MASTER

Assessment of very large area atom interferometers

Meesters, A.C.H.

Award date:
2003

Link to publication
Assessment of Very Large Area Atom Interferometers

A.C.H. Meesters

AQT 03-08
Oktober 2003
Summary

An atom interferometer is the direct equivalent of a standard optical interferometer, utilizing the wave nature of the center-of-mass motion of the atoms. Essentially, it is thus an instrument to measure phase changes in the motional wavefunction of an atom. The primary use of the atom interferometer that is presently being built up at TU/e is in a cavity Quantum Electrodynamics (QED) experiment to directly measure the quantization of the electromagnetic field. In the TU/e atom interferometer the splitting angle and the separation between the two paths the atom can follow will be extremely large, up to 10 mm, as compared to the 100 μm separation in present state of the art atom interferometers. The reflection and splitting of the atom beam will be accomplished by means of Bragg diffraction on a standing light wave. A combination of three such standing light waves results in an atom interferometer in the Mach-Zehnder configuration.

Before building this atom interferometer an assessment is made of the potential problems that can be expected. The investigation is divided into four separate sections: 1) Misalignment sensitivity of Bragg diffraction, 2) Optical Components Misalignment, 3) Electric and Magnetic fields, and 4) Atom Interferometer Motions.

Bragg diffraction is most efficient when the Bragg condition is met for the angle between the standing light wave and the atom beam. A deviation from the Bragg condition, caused by possible misalignments of the standing light waves and the velocity spread of the atoms, will reduce the efficiency of the beam splitting, thereby influencing the performance of the atom interferometer. When fifth order Bragg diffraction is used, the Bragg-angle deviation should not be larger than 20 μrad. This becomes smaller for higher orders. The velocity spread of the atoms also has a significant effect on the atom interferometer performance. When tenth order Bragg diffraction is used, it is shown that with the present quality of the atom beam the visibility of the atom interferometer signal will drop to 0.5. This becomes even worse for higher order Bragg diffraction.

Misalignment of the optical components has significant effects on the atom interferometer’s performance. These components create the three standing light waves which are aligned parallel to each other. It turns out that these standing light waves have to be parallel to each other to within 14 μrad to get a high quality atom interferometer. Also, as long as the misalignment of the laser’s incident angle on the mirror is smaller than 0.3 mrad the overlap of the laser beams remains large enough to create an acceptable standing light wave.

Due to the magnetic dipole of the metastable Helium atoms used in the experiments, the atom interferometer becomes susceptible to magnetic fields, which can therefore influence the measurements with the atom interferometer. It is calculated that the magnetic field gradient must be stable to within 84 μG/m and 1.9 μG/m for respectively the 0.3 m test interferometer and the full 2 m atom interferometer that are being constructed.

It was already shown theoretically and experimentally that atom interferometers are susceptible to motions of the setup. Therefore the effects of motions on the setup are also
investigated, resulting in a set of requirements. These requirements are compared with
the vibrations measured on the present setup, on which the atom interferometer will be
built. From this comparison the conclusion is that the 0.3 m atom interferometer does
not need a strong isolation (noise reduction > 5) from the environment, provided that the
strength of the mechanical resonances of the present setup is reduced by two orders of
magnitude. The environmental isolation of the 2 m atom interferometer has to be at least
6 times stronger than that of the 0.3 m atom interferometer.

The sensitivity for distortions is the same for both atom interferometers and very
large. To meet the requirements the mirror displacement from a straight line must be
< 8.6 pm. Displacements due to stress and/or temperature changes in the system can
easily drift hundreds of nanometers. Therefore a method is developed to compensate for
these distortions. An optical interferometer is added to measure the distortion of the atom
interferometer setup. The signal of the optical interferometer is then used to actively
position one of the mirrors with the use of an electronic feedback system, such that the
distortions effects are compensated as much as possible. The effectiveness together with
the stability and the possible introduction of extra disturbances of this compensation
method are studied. For this study the mirror setup of the atom interferometer with the
complete distortion compensation (optical interferometer and electronic feedback system)
has been built, separate from the present atom beam setup. Measurements on this setup
showed a significant reduction of a low frequency 1/15 Hz distortion with a amplitude of
8 nm. Calculations give a reduction of about 15000 times, which would result in a residual
amplitude of about 0.3 pm.
Contents

1 Introduction 1

2 The Atom Interferometer 5
   2.1 Introduction 5
   2.2 Statistics of the Atom Interferometer 5
   2.3 Bragg diffraction 7
      2.3.1 Bragg diffraction experiment 10
   2.4 He* atoms in atom interferometer 11
   2.5 Configuration of the atom interferometer 12

3 Design Considerations 15
   3.1 Bragg diffraction efficiency 17
      3.1.1 Misalignment of the standing light wave 17
      3.1.2 Atom velocity spread 20
   3.2 Alignment of optical components 22
      3.2.1 Laser-mirror misalignment 22
      3.2.2 Mirror misalignment 23
      3.2.3 Position of the standing light waves on the Z-axis 26
   3.3 Relative mirror positions 27
   3.4 Motions of the atom interferometer 28
      3.4.1 Theory of motions 29
      3.4.2 Oscillating motion 35
      3.4.3 Noise analysis 39
      3.4.4 Preliminary vibration measurements QND-setup 39
   3.5 Magnetic fields 40
      3.5.1 Magnetic field variations 42
      3.5.2 Oscillating magnetic field 43
      3.5.3 Static magnetic field gradient 44
   3.6 Electric Fields 45
   3.7 Conclusions 46

4 Stabilization of the atom interferometer mirrors 47
   4.1 Method of stabilization 47
   4.2 Distortion measurement 48
      4.2.1 Misalignment sensitivity 49
      4.2.2 Laser beam direction sensitivity 50
   4.3 Feedback control 53
      4.3.1 Laplace domain 55
Chapter 1

Introduction

An atom interferometer’s basic operation is to let two atom beams interfere with each other, measuring the phase difference between the two atom beams. This gives the atom interferometer many possible applications. The primary application of the atom interferometer in the group lead by Ton v. Leeuwen will be for experiments related to so called Quantum Non Demolition experiments. The objective of these experiments is to give answers to fundamental questions on for instance photon locality and quantization of light.

The first objective of the QND-experiment is to directly observe the quantized behavior of the light field, or better the photon field. The hypothesis of the quantization of a radiation field was first proposed by Planck. Shortly, after Einstein introduced the idea of photons (energy packages with a fixed energy content) with a particle-like behavior. Later in quantum mechanics it was possible to actually describe any electric and magnetic field as a photon field. Most experiments realized sofar however, can be completely described by a classical (i.e. non quantized) light field with a quantum mechanical atom. So the question whether or not electromagnetic fields are actually quantized was not answered by these experiments. The quantization of the light fields was first observed directly by M. Brune et. al. [1]. Here a cavity is used to capture a limited number of photons in a confined space. Atom are then past through this cavity, so that they can interact with the created radiation field. The number of radiation photons in the cavity was not always preserved by the interaction. In the QND-experiment near infra-red photons will be captured in a cavity. The atom will interact with the photons nondestructively, preserving the number of photons. The atom will remain in the ground state. The momentum and/or the phase of the atoms can still change due to the nondestructive interaction with the photons.

The change of the atom’s momentum can easily be measured by measuring the deviation from its original flight path. But for very weak interactions the momentum will not change, leaving only a phase change. To measure the phase change, an atom interferometer will be used. The basic design of the atom interferometer consists in splitting the atom beam coherently and then bringing both atom beams back together again so that the two beams can interfere with each other. It is then possible to let each atom beam interact with different fields which results in a phase change, which the atom interferometer translates into an alteration in the intensity of the emerging atom beam. To be able to perform the QND-experiments the physical separation of the two beams must be large enough so that it is possible that only one of the two beams interact with the photon field in the cavity.

The atom beam can consist of one atom at a time, which is sent through the atom interferometer. Therefore, in the atom interferometer, experiments are carried out with
highly delocalized atoms, which makes also other experiments than QND possible. These experiments can give answers to fundamental questions of quantum mechanical problems, which can be very useful for other investigations like Quantum Information, Quantum Computing, Quantum Teleportation, which are key studies for developing new technologies.

The atom interferometer itself, without the cavity, can also be used for other measurements, for instance:

- **Measuring rotations and accelerations.** The sensitivity of the atom interferometer for rotation measurements depends on the atom mass and the surface area enclosed by the two arms of the atom interferometer. Because of the large separation of the two arms the surface area becomes very large, making it very sensitive to rotations and accelerations.

- **Atom Interactions** with other atoms or electromagnetic fields to study the characteristics of the used atom in even more detail. The large separation of the atom beams makes setting up such experiments much easier.

### QND experimental setup

The QND-experiments will be performed with metastable Helium atoms (He*) with a well defined momentum. The total setup itself consists of three distinguishable parts:

1. **Atom beam preparation.** Here the atoms are forced into a cold atom beam needed for the experiments.

2. **Atom interferometer** which measures the phase change of the atom due to the interaction with the photons.

3. **Cavity** which stores the photon.

The preparation of the atom beam is accomplished with an existing setup (see [2]). Here, He* atoms are emerging from a helium source by means of supersonic expansion. The divergence and the longitudinal velocity distribution of the atom beam emerging from this source are both too large to be useful for the experiments. Required is a narrow atom beam with a very small transverse and longitudinal velocity spread. Therefore the atom beam needs some preparation. The preparation is divided into five separate sections:

- **Collimation.** To prevent a large atom loss during the next stage, the strongly divergent He* atoms emerging from the source are collimated in a parallel atom beam.

- **Zeeman slower.** Here the atoms in the beam are slowed down to a well defined velocity with a very small spread, or in other words, to a cold atom beam. The result is a divergent and wide atom beam.

- **Magneto Optic Lens (MOL).** This section focuses the atom beam. In the focal point the next section will be placed.

- **Magneto Optic Collimator (MOC).** This section transforms the divergent atom beam into a narrow and well collimated atom beam.

- **Collimation Apertures.** Two apertures of 60 µm and 25 µm with 2 m separation make a selection of the transverse velocity distribution.
The atom interferometer consists of two elements that split the atom beam and one element for the reflection of the atoms. The splitting and the reflection of the atom beam will be accomplished by means of what is known as Bragg scattering on a standing light wave. These standing light waves are created by reflecting the laser beam back from a mirror such that the reflected beam overlaps with the incident beam. This method gives the possibility to create in a simple way a large splitting angle of the atom beam. Armand Koolen successfully proved in his experiments that Bragg diffraction up until the 8th order is possible, and that the atom beam is split in two beams only [2].

The cavity is used to capture photons long enough to do experiments with them. This cavity mainly consists of two concave mirrors aligned in such a way that the cavity resonance corresponds to the light frequency used for the experiments. The alignment of the mirrors has to be extremely accurate. The design of the alignment stabilization of the mirrors is still in progress.

The status of the QND-experimental setup is that there is a well collimated cold atom beam available. The next step is to design and build the atom interferometer. The goal is to build an atom interferometer with a length of 2 meter, which is the distance between the first and the last beam splitter. To get some experience with atom interferometers a 0.3 meter atom interferometer will be built first. Before actually building the atom interferometer an assessment of the potential problems will be made.

Contents report

Before we can continue with the actual assessment of the potential problems some understanding of the principles involved in the atom interferometer is needed. This is the content of the second chapter. In this chapter the theory related to and the basic setup of the atom interferometer are discussed. Next the characteristics of the atom beam and the two Bragg diffraction experiments by Armand Koolen will be discussed in more detail in this chapter.

In the third chapter we will investigate the potential problems that can influence the measurements done with the atom interferometer. They can be divided into four groups:

- **Sensitivity of the Bragg diffraction** for misalignment of the standing light waves and for the atoms' velocity spread,

- **Misalignment of the Optical Components,**

- **Atom Interferometer Motions,** and

- **Electric and Magnetic fields** which affect the phase of the atom beams.

These problems will be investigated in depth, and a limit to their allowed values will be given.

From the investigations on mechanical vibrations it became clear that the atom interferometer is very sensitive to the distortions of the setup. To compensate for these distortions a stabilization method is designed. It measures the distortion using a light interferometer which behaves like the atom interferometer. The signal from this light interferometer is then used to actively control the position of one of the mirrors, compensating for the distortion effects as much as possible. The design, construction and testing of this mirror stabilization are discussed in the fourth chapter.
Chapter 2

The Atom Interferometer

2.1 Introduction

For QND-experiments a Mach-Zehnder type atom interferometer will be built. The basic design consists of two atom beam splitters and one reflector, each consisting of standing light waves. The first beam splitter splits the incoming atom beam in two beams. These two beams are brought back together again by the atom beam reflector. At the position where the two beams intersect, the second beam splitter is placed resulting in two beams, each produced by two beams interfering with each other (see Figure 2.1). The splitting and reflection of the atom beam are accomplished by means of Bragg diffraction on a standing light wave. The advantages of this method are: 1) coherent deflection (or splitting) of the atom beam, 2) high efficiency because the atoms are mainly redistributed over two diffraction orders while all other orders remain almost empty, 3) only one laser beam is needed which is reflected by a mirror to create a standing light wave.

2.2 Statistics of the Atom Interferometer

The atom interferometer has two output signals from detectors $A$ and $B$. Let $P_A$ be the chance the atom hits detector $A$. Because in the ideal situation the total number of particles detected on both detectors is preserved, the chance the atom hits detector $B$ is

![Standing wave diagram](image)

Figure 2.1: The basic setup of the atom interferometer. The atom interferometer setup consists of three standing light waves produced by reflecting the incoming light beam from mirrors. It gives two output signals on two detectors indicated by $A$ and $B$. 


1 - \( P_A \). The value of \( P_A \) is determined by the phase difference between the two interfering atom beams. If the phase difference between these two interfering atom beams hitting detector \( A \) is \( \varphi \), the combined wave function becomes:

\[
\Psi_A = \frac{1}{2} e^{(i \vec{p} \cdot \vec{x} - E t / \hbar)} (1 + e^{i \varphi})
\]

(2.1)

with \( \hbar \vec{p} \) the momentum and \( E \) the energy of the atoms. The intensity of the atom beam is:

\[
I_A = \frac{I_0}{2} |\Psi|^2 = \frac{I_0}{2} (1 + \cos \varphi)
\]

(2.2)

with \( I_0 \) the total number of atoms per second entering the atom interferometer. This is valid for an ideal situation in which both interfering beams have the same intensity and is a quarter of that of the beam that entered the atom interferometer. This equation shows constructive interference for \( \varphi = 2m \pi \) and destructive interference for \( \varphi = (2m + 1) \pi \), where \( m \) is an integer. Because of the conservation of particles the intensity of the beam on detector \( B \) must be \( I_B = 1 - I_A = \frac{1}{2} I_0 (1 - \cos \varphi) \). The value of \( P_A \) is then the ratio between \( I_A \) and \( I_0 \):

\[
P_A = \frac{1}{2} (1 + \cos \varphi).
\]

(2.3)

When the atom interferometer is used as a measurement device, a situation should be created where \( \varphi \) is dependent on the physical quantity to be measured. To get the most accurate results it is advisable to perform the measurements where \( \partial P_A / \partial \varphi \) has a maximum, which is near \( \varphi = \frac{\pi}{2} \). In this situation \( P_A = \frac{1}{2} \).

The ratio \( P_A \), mentioned previously only holds when there is an infinite number of atoms. If the number of atoms is limited, there is a chance of getting a different ratio. The chance for a certain outcome can be calculated with the following formula,

\[
P_{P_A} (N_a) = P_A^{N_a} (1 - P_A)^{N - N_a} \binom{N}{N_a}
\]

(2.4)

where \( N_a \) is the number of atoms measured on detector \( A \) out of the total of \( N \). This is a standard statistical expression. The first part of the expression \( P_A^{N_a} (1 - P_A)^{N - N_a} \) indicates the chance of getting one particular sequence of \( N_a \) atoms hitting detector \( A \). The second expression \( \binom{N}{N_a} \) is the Newton’s binomial which gives the number of possible different sequences with a total of \( N \) atoms and \( N_a \) hitting detector \( A \).

The relation between the intensity and the total number of atoms is \( N = I_0 \tau \) which is equal to the total number of atoms measured over a period of time \( \tau \). The average and the standard deviation (spread) of the outcome can be calculated by:

\[
\bar{N}_a = \sum_{N_a} N_a P_A (N_a) = P_A N,
\]

(2.5)

\[
\sigma_{\bar{N}_a}^2 = \sum_{N_a} N_a^2 P_A (N_a) - \bar{N}_a^2 = NP_A (1 - P_A),
\]

(2.6)

\[
\sigma_{P_A} = \sigma_{N_a} / N = \sqrt{\frac{P_A (1 - P_A)}{N}},
\]

(2.7)
where $\sigma_{PA}$ is the smallest achievable resolution of the measurement. In the region of highest accuracy this limit becomes $\sigma_{PA} = 1/(2\sqrt{N})$. This indicates that the minimum number of atom to be measured to get a required resolution of $\xi$ becomes $1/(4\xi^2)$, which means that a measurement time $\tau_{exp} > 1/(4\xi^2I_0)$ is required. Therefore if the intensity of the atom beam is low, it is necessary to take long measuring times to get accurate measurements. The parameters of the atom interferometer which influence the measurements have to remain stable during the entire measurement.

2.3 Bragg diffraction

Previously in this chapter it was mentioned that the splitting or deflection of the atom beam will be accomplished by what is known as Bragg diffraction on a standing light wave. This standing light wave, $\vec{E}_{st}$, is produced by two counterpropagating monochromatic light waves with the same frequency. This can be described as

$$\vec{E}_{st} = \vec{E}_1 + \vec{E}_2$$

with

$$\vec{E}_i = \frac{E_0}{2} \hat{e}_i \left\{ e^{i(\vec{k}_i \cdot \vec{x} - \omega t + \phi_i)} + \text{c.c.} \right\} \quad \text{(for } i = 1, 2\text{)},$$

in which $\vec{k}_i$, $\hat{e}_i$ and $\omega$ are the wave vector, the polarization direction and the frequency of the light, $\vec{E}_1$ and $\vec{E}_2$ are the electric fields of the two light waves for a plane standing wave, with $\vec{k}_1 = -\vec{k}_2$ and $\hat{e}_1 = \hat{e}_2$, and $\phi_i$ is the phase at time $t = 0$ and $\vec{x} = 0$. For the atoms that interact with the standing light wave a two level atom is considered described by the general expression:

$$|\Psi(t)\rangle = \int d\vec{p}' |\vec{p}'\rangle \left\{ C_g(\vec{p}', t)e^{-i\omega t}|g\rangle + C_e(\vec{p}', t)e^{-i\omega t}|e\rangle \right\}.$$ 

(2.9)

where $C_g(\vec{p}', t)$ and $C_e(\vec{p}', t)$ are the expansion coefficients of the atom’s wave function. The energy levels $\hbar \omega_g$ and $\hbar \omega_e$ correspond to the energy levels of the internal ground state $|g\rangle$ and the excited state $|e\rangle$. The state vector $|\vec{p}'\rangle$ describes the momentum state of the atom.

The time dependent Schrödinger equation that includes the interaction with the standing light field is:

$$i\hbar \frac{d|\Psi\rangle}{dt} = (H_0 + H_{int})|\Psi\rangle$$

(2.10)

with

$$H_0|\Psi_{i, \vec{p}'}\rangle = [\hbar \omega_i + E_{kin}(\vec{p}')]|\Psi_{i, \vec{p}'}\rangle \quad \text{(for } i = e, g\text{)}$$

$$H_{int} = -\vec{E}_{st} \cdot \vec{\mu},$$

$$E_{kin}(\vec{p}') = \frac{p'^2}{2m}.$$ 

In these expressions $H_{int}$ is the interaction Hamiltonian, $\vec{\mu}$ is the electric dipole operator, $m$ the mass of the atom and $p'$ the magnitude of the momentum of the atom. Multiplication
of the time dependent Schrödinger equation (Eq. 2.10) from the left with the state vectors \( \langle g, \vec{p} \rangle \) and \( \langle e, \vec{p} \rangle \) and applying the rotating wave approximation results in the following infinite set of coupled equations:

\[
\begin{align*}
\dot{C}_g(\vec{p}, t) &= -\frac{i}{\hbar} E_{\text{kin}}(\vec{p}) C_g(\vec{p}, t) + i\tilde{\Omega}^* \cdot \{ \dot{\epsilon}_1 C_e(\vec{p} + \vec{p}_1, t) e^{-i\varphi_1} + \\
&\quad \dot{\epsilon}_2 C_e(\vec{p} + \vec{p}_2, t) e^{-i\varphi_2} \} e^{i(\omega - \omega_0)t}, \\
\dot{C}_e(\vec{p}, t) &= -\frac{i}{\hbar} E_{\text{kin}}(\vec{p}) C_e(\vec{p}, t) + i\tilde{\Omega} \cdot \{ \dot{\epsilon}_1 C_g(\vec{p} - \vec{p}_1, t) e^{i\varphi_1} + \\
&\quad \dot{\epsilon}_2 C_g(\vec{p} - \vec{p}_2, t) e^{i\varphi_2} \} e^{-i(\omega - \omega_0)t},
\end{align*}
\]

(2.11a) \hspace{1cm} (2.11b)

with

\[
\begin{align*}
\vec{p}_i &= \hbar \vec{k}_i \quad (i=1,2) \\
\tilde{\Omega} &= \frac{E_0}{2\hbar} \langle e | \vec{p} | g \rangle \\
\omega_0 &= \omega_e - \omega_g,
\end{align*}
\]

where \( \hbar \omega_0 \) the transition energy from the ground to the excited state and \( \tilde{\Omega}^* \) the complex conjugate of \( \tilde{\Omega} \). These equations describe the process of absorption and stimulated emission of a photon in the standing light wave. At each absorption the momentum of the photon is transferred to the atom, for stimulated emission this process is reversed. Thus, each process can alter the atom’s momentum only with \( \vec{p}_1 \) and \( \vec{p}_2 \). This makes the following substitutions possible:

\[
C_{e/g}(\vec{p}, t) \rightarrow C_{e/g}(\vec{p}, t)e^{i[\pm \frac{1}{2} \hbar + \varphi]}
\]

(2.12)

with

\[
\begin{align*}
\vec{p} &= \vec{p}_0 + N_1 \vec{p}_1 + N_2 \vec{p}_2, \\
\varphi &= N_1 \varphi_1 + N_2 \varphi_2, \\
\delta &= \omega_0 - \omega,
\end{align*}
\]

and the set of coupled equations simplifies to

\[
\begin{align*}
\dot{C}_g(\vec{p}, t) &= -\frac{i}{\hbar} \left[ E_{\text{kin}}(\vec{p}) - \frac{\hbar \delta}{2} \right] C_g(\vec{p}, t) + \\
&\quad i\tilde{\Omega}^* \cdot \{ \dot{\epsilon}_1 C_e(\vec{p} + \vec{p}_1, t) + \dot{\epsilon}_2 C_e(\vec{p} + \vec{p}_2, t) \} \\
\dot{C}_e(\vec{p}, t) &= -\frac{i}{\hbar} \left[ E_{\text{kin}}(\vec{p}) + \frac{\hbar \delta}{2} \right] C_e(\vec{p}, t) + \\
&\quad i\tilde{\Omega} \cdot \{ \dot{\epsilon}_1 C_g(\vec{p} - \vec{p}_1, t) + \dot{\epsilon}_2 C_g(\vec{p} - \vec{p}_2, t) \}.
\end{align*}
\]

(2.13a) \hspace{1cm} (2.13b)

Here \( \vec{p}_0 \) represents the initial momentum of the atom before it entered the standing light wave, and \( N_1 \) and \( N_2 \) are integers. The integers \( N_1 \) and \( N_2 \) each are the differences between the number of photons absorbed from and the number of photons emitted by stimulated emission into the specific laser beam. As an example, let’s start with an atom in the ground state that did not have any interaction with the light field sofar. In that particular situation both integers are zero. When the atom absorbs a photon from, for example, laser
beam 1, integer \( N_1 \) is increased by 1. The atom is now in the excited state and it cannot absorb another photon from either of the two laser beams. Stimulated emission brings the atom back to the ground state. A photon is then emitted in the direction of either laser beam 1 or 2. The integer related to the laser beam the photon is sent into is then decreased by 1. This is repeated during the interaction time of the atom with the light field. The sum of both integers is the difference between the number of photons absorbed and the number of photons emitted via stimulated emission resulting in \( N_1 + N_2 = 0 \) for atoms in the ground state and \( N_1 + N_2 = 1 \) for atoms in the excited state. This is an idealized picture in which spontaneous emissions are completely ignored.

