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Wave propagation in a stratified medium described with nonideal MHD

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Wave propagation in a stratified medium described with nonideal MHD

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Summary

In a stratified medium and under certain conditions, the ideal MHD equations possess two singularities: the cusp singularity and the Alfvén singularity. Near these singularities Ideal MHD fails to describe magnetoacoustic-gravity waves and Alfvén waves.

In the present report we tried to describe these waves by nonideal MHD. It's shown that the cusp singularity and the Alfvén singularity are resolved by including viscosity or resistivity. Thermal conductivity resolves only the cusp singularity. Also the inclusion of various terms in a generalized Ohm's law is shown to resolve the cusp singularity and the Alfvén singularity.

In ideal MHD, magnetoacoustic-gravity waves are described by a second order singular differential equation. By including the nonideal terms we mentioned, we obtain a fourth-order differential equation. In the neighbourhood of the cusp resonance and by using boundary layer theory, this equation can be approximated by the so-called inner equation. It's shown, that the nonideal terms in concern lead to the same inner equation. This leads us to state that the mode that is generated near the cusp resonance is the same for all these terms that we considered. Indeed, by using the WKB method, it's demonstrated that mode conversion takes place near the cusp resonance and that the new mode that is generated is always a modified slow magneto-acoustic wave.
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Chapter 1

Introduction

In the field of wave propagation phenomena, we often consider the medium where the propagation takes place as uniform. This consideration is valid only if the wavelength is much smaller than the length-scale for variations in the medium. The mathematical advantage of considering a uniform medium is that the fundamental differential equations describing the wave in concern reduce to algebraic equations. The wave properties are then described by a dispersion relation. However, if, for example, the medium is structured in the z-direction, the perturbation equations reduce to ordinary differential equations in z. The main agent for creating inhomogeneity on, for example, the sun, is gravity. Gravity causes the pressure to increase inwards towards the solar centre.

The solar atmosphere can be considered as a fully ionized plasma. In such a plasma, there are typically four modes of wave motion, driven by different restoring forces. The magnetic tension can drive so-called Alfven waves [1][2]. The magnetic pressure, the plasma pressure and gravity can act separately and generate compressional Alfven waves [1][2], sound waves and gravity waves [3][4], respectively; but, when acting together, these three forces produce so-called magneto-acoustic-gravity modes [5].

The waves mentioned above can be described by ideal MHD [1][2]. In this model which considers the plasma as one fluid, the behaviour of a plasma is governed by the Maxwell equations together with the momentum equation, the continuity equation, the Ohm’s law and the adiabatic equation of state. Using these equations, one can derive a second order differential equation governing magnetoacoustic-gravity waves in a stratified atmosphere permeated by a horizontal magnetic field [6][7]. It can be shown, that such a differential equation is singular at a certain location called the cusp location. In the same way it can be shown that the differential equation governing Alfven waves is singular at the Alfven resonance[8]. This implies that field quantities become infinite at
the location of these singularities.

Miles (1961) [9] showed that there is a discontinous drop in the wave energy flux across a singular point. Rae and Roberts (1982) [10] suggest that at the location of a singularity resonant absorption takes place and postulate that this may be of importance in the heating of the solar corona [5][8], which has been a subject in solar physics. Before 1940 it was thought, quite naturally, that the temperature decreases as one goes away from the solar surface. But, since then, it has been realised that, after falling from about 6600 K (at the bottom of the photosphere) to a minimum value of about 4300 K (at the top of the photosphere), the temperature rises slowly through the lower chromosphere and then dramatically through the transition region to a few million degrees in the corona (figure 1.1). There after, the temperature falls slowly in the outer corona, which is expanding outwards as the solar wind, to a value of $10^5 K$. The reason for the temperature rise above the photosphere has been one of the major problems in solar physics. Many suggestions were made to explain the heating of the upper atmosphere of the sun (the chromosphere and corona) [5]. One of these suggestions is the resonant absorption of waves. This resonant absorption is described by a singular equation. At the singular points the velocity can possess a logarithmic singularity and energy is therefore accumulated ad infinitum. It is obvious that dissipation is impossible within the framework of ideal MHD. A fact which is not surprising in view of the equation’s ignorance of the fluid kinetic properties on which thermal heating must depend [11]. The ideal MHD equations describe a conservative system, and they neither permit dissipation modes nor provide any information on dissipation that might result from a kinetic treatment.

