MASTER

Modelling aspects of stress analysis for abdominal aortic aneurysms

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Nomenclature

\[(x, y, z)\] Cartesian coordinates (Eulerian description)

\[(X, Y, Z)\] Cartesian coordinates (Lagrangian description)

\[(r, \theta, z)\] Cylindrical coordinates (Eulerian description)

\[(R, \Theta, Z)\] Cylindrical coordinates (Lagrangian description)

\[e_1, e_2, e_3\] Unit Cartesian vectors (Eulerian description)

\[E_A, E_B, E_C\] Unit Cartesian vectors (Lagrangian description)

\[e_r, e_\theta, e_z\] Unit cylindrical vectors (Eulerian description)

\[E_r, E_\theta, E_z\] Unit cylindrical vectors (Lagrangian description)

\[\mathcal{F}\] Deformation gradient tensor

\[\mathcal{C}\] Right Cauchy-Green strain tensor

\[\mathcal{B}\] Left Cauchy-Green strain tensor

\[\mathcal{E}\] Lagrangian deformation tensor

\[E\] Elasticity Modulus

\[\mu\] Shear Modulus

\[\nu\] Poisson ratio

\[\rho\] Density

\[\lambda\] Stretch ratio

\[\sigma_{ij}\] Cauchy stresses

\[\sigma_{ij} \sum_{j=1}^{N} \frac{\partial \sigma_{ij}}{\partial x_j} \text{(summation convention)}\]
Acknowledgements

This report is a result of the 9 months final internship of Ellen van Nunen at Philips Medical Systems in Best, at the department 'Advanced development' in the context of the Hemodyn project. The Hemodyn project is concerned with patient specific modelling of aneurysms in order to determine the wall stress which might be a good rupture predictor.

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An Abdominal Aortic Aneurysm (AAA) is a permanent dilatation of the aorta. Rupture of an AAA is lethal in most cases. Nowadays, the diameter of an AAA is used as the parameter for rupture prediction, but this can be improved by determining the wall stress using a computational method instead.

There are some difficulties in determining the stress. First, the stress depends on the shape of the aneurysm, the internal pressure and the properties of the wall material. It is hard or even impossible to determine these aspects exactly. Consequently, errors occur in the stress. One goal of this study is to determine the influence of perturbations in shape, internal pressure and properties of the wall material on the stress computed.

To be able to handle this problem analytically a simplification is necessary. For this reason a cylindrical tube will be considered. The choice of the elasticity model is a linear elastic model with small deformations. Making use of perturbation methods we can conclude that small perturbations in internal pressure and properties of the wall material have little influence on the stress. But the shape does have much influence on the stress. So the shape is an important parameter in predicting stress.

Another difficulty of determining the stress are modelling aspects, such as modelling thrombus, patient specific shape, 3D or 2D.

The second part of this thesis focuses on the role of the spatial dimension in the model, by considering a cross sectional 2D model versus a full 3D model. Several investigations are done in 2D [Di Martino et al., 1998], [Okamoto et al., 2003]. Some questions arise. Do conclusions for 2D modelling still hold for 3D geometries? What is the influence of the curvature of the shape in longitudinal direction? To answer these questions, 3D modelling is done on four aneurysms. The stress is determined, the slices on which the highest stress occur are compared to results of 2D modelling (plane strain). Results show a large difference between the values of stress in the 2D and 3D modelling. The difference is mainly caused by the curvature of the aneurysm in the longitudinal direction.

The third part deals with the influence of thrombus, which is a weak material, clotted at the vessel wall. Is it necessary to model thrombus to determine whether or not an AAA will rupture? This investigation is done by simulations. Since there is no 3D model available within the project which handles thrombus, the investigation has to be done in 2D. Results show that the stresses in the model without thrombus are significantly higher then the stresses in the model with thrombus, implying that thrombus decreases the stress.

Another aspect which is considered is the patient specific shape. Is the patient specific shape necessary or would an axisymmetric shape be sufficient in order to determine whether or not an AAA will rupture? The 2D simulations of the patient specific geometry are compared to 2D analytical plane strain calculations on an axisymmetric shape. Results show that the stresses in the model with patient specific shape are significantly higher than the stresses in the model with axisymmetric shape.

We need to be careful to conclude the same influences for the 3D modelling, but it is likely that axisymmetry and the presence of thrombus reduce the stresses in 3D as well.
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Chapter 1

Introduction

1.1 Background

An Abdominal Aortic Aneurysm (AAA) is a permanent dilatation of the aorta. The word "aneurysm" is derived from the Greek word "aneurysma" meaning "a widening". Most aortic aneurysms are fusiform, they are shaped like a spindle (fusus means spindle in Latin) with widening all around the circumference of the aorta (Figure 1.1). Saccular aneurysms occur most often in the head.

![Figure 1.1: Saccular and fusiform aneurysms.](image)

Most fusiform aortic aneurysms are located below the renal arteries and above the aortic bifurcation to the iliac arteries as can be seen in Figure 1.2. This is called an infra-renal AAA.
Abdominal Aortic Aneurysm.

Because of the high pressure caused by the pumping of the heart, there are significant stresses acting on the AAA wall. When the wall stresses exceed the stresses the tissue can withstand the aorta will rupture. AAA rupture is the 13th most common cause of death in the USA. A reliable predictor for rupture has not been found yet, and since the surgery -which actually consists of placing a prosthesis for the normal aortic lumen [van Oijen, 2003]- has a high risk, it is very important to find a reliable predictor.

Nowadays the diameter of an AAA is used as predictor. Surgery is done when the diameter exceeds 5 cm. However, this is not a fail-safe criterion since some AAA rupture at a size smaller than 5 cm and other have grown to as large as 8 cm without rupturing [Raghavan and Vorp, 2000].

Often, calcifications and thrombus are found in an AAA. Calcifications are atherosclerotic hardenings in the vessel wall. Thrombus is a weak material, clotted at the vessel wall. Research has been performed on the material properties of thrombus [Di Martino et al., 1998], but these properties seem to differ a lot.

In Latin, blood is called "lumen". This word will be used in this report to denote the volume in which blood flows. A possible cross-section of an AAA is drawn in Figure 1.3.

![Figure 1.2: Abdominal Aortic Aneurysm.](image)

![Figure 1.3: A possible cross section of an AAA.](image)
In the Hemodyn project, which is a cooperation between the Technische Universiteit Eindhoven, the Thorax centre (Erasmus university Rotterdam) and Philips Medical Systems in Best, we want to model the wall mechanics and fluid mechanics in order to determine whether a specific AAA shall rupture or not. Thus we need to know the stress in the AAA wall.

1.2 Problem

In order to predict whether an AAA will rupture or not, we determine the stress in the AAA wall.

There are several difficulties in determining the stress. First problem is to get the exact information which is needed to predict the stress. Material properties (such as the shear modulus), geometrical properties (such as the shape, the thickness of the wall) and boundary conditions (for example pressure) are needed. Information like the geometry, thickness of the wall, velocity of the blood can be obtained by CTA (CT angiography) or MRA (MR angiography). For example movement of a patient (breathing) or the inhomogeneity of the magnetic field may cause noise in the scan-images and therefore the images may not be very accurate. The blood pressure is measured just before and after a patient is scanned, but on the moment of scanning itself it is an unknown. So only an estimation for the pressure can be given.

Second problem comes to the modelling. We want to know which aspects are important to take in account for predicting the stress as good as possible. We question ourselves whether or not it is necessary to model thrombus. When the stresses are (almost) the same it would not be useful to make a simulation program which is able to model thrombus. We also want to know whether or not we need the patient specific shape in stead of an axisymmetric shape. Further, maybe 2D modelling gives enough information about the stress.

1.3 Literature review

Many articles are written on AAA’s. Most of them agree that stress is a good rupture predictor, for example [Elger et al., 1996, Raghavan and Vorp, 2000] has shown that the geometry of the AAA has a large influence on the wall mechanics and that the strains and stresses in
the wall are a significant predictor for rupture.

Often results from 2D modelling are used for 3D geometries. Examples are [Di Martino et al., 1998] and [Okamoto et al., 2003].

Di Martino et al concluded that well organized thrombus reduces the effect of the pressure load on the aneurysmal aortic wall. Results should show how the presence of the thrombus, the mechanical properties of the thrombus and the eccentricity of the patient lumen influence the aortic wall stress distribution. A 2D simplified model was used for this investigation.

Okamoto et al used a cylindrical mathematical model of the aorta in order to predict distensibility and wall stress which may be contributing factors to the risk of aortic rupture and dissection.

On the influence of thrombus is also some research done. There has been an investigation on the pressure within the aneurysmal thrombus compared to the systemic pressure, [Schurink et al., 2000]. Results showed that thrombus within the aneurysm does not reduce both the mean and the pulse pressure near the aneurysmal wall and thus will not reduce the risk of rupture of the aneurysm. There is some discussion about this article. Many other articles do not agree with the results ([Inzoli et al., 1993], [Di Martino et al., 1998], [Mower et al., 1997], [Satta et al., 1996]).

1.4 Goals

The main goal of this project is to determine important modelling aspects in order to predict stress in the wall of an aneurysm.

The main goal is split into three problem descriptions:

1. *Sensitivity analysis*; we want to know the influences of small perturbations of internal pressure, properties of wall material and shape on the stress in a simplified model.

2. *3D versus 2D*: we want to compare stress in a 3D geometry with the stress in a 2D plane strain problem (Figure 1.5).

3. *Modelling thrombus and using patient specific shape*; we want to compare stresses when modelling with and without thrombus and when modelling with patient specific shape and axisymmetric shape in 2D (Figure 1.6).
1.5 Overview

Some theory on elasticity (and continuum mechanics) is given in Chapter 2. The methods which are used to solve the problems are described in Chapter 3. Since there are three goals, there are three methods described. The results and discussions of the three problems are described in Chapter 4, followed by the conclusions and recommendations in Chapter 5.
Chapter 2

Continuum mechanics and theory of elasticity; a survey

In order to predict the stresses of an AAA we need to understand the basic aspects of mechanics. The wall of an AAA is assumed to be an elastic medium. In this Chapter theory of elasticity for small and large deformations is derived based on the fundamentals of continuum mechanics. The resulting linear elastic and non-linear elastic (Neo-Hookean) models will be used in the methods described in Chapter 3 to derive deformation and stress in aneurysm models.

2.1 Fundamentals of continuum mechanics

Continuum mechanics aims at describing the motion and deformation of continuous material bodies undergoing forces (or other sources, such as heating or electromagnetic fields). The basic of classical continuum mechanics is formed by a set of global balance laws for mass, momentum, moment of momentum, and energy. These laws hold universally, i.e. irrespective of the specific material of the body. The specific type of material (elastic solid, viscous fluid, etc.) is described by so called constitutive equations. In this report, we will consider an elastic solid. A well-known constitutive equation for a linear elastic solid is Hooke’s law. We will consider nonlinear elastic solids as well. The derivations of the equations is based on the manuscripts of [van de Ven, 2001] and [Mase and Mase, 1999].

In continuum mechanics, three fundamental assumptions are made:

1. The substance of material bodies is distributed uniformly and completely fills the space it occupies.
   This assumption enables us to assign a positive density $\rho$ to every point of a material body.

2. Two material points, initially infinitesimally close together remain infinitesimally close during the deformation.

3. Internal or boundary points (at the surface of the body) remain internal or boundary points, respectively, during the deformation of the body.
   The latter two assumptions exclude cracks and ensures that the deformations remain smooth (i.e. the mapping representing the motion of a deformable body is a bijective mapping).
In continuum mechanics, we define a material body $\mathcal{B}$ as a set of elements $X$, called material points. The set of the positions $x$ of all $X \in \mathcal{B}$ with respect to a fixed origin at some instant of time is said to define the configuration $G \subset \mathbb{R}^3$ of the body at that instant:

$$G := \{ x \in \mathbb{R}^3 | x = x(X,t), X \in \mathcal{B}, t \in \mathbb{R} \}.$$  

A change of configuration is the result of the motion of the body. A motion of a body $\mathcal{B}$ is a continuous following of displacements. To describe the motion of $\mathcal{B}$ in a geometrical sense, that is by a mapping from $\mathbb{R}^3 \rightarrow \mathbb{R}^3$, we label each material point $X \in \mathcal{B}$ with a reference position $X \in \mathbb{R}^3$ and we introduce the reference configuration $G_r$ as the set of $X$ for all $X \in \mathcal{B}$. Then, $X$ is the position of the material point $X$ in the reference configuration. This vector $X$ does not change during the motion of $X$ (as it is a label for $X$). The motion of $X$ from $X$ to $x$ is described by the bijective mapping from $\mathbb{R}^3 \rightarrow \mathbb{R}^3$:

$$\forall X \in G_r, t \in \mathbb{R} : x = x(X, t).$$

In this report, we will use two different types of notations:

1. **Tensor notation.** This notation is independent of any basis and is specially used for general formulations. Here, a scalar $\alpha$ is a 0-tensor ($\alpha \in \mathbb{R}$), a vector $a$ is a 1-tensor ($a \in \mathbb{R}^3$) and a linear mapping $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a 2-tensor.

2. **Index notation.** This is a notation in components with respect to a basis $\{Oe_1,e_2,e_3\}$, especially used for explicit equations with respect to this basis. In this notation, summation convention is applied (automatic summation over double indices). For more details of these notations, see [van de Ven, 2001].

### 2.2 Balance laws

In this section, four mechanical balance laws will be postulated. However, before doing this, we first introduce a material partial volume $b \subseteq \mathcal{B}$. The configuration of any subset $b \subseteq \mathcal{B}$ at time $t$ will be denoted by $g(b, t)$ and its boundary by $\partial g(b, t)$. The general expression of a global balance law is of the form

$$\forall b \subseteq \mathcal{B} \quad \frac{d}{dt} \int_{g(b,t)} \rho \Phi dV = \int_{g(b,t)} \rho s dV + \int_{\partial g(b,t)} t(n) dS. \quad (2.1)$$

Here, $\rho \Phi$ denotes the physical quantity considered in the balance, with $\Phi$ being either a scalar or a vector, $\rho s$ is the source per unit of time of $\rho \Phi$ and $t(n)$ is the flux per unit of time over the boundary of $g(t)$, where $n$ denotes the unit outward normal to $\partial g(t)$. The field $\Phi$ can be either a function of the local coordinates $x$ and $t$ ($\Phi = \bar{\Phi}(x, t)$; Euler formulation), or of the material coordinates $X$ and $t$ ($\Phi = \tilde{\Phi}(X, t)$; Lagrangian formulation). Besides a function of $x$ and $t$, or $X$ and $t$, the flux $t(n)$ is also a function of $n$. Thus we have to distinguish between two time derivatives:

1. the local time derivative (notation $\frac{\partial}{\partial t}$)

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \bar{\Phi}(x, t)}{\partial t}.$$
2. the material (sometimes called total) time derivative (notation \( \frac{d}{dt} \) or a superimposed \( \cdot \))

\[
\dot{\Phi} = \frac{d\Phi}{dt} = \frac{\partial \tilde{\Phi}(X, t)}{\partial t}.
\]

The relation between the two operators is

\[
\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + (\text{grad}\Phi, v),
\]

where \( v = \dot{x} \) is the velocity of the material point \( X \subset \mathcal{B} \). The operator ‘grad’ is defined with respect to \( x \), so for \( \Phi \in \mathbb{R} \),

\[
(\text{grad}\Phi)_i = \frac{\partial \Phi}{\partial x_i} \quad \text{and} \quad (\text{grad}\Phi, v) = v_i \frac{\partial \Phi}{\partial x_i},
\]

but for \( \Phi \equiv \Phi \in \mathbb{R}^3 \)

\[
(\text{grad}\Phi)_{ij} = \frac{\partial \Phi_i}{\partial x_j}, \quad \text{and} \quad (\text{grad}\Phi, v)_i = \frac{\partial \Phi_i}{\partial x_j} v_j.
\]

Continuum mechanics is build upon a fundament of four global conservation laws, which serve as basic postulates, namely

1. the law of conservation of mass,
2. the law of conservation of momentum,
3. the law of conservation of moment of momentum,
4. the law of conservation of energy.

We can evaluate these global laws in a standard way to local balance equations (partial differential equations) together with the associated jump or discontinuity conditions; see [van de Ven, 2001]. For this, two essential steps are necessary:

1. In the left-hand side of (2.1), the order of differentiation with respect to \( t \) and integration has to be changed. This is done by use of a transport theorem, which is based on the conservation of mass of a material volume element: \( \rho dV = \rho_0 dV_r \), and states that

\[
\frac{d}{dt} \int_g \rho \Phi dV = \int_g \rho \dot{\Phi} dV.
\]

2. The surface integral in the right-hand side of (2.1) has to be changed into a volume integral. For this, it is first used that \( t_n \) is linear in \( n \) (this can be proved by taking in (2.1) for \( b \) an infinitesimally small tetrahedron, (see [van de Ven, 2001], Section 3.4), so \( t_n(x, t, n) = T(x, t) n \), and then applying Gauss’ theorem, by which

\[
\int_{\partial g} t_n dS = \int_{\partial g} T n dS = \int_g \text{div} T dV.
\]

We mention here that in case \( t_n \in \mathbb{R} \), a scalar, \( T \) is a vector, whereas in case \( t_n \in \mathbb{R}^3 \), a vector, \( T \) is a 2-tensor. In the latter case, \( \text{div} T \) is such that

\[
(\text{div} T)_i = \frac{\partial T_{ij}}{\partial x_j}.
\]
We can now write (2.1) as one volume integral over an arbitrary material partial volume \( b \subseteq \mathcal{B} \). But then (2.1) must also hold locally, yielding the general local balance law

\[ \forall x \in G : \rho \dot{\Phi} = \rho s + \text{div} T. \]

Let \( \Sigma \) be a material discontinuity surface for \( \Theta \) within \( \mathcal{B} \) (\( \Sigma \subset \mathcal{B} \subset \mathbb{R}^3 \)). Then, this localization procedure also yields the associated general discontinuity condition (under the exclusion of surface sources)

\[ [T_n] = T^+ n - T^- n = 0, \quad \text{across } \Sigma, \]

where \( n \) is the normal on \( \Sigma \), \( T^+ \) the value of \( T \) just behind \( \Sigma \) (in the direction of \( n \)) and \( T^- \) the value of \( T \) just before \( \Sigma \), see Figure 2.1.

![Figure 2.1: Material discontinuity surface \( \Sigma \).](image)

We now proceed by postulating the four global conservation laws mentioned above, followed by the local balance laws and the discontinuity conditions. In all these laws, \( b \subseteq \mathcal{B} \) is an arbitrary partial material volume of \( \mathcal{B} \).

1. **Conservation of mass** implies, (\( \phi = 1 \)),

\[ \forall b \subseteq \mathcal{B} \quad \frac{d}{dt} \int_b \rho dV = 0, \]

yielding the local balance equation of mass or the *equation of continuity*

\[ \frac{dp}{dt} + \rho \text{div} \mathbf{v} = \frac{\partial \rho}{\partial t} + \text{div} (\rho \mathbf{v}) = 0. \]

Since there is no mass flux across \( \partial g \), no discontinuity condition applies here.