From the substitution with Eq. 2.12 the phase correction \( \varphi = N_1 \varphi_1 + N_2 \varphi_2 \) implies that the phase of the laser beams is transferred to the atom when a photon is absorbed while this phase is extracted from the atom when a photon is transmitted through stimulated emission.

The momentum of the atom is given by \( \vec{p} = \vec{p}_0 + N_1 \vec{p}_1 + N_2 \vec{p}_2 \) (Eq. 2.12). When the atoms are in the ground state \( N_1 = -N_2 = N \) while for atom in the excited state \( N_1 = 1 - N_2 = M \) where both \( N \) and \( M \) are integers. For the ground and the excited state the momentum changes become:

\[
\begin{align*}
\vec{p}_g &= \vec{p}_0 + N_1 \vec{p}_1 + N_2 \vec{p}_2 \\
\vec{p}_e &= \vec{p}_0 + N_2 \vec{p}_2 + M_1 \vec{p}_1 + N_2 \vec{p}_2
\end{align*}
\] (2.14) (2.15)

Equation Eq. 2.13 gives an infinite set of coupled equations. Using an idealized situation where \( \vec{p}_1 = -\vec{p}_2 \) the set of coupled equations gets a familiar matrix form already described by others in [3, 4] where they also included spontaneous emission.

Laser beam 2 can be produced by reflecting laser beam 1 from a mirror. The phase relation \( \varphi_m \) and the amplitude relation between both laser beams at the mirror surface is constant. By using high reflectivity mirrors the amplitudes of the both beams can be taken approximately equal. To ensure a perfect overlap of both beams the wave vector of the incident beam must be perfectly perpendicular to the mirror surface so that \( \vec{k}_2 = -\vec{k}_1 = \vec{k}_s \) and \( \vec{e}_1 = \vec{e}_2 = \vec{e} \). Under these assumptions the electric field of the standing light wave can be described as

\[
\vec{E} = 2E_0 \vec{e} \cos \left( \frac{\varphi_1 + \varphi_2}{2} - \omega t \right) \cos \left( \vec{k}_s \cdot \vec{x} + \frac{\varphi_1 - \varphi_2}{2} \right). \] (2.16)

The phase relation between the incident and the reflected laser beam at the mirror surface is given by \( \varphi_m = 2 \vec{k}_s \cdot \vec{x}_{mirror} + \varphi_2 - \varphi_1 = 2 |\vec{k}_s| d_m + \varphi_2 - \varphi_1 \), with \( d_m \) the distance between the intersection of the atom beam with the standing light wave, and the mirror surface, then \( 2 |\vec{k}_s| d_m = \varphi_1 - \varphi_2 + \varphi_m \). The phase factor \( \varphi_m \) is the phase change of the reflected light beam due to the characteristics of the mirror surface, which is a constant and can therefore be left out of the equation. The phase change of the diffracted atoms due to a position change of the mirror surface \( \Delta d_m \) is then given in Eq. 2.12 which results in the relationship:

\[
\Delta \varphi_{atom} = N \Delta (\varphi_1 - \varphi_2) = 2N |\vec{k}_s| \Delta d_m \] (2.17)

with \( N \) indicating the momentum change \( (\Delta \vec{p} = 2N \hbar \vec{k}_s) \) \[5\], also known as the order of the Bragg diffraction. This relation is of importance for defining the mechanical stability requirements of the setup.
2.3.1 Bragg diffraction experiment

Bragg diffraction of atoms in a standing light wave has a close correlation with the scattering of X-ray photons on a crystal. The X-ray photons see these crystal as a medium with a periodically modulated refractive index. This modulation form crystal planes, resulting in the X-ray photons being either transmitted through or reflected from these planes. In Bragg diffraction the planes of equal intensity in the standing light wave form a periodic potential. This potential is for the atoms passing through the standing light wave equivalent to the crystal planes in the crystal for the X-ray photons.

In a standing light wave the atom beam is split into two beams or completely reflected depending on the intensity of the laser beam. Calculations show that Bragg diffraction is most efficient when the Bragg condition

$$N \, P_{\text{light}} = p_{\text{atom}} \sin \Theta$$

(see Figure 2.2) and the adiabatic condition

$$\frac{1}{E_n} \frac{\partial E_n}{\partial t} \ll \frac{\Delta_n}{\hbar}$$

(2.19)

are met. Here $p_{\text{atom}}$ and $p_{\text{light}}$ are the magnitude of the momenta of respectively the atom and the photons in the standing light wave, $N$ is an integer indicating the order of the Bragg diffraction, and $E_n$ the energy of the $n$th adiabat and $\Delta_n$ the energy difference between the nearest neighboring eigenstates in the standing light wave [4].

Armand Koolen performed experiments with Bragg diffraction with large diffraction angles and compared his results with the theoretical predictions [2]. In these experiments a laser with a maximum power of 5 mW with a $\sigma^-$ polarization is used. This laser was red detuned with $185 \cdot 2\pi$ Mrad/s from the transition frequency of the $|J, m_J : 1, -1 \rangle \rightarrow |2, -2 \rangle$ transition. In the first experiment (see Figure 2.3) the numbers of deflected and non-deflected atoms are measured as a function of the laser intensity. Armand Koolen showed that his predictions correspond with the experiments. Figure 2.3 shows an oscillation between maximal and minimal diffraction of the atom beam as a function of the intensity of the standing light field. This so called Pendellösung oscillation makes it possible to adjust the laser intensity to make a 50-50% atom beam splitter or an almost complete reflector which are needed both to make a atom interferometer.

The second experiment (Figure 2.4) showed the possibility of higher order Bragg diffraction. To maintain a maximal deflection of atoms the laser intensity is continuously tuned. Figure 2.4 clearly shows that the atom beam is split mainly into two beams. The beam intensities at other possible diffraction angles remain very small. The angle between the
two exiting beams is determined by the incoming Bragg-angle, and is twice this angle. This corresponds to the difference between transmission and reflection from the wave fronts of the standing light wave.

### 2.4 He* atoms in atom interferometer

The experiments related to QND will be carried out with metastable Helium atoms (He*). It is therefore important to investigate the interaction of these atoms with the standing light wave in more detail. In Bragg diffraction experiments the transition $^3S_1 \rightarrow ^3P_2$ is used. Transitions to states with a quantum number other than $J = 2$ are not likely because of the large fine splitting energy.

For the atom interferometer atoms in one particular magnetic substate will be used. This results in lower atom loss for the simple reason that Bragg diffraction strongly depends on the interaction strength of the atoms with the light field, which is different for each magnetic substate (Clebsch-Gordan coefficients). To keep the atoms polarized a magnetic field has to be present. This magnetic field also has to be aligned with the wave vector of the standing light waves. If not, the precession of the atom's magnetic dipole in the magnetic field will change the magnetic substate distribution in relation to the standing light wave, which in turn will result in efficiency loss.

The metastable Helium atoms are emerging from the source by supersonic expansion. At this stage the atomic beam's divergence and large velocity spread are unacceptably large. For our experiments a narrow, well collimated atom beam is required with a small velocity spread. Preparing the atom beam is therefore necessary. The preparation of the beam is divided into five stages. In the first laser cool section, the divergent beam is collimated. Next, the Zeeman slower slows the atoms down to the desired velocity with the aid of a counter-propagating laser beam [6]. The velocity capture range of the in the laser cooling process is very small, resulting in a large atom loss if no precautions are taken. This is solved by using the Zeeman shift between two magnetic substates of the atoms in a magnetic field with varying strength along the path the atoms travel. The
output of this stage is a wide and again divergent beam that has to be compressed and collimated. This is accomplished in the last two stages, the MOL (Magneto Optic Lens) and MOC (Magneto Optic Compressor). Both stages consist of a quadrupole magnetic field and two sets of two counter-propagating laser beams perpendicular to the atom beam. In the final stage the atoms pass through two pinholes separated by 2 m with 60 μm and 25 μm diameter respectively. The velocity of the atoms in the atom beam then becomes equal to $v_\parallel = 247 \text{ m/s}$ with a longitudinal and transverse velocity spread of respectively $\sigma_{v\parallel}/v_\parallel = 1.5 \cdot 10^{-2}$ and $\sigma_{v\perp} = 9 \cdot 10^{-3} \text{ ms}^{-1}$.

The state in which the atoms leave the compressor section is mainly $|J, m_J\rangle = |1, -1\rangle$. Armand Koolen showed [2] that the distribution of the magnetic substates is about 75% $|J, m_J\rangle = |1, -1\rangle$, 25% $|J, m_J\rangle = |1, 0\rangle$ and no $|J, m_J\rangle = |1, +1\rangle$. The Clebsch-Gordan coefficient for transition $|1, 0\rangle \rightarrow |2, -1\rangle$ is $1/\sqrt{2}$ while that of transition $|1, -1\rangle \rightarrow |2, -2\rangle$ is equal to 1. For the former transition, this has the same effect as lowering the intensity of the standing light wave by a factor 2. When the intensity of the standing light wave is tuned to the first maximum of the Pendellösung oscillation (see Figure 2.3) for the $|1, -1\rangle \rightarrow |2, -2\rangle$ transition, the atoms in substate $|J = 1, m_J = 0\rangle$ will hardly be diffracted. The result is that the majority of atoms in the $|J = 1, m_J = 0\rangle$ state will pass straight through the interferometer setup and will not end up on one of the detectors. This will not effect the performance of the atom interferometer other than that the signal intensity will drop by 25%.

The conclusion is that for the atom interferometer only the transition $|J, m_J\rangle, |1, -1\rangle \rightarrow |2, -2\rangle$ needs to be considered.

### 2.5 Configuration of the atom interferometer

Figure 2.5 shows the design of the optical setup of the atom interferometer. As the figure shows it will basically consist of three mirrors that build up the three standing light waves splitting or reflecting the atom beam, three polarizing beam splitter cubes with...
halfwave plates and three quarterwave plates. The mirrors have to be placed in such a way that the created standing light waves are correctly aligned in relation to the atom beam (Bragg condition). They also have to be positioned such that the third standing light wave intersects with the crossing of the two atom paths in the atom interferometer. This means that the successive distances of these light beams have to be equal. The position of these light beams are also set by the positions of the polarizing beam splitter cubes. These cubes have to be aligned to the mirrors in such a way that the reflected and the incident light beams overlap each other, so that standing light waves are formed. This means that if the mirrors are realigned, the cubes also have to be realigned to the new position of the mirrors. The combination of the polarizing beam splitter cubes with the halfwave plates gives the possibility of adjusting the intensity of the standing light waves. From Figure 2.5 it becomes clear that this setup requires a fixed sequence of adjusting the intensities of the standing light beams. Changing the position of the first halfwave plate the laser passes means changing the intensity of not only the standing light wave related to mirror 3, but also those related to mirrors 1 and 2.

A quarterwave plate is used to change the linearly polarized light from the polarizing beam splitter cubes into the required circular polarization. The light coming back from the mirrors will retain, preserving its circular polarization. After passing the quarterwave plate the light will again be linearly polarized again, except the polarization is now perpendicular to that of the incoming beam. This means that the light will pass straight through the polarizing splitting cube, and can therefore not couple back into the laser.

For stability reasons the mirrors have to be positioned on a stable mount as indicated. In the next chapter the stability requirements will be investigated. From theory and experiences in groups that already designed atom interferometers, it is also clear that the total setup needs to be isolated from vibrations, because the atom interferometer is very sensitive to accelerations and rotations.
Chapter 3

Design Considerations

In this chapter a complete assessment is made of possible errors that influence the atom interferometer during measurements. This assessment results in a set of requirements for the atom interferometer to meet a minimum desired signal quality. To get some experience with atom interferometers, first a smaller prototype atom interferometer with a length of 0.3 meter will be designed and built. The length of the atom interferometer refers to the distance between the first and the third standing light wave. The knowledge gained from this prototype atom interferometer will be used in the design of the atom interferometer that will be used for QND-experiments with a length of 2 m. The assessment in this chapter will involve both atom interferometers, resulting in 2 sets of requirements.

The interferometer's basic design consists of three standing light waves that are aligned to the atom beam such that the Bragg condition \( p_{\text{atom}} \sin \Theta = p_{\text{photon}} N \) is met. The atoms in the atom beam used for the QND-experiments have a velocity of 247 m/s resulting in a momentum of \( 1.66 \cdot 10^{-24} \) Ns. The wave-length used for the standing light waves \( \lambda_s = 1083 \) nm which results in a photon momentum of \( 6.11 \cdot 10^{-28} \) Ns. Because \( \Theta \) is very small, the Bragg condition can be approximated by \( p_{\text{atom}} \Theta = p_{\text{photon}} N \). This means that the difference between neighboring Bragg-angles is equal to 0.368 mrad. Such small angles need careful alignment of the standing light waves. It will be shown that a deviation from the Bragg condition affects the Bragg diffraction which has a significant negative effect on the performance of the atom interferometer. This deviation can be a result of standing light wave misalignments or the velocity spread of the atoms in the atom beam.

Bragg diffraction also introduces an extra phase change between the two different arms in the atom interferometer due to the displacement of the mirrors as shown in section 2.3. In general interferometers are very sensitive for distortions of the setup. Our atom interferometer is not any different. This has large implications on its operation. Atom interferometers are also very sensitive to accelerations and rotations of the entire setup without distorting it which was shown in [7, 8]. This makes the atom interferometer very useful to accurately measure very small motions. However in the QND-experiments the atom interferometer will not be used for motion detection, and the signal resulting from the atom interferometer motions will be considered measurements errors. Therefore, it is very important to look at the effect of these different motions on the atom interferometer signal. It will be shown that distortions, accelerations and rotations are closely related which makes it possible to examine them together. Gravity will only be discussed very briefly because the diffraction of the atom beams takes place in the horizontal plane perpendicular to the direction of the gravity force, effectively eliminating its influence. This way the
Figure 3.1: Output signal as function of the phase difference $\Delta \varphi$ between the paths of the atom beam. $N$ is the actual measured number of atom at a detector while $N_0$ is the total number of atoms. Visibility is calculated by $V = (I_{\text{max}} - I_{\text{min}})/(I_{\text{max}} + I_{\text{min}})$

effects of gravity on the two paths of the atom beam are identical, not causing any phase difference between the two paths. If however, the atom interferometer setup is tilted, the gravitation will introduce a constant phase difference between the two paths, but because gravity itself is constant, this effect can easily be compensated for.

The fact that He$^*$ has a magnetic moment makes it susceptible to magnetic fields. Magnetic fields can be created by electronic devices and (ferro)magnetic materials. It is therefore important to look into how sensitive the atom interferometer is to these fields. Electric fields (low frequency compared to light fields) do not have any significant effect on the He$^*$ atoms because they are neutral and will not be easily polarized.

Errors in or imposed on the atom interferometer can affect the measurement in two ways. It can directly result in an error $\varphi_{\text{err}}$ in the desired phase change, and/or the quality of the signal decreases. The latter effect, the signal quality, can be expressed in a quality factor called the visibility $V = (I_{\text{max}} - I_{\text{min}})/(I_{\text{max}} + I_{\text{min}})$ (see Figure 3.1) with $I_{\text{max}}$ and $I_{\text{min}}$ the maximum and minimum measured intensity at the detector respectively. Visibility $V$ is a measure for the amplitude of the fringes in the output signal, relative to the total output signal from both detectors in the atom interferometer. It is quite clear that if the visibility diminishes, the measurement becomes less accurate ($\sigma_{P_A} = 1/(V\sqrt{N})$).

The output signal depends on the chance that the atom hits detector $A$ or $B$ (see Figure 2.1), indicated by $P_A$ (or $1 - P_A$) in the previous chapter. Due to disturbances this factor may no longer be considered constant over time. The output of the measurement is the integration of the signal over time. For instance, for detector $A$ the output $U$ becomes:

$$U = \int_0^\tau P_A(\Delta \varphi(t))dt,$$

where $\tau$ represents the measurement time. The chance $P_A$ is a function of $\Delta \varphi(t)$ which is a function of time. The relation between $P_A$ and $\Delta \varphi$ is given in Figure 3.1. This figure shows
that if the amplitude of $\Delta \varphi(t)$ is much larger than $2\pi$, the integrated output $U_{\text{fringe}} \rightarrow 0$. No accurate measurements are possible anymore.

It is not always possible to define $\Delta \varphi$ as a function of time. In some situations there is only information on the spread of $\Delta \varphi$ indicated by $\sigma_{\varphi}$. In this situation a Gaussian function $P_{\varphi}(\Delta \varphi)$ is used, which is a function of $\Delta \varphi$ with a maximum at $\Delta \varphi = 0$ and a standard deviation of $\sigma_{\varphi}$. The signal output will then be averaged over the Gaussian function:

$$U = \int_{-\infty}^{\infty} P_{\varphi}(\Delta \varphi)P_{\varphi}(\Delta \varphi) d\Delta \varphi. \quad (3.2)$$

To determine the requirements for the various possible errors affecting the atom interferometer, the following criteria will be used throughout the investigation:

$$\varphi_{\text{err}} < \frac{1}{1000} \quad (3.3)$$

$$V > 0.99. \quad (3.4)$$

### 3.1 Bragg diffraction efficiency

Bragg diffraction can be influenced by a misalignment of the standing light wave from its Bragg condition and also by a velocity change of the atoms, as the Bragg-angle is velocity dependent. In the ideal atom interferometer the 50–50% beam splitters diffract the atom beam into two separate beams with equal intensity. Also it will be assumed that initially, when there are no misalignments, the Bragg condition is met. It is possible to make a 50–50% beam splitter without meeting the Bragg condition, but this makes the atom interferometer even more susceptible to misalignment errors. Because of misalignment errors, the intensity of both beams is no longer equal. This affects the quality of the atom interferometer signal. Any deviation from the Bragg condition will be treated as a misalignment error. This means that a velocity change will also lead to a misalignment error of the standing light wave. In this section the effects of the standing light wave misalignment will be studied first. These results are then used to study the effect of the velocity distribution of the atoms on the atom interferometer.

#### 3.1.1 Misalignment of the standing light wave

In the previous chapter it was already mentioned that Bragg diffraction is most efficient when the Bragg condition

$$p_{\text{atom}} \sin \Theta = N p_{\text{light}} \quad (3.5)$$

is met. Due to misalignments in the X-Z plane (see Figures 3.3 and 3.2) the Bragg condition is no longer met resulting in a change in the distribution between the transmitted and the deflected atom beam. Figure 3.2 shows the result of simulations carried out with a program based on the one used by Armand Kooien, which shows that for small alignment errors ($\Delta \Theta < 100 \mu\text{rad}$) all the atoms remain in two beams and the intensity of all other orders of diffraction remains negligible. It also shows that the splitting ratio changes. When the alignment error reaches about 180 $\mu\text{rad}$ (halfway between the two orders) there is hardly any splitting of the atom beam. The intensity of the diffracted atom beam after a 50-50% beam splitter as function of the angular misalignment, the lower distribution of Figure 3.2, can be approximated by:

$$G_N(\Delta \Theta) = 0.5 e^{-\frac{1}{2}((\Delta \Theta/\sigma)^2)} \quad (3.6)$$
Figure 3.2: Results of the simulations. For the simulations a standing light wave with a Gaussian intensity profile and a diameter of 0.85 mm is assumed. The simulations show the change of the distribution of the deflected and transmitted atoms due to misalignment of the standing light wave. Upper distribution belongs to the beam deflector section, the lower to the 50-50% beam splitter. For both the order of Bragg diffraction \( N = 5 \). The curve marked "-5" belongs to the deflected beam, the one marked "5" belongs to the transmitted beam.

Figure 3.3: The standing light wave misaligned with an angle \( \Delta \Theta \) from the Bragg-angle \( \Theta \).

where \( \varsigma_r \) is the standard deviation and is a measure for the maximum allowed alignment error, and \( N \) is the order of the Bragg diffraction. This approximation is quite accurate, the error was smaller than 3%. From the simulation results for the 1st, the 5th and the 10th order of Bragg diffraction the values for \( \varsigma_r \) are determined. These values are presented in the following table:

<table>
<thead>
<tr>
<th>Order Bragg diffraction</th>
<th>( \varsigma_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>138 ( \mu \text{rad} )</td>
</tr>
<tr>
<td>5</td>
<td>66 ( \mu \text{rad} )</td>
</tr>
<tr>
<td>10</td>
<td>48 ( \mu \text{rad} )</td>
</tr>
</tbody>
</table>
These values will be used for further calculations.

