_a}} e^{jk_x x} e^{jk_y y} \left[ J_0(k_0 \beta_{1b}) - J_0(k_0 \beta_{1a}) \right].
\end{align*}
\]

(4.41)

For \(y\)-directed patch basis functions a similar expression can be found, i.e.

\[
V_{iy}^p = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{J}_i^* \left[ Q_{yx}^E M_{\text{frill}}^x + Q_{yy}^E M_{\text{frill}}^y \right],
\]

(4.42)

with

\[
\begin{align*}
Q_{yx}^E M_{\text{frill}}^x + Q_{yy}^E M_{\text{frill}}^y &= \\
-\frac{N'_m(z_p)}{T_m} \frac{2\pi j k_y U}{k_1^2 \beta_1 \ln \frac{\tau_k}{\tau_a}} e^{jk_x x} e^{jk_y y} \left[ J_0(k_0 \beta_{1b}) - J_0(k_0 \beta_{1a}) \right].
\end{align*}
\]

(4.43)

Because the patches never touch the ground plane, the asymptotic values of \(V^p\) can be made zero.
Chapter 5

Properties of EMC Microstrip Arrays

5.1 Introduction

A software package has been developed to analyse the array properties of the electromagnetic coupled microstrip antennas such as input impedance and reflection coefficient. Initially, we look at the convergence problem raised from the infinite summation in formulae derived in the previous chapter.

There are two separate convergence issues which need to be addressed in the present solution. First, the solution to the problem of an infinite array is formulated in terms of the $[Z]$ matrix and the $[V^t]$ vector elements, which are expressed as double infinite summations. Each term in the infinite summation corresponds to a Floquet mode of the infinite periodic structure. From earlier studies [3, 6], it was suggested that results with excellent accuracy can be obtained by using about 121 Floquet modes ($-60 \leq m, n \leq 60$) in each series.

Second, an appropriate choice of the expansion and test modes of the form of equations (4.20) and (4.25). Through the numerical convergence checks performed in section 3.2.2, it was found that five rooftop basis functions are sufficient to give stable solutions. From [3] it can be concluded that the modes $1, 3, 5, 7$ $x$-directed currents and the modes $1, 2$ $y$-directed currents yield good results for linearly polarized patches. More expansion modes do not change the results significantly. Hence, all of the rigorous probe feed model calculations presented here use at least eleven expansion modes.

As was found in [7], the input impedance becomes increasingly inductive as the substrate becomes electrically thicker. In some cases a resonance with a real-valued $Z_{in}$ cannot be obtained. By placing the feed probe at the edge of the patches, a real-valued input impedance can generally be obtained. When the input impedance is known, the reflection coefficient can be calculated using equation (2.92). Additionally, the reflection coefficients plotted in the figures of this chapter are referenced to a $50\Omega$ system.

The effect of placing a rectangular patch above the ground plane and probe on the scan
performance can be demonstrated by analysing the following antenna. The width and length of each patch are 16.0 mm. The patches are separated at distance \( a = b = 60.6 \) mm. The thickness \( d \) of the substrate and the height \( z_p \) of the patch are equal to 10.0 mm. Each probe, with length \( z_f = 9.8 \) mm, is located at \( x_s = 7 \) mm, \( y_s = 0 \). The dielectric constant of the substrate is 2.50 and the loss tangent is 0.0005. The inner and outer radius of the coaxial cable are respectively 0.565 mm and 1.853 mm. Again the analysis is performed using the waveguide simulation in the D-plane. It means that the scan angle is varied with frequencies according to the relation \( \sin \theta = \frac{\lambda_0}{\alpha s^2} \). Figure 5.1 shows the input impedance and reflection coefficient of this array with \( \phi = 45^\circ \). The characteristic impedance for the Smith chart is 50Ω. The effect of placing patches above probes can be seen from this figure. The reflection coefficient of the EMC array is lower than that of the array consisting of only the probes. The presence of the patches causes a resonant structure in the substrate that leads to the increase of the real part of the input impedance.

5.2 Bandwidth of EMC Microstrip Arrays

The bandwidth of a microstrip array can be calculated if the reflection coefficient is known. The reflection coefficient is calculated via equation (2.92). The bandwidth of an array is defined as the frequency band for which the reflection coefficient is less than \( \frac{1}{3} \). These values
5.2. BANDWIDTH OF EMC MICROSTRIP ARRAYS

Figure 5.2: Input impedance (a) and reflection coefficient (b) at broadside.

correspond with a voltage standing-wave ratio (VSWR) of less than 2. The bandwidth is often expressed as a fraction of the center frequency. Figure 5.2 gives the reflection coefficient of an array of EMC microstrip antennas with the following parameters: \( \varepsilon_r = 2.33, \tan \delta = 0.001, d = z_p = 6.61 \text{ mm}, z_f = 6.36 \text{ mm}, W_x = W_y = 11.5 \text{ mm}, r_a = 0.635 \text{ mm}, r_b = 2.1 \text{ mm}, x_s = W_x/2, y_s = 0 \) and \( a = b = 32 \text{ mm} \). The input impedance is calculated in a frequency range of 5.5 - 7.7 GHz, which implies that \( 0.185 \leq d/\lambda \leq 0.259 \). The Smith chart shows that the introduction of a loop leads to an improvement of bandwidth. The bandwidth of this array at broadside is approximately 25%. Table 5.1 shows the bandwidth as the probe length \( z_f \) varies. As can be observed, increasing the distance between the patches and probes causes a declining bandwidth.