2. **Conservation of momentum** states that the rate of change of the momentum equals the resultant force acting on the body (Newton’s law). This resultant force is the sum of body forces \( b \) and surface tractions \( t_n \). In other words

\[ \frac{d}{dt} \int_{g(b,t)} \rho \mathbf{v} dV = \int_{g(b,t)} \rho \mathbf{v} dV + \int_{\partial g(b,t)} t_{(n)} dS, \quad \text{for all } b \subseteq \mathcal{B}. \]

To convert this expression to its local version, we need to express \( t_n \) in \( n \), according to \( t_{(n)}(x, t, n) = T(x, t)n \). In continuum mechanics, this relation is referred to as Cauchy’s...
stress law. The tensor $\mathbf{T}$ is called the *Cauchy stress tensor*. The surface traction $\mathbf{t}(\mathbf{n})$ is also called the *stress vector*; it represents the force per unit of area acting on a surface element with outward normal $\mathbf{n}$. The components of $\mathbf{T}$, denoted by $\sigma_{ij}$,

\[ \sigma_{ij} = (\mathbf{e}_i, \mathbf{T} \mathbf{e}_j), \quad i, j \in (1, 2, 3) \]

generate the *stresses*; $\sigma_{ij}$ represents the force per unit of area in $\mathbf{e}_i$-direction acting on a surface element with outward normal in $\mathbf{e}_j$-direction. In component form, Cauchy’s law reads

\[ \mathbf{t}_i = (\mathbf{t}(\mathbf{n})(\mathbf{x}, t, \mathbf{n}), \mathbf{e}_i) = \sigma_{ij} \mathbf{n}_j. \]

From this relation, it follows immediately that

\[ \sigma_{ij} = (\mathbf{t}(\mathbf{n})(\mathbf{x}, t, -\mathbf{e}_j), -\mathbf{e}_i), \]

in accordance with the basic mechanical principle *‘Action is reaction’*. The local balance law of momentum, or *equation of motion*, states

\[ \forall \mathbf{x} \in \mathcal{G} \quad \text{div} \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}} \quad \leftrightarrow \quad \sigma_{ij,j} + \rho b_i = \rho \dot{v}_i, \quad (2.2) \]

and the associated discontinuity condition expresses the continuity of the stress vector across a material discontinuity surface $\Sigma$,

\[ [\mathbf{T} \mathbf{n}] = 0 \quad \leftrightarrow \quad [\sigma_{ij} \mathbf{n}_j] = 0 \quad \text{at} \ \Sigma. \]

In case of a static situation, when $\rho \dot{\mathbf{v}} = 0$, the equation of motion reduces to the *equation of equilibrium*:

\[ \text{div} \mathbf{T} + \rho \mathbf{b} = 0 \quad \leftrightarrow \quad \sigma_{ij,j} + \rho b_i = 0. \]

3. **Conservation of moment of momentum** states that the rate of change of moment of momentum of a body with respect to a given point is equal to the moment of the surface and body forces with respect to that point:

\[ \frac{d}{dt} \int_{g(b,t)} (\mathbf{x} \times \mathbf{v}) \rho dV = \int_{g(b,t)} (\mathbf{x} \times \mathbf{b}) \rho dV + \int_{\partial g(b,t)} (\mathbf{x} \times \mathbf{t}(\mathbf{n})) dS, \quad \text{for all} \ b \subset \mathcal{B}. \]

We can also write this in component form as

\[ \frac{d}{dt} \int_{g(b,t)} \rho \varepsilon_{ijk} x_j v_k dV = \int_{g(b,t)} \rho \varepsilon_{ijk} x_j b_k dV + \int_{\partial g(b,t)} \varepsilon_{ijk} x_j (\mathbf{t}(\mathbf{n}))_k dS, \quad \text{for all} \ b \subset \mathcal{B}, \quad (2.3) \]

with $\varepsilon_{ijk}$ the permutation symbol:

\[ \varepsilon_{ijk} = \begin{cases} 1, & \text{if the numerical values of } ijk \text{ are an even permutation of 1,2,3;} \\ -1, & \text{if the numerical values of } ijk \text{ are an odd permutation of 1,2,3;} \\ 0, & \text{if numerical values of } ijk \text{ appear in any other sequence.} \end{cases} \]

Making use of the equation of motion in the localized version of $(2.3)$, we arrive at

\[ \varepsilon_{ijk} \sigma_{kj} = 0 \quad \Rightarrow \quad \sigma_{ji} = \sigma_{ij} \leftrightarrow \mathbf{T}^T = \mathbf{T}. \]

In words: the stress tensor is symmetric.
4. Conservation of energy states that for every subset \( b \) of \( \mathcal{B} \) the material time derivative of the kinetic (\( \dot{K} \)) plus internal (\( \dot{W} \)) energies equals the sum of the power of the mechanical forces (\( P \)) and the added heat per unit of time (\( Q \)) (first law of thermodynamics):

\[
\dot{K} + \dot{W} = P + Q, \quad \text{for all } b \subset \mathcal{B}.
\]  

(2.4)

By definition, we have

\[
K(b) = \int_{g(b,t)} \frac{1}{2} \rho(v,v) dV, \quad \text{(2.5)}
\]

\[
W(b) = \int_{g(b,t)} \rho E_{int} dV, \quad \text{(2.6)}
\]

\[
P(b) = \int_{g(b,t)} (b,v) \rho dV + \int_{\partial g(b,t)} (t(n), v) dS, \quad \text{(2.7)}
\]

\[
Q(b) = \int_{g(b,t)} pr dV - \int_{\partial g(b,t)} (q, n) dS, \quad \text{(2.8)}
\]

with \( E_{int} \) the internal energy density per unit of mass, \( q \) the heat flux vector and \( \rho r \) the heat source. Substitution of (2.5)-(2.8) in (2.4) gives the equation of conservation of energy

\[
\frac{d}{dt} \int_{g(b,t)} \left( \frac{1}{2} (v,v) + E_{int} \right) \rho dV = \int_{g(b,t)} ((b,v) + r) \rho dV + \int_{\partial g(b,t)} ((t(n), v) - (q, n)) dS.
\]  

(2.9)

With use of the equation of motion, the local energy balance can be reduced to its final form:

\[
\rho \dot{E}_{int} = \text{tr}(TD) + \rho r - \text{div} q \quad \leftrightarrow \quad \rho \dot{E}_{int} = \sigma_{ij} D_{ij} + \rho r - q_{i,i}, \quad \text{(2.10)}
\]

where \( \text{tr}(.) \) stands for 'trace', and \( D \) is the rate of deformation tensor, having as components

\[
D_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).
\]

The associated discontinuity condition expresses the continuity of heat flux, i.e.

\[
[(q, n)] = 0 \leftrightarrow [q, n_i] = 0, \quad \text{at } \Sigma.
\]

2.3 Theory of elasticity

2.3.1 Introduction of an elastic medium

Elasticity is a subject that has interested people for many years. In Love’s treatise on elasticity theory (1893) it is stated:

"The property of recovery of an original size and shape is the property that is termed elasticity. The changes of size and shape are expressed by specifying strains."

Elastic behavior is characterized by the following two conditions:
- The stress in a material is a unique function of the strain.
- The material has the property for complete recovery to a "natural" shape upon removal of the applied forces.

The relations between stress and strain depend on the material. Nowadays it is not yet agreed what kind of model is valid for an AAA-wall. Therefore, two models will be presented:

- Linear elastic model, assuming small deformations and stresses.
- Linearly elastic, but geometrically nonlinear, model, allowing large deformations, but maintaining a linear relationship between deformations and stresses.

A third model, for modelling a healthy aortic wall is presented in Appendix B. It considers a nonlinear elastic model, allowing for large deformations.

In the previous section we derived the balance laws. These laws form an incomplete system of equations for the unknown field variables introduced there. To make the system complete, it has to be supplemented by a set of constitutive equations, in this case for a linearly elastic medium. These constitutive equations relate the stresses to the deformations.

The stress tensor $\mathbf{T}$ is introduced by means of Cauchy’s stress formula

$$t_{(n)} = \mathbf{T} \mathbf{n},$$

revealing for the components $\sigma_{ij}$ of $\mathbf{T}$,

$$\sigma_{ij} = (e_i, t_{(n)}(\mathbf{x}, t, e_j)).$$

For a visualization of the stress components in Cartesian components, see Figure 2.2.

![Figure 2.2: Cauchy stress in Cartesian coordinates.](image-url)
The Cauchy stress tensor can be represented by a matrix. The eigenvalues of this matrix are called the principal stresses. The eigenvectors are called principal stress directions. The von Mises stress is defined as

$$\sigma_{vm} = \sqrt{\frac{1}{2} \left( \sum_{i=1}^{n} (\sigma_i^2 - \sigma_{ij}^2) \right)},$$

with $\sigma_i$ the eigenvalues of the stress matrix and $n$ the dimension of the stress matrix.

The stress vector on a surface through a material point can be written as the sum of two vectors, a vector perpendicular to the surface and a vector tangential to the surface, according to

$$t(n) = (n, Tn)n + (Tn - (n, Tn)n) = \sigma_n n + t_s,$$

with

$$\sigma_n = (n, Tn), \quad \text{and} \quad t_s = Tn - \sigma_n n = t(n) - \sigma_n n.$$

These vectors are called normal stress vector and shear stress vector, respectively. Their respective magnitudes are called normal stress and shear stress.

A material body can deform, i.e. it can change in shape and volume. To describe this deformation, we introduce the deformation gradient tensor $F$ as

**Definition 2.1.**

$$F = \frac{\partial x}{\partial X}. \quad (2.11)$$

Here, $x = x(X,t)$ denotes the current configuration and $X$ the reference configuration. For elastic media, the reference configuration $G_r$ is always chosen equal to the undeformed state. The tensor $F$ characterizes the local deformation in a material point $X$. Because of the bijective relation between the current and initial configuration, the inverse of the deformation tensor exists as (now $X = X(x,t)$)

$$F^{-1} = \frac{\partial X}{\partial x}.$$

We want an expression for the stress tensor $T$ related to the deformation. When we assume $T$ to be a function of the deformation gradient tensor, it would imply that a rigid rotation of a body produces stress. This is not realistic, so we need to find a characteristic for the deformation that is rotation invariant. Therefore, rotation-invariant deformation tensors will be introduced.

A deformation tensor is based upon the change of the distance between material points during deformation; for instance

$$||dx||^2 - ||dX||^2 = (\mathcal{F}dx, \mathcal{F}dx) - (dX, dX) = ((\mathcal{F}^T \mathcal{F} - I)dX, dX).$$

**Definition 2.2.** The right Cauchy-Green strain tensor $C$ is defined as

$$C = \mathcal{F}^T \mathcal{F}. \quad (2.12)$$

Note that this tensor is symmetric.

The tensor $C$ is rotation invariant, because substitution of $\mathcal{F} = R\mathcal{U}$ with $R$ a rotation matrix gives $C = (R\mathcal{U})^T (R\mathcal{U}) = U^T R^T R \mathcal{U} = U^T \mathcal{U}$, which is independent of the rotation.
Definition 2.3. We define the Lagrangian deformation tensor $E$ as
\[ E = \frac{1}{2}(\mathcal{C} - \mathcal{I}). \] (2.13)

Definition 2.4. Besides the right, also the left Cauchy-Green strain tensor can be defined as
\[ B = \mathcal{F} F^T. \]

Note that also $B$ is symmetric and rotation invariant.

Definition 2.5. The displacement in material description is defined as (we omit here and in the sequel of this Chapter the explicit dependence of $t$)
\[ u(X) = x(X) - X. \] (2.14)

Introducing the component notation $X = X_A e_A$, $A \in (1, 2, 3)$, and $x = x_i e_i$, $i \in (1, 2, 3)$, we can write (2.14) as
\[ u_A = x_i \delta_{iA} - X_A. \]

Substitution of (2.12) and (2.14) in (2.13) gives
\[ E = \frac{1}{2} \left( \left( \frac{\partial u}{\partial x} + \mathcal{I} \right)^T \left( \frac{\partial u}{\partial x} + \mathcal{I} \right) - \mathcal{I} \right) \]
\[ = \frac{1}{2} \left( \frac{\partial u}{\partial X} + \left( \frac{\partial u}{\partial X} \right)^T + \left( \frac{\partial u}{\partial X} \right)^T \left( \frac{\partial u}{\partial X} \right) \right), \]
or, in components,
\[ E_{AB} = \frac{1}{2} \left( \frac{\partial u_A}{\partial X_B} + \frac{\partial u_B}{\partial X_A} + \frac{\partial u_C}{\partial X_A} \frac{\partial u_C}{\partial X_B} \right) \]
\[ = \frac{1}{2}(u_{A,B} + u_{B,A} + u_{C,A} u_{C,B}), \]

where we have introduced the notation
\[ A = \frac{\partial}{\partial X_A}. \]

Likewise we will use
\[ i = \frac{\partial}{\partial x_i}. \]

Definition 2.6. The spatial velocity gradient $L$ is defined by
\[ \mathcal{L} = \text{grad} v \leftrightarrow L_{ij} = \frac{\partial v_i}{\partial x_j} = v_{i,j}, \]
with $v = v(x, t)$ the velocity; note that $v = \dot{u}$. The tensor $L$ is related to $\mathcal{F}$ by
\[ \mathcal{L} = \dot{\mathcal{F}} \mathcal{F}^{-1}. \]
A decomposition of this tensor into its symmetric and skew-symmetric parts is

\[ \mathcal{L} = \mathcal{D} + \mathcal{W} \quad \leftrightarrow \quad L_{ij} = D_{ij} + W_{ij}, \]

with

\[ D = \frac{1}{2}(\mathcal{L} + \mathcal{L}^T) \quad \leftrightarrow \quad D_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \]

\[ W = \frac{1}{2}(\mathcal{L} - \mathcal{L}^T) \quad \leftrightarrow \quad W_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right). \]

Here \( \mathcal{D} \) is called the rate of deformation tensor and \( \mathcal{W} \) the vorticity or spin tensor. We are now able to define the concept elastic medium. An elastic medium is defined as a medium in which the stress tensor \( T \), as well as the internal energy density \( E_{\text{int}} \), only depends on the Cauchy-Green tensor \( C \) (or \( B \)),

\[ T = \mathcal{G}(C). \tag{2.15} \]

Especially for linear elastic media, it is more appropriate to replace \( C \) by \( \mathcal{E} \), as we will see in the next section. Requirements of material objectivity (see [Geers et al., 2003], Chapter 6.2.3) induce the replacement of \( C \) by \( \mathcal{B} \) in (2.15), i.e.

\[ T = \mathcal{H}(\mathcal{B}). \tag{2.16} \]

The Cayley-Hamilton theorem states that a 2-tensor \( \mathcal{B} \in \mathbb{R}^3 \times \mathbb{R}^3 \) satisfies its own characteristic equation:

\[ \mathcal{B}^3 - I_B \mathcal{B}^2 + II_B \mathcal{B} - III_B \mathcal{I} = 0, \]

where \( I_B \), \( II_B \) and \( III_B \) are the invariants of \( \mathcal{B} \).

Assuming that \( \mathcal{H}(\mathcal{B}) \) can be expressed as a polynomial in \( \mathcal{B} \), we can eliminate all terms with powers of \( \mathcal{B} \) equal to or higher than 3 in favour of \( \mathcal{B}^2 \), \( \mathcal{B} \), and \( \mathcal{I} \). The most general form of (2.16) thus becomes

\[ T = \beta_0 \mathcal{I} + \beta_1 \mathcal{B} + \beta_2 \mathcal{B}^2, \tag{2.17} \]

where \( \beta_0 \), \( \beta_1 \) and \( \beta_2 \) are functions of the invariants \( I_B \), \( II_B \) and \( III_B \).

### 2.3.2 Linear elasticity

With linear elasticity we mean a linear relationship between the stresses and the strains. Still the material can deform geometrically in a nonlinear way. This happens when large deformations arise; an example is given in Appendix B.

For small deformations the elements of \( \mathcal{E} \) become small, i.e. the norm of \( \mathcal{E} \) is of order \( \varepsilon \), with \( 0 < \varepsilon << 1 \). So, under the neglect of terms of \( \mathcal{O}(\varepsilon^2) \), the constitutive equation for linear elastic behavior can be written as

\[ T = \mathcal{C} \mathcal{E} \quad \leftrightarrow \quad \sigma_{ij} = C_{ijkl} \varepsilon_{km}. \tag{2.18} \]

This equation is the most fundamental form of Hooke’s law; the 4-tensor \( \mathcal{C} \) is called the elasticity tensor.

When assuming the displacement gradients to be small everywhere compared with unity the distinction between the Eulerian and Lagrangian descriptions is negligible. We can now introduce an infinitesimal strain tensor as

\[ \varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right). \tag{2.19} \]
In order to gather more information about the tensor $\mathcal{C}$ in (2.18), we have to introduce the elastic energy density. The energy balance (2.10) combined with the neglect of thermal effects gives

$$\dot{E}_{\text{int}} = \frac{1}{\rho} \sigma_{ij} D_{ij}.$$  

For small-deformation theory we may take $D_{ij} = \dot{\varepsilon}_{ij}$, so

$$\dot{E}_{\text{int}} = \frac{1}{\rho} \sigma_{ij} \dot{\varepsilon}_{ij}. \quad (2.20)$$

In linear elasticity theory, the density $\rho$ may be assumed to be constant ($\rho = \rho_0(1 + O(\varepsilon))$). From now on we say $\rho = \rho_0$. We introduce the elastic energy density $W$ per unit of volume by

$$W = \rho E_{\text{int}}.$$  

Note that $W$ is a function of the deformation only ($W = W(\varepsilon_{ij})$), so we obtain from (2.20)

$$\sigma_{ij} = \rho \frac{\partial E_{\text{int}}}{\partial \varepsilon_{ij}} = \frac{\partial W}{\partial \varepsilon_{ij}}. \quad (2.21)$$

An expansion of $W$ in terms of $\varepsilon_{ij}$, substituted in (2.21) gives

$$\sigma_{ij} = \frac{\partial W(0)}{\partial \varepsilon_{ij}} + \frac{\partial^2 W(0)}{\partial \varepsilon_{ij} \partial \varepsilon_{km}} \varepsilon_{km} + ...$$

Since there are no residual stresses in the undeformed state of the material, we get

$$\sigma_{ij} = \frac{\partial^2 W(0)}{\partial \varepsilon_{ij} \partial \varepsilon_{km}} \varepsilon_{km} = C_{ijkm} \varepsilon_{km}.$$  

Changing the order of the partial derivatives gives $C_{ijkm} = C_{kmij}$. If a body’s elastic properties are the same in every set of reference axes, we call the elastic material isotropic. For such materials, the constitutive equation has only two independent elastic constants. The most general fourth-order isotropic tensor can be written in terms of Kronecker deltas as

$$C_{ijkm} = \lambda \delta_{ij} \delta_{km} + \mu (\delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}) + \beta (\delta_{ik} \delta_{jm} - \delta_{im} \delta_{jk}),$$

where $\lambda, \mu$ and $\beta$ are scalars. From the symmetry of tensor $\mathcal{C}$, it follows that $\beta = -\beta = 0$. Substitution finally gives Hooke’s law:

$$T = \lambda \text{tr}(\varepsilon) I + 2\mu \varepsilon \quad \leftrightarrow \quad \sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}. \quad (2.22)$$

The Lamé constants, $\lambda$ and $\mu$, are material coefficients. Also other material constants can be introduced:

**Definition 2.7.** The *elasticity modulus*, or *Young’s modulus*, $E$, relates the elongation in a certain direction to the tensile stress in that direction. The Young’s modulus is related to the Lamé parameters according to

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}. \quad (2.23)$$
Definition 2.8. Poisson’s ratio, $\nu$, is defined as the measure for lateral contraction, and is related to $\lambda$ and $\mu$ according to

$$\nu = \frac{\lambda}{2(\lambda + \mu)}.$$  

(2.24)

Definition 2.9. The shear modulus, $G$, is defined as the ratio between shear stress and shear, and satisfies the relations

$$G = \frac{E}{2(1 + \nu)} = \mu.$$  

(2.25)

Definition 2.10. The bulk modulus, $K$, describes the resistance against compression and satisfies

$$K = \frac{1}{3}(2\mu + 3\lambda) = \frac{E}{2(1 - 2\nu)}.$$  

(2.26)

In linear theory with small deformations all terms of $O(\varepsilon^2)$ will be neglected. Using the chain rule and (2.14) gives

$$\frac{\partial u}{\partial \mathbf{X}} = \frac{\partial u}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial u}{\partial \mathbf{x}} (I + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}) = \frac{\partial u}{\partial \mathbf{x}} (1 + O(\varepsilon)).$$

Because $||\frac{\partial u}{\partial \mathbf{X}}|| = O(\varepsilon)$ it follows immediately that

$$\frac{\partial u}{\partial \mathbf{X}} = \frac{\partial u}{\partial \mathbf{x}}.$$

So we do not need to distinguish the Eulerian and Lagrangian description and we may write

$$\mathbf{F} = I + \text{grad}(\mathbf{U}) \quad \leftrightarrow \quad F_{ij} = \delta_{ij} + u_{i,j},$$

in which $\text{grad}(\mathbf{u}) = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$.