The visibility of the atom interferometer is not affected by a change in the diffraction ratio of the beam reflector section. It does however influence the intensity of the two output beams of the atom interferometer. The diffraction ratios of the two beam splitters do influence the visibility because the relative intensity of the two beams in the atom interferometer will change. Therefore the amplitudes of the two interfering beams will no longer be equal. With one beam splitter misaligned, the visibility becomes:

\[ V = \frac{2R}{R^2 + 1} \]  

(A.16)

with \( R \) the amplitude ratio of the wave functions of the two split beams (the complete calculation of visibility is given in appendix A). The way the visibility changes as function of the misalignment angle is shown in Figure 3.4. This figure shows two graphs. One graph shows the change in the visibility on both detectors due to a alignment error in one of the beam splitters starting from a ideal situation with perfect 50-50% beam splitters. The second graph shows the worst case scenario for detector \( A \) when the two beam splitters have the same misalignment. The reaction of the two detector signals on these misalignments is very different. The visibility on detector \( B \) actually becomes larger if both splitters are misaligned while the visibility on detector \( A \) becomes smaller. Increased visibility on detector \( B \) does not necessary mean a more accurate measurement, because the chance of measuring an atom on this detector decreases.
3.1.2 Atom velocity spread

The velocity distribution of the atoms in the atom beam also has a considerable effect on the measurements. Its influence is mainly due to the change of the Bragg-angle when the velocity of the atoms changes, which has the same result as the misalignment of the standing light waves. The Bragg condition for small angles states that

\[ \Theta = N p_{\text{ph}} / p_{\parallel} \]

with \( p_{\parallel} \) and \( p_{\text{ph}} \), the magnitude of the momenta of respectively the atom and the photon. To first order, the change in the Bragg-angle due to the change of longitudinal velocity becomes

\[ \Delta \Theta_{\parallel} = -\Delta p_{\parallel} N p_{\text{ph}} / p_{\parallel}^2 = -\Theta \Delta v_{\parallel} / v_{\parallel}. \]

The spread in the deviation from the Bragg-angle due to the longitudinal velocity \( \sigma_{\Theta_{\parallel}} = \Theta \sigma_{v_{\parallel}} / v_{\parallel} \). The deviation from the Bragg-angle due to the transversal velocity is \( \Delta \Theta_{\perp} = \Delta v_{\perp} / v_{\parallel} \). So the spread in the deviation from the Bragg-angle due to the transversal velocity spread \( \sigma_{\Theta_{\perp}} = \sigma_{v_{\perp}} / v_{\parallel} \).

Because the velocity spreads \( \sigma_{v_{\parallel}} \) and \( \sigma_{v_{\perp}} \) are independent variables, the total spread of the Bragg-angle deviation becomes:

\[ \sigma_{\Theta} = \sqrt{\left( \sigma_{v_{\parallel}} \right)^2 + \left( \sigma_{v_{\perp}} \right)^2 / v_{\parallel}}. \quad (3.7) \]

Filling in the characteristics of the atom beam (see chapter 2) leads to the following values for \( \sigma_{\Theta} \):

<table>
<thead>
<tr>
<th>Order Bragg diffraction</th>
<th>( \sigma_{\Theta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>37 ( \mu )rad</td>
</tr>
<tr>
<td>5</td>
<td>45 ( \mu )rad</td>
</tr>
<tr>
<td>10</td>
<td>66 ( \mu )rad</td>
</tr>
</tbody>
</table>

Comparing these values to at Figures 3.2 and 3.4, we conclude that these values can not be neglected.

The change of the time of flight through the standing light wave \( T \) has only a minor effect on the diffracted and transmitted population of the atoms. From [4] these populations can be approximated by:

\[
T = \frac{1}{2}(1 + \cos \varphi_p) \\
D = \frac{1}{2}(1 - \cos \varphi_p) \quad (3.8)
\]

in which \( \varphi_p \) is the phase difference between the states \( 1/\sqrt{2}(|p\rangle \pm | -p\rangle) \) due to the interaction with the standing light wave. \( p \) is the momentum in the x-direction of the atom entering the standing light wave. The phase difference is given by:

\[ \varphi_p = \int_{T_s} \frac{\Delta E(t)}{h} \, dt. \]

\( \Delta E(t) \) is the energy difference between the two eigenstates of the atom when the atom travels adiabatically through the standing light wave. Because \( t = z/v_z \) this equation can be rewritten as:

\[ \varphi_p = \frac{1}{h v_z} \int \Delta E(z) \, dz. \quad (3.9) \]

The profile of the standing light wave intensity does not change when the atom’s velocity changes. Therefore the change in the phase difference \( \Delta \varphi_p = -\varphi_p \Delta v_z / v_z \approx -\varphi_p \Delta v_{\parallel} / v_{\parallel} \)
because of the small Bragg-angle \((v_z \approx v_1)\). The effects of the changes in the transverse velocity can be neglected because the effect of these changes on \(v_z\) are of second order. For small deviations in \(\varphi_p\), the change in the diffracted population \(\Delta D = \frac{1}{2} \Delta \varphi_p \sin \varphi_p = \Delta \varphi_p / 2\) for a 50-50% beam splitter with \(\varphi_p = \frac{1}{2} \pi\), resulting in a diffraction spread \(\sigma_D = \pi \sigma_{\varphi_1} / 4 v_{||}\). Filling in the information from the previous chapter, we find \(\sigma_D = 0.012\) for \(D = \frac{1}{2}\) in the optimal situation. The deviation is only about 2.5%, which is very small.

Because the atoms don’t have exactly the same velocities, the measured signal will not be constant over time. Rather the measured signal will be the averaged intensity:

\[
\overline{I} = \int_{-\Delta \Theta}^{\Delta \Theta} P(\Delta \Theta) I(\Delta \Theta) d\Delta \Theta
\] (3.10)

with

\[
P(\Delta \Theta) = \frac{1}{\sqrt{2 \pi \sigma_{\Theta}}} e^{-\frac{(\Delta \Theta)^2}{2 \sigma_{\Theta}^2}}.
\] (3.11)

Here the assumption is made that the distribution of the Bragg-angle deviation has a Gaussian profile. The intensity on detector \(A\) is:

\[
I_A = 1 - 2 G_N(\Delta \Theta) [1 - G_N(\Delta \Theta)] [1 + \cos \varphi] = 1 - f(\Delta \Theta) - f(\Delta \Theta) \cos \varphi,
\] (3.12)

where the expression of \(G_N(\Delta \Theta)\) is given by Eq. 3.6 and \(f(\Delta \Theta) = 2 G_N(\Delta \Theta) [1 - G_N(\Delta \Theta)]\). This expression is derived in appendix A. The average of this intensity can then be described by:

\[
\overline{I}_A = 1 - \overline{f} [1 + \cos \theta]
\] (3.13)

with

\[
\overline{f} = \int_{-\Delta \Theta}^{\Delta \Theta} P(\Delta \Theta) f(\Delta \Theta) d\Delta \Theta.
\]

The visibility of the signal of detector \(A\) then becomes \(V_A = \overline{f} / (1 - \overline{f})\). After integration the expression for \(\overline{f}\) becomes:

\[
\overline{f} = \frac{c_n}{2 \sqrt{\sigma_{\Theta}^2 + c_n^2}} = \frac{c_n}{4 \sqrt{2 \sigma_{\Theta}^2 + c_n^2}},
\] (3.14)

with \(\sigma_{\Theta}\) in \(\mu\)rad. Substituting for \(\sigma_{\Theta}\) and \(c_n\) the values previously determined for the 1st, the 5th and the 10th order Bragg diffraction results in the following values for the visibility:

<table>
<thead>
<tr>
<th>Order Bragg diffraction</th>
<th>(V_A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.994</td>
</tr>
<tr>
<td>5</td>
<td>0.8</td>
</tr>
<tr>
<td>10</td>
<td>0.5</td>
</tr>
</tbody>
</table>

\(V_B\) remains 1. It is clear that the visibility of detector signal \(A\) deteriorates for the higher order Bragg diffractions. This means that for higher order Bragg diffraction a higher quality cold atom beam could be necessary.
3.2 Alignment of optical components

The basic setup of the atom interferometer consists of three optical mirrors with each one laser beam, creating the three standing light waves required. These mirrors and laser beams have to be carefully aligned in relation to each other and the atom beam. In this section the requirements for the alignment accuracies are determined. To determine the requirements the effects of misalignments from the optimum situation are studied. It is obvious that the standing light waves must cross the atom beam otherwise no interaction will take place. This will not be further investigated. The study of the alignment of the optical components is divided into three separate studies.

First the effects of the alignment between the laser and the mirror are studied. This alignment affects the creation of the standing light waves. To create a perfect standing light wave the laser beam have to be aligned perpendicular to the mirror surfaces. Deviations from this alignment results in an altered standing light wave. This study will show that the direction of the standing light wave is completely determined by the position of the mirror alone.

The second part is a study concerning the alignment of the mirrors, determining the direction the three standing light waves. These standing light have to be aligned parallel to each other. Deviations from this parallel alignment have significant effects on the atom interferometer output signal. This results in a requirement for the mirror alignment.

The last part investigates the effects of the standing light wave positions on the atom interferometer. It is required that the third standing light wave is positioned at the intersection of the atom beams in the atom interferometer when they come together again. The optimum is the position where the two atom beams overlap each other completely. Deviation from this position results in lower signal quality, which has a negative effect on the measurement accuracy.

3.2.1 Laser-mirror misalignment

The distance between the mirrors and the atom beam will be about 15 cm because the mirrors are placed outside the vacuum. That means that the overlap of the incident and the reflected light beam at the position of the atom beam becomes less when there is an angular displacement of the incident laser beam. The relative overlap \( O \approx 1 - 2d \tan \phi/w \) with \( d \) the distance between the mirror and the atom beam, and \( w \) the waist of the laser beam. The waist of the laser beam will be about 1 mm. This means that \( \phi < 0.3 \text{ mrad} \) is needed to get an overlap of at least 90%.

For the reflection from the mirror it will be assumed that the incident angle \( \phi \) on the mirror is much smaller than 0.3 mrad. For a high reflection dielectric mirror with a reflectivity of more than 99.9% the electric field component of the reflected laser light will get a phase shift of \( \pi \) radians. The laserlight used in the atom interferometer is \( \sigma^- \) circularly polarized with a wavelength of 1083 nm. The electric field can be described by

\[
E_i = E_0 \left\{ \hat{\xi}_{iy} \cos(h_i \cdot \vec{x} - \omega t) + \hat{\xi}_{iz} \sin(h_i \cdot \vec{x} - \omega t) \right\}
\]

in which the unit vectors \( \hat{\xi}_{iy} \) and \( \hat{\xi}_{iz} \) are perpendicular to the wave vector of the light beam and to each other. Because of the misalignment the wave vector will this time not be perpendicular to the mirror surface. Therefore, the standing light wave can be divided into three separate standing light waves with linear and circular polarizations. The reference axis for the polarizations of the resulting standing light field is chosen perpendicular to
the mirror surface. The amplitudes of the polarizations are:

<table>
<thead>
<tr>
<th>Polarisation</th>
<th>Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>$\sin \phi$</td>
</tr>
<tr>
<td>$\sigma^-$</td>
<td>$1 + \cos \phi$</td>
</tr>
<tr>
<td>$\sigma^+$</td>
<td>$\frac{1 - \cos \phi}{2}$</td>
</tr>
</tbody>
</table>

This combination of polarization is only valid for the light field in the overlapping region of the incident and the reflected laser beam. The misalignment will be very small, and therefore the standing light waves with polarizations other than the $\sigma^-$ can safely be disregarded. The amplitude modulation of the light field has the form $\sin(\Delta \vec{k} \cdot \vec{x}/2)$ with $\Delta \vec{k} = \vec{k}_r - \vec{k}_l$. $\Delta \vec{k}$ is always perpendicular to the mirror surface, and the length of this vector $|\Delta \vec{k}| = 2|k_i| \cos \phi$ (see Figure 3.5). The wavelength of the amplitude modulation of the standing light wave is the same for the three polarizations and changes only in the second order with the incident angle of the laser beam. Because $\phi < 0.3 \text{ mrad}$ this effect is negligible.

The incident angle also has an effect on the deflection angle of the diffracted atom beam. In the previous chapter it was shown that the deflected atoms in the ground state will have a momentum change $\hbar \vec{k}_d = -N \hbar \Delta \vec{k}$ with $N$ the order of the Bragg diffraction, showing that the momentum change and the wave vector of the standing light wave are directly related by $N\hbar$. The change in the deflection angle will therefore also be of second order in the incident angle.

### 3.2.2 Mirror misalignment

In the previous subsection we found that the direction of the standing light wave is defined by the angular position of the of the mirrors. Bragg diffraction is very dependent on the angular alignment of the standing light wave in relation to the atom beam and therefore determines the position of the mirror in the X-Z plane. The performance of the atom interferometer is also dependent on the mutual misalignments in the X-Y plane of the three standing light waves, just like the gratings in a light interferometer. This basically means that these standing light waves must be aligned parallel to each other. Because the allowed misalignment angles are very small, these misalignments in the two different planes can be investigated separately.
Angular misalignment X-Z plane

The misalignment of the mirrors in the X-Z plane mainly affects the Bragg diffraction of the atom beam (see Figure 3.3). Because of this misalignment the Bragg condition will not be met. The allowed misalignment angle $\Delta \Theta$ depends on the sensitivity of the Bragg diffraction to deviations from the Bragg condition. This was discussed in section 3.1.

Angular misalignment X-Y plane

The second possible misalignment is in the X-Y plane (see Figure 3.7). The effect of this misalignment on the Bragg diffraction efficiency can be neglected because the change in the angle between the atom beam and the standing light wave remains very small. It does however still affect the visibility of the atom interferometer because this type of misalignment causes the two interfering atom beams at the detectors $A$ and $B$ to create intensity fringes as the two interfering beams are no longer parallel to each other. The interfering beams can be described by the following wave functions:

$$\Psi_I = A_0 e^{i(k_I \cdot \vec{x} - \omega t)},$$  \hspace{1cm} (3.17)

$$\Psi_{II} = A_0 e^{i(k_{II} \cdot \vec{x} - \omega t - \varphi)}$$ \hspace{1cm} (3.18)

with $A_0$ the amplitude of the wave function, which is taken equal for both interfering beams, and $\varphi$ the phase difference between the two interfering atom beams. The total beam wave function then becomes $\Psi_T = \Psi_I + \Psi_{II}$. The intensity of the beam $I = |\Psi_T|^2$ which becomes:

$$I(\vec{x}) = 2A_0^2 \{1 + \cos[(k_I - k_{II}) \cdot \vec{x} + \varphi]\}.$$  \hspace{1cm} (3.19)

Because of the small Bragg-angle the deviation from the Bragg condition remains negligibly small for small rotations of the standing light waves in the X-Y-plane. Therefore it can be assumed that both $k$-vectors will have the same length, which results in the fact that $(k_I + k_{II}) \perp (k_I - k_{II})$. The surface of the detector is usually aligned perpendicular to the sum of the two $k$-vectors, which is the effective propagation direction of the two beams combined. To get the signal measured by the detector, an integration over the surface array must be performed. Thus the output signal becomes:

$$U = \iint_S I(\vec{x})dS$$

$$= 2I_0 S \left\{1 + \frac{\sin(k_x D_x/2) \sin(k_y D_y/2)}{(k_x D_x/2) (k_y D_y/2)} \cos \varphi \right\}.$$ \hspace{1cm} (3.20)

Here $I_0 = A_0^2$ is the intensity of the separate interfering beams. $k_x$ and $k_y$ are the components of vector $(\vec{k}_I - \vec{k}_{II})$ in respectively the x- and the y-direction of the atom interferometer. $S = D_x D_y$ is the total surface area on the detector covered by the atom beam which is approximated by a rectangle. The visibility $V$ then becomes:

$$V = \frac{U_{\text{max}} - U_{\text{min}}}{U_{\text{max}} + U_{\text{min}}}$$

$$= \frac{\sin(k_x D_x/2) \sin(k_y D_y/2)}{(k_x D_x/2) (k_y D_y/2)}.$$ \hspace{1cm} (3.21)

The values of $k_x$ and $k_y$ are determined by the misalignment of the three standing light waves. Each time the beam is deflected either partially or completely, the deflected beam...
Figure 3.6: The atom interferometer with three standing light waves indicated with numbers 1, 2 and 3. The beam is split into 2 beams indicated by \( U \) the Upper and \( L \) the Lower beam. They both come together at the last beam splitter.

Figure 3.7: Misalignment of the mirrors (and therefore also the standing wave light fields) in the X-Y plane indicated with the angles \( \alpha_2 \) and \( \alpha_3 \).

changes its momentum by \( 2N \hbar \bar{k}_s \) with \( N \) the order of the Bragg diffraction and \( \hbar \bar{k}_s \) the momentum of the photons in the standing light wave. The initial beam entering the atom interferometer will be split up into four separate beams (see Figure 3.6) which will be given the labels \( UA, UB, LA \) and \( LB \). The first letter of the label refers to the path the atom took, and the second letter to the detector it hits. The momenta of the four beams then become:

\[
\begin{align*}
\vec{p}_{UA} &= \hbar[\bar{k}_0 + 2N(\bar{k}_1 - \bar{k}_2 + \bar{k}_3)], \\
\vec{p}_{LA} &= \hbar[\bar{k}_0 + 2N\bar{k}_2], \\
\vec{p}_{UB} &= \hbar[\bar{k}_0 + 2N(\bar{k}_1 - \bar{k}_2)], \\
\vec{p}_{LB} &= \hbar[\bar{k}_0 + 2N(\bar{k}_2 - \bar{k}_3)].
\end{align*}
\]

(3.22) (3.23) (3.24) (3.25)

\( \hbar \bar{k}_1, \hbar \bar{k}_2 \) and \( \hbar \bar{k}_3 \) are the momenta of the photons in the three standing light waves. Their magnitudes are the same, they differ only in direction. The difference in the wave-vectors

25
between the two interfering atom beams $\vec{k}_I - \vec{k}_{II} = 2N(\vec{k}_1 - 2\vec{k}_2 + \vec{k}_3)$ which is the same for both detectors. The next equations give the coordinates of the wave-vectors when the standing light waves 2 and 3 are misaligned compared to the first standing light wave:

$$\vec{k}_1 = \frac{4\pi N}{\lambda_s}(1, 0) \quad (3.26a)$$

$$\vec{k}_2 = \frac{4\pi N}{\lambda_s}(\cos \alpha_2, \sin \alpha_2) \quad (3.26b)$$

$$\vec{k}_3 = \frac{4\pi N}{\lambda_s}(\cos \alpha_3, \sin \alpha_3), \quad (3.26c)$$

where $\lambda_s$ is the wave length of the photon in the standing light wave and $\alpha_2$ and $\alpha_3$ are the angles of misalignment in the X-Y plane of the standing light waves 2 and 3 (see Figure 3.7). The wave vector difference $\vec{k}_I - \vec{k}_{II}$ then becomes:

$$(k_x, k_y) = 2N\lambda_s(1 - 2 \cos \alpha_2 + \cos \alpha_3, [-2 \sin \alpha_2 + \sin \alpha_3])$$

$$\approx \frac{4\pi N}{\lambda_s}(\alpha_2 - \frac{\alpha_3^2}{2}, [\alpha_3 - 2 \alpha_2]). \quad (3.27)$$

The only time that both $k_x$ and $k_y$ are zero is when $\alpha_2 = \alpha_3 = 0$. The magnitude of $k_x$ is of second order in the misalignment and therefore much less sensitive for misalignments than the magnitude of $k_y$ which is of the first order. For small misalignments it can be assumed that $k_x \ll k_y$ and therefore Eq. 3.21 can be simplified to

$$V \approx \sin(k_y D_y/2) / (k_y D_y/2). \quad (3.28)$$

For a visibility $V > 99\%$ it is required that $k_y D_y/2 < 0.245$. With $\lambda_s = 1083$ nm, $D_y \approx 50 \mu$m and tenth order Bragg diffraction ($N = 10$) this means:

$$|\alpha_3 - 2 \alpha_2| < 42 \mu\text{rad}. \quad (3.29)$$

For small values of $k_y$, the visibility can also be approximated by $1 - V \approx (k_y D_y)^2/24$. For these small angles the displacement in the $y$-direction of the atom beams, when they hit the detectors, will be very small. This displacement can be calculated by $d \approx L \Theta \alpha_2/2$ where $L$ is the length of the atom interferometer and $\Theta$ is the angle of the Bragg diffraction. Stating that the misalignment angle $\alpha_2 < 30 \mu\text{rad}$ and the Bragg-order $N = 10$, the displacement $d < 16.7$ nm. This is much smaller than the width of the atom beam, and its effect can therefore be neglected.

### 3.2.3 Position of the standing light waves on the Z-axis

The third standing light wave has to be positioned at the intersection of the two atom beams. The ideal position of the third standing light wave is at the position where the overlap of the two atom beams is maximum (see Figure 3.8). There is no interference for the atoms in the non overlapping regions. When the standing light wave is displaced by $\Delta z$ from this ideal position, the overlapping area between the two beams becomes less. To get an impression of the effects on the visibility of the atom interferometer, atom beams with a circular profile with diameter $D$ and uniform intensity are assumed. The relative overlap is then described by:

$$Q = S_{\text{overlap}} / S_{\text{total}}$$

$$= \frac{2 \xi - \sin(2 \xi)}{\pi} \quad (3.30)$$
Figure 3.8: Intersection area of the two atom beams. The size of this area is indicated by $L_{\text{intersect}}$. This area is determined by the Bragg-angle $\Theta$ and the atom beam diameter $D$. $\Delta z$ is the displacement of the standing light wave from its ideal position where to get two interfering atom beams at the outputs of the atom interferometer.

with

$$\cos \xi = \frac{\Delta z}{D} \tan \Theta.$$  

where $S_{\text{overlap}}$ is the surface area of overlap of the two interfering atom beams and $S_{\text{total}}$ is the sum of both areas of the interfering atom beams. $\Theta$ is the Bragg-angle. The signal from the detector of the atom interferometer is then given by:

$$U = \frac{U_0}{2} (1 + Q \cos \varphi). \quad (3.31)$$

The visibility belonging to this signal then becomes:

$$V = Q. \quad (3.32)$$

With the assumptions that $\Theta \ll 1$, and that $1 - V = \Delta V \ll 1$, $\Delta V$ can be approximated by

$$\Delta V = \frac{4 \Delta z \Theta}{\pi D}. \quad (3.33)$$

For the requirement that $V > 0.99$, or $\Delta V < 0.01$, $\Delta z < \Delta V \pi D/(4 \Theta) = 127 \mu m$ for tenth order Bragg diffraction.