<table>
<thead>
<tr>
<th>( z_f ) (mm)</th>
<th>6.36</th>
<th>6.11</th>
<th>5.86</th>
<th>5.61</th>
<th>5.36</th>
<th>5.11</th>
<th>4.86</th>
<th>4.61</th>
<th>4.36</th>
</tr>
</thead>
<tbody>
<tr>
<td>BW (%)</td>
<td>25.0</td>
<td>22.1</td>
<td>20.4</td>
<td>18.8</td>
<td>17.3</td>
<td>15.7</td>
<td>14.2</td>
<td>9.9</td>
<td>2.8</td>
</tr>
</tbody>
</table>

Table 5.1: Bandwidth at broadside scan for various \( z_f \).

In the above calculation we have used an element spacing that is greater than \( \lambda_0/2 \). This choice of the element spacing is made to diminish the mutual coupling between the array elements. This coupling decreases with increasing grid spacing. Another possible solution to reduce the mutual coupling is to choose an appropriate spacing so that the surface wave
Figure 5.3: Input impedance (a) and reflection coefficient (b) at broadside. The frequency is increasing clockwise in the Smith chart.

coming from one element quenches that from another element. This quenching is a result of a phase difference between the couplings. For that purpose, the configuration of figure 5.2 is examined at a larger frequency range for several element spacings. The calculated input impedance and reflection coefficients at broadside scan are given in figure 5.3. The first element spacing is less than $\lambda_0/2$ over the frequency range. Its reflection coefficient is extremely high. It seems that the coupling coefficients are accumulated over the entire frequencies. The maximum bandwidth for the other element spacings are 10.5% and 26.5% respectively.

Next, the scan performance of an array of microstrip antennas will be investigated. To this aim, the antenna of figure 5.3 with $a = b = 28.5$ mm is used as an example. The frequency is 7 GHz. The reflection coefficient magnitude of this antenna for three principal planes is shown in figure 5.4. We define the scan range of an antenna as the scan angles for which the reflection coefficient is less than $1$. The scan range is approximately $17^\circ$, $35^\circ$ and $19^\circ$ for respectively $E$-, $D$- and $H$-scan plane. From this figure it can be seen that the calculated blind spot of the $E$- and $H$-plane occur almost at the same angle.

Finally, a substrate with a higher dielectric constant is investigated. High dielectric substrates are often used in order to achieve a high gain radiation pattern of a microstrip array or to enlarge the scan range. As an example of a microstrip array with a relatively high
permittivity, an antenna with the following configuration is examined:

- patch location $z_p = 4.1$ mm,
- substrate thickness $d = 4.1$ mm,
- probe length $z_f = 3.9$ mm,
- permittivity $\varepsilon_r = 6.15$ and $\tan \delta = 0.0005$,
- patch dimension $W_x = W_y = 7.25$ mm,
- dimension of coaxial cable $r_a = 0.635$ mm, $r_b = 2.1$ mm,
- excitation point $x_s = 3.5$ mm, $y_s = 0$,
- grid spacing $a = 22$ mm, $b = 26$ mm.

Note that the above configuration is obtained by multiplying the parameter set of figure 5.2 with $0.616 (= \sqrt{2.33/6.15})$ and by changing the grid spacing so that the optimum bandwidth is achieved. The input impedance at broadside scan for various frequencies is given in figure 5.5. The reflection coefficient plot shows a bandwidth of about 19.5% which is clearly smaller than the bandwidth of figure 5.2. The reflection coefficient magnitudes versus scan angle for three principal planes are displayed in figure 5.6. The used frequency is 6.5 GHz. The usable scan range is 30°, 52° and 27° respectively for $E$-, $D$- and $H$- scan plane. These values of the scan range are greater than the values of figure 5.4.

Figure 5.4: Active reflection coefficient magnitudes: $d = z_p = 0.154\lambda_0 (= 0.236\lambda)$, $z_f = 0.148\lambda_0 (= 0.226\lambda)$, $W_x = W_y = 0.268\lambda_0$ and $a = b = 0.665\lambda_0$
In [1, 6] the influence of the tolerance of thin substrate materials has been examined. It is shown that the bandwidth properties are strongly reduced if the permittivity is slightly varied. Figure 5.7 shows the input impedance \( Z_{in} = R_{in} + jX_{in} \) of the antenna of figure 5.5 for several values of \( \varepsilon_r \). It can be seen that the input impedance, and therefore the available
bandwidth are almost unchanged. Also a slight shift in the resonant frequency is observed in the direction of lower frequencies. For off-broadside scan the calculated input impedance and reflection coefficient against frequency are shown in figure 5.8 and 5.9. The calculations are made for three scan angles, i.e. $\theta = 20^\circ$, $\theta = 40^\circ$ and $\theta = 60^\circ$. The array properties deteriorate rapidly as the main beam is steered from the broadside.
Figure 5.7: Input resistance $R_{in}$ (a) and input reactance $X_{in}$ (b) at broadside.
Figure 5.8: Input impedance and reflection coefficient for $\phi = 0^\circ$,
(a) Re($Z_{in}$), (b) Im($Z_{in}$), (c) $|R|$
Figure 5.9: Input impedance and reflection coefficient for $\phi = 90^\circ$,
(a) $\text{Re}(Z_{in})$, (b) $\text{Im}(Z_{in})$, (c) $|R|$
Chapter 6

Conclusions and Recommendations

In this report the use of electrically thick substrates in infinite arrays of printed antennas has been investigated with regard to the scan blindness effect, impedance mismatch and grating lobes.