We recall the conservation of mass of a material volume element: $\rho dV = \rho_0 dV_r$. Define $J = \det(\mathbf{F})$ as a measure for the change of volume. We can write

$$\frac{\rho}{\rho_0} = \frac{d\tau_r}{d\tau} = \frac{1}{J}.$$  

A medium is called incompressible if for each stress distribution the volume does not change, implying $J = 1$. We have for a linear elastic medium (using (2.27))

$$\frac{d\tau - d\tau_r}{d\tau_r} = J - 1 = \det(I + \text{grad}(\mathbf{U})) = 1 + \text{tr}({\text{grad}(\mathbf{U}})) + O(\varepsilon^2) - 1 = \varepsilon_{kk} + O(\varepsilon^2) = \varepsilon_{kk}.$$

So for a linear elastic medium $J = 1$ implies:

- $\varepsilon_{kk} = 0$.
- $K \to \infty$.
- $\nu \to \frac{1}{2}$.
- $\lambda \to \infty$, 

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implying that the product \( \lambda \varepsilon_{kk} \) in (2.22) is undetermined. Let us call this product \(-P\). Substitution in Hooke’s law gives

\[
\sigma_{ij} = -P \delta_{ij} + 2\mu \varepsilon_{ij}.
\]

The trace of the stresses equals \( \sigma_{kk} = -3P \) and thus \( P = -\frac{1}{3} \sigma_{kk} \), which equals the hydrostatic pressure. In this case, \( P = P(x) \), is an extra unknown field variable (needed to compensate the extra relation \( \varepsilon_{kk} = 0 \)).

Sometimes simplifications are necessary to be able to handle the problem analytically. When assuming specific body geometry and loading patterns the equations of elasticity can reduce to a two-dimensional form. These situations are referred to as plane elasticity. In plane stress problems, the geometry of the body is that of a thin plate with one dimension very much smaller than the other two. In plane strain problems, the geometry is that of a prismatic cylinder having one dimension very much larger than the other two and having the loads perpendicular to and distributed uniformly with respect to this large dimension. In this case, because conditions are the same at all cross sections, the analysis may be focused on a thin slice of the cylinder.

### 2.3.3 Geometrically nonlinear model

The model following from (2.17) with \( \beta_0 = -P \), \( \beta_1 = 1 \) and \( \beta_2 = 0 \) is called the Neo-Hookean model. It is often used to model rubber-like material.

The initially isotropic material has to obey certain symmetries with regard to the functional form of the strain energy function. Assume the strain energy per unit volume \( W \) to be an isotropic function of the strain in the form of the left deformation tensor invariants \( I_1, I_2 \) and \( I_3 \). Denoting the components of \( B \) with \( B_{ij} \), the invariants of the left deformation tensor equal:

\[
\begin{align*}
I_1 &= \text{tr}(B) \quad \leftrightarrow \quad I_1 = B_{ii}, \\
I_2 &= \frac{1}{2}((\text{tr}(B))^2 - \text{tr}(B^2)) \quad \leftrightarrow \quad I_2 = \frac{1}{2}(B_{ii}B_{jj} - B_{ij}B_{ij}), \\
I_3 &= \text{det}(B) \quad \leftrightarrow \quad I_3 = \frac{1}{6}e_{ijk}e_{rst}B_{ir}B_{js}B_{kt}.
\end{align*}
\]

The simplest form of the strain energy for a rubber-like material is a one-parameter model, namely the Neo-Hookean model. The strain energy only depends on the first invariant of \( B \):

\[
W = \mu(I_1 - 3).
\]

We can write \( I_1 = F_{iA}F_{iA} \). Now we want to determine the Cauchy stress. Assume that the stress is formed by adding two stress components,

\[
\sigma_{ij} = \hat{\sigma}_{ij} + \dot{\sigma}_{ij}.
\]

Here we assume

\[
\hat{\sigma}_{ij} = -P \delta_{ij},
\]

with \( P \) the hydrostatic pressure. The other term, \( \dot{\sigma} \), can be determined from the constitutive response for Cauchy stress,

\[
\dot{\sigma}_{ij} = F_{iA} \frac{\partial W}{\partial F_{jA}}.
\]
Writing

$$\frac{\partial W}{\partial F_{jA}} = \frac{\partial W}{\partial I_1} \frac{\partial I_1}{\partial F_{jA}},$$

we finally obtain the Neo-Hookean model

$$\mathcal{T} = -P I + 2\mu \mathcal{E} \quad \leftrightarrow \quad \sigma_{ij} = -P \delta_{ij} + \frac{\partial W}{\partial I_1} B_{ij} = -P \delta_{ij} + \mu B_{ij}. \quad (2.28)$$
Chapter 3

Methods

In this Chapter the methods of the three goals mentioned in Section 1.4 will be described. The method of the first goal, which was to determine influences of small perturbations of internal pressure, material properties and shape on the stress, is described in Section 3.2. The method of the second goal, which was to compare 3D stress versus 2D stress, is described in Section 3.3. Finally, the method of the third goal, which was to compare stress when modelling with and without thrombus and patient specific versus axisymmetric in 2D, is described in the last Section of this Chapter, namely Section 3.4.

But first we present some general assumptions which are needed in order to realize the project.

3.1 General assumptions
The general assumptions which are made:

- Blood flow will be neglected. It can be proven (appendix A) that the blood flow has negligible influence on the mechanical stresses in the wall.

- Gravitation is not taken in account. Gravitation is negligible with respect to the blood pressure.

- Thermal effects will be neglected. The temperature of a human body does not have much influence on the stresses.

- The AAA wall is assumed to be purely solid which means that the AAA wall is incompressible. The AAA wall actually consists of two-phase material, which means that when the wall is compressed a fluid gets separated from the wall material. This implies compressibility.

- We do not take in account the whole vessel tree. The vessel tree will have some influences on the stress because of for example reflection of waves at vessel bifurcations.

- The analytical part of this report describes a static system, only one time step will be considered.

- Initial stress (often also called residual stress) is not taken in account. Research [van Oijen, 2003] showed this aspect might be important, but it is still not verified how to estimate or determine these initial stresses.
3.2 Sensitivity Analysis

This section describes an analytical method to investigate influences of small perturbations in internal pressure, shear modulus and shape on the stress.

To be able to handle the problem analytically the following extra assumptions are made:

- The shape will be simplified into an infinitely long tube.
- No thrombus will be taken into account.
- The material does not move in longitudinal direction (plane strain problem).

We will first consider a pressure acting on the inside of an infinitely long tube (section 3.2.1) using a linear elastic model with small deformations. The tube will deform as a result of the internal pressure (Figure 3.1).

![Figure 3.1: Uniform load in an elastic, infinitely long tube.](image)

In Sections 3.2.5 - 3.2.6, we will perturb some material and geometrical aspects, i.e. shape, internal pressure and shear modulus.

Perturbation on the shear modulus will be done by multiplying the shear modulus with $(1 + \varepsilon)$ in which $0 < \varepsilon \ll 1$. It is useful to do a perturbation depending on the angle as well. To achieve this, the shear modulus will be multiplied with $(1 + \varepsilon \cos(\theta))$.

Perturbation on the internal pressure will be realized by multiplying the value of the internal pressure with $(1 + \varepsilon)$. The pressure is assumed to be a normal load.

There are several aspects of shape which can be perturbed. The ratio of inner and outer radius will be perturbed in the same way as the pressure. We can also perturb the boundaries over the angle, by multiplying both the inner and the outer radius with $(1 + \varepsilon \cos(2\theta))$. The shape will change as drawn in Figure 3.2. The blue lines show the boundaries of the tube without perturbation, the red lines show the perturbed boundaries.
Finally, we will consider the Neo-Hookean model and compare this model to the linear elastic model with small deformations.

We will reduce the elasticity models by scaling with

\[ p = \frac{P}{P_0}, \]
\[ \tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{\sigma_0}. \]

For example, the dimensionless Hookes law for an incompressible medium becomes

\[ \tilde{\sigma}_{ij} = -p\delta_{ij} + \varepsilon_{ij}. \]

This Chapter contains calculations in cylindrical coordinates. The inner radius and outer radius of the tube will be denoted by \( R_{in} \) and \( R_{out} \) respectively. The radius also needs to be scaled, say \( r = R/R_{in} \), where \( R \) denotes the dimensional radial coordinate. The ratio \( R_{out}/R_{in} \) will be denoted by \( r_o \). Thus \( 1 < r < r_o \) means in dimensions \( R_{in} < R < R_{out} \).

### 3.2.1 Uniform load in a linear elastic tube

This section describes the calculation of the stress in a linear elastic tube on which an internal pressure acts. There will be no perturbations done. We will need this calculation of stress in the Sections 3.2.5 - 3.2.6.

The following assumptions will be made:

- small deformations,
- incompressibility,
- rotational symmetry.

The boundary conditions are:

\[ (\mathbf{T} \mathbf{n}_{in}, \mathbf{e}_r)|_{r=1} = -\hat{\sigma}_{rr}(1) = -p_0, \]
\[ (\mathbf{T} \mathbf{n}_{out}, \mathbf{e}_r)|_{r=r_o} = \hat{\sigma}_{rr}(r_o) = 0. \]
We denote the displacement in $r$-direction with $u_r = u(r)$. The deformations that are unequal to zero are:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r},$$

$$\varepsilon_{\theta\theta} = \frac{u}{r}.$$

**Remark 3.1.** The appearance of the factor $1/r$ implies that there is a singular point in the origin. However, the origin is not of special interest in this report, because the wall of the tube starts at point $r = 1 > 0$. Deformations will occur only in the outer direction, so the wall never reaches the origin.

The non-zero stresses are:

$$\hat{\sigma}_{rr} = -p(r) + \frac{\partial u}{\partial r},$$

$$\hat{\sigma}_{\theta\theta} = -p(r) + \frac{u}{r},$$

$$\hat{\sigma}_{zz} = -p(r).$$

The equilibrium equation is

$$\frac{\partial \hat{\sigma}_{rr}}{\partial r} + \frac{1}{r}(\hat{\sigma}_{rr} - \hat{\sigma}_{\theta\theta}) = 0.$$

Because of incompressibility we have

$$\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = \frac{\partial u}{\partial r} + \frac{u}{r} = \frac{1}{r} \frac{\partial}{\partial r}(ru) = 0. \quad (3.1)$$

So we have five equations for the five unknowns $(u, \hat{\sigma}_{rr}, \hat{\sigma}_{\theta\theta}, \hat{\sigma}_{zz}$ and $p)$. The solutions are:

$$u(r) = \frac{C_1}{r},$$

$$\hat{\sigma}_{rr}(r) = -\frac{C_1}{r^2} + C_2,$$

$$\hat{\sigma}_{\theta\theta}(r) = \frac{C_1}{r^2} + C_2,$$

$$\hat{\sigma}_{zz}(r) = -p(r) = C_2.$$

The constants of integration are determined by the boundary conditions:

$$C_1 = \frac{p_0 r_o^2}{r_o^2 - 1}, \quad (3.2)$$

$$C_2 = \frac{p_0}{r_o^2 - 1}. \quad (3.3)$$

So now we have expressions for the (scaled) stresses. We can use them in the following sections.
3.2.2 Perturbation of the internal pressure

When a small error is made in the internal pressure we will obtain an error in the stresses as well. We want to know how large this error in the stress will be, so this will be investigated in this section.

The scaled value of the pressure is perturbed with $\varepsilon$, such that $p_0$ becomes $p_0(1 + \varepsilon)$. The stresses have almost the same expressions as described in the previous section. Only the boundary conditions change into

$$\hat{\sigma}_{rr}(1) = -p_0(1 + \varepsilon),$$
$$\hat{\sigma}_{rr}(r_o) = 0.$$

The equivalent constants of (3.2) and (3.3) now change into

$$\hat{C}_1 = \frac{p_0 r_o^2 (1 + \varepsilon)}{r_o^2 - 1},$$
$$\hat{C}_2 = \frac{p_0 (1 + \varepsilon)}{r_o^2 - 1}.$$

In Section 4.1 the influence of this perturbation on the stresses will be shown.

3.2.3 Perturbation of the shear modulus

Perturbation of the shear modulus with $\varepsilon$, such that $\mu_0$ becomes $\mu_0(1 + \varepsilon)$ gives for the stresses

$$\hat{\sigma}_{rr} = -p + (1 + \varepsilon) \varepsilon_{rr} = -p + (1 + \varepsilon) \frac{\partial u}{\partial r},$$
$$\hat{\sigma}_{\theta\theta} = -p + (1 + \varepsilon) \varepsilon_{\theta\theta} = -p + (1 + \varepsilon) \left( \frac{u}{r} \right),$$
$$\hat{\sigma}_{zz} = -p. \tag{3.6}$$

The stresses, hydrostatic pressure and displacement are now all dependent on $\varepsilon$ and $r$ only (since the problem is still axisymmetric). To obtain a first order correction, we substitute

$$p = p_1(r) + \varepsilon p_2(r), \tag{3.7}$$
$$u = u_1(r) + \varepsilon u_2(r), \tag{3.8}$$

in the stress equations (3.4 - 3.6). The solution for $\varepsilon = 0$ has been found in Section 3.2.1 and so $p_1(r)$ and $u_1(r)$ are known. Substitution of (3.7) and (3.8) in the stress equations (3.4 - 3.6) followed by substitution of the stresses in the equilibrium equation together with the incompressibility condition yields differential equations for $p_2$ and $u_2$. These equations are solved in a straightforward way. The result is:

$$u_2(r) = \frac{K_1}{r},$$
$$p_2(r) = K_2,$$

where the constants $K_1$ and $K_2$ follow from the boundary conditions. We find:

$$K_1 = -C_1,$$
$$K_2 = 0.$$
So the final expressions for the stresses are:

\[
\dot{\sigma}_{rr} = -p + (1 + \varepsilon) \frac{\partial u}{\partial r} = C_2 - \frac{C_1}{r^2}, \\
\dot{\sigma}_{\theta\theta} = -p + (1 + \varepsilon) \left( \frac{u}{r} \right) = C_2 + \frac{C_1}{r^2}, \\
\dot{\sigma}_{zz} = -p = C_2,
\]

with \(C_1\) and \(C_2\) as in (3.2) and (3.3).

**Remark 3.2.** The stresses are the same as in Section 3.2.1, but the displacement in \(r\)-direction changes into

\[
u(r, \varepsilon) = \frac{C_1}{r} - \varepsilon \frac{C_1}{r}.
\]

In Section 4.1 the influence of this perturbation on the stresses will be shown.

**3.2.4 Angle dependent perturbation of the shear modulus**

So far, only uniform perturbations were studied. Let us now consider the following perturbation

\[
\mu(\theta, \varepsilon) = \mu_0 (1 + \varepsilon \cos(\theta)).
\]

The shear modulus now varies over the angle \(\theta\) and thus the problem is no longer axisymmetric.

We will denote \(u(r, \theta, \varepsilon) = u_r\) and \(v(r, \theta, \varepsilon) = u_\theta\). The deformations which are unequal to zero are:

\[
\varepsilon_{rr} = \frac{\partial u}{\partial r}, \\
\varepsilon_{r\theta} = \frac{1}{2} \left( \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r} \right), \\
\varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r}.
\]

The stresses which are unequal to zero are:

\[
\dot{\sigma}_{rr} = -p + (1 + \varepsilon \cos(\theta)) \frac{\partial u}{\partial r}, \\
\dot{\sigma}_{r\theta} = (1 + \varepsilon \cos(\theta)) \frac{1}{2} \left( \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r} \right), \\
\dot{\sigma}_{\theta\theta} = -p + (1 + \varepsilon \cos(\theta)) \left( \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \right), \\
\dot{\sigma}_{zz} = -p.
\]

The equilibrium equations are:

\[
\frac{\partial \dot{\sigma}_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \dot{\sigma}_{r\theta}}{\partial \theta} + \frac{1}{r} (\dot{\sigma}_{rr} - \dot{\sigma}_{\theta\theta}) = 0, \quad (3.9) \\
\frac{\partial \dot{\sigma}_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \dot{\sigma}_{\theta\theta}}{\partial \theta} + \frac{2}{r} \dot{\sigma}_{r\theta} = 0. \quad (3.10)
\]
Incompressibility implies
\[ \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} = 0. \]  
(3.11)

From the latter equation we obtain
\[ \frac{\partial v}{\partial \theta} = -\frac{\partial}{\partial r}(ru). \]  
(3.12)

Introduce \( \phi = \phi(r, \theta, \varepsilon) \) as
\[ u = \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \]  
(3.13)
\[ v = -\frac{\partial \phi}{\partial r}. \]  
(3.14)

Now the incompressibility condition is automatically satisfied which implies that we have a system of 6 equations with 6 unknowns \( (\phi, \hat{\sigma}_{rr}, \hat{\sigma}_{r\theta}, \hat{\sigma}_{\theta\theta}, \hat{\sigma}_{zz}, p) \).

Remark 3.3. The function \( \phi \) is introduced to assure the divergence of \( (u, v) \) equals zero, so that the incompressibility condition is automatically satisfied. This is a mathematical tool to solve the system of equations.

Because of the perturbation, the equations are dependent on the variable \( \varepsilon \). For the shear modulus \( \mu(\theta) = \mu(-\theta) \), so the problem is even in \( \theta \). This implies that
\[ \phi, u, \hat{\sigma}_{r\theta}, \varepsilon_{r\theta} \sim \sin(\theta), \]  
\[ p, u, \hat{\sigma}_{rr}, \varepsilon_{rr}, \varepsilon_{\theta\theta} \sim \cos(\theta). \]

Substitution of
\[ \phi(r, \theta, \varepsilon) = C_1 \theta + \varepsilon \phi_1(r) \sin(\theta), \]  
(3.15)
\[ p(r, \theta, \varepsilon) = -C_2 + \varepsilon p_1(r) \cos(\theta), \]  
(3.16)
will lead to a solution, (neglecting \( O(\varepsilon^2) \)) namely:
\[ \phi_1 = -\frac{2C_1}{3} + \frac{C_3}{r} + C_4 r + C_5 r^3 + C_6 r \log(r), \]  
(3.17)
\[ p_1 = -\frac{2C_1}{3r^2} - \frac{C_6}{r} + 4C_5 r. \]  
(3.18)

Here, \( C_1 \) and \( C_2 \) are as given in (3.2) and (3.3), \( C_4 \) represents a rigid-body translation (see at the end of this section), and \( C_3, C_5 \) and \( C_6 \) will follow from the boundary conditions. These representations are such that \( \phi_1 \) is a biharmonic function \( (\Delta \Delta \phi_1 = 0) \), and \( p_1 \) is a harmonic function \( (\Delta p_1 = 0) \).

The stresses equal
\[ \hat{\sigma}_{rr} = -\frac{C_1}{r^2} + C_2 + \varepsilon \cos(\theta)(-\frac{2C_3}{r^3} + \frac{C_1}{3r^2} + \frac{2C_6}{r} - 2C_5 r), \]  
(3.19)
\[ \hat{\sigma}_{r\theta} = \varepsilon \sin(\theta)(\frac{2C_3}{r^3} + \frac{C_1}{3r^2} - 2C_5 r), \]  
(3.20)
\[ \hat{\sigma}_{\theta\theta} = \frac{C_1}{r^2} + C_2 + \varepsilon \cos(\theta)(\frac{2C_3}{r^3} - 6C_5 r), \]  
(3.21)
\[ \hat{\sigma}_{zz} = C_2 + \varepsilon \cos(\theta)(-\frac{2C_1}{3r^2} - \frac{C_6}{r} + 4C_5 r). \]  
(3.22)

There are two types of boundary conditions:
• The stresses at the inner and outer radius have to equal to the pressure acting at that point.

• There is no rigid-body motion.