### 3.3 Relative mirror positions

The relative mirror positions have a very strong effect on the output signal of the atom interferometer. In section 2.3 it is already mentioned that the phase difference between the transmitted and the diffracted atom beams depends on the position of the mirror. The expression for the phase is

$$\varphi = 2N|\vec{k}_s|d = 4\pi N d/\lambda_s.$$  

The total phase difference between the two interfering beams due to the displacement of the mirror is described in [9] which is

$$\varphi_{ai} = \frac{4\pi N}{\lambda_s} (d_1 - 2d_2 + d_3) \quad (3.34)$$

27
Figure 3.9: Every displacement of the three mirrors of the atom interferometer is a combination of the three displacement types given in this picture. The upper two do not result in any phase change. Only the lower one does when the displacement has a curvature.

(the derivation of this equation is given in appendix A). This means that if the mirrors are positioned on a straight line, $\varphi_{ai} = 0$ as shown in Figure 3.9. The next section will show that this is only true for very slow displacements. When the displacement has a curvature like the lowest figure, the phase change for $N = 10$ will become $\varphi_{ai} = 1.16 \cdot 10^8 d_3$ rad, with $d_3$ the displacement of the third mirror in meters. The other two mirrors remain in position. This means that the phase changes $2\pi$ rad with a displacement of $d_3 \approx 50$ nm. To do accurate measurements the displacement must remain much smaller than that during the complete measurement which can take up to several hours. Because of mechanical stress on the setup and possible temperature changes the displacement can easily drift by several hundreds of nanometers. Therefore an active position control of the relative position of the mirrors is necessary. The method of active stabilization of the atom interferometer mirrors will be discussed in Chapter 4.

3.4 Motions of the atom interferometer

The environmental vibrations impose forces on every instrument. This gives the atom interferometer possible rotations, accelerations and distortions which could have a significant effect on the atom interferometer output signal. In Refs. [7, 8] the sensitivity of atom interferometers for these motions is studied. Before determining the effects of the different motions, first the theory concerning these motions will be studied. After the theoretical study the effects will be calculated for oscillating and noise-like motions of the interferometer. These effects will be calculated for atom interferometers of length 2 and 0.3 meter assuming a tenth order Bragg diffraction ($N = 10$). The velocity of the atoms is 247 m/s, which means that the time of flight through the atom interferometer $T = 1.21$ ms for the 0.3 m and $T = 8.10$ ms for the 2 m atom interferometer. We first analyze the effects of single-frequency oscillating motions, which give insight in
the frequency dependence of the vibrations. Next, we briefly discuss the effect of noise-like motions. This is useful when the motions cover a wide range of frequencies, so that also for these motions types a simple limit can be defined.

3.4.1 Theory of motions

In this theoretical assessment, the equations will be derived needed to calculate the effects of the atom interferometer motions. Because of these motions, the term describing the interaction of the atoms with the standing light wave \( V(x', t) \), changes with the movement of the atom interferometer. Therefore the first step is to translate the time dependent Schrödinger equation to a frame moving with the atom interferometer. This way \( V(x', t) \) becomes stationary. This will be accomplished for each motions type. The next step is to determine the effect of the atom interferometer motions on the Bragg diffraction. The next step is deriving the atom interferometer motion effect equations. These equations will be derived for each motion type. However it will become clear that the three derived equations are identical.

Framework definitions

In this subsection the time dependent Schrödinger equations are transformed into frameworks for accelerating and rotatory motions, making \( V(x', t) \) effectively stationary. This translation will not be needed for the distorting motions.

**Acceleration**

Here the three mirrors move simultaneously. When the movement has a constant velocity, this does not influence the atom interferometer signal. To examine the influence of accelerations on the atom interferometer it is best to move from the laboratory to a reference frame that moves with the atom interferometer-setup. The general time dependent Schrödinger equation for the atoms has the form:

\[
\frac{i\hbar}{\partial t} \frac{\partial \Psi(x', t)}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(x', t) \right) \Psi(x', t).
\]  

(3.35)

The potential \( V(x', t) \) describes the interaction of the atoms with the standing light wave which is influenced by the displacement of the standing light wave (equal to the mirror displacement). By performing the transformation to the frame moving with the atom interferometer, with aid of the substitutions

\[
\begin{align*}
\bar{x}' &\rightarrow \bar{x} + \bar{d}(t) \\
\Psi(\bar{x}', t) &\rightarrow \Phi(\bar{x}, t)e^{i\chi(t) + \Delta \bar{p} \cdot \bar{x}}/\hbar
\end{align*}
\]

(3.36)

with \( \bar{d}(t) \) the displacement and

\[
\chi(t) = \int_0^t \frac{|\Delta \bar{p}|^2}{2m} dt'
\]

\[
\Delta \bar{p} = m \frac{d}{dt} \bar{d}(t)
\]

the time dependent Schrödinger equation transforms into:

\[
\frac{i\hbar}{\partial t} \frac{\partial \Phi(\bar{x}, t)}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + m\bar{a}(t) \cdot \bar{x} + V(\bar{x}, t) \right) \Phi(\bar{x}, t).
\]  

(3.37)

29
where \( \dddot{a}(t) = \frac{d^2 \ddot{a}(t)}{dt^2} \) is the accelerative motion of the atom interferometer and \( m \) the mass of the Helium atoms in the atom interferometer. The potential \( V(\vec{x}, t) \) has now become effectively independent of the accelerative motion of the setup.

**Rotation** For rotations a similar approach is possible. Here it is best to work in a reference frame rotating with atom interferometer-setup such that the standing light wave becomes stationary. To move to this frame, the azimuth angle is substituted by 

\[
\phi' = \phi + \Delta \phi(t), \quad \text{and} \quad \Psi(\vec{x}', t) = \Phi(\vec{x}, t).
\]

(3.38)

\( \Delta \phi \) is the rotational motion of the atom interferometer. The time dependent Schrödinger equation will then be transformed to:

\[
i\hbar \frac{\partial \Phi(\vec{x}, t)}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 - \vec{\Omega}(t) \cdot \vec{L} + V(\vec{x}, t) \right) \Phi(\vec{x}, t),
\]

(3.39)

with \( \vec{\Omega}(t) = (d\Delta \phi(t)/dt)\hat{\epsilon}_\Omega \) the angular velocity vector and \( \vec{L} \) the angular momentum operator. The standing light wave interaction \( V(\vec{x}, t) \) has now effectively become independent of the rotation.

**Distortion** In the situation of distortions, only one standing light wave will move. The other two remain static. To determine the equation to calculate the phase error due to distortions, it is not necessary to define a special framework. These frameworks will be used to derive the equations needed to calculate the phase error induced by the different motions.

**Influence on Bragg scattering**

Before deriving the equations needed to calculate the phase error induced by the different motions, we first have to determine the influence of the three motion types, acceleration, rotation and distortion, on Bragg diffraction in the deflector and beam splitter sections of the atom interferometer. In this study we look at the influence on the Bragg diffraction of the movements the standing light wave can make due to the different motion types. The movements of the standing light wave can be divided into a acceleration \( \vec{a} \), a velocity \( \vec{v} \) and a rotation \( \phi \). The first two are induced by all three atom interferometer motions types while rotation of the standing light wave can only be induced by the atom interferometer’s rotating motion.

**Acceleration of the standing light wave**

To determine the significance of the acceleration of the standing light wave on influencing Bragg diffraction, the time dependent Schrödinger equation in an accelerated frame is used. In this frame the standing light wave has become stationary, and a extra potential term \( m\dddot{a}(t) \cdot \vec{x} \) is added (see Eq. 3.37). The atoms interacting with the standing light wave experience a change in the potential equal to the energy shift \( \Delta E = \hbar \gamma^2 I_1/(2\delta I_s) \) [6] for \( \delta \gg \gamma \) with \( \gamma \) the linewidth of He*, \( I_1 \) is the intensity of the incoming laser beam, \( \delta \) the detuning, and \( I_s \) the saturation intensity. This will then be compared with the change in potential due to \( m\dddot{a}(t) \cdot \vec{x} \).

In our case, the parameters in the energy shift get the values:
resulting in an energy shift $\Delta E = 2.76 \cdot 10^{-27} \text{J}$.

An upper bound for the energy change $\Delta E_a$ due to the term $m\vec{a} \cdot \Delta \vec{x}$ is given by:

$$\Delta E_a \leq m|\vec{a}|w/\cos(\Theta_{10})$$

with $w$ the waist of the laser beam and $\Theta_{10}$ the tenth order Bragg-angle of the atom beam in the atom interferometer. The term $w/\cos(\Theta_{10})$ is the path length of the atom through the standing light wave. When $\Delta E_a \ll \Delta E$ acceleration effects on the Bragg diffraction can be neglected. This is true for $|\vec{a}| \ll 5 \cdot 10^5 \text{m/s}^2$ for a waist $w = 0.85 \text{mm}$ and tenth order of Bragg diffraction. Later it will be shown that the accelerations, the standing light wave is allowed to make, are much smaller than this. It can therefore be assumed that the effects of these accelerations can be neglected within allowed acceleration limits.

Velocity of the standing light wave A constant velocity of the standing light wave changes because of the Doppler shift the detuning of the laser beam with $\Delta \delta = v_x k_w$. This detuning change directly influences the Bragg diffraction. This is equivalent to changing the laser intensity with $I \Delta \delta / \delta [2]$, which effects the diffraction ratio (see Figure 2.3). Therefore it is advisable that the relative change of the intensity $\Delta I/I = \Delta \delta / \delta \ll 10^{-3}$. The Doppler shift change should then not be more than 0.1 MHz, so it is required that $\Delta \delta \ll 0.2 \pi \text{Mrad/s}$ which means that $v_x \ll 0.1 \text{m/s}$. This is also be much larger than the allowed velocity of the standing light wave.

Because of the constant velocity the standing light wave will experience the atom beam as slightly rotated, resulting in a deviation from the Bragg-angle $\Delta \Theta_{\text{bragg}} = v_x / v_{\text{atom}}$. To be able to neglect this effect it is required that $\Delta \Theta < 1 \mu \text{rad}$ (see Figure 3.2), resulting in the requirement $v_x < 250 \mu \text{m/s}$, which is still much larger than the allowed standing light wave velocity.

This shows that within the allowed velocity limits, the effects of the standing light wave velocity are negligible.

Rotation of the standing light wave Rotation of the standing light wave relative to the atom beam results directly in a deviation from the Bragg-angle $\Delta \Theta_{\text{bragg}} = \phi$. A rotation is directly translated to a displacement by the relation $d = \phi L/2$. Using the same requirement as before $\Delta \Theta_{\text{bragg}} = \phi < 1 \mu \text{rad}$, the displacement of the standing light wave $d < 150 \mu \text{m}$ for a 0.3 m atom interferometer. The allowed limit of the displacement due to rotations is only larger than this for frequencies $f < 1/100 \text{Hz}$ (this will be shown later). The optical setup of the atom interferometer will be kept aligned with the atom beam. This means that the displacements in relation to the atom beam (due to the rotations of the setup) with frequencies up until a few Herz, will be kept smaller than the allowed limit.

Within the limit of the maximum allowed rotation velocity of the atom interferometer, the standing light wave rotates less than 0.012 $\mu \text{rad}$, during the time the atom travels through the atom interferometer. This will not be noticed by the Bragg diffraction.

The conclusion can be made that, within the allowed limits of movements of the atom interferometer, the influence of atom interferometer motions on the splitting ratio of Bragg diffraction is negligible. Also can be stated that the momentum change of the atoms due
to Bragg diffraction is not influenced by the atom interferometer motions.

**Motion induced phase error**

In this paragraph the equations are derived needed to calculate the phase error induced by the different motions. They will be derived for each motion individually. The end result will be one general equation, that can be used for the phase error calculations of the three motion types.

**Acceleration induced phase error** The starting point for the phase difference calculations will be an atom interferometer without misalignments of the standing light waves. In section 2.3 it was shown that when there are no vibrations the difference in momenta between the transmitted and the diffracted atom beams in the atom interferometer $\vec{p}_D - \vec{p}_T = 2N\hbar\vec{k}_s$ with $N$ the order of the Bragg diffraction (see Figure 3.10) and $\vec{k}_s$ the wave vector of the standing light field which is perpendicular to the mirror surface, parallel to the x-axis. The momenta in the other directions will not be influenced.

The previous paragraph showed that the acceleration effects on the splitting ratio of Bragg diffraction can be neglected. Due to the potential given by $m\ddot{a} \cdot \vec{x}$ from Eq. 3.37 the velocity of the atom in the moving frame will change, but for small deviations from the Bragg condition it can be assumed that the momentum change due to Bragg diffraction itself will not. Therefore it can be stated that the momentum change due to the potential $m\ddot{a} \cdot \vec{x}$, $\Delta \vec{p} = m\dot{d}(t)/\dot{t}$ is independent of the position of the atom in the atom interferometer.

The atoms in the laboratory frame don't experience any vibrations and therefore the phase factor of the plane wave equation in the laboratory frame is

$$\varphi(t) = \frac{1}{\hbar} \left[ \vec{p}_j \cdot \vec{x}' - \frac{|\vec{p}_j|^2}{2m} t \right]$$

with $j = I, II, III, IV$, the roman numbers for the different path sections as indicated in Figure 3.10. This phase factor can easily be transformed to the phase factor for the accelerated frame by using the substitution equations of Eq. 3.36 with the result:

$$\varphi(t) = \frac{1}{\hbar} \left[ \vec{p}_j \cdot (\vec{x} + \vec{d}(t)) - \Delta \vec{p}(t) \cdot \vec{x} - \frac{|\vec{p}_j|^2}{2m} t - \chi(t) \right] $$

$$= \frac{1}{\hbar} \left[ (\vec{p}_j - \Delta \vec{p}(t)) \cdot \vec{x} - \int \frac{|\vec{p}_j - \Delta \vec{p}(t)|^2}{2m} dt \right]$$

The interaction of the atom with the standing light wave will be very short compared to the rate of change of this phase. The interaction time is about $3.5\mu s$, which makes it possible to assume the displacement, due to vibrations with frequencies far lower than $300\,kHz$, constant in time during the interaction. The frequencies of interest are much smaller than that.

The widths of the atom beams are much larger than the atoms' de Broglie wavelengths, and therefore the atom beam can be treated as a plane wave. To simplify the calculation the width of the atom beam can be expanded to infinity such that the atom beams always overlap each other. This effectively eliminates the need for positions on the standing light wave in the calculations. For the calculation only the phase changes due to the motion are of interest. Thus all the terms in Eq. 3.41 without $\Delta \vec{p}(t)$ or $\ddot{d}(t)$ can be eliminated. The term $\chi(t)$ can be omitted because it is independent of the paths in the atom interferometer and therefore does not result in a phase difference between the two paths. $\Delta \vec{p} \cdot \vec{x}/\hbar$ also
Figure 3.10: A schematic setup of the atom interferometer. For calculations the atom interferometer can be divided into four section indicated with roman numbers. A distinction is made between the upper and the lower path because the phase difference of these two paths will be measured. \( t_1, t_2 \) and \( t_3 \) indicate moments in the time the atom interacts with the three standing light waves and \( \Omega \) indicates the angular velocity of the atom interferometer.

drops out because at the third standing light wave only the overlapping parts of the atom beam will interfere and \( \Delta \phi \) is the same for both paths. What remains is the term \( \vec{p}_j \cdot \vec{d}(t)/\hbar \) which is independent of \( \vec{x} \). The resulting phase change caused by the acceleration in the time the atom travels from standing light wave 1 to 2 or 2 to 3 becomes:

\[
\Delta \phi_{t_{2/3}}^{t_{1/2}} = \frac{\vec{p}_j \cdot [\vec{d}(t_{2/3}) - \vec{d}(t_{1/2})]}{\hbar} \quad j = \{I, II, III, IV\}.
\]

(3.42)

In total there are four path sections, two for each arm of the atom interferometer as shown in Figure 3.10. By subtracting the phase change of the upper path from that of the lower path, the total phase change due to the accelerations becomes:

\[
\Delta \phi_{\text{acc}} = \frac{1}{\hbar} \left\{ \left[ \vec{p}_{\text{III}} - \vec{p}_i \right] \cdot \left[ \vec{d}(t_2) - \vec{d}(t_1) \right] + \left[ \vec{p}_{\text{IV}} - \vec{p}_i \right] \cdot \left[ \vec{d}(t_3) - \vec{d}(t_2) \right] \right\}
\]

(3.43)

\[
= 2Nk s \cdot \left[ \vec{d}(t_1) - 2\vec{d}(t_2) + \vec{d}(t_3) \right] = \frac{4\pi N}{\lambda_s} \left[ dx(t_1) - 2dx(t_2) + dx(t_3) \right],
\]

where \( t_1, t_2 \) and \( t_3 \) are the times when the atom interacted with standing light waves 1, 2 and 3. The last approximation in Eq. 3.43 is possible because the previous sections showed that the allowed misalignments of the mirrors are very small. This shows it is sufficient to know the displacement in the x-direction of the atom interferometer at the moment the atom interacts with the standing light waves. Comparing Eq. 3.43 with Eq. 3.34 reveals that the atom experiences the atom interferometer as being distorted because of the accelerative motion of the setup.

**Rotation induced phase error** A rotation induced phase error in an atom interferometer is better known as the Sagnac effect. Clauser [7] showed that the influence of the rotation on the phase shift is:

\[
\varphi_{\text{rot}} = \frac{2mO_{\text{at}} \Omega}{\hbar},
\]

(3.44)
where \( m \) is the mass of the atoms used, \( O_{ai} \) the area enclosed by the two arms of the atom interferometer, and \( \Omega \) the atom interferometer’s rotation velocity. This equation is valid for constant rotation velocities. It is inadequate for calculating the effects of vibrations, where the rotation velocity varies.

To determine the effects of rotatory vibrations, it is possible to use the same approach as for accelerations. The starting point is again the phase factor of the plane wave in the laboratory frame which was given by (see Eq. 3.40)

\[
\varphi(t) = \frac{1}{\hbar} \left[ \vec{p}_j \cdot \vec{r}' - \frac{|\vec{p}_j|^2}{2m} t \right] \cdot j = \{I, II, III, IV\}. \tag{3.45}
\]

By using the substitution from Eq. 3.38 the phase factor of the wave function in the rotated frame transforms to:

\[
\varphi(t) = \frac{1}{\hbar} \left[ \vec{p}_j \cdot [\vec{r} + \Delta \vec{r}(t)] - \frac{|\vec{p}_j|^2}{2m} t \right] \quad j = \{I, II, III, IV\} \tag{3.46}
\]

with \( \Delta \vec{r}(t) \approx \vec{r} \times \dot{\vec{e}}_\Omega \Delta \phi(t) \) where \( \dot{\vec{e}}_\Omega \) indicates the direction of the rotation vector and \( \Delta \vec{r}(t) \) the displacement due to rotation. The arguments for the acceleration are also valid here. For those reasons the position on the standing light wave does not play any role in the calculation. For this calculation only the phase changes due to the rotational motion are of interest. So all the terms in Eq. 3.41 without \( \Delta \vec{r}(t) \) can be omitted. This eliminates all but one term. So the resulting phase difference due to the rotation while the atom is traveling from standing light wave 1 to 2 or 2 to 3 then becomes the same as Eq. 3.42

\[
\Delta \varphi_{t_{2/3}} = \frac{1}{\hbar} \left[ \vec{p}_j \cdot \Delta \vec{r}_{2/3}(t_{2/3}) - \Delta \vec{r}_{1/2}(t_{1/2}) \right] \quad j = \{I, II, III, IV\} \tag{3.47}
\]

where \( \Delta \vec{r}_n(t_n) \) \((n = \{1, 2, 3\})\) is the displacement of standing light wave \( n \) in the laboratory frame. To calculate the total phase difference the same procedure has to be followed here as for the acceleration. So the subtraction of the phase change of the upper path from that of the lower path leads to:

\[
\Delta \varphi_{\text{rot}} = \frac{1}{\hbar} \left[ \Delta \vec{r}_1(t_1) \cdot [\vec{p}_l - \vec{p}_u] + \Delta \vec{r}_{2,u}(t_{2,u}) \cdot [\vec{p}_u_1 - \vec{p}_v] + \Delta \vec{r}_{2,l}(t_{2,l}) \cdot [\vec{p}_u - \vec{p}_v] + \Delta \vec{r}_{3}(t_3) \cdot [\vec{p}_v - \vec{p}_u] \right] \tag{3.48}
\]

where \( t_1, t_{2,u}, t_{2,l}, \) and \( t_3 \) are the times the atom interacted with standing light waves 1, 2, and 3. Here a distinction is made between the position of the upper and the lower paths, indicated with indices \( u \) and \( l \), on the second standing light wave. For the acceleration this was not necessary because the standing light waves were displaced in the direction of the laser beam with the result that the time of interaction of the atom with the standing light waves did not change. For the rotation the standing light waves rotate which results in a position difference of the standing light waves in the direction of the atom beams for the two paths in the atom interferometer, resulting in two different times of interaction at the second standing light wave. However the allowed angular displacement is small enough to make the assumption that the two interaction times with the second standing are the same. Using this the phase change due to the rotation results basically in the same expression as Eq. 3.43:

\[
\Delta \varphi_{\text{rot}} \approx \frac{4\pi N}{\lambda_s} \left[ d_{x,1}(t_1) - 2d_{x,2}(t_2) + d_{x,3}(t_3) \right]. \tag{3.49}
\]
The expressions \(d_{x,1}(t_1), d_{x,2}(t_2)\) and \(d_{x,3}(t_3)\) are the displacements of mirrors 1, 2 and 3 at the times the atom interacts with the specific standing light waves. In this equation the center of rotation can be taken completely arbitrarily. The motion can be divided into a rotation, with the center in the middle of the atom interferometer as indicated in Figure 3.10 and an acceleration, which was already discussed in the previous paragraph. Therefore in the next paragraph when the influence of rotations on the atom interferometer output signal is discussed, the center of rotation will be placed in the center of the atom interferometer. This means that the displacement \(d_{x,2}(t) = 0\).

From Eq. 3.49 the effect of a constant rotation velocity is derived by taking:

\[
\begin{align*}
\dot{d}_{x,1}(t) &= -\frac{\Omega L}{2} t, \\
\dot{d}_{x,2}(t) &= 0, \quad \text{and} \\
\dot{d}_{x,3}(t) &= \frac{\Omega L}{2} t
\end{align*}
\]

where \(L\) is the length of the atom interferometer. The atom passes the first standing light wave at \(t_1 = 0\) and the last standing light wave at \(t_3 = \frac{L}{v_z}\) where \(v_z\) is the velocity of the atom in the z-direction. Also the surface area enclosed by the two arm of the atom interferometer \(O_{ai} = \pi N L^2/(m \lambda_s v_z)\). Using these times and Eq. 3.50 in Eq. 3.49 results in:

\[
\Delta \phi_{\text{rot}} = \frac{4\pi N L^2}{\lambda_s v_z} \Omega = \frac{2m O_{ai} \Omega}{h},
\]

which is equal to Eq. 3.44.