The analysis is performed with the use of a rigorous spectral domain-moment method and exact Green's functions. In order to model the coaxial aperture in the ground plane that excites the patches and probes, a frill of magnetic current is used as a source. The induced currents on the patches are modelled with entire-domain sinusoidal functions and the currents on the probes with subdomain rooftop functions. Results are presented for several specific examples of arrays of vertically printed monopoles and electromagnetically coupled probe-fed rectangular microstrip patches. However, the theory described in this report can easily be extended to the case where probes and patches are physically connected by introducing an attachment mode at the probe-patch transition.

Initially, an infinite array of vertically printed monopoles is investigated. The theoretical results are verified with measured data from the waveguide simulator presented in [8]. The agreement is reasonably good. Five rooftops and 161 Floquet modes are sufficient to produce stable results. The influence of the substrate permittivity on the scan performance is seen to produce a serious input impedance mismatch by a 50Ω feed network and to cause a scan blindness phenomenon. It is shown that the scan blindness is accurately predicted using a modified grating lobes diagram. Comparison between the finite array results as presented in [11] and our infinite array approach is made. We have found that a significant difference is observed, even for large arrays except for the case of an air-filled substrate.

In the analysis of infinite arrays of the electromagnetically coupled (EMC) microstrip patches, we have found that electrically thick substrates can significantly improve the available bandwidth (≈ 27%), but the price paid for this improvement is the reduction of the scanning range. If the dielectric substrate becomes thicker, the surface wave effect becomes important and therefore the grid spacing must be taken larger than λ₀/2 in order to obtain a good impedance matching, thus low reflection coefficients at the input port of antennas.

Finally, the performance of an infinite array with higher permittivities is investigated. It is found that the high dielectric constants have a considerable effect on the impedance
matching at broadside. Generally, a higher permittivity of dielectric slab results in smaller bandwidth at broadside but with a larger scan range. For off broadside scan, the bandwidth declines rapidly when increasing the scan angle towards endfire ($\theta = 90^\circ$).

As mentioned earlier in this report, we have limited our analysis to infinite arrays consisting of identical elements excited by a uniform amplitude of current sources. The in-phase coupling of the surface waves is seen as a primary cause of the degradation of the array performance, especially for small grid spacings. Subarraying (figure 6.1a) and amplitude tapering of current sources could suppress the mutual couplings caused by the surface waves.

In the calculations of the array properties, we have used substrates with low dielectric losses. The influence of a relatively high dielectric loss in the substrate layer on the array performance has not yet been examined.

Stacked antennas (figure 6.1b) with a two-layer substrate have received much attention in the recent literature, because of their broadband properties compared to single patch antennas which have been treated in this work. A next step in the research could be made by the extension of the present solution to the problem containing stacked antennas in an infinite array environment.

![Figure 6.1: Recommendation for a further research:](image-url)

(a) Subarray, (b) Stacked antenna
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Appendix A

Green’s Function of a z-Directed Electric Dipole

In the past, microstrip antennas were analyzed using simple models, such as the cavity model and the transmission line model. These approaches simplify the problem in such a way that there is a limited capacity in handling the characteristics of microstrip antennas such as surface wave effects and different substrate configurations. Therefore a rigorous spectral domain full wave analysis is used in this section to calculate the exact Green’s function. Because of the z-direction of the currents on the monopole’s surface, the Green’s function of a z-directed electric dipole in a dielectric substrate will be determined. The dipole is located in the substrate at \((x', y', z')\). With a time-dependence \(e^{j\omega t}\), Maxwell’s equations in the space domain are of the following form

\[
\begin{align*}
\nabla \times \mathbf{E} &= -j\omega \mu_0 \mathbf{H}, \\
\nabla \times \mathbf{H} &= j\omega \varepsilon \mathbf{E} + \mathbf{J}, \\
\nabla \cdot \mathbf{H} &= 0, \\
\nabla \cdot \mathbf{E} &= \frac{\rho_c}{\varepsilon}.
\end{align*}
\]

(A.1)

where \(\omega\) is the radial frequency, \(\varepsilon\) is the permittivity, \(\mu_0\) the permeability, and \(\rho_c\) the charge density. \(\mathbf{J}\) is the current density. The electric and magnetic fields can be expressed in terms of the electric vector potential \(\mathbf{A}_e\) and scalar potential \(\phi_e\)

\[
\begin{align*}
\mathbf{H} &= \frac{1}{\mu_0} \nabla \times \mathbf{A}_e, \\
\mathbf{E} &= -j\omega \mathbf{A}_e - \nabla \phi_e.
\end{align*}
\]

(A.2)
The scalar potential \( \phi_e \) can be related to the divergence of the vector potential \( \vec{A}_e \) using the Lorentz gauge

\[
\nabla \cdot \vec{A}_e = -j\omega\mu_0\phi_e. \tag{A.3}
\]