The first condition implies for the inner wall:

\[ T n \big|_{r=1} = -p_0 e_r. \]

For the outer wall we assume no pressure:

\[ T n \big|_{r=r_o} = 0. \]

All this implies:

\[
\begin{align*}
\hat{\sigma}_{rr}(1, \theta, \varepsilon) &= -p_0, \quad (3.23) \\
\hat{\sigma}_{r\theta}(1, \theta, \varepsilon) &= 0, \quad (3.24) \\
\hat{\sigma}_{rr}(r_o, \theta, \varepsilon) &= 0, \quad (3.25) \\
\hat{\sigma}_{r\theta}(r_o, \theta, \varepsilon) &= 0. \quad (3.26)
\end{align*}
\]

yielding

\[
\begin{align*}
C_3 &= \frac{C_1}{6} - \frac{C_1}{6(1 + r_o + r_o^2 + r_o^3)}, \\
C_5 &= \frac{C_1}{6(1 + r_o + r_o^2 + r_o^3)}, \\
C_6 &= 0.
\end{align*}
\]

with \( C_1 \) as given by (3.2).

**Remark 3.4.** There are three unknowns that can be solved from the four boundary conditions. This is no problem because boundary condition (3.23) gives the same equation as (3.24) when \( C_6 = 0 \) and boundary condition (3.25) gives the same equation as (3.26) when \( C_6 = 0 \).

The displacement can be written as a sum of displacement caused by rigid-body motion and a displacement caused by deformation, say

\[
\begin{align*}
\mathbf{u} &= u^{\text{rigid}} + u^{\text{deform}}, \\
\mathbf{v} &= v^{\text{rigid}} + v^{\text{deform}}.
\end{align*}
\]

The displacements caused by rigid-body motion can be written as

\[
\begin{align*}
u^{\text{rigid}} &= A \cos(\theta) + B \sin(\theta), \\
v^{\text{rigid}} &= -A \sin(\theta) + B \cos(\theta),
\end{align*}
\]

with A and B constants. The constants A and B in the following equation are responsible for the rigid-body motion in our problem:

\[ \phi = Ar \sin(\theta) - Br \cos(\theta). \]

By excluding rigid-body motion, we can take

\[ C_4 = 0. \]

In Section 4.1 the influence of this perturbation on the stresses will be shown.
3.2.5 Perturbation of the ratio between outer and inner radius

When the ratio between the outer and inner radius is perturbed with \( \varepsilon \), such that \( r_o(=R_{out}/R_{in}) \) becomes \( r_o(1+\varepsilon) \), the boundary conditions change into

\[
\hat{\sigma}_{rr}(1) = -p_0, \\
\hat{\sigma}_{rr}(r_o(1+\varepsilon)) = 0,
\]

causing the equivalent constants (3.2) and (3.3) to change into

\[
\hat{C}_1 = \frac{p_0 r_o^2 (1 + 2\varepsilon)}{r_o^2 (1 + 2\varepsilon) - 1}, \\
\hat{C}_2 = \frac{p_0 r_o^2 (1 + 2\varepsilon)}{r_o^2 (1 + 2\varepsilon) - 1},
\]

when \( O(\varepsilon^2) \) is neglected (with \( 0 < \varepsilon << 1 \)).

In Section 4.1 the influence of this perturbation on the stresses will be shown.

3.2.6 Angle dependent perturbation on shape

In this section we will perturb the boundaries of the tube.

Consider an infinitely long, linear elastic homogeneous tube having the following form

\[
R_{in}(1 + \varepsilon \cos(2\theta)) < R < R_{out}(1 + \varepsilon \cos(2\theta)),
\]

or after scaling

\[
(1 + \varepsilon \cos(2\theta)) < r < r_o(1 + \varepsilon \cos(2\theta)).
\]

When \( \varepsilon = 0 \) the tube is axisymmetric, but for \( \varepsilon > 0 \) the problem becomes dependent on the angle. We seek solutions of the following form (upon neglecting \( O(\varepsilon^2) \)):

\[
u_r(r, \theta, \varepsilon) = C_1 r + \varepsilon \cos(2\theta) u_1(r), \\
u_\theta(r, \theta, \varepsilon) = \varepsilon \sin(2\theta) v_1(r), \\
p(r, \theta, \varepsilon) = -C_2 + \varepsilon \cos(2\theta) p_1(r),
\]

with \( C_1 \) and \( C_2 \) as in (3.2) and (3.3). Substitution of (3.37) - (3.39) into the equilibrium equations (3.9) and (3.10) and the incompressibility condition (3.11) yields

\[
u_1(r) = \frac{C_3}{r^3} + \frac{C_4}{r} + r C_5 + r^2 C_6, \\
v_1(r) = \frac{C_3}{r^3} - r(C_5 + 2C_6 r^2), \\
p_1(r) = \frac{C_4 + 3C_6 r^4}{r^2}.
\]

The non-zero stresses are

\[
\sigma_{rr}(r, \theta, \varepsilon) = -\frac{C_1}{r^2} + C_2 + \varepsilon(-3C_3 r^{-4} - 2C_4 r^{-2} + C_5) \cos(2\theta), \\
\sigma_{r\theta}(r, \theta, \varepsilon) = -\varepsilon(3C_3 r^{-4} + C_4 r^{-2} + C_5 + 3C_6 r^2) \sin(2\theta), \\
\sigma_{\theta\theta}(r, \theta, \varepsilon) = \frac{C_4}{r^2} + C_2 + \varepsilon(-C_5 + \frac{3}{r^2} C_3 - 6C_6 r^2) \cos(2\theta), \\
\sigma_{zz}(r, \theta, \varepsilon) = C_2 - \varepsilon(C_4 r^{-2} + 3C_6 r^2) \cos(2\theta).
\]
We have the following boundary conditions
\[ (T \mathbf{n}_{\text{in}})|_{r=(1+\varepsilon \cos(2\theta))} = -p_0 \mathbf{e}_r, \] (3.47)
\[ (T \mathbf{n}_{\text{out}})|_{r=r_o(1+\varepsilon \cos(2\theta))} = 0. \] (3.48)

The normal at the inner and outer radius respectively equals
\[ \mathbf{n}_{\text{in}} = \mathbf{e}_r + 2\varepsilon \sin(2\theta) \mathbf{e}_\theta. \] (3.49)
\[ \mathbf{n}_{\text{out}} = \mathbf{e}_r + r_o 2\varepsilon \sin(2\theta) \mathbf{e}_\theta. \] (3.50)

Taking the inner product of (3.47) and \( \mathbf{e}_r \) renders
\[ (T \mathbf{n}_{\text{in}}, \mathbf{e}_r)|_{r=(1+\varepsilon \cos(2\theta))} = (T(1 + \varepsilon \cos(2\theta), \theta)(\mathbf{e}_r + 2\varepsilon \sin(2\theta) \mathbf{e}_\theta), \mathbf{e}_r) \]
\[ = \sigma_{rr}(1 + \varepsilon \cos(2\theta), \theta) + 2\varepsilon \sin(2\theta) \sigma_{r\theta}(1 + \varepsilon \cos(2\theta), \theta) \]
\[ = -p_0. \] (3.51)

Applying a Taylor expansion with respect to \( r \) at \( r = 1 \) of equation (3.51) and neglecting order \( \varepsilon^2 \) we obtain
\[ \sigma_{rr}(1, \theta) + \varepsilon \cos(2\theta) \frac{\partial \sigma_{rr}}{\partial r}|_{r=1} + 2\varepsilon \sin(2\theta) \sigma_{r\theta}(1, \theta) = -p_0. \] (3.52)

At the outer radius we have
\[ (T \mathbf{n}_{\text{out}}, \mathbf{e}_r)|_{r=r_o(1+\varepsilon \cos(2\theta))} = (T(r_o(1 + \varepsilon \cos(2\theta)), \theta)(\mathbf{e}_r + r_o 2\varepsilon \sin(2\theta) \mathbf{e}_\theta), \mathbf{e}_r) \]
\[ = 0. \] (3.53)

A Taylor expansion with respect to \( r \) at \( r = r_o \) of equation (3.54) gives
\[ \sigma_{rr}(r_o, \theta) + \varepsilon \cos(2\theta) \frac{\partial \sigma_{rr}}{\partial r}|_{r=r_o} + r_o 2\varepsilon \sin(2\theta) \sigma_{r\theta}(r_o, \theta) = 0. \] (3.54)

A similar procedure, now with respect to \( \mathbf{e}_\theta \), gives
\[ (T \mathbf{n}_{\text{in}}, \mathbf{e}_\theta)|_{r=(1+\varepsilon \cos(2\theta))} = (T(1 + \varepsilon \cos(2\theta), \theta)(\mathbf{e}_r + 2\varepsilon \sin(2\theta) \mathbf{e}_\theta), \mathbf{e}_\theta) \]
\[ = \sigma_{r\theta}(1 + \varepsilon \cos(2\theta), \theta) + 2\varepsilon \sin(2\theta) \sigma_{\theta\theta}(1 + \varepsilon \cos(2\theta), \theta) \]
\[ = \sigma_{r\theta}(1, \theta) + \varepsilon \cos(2\theta) \frac{\partial \sigma_{r\theta}}{\partial r}|_{r=1} + 2\varepsilon \sin(2\theta) \sigma_{\theta\theta}(1, \theta) \]
\[ = 0, \] (3.55)

and for the outer wall
\[ (T \mathbf{n}_{\text{out}}, \mathbf{e}_\theta)|_{r=r_o(1+\varepsilon \cos(2\theta))} = (T(r_o(1 + \varepsilon \cos(2\theta)), \theta)(\mathbf{e}_r + r_o 2\varepsilon \sin(2\theta) \mathbf{e}_\theta), \mathbf{e}_\theta) \]
\[ = \sigma_{r\theta}(r_o(1 + \varepsilon \cos(2\theta)), \theta) + r_o 2\varepsilon \sin(2\theta) \sigma_{\theta\theta}(r_o(1 + \varepsilon \cos(2\theta)), \theta) \]
\[ = \sigma_{r\theta}(r_o, \theta) + \varepsilon \sigma_{r\theta}(0) \frac{\partial \sigma_{r\theta}}{\partial r}|_{r=r_o} + 2r_o \varepsilon \sin(2\theta) \sigma_{\theta\theta}(r_o, \theta) \]
\[ = 0. \] (3.56)
These four boundary conditions yield for the remaining constants:

\[
C_3 = \frac{2p_0(r_o^2 + 3r_o^3 + r_o^4 + r_o^5)}{3(1 + r_o - 2r_o^2 - 2r_o^3 + r_o^4 + r_o^5)},
\]

\[
C_4 = \frac{2p_0(-r_o^2 + r_o^3 + r_o^4 + 3r_o^5)}{1 + r_o - 3r_o^2 - 3r_o^3 + 3r_o^4 - r_o^5 - r_o^7},
\]

\[
C_5 = \frac{2p_0(-r_o^2 + 3r_o^3 + r_o^4 + r_o^5)}{1 + r_o - 3r_o^2 - 3r_o^3 + 3r_o^4 + 3r_o^5 - r_o^7},
\]

\[
C_6 = -\frac{2p_0(1 + r_o - 3r_o^2 + 3r_o^3 + 2r_o^4)}{3(1 + r_o - 3r_o^2 - 3r_o^3 + 3r_o^4 + 3r_o^5 - r_o^6 - r_o^7)}.
\]

We use these calculations to determine the influence of the perturbations of the aspects mentioned on the principal stresses. The results are shown in Section 4.1.

### 3.2.7 Uniform load in a non-linear elastic tube

In order to understand the Neo-Hookean model and the complexity of this model we will present an example of calculating the stresses in a non-linear elastic tube on which an internal normal load acts.

Consider the case of plane strain, implying there will be no strain in the \( z \)-direction, so

\[
r = r(R, \Theta), \quad \theta = \theta(R, \Theta), \quad z = Z.
\]

(3.56)

Here \((r, \theta, z)\) denotes the deformed state and \((R, \Theta, Z)\) the undeformed state. The deformation gradient equals

\[
\mathcal{F} = \begin{pmatrix}
\frac{\partial r}{\partial R} & \frac{1}{R} & \frac{1}{R} \\
\frac{\partial \theta}{\partial R} & \frac{\partial \theta}{\partial \Theta} & 0 \\
\frac{\partial z}{\partial R} & \frac{\partial z}{\partial \Theta} & 0
\end{pmatrix} = \begin{pmatrix}
\frac{\partial r}{\partial R} & \frac{1}{R} \frac{\partial r}{\partial \Theta} & 0 \\
\frac{r}{\partial R} & \frac{\partial \theta}{\partial \Theta} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(3.57)

So the left deformation tensor equals

\[
\mathcal{B} = \begin{pmatrix}
(r^2 + \frac{1}{R^2})^2 & \frac{r^2}{R^2} \frac{\partial r}{\partial \Theta} & 0 \\
\frac{r^2}{R} \frac{\partial \theta}{\partial R} & \frac{r^2}{R^2} (\frac{\partial \theta}{\partial \Theta})^2 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(3.58)

Substitution in equation (2.28) and scaling by dividing through \( \mu \) gives:

\[
\sigma_{rr} = -p + \frac{1}{2} \left( \frac{\partial r}{\partial R} \right)^2 + \frac{1}{R^2} \left( \frac{\partial r}{\partial \Theta} \right)^2,
\]

(3.59)

\[
\sigma_{\theta\theta} = -p + \frac{1}{2} \left( r^2 \left( \frac{\partial \theta}{\partial R} \right)^2 + \frac{r^2}{R^2} \left( \frac{\partial \theta}{\partial \Theta} \right)^2 \right),
\]

(3.60)

\[
\sigma_{r\theta} = \frac{1}{2} \left( r \frac{\partial r}{\partial R} \frac{\partial \theta}{\partial \Theta} + r \frac{\partial r}{\partial \Theta} \frac{\partial \theta}{\partial R} \right),
\]

(3.61)

\[
\sigma_{zz} = -p.
\]

(3.62)

The incompressibility condition \((\text{det}(\mathcal{F}) = 1)\) can be written as

\[
r \frac{\partial \theta}{\partial R} \frac{\partial r}{\partial \Theta} - r \frac{\partial \theta}{\partial \Theta} \frac{\partial r}{\partial R} = 1.
\]

(3.62)
The inverse of the deformation gradient equals:

\[
\mathbf{F}^{-1} = \begin{pmatrix}
\frac{\partial R}{\partial r} & \frac{1}{r} \frac{\partial R}{\partial \theta} & \frac{\partial R}{\partial z} \\
\frac{\partial R}{\partial r} & \frac{1}{R} \frac{\partial \Theta}{\partial \theta} & \frac{\partial R}{\partial z} \\
\frac{\partial R}{\partial \theta} & \frac{1}{r} \frac{\partial \Theta}{\partial \theta} & \frac{\partial R}{\partial z}
\end{pmatrix} = \begin{pmatrix}
\frac{r}{R} \frac{\partial \Theta}{\partial r} & -\frac{1}{R} \frac{\partial r}{\partial \Theta} & 0 \\
-r \frac{\partial \Theta}{\partial r} & \frac{\partial \Theta}{\partial r} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

So the following relations must hold:

\[
\frac{\partial R}{\partial r} = \frac{r}{R} \frac{\partial \Theta}{\partial r}, \quad \frac{1}{r} \frac{\partial R}{\partial \theta} = -\frac{1}{R} \frac{\partial \Theta}{\partial r}, \quad \frac{\partial R}{\partial z} = -\frac{\partial \Theta}{\partial z} \frac{\partial r}{\partial R}.
\] (3.63)

When axisymmetry is assumed, \( \theta = \Theta \) and \( r(R) \). The problem then reduces and is easier to solve. Introduce a function \( \lambda(r) \) and say \( R = r/\lambda(r) \) or \( r = \lambda(R)R \).

Consider the following relations:

- \( \frac{R}{r} = \frac{\partial r}{\partial r} \) implies \( \frac{1}{\lambda(R)} = \frac{\partial \lambda(R)}{\partial r} + \lambda(R) \),

- \( \frac{R}{r} = \frac{\partial R}{\partial r} \) implies \( \lambda(R) = \frac{\partial R}{\partial r} \).

From the first relation \( \lambda(R) \) can be determined.

\[
\lambda(R) = \sqrt{\frac{K_1^2}{R^2} + 1} \quad (3.64)
\]

The second relation can be used to determine the inverse of the deformation gradient tensor for an axisymmetric problem with \( \lambda \) substituted:

\[
\mathbf{F} = \begin{pmatrix}
\frac{1}{\lambda(R)} & 0 & 0 \\
0 & \lambda(R) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (3.65)

The left deformation tensor equals

\[
\mathbf{B} = \begin{pmatrix}
\left(\frac{1}{\lambda(R)}\right)^2 & 0 & 0 \\
0 & \left(\lambda(R)\right)^2 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (3.66)

So the stresses become:

\[
\hat{\sigma}_{rr}(R) = -P(R) + \frac{1}{2} \lambda(R)^2, \quad (3.67)
\]

\[
\hat{\sigma}_{\theta\theta}(R) = -P(R) + \frac{1}{2} \lambda(R)^2, \quad (3.68)
\]

\[
\hat{\sigma}_{zz}(R) = -P(R) + \frac{1}{2}. \quad (3.69)
\]

The equilibrium equation is

\[
\frac{\partial \hat{\sigma}_{rr}}{\partial R} + \frac{1}{R} (\hat{\sigma}_{rr} - \hat{\sigma}_{\theta\theta}) = 0. \quad (3.70)
\]

Substitution of \( \lambda(R) \) and the stresses in equation (3.70) yields a solution for \( P(R) \), namely

\[
P(R) = k_2 + \frac{k_1^2}{4R^2} - \frac{k_1^2}{2(k_1^2 + R^2)} - \frac{1}{2} \log(R) + \frac{1}{4} \log(k_1^2 + R^2). \quad (3.71)
\]

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The boundary conditions equal:

\[
\begin{align*}
\dot{\sigma}_{rr}(R = 1) &= -p_0, \\
\dot{\sigma}_{rr}(R = R_0) &= 0.
\end{align*}
\] (3.72) (3.73)

For \(\dot{\sigma}_{rr}(R)\) we have the following equation:

\[
\dot{\sigma}_{rr}(R) = -k_2 + \frac{1}{2(1 + k_1^2)} \left( -\frac{k_1^2}{4R^2} + \frac{k_1^2}{2(k_1^2 + R^2)} + \frac{1}{2} \log(R) - \frac{1}{4} \log(k_1^2 + R^2) \right).
\]

The boundary conditions imply

\[
y_1(k_1) = y_2(k_1)
\] (3.74)

with

\[
\begin{align*}
y_1(k_1) &= -\frac{k_1^2}{4} + p_0 + \frac{k_1^2}{4R_0^2}, \\
y_2(k_1) &= \frac{1}{2} \log(R_0) - \frac{1}{4} \log(k_1^2 + R_0^2) + \frac{1}{4} \log(1 + k_1^2).
\end{align*}
\]

**Statement**

There exists a unique solution for (3.74) in the interval \(I = [0, \infty)\).

**Proof**

The functions \(y_1\) and \(y_2\) are continuous in \(I\).

\[
y_1(0) = p_0 > 0 \quad \text{and} \quad y_2(0) = 0
\]

\[
y_1'(k_1) = -\frac{k_1^2}{4} + \frac{k_1^2}{2R_0^2} < 0 \quad \text{for} \quad k_1 \in I
\]

\[
y_2'(k_1) = \frac{k_1}{2(1 + k_1^2)} - \frac{k_1}{2(k_1^2 + R_0^2)} > 0 \quad \text{for} \quad k_1 \in I
\]

\[
\Rightarrow \text{there has to be a unique intersection point and thus a solution.}
\]

There exists no analytical solution for the constants. We can only approximate the constants (for example with a Taylor expansion). So the Neo-Hookean model will only be used numerical.

### 3.2.8 Neo-Hookean model versus linear model with small deformations

We want to know the theoretical difference between the Neo-Hookean model and the linear elasticity model with small deformations. We will show that the models are the same for small deformations.