**Distortion induced phase error** Distortion is defined as a motion where only one standing light wave moves. The movements of the standing light wave has a negligible influence on the splitting ratio of Bragg diffraction. The momentum change induced by this process is also not influenced. Therefore it is possible to directly use Eq. 3.34 to determine the phase error induced by a distortion in the atom interferometer. Thus

\[
\Delta \phi_{\text{dist}} = \frac{4\pi N}{\lambda_s} \left[ d_{x,1}(t_1) - 2d_{x,2}(t_2) + d_{x,3}(t_3) \right],
\]

\[
\Delta \phi = \Delta \phi_{\text{rot}}
\]

**Result** The conclusion is that for the determination of the phase error induced by the three motion types, only one equation has to be defined which is valid for all three motion types. This phase error equation is:

\[
\Delta \phi = \frac{4\pi N}{\lambda_s} \left[ d_{x,1}(t_1) - 2d_{x,2}(t_2) + d_{x,3}(t_3) \right].
\]

This means that the three motion types can be examined together.

### 3.4.2 Oscillating motion

A movement of the atom interferometer means that the mirrors and hence the standing light waves, are moving. The atoms themselves perform a free flight when not interacting
with the standing light waves. The only force these atoms experience is the gravitational force which is perpendicular to the plane spanned by the two arms of the atom interferometer (i.e., in the y-direction), causing the atoms to move down slightly during the time of flight \( T \), in the atom interferometer-setup. An estimate of the minimum deviations from a straight line is \( \Delta y = gT^2/8 = gL^2/(8v_0^2) = 1.8 \mu m \) for the 0.3 m and \( \Delta y = 80 \mu m \) for the 2 m atom interferometer. Because of this drop the energy of the atom increases and therefore its phase changes. This happens in both arms of the atom interferometer, so that effectively the phase difference between the two arms will not change. This is also true for all kinds of disturbances in the y-direction and the z-direction.

Oscillating motions can be divided into three displacement types (see Figure 3.9). One type is an oscillating motion in which all mirrors move simultaneously in phase with equal amplitude, which is called the accelerating motion. In the second motion the two outer mirrors move in the opposite direction, while the middle mirror stands still. This is called the rotational motion. In the third type of motion only one of the mirrors moves. This can happen if the system is distorted.

The assessment in the previous subsection showed that for the calculation of the error in the interferometer output signal, the same equation can be used for all three types of motion. For each standing light wave the displacement has to be determined, which corresponds to the displacement of the mirrors, at the moment the atom passes it. Because the atom needs time \( T/2 \) to travel to the next standing light wave, that even though in some situations the mirrors are moving such that they stay in a straight line, the atoms experience a different displacement of these mirrors.

In this study we give the mirrors (with the standing light wave) that move, an oscillating motion described by \( D(t) = D_0 \cos(\omega t) \), where \( D_0 \) is the amplitude and \( \omega \) the oscillating frequency of the mirrors motion. Because of the limited velocity of the atoms, these atoms reach the position of the different mirrors at different times \( 0, T/2 \) and \( T \) with \( T \) the time of flight through the atom interferometer. At the moment the atom reaches the standing light waves they will have, from the point of view of the atom, the following displacements for the three different motions:

<table>
<thead>
<tr>
<th>Mirror position</th>
<th>Motion</th>
<th>Acceleration</th>
<th>Rotation</th>
<th>Distortion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_{x,1}(t_1) = D(t_0) )</td>
<td>( D(t_0) )</td>
<td>( D(t_0) )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( d_{x,2}(t_2) = D(t_0 + T/2) )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( d_{x,3}(t_3) = D(t_0 + T) )</td>
<td>( -D(t_0 + T) )</td>
<td>( D(t_0 + T) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where \( t_0 \) is the time at which the atom entered the atom interferometer. In the previous paragraphs we already determined the phase changes due to the displacement of the mirrors. Filling in the expressions for \( d_{x,1}(t_1) \), \( d_{x,2}(t_2) \) and \( d_{x,3}(t_3) \) in Eq. 3.54 the expressions for the phase changes become:

\[
\varphi_{ac} = \frac{8\pi ND_0}{\lambda_s} \left[ \cos(\omega T/2) - 1 \right] \cos(\omega(T/2 + t_0)), \tag{3.56}
\]

\[
\varphi_{ro} = \frac{8\pi ND_0}{\lambda_s} \sin(\omega T/2) \sin(\omega(T/2 + t_0)), \tag{3.57}
\]

\[
\varphi_{di} = \frac{4\pi ND_0}{\lambda_s} \cos(\omega(T + t_0)). \tag{3.58}
\]

Eqs. 3.56 and 3.57 show that at certain frequencies the time independent amplitude of these equations goes to 0. This means that at \( t = T/2 \) and \( t = T \) the mirrors return
to their original position of $t = 0$. The resulting visibility (derived in appendix B) can be calculated by:

$$V_{A/B} = \left(1 - \frac{A^2}{8}\right)^2,$$

with $A$ that part of either Eq. 3.56, Eq. 3.57 or Eq. 3.58, that is independent of $t_0$. For small values of $A$ the visibility scales with this amplitude quadratically. To get a minimum visibility of $V_{\text{min}}$, it is required that $A^2 < 8(1 - \sqrt{V_{\text{min}}})$. So for $V > 0.99$ is required that $A < 0.2$. Damping the vibrations is difficult for low frequencies. For these low frequencies the amplitudes of the first two motions types can be approximated by:

$$A_{\text{acc}} \approx \frac{\pi ND_0 \omega^2 T^2}{\lambda_s} = \frac{\pi N a_0 T^2}{\lambda_s} = \frac{m O_{\text{air}} a_0}{\hbar}, \quad (3.60)$$

$$A_{\text{rot}} \approx \frac{4\pi ND_0 \omega T}{\lambda_s} = \frac{4\pi N v_0 T}{\lambda_s} = \frac{2m O_{\text{air}} \Omega}{\hbar}. \quad (3.61)$$

with $a_0$ the amplitude of the acceleration, because $a(t) = d^2 D(t)/dt^2 = -\omega^2 D(t)$, and $v_0$ the amplitude of the velocity because the velocity $v(t) = \partial D(t)/\partial t = \omega D(t)$. Here the relations $v_0 = \Omega L/2$ and $O_{\text{air}} = \pi \hbar N L^2/(m \lambda_s v_2)$ are used. From Eqs. 3.60 and 3.61 it is clear that rotational oscillations are dominant for $\omega < 4/T$ when the displacement amplitudes are the same. The amplitude of the distorting motion remains the same. The requirement $A < 0.2$ results in the following requirements for the different motions:

<table>
<thead>
<tr>
<th>Motion type</th>
<th>Parameter</th>
<th>Dimension</th>
<th>0.3m</th>
<th>2m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acceleration</td>
<td>$a_0$</td>
<td>$\mu$m/s$^2$</td>
<td>$&lt; 4670$</td>
<td>$&lt; 105$</td>
</tr>
<tr>
<td>Rotation</td>
<td>$\Omega$</td>
<td>$\mu$rad/s</td>
<td>$&lt; 9.47$</td>
<td>$&lt; 0.21$</td>
</tr>
<tr>
<td>Distortion</td>
<td>$D_0$</td>
<td>nm</td>
<td>$&lt; 1.72$</td>
<td></td>
</tr>
</tbody>
</table>

As long as the measurement time $\tau$ is much longer than the oscillation period $1/\omega$, the measurement error becomes very small because the signal error is averaged out during the measurement. This is however not the case when $\omega \tau < 1$. It can be derived (see appendix B) that

$$\varphi_{\text{error}} < A \frac{\sin(\omega \tau/2)}{\omega \tau/2}.$$

A typical measurement time is in the order of minutes. For these long measurement times the expression $\sin(\omega \tau/2)/(\omega \tau/2)$ becomes very small for even very low frequencies. If it is required to have an error of $\varphi_{\text{error}} < 1/1000$, then $\omega \tau > 40$ when using the previous requirement $A < 0.2$. For $\tau = 60$s all frequencies $\omega < 2\pi$rad/s must have a smaller acceleration, velocity or a displacement amplitude than previously indicated. A complete picture is given in Figure 3.11. To be able to make a comparison in these graphs, the frequency dependent oscillation limits are translated to an amplitude limit of the displacement(limit of $D_0$) of the last mirror. In the graphs, four regions can be distinguished by the sharp bend in the otherwise straight line. The frequencies at the
Figure 3.11: This figure shows the limits of the three different motions as function of the frequency for the 0.3 m and the 2 m interferometer. The graphs show a change in the frequency behavior at the frequencies indicated with \( f_1 \), \( f_2 \), \( f_3 \) and \( f_4 \). The first two \( f_1 \) and \( f_2 \) are correlated with the measurement time. The third \( f_3 \) belongs to the rotational and the fourth \( f_4 \) to the accelerative motion. These graphs are calculated for \( A_{\text{sys}} = 0.2 \), \( \varphi_{\text{error}} = 1/1000 \), \( u_{\text{atom}} = 247 \, \text{m/s} \) and \( \tau = 60 \, \text{s} \).
bends as indicated in Figure 3.11 are related by

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1 )</td>
<td>( \frac{2}{\tau} )</td>
</tr>
<tr>
<td>( f_2 )</td>
<td>( 2A_{vis}/(\varphi_{error}\tau) )</td>
</tr>
<tr>
<td>( f_3 )</td>
<td>( \frac{2}{T} )</td>
</tr>
<tr>
<td>( f_4 )</td>
<td>( \frac{4}{T} )</td>
</tr>
</tbody>
</table>

The value of \( A_{vis} \) is the maximum allowed value of \( A \) for a minimum required visibility \( V \). For Figure 3.11 \( A_{vis} = 0.2 \). If the vibration frequency is lower than \( f_1 \), the signal error will not be averaged out because the signal error hardly varies during the measurement. The averaging is effective if the measurement time is longer than the period of the vibration decreasing the sensitivity of the measurement for increasing vibration frequencies. At frequencies \( \omega = 2\pi M/T \text{ rad/s} \) with \( M \) an integer, the sensitivity can even become zero because of averaging over an integer number of oscillation cycles of the motion. This is not shown in Figure 3.11 because the graphs only show the outline of the upper limits. After frequency \( f_2 \) the requirements on the visibility becomes dominant over those of the measurement error. For increasing frequencies the velocity and the acceleration grow because these are the first and the second time derivatives of the displacement. This means that the sensitivity of the visibility grows with increasing frequency. The rotation and acceleration induced error reach their maximum at frequencies \( f_3 \) and \( f_4 \). In these regions it was shown before that at certain frequencies \( \omega = 4\pi M/T \text{ rad/s} \) for accelerations and \( \omega = 2\pi M/T \text{ rad/s} \) for rotations) the induced error becomes zero. Again only the outline of the limits are given Figure 3.11.

### 3.4.3 Noise analysis

The analysis of random types of motion, which is called noise, needs a slightly different approach (see appendix B). It will be assumed that the measurement time is much longer than the time of flight of the atom through the atom interferometer, so that higher frequencies have little influence on the measurement. Therefore the different motions can be assumed constant over the time of flight. The effect of the different motions on the change in the phase difference between the two arms of the atom interferometer are given in the previous paragraph by Eqs. 3.60, 3.61 and 3.58. The expression for the visibility is given by equation (see appendix B)

\[
V = e^{-\frac{\sigma_i^2}{2}} \approx 1 - \frac{\sigma_i^2}{2}
\]

with \( \sigma_i \) the standard deviation of either \( A_{acc}(a_0) \), \( A_{rot}(v_0) \) or \( A_{dist}(D_0) \). Comparing Eqs. 3.59 and 3.65 shows that for small values of \( A \) and \( \sigma_i \) these two parameters are related by \( \sigma_i \approx A/\sqrt{2} \). To get the limits for the noise, the values of Eq. 3.62 can be used.

### 3.4.4 Preliminary vibration measurements QND-setup

To get an impression of how large the vibrations are, measurements were carried out on the present setup with an acceleration probe. The results of the measurements are given in Figure 3.12. The amplitude of the vibrations \( a_{ vib} \approx 60 \text{ mm/s}^2 \). The vibrations are dominated by the 20 Hz component and can therefore be approximated by a sinusoidal signal.
The measurement does not show whether the motion is an acceleration or a rotation so also the requirements for the rotation has to be considered. The rotation velocity can be calculated by \( \Omega_{\text{vib}} = 2a_{\text{vib}}/(\omega_{\text{vib}}L) = 3.2 \text{ mrad/s} (L = 0.3 \text{ m}) \) or \( 0.48 \text{ mrad/s} (L = 2 \text{ m}) \). Comparing both vibration values \( a_{\text{vib}} \) and \( v_{\text{vib}} \) with the values in Eq. 3.62 shows that they are too large. In the next table the measured vibrations and the requirements are compared:

<table>
<thead>
<tr>
<th>Motion type</th>
<th>Parameter</th>
<th>atom interferometer</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.3 m</td>
</tr>
<tr>
<td>Acceleration</td>
<td>( a_{\text{disturb}}/a_{0,\text{max}} )</td>
<td>13</td>
</tr>
<tr>
<td>Rotation</td>
<td>( v_{\text{disturb}}/v_{0,\text{max}} )</td>
<td>340</td>
</tr>
</tbody>
</table>

This comparison shows that for the 2 m atom interferometer more attention has to be paid to vibration isolation. The vibrations measured are caused by the mechanical resonances in the QND-setup, which amplify the 10 and 20 Hz oscillations tremendously. The problem becomes less serious if the mechanical resonances in the QND-setup could be eliminated. The measured background noise level already suggests that when these resonances are absent the 0.3 m atom interferometer just needs little vibration damping, (reduction factor > 5) although this background noise is probably caused by the detector itself. For the 2 m atom interferometer a more sensitive detector is needed to be able to draw any conclusions. The background noise level of the 2 m atom interferometer has to be at least 6 times lower than required for the 0.3 m atom interferometer.

### 3.5 Magnetic fields

One of the characteristics of He* is that it has a magnetic dipole making it susceptible to magnetic fields. The magnetic dipole is generated by the alignment of the electron spins along the same direction. This creates an atomic triplet state indicated by \( ^3S_1 \). In a magnetic field \( B \) the energy levels are Zeeman-shifted \([10]\) by an amount:

\[
\Delta E_{\text{zm}} = \mu_\text{e} g_\text{i} B m_\text{j},
\]

where \( \Delta E_{\text{zm}} \) is called the Zeeman shift, \( \mu_\text{e} \) the Bohr magneton and \( g_\text{i} = 2 \) the Landé factor. In the atom interferometer the state \( |J,m_\text{j}\rangle = |1,-1\rangle \) is used as explained in chapter 2. It is assumed that the magnetic dipole keeps its alignment with the magnetic field during its flight in the interferometer. This is true if the magnetic dipole can follow the magnetic field adiabatically. That means that the adiabatic condition \([10]\) must be met:

\[
\frac{1}{E} \frac{\partial E}{\partial t} \ll \frac{\Lambda_n}{\hbar} \quad \text{which results in}
\]

\[
\frac{\partial B}{\partial t} \ll \frac{B^2 \mu_\text{e} g_\text{i}}{\hbar}
\]

Here \( \Lambda_n \) is the energy difference between two neighboring magnetic substates of the \( ^3S_1 \) state. The condition is valid from the atom’s point of view. That means that an inhomogeneous field will result in a time varying field because of the velocity of the atom. To keep the atoms magnetically polarized a static homogeneous magnetic field \( B_0 \) will be applied. Time independent variation (position dependent) on this magnetic field from the point of view of the atoms can not violate the adiabatic condition, a time varying one can. To obey the adiabatic condition it is required that the amplitude of the magnetic field variation
Figure 3.12: The vibrations measured at the present QND-setup. The graphs give the vibrations in the x-direction. This is the direction the atom interferometer is most sensitive of. The upper graph is the frequency spectrum of the vibrations up until 400 Hz. It shows that there are two dominant frequencies present. The lower graphs shows the time plot of the vibrations. Here two lines are drawn which indicate the amplitude of the vibrations. All signals with frequencies higher then 400 Hz are filtered out during the measurement.
\( \Delta B \) meets the following conditions:

\[
\Delta B \ll \frac{B_0}{2},
\]

and

\[
(\omega + \vec{v} \cdot \nabla) \Delta B \ll \frac{B_0^2 \mu_0 g_I}{\hbar}
\]

with \( B_0 \) the strength of the homogenous static magnetic field. Only for frequencies \( \omega > 2\pi \cdot 5.6 \cdot 10^6 B_0 \text{rad/s} \) or a wavelength of the gradient \( \lambda_{\nabla B} < 44/B_0 \text{µm} \) a field variation starts to play a role. This means that by applying a magnetic field, which does not have to be large, the adiabatic condition is easily kept. This makes the calculations a lot easier because the direction of the magnetic field can now be neglected, and only the magnitude of the magnetic field is of importance.

### 3.5.1 Magnetic field variations

The inquiry will be divided into a static and a time varying magnetic field, both in relation to the laboratory frame. It will be shown that a time varying inhomogeneous magnetic field will influence the measurement and a maximum allowed field variation will be given. A inhomogeneous static magnetic field, will only in combination with velocity variations introduce some errors in the measured signal. Without such velocity variation this static inhomogeneous magnetic field will only create an offset which can easily be compensated for. The induced errors are averaged over a measurement time \( T \). This can reduce the sensitivity of the atom interferometer to magnetic field variations for higher frequencies considerably. To gain insight in the frequency dependence of this sensitivity, the time varying magnetic field is assumed to oscillate with a fixed frequency \( \omega \).

In general the phase difference between the two arms of the atom interferometer depends on the energy difference of the atom states in these two arms:

\[
\Delta \varphi_{ai} = \int_0^T \frac{E_1(t) - E_2(t)}{\hbar} \text{dt},
\]

with \( T \) the time of flight through the atom interferometer. Two considerations can be taken into account. First the only energy difference of interest is the Zeeman energy shift due to the magnetic field shown by Eq. 3.67, and second the atom is localized so that it only experiences the magnetic field at a small region at the atom’s position, justifying the substitution \( t = z/v_z \).

The combination of these two considerations results in:

\[
\Delta \varphi_{ai} = \frac{m_1 \mu_0 g_I}{\hbar v_z} \int_0^L [B_1(z) - B_2(z)] \text{dz}.
\]

\( B_1(z) \) and \( B_2(z) \) are the two magnetic fields experienced by the atoms in the two different arms. The general description of the magnetic field used in the following calculation is a Taylor sequence for the magnetic field which is a function of \( x, y, z \) and \( t \). The magnetic field then gets the form:

\[
B(\vec{x}, t) = B_0 + (\vec{x} \cdot \nabla) B + \frac{\partial B}{\partial t} \frac{z}{v_z} + O(x^2).
\]

For this inquiry a first order approximation in \( \vec{x} \) is sufficient to get a good impression of the limits of the magnetic field variations. This means that in principle the gradient of
the magnetic field is taken independent of $\vec{x}$. Then the simplified magnetic field gets the form:

$$B(\vec{x}) = B_0(t) + [(\vec{x} \cdot \nabla)B]_{st} + [(\vec{x} \cdot \nabla)B]_{osc} \sin(\omega z/v_z + \omega t_0 + \varphi(z)).$$

(3.74)

The phase $\omega t_0$ is the time dependent phase at the moment the atom enters the atom interferometer, and $\varphi(z)$ is the position dependent phase. The oscillating magnetic field gradient is in principle no longer independent of the $z$-direction. $\varphi(z)$ is added because from the point of view of the atom the magnetic field becomes via substitution $z' \rightarrow z - v_z t$:

$$B(\vec{x}) = B_0(t) + [(\vec{x} \cdot \nabla)B]_{st} + [(\vec{x} \cdot \nabla)B]_{osc} \sin(\omega(z'/v_z + t) + \omega t_0 + \varphi(z'/v_z t)).$$

(3.75)

Because of the motion of the atom in the atom interferometer the position dependent phase has also become time dependent. The oscillation frequency the atom experiences changes. It is even conceivable that the atom experiences a static magnetic field gradient when $\omega(z'/v_z + t) + \varphi(z'/v_z t)$ is constant. This would be the most pessimistic situation. When this is not a constant, the effects of the time varying magnetic field gradient will become smaller because it will be averaged over the flight time of the atom through the atom interferometer. To determine the upper limit of the allowed magnetic field variations we will use the most pessimistic scenario, thus $\omega(z'/v_z + t) + \varphi(z'/v_z t)$ is constant. The magnetic field in the laboratory frame for the most pessimistic approach would then become:

$$B(\vec{x}) = B_0(t) + [(\vec{x} \cdot \nabla)B]_{st} + [(\vec{x} \cdot \nabla)B]_{osc} \sin(\omega t_0).$$

(3.76)

The atom beam used in the experiments is very narrow ($\approx 50 \mu m$) compared to the displacement of the diffracted beams in the atom interferometer ($> 1$ mm). Therefore the atoms can also considered to be very localized in the x- and the y-direction. The atoms travels in a straight line which simply means that the coordinates $x$ and $y$ are linear functions of $z$. First $z_1 = z_2 = z$, $y_1,2(z) = 0$ and $x_1(z) = -x_2(z) = x(z)$ with

$$x(z) = \begin{cases} 
  z \sin \Theta & 0 \leq z \leq L/2, \\
  (L - z) \sin \Theta & L/2 < z \leq L,
\end{cases}$$

where $\Theta$ is the Bragg-angle of the transmitted and the deflected atom beam (the two arms of the atom interferometer). Using this and Eq. 3.74 to substitute $B_1(z)$ and $B_2(z)$ in Eq. 3.72, and solving the integration, results in:

$$\Delta \varphi_{ai} = \frac{m_1 \mu g_1}{\hbar v_z} \frac{O_{ai}}{\partial x} \left\{ \frac{\partial B_{st}}{\partial x} + \frac{\partial B_{osc}}{\partial x} \sin(\omega t_0) \right\}. \tag{3.77}$$

with $O_{ai} = L^2 \tan(\Theta)/2$ the surface area enclosed by the two arms of the atom interferometer. The effects of the static and the oscillating part will be studied separately.