Substituting the Lorentz gauge in (A.3) yields

\[
\check{\mathcal{H}} = \frac{1}{\mu_0} \nabla \times \vec{A}_e, \tag{A.4}
\]

\[
\vec{E} = -\frac{j\omega}{k^2} \left[ k^2 \vec{A}_e + \nabla (\nabla \cdot \vec{A}_e) \right],
\]

where \( k = \omega\sqrt{\varepsilon\mu_0} \) is the wave number in the dielectric substrate. Equation (A.4) is evaluated in rectangular components

\[
\check{\mathcal{H}} = \frac{1}{\mu_0} \begin{bmatrix}
\partial_y A_{ez} - \partial_z A_{ey} \\
\partial_z A_{ex} - \partial_x A_{ez} \\
\partial_x A_{ey} - \partial_y A_{ex}
\end{bmatrix}, \tag{A.5}
\]

\[
\vec{E} = -j\omega \begin{bmatrix}
A_{ex} \\
A_{ey} \\
A_{ez}
\end{bmatrix} - \frac{j\omega}{k^2} \begin{bmatrix}
\partial_x A_{ez} + \partial_y A_{ey} + \partial_z A_{ex} \\
\partial_y A_{ez} + \partial_z A_{ey} + \partial_x A_{ex} \\
\partial_z A_{ez} + \partial_x A_{ey} + \partial_y A_{ex}
\end{bmatrix}. \tag{A.6}
\]

Transforming equations (A.5) and (A.6) to the spectral domain, yields

\[
\check{\mathcal{H}} = \frac{1}{\mu_0} \begin{bmatrix}
-jk_y A_{ez} - \partial_z A_{ey} \\
\partial_z A_{ex} + jk_x A_{ez} \\
-jk_x A_{ey} + jk_y A_{ex}
\end{bmatrix}, \tag{A.7}
\]

\[
\vec{E} = -\frac{j\omega}{k^2} \begin{bmatrix}
(k^2 - k_x^2) A_{ex} - k_x k_y A_{ey} - jk_y \partial_z A_{ez} \\
(k^2 - k_y^2) A_{ey} - jk_y k_x A_{ex} - jk_x \partial_z A_{ez} \\
(k^2 + \partial_z^2) A_{ez} - jk_z \partial_x A_{ex} - jk_x \partial_z A_{ey}
\end{bmatrix}. \tag{A.8}
\]

Substituting (A.4) into (A.1), results in the Helmholtz equation for the vector potential \( \vec{A}_e \)

\[
\nabla^2 \vec{A}_e + k^2 \vec{A}_e = -\mu_0 \vec{J}. \tag{A.9}
\]
The vector potential can be expressed in terms of the dyadic Green’s function $\tilde{G}$

$$\mathcal{A}_e(\mathbf{r}) = \int_\infty^{-\infty} \int_\infty^{-\infty} \int_\infty^{-\infty} \tilde{G}(\mathbf{r}, \mathbf{r}') \cdot \mathcal{J}(\mathbf{r}') \, dx' \, dy' \, dz'. \quad (A.10)$$

$\tilde{G}(\mathbf{r}, \mathbf{r}')$ is a $3 \times 3$ matrix with general form

$$\tilde{G}(\mathbf{r}, \mathbf{r}') = \begin{pmatrix} G_{xx} & G_{xy} & G_{xz} \\ G_{yx} & G_{yy} & G_{yz} \\ G_{zx} & G_{zy} & G_{zz} \end{pmatrix}.$$ \quad (A.11)

In (A.11) $G_{xz}$, for instance, is the $x$-component of the Green’s function excited by a $z$-directed dipole located at $\mathbf{r}'$. The subscripts 1 and 2 shall be used to indicate whether the quantity is defined in medium 1 (=substrate) or medium 2 (=free space). The boundary conditions that have to be taken in order to find the vector potential are [13]

$$\begin{align*}
\hat{e}_z \times \hat{e}_1 &= 0 \quad \text{for } z = 0, \\
\hat{e}_z \times \hat{e}_1 &= \hat{e}_z \times \hat{e}_2 \quad \text{for } z = d.
\end{align*} \quad (A.12)$$

However, it is not possible to find a closed-form expression for the vector potential in the space domain. In the spectral domain the above boundary value problem can easily be solved. Using the inverse Fourier transformation, equation (A.10) can be expressed in terms of the spectral dyadic Green’s function $\tilde{G}$

$$\mathcal{A}_e = \int_\infty^{-\infty} \int_\infty^{-\infty} \int_\infty^{-\infty} \tilde{G}(\mathbf{r}, \mathbf{r}') \cdot \mathcal{J}(\mathbf{r}') \, dx' \, dy' \, dz'$$

$$= \int_\infty^{-\infty} \int_\infty^{-\infty} \int_\infty^{-\infty} \left[ \frac{1}{4\pi^2} \int_\infty^{-\infty} \tilde{G}(k_x, k_y, z, \mathbf{r}') e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y \right] \cdot \mathcal{J}(\mathbf{r}') \, dx' \, dy' \, dz'$$

$$= \frac{1}{4\pi^2} \int_\infty^{-\infty} \int_\infty^{-\infty} \int_\infty^{-\infty} \tilde{G}(k_x, k_y, z, \mathbf{r}') \cdot \mathcal{J}(\mathbf{r}') \, dx' \, dy' \, dz' \, e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y, \quad (A.13)$$

where

$$\tilde{G} = \begin{pmatrix} G_{xx} & G_{xy} & G_{xz} \\ G_{yx} & G_{yy} & G_{yz} \\ G_{zx} & G_{zy} & G_{zz} \end{pmatrix}. \quad (A.14)$$
Assume that the current source is a z-directed infinitesimal dipole located at \( \mathbf{r}_0 = (x_0, y_0, z_0) \) in medium 1, thus
\[
\mathbf{J}(\mathbf{r}) = J_z \mathbf{e}_z = \delta(\mathbf{r} - \mathbf{r}_0) \mathbf{e}_z,
\]
then equation (A.13) can be written as
\[
\vec{A}_e = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{G}(k_x, k_y, z, \mathbf{r}) \cdot \delta(\mathbf{r} - \mathbf{r}_0) \mathbf{e}_z \, dz' \, dy' \, dx' \, e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y
\]
\[
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{G}(k_x, k_y, z, z_0) \cdot \mathbf{e}_z \, e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y.
\]