The stress \(\sigma_{ij}^L\) or \(T^L\) can be described by a linear elastic model as

\[
\sigma_{ij}^L = -P^L \delta_{ij} + 2\mu \varepsilon_{ij} \quad \leftrightarrow \quad T^L = -P^L \mathcal{I} + 2\mu \varepsilon,
\]

with \(P^L\) the hydrostatic pressure (treated as a variable). Using the definition of displacement we can write

\[
T^L = -P^L \mathcal{I} + \mu((\mathcal{F} - \mathcal{I}) + (\mathcal{F} - \mathcal{I})^T).
\]

The stress \(T^N\) can be described by the Neo-Hookean model as

\[
T^N = -P^N \mathcal{I} + \mu \mathcal{B} = -P^N \mathcal{I} + \mu \mathcal{F} \mathcal{F}^T.
\]
Writing $\mathcal{F} = \mathcal{I} + \mathcal{A}$ and assuming small deformations such that $\mathcal{A}\mathcal{A}^T \to 0$ gives

$$
T^N = -P^N\mathcal{I} + \mu(\mathcal{I} + \mathcal{A})(\mathcal{I} + \mathcal{A})^T,
$$

$$
= -P^N\mathcal{I} + \mu(\mathcal{I} + \mathcal{A} + \mathcal{A}^T + \mathcal{A}\mathcal{A}^T),
$$

$$
\approx -P^N\mathcal{I} + \mu(\mathcal{I} + \mathcal{A} + \mathcal{A}^T),
$$

$$
= -(P^N - \mu)\mathcal{I} + \mu(\mathcal{A} + \mathcal{A}^T),
$$

$$
= -(P^N - \mu)\mathcal{I} + \mu((\mathcal{F} - \mathcal{I}) + (\mathcal{F} - \mathcal{I})^T).
$$

So when $P^L = P^N - \mu$ the stresses of both models are equal ($T^L \approx T^N$) for small deformations. From the equilibrium equations and incompressibility condition the condition $P^L = P^N - \mu$ will follow.
3.3 3D modelling versus 2D modelling

In this section we want to investigate whether results (e.g. influences of thrombus and axisymmetry) obtained from 2D modelling can be used for a 3D geometry. Therefore we want to know the influence of the shape of the aneurysm in longitudinal direction.

We will present two ways to determine whether it is necessary to do 3D modelling;

- numerical approach: determine stress with sepran for 3D geometry and compare it to the stress for a cross-section (2D),
- analytical approach: investigate the theoretical difference between the 3D problem and a plane strain problem.

In the Hemodyn project four patient data sets are available. Also a program [Breeuwer et al., 2004] to determine the stress for a 3D geometry caused by an internal pressure is available, but only for modelling without thrombus (see Figure 3.3).

![Figure 3.3: Courtesy of Catharina Hospital Eindhoven, CT: PMS AVEU. An aneurysm, meshed without thrombus.](image)

The following assumptions are made:

- The Neo-Hookean model is used.
- As boundary conditions we assume the upper and lower slice to be fixed in all directions.
- The internal pressure has a value of 5.3 kPa (which equals the difference between the highest blood pressure and the lowest blood pressure of a healthy human).
• The thickness of the wall equals 2 mm.

Using the 3D program we will determine the highest stresses and the slices (cross section in longitudinal direction) on which these stresses occur for each patient. These slices will be used for the 2D modelling (plane strain). An example of a slice can be seen in Figure 3.4. With Sepran (for more information on this package, see Appendix C) the stress will be calculated.

![Figure 3.4: A slice of an aneurysm, meshed without thrombus.](image)

3.3.1 Numerical approach

Model

The Neo-hookean elastic model handles large deformations and is implemented in a finite element package.

An internal pressure will act on the aneurysm which will deform and undergo stresses. The upper and lower slices of the aneurysm will be fixed in all directions to avoid rigid body motion. The stresses will be calculated with a finite element package called Sepran (for more information on this package see Appendix C).

The equations which describe the Neo-hookean elastic model (2.28), together with the equilibrium equation, incompressibility-condition and the boundary conditions, can be written as a matrix-representation [SEPRA, 2000]. Let us write the problem as \( A\mathbf{x} = \mathbf{b} \) in which \( \mathbf{x} \)
denotes the stresses:

\[
\mathbf{x} = \begin{pmatrix}
\sigma_{xx} \\
\sigma_{xy} \\
\sigma_{xz} \\
\sigma_{yx} \\
\sigma_{yy} \\
\sigma_{yz} \\
\sigma_{zx} \\
\sigma_{zy} \\
\sigma_{zz}
\end{pmatrix}
\]

We want to determine the vector \( \mathbf{x} \) in a numerical way. There are several ways to determine this vector. We used ILU as preconditioner and BICGSTAB to solve the preconditioned problem. For more details we refer to Appendix D. Finally, we determine the eigenvalues of

\[
\sigma = \begin{pmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{xz} & \sigma_{yz} & \sigma_{zz}
\end{pmatrix}
\]

to be able to determine the maximal principal stress and the von Mises stress.

**Data**

The meshes (structurized collections of elements) of the lumen of the patient data which is used can be seen in Figure 3.5 - Figure 3.8. Patient 3 does not have any thrombus.

![Figure 3.5: Mesh of the lumen of the AAA of patient number 1.](image)
Figure 3.6: Mesh of the lumen of the AAA of patient number 2.

Figure 3.7: Mesh of the lumen of the AAA of patient number 3.
Figure 3.8: Mesh of the lumen of the AAA of patient number 4.

The data which is used for the 3D simulations is shown in Table 3.1. More details can be found in Appendix E.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear modulus</td>
<td>333 kPa</td>
</tr>
<tr>
<td>Internal pressure</td>
<td>5.3 kPa</td>
</tr>
<tr>
<td>Thickness wall</td>
<td>2 mm</td>
</tr>
</tbody>
</table>

Table 3.1: Model parameters which are used for the 3D simulations.

The results for all the investigated slices can be found in Section 4.2.

3.3.2 Analytical approach

In order to compare the stress in a 3D geometry with the stress in a 2D plane strain problem, we must know the difference between the models.

The plane strain problem for an axisymmetric shape assumes that there is no displacement in the longitudinal direction. The displacement in the radial direction only depends on the radius.

In a 3D axisymmetric geometry there is displacement in the radial direction and in the longitudinal direction. These displacements both depend on the radius and the longitudinal coordinate.

So when the displacement of the 3D geometry in the longitudinal direction (almost) equals zero and when the displacement in the radial direction is not dependent of the longitudinal coordinate, the 3D stress will be equal to the 2D plane strain stress.
3.4 Modelling thrombus and patient specific shape

This section describes a method to compare the stress when modelling with and without thrombus and to compare the stress when modelling with patient specific shape and axisymmetric shape.

Since it is not yet possible to model thrombus in the 3D meshing application which is available, we will do this investigation in 2D. For patient specific shape modelling (described in Section 3.4.1) we need a finite element package (sepran) and for the axisymmetric shape modelling we choose for an analytical approach (Section 3.4.2).

Summarizing, we compare the following stresses (Figure 3.9):

- stress in patient specific shape without thrombus,
- stress in patient specific shape with thrombus,
- stress in axisymmetric shape without thrombus,
- stress in axisymmetric shape with thrombus.

![Diagram showing 2D-problem in Sepran and 2D-problem analytical with patient specific shape and axisymmetric shape, with and without thrombus.]

**Figure 3.9**: The stress of several modelling aspects will be compared.

The following assumptions are made:

- The thrombus is fixed to the aortic wall, we assume there is no free movement possible between thrombus and wall.
- The Neo-Hookean model is used for the numerical part and the linear elasticity model is used for the analytical part since this is easy to handle. The deformations are small when the shape is axisymmetric so we do not need an Neo-Hookean model.
3.4.1 Numerical approach

The slices on which the highest stress occur are determined by the 3D modelling. A matlab script (made by Radj Baldewsing) which we adjusted makes it possible to mesh the slices. An example of a mesh of a slice which is used can be seen in Figure 3.10 and 3.11.

Figure 3.10: Mesh of the lumen of the AAA of a slice.
An internal pressure of 5.3 kPa deforms the slice and causes stresses which will be calculated using a plane strain model (one point fixed in all directions to avoid rigid body motion).

For the 2D simulations we used the data written in Table 3.2.

<table>
<thead>
<tr>
<th>Property</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear modulus wall</td>
<td>333 kPa</td>
</tr>
<tr>
<td>Shear modulus thrombus</td>
<td>131 kPa</td>
</tr>
<tr>
<td>Internal pressure</td>
<td>5.3 kPa</td>
</tr>
<tr>
<td>Thickness wall</td>
<td>2 mm</td>
</tr>
</tbody>
</table>

Table 3.2: Data which is used for the 2D simulations.

The results for each investigated slice can be found in Section 4.3.

3.4.2 Analytical approach

We already derived an analytical 2D model for the modelling of one layer (without thrombus) in Section 3.2.1. Now we only need a model to determine stress for a two layer model.

We will simplify the shape again by assuming an axisymmetric tube consisting of two layers (wall and thrombus). We will use the linear elastic model for small deformations because it is easy to handle and the deformations are small when the shape is axisymmetric (maximum displacement is of order $10^{-2}$ mm). The two layers will have a different shear modulus as drawn in Figure 3.12. The radius is scaled on the internal radius. An internal pressure causes
deformations. The scaled value of this internal pressure equals $p_0$. The shear moduli $\mu_1$ and $\mu_2$ are scaled as well. These values will all be scaled on $\mu_1$. We assume axisymmetry and incompressibility. Now we want to determine the stresses.

![Two layer model](image)

**Figure 3.12**: Two layer model.

We denote the stresses of the inner layer with $\sigma_1$ and the stresses of the outer layer with $\sigma_2$. We will use the same notation for the displacement and the hydrostatic pressure. Now we have for $1 < r < R_1$:

\[
\begin{align*}
\sigma_{1rr} &= -p_1 + 2\mu_1 \frac{\partial u_{1r}}{\partial r}, \\
\sigma_{1\theta\theta} &= -p_1 + 2\mu_1 \frac{u_{1r}}{r}, \\
\sigma_{1zz} &= -p_1.
\end{align*}
\]

(3.75) (3.76) (3.77)

And for $R_1 < r < R_2$ we have

\[
\begin{align*}
\sigma_{2rr} &= -p_2 + 2\mu_2 \frac{\partial u_{2r}}{\partial r}, \\
\sigma_{2\theta\theta} &= -p_2 + 2\mu_2 \frac{u_{2r}}{r}, \\
\sigma_{2zz} &= -p_2.
\end{align*}
\]

(3.78) (3.79) (3.80)

From the incompressibility condition and from the equilibrium equation we get:

\[
\begin{align*}
u_{1r} &= \frac{C_1}{r}, \\
u_{2r} &= \frac{\tilde{C}_1}{r}, \\
p_1 &= \tilde{C}_2, \\
p_2 &= C_3.
\end{align*}
\]

(3.81) (3.82) (3.83) (3.84)

The displacement in $r$ direction at $r = R_1$ should be the same for both layers, implying that $u_{1r} = u_{2r} = u_r$ for every $r$, and thus $C_1 = \tilde{C}_1$. 

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The boundary conditions can be derived making use of the jump conditions.

\[
(T_n_{1in}, e_r)|_{r=1} = -\sigma_{1rr}(1) = p_0, \\
(T_n_{1out}, e_r)|_{r=R_1} = \sigma_{1rr}(R_1) = (T_n_{2in}, e_r)|_{r=R_1} = \sigma_{2rr}(R_1), \\
(T_n_{2out}, e_r)|_{r=R_2} = \sigma_{2rr}(R_2) = 0.
\] (3.85)

With these three boundary conditions we can solve the constants \(C_1, C_2\) and \(C_3\):

\[
C_1 = \frac{-p_0 R_1^2 R_2^2}{2\mu_2 R_1^2 - 2\mu_2 R_2^2 - 2\mu_1 R_2^2 + 2\mu_1 R_1^2 R_2^2}, \\
C_2 = p_0 - \mu_1 \frac{p_0 R_1^2 R_2^2}{-\mu_2 R_1^2 + \mu_2 R_2^2 - \mu_1 R_2^2 + \mu_1 R_1^2 R_2^2}, \\
C_3 = -\frac{p_0 R_1^2 \mu_2}{-\mu_2 R_1^2 + \mu_2 R_2^2 + \mu_1 R_2^2 + \mu_1 R_1^2 R_2^2}. 
\] (3.88)

(3.89)

(3.90)

Since we know the surfaces of the lumen, thrombus and wall we can determine radius \(R_1\) and \(R_2\) (and the scaling factor \(R_{in}\)). Assuming a pressure of 5.3 kPa, a shear modulus of 131 kPa for the thrombus and a shear modulus of 333 kPa for the wall gives us the results for the stress as can be seen in Section 4.3.
Chapter 4

Results

This Chapter presents the results of the perturbations in Section 3.2.2 - 3.2.6, the results of the comparison of the 3D stress and 2D stress in Section 3.3 and the results of the comparison of the modelling with and without thrombus and the modelling with patient specific shape versus axisymmetric shape in Section 3.4. These results will be discussed in Section 4.4.

4.1 Results of analytical perturbations

Perturbing a parameter $A$ by multiplication of $(1 + \epsilon)$ has influence on the stress. As drawn in Figure 4.1 the principal stress will be perturbed by a same kind of multiplication. In this Section we determine $x_1$, $x_2$ and $x_3$ for each perturbed parameter. We will denote the principal stresses with $\sigma_1$, $\sigma_2$ and $\sigma_3$ and the value of the unperturbed principal stresses with $\sigma_{10}$, $\sigma_{20}$ and $\sigma_{30}$.

Aspect $A_0$ is perturbed $A(\epsilon) := A_0(1 + f(\epsilon))$

\[ \begin{align*}
\text{Linear elasticity model} \\
\text{The principal stresses:} \\
\sigma_1(\epsilon) &= \sigma_{10}(1 + x_1 f(\epsilon)) \\
\sigma_2(\epsilon) &= \sigma_{20}(1 + x_2 f(\epsilon)) \\
\sigma_3(\epsilon) &= \sigma_{30}(1 + x_3 f(\epsilon))
\end{align*} \]

Figure 4.1: Influence of perturbation of parameter $A$ on stress $\sigma$. 
4.1.1 Perturbation of the internal pressure

For the perturbation of the internal pressure \((p_0(1 + \varepsilon))\) we obtain for all principal stresses the same dependency;

\[
\begin{align*}
\sigma_1(\varepsilon) &= \sigma_{10}(1 + \varepsilon), \\
\sigma_2(\varepsilon) &= \sigma_{20}(1 + \varepsilon), \\
\sigma_3(\varepsilon) &= \sigma_{30}(1 + \varepsilon).
\end{align*}
\]

So when an error of \(\varepsilon = 0.01 = 1\%\) is made in the internal pressure there will also be an error of \(\varepsilon = 0.01 = 1\%\) in the stresses.

4.1.2 Angle dependent perturbation of the shear modulus

The angle dependent perturbation of the shear modulus was done by multiplying the shear modulus with \((1 + \varepsilon \cos(\theta))\). For the three principal stresses we obtain the following results:

\[
\begin{align*}
\sigma_1(\varepsilon) &= \sigma_{10}(1 + \varepsilon \cos(\theta)) \frac{r_o^2(-3r^4 + r_o + r_o^3 + r_o^3)}{3r(r^2 + r_o^2)(1 + r_o + r_o^2 + r_o^3)}, \\
\sigma_2(\varepsilon) &= \sigma_{20}(1 - \varepsilon \cos(\theta)) \frac{2r_o^2(1 - r^3 + r_o + r_o^2 + r_o^3)}{3r^2(1 + r_o + r_o^2 + r_o^3)}, \\
\sigma_3(\varepsilon) &= \sigma_{30}(1 - \varepsilon \cos(\theta)) \frac{(r - 1)r_o^2(1 + r + r^2 + r_o + rr_o + r_o^2)}{3r(r + r_o)(1 + r_o + r_o^2 + r_o^3)}.
\end{align*}
\]

Substituting the values \(r_o = \frac{0.03}{0.02} = 1.5\), \(p_0 = 5.3kPa/2 \times 333kPa = 0.16\) gives at the inner radius \(r = 1\):

\[
\begin{align*}
\sigma_1(\varepsilon) &= 0.4(1 + 0.1\varepsilon \cos(\theta)), \\
\sigma_2(\varepsilon) &= 0.1(1 - 1.3\varepsilon \cos(\theta)), \\
\sigma_3(\varepsilon) &= -0.2.
\end{align*}
\]

4.1.3 Perturbation of the ratio between outer and inner radius

The perturbation of the ratio between outer and inner radius is done by multiplication with \((1 + \varepsilon)\). The principal stresses become

\[
\begin{align*}
\sigma_1(\varepsilon) &= \sigma_{10}(1 - \varepsilon \frac{2(r^2 + 1)r_o^2}{(r_o^2 - 1)(r^2 + r_o^2)}), \\
\sigma_2(\varepsilon) &= \sigma_{20}(1 - \varepsilon \frac{2r_o^2}{r_o^2 - 1}), \\
\sigma_3(\varepsilon) &= \sigma_{30}(1 - \varepsilon \frac{2(r^2 - 1)r_o^2}{(r_o^2 - r_o^2)(r^2 - 1)}).
\end{align*}
\]

Substituting the values \(r_o = \frac{0.03}{0.02} = 1.5\), \(p_0 = 5.3kPa/2 \times 333kPa = 0.16\) gives at the inner radius \(r = 1\):

\[
\begin{align*}
\sigma_1(\varepsilon) &= 0.4(1 - 2.2\varepsilon), \\
\sigma_2(\varepsilon) &= 0.1(1 - 3.6\varepsilon), \\
\sigma_3(\varepsilon) &= -0.2.
\end{align*}
\]
4.1.4 Angle dependent perturbation on shape

The perturbations of the boundaries \((R_{\text{out}}(1 + \varepsilon \cos(2\theta)))\) and \((R_{\text{in}}(1 + \varepsilon \cos(2\theta)))\) give the following results for the principal stresses:

\[
\begin{align*}
\sigma_1(\varepsilon) &= \sigma_0(1 + \varepsilon \cos(2\theta)) \frac{1}{C_2r^2 + C_1} (-3C_6r^4 + \frac{3C_3}{r^2} - C_5r^2 - 3C_6r^4), \\
\sigma_2(\varepsilon) &= \sigma_0(1 - \varepsilon \cos(2\theta)) \frac{C_4 + 3C_6r^4}{C_2r^2}, \\
\sigma_3(\varepsilon) &= \sigma_0(1 + \varepsilon \cos(2\theta)) \frac{1}{C_1 - C_2r^2} (2C_4 + 3C_6r^4 + \frac{3C_3}{r^2} - C_5r^2 - 3C_6r^4),
\end{align*}
\]

with the constants as mentioned in section 3.2.6. Substituting the values \(r_o = \frac{0.03}{0.02} = 1.5\), \(p_0 = 5.3kPa/2 \times 333kPa = 0.16\) gives at the inner radius \(r = 1\):

\[
\begin{align*}
\sigma_1(\varepsilon) &= 0.4(1 + 5.5\varepsilon \cos(2\theta)), \\
\sigma_2(\varepsilon) &= 0.1(1 + 6.6\varepsilon \cos(2\theta)), \\
\sigma_3(\varepsilon) &= -0.2(1 + 3.7\varepsilon \cos(2\theta)).
\end{align*}
\]
### 4.2 Results 3D modelling versus 2D modelling

For the slices of the 3D geometry on which the highest stress occurs we calculated the stress when assuming a plane strain problem. The results for the stress of the 3D simulations and the 2D simulations are summarized in the following tables.