### 3.5.2 Oscillating magnetic field

An oscillating magnetic field gradient results in an oscillating error in the measurement signal of the atom interferometer with an amplitude given by (see Eq. 3.77):

$$A_{osc} = \frac{m_1 \mu g_1}{\hbar v_z} \frac{O_{ai}}{\partial x} \frac{\partial B_{osc}}{\partial x}. \tag{3.78}$$
In section 3.4 the effect of an oscillating phase error was already investigated. The phase error due to the oscillating magnetic field has the same effect on the measurements. So the same formulas can be applied here. In section 3.4 it was shown that it is required that $A < 0.2$ for a visibility $V > 0.99$. Using the same parameters as before for the 0.3 m and the 2 m atom interferometer the requirements for the magnetic field gradient become:

$$\frac{\partial B_{osc}}{\partial x} < \begin{cases} 17 \text{ mG/m} & 0.3 \text{ m at.int.}, \\ 0.38 \text{ mG/m} & 2 \text{ m at.int.} \end{cases} \quad (3.79)$$

For a measurement error $\varphi_{error} < 1/1000$ and using the same measurement time it was calculated that only for $\omega < 2\pi \text{ rad/s}$ the signal error requirement becomes dominant. The complete picture is given in Figure 3.13. The bends in this graph coincides with the the frequencies $f_1$ and $f_2$ from Figure 3.11. Their relations are given in Eq. 3.64.

### 3.5.3 Static magnetic field gradient

Previously the static part of Eq. 3.77 showed that the phase change is dependent on the time of flight, the magnetic field gradient and the surface area enclosed by the two arms of the atom interferometer. As long as these three parameters remain constant, it causes a constant phase shift which can easily be compensated. Two parameters, the enclosed surface area and the time of flight, will be altered when the velocity of the atom through the atom interferometer changes. A transversal velocity change has, due to the small Bragg-angles, a negligible influence on these parameters, compared to the influence of the longitudinal velocity change. The phase shift between the two arms of the atom
The phase shift due to the velocity change is given by \( \Delta \varphi \approx \frac{2m_1 \mu_B g_1}{h v_s} O_{ai} \frac{\Delta v \cdot \vec{B}_{st}}{v} \). Because of the change in the longitudinal velocity, the angle of the deflected beam alters, resulting in a variation in the surface area \( \Delta O_{ai} \approx -O_{ai} \Delta v/v \). Also \( \partial v_s/\partial v \approx 1 \) because the Bragg-angle is very small. Using this gives the result

\[
\Delta \varphi \approx \frac{2m_1 \mu_B g_1}{h v_s} O_{ai} \frac{\Delta v \cdot \vec{B}_{st}}{v}. 
\]

The velocity distribution is assumed to be Gaussian with a relative spread of \( \sigma_v/v = 1.5 \cdot 10^{-2} \) and the center at \( v = 247 \text{ m/s} \). The calculations needed for this type of phase error are the same as for the calculations for noise. From equation

\[
V = e^{-\frac{\sigma_v^2}{2}}, 
\]

it can be easily derived that for \( V > 0.99 \) it is required that \( \sigma_v < 0.142 \). This means that:

\[
\frac{\partial B}{\partial x} < \frac{66.5 \cdot 10^{-6}}{O_{ai}} = \begin{cases} 399 \text{ mG/m} & 0.3 \text{ at. interf}, \\ 9 \text{ mG/m} & 2 \text{ m at. interf}. \end{cases} 
\]

### 3.6 Electric Fields

The atom itself has no electric dipole when no electric field is applied. Only when the atom is placed in an electric field it will get a dipole. The effect is negligible for DC fields, and only appreciable for AC fields that are almost resonant with an atomic transition. (AC Stark shift). The potential induced by an almost resonant AC electric field, is given by [11]

\[
U = \hbar (\omega - \omega_0) \ln[1 + p(\omega)], 
\]

with

\[
p(\omega) = I \frac{\gamma^2/4}{I_{sat} (\omega - \omega_0)^2 + \gamma^2/4},
\]

where \( I = \frac{1}{2} \varepsilon_0 E^2 \) is the intensity of the electric field, \( I_{sat} \), \( \gamma \) and \( \omega_0 \) are respectively the saturation intensity, the natural line width and the resonant transition frequency of \( \text{He}^* \), and \( \omega \) the frequency of the electric field. To simplify the equation it is valid to make the assumptions that \( \omega \ll \omega_0 \) and \( \omega_0 \gg \gamma \). This simplifies the equation to:

\[
U = \hbar \gamma^2 \frac{\frac{1}{2} \varepsilon_0 E^2}{4 \omega_0 I_{sat}}, 
\]

\[
= 1.95 \cdot 10^{-40} E^2. 
\]

To notice any effect, the electric field must be very strong. So the effects of an electric radiation field, even for frequencies almost resonance with an atomic transition \(< 1.7 \cdot 10^{15} \text{ rad/s} \), can safely be neglected.
### Parameters Values

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>$L = 0.3 \text{ m}$</th>
<th>$L = 2 \text{ m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Misalignment standing light wave</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Angular position</td>
<td>$\alpha_2, \alpha_3$</td>
<td>$&lt; 14 \mu\text{rad}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\Delta z$</td>
<td>$&lt; 127 \mu\text{m}$</td>
<td></td>
</tr>
<tr>
<td>Motions atom interferometer</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Acceleration</td>
<td>$a_0$</td>
<td>$&lt; 4670 \mu\text{m/s}^2$</td>
<td>$&lt; 105 \mu\text{m/s}^2$</td>
</tr>
<tr>
<td>Rotation</td>
<td>$\Omega$</td>
<td>$&lt; 9.47 \mu\text{rad/s}$</td>
<td>$&lt; 0.21 \mu\text{rad/s}$</td>
</tr>
<tr>
<td>Distortion (visibility)</td>
<td>$D_0$</td>
<td>$&lt; 1.72 \text{nm}$</td>
<td></td>
</tr>
<tr>
<td>Distortion ($\varphi_{\text{error}}$)</td>
<td>$D_0$</td>
<td>$&lt; 8.6 \text{pm}$</td>
<td></td>
</tr>
<tr>
<td>Magnetic field variations</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gradient (visibility)</td>
<td>$\partial B_{\text{osc}}/\partial x$</td>
<td>$&lt; 17 \text{mG/m}$</td>
<td>$&lt; 0.38 \text{mG/m}$</td>
</tr>
<tr>
<td>Gradient ($\varphi_{\text{error}}$)</td>
<td>$\partial B_{\text{osc}}/\partial x$</td>
<td>$&lt; 84 \mu\text{G/m}$</td>
<td>$&lt; 1.9 \mu\text{G/m}$</td>
</tr>
</tbody>
</table>

- Waist atom beam $D \approx 50 \mu\text{m}$
- Velocity atoms $v_\parallel = 247 \text{m/s}$
- Longitudinal velocity spread $\sigma_{v_\parallel} = 1.5 \cdot 10^{-2} v_\parallel$
- Order Bragg diffraction $N = 10$
- Waist laser beam $w = 0.85 \text{mm}$
- Wavelength laser beam $\lambda_\text{s} = 1083 \text{nm}$
- Visibility $V > 99\%$
- Measurement error $\varphi_{\text{error}} < 1/1000 \text{rad}$

Table 3.1: The error sources which need the most attention. They were discussed in the previous paragraphs. Beneath the table the parameters are given used to determine the values in the table.

#### 3.7 Conclusions

From the previous calculations it shows that the atom interferometer is very sensitive to a number of error sources. In Table 3.1 the error sources are given that need special attention.

Environmental vibrations affecting the atom interferometer can be isolated by using damping materials to isolate the atom interferometer setup from the outside world. For a proper operation of the 0.3 m atom interferometer, a simple vibration isolation can be used. For the 2 m atom interferometer more attention must be given to this matter.

Distortions of the atom interferometer setup will result in very large measurement errors. Such distortions can also occur due to temperature changes. It is therefore necessary to add an active stabilization to compensate for possible mechanical changes in the atom interferometer setup. This will be discussed in the next chapter.

Table 3.1 shows that the applied magnetic field gradient should be very stable or very small, otherwise the measurement error becomes to large to make accurate measurements.
Chapter 4

Stabilization of the atom interferometer mirrors

In the previous chapter it became clear that distortion of the atom interferometer setup has to be compensated actively, otherwise the measurement errors will become so large that an accurate measurement is becomes impossible. Table 3.1 listed the parameters that have significant effects on the quality of the atom interferometer. This table shows that the displacement of the last (or the first) mirror relative to the other two mirrors should remain less than 8.6 pm to obtain the required accuracy.

4.1 Method of stabilization

The goal of the mirror stabilization is to compensate for the effects of distortion in the atom interferometer setup. To do this, a method to measure this effect is needed. The method we will use is based on an optical interferometer, that responds in the same way to distortions as the atom interferometer setup, and makes accurate distortion measurements possible. However, the sensitivity of the optical interferometer to other motions such as accelerations and rotations is extremely low compared to that of the atom interferometer. Therefore the optical interferometer is not suitable to measure these kinds of motions.

The response of the atom interferometer to distortions is given by

\[ \Delta \varphi_{ai} = \frac{4\pi N}{\lambda} (d_{x,1} - 2d_{x,2} + d_{x,3}), \]  

as shown in the previous section. In this expression \( \Delta \varphi_{ai} \) is the measured phase change between the two interfering atom beams. The displacements \( d_{x,1}, d_{x,2} \) and \( d_{x,3} \) are the displacements of the respective mirror in the direction perpendicular to the mirror surfaces. The sensitivity to these displacements depends on the order of the Bragg diffraction \( N \) and the wavelength of the light \( \lambda \), to make the standing light waves for the diffraction of the atom beam. Eq. 4.1 shows that the position of only one mirror needs to be actively controlled in order to compensate for the distortions. This can introduce an extra rotation and acceleration to the mirror setup of the atom interferometer. However for very low frequencies (< 1 Hz depending on the measurement time \( \tau \), see Figure 3.11) the atom interferometer is less sensitive to these two motions than to a distortion. The distortion compensation is achieved by feeding the by the optical interferometer measured distortion back to the actively controlled mirror, creating a feedback control system which has to
be stable. The latter remark may seem trivial at first, but to achieve this is not always easy. Therefore some elementary knowledge of feedback control systems is necessary. The schematic setup is given in Figure 4.1. The intensity of the laser will also be measured, to correct for possible intensity variations in the laser beam itself.

### 4.2 Distortion measurement

For the measurement of the distortion of the atom interferometer an optical interferometer setup will be used. Just as in the atom interferometer the light beam is first split and then brought back together again so that two light beams can interfere with each other as indicated in Figure 4.1. For the optical interferometer three identical optical gratings will be used to split the light beams. In general all gratings have the same behavior. They only differ in the intensity distribution over the different diffraction modes. The general behavior of an optical grating is shown in Figure 4.2. Here, the indicated angles $\phi_i$ and $\phi_o$ are the angles of incident and diffraction of the laser beam interacting with the grating. The beam is split because of the repetitive structure of the grating (parallel lines), with a certain repetition length indicated with $s$. The relation between the angles of the incident and the refracted laser beams is described by:

$$\frac{N\lambda_o}{s} = (n_1 \sin \phi_i - n_2 \sin \phi_o),$$

where $\lambda_o$ is the wavelength of the laser beam in vacuum, and $N$ the diffraction order of the diffracted laser beam. The parameters $n_1$ and $n_2$ are the refractive indices of the material on either side of the grating. (A complete derivation of equation Eq. 4.2 is given in appendix C).

The phase change of the diffracted beam depends on the displacement of the grating $d$ which is described by $\varphi = 2N\pi d/s$. In Figure 4.1 only the first order of diffraction is drawn, the other orders will not be used. To keep the optical interferometer within acceptable dimensions (1cm displacement of the diffracted beam) the diffraction angle has to be kept small. Also the angle of incidence of the light beam will be set to $\phi_i = 0$. This simplifies
the expression for the diffraction angle to $\phi_0 = \lambda_O / s$. The formulas that apply to the atom interferometer can also be applied here. This means that the relative phase change between the two interfering beams is equal to $\Delta \phi_{st} = 2\pi(d_{x,1} - 2d_{x,2} + d_{x,3})/s$ when the first diffraction order is used. Here, parameters $d_{x,1}$, $d_{x,2}$ and $d_{x,3}$ are the displacements of the three gratings in the x-direction.

The HeNe laser has an optical wave length $\lambda_O = 632\text{nm}$ and the grating a line density of 110 lines/mm. The diffraction angle of the first diffraction order then becomes $\phi_0 \approx 70\text{ mrad}$ and the sensitivity of the optical interferometer for the grating displacement $220\pi \text{ rad/mm}$.

Just like the atom interferometer the optical interferometer is sensitive to misalignments of the optical components used which, in this situation, are the optical gratings.

4.2.1 Misalignment sensitivity

Basically each the grating has four degrees of freedom for misalignments which are the displacement in the Z-direction (in the direction of the incoming laser beam) and rotations about the X-, Y- and the Z-axis.

The displacement in the Z-axis diminishes overlap of the interfering light beams, just like in the situation of the atom interferometer, described in the subsection on page 26. By using a beam with a large waist, the light interferometer can be made less sensitive to this displacement. To get an impression of the accuracy with which the third grating has to be positioned, the following expression can be used: $\Delta z < \Delta V \pi w/(2\phi_o)$ (see Eq. 3.33) with $\Delta V$ the deviation of the fringe visibility from 1, $w$ the waist of the laser beam, and $\phi_o$ the diffraction angle of the laser beam by the grating. Using a beam diameter of 2mm gives $\Delta z < 40 \cdot \Delta V \text{ mm}$. For the purpose of feedback control a $\Delta V = 0.2$ is sufficient which means that $\Delta z < 8 \text{ mm}$, which is very tolerant.

The angular misalignments with the X- or Y-axis as the rotation axis influences the optical interferometer only in the third order of the angular misalignments which can therefore can be neglected. For the angular displacement with the Z-axis as the rotation axis, the optical interferometer is much more sensitive.
To be able to describe the effects, a more general description of the grating behaviour given by Eq. 4.2 has to be used. This is:

$$k_{ix} - k_{ox} = -\frac{2\pi N}{s}.\quad (4.3)$$

The parameters $k_{ix}$ and $k_{ox}$ are the x-components of the wave vectors of the incoming and the outgoing laser beam $k_{i}$ and $k_{o}$, respectively. The relation between the incoming and the outgoing wave vector of the laser beam then becomes

$$ (k_{ox}, k_{oy}, k_{oz}) = \left( k_{ix} + 2\pi N/s, k_{iy}, \sqrt{k^2 - k_{iy}^2 - (k_{ix} + 2\pi N/s)^2} \right), \quad (4.4) $$

where $N = -1, 0, +1$. For small angular misalignments ($\ll 1$ rad), a valid approximation is to only rotate the difference between the incoming and the outgoing wave vectors $\Delta \vec{k} = \vec{k}_{i} - \vec{k}_{o}$. The wave vector then will transform into:

\begin{align*}
  k_{ox} &= k_{ix} + \frac{2\pi N}{s} \cos \alpha \approx k_{ix} + \frac{2\pi N}{s} \left( 1 - \frac{\alpha^2}{2} \right) \\
  k_{oy} &= k_{iy} + \frac{2\pi N}{s} \sin \alpha \approx k_{iy} + \frac{2\pi N}{s} \alpha \\
  k_{oz} &= \sqrt{k^2 - k_{oz}^2 - k_{oy}^2} \approx k.
\end{align*} \quad (4.5a, b, c)

Here $\alpha$ indicates the relative angular displacement of the grating compared to the other gratings. To calculate the effect of this misalignment, the two misalignment angles of gratings 2 ($\alpha_2$) and 3 ($\alpha_3$), in relation to the first grating, will be used. In the optical interferometer the laser beams follow different paths, of which only two are used which makes it similar to the atom interferometer. The grating constant $s$ is in this case equivalent to the factor $\lambda_s/2$ in equation Eq. 3.27 with $N = \pm 1$. One beam is deflected in the positive x-direction at the first grating ($N = 1$) and then deflected back again at the second grating ($N = -1$). The other laser beam is deflected the same way with the same sequence, but with the second and the third grating. This strongly corresponds to output B of the atom interferometer. Following the same reasoning as for the atom interferometer the difference in wave vectors of the interfering laser beams then becomes:

$$ (\Delta k_x, \Delta k_y) \approx \frac{2\pi}{s} \left( \frac{\alpha_2^2 - \alpha_3^2}{2}, [\alpha_3 - 2\alpha_2] \right). \quad (4.6) $$

The visibility (from equation Eq. 3.28) of the optical interferometer is:

$$ V \approx \frac{\sin(\Delta k_y w_y/2)}{(\Delta k_y w_y/2)}, \quad (4.7) $$

where $w_y$ is the waist of the laser beam in the y-direction. Demanding that $V > 0.8$ gives $\alpha_3/\alpha_2 < 546 \mu$rad. This is a small angle. Therefore, two gratings have been given the possibility to rotate in the XY-plane relative to the mirror. The mechanical contraction needed to implement this, is placed on the mirror mounts of the two mirrors which are not actively controlled.

### 4.2.2 Laser beam direction sensitivity

When the alignment of the three gratings in the optical interferometer is optimal, changing the direction of the incoming laser beam with $\Delta \theta$ should have no effect on the optical
interferometer. However, when we look at the problem more carefully and take a possible misalignment in the z-direction into account (such that the two interfering laser beams, emerging from the optical interferometer, no longer perfectly overlap), this changes. This sensitivity results in a displacements of the actively controlled mirror, which in turn will induce measurement errors. Since the atom interferometer is extremely sensitive to these displacements, it is worthwhile to look into this further. The setup of the optical interferometer is shown in Figure 4.3.

For the calculation of the influence of a change in $\theta_1$, i.e. the angle of the incident laser beam, as indicated in Figure 4.3, the laser beam can be regarded as a plane wave which can be expanded into an infinite wide beam. Only the phase difference between the two beams emerging from the optical interferometer is of interest. This phase difference is the same at every position at the last grating because these two laser beams are still parallel. Here is assumed that the gratings are properly aligned. The wave vector can be described as

$$ \vec{k}_{O,j} = k_{O,j} (\sin \theta_j, 0, \cos \theta_j), \quad (4.8) $$

with $k_{O,j}$ the length of the wave vector of the laser beam in the medium (air or grating substrate). Angle $\theta_j$ is the angle indicated in Figure 4.3 which is the angle in the medium, where index $j$ indicates the different angles the laser beam makes in the optical interferometer in the different media. This can be converted into an angle in vacuum by $\sin \theta_{vac,j} = n_j \sin \theta_{med,j}$ with $n_j$ the refractive index of the medium. By this conversion the equations can be made medium independent. The vacuum angle has only two different values, $\theta_1$ and $\theta_2$, in the optical interferometer with $\theta_1$ the initial angle of the laser beam entering the optical interferometer and $\theta_2$ the angle of the first order diffracted laser beam after the first grating. The same angles will emerge after the second and the third grating as shown in Figure 4.3. The optical interferometer is made up of three gratings on glass substrates with thicknesses of $D_1$ and $D_2$ for the first two, placed in air separated by distances of $L/2 + \Delta/2 - D_1$ and $L/2 - \Delta/2 - D_2$ with $\Delta/2$ the displacement of the middle grating from an ideal position. The refractive index of air $n_a \approx 1$ and that of glass $n_g \approx 1.5$. In the following calculations only the angles in vacuum will be used which means that the wave vector from Eq. 4.8 has to be converted to:

$$ \vec{k}_{O,i,m} = k_O \left( \sin \theta_i, 0, \sqrt{n_m^2 - \sin^2 \theta_i} \right), \quad i = \{1, 2\}, \ m = \{a, g\}. \quad (4.9) $$
The total phase difference can easily be calculated. First for the two paths taken in the optical interferometer separately, all the phase changes are added. The effects of the gratings themselves also have to be taken into account. These are given by the relationships:

\[
\frac{2\pi N}{s} = k_{x,\text{in}} - k_{x,\text{out}},
\]

\[
= k_0 (\sin \theta_{\text{in}} - \sin \theta_{\text{out}}), \quad \text{and}
\]

\[
\varphi = (k_{x,\text{in}} - k_{x,\text{out}}) (d_x - x)
\]

\[
= \frac{2\pi N}{s} (d_x - x),
\]

with \( N = \pm 1 \), \( x \) the position on the grating crossed by the path taken, and \( d_x \) the displacement of the grating, both in the x-direction. The paths can be taken completely arbitrarily with the restriction that the two paths for the two laser beams are continuous and that both paths start and end at the same points on the first and the third grating respectively. Because the interfering laser beams are parallel, the phase difference is the same at every position after the third grating. The total phase difference then becomes:

\[
\varphi_{\text{oi}} = \frac{2\pi}{s} (d_{x,1} - 2d_{x,2} + d_{x,3}) + k_0 \left\{ (\cos \theta_2 - \cos \theta_1) \Delta + \left( \cos \theta_2 - \cos \theta_1 + \sqrt{n_g^2 - \sin^2 \theta_1} - \sqrt{n_g^2 - \sin^2 \theta_2} \right) (D_1 - D_2) \right\}.
\]

The parameters \( d_{x,k} \) with \( k = \{1, 2, 3\} \) are the displacements of the three gratings. The angles \( \theta_{1,2} \ll 1 \), and therefore the deviation of these angles, due to small changes of the direction of the incoming laser beam, are approximately the same \( \Delta \theta_1 = \Delta \theta_2 = \Delta \theta \ll 1 \). The first order deviation on the phase change \( \varphi_{\text{oi}} \) due to \( \Delta \theta \) is:

\[
\Delta \varphi_{\text{oi}} = k_0 \Delta \theta \left\{ (\sin \theta_1 - \sin \theta_2) \Delta + \left[ \sin \theta_1 \left( 1 - \frac{\cos \theta_1}{\sqrt{n_g^2 - \sin^2 \theta_1}} \right) - \sin \theta_2 \left( 1 - \frac{\cos \theta_2}{\sqrt{n_g^2 - \sin^2 \theta_2}} \right) \right] \times (D_2 - D_1) \right\}.
\]

Because the angles \( \theta_{1,2} \ll 1 \), \( \cos \theta_{1,2} \approx 1 \) and \( \sin \theta_{1,2} \ll n_g \). Then Eq. 4.12 can be simplified to:

\[
\Delta \varphi_{\text{oi}} \approx \frac{2\pi \Delta \theta}{s} \left\{ \left( 1 - \frac{1}{n_g} \right) (D_2 - D_1) + \Delta \right\}.
\]

The relation between the phase changes in the optical and the atom interferometer is given by \( \Delta \varphi_{\text{ai}} = 2 N_{\text{ai}} \Delta \varphi_{\text{oi}} / \lambda_s \) with \( N_{\text{ai}} \) the order of the Bragg diffraction and \( \lambda_s \) the wavelength of the standing light waves. This then results in:

\[
\Delta \varphi_{\text{ai}} = \frac{4\pi N_{\text{ai}}}{\lambda_s} \Delta \theta \left\{ \frac{1}{3} (D_2 - D_1) + \Delta \right\}.
\]
This is completely independent of the grating period $s$. From the specifications of the considered HeNe lasers the long term pointing stability $\Delta \theta < 30 \mu$rad. A Bragg-order of $N_{oi} = 10$ together with the requirement $\Delta \varphi_{oi} < 1/1000$ results in the requirement

$$\frac{1}{3} (D_2 - D_1) + \Delta < 0.3 \mu m$$

(4.15)

This requires a method to accurately position the gratings in the $z$-direction.