Three components of \( \vec{A}_e \) can be separated from equation (A.16):
\[
\begin{align*}
A_{ex} &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_{ex} \, e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y \quad \text{(A.17)} \\
&= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{xz}(k_x, k_y, z, z_0) \, e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y, \\
A_{ey} &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_{ey} \, e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y \quad \text{(A.18)} \\
&= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{yz}(k_x, k_y, z, z_0) \, e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y, \\
A_{ez} &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_{ez} \, e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y \quad \text{(A.19)} \\
&= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{zz}(k_x, k_y, z, z_0) \, e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y.
\end{align*}
\]

First the boundary value problem will be solved to find the exact expression for the Green’s function. If the Green’s function is found, then the vector potential can easily be calculated using equations (A.17), (A.18) and (A.19). Substitution of (A.4) in (A.1) results in the inhomogeneous Helmholtz equation. The Helmholtz equations in the two various media are
\[
\nabla^2 \vec{A}_{te} + k^2 \vec{A}_{te} = -\mu_0 \mathbf{J}, \quad k^2 = \omega^2 \mu_0 \varepsilon_\varepsilon_0,
\]
\[
\nabla^2 \vec{A}_{ce} + k^2_0 \vec{A}_{ce} = 0, \quad k^2_0 = \omega^2 \mu_0 \varepsilon_0.
\]
Contrary to the derivation of the Green’s function due to an \( x \) or \( y \)-directed dipole in [3] \( \vec{A}_{ex} \) and \( \vec{A}_{ey} \) can be made zero to satisfy all the boundary conditions within the microstrip.
structure with only a $z$-component in $\tilde{\mathbf{A}}$. Thus, $\tilde{\mathbf{A}} = A_{ez} \hat{e}_z$. The combination of equation (A.20) with equation (A.10) yields

$$\nabla^2 \mathcal{G}_{1zz} + k^2 \mathcal{G}_{1zz} = -\mu_0 \delta(x - x') \delta(y - y') \delta(z - z'),$$  \hspace{1cm} (A.21)

$$\nabla^2 \mathcal{G}_{2zz} + k_0^2 \mathcal{G}_{2zz} = 0.$$  

The combination of equation (A.21) and the boundary conditions gives the solution of the Green's function [5]. An easy method for the solutions of (A.21) can be obtained by using the Fourier transformation. Using the Fourier transformation, equation (A.21) is transformed from the space domain to the spectral domain

$$\partial_z^2 \mathcal{G}_{1zz} + k_1^2 \mathcal{G}_{1zz} = -\mu_0 \delta(z - z') e^{jk_z z'} e^{jk_y y'},$$  \hspace{1cm} (A.22)

$$\partial_z^2 \mathcal{G}_{2zz} + k_2^2 \mathcal{G}_{2zz} = 0,$$

where

$$k_1^2 = k^2 - k_x^2 - k_y^2,$$

$$k_2^2 = k_0^2 - k_x^2 - k_y^2 \quad \text{with } \text{Im}(k_2) < 0.$$  \hspace{1cm} (A.23)

The restriction that $\text{Im}(k_2) < 0$ follows from the radiation condition that the fields are outward propagating waves, decaying with distance from the source. The general solutions of the inhomogeneous Helmholtz equations in the spectral domain are of the form

$$G_{1zz} = \begin{cases}  
C_1 e^{jk_1 z} + C_2 e^{-jk_1 z}, & 0 \leq z \leq z', \\
C_1 e^{jk_1 z} + C_2 e^{-jk_1 z} - \frac{\mu_0}{k_1} e^{jk_x x'} e^{jk_y y'} \sin k_1 (z - z'), & z' \leq z \leq d,
\end{cases}$$  \hspace{1cm} (A.24)

$$G_{2zz} = C_3 e^{-jk_2 z}, \quad z \geq d,$$

where $C_i$, $i = 1, 2, 3$ are constants that have to be determined via the boundary conditions (A.12). Using equations (A.7) and (A.8) the boundary conditions in equation (A.12) can now be written in the spectral domain

$$\partial_z G_{1zz} = 0 \quad \text{for } z = 0,$$

$$\partial_z G_{1zz} = \varepsilon_r \partial_z G_{2zz} \quad \text{for } z = d,$$

$$G_{1zz} = G_{2zz} \quad \text{for } z = d.$$  \hspace{1cm} (A.25)

The combination of (A.24) and (A.25) results in
APPENDIX A. GREEN'S FUNCTION OF A Z-DIRECTED ELECTRIC DIPOLE

Medium 1, \(0 \leq z \leq d\)

\[ G_{1zz} = \mu_0 G_3 \, e^{jk_z x'} \, e^{jk_y y'} . \]  
\[ (A.26) \]