#### Patient 1

**Slice 20**

<table>
<thead>
<tr>
<th></th>
<th>Von Mises stress</th>
<th>(x,y)-coordinates</th>
<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D simulation</td>
<td>265</td>
<td>(-13.5, -167.7)</td>
<td>303</td>
<td>(-13.5, -167.7)</td>
</tr>
<tr>
<td>2D patient specific without thrombus</td>
<td>239</td>
<td>(12.3, -169.9)</td>
<td>253</td>
<td>(12.3, -169.9)</td>
</tr>
</tbody>
</table>

**Slice 21**

<table>
<thead>
<tr>
<th></th>
<th>Von Mises stress</th>
<th>(x,y)-coordinates</th>
<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D simulation</td>
<td>304</td>
<td>(-12.9, -167.0)</td>
<td>340</td>
<td>(-12.9, -167.0)</td>
</tr>
<tr>
<td>2D patient specific without thrombus</td>
<td>235</td>
<td>(-14.9, -163.2)</td>
<td>244</td>
<td>(-14.9, -163.2)</td>
</tr>
</tbody>
</table>

**Slice 22**

<table>
<thead>
<tr>
<th></th>
<th>Von Mises stress</th>
<th>(x,y)-coordinates</th>
<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D simulation</td>
<td>295</td>
<td>(-14.9, -168.9)</td>
<td>335</td>
<td>(-14.9, -168.9)</td>
</tr>
<tr>
<td>2D patient specific without thrombus</td>
<td>270</td>
<td>(-17.8, -190.1)</td>
<td>277</td>
<td>(-17.8, -190.1)</td>
</tr>
</tbody>
</table>
### Patient 2

#### Slice 18

<table>
<thead>
<tr>
<th></th>
<th>Von Mises stress</th>
<th>(x,y)-coordinates</th>
<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D simulation</td>
<td>115</td>
<td>(0.5, 96.1)</td>
<td>130</td>
<td>(0.5, 96.1)</td>
</tr>
<tr>
<td>2D patient specific without thrombus</td>
<td>135</td>
<td>(6.5, 118.5)</td>
<td>134</td>
<td>(7.0, 118.8)</td>
</tr>
</tbody>
</table>

#### Slice 19

<table>
<thead>
<tr>
<th></th>
<th>Von Mises stress</th>
<th>(x,y)-coordinates</th>
<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D simulation</td>
<td>113</td>
<td>(0.3, 96.2)</td>
<td>138</td>
<td>(0.3, 96.2)</td>
</tr>
<tr>
<td>2D patient specific without thrombus</td>
<td>131</td>
<td>(7.9, 117.9)</td>
<td>131</td>
<td>(7.9, 117.9)</td>
</tr>
</tbody>
</table>

#### Slice 26

<table>
<thead>
<tr>
<th></th>
<th>Von Mises stress</th>
<th>(x,y)-coordinates</th>
<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D simulation</td>
<td>115</td>
<td>(13.7, 95.2)</td>
<td>135</td>
<td>(14.7, 95.9)</td>
</tr>
<tr>
<td>2D patient specific without thrombus</td>
<td>107</td>
<td>(16.8, 95.6)</td>
<td>108</td>
<td>(16.8, 95.6)</td>
</tr>
</tbody>
</table>
### Patient 3

#### Slice 4

<table>
<thead>
<tr>
<th></th>
<th>Von Mises stress</th>
<th>(x,y)-coordinates</th>
<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D simulation</td>
<td>71</td>
<td>(23.3, 80.1)</td>
<td>81</td>
<td>(23.3, 80.1)</td>
</tr>
<tr>
<td>2D patient specific without thrombus</td>
<td>123</td>
<td>(47.9, 109.2)</td>
<td>123</td>
<td>(47.9, 109.2)</td>
</tr>
</tbody>
</table>

#### Slice 15

<table>
<thead>
<tr>
<th></th>
<th>Von Mises stress</th>
<th>(x,y)-coordinates</th>
<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D simulation</td>
<td>75</td>
<td>(22.9, 98.7)</td>
<td>80</td>
<td>(22.9, 98.7)</td>
</tr>
<tr>
<td>2D patient specific without thrombus</td>
<td>75</td>
<td>(37.3, 104.6)</td>
<td>73</td>
<td>(37.3, 104.6)</td>
</tr>
</tbody>
</table>

#### Slice 16

<table>
<thead>
<tr>
<th></th>
<th>Von Mises stress</th>
<th>(x,y)-coordinates</th>
<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
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<tr>
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<td>76</td>
<td>(21.8, 99.1)</td>
<td>82</td>
<td>(21.8, 99.1)</td>
</tr>
<tr>
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<td>82</td>
<td>(20.3, 98.7)</td>
<td>80</td>
<td>(20.3, 98.7)</td>
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</tbody>
</table>

#### Slice 18

<table>
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<th>(x,y)-coordinates</th>
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</thead>
<tbody>
<tr>
<td>3D simulation</td>
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<td>(17.0, 81.0)</td>
<td>83</td>
<td>(17.0, 81.0)</td>
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<tr>
<td>2D patient specific without thrombus</td>
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<td>(14.7, 80.0)</td>
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Patient 4

Slice 14

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<td>(-8.9, 104.2)</td>
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<td>(31.8, 88.0)</td>
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<tr>
<td>2D patient specific</td>
<td>169</td>
<td>(-8.1, 102.5)</td>
<td>169</td>
<td>(-8.1, 102.5)</td>
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Slice 15

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<th>(x,y)-coordinates</th>
</tr>
</thead>
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<tr>
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<td>(-10.5, 103.8)</td>
<td>109</td>
<td>(-10.5, 103.8)</td>
</tr>
<tr>
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<td>(30.5, 92.4)</td>
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<td>(30.5, 92.4)</td>
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<td>without thrombus</td>
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Slice 16

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<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D simulation</td>
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<td>(19.5, 115.1)</td>
<td>105</td>
<td>(19.5, 115.1)</td>
</tr>
<tr>
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<td>146</td>
<td>(-14.4, 101.2)</td>
<td>147</td>
<td>(-14.4, 101.2)</td>
</tr>
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<td>without thrombus</td>
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Slice 17

<table>
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<th>(x,y)-coordinates</th>
<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
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<td>(18.0, 114.8)</td>
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<tr>
<td>2D patient specific</td>
<td>137</td>
<td>(9.1, 116.5)</td>
<td>140</td>
<td>(9.1, 116.5)</td>
</tr>
<tr>
<td>without thrombus</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We will determine the following percentage for each slice:

\[
\frac{\text{Max. p-stress in 3D simulations} - \text{max. p-stress in 2D simulation}}{\text{Max. p-stress in 3D simulations}} \times 100\%
\]

for comparing the stresses in the two different models. The differences (in percentages) between the aspects which can be compared are plotted on the next pages.
Figure 4.2: Difference in von Mises stress between 2D and 3D simulations.

Figure 4.3: Difference in maximal principal stress between 2D and 3D simulations.
### 4.3 Results modelling thrombus and patient specific shape

The values of the maximal principal stress and Von Mises stress and the coordinates on which they occur are calculated and summarized in tables.

**Patient 1**

**Slice 20**

<table>
<thead>
<tr>
<th></th>
<th>Von Mises stress</th>
<th>(x,y)-coordinates</th>
<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>2D patient specific without thrombus</td>
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<td>(12.3, -169.9)</td>
<td>253</td>
<td>(12.3, -169.9)</td>
</tr>
<tr>
<td>2D patient specific with thrombus</td>
<td>77</td>
<td>(-17.2, -164.5)</td>
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<td>(-17.2, -164.5)</td>
</tr>
<tr>
<td>2D axisymmetric shape without thrombus</td>
<td>87</td>
<td>-</td>
<td>80</td>
<td>-</td>
</tr>
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<td>2D axisymmetric shape with thrombus</td>
<td>19</td>
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**Slice 21**

<table>
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<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
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<td>(-14.9, -163.2)</td>
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<tr>
<td>2D patient specific with thrombus</td>
<td>45</td>
<td>(-16.9, -163.9)</td>
<td>43</td>
<td>(-16.9, -163.9)</td>
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<td>2D axisymmetric shape with thrombus</td>
<td>22</td>
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**Slice 22**

<table>
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<td>51</td>
<td>(23.3, -186.1)</td>
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<tr>
<td>2D axisymmetric shape without thrombus</td>
<td>88</td>
<td>-</td>
<td>81</td>
<td>-</td>
</tr>
<tr>
<td>2D axisymmetric shape with thrombus</td>
<td>22</td>
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### Patient 2

#### Slice 18

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<td>(6.5, 118.5)</td>
<td>134</td>
<td>(7.0, 118.8)</td>
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<td>without thrombus</td>
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<td></td>
<td></td>
<td></td>
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<tr>
<td>2D patient specific</td>
<td>56</td>
<td>(6.9, 118.9)</td>
<td>55</td>
<td>(6.4, 118.9)</td>
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<td>with thrombus</td>
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<td></td>
<td></td>
<td></td>
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<td>16</td>
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#### Slice 19

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<th>(x,y)-coordinates</th>
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<tbody>
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<td>(7.9, 117.9)</td>
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<td>(7.9, 117.9)</td>
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<td></td>
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<tr>
<td>2D patient specific</td>
<td>49</td>
<td>(16.2, 120.2)</td>
<td>50</td>
<td>(16.2, 120.2)</td>
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<td></td>
<td></td>
<td></td>
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<td>39</td>
<td>-</td>
<td>40</td>
<td>-</td>
</tr>
<tr>
<td>without thrombus</td>
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<td>18</td>
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#### Slice 26

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<td>(16.8, 95.6)</td>
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<td></td>
<td></td>
<td></td>
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<td>(11.8, 121.5)</td>
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<tr>
<td>with thrombus</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2D axisymmetric shape</td>
<td>40</td>
<td>-</td>
<td>41</td>
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<td>without thrombus</td>
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<td></td>
<td></td>
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<td>-</td>
<td>13</td>
<td>-</td>
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### Patient 3

##### Slice 4

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<td>123</td>
<td>(47.9, 109.2)</td>
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<td>53</td>
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##### Slice 15

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<td>(37.3, 104.6)</td>
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<td>45</td>
<td>-</td>
<td>46</td>
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##### Slice 16

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<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>2D patient specific without thrombus</td>
<td>82</td>
<td>(20.3, 98.7)</td>
<td>80</td>
<td>(20.3, 98.7)</td>
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<td>48</td>
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##### Slice 18

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<th>(x,y)-coordinates</th>
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</thead>
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<td>(14.7, 80.0)</td>
<td>112</td>
<td>(14.7, 80.0)</td>
</tr>
<tr>
<td>2D axisymmetric shape without thrombus</td>
<td>46</td>
<td>-</td>
<td>48</td>
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### Patient 4

#### Slice 14

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<tbody>
<tr>
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<td>169</td>
<td>(-8.1, 102.5)</td>
<td>169</td>
<td>(-8.1, 102.5)</td>
</tr>
<tr>
<td>2D patient specific with thrombus</td>
<td>86</td>
<td>( 35.9, 89.5)</td>
<td>86</td>
<td>( 35.9, 89.5)</td>
</tr>
<tr>
<td>2D axisymmetric shape without thrombus</td>
<td>61</td>
<td>-</td>
<td>66</td>
<td>-</td>
</tr>
<tr>
<td>2D axisymmetric shape with thrombus</td>
<td>30</td>
<td>-</td>
<td>32</td>
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#### Slice 15

<table>
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<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
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<tbody>
<tr>
<td>2D patient specific without thrombus</td>
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<td>(30.5, 92.4)</td>
<td>147</td>
<td>(30.5, 92.4)</td>
</tr>
<tr>
<td>2D patient specific with thrombus</td>
<td>70</td>
<td>( 36.1, 88.9)</td>
<td>70</td>
<td>( 36.4, 90.0)</td>
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<tr>
<td>2D axisymmetric shape without thrombus</td>
<td>60</td>
<td>-</td>
<td>64</td>
<td>-</td>
</tr>
<tr>
<td>2D axisymmetric shape with thrombus</td>
<td>40</td>
<td>-</td>
<td>43</td>
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#### Slice 16

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<th>(x,y)-coordinates</th>
</tr>
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<td>146</td>
<td>(-14.4, 101.2)</td>
<td>147</td>
<td>(-14.4, 101.2)</td>
</tr>
<tr>
<td>2D patient specific with thrombus</td>
<td>168</td>
<td>( 26.4, 78.6)</td>
<td>92</td>
<td>( 26.8, 78.9)</td>
</tr>
<tr>
<td>2D axisymmetric shape without thrombus</td>
<td>55</td>
<td>-</td>
<td>58</td>
<td>-</td>
</tr>
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<td>2D axisymmetric shape with thrombus</td>
<td>28</td>
<td>-</td>
<td>30</td>
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#### Slice 17

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<th>Max principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>2D patient specific without thrombus</td>
<td>137</td>
<td>( 9.1, 116.5)</td>
<td>140</td>
<td>( 9.1, 116.5)</td>
</tr>
<tr>
<td>2D patient specific with thrombus</td>
<td>64</td>
<td>( -4.2, 122.2)</td>
<td>65</td>
<td>( -4.2, 122.2)</td>
</tr>
<tr>
<td>2D axisymmetric shape without thrombus</td>
<td>56</td>
<td>-</td>
<td>60</td>
<td>-</td>
</tr>
<tr>
<td>2D axisymmetric shape with thrombus</td>
<td>32</td>
<td>-</td>
<td>34</td>
<td>-</td>
</tr>
</tbody>
</table>
We will determine the following percentages for each slice; For the simulations of patient specific shape:

\[
\frac{\text{Max. p-stress in 2D with thrombus} - \text{max. p-stress in 2D without thrombus}}{\text{Max. p-stress in 2D with thrombus}} \times 100\%.
\]

For the analytical part assuming axisymmetric shape:

\[
\frac{\text{Max. p-stress in 2D with thrombus} - \text{max. p-stress in 2D without thrombus}}{\text{Max. p-stress in 2D with thrombus}} \times 100\%.
\]

For comparing the analytical part (axisymmetric shape, without thrombus) with the simulations (patient specific shape):

\[
\frac{\text{Max. p-stress patient specific shape} - \text{max. p-stress axisymmetric shape}}{\text{Max. p-stress patient specific shape}} \times 100\%.
\]

For comparing the analytical part (axisymmetric shape, with thrombus) with the simulations (patient specific shape):

\[
\frac{\text{Max. p-stress patient specific shape} - \text{max. p-stress axisymmetric shape}}{\text{Max. p-stress patient specific shape}} \times 100\%.
\]

The differences (in percentages) between the aspects which can be compared are plotted on the next pages.
Figure 4.4: Difference in von Mises stress in 2D simulations between with and without thrombus.

Figure 4.5: Difference in maximal principal stress in 2D simulations between with and without thrombus.
Figure 4.6: Difference in von Mises stress in 2D axisymmetric shape between with and without thrombus.

Figure 4.7: Difference in maximal principal stress in 2D axisymmetric shape between with and without thrombus.
Figure 4.8: Difference in von Mises stress in modelling without thrombus between patient specific and axisymmetric.

Figure 4.9: Difference in maximal principal stress in modelling without thrombus between patient specific and axisymmetric.
Figure 4.10: Difference in von Mises stress in modelling with thrombus between patient specific and axisymmetric.

Figure 4.11: Difference in maximal principal stress in modelling with thrombus between patient specific and axisymmetric.
4.4 Discussion

The most important general assumption which is made is that we neglect the initial stress in the AAA wall. This means that we do not model real life, but we try to get close to reality in order to determine whether an AAA may rupture or not. Some research is performed about simplifying models [Ambler et al., 2002]. Sometimes a simplified model already describes the reality quite well.

For the analytical perturbations even more assumptions are made. We assume a simplified shape, but many articles point out that shape has much influence on stress (for example [Elger et al., 1996]).

The advantage of the analytical work which will be done is that no assumptions need to be made about the numerical values of parameters as radius, shear modulus and internal pressure. This makes it more general.

For the comparing of stress in 2D and 3D we do not only measure the influence of the curvature of the longitudinal direction of the shape. The boundary conditions for 3D modelling differ from the boundary conditions for 2D modelling. Plane strain means assuming an infinitely long tube while for the 3D modelling we assume the upper and lower slices to be fixed in all directions. It can be shown that this assumption of boundary conditions has some influence on the stresses (for example by comparing the results with another 3D simulation in which the upper and lower slices are fixed in only the longitudinal direction, Appendix F). So we cannot say that the difference in stress in the 2D and 3D modelling is only caused by the curvature of the longitudinal direction of shape.

In the last part of the results we compared the stress when modelling a patient specific shape versus modelling an axisymmetric shape.

The stress which is calculated with Sepran (for a patient specific shape) is obtained from a Neo-Hookean model. The stress which is calculated analytically (for an axisymmetric shape) is obtained from a linear elastic model for small deformations. This will also cause (a small) difference in the stress. This difference will not be very large because the deformations in an axisymmetric shape are very small and thus the models are (almost) similar.

When an axisymmetric shape is used in stead of a patient specific shape the stress decreases, but this might be a logical consequence of the better distribution of material. Also, when thrombus is added in the model we obtain a decrease in stress. This might have been expected because there is more material available to receive the load and thus the stress will decrease. But now we have an idea of how much the stress decreases.
Chapter 5

Conclusions

5.1 Conclusion sensitivity analysis
Simplifying the shape, assuming incompressibility and all the other assumptions mentioned in Section 3.2 make it possible to use perturbation methods in order to determine whether the stresses in this model are relevantly influenced by shape, pressure or shear modulus. Small perturbations of shear modulus and pressure do not have much influence on the stresses. Also an angular perturbation (in cosinus-form) of the shear modulus does not have much influence on the stresses in this model. But perturbation of the ratio between outer and inner radius does have much influence on the stress. Also an angular perturbation of the boundaries has much influence on the stress. Therefore, it is important to model the shape of an aneurysm as good as possible. When small errors occur in the internal pressure or the material properties the stress is not influenced much, so these parameters do not have to be very accurate.

5.2 Conclusion 3D modelling versus 2D modelling
The values of the stress (maximal principal stress and von Mises stress) predicted by the 3D model differ a lot from the stresses predicted by the 2D model. In extreme cases the values of the 2D-stresses are twice as large as the 3D-stresses. The position of the highest stress (in (x,y)-coordinates) is sometimes almost the same. The curvature of the longitudinal coordinate is thus very important. The boundary conditions of the 3D simulations and 2D simulations do not match which will also cause difference in stress. There seems to be no correlation between the stresses predicted by the 2D model and the stresses predicted by the 3D model.

5.3 Conclusion of modelling thrombus and patient specific shape
Thrombus significantly reduces the stress in 2D. From the analytical part of the investigation we can also conclude that thrombus highly decreases the stress as well. Modelling an axisymmetric shape in stead of a patient specific shape also decreases the stresses in 2D. Thrombus and axisymmetry seem to have influence on the stresses for the 2D modelling, in such a way that the stress decreases.
Therefore, it is important to model the patient specific shape instead of an axisymmetric shape. Also the modelling of thrombus is important.

5.4 Recommendations

It would be sensible not to use information obtained from 2D simulations in 3D simulations. Therefore it is useful to improve the existing 3D model in the Hemodyn project by adding the thrombus in the model. The third dimension is needed in order to determine some sensitivities. The influence of perturbations can then be determined by performing a lot of simulations. For example, a random perturbation of the shear modulus with a maximum of 10% can be done. A few runs have already been performed in the model without thrombus, so one can easily proceed this work.

Maybe it is also possible to estimate certain material properties when the 3D model handles thrombus and maybe calcifications. By varying the values of for example shear moduli and internal pressure, the deformations will vary as well. A realistic deformation will be caused by realistic values of shear moduli and internal pressure. Unfortunately there are probably many combinations of values possible which cause a realistic deformation.

It might be interesting as well to investigate the influence on the stresses when calcifications are not recognized. This can be done by simulations in which the shear modulus has two different values (over two areas).

To determine the influences of axisymmetry in a 3D geometry one can simulate a 3D tube which has axisymmetric slices. To avoid influences of boundary conditions one can elongate the aneurysm with the upper and lower slice.
Appendix A

Justification of neglecting flow effects for AAA rupture prediction

Consider a tube with distensible wall and cross-sectional area $A(z, t)$ with a flow $q(z, t)$ in the longitudinal direction. Assume the relation between the pressure $p(z, t)$ and area $A(z, t)$ to be independent of $z$ for the undeformed state. The integrated momentum equation in linearized form is

$$\rho \frac{\partial q}{\partial t} + A_0 \frac{\partial p}{\partial z} = 0. \quad (A.1)$$

When no friction acts on the system and when $V/c << 1$ and $2\pi a_0/\lambda << 1$. $A_0$ denotes the linearized form of $A(z, t)$.