In the recent past, a number of atom interferometers were successfully build with use of an atom beam which is not cooled first as in Ref. [12]. The velocity spread of the atoms in such atom beam is very large, resulting in a considerable spread in the direction. This made us assume that the combination of a displacement of the third standing light wave in the $z$-direction with a velocity spread of the atoms in the atom beam has no or very little effect on the atom interferometer’s performance. Therefore such combination has not been further investigated.

4.3 Feedback control

The basic setup for the stabilization was already given by Figure 4.1. It shows that the measured position signal from the optical interferometer is fed back into the controller. Here the position signal is compared to a reference voltage creating a position error signal. This signal is then used to drive the mirror in such a way that this error signal becomes as small as possible. This process is given in a diagram in Figure 4.4. As shown the complete stabilization can be divided into three major subsystems:

1. **The controller** ($Ct$) conditions the error signal such that this signal is reduced as much as possible in a closed loop, while at the same time ensuring the stability of the mirror stabilization.

2. **The Piezo Driver + Piezo** ($Pd$) serves as the actuator for the mirror. This system with the mirrors and mirror mounts has many mechanical resonances frequencies. Only one dominant resonance frequency needs to be considered because the others seem much weaker.

3. **The optical interferometer** ($Oi$) measures the displacement of the mirror as shown in the previous section. The light intensity from the interferometer is measured by
a light sensitive detector and translated into an electrical signal which serves as the input for the controller (Position Signal).

The characteristics of these subsystems are indicated by operators $C_t$, $D_p$ and $O_i$ which transform the incoming signal. The relation between the signal from the optical interferometer and the mirror displacement can be taken linear with a constant amplification factor since the displacements will be very small and the detector is much faster than the feedback control system. The amplification factor of the optical interferometer does however depend on the position on the signal fringe. This amplification is defined as $A_{oi} = \frac{\partial U(d_x)}{\partial d_x}$ with $U(d_x)$ the signal from the optical interferometer and $d_x$ the displacement of the grating (equivalent to the mirror position) in the $x$-direction. For the optical interferometer the signal has the form:

$$U(x) = U_{offset} + U_0 \sin\left(\frac{2\pi d_x}{s}\right),$$

resulting in

$$A_{oi} = -\frac{2\pi U_0}{s} \cos\left(\frac{2\pi d_x}{s}\right),$$

where $U_{offset}$ is the offset in the output signal and $U_0$ the amplitude of the signal fringe. It becomes clear that for large displacements the amplification factor $A_{oi}$ will change. It also shows that the sign of $A_{oi}$ can be either positive or negative depending on the position on the fringe. Maximum amplification is achieved in the middle of the fringe where $U(x) = U_{offset}$.

Furthermore the ratio between the fringe periods of this optical and the atom interferometer is $2N_{ais}/\lambda_s = 167.9$ which means that a displacement over one fringe of the optical interferometer is equivalent to 167.9 fringes of the atom interferometer for $N_{ais} = 10$. So when the mirror is displaced over one fringe of the atom interferometer, $A_{oi}$ will remain almost constant. This gives the possibility to scan the atom interferometer over several complete fringes of the atom interferometer without significantly affecting the behavior of the mirror stabilization.

The characteristics of the controller and the combination Piezo + Piezo Driver can not be described as a constant amplification of the incoming signal. However, because these characteristics are linear, they can be described as a frequency response, $C_t(\omega)$ and $P_d(\omega)$ defining the frequency dependent amplification and extra induced phase on the input signal. These frequency responses allow us to look at the behavior of each frequency component of the signals in the feedback control loop of Figure 4.4. The frequency response $U(\omega)$ on the mechanical vibrations $I(\omega)$ of the closed loop system can then be easily calculated by:

$$U(\omega) = I(\omega) - D_p(\omega) C_t(\omega) A_{oi} U(\omega)$$

leading to

$$U(\omega) = \frac{I(\omega)}{1 + D_p(\omega) C_t(\omega) A_{oi}}$$

where $U(\omega)$ and $I(\omega)$ are the frequency spectra of the mirror position and mechanical vibrations respectively. Eq. 4.18 completely describes the response of the closed loop system to the mechanical vibrations. However, analyzing the stability of the closed loop system by means of this frequency response has always been very difficult. A better method is using the Laplace transformed functions and characteristics.
4.3.1 Laplace domain

To analyze the behavior of feedback control systems, the signals and the intrinsic behavior of the systems are often transformed to the Laplace domain. Many theories involving stability analyses use this transformation [13], and this subsection summarizes the main features.

The conversion from the time domain to the Laplace domain is defined by:

\[ \mathcal{L}(f(t))(s) = \int_0^\infty f(t)e^{-st}dt, \]  

(4.19)

where \( s \) is now a new parameter belonging to the Laplace domain.

In the time domain, the response of a linear subsystem in the feedback control system \( U(t) \) to its input signal \( I(t) \) can be described as a linear combination of time integration, differentiation and constant terms. The time integrated function:

\[ g(t) = \int_0^t i(t')dt' \]

will be converted to

\[ \mathcal{L}(g(t))(s) = -\frac{1}{s}g(0) + \frac{1}{s}i(s). \]  

(4.20)

This shows that an integration in the time domain simply means a division by \( s \) in the Laplace domain. It also gives an extra constant \( g(0) \) which is the starting condition of the integrated input signal. Exactly the same approach can be used for the differentiation operations. The conversion for these operations become:

\[ i(t) = \frac{1}{s}i(0) + si(s). \]  

(4.21)

This shows that differentiations transforms into a multiplication with \( s \). Also for the differentiations the starting conditions \( i(0) \) will appear in the conversion. The starting conditions are often used for the examination of switch-on characteristics of the system. These starting conditions for both the integration and differentiation operations will be left out because they don’t play any role in the stabilization analysis. Leaving out these starting conditions, the behavior of a subsystem can always generalized to:

\[ \begin{align*}
\left\{ \sum_n A_{d,n} \left( \frac{d}{dt} \right)^n + A_c \right\} U(t) &= \left\{ \sum_m B_{d,m} \left( \frac{d}{dt} \right)^m + B_c \right\} I(t) \\
\downarrow \mathcal{L} & \\
\left\{ \sum_n A_{d,n} s^n + A_c \right\} \tilde{U}(s) &= \left\{ \sum_m B_{d,m} s^m + B_c \right\} \tilde{I}(s).
\end{align*} \]

(4.22)

Here \( A_{d,n}, A_c, B_{d,m}, \) and \( B_c \) are real constants. The simple multiplications and divisions with \( s \) for the differentiation and integration operations means also that concatenating different linear systems \( H_1 \) and \( H_2 \) is equivalent to multiplying there characteristics in the Laplace domain. Also, in the Laplace domain, the output signal \( \tilde{U}(s) = \tilde{H}(s) \cdot \tilde{I}(s) \). Thus

\[ H_2[H_1 I(t)] \rightarrow \tilde{H}_2(s) \tilde{H}_1(s) \tilde{I}(s). \]  

(4.23)

This makes the analysis of the feedback control system much easier.
4.3.2 Stability

To study the stability of the mirror stabilization it is, in principle, sufficient to look at the system's reaction to a Dirac-pulse. The frequency spectrum of the Dirac-pulse contains all frequencies from zero to infinity, all with the same magnitude. This way, the frequencies are excited. The Laplace function for a dirac-pulse is \( \mathcal{L}(s) = 1 \). This means that the characteristic of the mirror stabilization system has the same form as Eq. 4.22. In this situation, \( \tilde{U}(s) \) becomes a fraction which can be rewritten as:

\[
\mathcal{H}(s) = \frac{U(s)}{n(s)} \tag{4.24}
\]

where \( n(s) \) is a polynomial in \( s \), and \( a_i \) and \( \omega_i \) are real constants. This equation consists of two polynomials, one in the numerator and one in the denominator. It can be split up into a set of equations with a basic form:

\[
\mathcal{C}(s) = \frac{n!}{2(s-a+i\omega)^{n+1}} + \frac{n!}{2(s-a-i\omega)^{n+1}}. \tag{4.25}
\]

The inverse Laplace transformation of this function is \( \mathcal{L}^{-1}(\mathcal{C}(s))(t) = t^n \exp(at) \cos(\omega t) \). This means that the response of the mirror stabilization system to a Dirac-pulse is a set of functions that can oscillate, each with its own decay parameter. The mirror stabilization system is considered stable when parameter \( a < 0 \). This means that the real part of all complex poles of the mirror stabilization characteristic has the be negative.

The Laplace description of the mirror stabilization system (see Figure 4.4) is given by

\[
\mathcal{H}(s) = \frac{1}{1 + \mathcal{O}(s)\mathcal{C}(s)\mathcal{D}(s)} \tag{4.26}
\]

This is the transfer function between the mechanical vibrations \( \tilde{I}(s) \) and the mirror position \( \tilde{U}(s) \). It becomes clear that the complete open loop characteristic \( \tilde{H}_O(s) = \mathcal{O}(s)\mathcal{C}(s)\mathcal{D}(s) \) contains the information concerning the stability of the feedback control system.

To get an impression of the error suppression by the feedback control system, the transfer function should be transformed to the frequency domain by substitution \( s = \omega \). The aim is to get the strongest suppression possible. This means that the amplitude of the open loop function \( |\tilde{H}_O(\omega)| \) should be made as large as possible.

4.3.3 Controller Design

As we saw, the stability of the mirror stabilization depends on the behavior of all components in the system. Therefore, to design the controller, the characteristics and influence on the stabilization of the remaining components have to be examined first. From this the requirements for the controller will be specified which then can be translated into a design.

Design considerations

The open loop characteristic of the mirror stabilization consists of the multiplication of the characteristics of the controller, the Piezo driver + Piezo and the optical interferometer. The optical interferometer was considered to be approximately linear with the displacement
for small displacements, and can therefore be assumed a constant amplification factor resulting in \( O_i(s) = A_{oi} \).

The combination Piezo driver + Piezo behaves as a damped mass-spring system, which can be described as

\[
\hat{D}_p(s) = A_P \frac{a_p^2 + \omega_p^2}{(s + a_p)^2 + \omega_p^2}
\]

(4.27)

where \( a_p \) is the damping, and \( \omega_p \) the resonance frequency of the Piezo-mirror system. From measurements performed on the optical interferometer, we find \( \omega_p \approx 9.2 \cdot 10^3 \, \text{rad/s} \) and \( a_p/\omega_p \approx 0.084 \). This is a simplified model of the reality, which has more resonance frequencies close to the indicated frequency. These resonances are much weaker, and can be neglected. The damping of the primary frequency \( 9.2 \, \text{kHz} \) is very low. This results in a large amplification at the resonance frequency and a phase change of \(-\pi\) over a very short frequency range.

To obtain a strong suppression of the low frequency vibrations, a very large low frequency amplification in the feedback loop is needed. For low frequencies all subsystems are virtually frequency independent. The maximum feedback amplification at higher frequencies is limited because of the mechanical resonances present in the system which can destabilize the mirror stabilization for high amplification factors. Therefore an electronic integrator is added to increase the amplification at low frequencies, and at the same time decrease the amplification at higher frequencies. The controller can then be described by the transfer function \( Ct(s) = A_{PI}/s \), where \( A_{PI} \) is the adjustable amplification factor. The total open loop function then becomes:

\[
\hat{H}_O(s) = \frac{A_{fb}}{s} \frac{a_c^2 + \omega_p^2}{(s + a_c)^2 + \omega_p^2},
\]

(4.28)

with \( A_{fb} = A_{oi}A_PA_{PI} \) the complete frequency independent open loop feedback amplification.

The stability can be determined by analyzing the complex roots of the polynomial \( 1 + \hat{H}_O(s) \). The open loop characteristic \( \hat{H}_O(s) \) is a third order polynomial in \( s \), which results in three different roots. All complex roots have a complex conjugated counterpart, this means that at least one root is real which happens to be always negative in this situation. The two remaining roots are complex and therefore complex conjugates. So only one root has to be considered. In Figure 4.5 the ratio is given between the real part of the relevant root \( a_{cl} \) and the damping factor \( a_p \) of the combination Piezo driver + Piezo. The reciprocal of the ratio given in the figure is approximately equal to the maximum amplification of the distortion induced signal error by the stabilization setup at resonance frequency (the imaginary part of the root). It shows that the mirror stabilization becomes unstable when increasing the feedback amplification beyond \( A_{fb} = 1.55 \cdot 2000\pi \) \((a_{cl} > 0)\). The objective is to eliminate the distortion influence on the measurements as much as possible by making the feedback as strong as possible. Looking at figure 4.5, it shows that with this setup a feedback amplification of more than \( A_{fb} > 2000\pi \) is not feasible, because otherwise the signal error amplification becomes to large (because the damping factor \( a_{cl} \) becomes very small).

A useful tool for stability analysis is the Nyquist plot. In the Nyquist plot the open loop characteristic of the mirror stabilization system \( \hat{H}_O(i\omega) \) (see Eq. 4.3.3) is visualized in a complex plot with \( \omega \) the running variable, as shown in Figure 4.6. According to Nyquist's theory the number of unstable poles in the closed-loop characteristic of mirror
Figure 4.5: This figure shows the expression $a_{x3}/a_{y}$ as function of the feedback amplification factor. The mirror stabilization is stable when $a_{x3}/a_{y} < 0$. The feedback amplification is chosen to be $A_{fl} = 2000\pi A$, such that the open loop amplification becomes 1 at frequency $\omega/(2\pi) = 1$ kHz.

Figure 4.6: Nyquist plot of the open loop function. The solid line is the characteristic of the open loop for $\omega = 0 \cdots \infty$, and the dashed line for $\omega = -\infty \cdots 0$. The solid and the dashed line combined is the complete Nyquist plot. The graph is normalized to the gain factor $A = A_{fl}/(2000\pi)$. The number of unstable poles is then equal to the number of encirclements of $-1/A$. Also given in the plots are the circles of unity gain for $A = 1$ and $A = 2$. 

58
stabilization $Z$ (these are the poles with a positive real part ($a_{ol} > 0$)) is given by \[ Z = N + P, \] (4.29)
where $N$ is the number of clockwise encirclements of the point $-1$ in the diagram, and $P$ the number of unstable poles of the open loop characteristic $H_o(s)$ ($a_p > 0$). The Nyquist plot in Figure 4.6 is normalized, so that the plots for the different open loop amplifications remains the same. Because of this normalization, the point $-1$ is shifted to $-2000\pi/A_{fb}$ in the figure. This means that $N$ becomes the number of encirclements of point $-2000\pi/A_{fb}$. This also means that increasing the amplification means shifting the $-1$ point towards $0$ on the real axis.

The open loop characteristic is considered stable, which means that it does not contain unstable poles. Therefore it can be stated that $P = 0$, so that $Z = N$. From Figure 4.6 can be deduced that as long as $A_{fb}/2000\pi < 1.55$, $N = Z = 0$. As soon as $A_{fb}/2000\pi > 1.55$, $N = Z = 2$. This means that the mirror stabilization becomes instable.

Electronic design of the controller

The design of the controller is given in Figure 4.7. The design shows that the incoming signals are conditioned so that electrical noise is eliminated as much as possible. These signal conditioners also give the electronic devices in the controller an over-voltage protection. So mistakes made with connections will not damage the electronic circuits. This is also the case for the output signals. For this purpose the error signal is sent via an electronic buffer. This buffer also prevents electrical noise from entering the position error signal.

The position signal entering the controller is, after the conditioning at the input, normalized to the intensity of the laser, which enters via a second input. This is accomplished by dividing the position signal with the intensity signal. In this way, gain any variation due to intensity fluctuations can be eliminated. After the normalization the position signal is compared with an adjustable position reference by subtraction, resulting in a position
error signal. This signal will be used to compensate for the distortion of the atom interferometer, and will be regulated to 0 when the feedback loop is closed. Finally, the position error signal is integrated before it is sent to the mirror actuator, to get a large suppression of the low frequency distortion effects on the atom interferometer signal. The amplification of the integrator is kept adjustable to have control over the stability of the complete feedback loop. Among the essential parts of the controller are two switches: One switch can invert the position error signal. This way the sign of the amplification of the optical interferometer can be set to obtain a stable feedback loop. The other switch opens the feedback loop. Due to offsets present in the electronic circuits, the integrator may integrate to a maximum voltage and saturate. This is no problem since the displacement of the mirror will be limited to a few micrometers which is shorter than one fringe period of the optical interferometer. So stabilizing the mirror after closing the feedback loop again should not be a problem.

An analog input is added to the controller for a test signal which can be added to the position error to simulate mechanical vibrations on the controlled mirror or to modulate the mirror position. With a switch this test signal can be disabled so that possible electrical noise from the test input is stopped from entering the controller. The reaction of the mirror is measured with the position error which is sent to the test output via a buffer. By comparing the test signal on the analog test input with the signal on the test output, the system's response can be analyzed.

### 4.4 Measurements

After building the optical interferometer and the electronics for the controller, the behavior of the feedback control is measured. For the measurements the test input is used to simulate the distortion on the optical interferometer by applying an electrical sine wave signal with variable frequency. The feedback loop will compensate for this applied signal by changing the mirror position. For this purpose the test signal on the test input is compared with the signal just after summation of the test and the position error signal, just before the integrator (see Figure 4.7). The latter signal provides a measure of the capability of the mirror stabilization to compensate for mechanical vibrations. In Figure 4.8 the amplitude ratio of the two compared signals are shown. The theoretically predicted curve is also given in this figure. The theoretical and the measured curve coincide quite well. At a frequency of about 2 kHz, there is some deviation from the theoretical curve due to the fact that there appears to be another mechanical resonance which was previously neglected. It appears at a frequency at which the open loop gain is close to unity. Small changes in the open loop amplification combined with a phase change close to this frequency can have a significant influence on the reduction of the signal error. This is not investigated any further.

The effect of the mirror position control on the reduction of errors induced by the mechanical distortion are shown in Figure 4.9. Here the induced error is translated to a displacement of the actively controlled mirror. The distortion is induced by the movements of the floor and the table which are conducted to the optical interferometer resulting in the output signal shown in the upper graph. This graph shows vibrations of very low and a higher frequencies. The higher frequency turned out to be about 500 Hz which is the resonance frequency of the two other mirror mounts which are not damped. Only the mirror mount with the piezo was damped to stabilize the mirror position control. Because of the low damping factor of the two remaining mirror mounts, the amplification of the 500 Hz vibrations become very high, which shows up as a 500 Hz oscillation. After activating the stabilization by closing the feedback loop the low frequency signal becomes very small,
which is shown in the lowest graph. The graph in the middle shows mainly the 500 Hz component, that only reduces to half of its original amplitude, which was to be expected. The measurements showed a significant reduction of a low frequency 1/15 Hz distortion with a amplitude of 8 nm. Calculations give a reduction of about 15000 times, which would result in a residual amplitude of about 0.3 pm. The lowest graph shows a low frequency error signal much larger than this. This could be an effect of the sampling rate of the digital oscilloscope.

4.5 Discussion

The method of stabilization with aid of a optical interferometer works very well. The stabilization is very robust, can be kept simple and is capable of compensating for very small distortions. There are however some concerns regarding the mechanical setup. The present setup consists of three separate mirror mounts. The mirror mounts all have a strong resonance frequency at about 500 Hz. Because of the strength of the resonance it was necessary to insert some damping material between the mirror mount with the actively controlled mirror and the optical table, otherwise this mirror mount's resonance did destabilize the mirror stabilization. The mechanical resonances of the other two mirror mounts, without the added damping material, amplified the 500 Hz vibrations coming from the outside world but didn't have any influence on the mirror stabilization. Although the atom interferometer can be effectively isolated from vibrations for frequencies higher then 20 Hz, it is still advisable to design a mechanical construction for the mirrors with the optical gratings without such resonances.
Figure 4.9: The upper graph shows the signal error on the optical interferometer. The lower two graphs show that after activating the mirror stabilization, the low frequency signal error is compensated. The lowest graph is derived from the middle graph by averaging over about 5 s, effectively filtering out the higher frequencies. The signal errors are translated to a displacement of the actively controlled mirror.
Appendix A

Calculus atom interferometer

The atom interferometer consists of two 50-50% beam splitters and one atom beam reflector. The basic component of the atom interferometer is shown in Figure A.1. This can be either an atom beam splitter or a reflector. The general mathematical description of these sections is given by the expression:

\[ \tilde{V}_o = \mathbf{T} \tilde{V}_i \]  

(A.1)

with

\[ \mathbf{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \]

which describes the transfer of the wave functions of the two input beams, indicated with \( \tilde{V}_i \), into that of the two output beams, indicated with \( \tilde{V}_o \). The transfer function is described as a matrix \( \mathbf{T} \). Because of the preservation of the number of particles \( |\tilde{V}_o|^2 = |\tilde{V}_i|^2 \).

Expanding this results in

\[ \tilde{V}_i^T \mathbf{T}^T \mathbf{T} \tilde{V}_i = \tilde{V}_o^T \tilde{V}_o. \]  

(A.2)

This must be true for all the vectors in the two dimensional complex space. This means that \( \mathbf{T}^T \mathbf{T} = \mathbf{I} \), the unity matrix, indicating that

\[ |T_{11}|^2 + |T_{21}|^2 = 1 \]
\[ |T_{12}|^2 + |T_{22}|^2 = 1 \]
\[ \frac{T_{11}^*}{T_{21}} = \frac{T_{22}}{T_{12}}. \]  

(A.3)

From this it can be deduced that \( |T_{11}| = |T_{22}| \) and \( |T_{21}| = |T_{12}| \). The transfer matrix can therefore be described in the following way:

\[ \mathbf{T} = \begin{pmatrix} e^{i\xi} \cos \zeta & -e^{-i\varphi} \sin \zeta \\ e^{i\varphi} \sin \zeta & e^{-i\xi} \cos \zeta \end{pmatrix}. \]  

(A.4)

The phase \( \xi \) is the phase change of the transmitted atom beams. This phase is not influenced by the motion in the x-direction of the mirror. Phase \( \varphi \) on the other hand is.