Medium 2, \((z \geq d)\)

\[ G_{2zz} = \mu_0 G_4 \, e^{jk_z x'} \, e^{jk_y y'} . \]  
\[ (A.27) \]

with

\[
G_3 = \begin{cases} 
\frac{N'_m \cos k_1 z}{T_m k_1}, & 0 \leq z \leq z', \\
\frac{N'_m \cos k_1 z}{T_m k_1} + \frac{\sin k_1 (z' - z)}{k_1}, & z' \leq z \leq d,
\end{cases} 
\]

\[
G_4 = \left[ \frac{N'_m \cos k_1 d}{T_m k_1} + \frac{\sin k_1 (z' - d)}{k_1} \right] e^{jk_2 (d - z)}, 
\]

\[ (A.28) \]

\[ N'_m = k_1 \cos k_1 d + jk_2 \varepsilon_r \sin k_1 d, \]

\[ T_m = k_2 \varepsilon_r \cos k_1 d + jk_1 \sin k_1 d. \]

The above expression for \(G_3\) can also be written in the following form

\[
G_3 = \begin{cases} 
\frac{\cos k_1 z}{T_m k_1} \, N_m(z'), & 0 \leq z \leq z', \\
\frac{\cos k_1 z'}{T_m k_1} \, N_m(z), & z' \leq z \leq d,
\end{cases} 
\]

\[ (A.29) \]

with

\[
N_m(z) = \varepsilon_r k_2 \sin k_1 (d - z) - jk_1 \cos k_1 (d - z). 
\]

\[ (A.30) \]

The vector potential in medium 1 \(A_{1z}\) and in medium 2 \(A_{2z}\) can now be written by using equations \((A.26)\) and \((A.27)\)

\[
A_{1z} = \frac{\mu_0}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_3(k_x, k_y, z, z')e^{-jk_x(x-x')}e^{-jk_y(y-y')} \, dk_x dk_y, 
\]

\[ (A.31) \]

\[
A_{2z} = \frac{\mu_0}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_4(k_x, k_y, z, z')e^{-jk_x(x-x')}e^{-jk_y(y-y')} \, dk_x dk_y. 
\]

\[ (A.32) \]
Appendix B

Fields from a z-Directed Dipole

In this section the electromagnetic fields distribution in a dielectric substrate will be given using the results from section A. The field distribution outside the substrate can be calculated in a similar way. The electric and magnetic fields in a dielectric substrate can be found via equations (A.7) and (A.8). Each component of these fields can easily be calculated in the spectral domain except for the z-component of E-field.

\[ E_{1z} = -\frac{j\omega \mu_0}{k^2} (-j k_x \partial_z G_3) e^{j k_x x'} e^{j k_y y'} = Q_{xz}^{EE} e^{j k_x x'} e^{j k_y y'}, \]  
\[ (B.1) \]

\[ E_{1y} = -\frac{j\omega \mu_0}{k^2} (-j k_y \partial_z G_3) e^{j k_x x'} e^{j k_y y'} = Q_{yz}^{EE} e^{j k_x x'} e^{j k_y y'}, \]  
\[ (B.2) \]

\[ H_{1z} = -j k_y G_3 e^{j k_x x'} e^{j k_y y'} = Q_{xz}^{HE} e^{j k_x x'} e^{j k_y y'}, \]  
\[ (B.3) \]

\[ H_{1y} = j k_x G_3 e^{j k_x x'} e^{j k_y y'} = Q_{yz}^{HE} e^{j k_x x'} e^{j k_y y'}, \]  
\[ (B.4) \]

\[ H_{1z} = 0. \]  
\[ (B.5) \]

The z-component of E-field has the following form

\[ E_{1z} = -\frac{j\omega }{k^2} \left( k^2 + \partial_z^2 \right) A_{1z} = -\frac{j\omega \mu_0}{k^2} \left( k^2 G_3 + \partial_z^2 G_3 \right) e^{j k_x x'} e^{j k_y y'}. \]  
\[ (B.6) \]
Due to the discontinuity of \( \partial_z G_3 \) for \( z = z' \), the second derivative of \( G_3 \) to \( z \) in (B.6) contains a \( \delta \)-function

\[
\partial_z^2 G_3 = -\delta(z - z') - k_z^2 G_3.
\] (B.7)

Substituting (B.7) in (B.6) yields

\[
E_{1z} = \frac{\mu_0}{k_z^2} \left( (k_z^2 - k_0^2) G_3 - \delta(z - z') \right) e^{jk_z z'} e^{jk_y y'}
\] (B.8)