Conservation of mass implies

$$\frac{\partial A}{\partial t} + \frac{\partial q}{\partial z} = 0. \quad (A.2)$$

We can write

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial p} \frac{\partial p}{\partial t} = C \frac{\partial p}{\partial t}, \quad (A.3)$$

with $C$ the compliance. This gives a system of two PDE’s:

$$\rho \frac{\partial q}{\partial t} + A_0 \frac{\partial p}{\partial z} = 0, \quad (A.4)$$

$$C^2 \frac{\partial q}{\partial t} + \frac{\partial q}{\partial z} = 0. \quad (A.5)$$

Differentiating the first equation with respect to $z$ and the second equation with respect to $t$ gives:

$$\rho \frac{\partial^2 q}{\partial z \partial t} + A_0 \frac{\partial^2 p}{\partial z^2} = 0, \quad (A.6)$$

$$C^2 \frac{\partial^2 p}{\partial t^2} + \frac{\partial^2 q}{\partial t \partial z} = 0. \quad (A.7)$$

Combining these formulas gives

$$\frac{A_0}{\rho} \frac{\partial^2 p}{\partial z^2} - C^2 \frac{\partial^2 p}{\partial t^2} = 0, \quad (A.8)$$
or,
\[
\frac{\partial^2 p}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0,
\]
(A.9)
with \(c = \sqrt{A_0/\rho C^2}\) the wave speed. For the abdominal aorta the wave speed equals more or less 6 m/s. The length of the aorta, \(L \approx 0.2\) m and the pressure in the aorta \(p_0 \approx 16\) kPa, implying
\[
\frac{\partial p}{\partial z} L << p_0.
\]
(A.10)
From this we can conclude that the boundary conditions of the pressure can be taken the same for every location in the aorta.

The shear stress for a Poiseuille flow depends on the blood velocity, the viscosity and the radius of the aorta. The peak velocity of the blood is about 2 m/s. The viscosity and radius of the aorta are respectively \(\eta = 3 \cdot 10^{-2}\) Pa.s and \(a = 0.02\) m.
\[
t_s = -\eta \frac{\partial v_z}{\partial r} \approx 3 \cdot 10^{-2}\ \text{Pa.s} \cdot \frac{2\ \text{m/s}}{0.02\ \text{m}} \approx 3\ \text{Pa}.
\]
(A.11)
The pressure in the aorta is about 16 kPa, so the stress caused by the flow may be neglected.

We can conclude two things:

- The pressure can be taken as a constant over the boundary.
- The shear stress caused by the flow is negligible.
Appendix B

Modelling a healthy aorta

The research described in this section has been performed by Chris van Oijen [van Oijen, 2003].

B.1 Composite material model

The aortic wall can be assumed to consist of matrix material and fibers (for example collagen, elastin). The matrix material stretches in another way than the fibers (Figure B.1). This implies that the stresses of the composite model are the sum of two parts.

\[
\psi = \psi(I_1, I_2, I_3, I_4, I_5),
\]

with

\[
I_1 = \text{tr}(\mathbf{C}),
\]

\[
I_2 = \frac{1}{2}((\text{tr}(\mathbf{C}))^2 - \text{tr}(\mathbf{C}^2)),
\]

\[
I_3 = \text{det}(\mathbf{C}),
\]

\[
I_4 = e_0 \cdot \mathbf{C} \cdot e_0 = \lambda^2,
\]

\[
I_5 = e_0 \cdot \mathbf{C}^2 \cdot e_0,
\]

with \(e_0\) the undeformed vector of the fiber direction. Holzapfel [Holzapfel et al., 2000] suggests a split of the isochoric strain-energy function into a part \(\psi_{\text{iso}}\) associated with isotropic deformations and a part \(\psi_{\text{aniso}}\) associated with anisotropic deformations. The collagen fibers are not active at low pressures so \(\psi_{\text{iso}}\) will be associated with the matrix material, which is assumed to be isotropic. The resistance to stretch at high pressures is almost entirely due
to collagenous fibers and this mechanical response is therefore governed by the anisotropic function $\psi_{aniso}$, namely

$$\psi(I_1, I_2, I_3, I_4, I_5) = \psi_{iso}(I_1, I_2) + \psi_{aniso}(I_1, I_2, I_3, I_4, I_5). \tag{B.7}$$

The Cauchy stress is derived from the strain energy function as

$$\sigma = \sqrt{\det(C)^{-1}} \mathcal{F} : 2 \frac{\partial \psi}{\partial C} : \mathcal{F}^T. \tag{B.8}$$

with $C$ the Green deformation tensor. Introducing the Cauchy-Green strain tensor $\mathcal{B} = \mathcal{F} : \mathcal{F}^T$ and applying the definition in equation (B.8) gives

$$\sigma = 2\sqrt{\det(C)^{-1}} ((I_2 \psi_2 + I_3 \psi_3)I + \psi_1 \mathcal{B} - J \psi_2 \mathcal{B}^{-1} + I_4 \psi_4 e^T e + I_4 \psi_5 (e^T \mathcal{B} e + e^T e)), \tag{B.9}$$

with $e$ the current fiber direction and $\psi_a = \partial \psi / \partial I_a$ and $a = 1, \ldots, 5$. When the material is incompressible; $\det(C) = 1$.

### B.1.1 Thick walled tube model

This model, introduced by Holzapfel, uses three material parameters, $\mu_0, k_1, k_2$. The free-energy function may be written as

$$\psi = \mu_0 (I_3 - 3) + \frac{k_1}{2k_2} \exp(k_2(I_4 - 1)^2) - 1). \tag{B.10}$$

Now the Cauchy stress in generalized form is

$$\sigma = -P I + \hat{\tau} + \tau_f ee^T + \tau_{f_2}(e^T \mathcal{B} e + e^T e). \tag{B.11}$$

In case of hyperelasticity

$$\hat{\tau} = 2(\psi_1 \mathcal{B} - \psi_2 \mathcal{B}^{-1}), \quad \tau_f = 2I_4 \psi_4, \quad \tau_{f_2} = 2I_4 \psi_5. \tag{B.12}$$

The term $\tau_{f_2}$ is not needed in practice to describe the most common features of transversely isotropic material. Therefore it is omitted. So a constitutive equation which sufficiently captures incompressible transversely isotropic material behavior, can be written as

$$\sigma = -P I + \hat{\tau} + \tau_f ee^T. \tag{B.13}$$

An extra fiber family will give an extra term in the form of $\tau_{fi} e_i e_i$ with the fiber stress $\tau_{fi}$ belonging to the fiber direction $e_i$.

### B.1.2 Volumetric participation

Chris van Oijen introduced a volumetric participation (cf Green and Naghdi, 1965) in the model of Holzapfel. A volume element that consists of both matrix and fiber material is represented by an element with averaged stress. However, if the anisotropic part of the stress is averaged over the total element any spatial information regarding the fiber is lost. This can be solved by using:

$$\sigma = -P I + \tau + n(\tau_f - e^T e) ee^T. \tag{B.14}$$

with $n$ the volumetric participation of the fiber-material.
Appendix C

Sepran

Sepran, developed by "Ingenieursbureau SEPRA", is a general finite element package. It enables simulations of a wide variety of problems, ranging from fluid mechanics, structural mechanics, electromechanics to lubrication. One of the main items in finite elements methods is the partitioning of the computational domain into elements. The collection of elements is called a mesh.

The problem that has to be solved can be defined. The physical problem consists of the set of partial differential equations and boundary conditions.
Appendix D

ILU and BICGStAB

As described in [Saad, 2000], the incomplete LU (ILU) factorization process computes a sparse lower triangular matrix \( L \) and a sparse upper triangular matrix \( U \) of a general sparse matrix \( A \) (with elements \( a_{ij} \)). The residual matrix \( R = LU - A \) satisfies certain constraints, such as having zero entries in some locations. The algorithm can be used as a preconditioning technique. For any zero pattern set \( P \), such that

\[ P \subset \{(i, j)|i \neq j, 1 \leq i, j \leq n\} \]

an ILU factorization can be computed as follows:

1. For \( k = 1, \ldots, n - 1 \) Do:
2. For \( i = k + 1, \ldots, n \) and if \( (i, k) \notin P \) Do
3. \( a_{ik} := a_{ik}/a_{kk} \)
4. For \( j = k + 1, \ldots, n \) and for \( (i, j) \notin P \) Do:
5. \( a_{ij} := a_{ij} - a_{ik}a_{kj} \)
6. EndDo
7. EndDo
8. EndDo

The Biconjugate Gradient Stabilized (BICGSTAB) algorithm can be used to solve \( Ax = b \):

1. Compute \( r_0 = b - Ax_0; r_0^* \) arbitrary;
2. \( p_0 := r_0 \)
3. For \( j = 0, 1, \ldots \) until convergence Do:
4. \( \alpha_j := (r_j, r_j^*)/(Ap_j, r_j^*) \)
5. \( s_j := r_j - \alpha Ap_j \)
6. \( w_j := (As_j, s_j)/(As_j, As_j) \)
7. \( x_{j+1} := x_j + \alpha_jp_j + w_js_j \)
8. \( r_{j+1} := s_j - w_jAs_j \)
9. \( \beta_j := (r_{j+1}, r_0^*)/(r_j, r_j^*) \times (\alpha_j/w_j) \)
10. \( p_{j+1} := r_{j+1} + \beta_j(p_j - w_jAp_j) \)
11. EndDo
Appendix E