As described in chapter 2 this phase is related to the mirror position as \( \varphi = 2N |\vec{k}_i| x \).
The transfer function of the complete atom interferometer can be described by:

$$\overline{T}_f = \overline{T}_3 \overline{T}_2 \overline{T}_1$$  \hspace{1cm} (A.5)

$\overline{T}_1$ and $\overline{T}_3$ are transfer matrices of the 50-50% atom beam splitters. In the ideal situation their transfer matrices look like:

$$\overline{T}_i = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\xi_i} & -e^{-i\xi_i} \\ e^{i\zeta_i} & e^{-i\zeta_i} \end{pmatrix}, \quad i=1,3$$  \hspace{1cm} (A.6)

The second transfer matrix describes the atom beam reflector. This becomes:

$$\overline{T}_2 = \sqrt{A} \begin{pmatrix} 0 & -e^{-i\varphi_2} \\ e^{i\varphi_2} & 0 \end{pmatrix}.$$  \hspace{1cm} (A.7)

Factor $A < 1$ is the reduction of the atom beam intensity due to possible misalignment. The diagonal elements of this transfer matrix are made 0 because the atoms that are not deflected will be lost, and therefore are no longer part of the atom interferometer. In general the total transfer matrix of the atom interferometer becomes:

$$\overline{T}_f = \begin{pmatrix} T_1 & -T_3^* \\ T_2 & T_1^* \end{pmatrix}$$  \hspace{1cm} (A.8)

with

$$T_1 = -\sqrt{A}e^{i(\varphi_1+\varphi_2+\varphi_3)} \cos \zeta_3 \sin \gamma_1 - e^{i(\varphi_2+\varphi_3+\xi_1)} \sin \zeta_3 \cos \gamma_1$$

$$T_2 = -\sqrt{A}e^{i(\varphi_1+\varphi_3-\varphi_2)} \sin \zeta_3 \sin \gamma_1 + e^{i(\varphi_2+\xi_1-\xi_3)} \cos \zeta_3 \sin \gamma_1.$$

The atom beam entering the atom interferometer, as indicated in Figure 2.1, will have the vector $V_{in} = (1,0)^T$. So the output vector will become $V_{out} = (T_1, T_2)^T$. The detectors $A$ and $B$ measure the intensity of the two exiting beams which are equal to $I_A = |V_{a2}|^2 = |T_2|^2$ for detector $A$ and $I_B = |V_{a1}|^2 = |T_1|^2$ for detector $B$. So the measured intensities by these detectors become:

$$I_A = A \left\{ \sin^2 \zeta_3 \sin^2 \gamma_1 + \cos^2 \zeta_3 \cos^2 \gamma_1 - 2 \cos \zeta_3 \sin \zeta_3 \cos \gamma_1 \sin \gamma_1 \sin \varphi_{ai} \right\}$$  \hspace{1cm} (A.9)

$$I_B = A \left\{ \cos^2 \zeta_3 \sin^2 \gamma_1 + \sin^2 \zeta_3 \cos^2 \gamma_1 + 2 \cos \zeta_3 \sin \zeta_3 \cos \gamma_1 \sin \gamma_1 \cos \varphi_{ai} \right\}.$$  \hspace{1cm} (A.10)
with

\[ \varphi_{ai} = \xi_3 - \xi_1 + \phi_1 + \varphi_3 - 2\varphi_2. \]

Here \( \varphi_{ai} \) is the phase difference between two interfering beams after the last beam splitter when there were no other interactions in the atom interferometer. In the ideal situation \( \cos \xi_1 = \sin \xi_1 = \cos \xi_3 = \sin \xi_3 = 1/\sqrt{2} \) and \( \xi_1 = \xi_3 \) which results in:

\[
I_{A/B} = \frac{A}{2} (1 \mp \cos(\varphi_1 + \varphi_3 - 2\varphi_2)).
\]

(A.11)

The amplitude of the fringe becomes 1/2. The visibility for this ideal situation is of course 1 for both detectors. This changes when the splitting ratios of the beam splitters and the reflector become less ideal. To describe the visibility it is easier to use a factor called the amplitude ratio indicated by \( R_i = \tan \xi_i \). The two intensities then get the following form:

\[
I_A = A \cos^2 \xi_1 \cos^2 \xi_3 (1 + R_1^2 R_3^2 - 2R_1 R_3 \cos \varphi_{ai})
\]

(A.12)

\[
I_B = A \cos^2 \xi_1 \cos^2 \xi_3 (R_1^2 + R_3^2 + 2R_1 R_3 \cos \varphi_{ai}).
\]

(A.13)

In this situation the fringe amplitude becomes \( 2A \cos^2 \xi_1 \cos^2 \xi_3 R_1 R_3 \). The visibility of the two detector signals described as \( V = (I_{\text{max}} - I_{\text{min}}) / (I_{\text{max}} + I_{\text{min}}) \) become:

\[
V_A = \frac{2R_1 R_3}{1 + R_1^2 R_3^2}
\]

(A.14)

\[
V_B = \frac{2R_1 R_3}{R_1^2 + R_3^2}.
\]

(A.15)

When one of the two beam splitters is perfect the two outputs react the same. In that case either \( R_1 \) or \( R_3 \) becomes equal to one and the other to \( R \). The visibility then becomes:

\[
V_A = V_B = \frac{2R}{1 + R^2}.
\]

(A.16)

The difference between the two output signals becomes apparent when the first and the last beam splitter behave the same way. Then \( R_1 = R_3 = R \) which results in

\[
V_B = 1
\]

(A.17)

\[
V_A = 2R^2 / (1 + R^4).
\]

(A.18)

That means that it is only possible to get visibility values \( V_A = V_B = 1 \) when \( R_1 = R_2 = 1 \), which means perfect 50-50% atom beam splitters. When the misalignment of the beam splitters are the same, it is sometimes more useful to perform the substitutions \( \sin^2 \xi_1 = g(\Delta \Theta) \) and \( \cos^2 \xi_3 = \cos^2 \xi_1 = 1 - g(\Delta \Theta) \) into equations Eq. A.9 and Eq. A.10, with \( g(\Delta \Theta) \) the intensity function of the diffracted atom beam as function of the angle of misalignment. Then the expressions for the intensity become:

\[
I_A/A = 1 - 2g(\Delta \Theta)[1 - g(\Delta \Theta)][1 + \cos \varphi_{ai}]
\]

(A.19)

\[
I_B/A = 2g(\Delta \Theta)[1 - g(\Delta \Theta)][1 + \cos \varphi_{ai}],
\]

(A.20)

which results in

\[
V_B = 1,
\]

(A.21)

\[
V_A = \frac{2g(\Delta \Theta)[1 - g(\Delta \Theta)]}{1 - 2g(\Delta \Theta)[1 - g(\Delta \Theta)]}.
\]

(A.22)
Appendix B

Signal quality calculations

The environment influences the atom interferometer, resulting in measurement errors, influencing the quality of the atom interferometer signal. Measurement errors can be approached in basically two different ways: 1) as an oscillating phase error which gives insight in the frequency sensitivity and 2) as a Gaussian distribution of the phase error. These two approaches have to be calculated differently.

B.1 Oscillations

The oscillating motion of the atom interferometer results in an oscillation in the phase difference between the two arms of the atom interferometer which results in variations in the output signal on the detectors. The phase difference can be described by \( \varphi_{\text{as}}(t) = \varphi_{\text{exp}} + A \cos(\omega t) \) with \( \omega \) the oscillation frequency, \( A \) the amplitude of the oscillation and \( \varphi_{\text{exp}} \) the phase difference due to the experiment, which is the desired value. The intensity on the detector then becomes \( I(t) = I_0 \left(1 + \cos(\varphi_{\text{as}}(t)) / 2 \right) \) with \( I_0 \) the total intensity of the two arms of the atom interferometer. Usually the measurement takes some time, so the output signal has to be averaged over the measurement time \( \tau \). The general description for the detector signal in the presence of oscillations then becomes:

\[
U = \int_{t_0}^{t_0+\tau} I(t) \, dt
\]

\[
= \frac{U_0}{2} \left\{ 1 + \frac{1}{\tau} \int_{t_0}^{t_0+\tau} \left[ \cos(\varphi_{\text{exp}}) \cos(\varphi(t)) - \sin(\varphi_{\text{exp}}) \sin(\varphi(t)) \right] \, dt \right\}
\]

(B.1)

with \( U_0 = I_0 \tau \) and \( \varphi(t) = A \cos(\omega t) \). The terms \( \cos(\varphi(t)) \) and \( \sin(\varphi(t)) \) can then be expanded into a Taylor expansion of the \( \varphi(t) \) term. Because \( A \ll 1 \) this Taylor expansion can be limited to the second order. After integration over \( \tau \) the result becomes:

\[
U \approx \frac{U_0}{2} \left( 1 + \cos(\varphi_{\text{exp}}) \left[ 1 - \frac{A^2}{4} - \frac{A^2 \sin(\omega \tau)}{4 \omega \tau} \cos(\omega \tau + 2\omega t_0) \right] - A \sin(\varphi_{\text{exp}}) \frac{\sin(\omega \tau / 2)}{\omega \tau / 2} \sin(\omega \tau / 2 + \omega t_0) \right) \]

(B.2)
For oscillation frequencies $\omega \gg 2\pi/\tau$, $\sin(\omega \tau/2)/(\omega \tau/2) \rightarrow 0$ and certainly $\sin(\omega \tau)/(\omega \tau) \rightarrow 0$, resulting in a more simple expression. In this situation expression Eq. B.1 remains acceptable simple when the expansion is increased to the fourth order of $\varphi(t)$ in the Taylor expansion. For a long measurement time $\tau$, $U$ becomes:

$$U \approx \frac{U_0}{2} \left\{ 1 + \cos(\varphi_{\exp}) \left[ 1 - \frac{A^2}{8} \right]^2 \right\}. \quad (B.3)$$

The maximum visibility that can be achieved by taking a long measuring time then becomes:

$$V = \left( 1 - \frac{A^2}{8} \right)^2. \quad (B.4)$$

The measurements are usually done in the region where $\cos(\varphi_{\exp}) = 0$ because this region is most sensitive to variations in $\varphi_{\exp}$. In that region the approximations $\cos(\varphi_{\exp}) \approx \varphi_{\exp}$ and $\sin(\varphi_{\exp}) \approx 1$ can be used. For $\omega \leq 2\pi/\tau$ and because $A < 1$ equation Eq. B.2 can be simplified to

$$U \approx \frac{U_0}{2} \left\{ 1 + \varphi_{\exp} - A \frac{\sin(\omega \tau/2)}{\omega \tau/2} \sin(\omega \tau/2 + \omega t_0) \right\}. \quad (B.5)$$

This shows that for low frequencies and/or short measurement times the measured phase error between the two arms of the atom interferometer become equal to the amplitude of the oscillating phase error. For high frequencies and/or long measurement times the influence of the oscillating phase error decreases by $\omega \tau/2$.

### B.2 Random Noise

The approach for random noise is somewhat different than the previous type of phase error. The starting point here is a Gaussian distribution of the phase error between the two arms of the atom interferometer, with a maximum at $\varphi = 0$ and a standard deviation (spread) of $\sigma_\varphi$. The intensity in this situation is the same as in the previous paragraph ($I(\varphi) = I_0(1 + \cos \varphi_{\text{at}}(\varphi))/2$). For this calculation $\varphi_{\text{at}}(\varphi) = \varphi_{\exp} + \varphi$ is used. The calculated detector signal then becomes:

$$U = \int_{-\infty}^{+\infty} \tau I(\varphi) P(\varphi) d\varphi$$

$$= \frac{U_0}{2} \left\{ 1 + \int_{-\infty}^{+\infty} [\cos(\varphi_{\exp}) \cos(\varphi) - \sin(\varphi_{\exp}) \sin(\varphi)] P(\varphi) d\varphi \right\}$$

$$= \frac{U_0}{2} \left\{ 1 + \cos(\varphi_{\exp}) e^{-\frac{\sigma_\varphi^2}{2}} \right\}. \quad (B.6)$$

Here $P(\varphi)$ is the Gaussian distribution of the phase error $\varphi$. The visibility then becomes:

$$V = e^{-\frac{\sigma_\varphi^2}{2}}. \quad (B.7)$$
Both estimates of the visibility for the oscillating and the noise like phase error are averages of the function \( f(\varphi) = \cos \varphi \approx 1 - \varphi^2 / 2 \). The variable \( \varphi \) of the oscillating phase error is the time dependent function \( \varphi_{osc}(t) = A \cos(\omega t) \), and \( f(\varphi_{osc}(t)) \) will then be averaged over time, while for the noise like phase errors the distribution of the value of \( \varphi_{noise} \) was made Gaussian with a spread of \( \sigma_\varphi \) and the averaging of \( f(\varphi_{noise}) \) will now take place over the Gaussian distribution. Comparing the outcome of both averaging methods the relation

\[
\sqrt{2(1 - V)} \approx \sigma_\varphi \approx \frac{A}{\sqrt{2}} \quad \text{(B.8)}
\]

for small values of \( \sigma_\varphi \) and \( A \) emerges.
Appendix C

Optical Grating Theory

What will be shown here is a simple approximation to derive the basic behavior of a grating. Starting point is a grating which consists of a structure of equally spaced parallel lines which influence light.

The incoming and the outgoing light fields can be described as:

\[ E_{\text{in}}(\vec{x}) = \frac{E_i}{2} G_i(x, y) \left\{ e^{i(\vec{k}_i \cdot \vec{x} - \omega t)} + \text{c.c.} \right\} \quad \text{and,} \]
\[ E_{\text{out}}(\vec{x}) = \frac{E_o}{2} G_o(x, y) \left\{ e^{i(\vec{k}_o \cdot \vec{x} + \varphi(\vec{k}_o) - \omega t)} + \text{c.c.} \right\} \]

with \( G_{i,o}(x, y) \) the Gaussian function

\[ G_j(x, y) = A e^{-\frac{(x^2/w_{jx}^2 + y^2/w_{jy}^2)}{j = i, o} \ }

where \( A \) is the normalization factor of the Gaussian function, \( x \) and \( y \) are two components of vector \( \vec{x} \), and \( w_{jx} \) and \( w_{jy} \) are the waists of the laser beams in the \( x- \) and the \( y- \) direction. In these expressions \( \vec{k}_i \) and \( \vec{k}_o \) are the wave vectors of the incoming and the emerging light beams. The phase \( \varphi(\vec{k}_o) \) is the extra phase of the emerging light beam depending on the direction of the beam.

For this calculation only the far field results are of interest. This means that only the interference of the parallel light waves will be examined. The phase difference between the emerging and the incoming light waves at the grating surface depends on the position on the grating surface. The strength of the electric field in the far field approximation is the integral of the electric fields of all the parallel light beams emerging from the grating surface. This can be described by

\[ E_{\text{out}}(\vec{x}) = \int_S f(x, \Delta \vec{k}) E_{\text{in}}^+(\vec{x}_g) e^{i \vec{k}_o \cdot |\vec{x} - \vec{x}_g|} dS + \text{c.c.,} \]

where \( S \) is the integration surface area on the grating, and \( \Delta \vec{k} = \vec{k}_i - \vec{k}_o \). \( E_{\text{in}}^+(\vec{x}_g) \) is the positive frequency part of the electric field of the incoming laser beam at the surface of the grating. The function \( f(x, \Delta \vec{k}) \) describes the influence of the grating on the light which can change the intensity and/or the phase of this electric field. As mentioned before the structure of the grating consists of equally spaced parallel lines which will be taken parallel to the \( Y- \) axis. The \( Z- \) axis will be taken perpendicular to the grating surface (see Figure C.1). Therefore the function \( f(x, \Delta \vec{k}) \) of the grating can be described as a Fourier
Here $s$ is the period of the line structure of the grating, and the terms $g_N(\Delta k)$ are complex constants depending both on the wave vectors of the incoming and the emerging laser beams. This function is independent of position parameters $y$ and $z$ and therefore equation Eq. C.3 can be expanded to:

$$E_{out}(x) = E_i e^{i[k_y z - \omega t]} \int_S G(x_9 - x_{1b}, y_9) e^{i y_9 (\Delta k_y)} dy_9 \times$$

$$f(x_9 - x_{gr}) e^{i x_9 (\Delta k_x)} dx_9 + c.c.$$  

(C.5)

where $\Delta k_x$ and $\Delta k_y$ are the $x$ and $y$-components of vector $\Delta k$, $x_{1b}$ and $x_{gr}$ are respectively the position where the center of incoming laser beam hits the grating surface and the position of the grating, both in the $x$-direction. The last section of the equation $f(x_9 - x_{gr}) \exp(i x_9 (\Delta k_x))$ can be expanded to

$$f(x_9 - x_{gr}) e^{i x_9 (\Delta k_x)} = e^{i x_9 (\Delta k_x)} \times$$

$$\sum_{N=-\infty}^{N=+\infty} g_N(\Delta k) e^{i(x_9 - x_{gr})(2\pi N/s + \Delta k_x)}.$$  

(C.6)

A general result of the integration of a Gaussian function with a complex exponential function, like in equation Eq. C.5, is

$$A \int_{-\infty}^{\infty} e^{i x_9 w} e^{-\frac{(x_9 - x_{1b})^2}{w^2}} dx_9 = e^{-\omega^2 w^2/4} e^{i \omega x_{1b}},$$  

(C.7)

where in the case of equation Eq. C.6 $\omega = 2\pi N/s + \Delta k_x$ and $w$ is the waist of the laser beam. When the waist of the laser beam is much wider than the line spacing of the grating, which is usually the case, then $w^2/2\pi/s \gg 1$. This means that $2\pi N/s + \Delta k_x \ll 2\pi/s$ otherwise the amplitude will go to zero. The same reasoning can be followed for the $y$-direction. Here equation Eq. C.7 can be used by replacing both $x_9$ and $x_9 - x_{1b}$ by $y_9$ and $y$-components of vector $\Delta k$. 


Figure C.1: Definitions of the axis orientation.

sequence with

$$f(x, \Delta k) = \sum_{N=-\infty}^{N=+\infty} g_N(\Delta k) e^{i x_9 (2\pi N/s)},$$  

(C.4)
also the waist of the laser beam is much wider than the wavelength of the laser light, so that \( w k_i,o \gg 1 \) with \( k_i,o \) the length of the wave vector of either the incoming or the emerging laser beam. This means that \( \Delta k_y \ll k_i,o \) to have any significance. Therefore the relations between the wave vectors of the incoming and the emerging laser beams become:

\[
\Delta k_x \approx -2\pi N / s, \quad k_{iy} \approx k_{oy},
\]

where \( N \) becomes the order of the diffraction. The resulting expression for the emerging beam is:

\[
E_{\text{out}}(k_0) = \delta(k_{iy} - k_{oy}) E_i g_h(\Delta k) e^{i[k_x - 2\pi N x_{gr} / s - \omega t]} + c.c.,
\]

with the delta function defined as:

\[
\delta(k_{iy} - k_{oy}) = \begin{cases} 
1 & k_{iy} = k_{oy} \\
0 & k_{iy} \neq k_{oy}
\end{cases}
\]

It shows that a displacement of the laser beam perpendicular to the laser direction has no effect on the phase of the diffracted laser beam. A displacement of the grating however results in a phase shift of \( \Delta \phi_{oi} = -2\pi N x_{gr} / s \).

The components of the wave vector can be described as

\[
k_x = k \sin \Phi \sin \alpha, \quad k_y = k \cos \Phi, \quad k_z = k \sin \Phi \cos \alpha,
\]

with \( k \) the length of the wave vector, \( \Phi \) is the polar angle with the \( y \)-axis, and \( \alpha \) the azimuth angle in the \( XZ \)-plane. All the parameters \( k, \Phi \) and \( \alpha \) belong to one medium. The wave vector is different for every medium the laser goes through. The gratings will be used such that angle \( \Phi = \pi / 2 \). Thus for both the incoming and the emerging laser beam \( k_{iy} = k_{oy} = 0 \). The relation between the incoming and the emerging wave vectors is described by equation Eq. C.8. Using equation Eq. C.12a and \( k_{i,o}/n_{i,o} = 2\pi / \lambda_0 \) with \( \lambda_0 \) the wavelength of the laser light in vacuum and \( n_{i,o} \) the refractive indices, this relation can be transformed to

\[
n_i \sin \alpha_i - n_o \sin \alpha_o = -\frac{N \lambda_0}{s},
\]

where \( \alpha_i \) and \( \alpha_o \) are the angles of respectively the incoming and the emerging laser beam. This equation describes the angular relation between the original and the diffracted laser beam.
Appendix D

Technology assessments

The atom interferometer that will be built will primarily be used in QND-experiments. These experiments can give answers to fundamental questions in physics involving radiation being quantized and whether or not photons can be localized. To be able to do these experiments the separation distance of the two beams in the atom interferometer must be large. Because of this large separation distance the atom becomes highly delocalized which makes this atom interferometer also a useful experimental device for studies on other fundamental questions of quantum mechanics, which can be useful towards other investigations.

The atom interferometer can also be used in experiments other than QND. For instance:

- **Measuring rotations and accelerations.** Because of the large separation of the two arms the enclosed surface area becomes very large, making it very sensitive to both rotations and accelerations.

- **Atom Interactions** with other atoms or electromagnetic fields to study the characteristics of the atom in even more detail. The large separation of the atom beams facilitates such experiments.

- **Which-Way experiments** which can give insight in the Heisenberg's uncertainty principle, which states that it is impossible to measure both the momentum and the position of the atom at the same time without small errors. Whether this is indeed impossible is still an open debate.

It is clear that an atom interferometer is a very versatile instrument, which can contribute to investigations in many fields of physics.
Appendix E

Drawing Electronics
Bibliography

Quantum Rabi Oscillation: A Direct Test of Field Quantization in a Cavity.
PRL 76, 1800 (1996)

[2] Armand Koolen
Dissipative Atom Optics with Cold Metastable Helium Atoms
Thesis for Promotion at Technical University Eindhoven (2000)

Coherent atomic deflection by resonant standing waves
PRA 23, 23 (1980)

Bragg Diffraction Of He* by a Standing Light wave
Graduation report TU/e AQT/B 01-03 (May 2001)

Atom Interferometer Based on Bragg Scattering from Standing Light Waves
PRL 75 2638 (1995)

Laser Cooling and Trapping

Ultra-high sensitivity accelerometers and gyroscopes using neutral atom matter-waves interferometry
Phisica B 151 262 (1988)

Rotation sensing with a dual atom-interferometer Sagnac gyroscope
Class. Quantum Grav. 17 2385 (2000)

[9] P.R. Berman
Atom Interferometery
Academic Press Inc. ISBN 0-12-092469-9

[10] D.J. Griffiths
Introduction to Quantum Mechanics
*Trapping of Atoms by Resonance Radiation Pressure*
PRL 40 729 (1978)

*High contrast Mach-Zehnder lithium atom interferometer in the Bragg regime*

*Feedback Control of Dynamic Systems*
Addison-Wesley Publ. Comp. ISBN 0-201-50862-1