For a general \( z \)-directed current distribution in the substrate, the electric and magnetic fields in the substrate can be expressed in terms of the Fourier transform of this current distribution

\[
\tilde{E}_1 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}_1 e^{-jk_z x} e^{-jk_y y} \, dk_x dk_y
\]

\[
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int \tilde{Q}^{EE}(k_x, k_y, z, z') \cdot J_z(k_x, k_y, z') \hat{\varepsilon}_z \, dz' \right] e^{-jk_z x} e^{-jk_y y} \, dk_x dk_y,
\] (B.9)

\[
\tilde{H}_1 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{H}_1 e^{-jk_z x} e^{-jk_y y} \, dk_x dk_y
\]

\[
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int \tilde{Q}^{HE}(k_x, k_y, z, z') \cdot J_z(k_x, k_y, z') \hat{\varepsilon}_z \, dz' \right] e^{-jk_z x} e^{-jk_y y} \, dk_x dk_y,
\] (B.10)

where the dyadic functions \( \tilde{Q}^{EE} \) and \( \tilde{Q}^{HE} \) can be represented by a \( 3 \times 3 \) matrix

\[
\tilde{Q}^{EE} = \begin{pmatrix}
0 & 0 & Q_{zz}^{EE} \\
0 & 0 & Q_{yz}^{EE} \\
0 & 0 & Q_{zz}^{EE}
\end{pmatrix}, \quad \tilde{Q}^{HE} = \begin{pmatrix}
0 & 0 & Q_{zz}^{HE} \\
0 & 0 & Q_{yz}^{HE} \\
0 & 0 & 0
\end{pmatrix}.
\] (B.11)

\( J_z \hat{\varepsilon}_z \) is the Fourier transform of the \( z \)-directed current density \( J_z \hat{\varepsilon}_z \). In general, the combination of the spectral dyadic \( Q \)-functions and the spectral current densities gives the solution of the spectral electromagnetic fields.
Appendix C

Software User Guide

In this appendix a description of the developed software is given. Because of the length of the programs, the source listings of these programs are not added to this report. All the programs are written in Microsoft (MS) Fortran-77 Power Station and use some extensions which are not supported by the standard Fortran-77 language. Therefore some modifications in the programs are necessary if one will use other compilers rather than the MS-Fortran compiler. Additionally, the software package is easy to use and can be easily adapted, because all the subroutines are provided with comments which the structure of the programs and subroutines is explained. Variables involved in the calculations are of type double precision or double complex.

Most of the computation time is used to determine the elements of the impedance matrix $[Z]$ and the excitation vector $[V]$. This computation time is dependent on the number of basis functions and the number of Floquet modes. According to the analysis performed in chapter 3 and 5, the number of the basis functions can be limited to 5 whereas 161 Floquet modes in each serie is sufficient to obtain accurate results for the case of a monopole array. As an example, the calculation time on a personal computer 486-33MHz is approximately 2.5 minutes for each frequency.

The main source code file "infarr.for" calculates the input impedance of an infinite array with the input parameters given in an input file called "datainf.ini". This input file must be present in the same directory as the one of the main program.

An example datainf.ini used to calculate the input impedance of an infinite array of EMC microstrip antennas is given hereunder:

```plaintext
&ARRAY
NX   = 5
NY   = 3
NZ   = 5
XMODE = 1, 3, 5, 7, 9, 11, 13, 15, 17, 19
YMODE = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
MMAX  = 80
NMAX  = 80
```

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The input parameters given above have the following meaning:

1. **NX** = number of entire domain basis functions in the x-direction (≤ 10);
2. **NY** = number of entire domain basis functions in the y-direction (≤ 10);
3. **NZ** = number of rooftop basis functions in the z-direction (≤ 10);
4. **XMODE** = i-th x-directed expansion current;
5. **YMODE** = i-th y-directed expansion current;
6. **MMAX** = number of m-Floquet modes (≤ 120);
7. **NMAX** = number of n-Floquet modes (≤ 120);
8. **XS [mm]** = x-coordinate of feed point;
9. **YS [mm]** = y-coordinate of feed point;
10. **RIN [mm]** = inner radius of coaxial cable;
11. **ROUT [mm]** = outer radius of coaxial cable;
12. $\text{EPSR}1$ = real part of permittivity of substrate;
13. $\text{TAND}$ = dielectric loss of substrate, $\tan \delta$;
14. $D$ [mm] = substrate thickness;
15. $ZP$ [mm] = patch location ($\leq D$);
16. $ZF$ [mm] = probe length ($< ZP$);
17. $WX$ [mm] = dimension of patch in the $x$-direction;
18. $WY$ [mm] = dimension of patch in the $y$-direction;
19. $F0$ [GHz] = used or start frequency;
20. $\text{NFOMAX}$ = number of frequencies used in calculation;
21. $\text{DF}0$ [GHz] = frequency interval;
22. $A$ [mm] = element distance in the $x$-direction;
23. $B$ [mm] = element distance in the $y$-direction;
24. $\text{PHI}$ [degrees] = scan plane, i.e. $E$-, $D$- or $H$-plane;
25. $\text{THETA}$ [degrees] = scan angle ($-90 \leq \theta \leq 90$);
26. $\text{ALPHA}$ [degrees] = skew angle of array grid ($0 \leq 90$): $\alpha = 60 \rightarrow$ triangular grid, $\alpha = 90 \rightarrow$ rectangular grid;
27. $\text{FREQ}$ has a boolean value. If $\text{FREQ} = \text{.TRUE.}$ then the calculation will be performed for $\text{NFOMAX}$ number of frequencies. Otherwise the scan angles at a equidistance $\text{DTHETA}$ will be used;
28. $\text{WGSIM}$ has a boolean value and indicates whether a waveguide simulation is used in the calculation;
29. $\text{DTHETA}$ [degrees] = interval of scan angles.

The output file $\text{zin.dat}$ contains the complex number of the input impedances. An input impedance curve can then be given in a Smith chart using $\text{smith.exe}$ and $\text{smith.dat}$. The resulting file $\text{data.dat}$ has a format which matches to the requirement of the PlotUtil software package. By typing $\text{plotutil smith.dat}$ a Smith chart plot of the input impedances will be shown.