Simulations in Sepran

The prb files are used to describe the problem in Sepran. Below the prb-file for the 3D modelling and the prb-file for the 2D modelling.

```plaintext
# aaapres.prb
#
# Problem file for AAA pressurisation.
#
# Berent Wolters, 24-06-04
# Ellen van Nunen, 17-7-04

constants
  integers
    1: solid = 1
    2: incr = 0
    3: mincr = 32
    4: bounsurf = 8
    5: insurf = 15
    6: outsurf = 16
    7: egrpwall = 2
    8: iwrite # trigger for writing text to standard out

reals
  1: dt
  2: tstart = 0.0
  3: tend = 1.0
  4: dw = 2d0
  5: eps = 1d-3

scalars
  1: incr
  2: mincr
  3: one
  4: dw

vector_names
  # The following vectors are needed (in this order)
  # +---+---------------------------------------------+
#  | 1  | u       | incremental solution |
#  | 2  | un      | old solution at increment n |
1: u      # intermediate solution
2: un      # last converged solution
3: uvec    # user vector
4: du      # iterative solution
5: wallpos # wall node position vector
6: displacm # general displacement vector
7: meshcoor # mesh coordinates
8: stress  # stress
9: strain  # strain

end

start

secomp = not # no output to sepcomp.out
name_mesh = 'meshoutput' # name of the mesh file
set output level = 0 # deactivate printing of array lengths
end

problem $solid

types
  elgrp1 = 210
  elgrp2 = 201
essbouncond
  degfd1, degfd2, degfd3 = surfaces (s15)
  degfd1, degfd2, degfd3 = surfaces (s16)
### Crouzeix-Raviart (201)
  renumber levels (1,2,3),(4,5,6,7)
end

matrix
  method = 6, problem = $solid
end

essential boundary conditions, sequence_number = $solid, problem = $solid
  surfaces(s15), degfd1, value = 0d0
  surfaces(s15), degfd2, value = 0d0
  surfaces(s15), degfd3, value = 0d0
  surfaces(s16), degfd1, value = 0d0
  surfaces(s16), degfd2, value = 0d0
  surfaces(s16), degfd3, value = 0d0
end

coefficients, sequence_number = $solid
  elgrp1 (nparm = 15)
    icoef1 = 1 # coordinate system for boundary condition
    # 0 - global system

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# 1 - local system
icoef3 = 0  # type of numerical integration
    # 0 - default value
d4 = (func=4)  # 1-component of stress
d7 = 0d0  # 2-component of stress
d8 = 0d0  # 3-component of stress
elgrp2 (nparm = 45)
icoef2 = 0  # type of stress-strain relation
    # 0 - full 3D
icoef3 = 0  # type of numerical integration
    # 0 - default value
    # 1 - Newton-Cotes
    # 3 - Gauss
icoef4 = 2  # constitutive law
    # 1 - compressible Neo-Hookean
    # 2 - incompressible Neo-Hookean
    # 4 - incompressible Mooney-Rivlin
    # 10 - composite material (t_hat + psi)
    # 11 - composite material t_hat + Theta...
    # 99 - user defined material
icoef5 = 0  # user flags, coef = iusrvec + 100*iusrflg
    # iusrvec = 0 - user vector is not filled
coefficient10 = 0.33333d3  # 0.174 d3 Neo-Hookean: shear modulus (kPa)
    # Mooney-Rivlin: material parameter c0
end

structure
### No output to sepcomp.out
no_output

### Create vectors
create_vector, vector %u, sequence_number = 1
create_vector, vector %un, sequence_number = 1
create_vector, vector %uvec, sequence_number = 1
create_vector, vector %du, sequence_number = 1

### Initialize wall transformation process
user_output, sequence_number = 7, //
extra_integers = ($bounsurf,$egrpwall)

### Transform mesh to AAA shape
create_vector, vector %displacm, sequence_number = 2
deform_mesh vector = %displacm

### Set wall thickness
scalar %dw = $dw

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### Initialize wall node position vector
create_vector, vector %wallpos, sequence_number = 3

### Get new wall node positions based on normals to lumen surface
user_output, sequence_number = 8, //
extra_integers = (%wallpos,$bounsurf,$insurf,$outsurf), extra_scalars = (%dw)

### Set new wall node positions
create_vector, vector = %displacm, sequence_number = 4
deform_mesh vector = %displacm

# Make isoparametric: correct edge/face midpoints and element centroid
# Extra_integer = (shape nr.)
user_output, sequence_number = 6, extra_integer = (14)

### Write definitive mesh
write_mesh, file = 'meshoutput_aaa'

### Set scalars
scalar %one = 1d0
scalar %nincr = $nincr

### Start time loop
start_time_loop

time_integration, sequence_number = 1
scalar %incr, func = 1

### Print time and increment number
user_output, sequence_number = 3

### Clear solution vector
create_vector, sequence_number = 1, vector %u

### Prescribe boundary conditions
prescribe_boundary_conditions, sequence_number = $solid, //
    vector = %du

### Start incremental solution loop
start_loop, sequence_number = 1

### Solve system of equations
solve_linear_system, seq_coef = $solid, seq_solve = $solid, //
    problem = $solid, vector = %du

### Write CPU-time used
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user_output, sequence_number = 4

### Update solution vector
compute_vector %u lin_combination //
   scalar %one vector %u scalar %one vector %du

### Clear incremental solution vector
create_vector, sequence_number = 1, vector %du

### End incremental solution loop
dend_loop

### Deform mesh
deform_mesh, vector = %u

### Update total solution vector
compute_vector %un lin_combination //
  scalar %one vector %un scalar %one vector %u

### Write total solution vector
user_output, sequence_number = 5, //
  extra_integers = (%un,$solid), extra_scalar = %incr

### Create mesh coordinate vector
create_vector, vector %meshcoor, sequence_number = 5

### Write mesh coordinate vector
user_output, sequence_number = 5, //
  extra_integers = (%meshcoor,$solid), extra_scalar = %incr

### Compute and write stress vector
derivatives, seq_deriv = 1, problem = $solid, vector = %stress
user_output, sequence_number = 5, //
  extra_integers = (%stress,$solid), extra_scalars = %incr

### Compute and write strain vector
derivatives, seq_deriv = 2, problem = $solid, vector = %strain
user_output, sequence_number = 5, //
  extra_integers = (%strain,$solid), extra_scalars = %incr

### End time loop
dend_time_loop

end

time_integration, sequence_number = 1
method = stationary
  tinit = $tstart
  tend = $tend
  tstep = $dt
end

loop_input, sequence_number = 1
  maxiter = 16
  miniter = 1
  accuracy = 1d-3
  criterion = absolute
  seq_vector = %u
  print_level = 2
  at_error= return
end

# Create solution vector for solid problem
create vector, sequence_number = 1, problem = $solid
  type = solution vector
  value = 0d0
end

# Create vector for main transformation
create vector, sequence_number = 2, problem = $solid
  type = vector of special structure 3
  old_vector = 1
end

# Initialize wall position vector
create vector, sequence_number = 3, problem = $solid
  type = vector of special structure 3
  value = 0d0
end

# Create wall node displacement vector
create vector, sequence_number = 4, problem = $solid
  volumes(v1), old_vector = 2, seq_vectors = v%wallpos
end

# Create coordinate vector
create vector, sequence_number = 5, problem = $solid
  type = vector of special structure 3
  old_vector = 3
end

derivatives, sequence_number = 1
  icheld = 6 # compute stress
element_groups = (2)
ix = 1
seq_input_vector = %un
numvec = 2
end

derivatives, sequence_number = 2
icheld = 8 # compute strain
element_groups = (2)
ix = 1
seq_input_vector = %un
numvec = 2
end

solve, sequence_number = $solid
   iteration_method = bicgstab, preconditioning = ilu, accuracy = $eps, //
   print_level = 0, termination_crit = rel_residual, start = old_solution
end

end_of_sepran_input
# prb-file for 2D patient specific modelling, plane strain problem.
#
# Ellen van Nunen, 17-7-04

CONSTANTS

integers
1: eltype = 201  # see PG 5.2.5
2: beltype = 210 # see PG 7.1.22
3: incr
4: nincr = 10

reals
1: dt = 0.1
2: tstart = 0.0
3: tend = 1.0
4: one = 1.0

variables
1: incr  # number of load/displacement increments
2: mp    # multiplication factor
3: nincr
4: one

vector_names
1: x     # intermediate displacement vector
2: x_n   # last converged total displacement vector
3: uvec  # uservector
4: dx    # iterative displacement-error vector
5: stress # stress vector
6: strain # strain vector
7: meshcoor # coordinates

END

START

renumber best band

END

PROBLEM

types
elgrp1 (type=$beltype)
elgrp2 (type=$eltype)

essbouncond
degfd2 = points(p2)
degfd1 = points(p2)
renumber levels (1,2)

END

essential boundary conditions, sequence_number = 1, problem = 1
points(p2), degfd2, value = 0d0
points(p2), degfd1, value = 0d0
### Initialize variables ###
scalar %mp = 1d0
scalar %incr = 0
scalar %nincr = $nincr

### Create some vectors ###
create_vector, sequence_number = 1, vector = %x
create_vector, sequence_number = 1, vector = %x_n
create_vector, sequence_number = 2, vector = %uvec
create_vector, sequence_number = 1, vector = %dx

plot_mesh

### Start time loop
start_time_loop
    time_integration, sequence_number = 1
    scalar %incr, func = 1

        ### Print time and increment number
        user_output, sequence_number = 3

            prescribe_boundary_conditions, sequence_number=1, //
                vector = %dx

### Start incremental solution loop
start_loop, sequence_number = 1

### Solve system of equations
solve_linear_system, seq_coef = 1, seq_solve = 1, //
    problem = 1, vector = %dx

### Write CPU-time used
user_output, sequence_number = 4

### Update solution vector
compute_vector %x lin_combination //
    scalar %mp vector %x scalar %mp vector %dx
### Clear incremental solution vector
create_vector, sequence_number = 1, vector %dx

### End incremental solution loop
end_loop

### Deform mesh
deform_mesh, vector = %x

plot_mesh

### Update total solution vector by: x_n = x_n + x ###
compute_vector %x_n lin_combination scalar%mp //
vector%x_n scalar%mp vector%x

output

### Create mesh coordinate vector!!!!!!!
create_vector, vector %meshcoor, sequence_number = 5

### Write mesh coordinate vector!!!!!!!
user_output, sequence_number = 5, //
extra_integers = (%meshcoor,1), extra_scalar = %incr

### Clear some vectors ###
create_vector, sequence_number = 1, vector = %x

### Calculate some initial derivatives ###
### intermediate x is cleared after updating ###
### to insure that derivatives are calculated with respect to total solution ###
derivatives, seq_coef = 1, seq_deriv = 1, vector = %stress
user_output, sequence_number = 5, //
extra_integers = (%stress,1), extra_scalars = %incr
derivatives, seq_coef = 1, seq_deriv = 2, vector = %strain
user_output, sequence_number = 5, //
extra_integers = (%strain,1), extra_scalars = %incr

### End time loop
de_time_loop

END

MATRIX
method = 1, problem = 1
time_integration, sequence_number = 1
method = stationary
tinit = $tstart
tend = $tend
tstep = $dt
toutinit = $tstart
toutend = $tend
toutstep = $dt
def

loop_input, sequence_number = 1
maxiter = 200
miniter = 3
accuracy = 1d-3
criterion = relative
seq_vector = %x
print_level = 2
at_error= return
def

COEFFICIENTS, sequence_number = 1
elgrp1 (nparm=25)
  icoef1= 1 # Prescribe stresses in normal and tangential direction
  icoef3= 0 # Newton-Cotes integration rule
  coef6 = (func = 3) # Load in normal direction
  coef7= 0 # Load in tangential direction
elgrp2 (nparm=45) Coefficients for solid nonlinear element
  icoef2 = 0 # type of stress-strain relation
              # 0 - full 3D
  icoef3 = 0 # type of numerical integration
              # 0 - default value
              # 1 - Newton-Cotes
              # 3 - Gauss
  icoef4 = 2 # constitutive law
              # 1 - compressible Neo-Hookean
              # 2 - incompressible Neo-Hookean
              # 4 - incompressible Mooney-Rivlin
              # 10 - composite material (t_hat + psi)
              # 11 - composite material t_hat + Theta...
**# 99 - user defined material**

icoef5 = 0 # user flags, coef = iusrvec + 100*iusrflg
# iusrvc = 0 - user vector is not filled

coeff10 = 333.3556d3 # shear modulus

END

SOLVE, sequence_number = 1
direct_solver
# iterative_method = cg, preconditioning = ilu
END

### Create solution type vector for elm200 ###
CREATE VECTOR, sequence_number = 1
   type = solution_vector
   value = 0d0
END

### Create vector of special structure for elm200 ###
CREATE VECTOR, sequence_number = 2
   type = vector of special structure 4
   value = 0d0
END

# Create coordinate vector!!!!!!!!!
create vector, sequence_number = 5, problem = 1
   type = vector of special structure 3
degfd1, func = 1
   degfd2, func = 2
   # old_vector = 3
end

### Calculate derivative : stress ###
DERIVATIVES, sequence_number=1
   icheld = 6 # stress vector
element_groups = (2)
   seq_input_vector = 2 # calculate derivative from vector V2 = x_n
   numvec = 2 # calculate derivative using also vector V(2+1) = uvec
END

### Calculate derivative : strain ###
DERIVATIVES, sequence_number=2
   icheld = 8 # strain vector
element_groups = (2)
   seq_input_vector = 2 # calculate derivative from vector V2 = x_n
   numvec = 2 # calculate derivative using also vector V(2+1) = uvec

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OUTPUT

    write 6 solutions

END

END_OF_SEPRAN_INPUT
Appendix F

Results 3D simulations

F.1 3D modelling 1
In the Hemodyn project, there are four patient specific data sets available. The coordinates of 32 points of lumen and thrombus can be derived from the scans with help of Easyscil. We assume the wall to have a thickness of 2 mm.
A 3D simulation which is made by Berent Wolters is used to determine the maximal principal stress, the von Mises stress and the coordinates on which these stresses occur. The three slices in which the highest stress occur are written in the tables below.
The upper and lower slice are fixed in all directions. When high stress values are located at these slices we do not take them in account, because these boundary conditions are an assumption. This assumption should not influence the highest stresses and positions.

<table>
<thead>
<tr>
<th>Patient 1</th>
<th>Von Mises stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>(-12.9, -167.0, 1612.0)</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(-14.9, -168.9, 1608.8)</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(-13.5, -167.7, 1615.3)</td>
<td>20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Patient 2</th>
<th>Von Mises stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>(-12.9, -167.0, 1612.0)</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(-14.9, -168.9, 1608.8)</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(-13.5, -167.7, 1615.3)</td>
<td>20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Patient 1</th>
<th>Principal stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>(-12.9, -167.0, 1612.0)</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(-14.9, -168.9, 1608.8)</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(-13.5, -167.7, 1615.3)</td>
<td>20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Patient 2</th>
<th>Principal stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>(0.5137, 96.0574, -107.7300)</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(13.7232, 95.1830, -132.7720)</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(0.2631, 96.2304, -110.9920)</td>
<td>19</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Patient 1</th>
<th>Principal stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>(0.2631, 96.2304, -110.9920)</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(14.6509, 95.8498, -132.6420)</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(0.5137, 96.0574, -107.7300)</td>
<td>18</td>
</tr>
</tbody>
</table>
### Patient 3

<table>
<thead>
<tr>
<th>Patient 3</th>
<th>Von Mises stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>76</td>
<td>(21.7635, 99.1162, -118.0570)</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>75</td>
<td>(22.8879, 98.6505, -115.4770)</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>74</td>
<td>(16.9502, 80.9581, -122.2430)</td>
<td>18</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Principal stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>83</td>
<td>(16.9502, 80.9581, -122.2430)</td>
</tr>
<tr>
<td>2</td>
<td>82</td>
<td>(21.7635, 99.1162, -118.0570)</td>
</tr>
<tr>
<td>3</td>
<td>81</td>
<td>(23.3437, 80.0602, -82.4137)</td>
</tr>
</tbody>
</table>

### Patient 4

<table>
<thead>
<tr>
<th>Patient 4</th>
<th>Von Mises stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>97</td>
<td>(-10.5255, 103.7700, -123.1690)</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>96</td>
<td>(19.5320, 115.0540, -130.5220)</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>93</td>
<td>(-8.9449, 104.2460, -120.4170)</td>
<td>14</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Principal stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>109</td>
<td>(-10.5255, 103.7700, -123.1690)</td>
</tr>
<tr>
<td>2</td>
<td>105</td>
<td>(19.5320, 115.0540, -130.5220)</td>
</tr>
<tr>
<td>3</td>
<td>101</td>
<td>(17.9854, 114.7660, -133.3290)</td>
</tr>
</tbody>
</table>
The boundary conditions can have some influence on the highest stresses, even when the upper and lower slices are not taken in account. The following simulations are done with Sepran.

The boundary conditions are changed; the upper and lower slice are only fixed in z-direction. One point is fixed in all directions. In this case the simulations are better comparable with the plane strain problem (2D simulations).

<table>
<thead>
<tr>
<th>Patient 1</th>
<th>Von Mises stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>308</td>
<td>(-12.1, -165.7, 1612.2)</td>
<td>21</td>
</tr>
<tr>
<td>2</td>
<td>289</td>
<td>(-13.8, -166.3, 1605.7)</td>
<td>22</td>
</tr>
<tr>
<td>3</td>
<td>267</td>
<td>(-12.6, -166.4, 1615.6)</td>
<td>20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Principal stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>345</td>
<td>(-12.1, -165.7, 1612.2)</td>
</tr>
<tr>
<td>2</td>
<td>325</td>
<td>(-13.8, -166.3, 1605.7)</td>
</tr>
<tr>
<td>3</td>
<td>307</td>
<td>(-12.6, -166.4, 1615.6)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Patient 2</th>
<th>Von Mises stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>115</td>
<td>(12.5417, 88.8803, -99.7036)</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>113</td>
<td>(17.8333, 92.1823, -89.7433)</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>112</td>
<td>(-0.0415, 95.9313, -105.6420)</td>
<td>17</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Principal stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>130.1430</td>
<td>(17.8333, 92.1823, -89.7433)</td>
</tr>
<tr>
<td>2</td>
<td>127.3173</td>
<td>(12.5417, 88.8803, -99.7036)</td>
</tr>
<tr>
<td>3</td>
<td>123.3753</td>
<td>(-0.2660, 95.9945, -109.0370)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Patient 3</th>
<th>Von Mises stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>80</td>
<td>(43.1646, 107.8090, -88.6329)</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>79</td>
<td>(59.2653, 91.4496, -76.7465)</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>78</td>
<td>(42.5265, 107.0810, -90.8834)</td>
<td>7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Principal stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>93</td>
<td>(28.8010, 106.9640, -76.4574)</td>
</tr>
<tr>
<td>2</td>
<td>91</td>
<td>(43.1646, 107.8090, -88.6329)</td>
</tr>
<tr>
<td>3</td>
<td>87</td>
<td>(42.5265, 107.0810, -90.8834)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Patient 4</th>
<th>Von Mises stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>112</td>
<td>(16.5018, 112.8450, -136.1960)</td>
<td>17</td>
</tr>
<tr>
<td>2</td>
<td>106</td>
<td>(14.3429, 113.3300, -138.9760)</td>
<td>18</td>
</tr>
<tr>
<td>3</td>
<td>102</td>
<td>(20.0123, 113.1510, -130.4590)</td>
<td>16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Principal stress</th>
<th>(x,y,z)-coordinates</th>
<th>Slice nr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>125</td>
<td>(16.5018, 112.8450, -136.1960)</td>
</tr>
<tr>
<td>2</td>
<td>121</td>
<td>(14.3429, 113.3300, -138.9760)</td>
</tr>
<tr>
<td>3</td>
<td>113</td>
<td>(20.0123, 113.1510, -130.4590)</td>
</tr>
</tbody>
</table>
Appendix G

Results 2D simulations

G.1 2D simulations

Using Sepran we calculated the stresses in 2D assuming plane strain. In this section the modelling is done without thrombus. The values of the three maximal principal stresses and the three maximal von Mises stresses are plotted in the following tabulars.

**Patient 1**

<table>
<thead>
<tr>
<th>slice 20</th>
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<tbody>
<tr>
<td><strong>Von Mises stress</strong></td>
<td><strong>(x,y)-coordinates</strong></td>
</tr>
<tr>
<td>1</td>
<td>239</td>
</tr>
<tr>
<td>2</td>
<td>224</td>
</tr>
<tr>
<td>3</td>
<td>222</td>
</tr>
<tr>
<td><strong>Principal stress</strong></td>
<td><strong>(x,y)-coordinates</strong></td>
</tr>
<tr>
<td>1</td>
<td>253</td>
</tr>
<tr>
<td>2</td>
<td>222</td>
</tr>
<tr>
<td>3</td>
<td>220</td>
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<table>
<thead>
<tr>
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<tbody>
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<td><strong>Von Mises stress</strong></td>
<td><strong>(x,y)-coordinates</strong></td>
</tr>
<tr>
<td>1</td>
<td>235</td>
</tr>
<tr>
<td>2</td>
<td>225</td>
</tr>
<tr>
<td>3</td>
<td>224</td>
</tr>
<tr>
<td><strong>Principal stress</strong></td>
<td><strong>(x,y)-coordinates</strong></td>
</tr>
<tr>
<td>1</td>
<td>244</td>
</tr>
<tr>
<td>2</td>
<td>229</td>
</tr>
<tr>
<td>3</td>
<td>225</td>
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<table>
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</thead>
<tbody>
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<td><strong>Von Mises stress</strong></td>
<td><strong>(x,y)-coordinates</strong></td>
</tr>
<tr>
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<td>270</td>
</tr>
<tr>
<td>2</td>
<td>238</td>
</tr>
<tr>
<td>3</td>
<td>235</td>
</tr>
<tr>
<td><strong>Principal stress</strong></td>
<td><strong>(x,y)-coordinates</strong></td>
</tr>
<tr>
<td>1</td>
<td>277</td>
</tr>
<tr>
<td>2</td>
<td>243</td>
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<tr>
<td>3</td>
<td>235</td>
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</table>
## Patient 2

### slice 18

<table>
<thead>
<tr>
<th></th>
<th>Von Mises stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>135</td>
<td>(6.5, 118.5)</td>
</tr>
<tr>
<td>2</td>
<td>135</td>
<td>(7.0, 118.8)</td>
</tr>
<tr>
<td>3</td>
<td>127</td>
<td>(21.4, 92.3)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Principal stress</th>
<th>(x,y)-coordinates</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>134</td>
<td>(7.0, 118.8)</td>
</tr>
<tr>
<td>2</td>
<td>131</td>
<td>(6.5, 118.5)</td>
</tr>
<tr>
<td>3</td>
<td>128</td>
<td>(21.4, 92.3)</td>
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### slice 19

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<th>Von Mises stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>131</td>
<td>(7.9, 117.9)</td>
</tr>
<tr>
<td>2</td>
<td>130</td>
<td>(16.9, 89.7)</td>
</tr>
<tr>
<td>3</td>
<td>127</td>
<td>(17.4, 89.8)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
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<th>Principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>131</td>
<td>(7.9, 117.9)</td>
</tr>
<tr>
<td>2</td>
<td>127</td>
<td>(19.3, 91.0)</td>
</tr>
<tr>
<td>3</td>
<td>127</td>
<td>(17.4, 89.8)</td>
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### slice 26

<table>
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<th>(x,y)-coordinates</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>107</td>
<td>(16.8, 95.6)</td>
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<tr>
<td>2</td>
<td>103</td>
<td>(16.4, 95.3)</td>
</tr>
<tr>
<td>3</td>
<td>97</td>
<td>(15.6, 94.6)</td>
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<th>Principal stress</th>
<th>(x,y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>108</td>
<td>(16.8, 95.6)</td>
</tr>
<tr>
<td>2</td>
<td>99</td>
<td>(16.4, 95.3)</td>
</tr>
<tr>
<td>3</td>
<td>94</td>
<td>(15.6, 94.6)</td>
</tr>
</tbody>
</table>
### Patient 3

#### slice 4

<table>
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<tr>
<th>Von Mises stress</th>
<th>(x, y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 123</td>
<td>(47.9, 109.2)</td>
</tr>
<tr>
<td>2 122</td>
<td>(34.6, 70.8)</td>
</tr>
<tr>
<td>3 116</td>
<td>(50.2, 71.8)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Principal stress</th>
<th>(x, y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 123</td>
<td>(47.9, 109.2)</td>
</tr>
<tr>
<td>2 123</td>
<td>(34.6, 70.8)</td>
</tr>
<tr>
<td>3 117</td>
<td>(50.2, 71.8)</td>
</tr>
</tbody>
</table>

#### slice 15

<table>
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<th>(x, y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 75</td>
<td>(37.3, 104.6)</td>
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<tr>
<td>2 72</td>
<td>(47.2, 76.3)</td>
</tr>
<tr>
<td>3 69</td>
<td>(37.8, 104.5)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Principal stress</th>
<th>(x, y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 73</td>
<td>(37.3, 104.6)</td>
</tr>
<tr>
<td>2 70</td>
<td>(47.2, 76.3)</td>
</tr>
<tr>
<td>3 66</td>
<td>(44.8, 74.2)</td>
</tr>
</tbody>
</table>

#### slice 16

<table>
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<th>Von Mises stress</th>
<th>(x, y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 82</td>
<td>(20.3, 98.7)</td>
</tr>
<tr>
<td>2 79</td>
<td>(44.6, 75.6)</td>
</tr>
<tr>
<td>3 73</td>
<td>(20.0, 98.2)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Principal stress</th>
<th>(x, y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 80</td>
<td>(20.3, 98.7)</td>
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<tr>
<td>2 77</td>
<td>(44.6, 75.6)</td>
</tr>
<tr>
<td>3 69</td>
<td>(20.0, 98.2)</td>
</tr>
</tbody>
</table>

#### slice 18

<table>
<thead>
<tr>
<th>Von Mises stress</th>
<th>(x, y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 112</td>
<td>(14.7, 80.0)</td>
</tr>
<tr>
<td>2 95</td>
<td>(14.4, 80.5)</td>
</tr>
<tr>
<td>3 90</td>
<td>(15.0, 79.6)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Principal stress</th>
<th>(x, y)-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 112</td>
<td>(14.7, 80.0)</td>
</tr>
<tr>
<td>2 90</td>
<td>(14.4, 80.5)</td>
</tr>
<tr>
<td>3 86</td>
<td>(15.0, 79.6)</td>
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Patient 4

**slice 14**

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<td>3</td>
<td>143</td>
<td>(33.3, 115.5)</td>
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**Principal stress (x,y)-coordinates**

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
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<td>170 (-8.1, 102.5)</td>
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<td>2</td>
<td>144 (33.3, 115.5)</td>
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<tr>
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<td>139 (-8.0, 103.1)</td>
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**slice 15**

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<td>(30.5, 92.9)</td>
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<td>(30.6, 93.5)</td>
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**Principal stress (x,y)-coordinates**

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<td>135 (30.5, 92.9)</td>
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**slice 16**

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<td>131</td>
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**Principal stress (x,y)-coordinates**

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<tr>
<td>2</td>
<td>133 (22.4, 79.8)</td>
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**slice 17**

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<td>3</td>
<td>126</td>
<td>(6.0, 113.5)</td>
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**Principal stress (x,y)-coordinates**

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<th></th>
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<td>2</td>
<td>125 (6.0, 113.5)</td>
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<td>123 (8.7, 116.1)</td>
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## G.2 2D simulations with thrombus

The stresses of the 2D plane strain problem can be seen in the following tables. In this section modelling is done with thrombus.

### Patient 1

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<td>51</td>
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<td>3</td>
<td>44</td>
<td>(22.6, -197.3)</td>
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### Patient 2

#### Slice 18

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<td>2 56</td>
<td>(6.4, 118.9)</td>
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<tr>
<td>3 56</td>
<td>(7.4, 119.0)</td>
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<td>(6.9, 118.9)</td>
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<td>3 54</td>
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#### Slice 19

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<td>(15.7, 120.4)</td>
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<td>(15.2, 120.5)</td>
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<td>(15.7, 120.4)</td>
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<td>3 49</td>
<td>(15.2, 120.5)</td>
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#### Slice 26

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<tr>
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<td>(11.8, 121.5)</td>
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<td>2 34</td>
<td>(15.6, 120.1)</td>
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<td>3 34</td>
<td>(11.6, 121.0)</td>
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<th>(x,y)-coordinates</th>
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<td>1 36</td>
<td>(11.8, 121.5)</td>
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<td>2 34</td>
<td>(15.6, 120.1)</td>
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### Patient 4

#### slice 14

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<td>85</td>
<td>(36.9, 91.4)</td>
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**Principal stress**

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<th>(x,y)-coordinates</th>
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<td>1</td>
<td>(35.9, 89.5)</td>
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<td>2</td>
<td>(36.4, 90.5)</td>
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<td>(36.9, 91.4)</td>
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#### slice 15

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<td>(36.4, 90.0)</td>
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<td>(35.8, 87.9)</td>
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**Principal stress**

<table>
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<td>2</td>
<td>(36.1, 88.9)</td>
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#### slice 16

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<td>(26.1, 78.2)</td>
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<td>(26.8, 78.9)</td>
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**Principal stress**

<table>
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<tr>
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<tr>
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<td>(6.7, 126.1)</td>
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<td>(7.8, 126.1)</td>
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#### slice 17

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**Principal stress**

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G.3 2D-analytical modelling of one layer structure

Using the model prescribed in section 3.2.1 we obtain for the maximal principal stress and the von misses stress the following:

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<tr>
<th>Patient 1</th>
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<th>maximal principal stress</th>
<th>von Mises stress</th>
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<tbody>
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The highest maximal principal stress occur in slices 5, 6 and 9. The highest von misses stress occur in the same slices.
### Patient 2

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The highest maximal principal stress occurs in slices 3, 7, and 9. The highest von Mises stress occurs in the same slices.
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The highest maximal principal stress occurs in slices 6, 7, and 8. The highest von Mises stress occurs in the same slices.
### Patient 4

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The highest maximal principal stress occurs in slices 13, 14 and 15. The highest von Mises stress occurs the same slices.
Bibliography