The output file $\text{refl.dat}$ and $\text{phase.dat}$ contain respectively the amplitude and the phase of the complex reflection coefficients. It is assumed that the microstrip array is fed by a $50\Omega$-feed network.
Appendix D

List of Symbols

\(a, b\) 
Element spacing in respectively \(x\)- and \(y\)-direction

\(\tilde{A}_e\) 
Electric vector potential

\(A_e\) 
Spectral domain electric vector potential

\(d\) 
Substrate thickness

\(\vec{E}\) 
Dyadic electric field

\(\vec{E}, \vec{E}_f, \vec{E}_{\text{inc}}, \vec{E}_{\text{scat}}, \vec{E}_{\text{tot}}\) 
Electric field vector

\(\vec{E}_{\text{tan}}\) 
Spectral domain electric field

\(\hat{e}_x, \hat{e}_y, \hat{e}_z\) 
Tangential component of electric field

\(\mathcal{F}\) 
Unity vectors

\(F\) 
General function

\(g_i, g_j\) 
General function in spectral domain

\(\vec{G}\) 
Rooftop basis function

\(\vec{G}_{xx}, \vec{G}_{xy}, \ldots, \vec{G}_{zz}\) 
Dyadic Green's function

\(G_{xx}, G_{xy}, \ldots, G_{zz}\) 
Spectral domain dyadic Green's function

\(G_1, G_2, G_3, G_4\) 
Elements of dyadic Green's function

\(\vec{H}\) 
Elements of spectral domain dyadic Green's function

\(H\) 
Green's functions coefficients

\(\vec{H}\) 
Magnetic field vector

\(\vec{h}\) 
Magnetic field vector in spectral domain

\(i, j\) 
Distance between successive rooftop basis functions

\([\mathcal{I}], [\mathcal{I}^1], [\mathcal{I}^2]\) 
i-th or j-th basis function

\(\vec{J}, \vec{J}_f, \vec{J}, \vec{J}_p, \vec{J}_p\) 
(Sub)Vector of unknown mode coefficient

\(\vec{J}_f, \vec{J}, \vec{J}_p, \vec{J}_p\) 
Electric current vector

\(\vec{J}^*\) 
Spectral domain electric current vector

\(J_0\) 
Complex conjugate of \(\vec{J}\)

\(\hat{J}\) 
Bessel function of zero order and first kind

\(\sqrt{-1}\) 
\(\sqrt{-1}\)

\(k_0\) 
Free space wave number

\(k_1, k_2\) 
Wave number in substrate and air, respectively
**APPENDIX D. LIST OF SYMBOLS**

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<th>Description</th>
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<td>Wave numbers</td>
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<tr>
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<td>Location of Floquet modes and patches</td>
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<tr>
<td>$n$</td>
<td>Normal vector perpendicular to conductor surface</td>
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<td>$N_m, N_{m'}, N_e$</td>
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<td>$N_x, N_y, N_z$</td>
<td>Number of basis functions in $x$-, $y$- and $z$-direction</td>
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<td>Number of Floquet modes in each direction</td>
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<td>Dyadic of the electric fields in the spectral domain</td>
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<tr>
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<tr>
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<tr>
<td>$R$</td>
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<tr>
<td>$r_a, r_b$</td>
<td>Inner and outer radius of coaxial cable</td>
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<tr>
<td>$S_f, S_{\text{frill}}, S_p$</td>
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<td>Coefficients in $G_1, \ldots, G_4$</td>
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<tr>
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<td>$V, V', V_{\text{frill}}$</td>
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<td>$W_x, W_y$</td>
<td>Dimension of patch conductor</td>
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<tr>
<td>$x, y, z$</td>
<td>Coordinates of field point</td>
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<td>Input admittance and input impedance</td>
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<tr>
<td>$[Z], [Z^f], [Z^{fp}], [Z^{pp}]$</td>
<td>Reaction (Impedance) matrix</td>
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<tr>
<td>$\alpha$</td>
<td>Skew angle of an array grid relative to the $x$-axis</td>
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<tr>
<td>$\beta$</td>
<td>Propagation constant of Floquet mode</td>
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<tr>
<td>$\varepsilon, \varepsilon', \varepsilon''$</td>
<td>Permittivity</td>
</tr>
<tr>
<td>$\varepsilon_r$</td>
<td>Relative permittivity of the substrate</td>
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<tr>
<td>$\varepsilon_0$</td>
<td>Permittivity of free space ($\approx \frac{10^{-9}}{36\pi} \text{C}^2 \text{N}^{-1} \text{m}^{-2}$)</td>
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<tr>
<td>$\lambda_0$</td>
<td>Free space wavelength</td>
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<tr>
<td>$\lambda$</td>
<td>Electrical wavelength</td>
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<tr>
<td>$\mu_0$</td>
<td>Permeability of free space ($= 4\pi \times 10^{-7} \text{Wb} \text{A}^{-1} \text{m}^{-1}$)</td>
</tr>
<tr>
<td>$\pi$</td>
<td>A constant ($\approx 3.141592$)</td>
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<tr>
<td>$\partial_x, \partial_y, \partial_z$</td>
<td>Derivation to $x, y, z$</td>
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<tr>
<td>$\rho_c$</td>
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<td>$\theta, \phi$</td>
<td>Scan angles</td>
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