![Fig 1.1. An illustrative model for the variation of the temperature with height in the solar atmosphere (Priest, 1982)
In the neighbourhood of singularities, linear ideal MHD is no more valid. So in general the input of new physics becomes necessary in order to resolve the singularities in concern. One of the possibilities to do that is to include some dissipation mechanism. This loss mechanism will dissipate the energy that is fed into the singularity. Consequently, the field quantity can’t grow indefinitely. At least a part of the energy that in the ideal case would be accumulated at the singularity is now used to create new modes. This process is called linear mode conversion. In an inhomogenous plasma, linear mode conversion is always involved to some extent in resolving every plasma resonance (Swanson [12]). In this case we deal with a differential equation that is higher order than the original singular differential equation implying that new solutions are allowed (new modes). It’s obvious that, even without taking into account any dissipation mechanism, a field quantity, could never grow indefinitely. Nonlinear effects will prevent that.

Our mean objective in this report, is to describe the behaviour of magnetoacoustic-gravity waves and Alfvén waves that in the ideal case are governed by a singular differential equation. In order to resolve the cusp singularity, we include different loss mechanisms: resistivity, viscosity and thermal conductance. We will show that these loss mechanisms resolve the cusp singularity by raising the order of the differential equation by two. We also consider the effect of including the Hall term and the $\frac{\mathbf{u} \cdot \mathbf{B}}{v_T}$ term in Ohm’s law on the cusp singularity. It will be shown that these terms also resolve the cusp resonance by raising the order of the singular second order differential equation by two.

The fourth order differential equations that we obtain are not easy to handle. Instead of trying to solve them exactly, we use singular perturbation techniques, or, more specifically, boundary layer theory [13][14][15], to analyze the effects of including the nonideal terms in concern. One important condition for the application of this technique is the assumption that these terms are only important in the neighbourhood of the cusp singularity. Far enough from this singularity, the ideal MHD solution is an excellent approximation to the actual solution. In our case boundary layer theory means that we have to distinguish between so-called outer region and inner region. In the outer region, the ideal MHD solution is an excellent approximation, while in the inner region the fourth order differential equation is approximated by so-called inner equation. We will show that all the nonideal terms we considered lead to the same inner equation. This result leads us to conclude that the new mode that is generated in the neighbourhood of the cusp resonance is the same, whatever the included nonideal term is. Indeed, using the WKB method we will show that mode conversion takes place near the cusp resonance and that an incoming magnetoacoustic-gravity wave is always converted into a slow magneto-acoustic wave in the limit of a large wave number normal to the magnetic field.
As we mentioned before, we also pay attention to Alfven waves. Due to the chosen geometry of our propagation problem, Alfven waves are described by a zeroth order "differential equation". By including the nonideal terms under consideration (except thermal conductivity), we obtain a second order differential equation. It will be shown, that the solutions of the obtained second order differential equation can be regarded as Alfven waves which are modified by the nonideal term in concern.

This report is organized as follows. Chapter 2 deals with some basic plasma characteristics. In section 2.3 the fluid description of a plasma is discussed. In Chapter 3 we discuss the waves that can exist in a uniform magnetized plasma. Chapter 4 gives a short description of wave motions in a unmagnetized stratified atmosphere. In chapter 5, we use ideal MHD to describe magnetoacoustic-gravity waves. We will show, that these waves are described by a second order singular differential equation. In chapter 6, we include resistivity. In chapter 7 we take into account the viscosity and in chapter 8 we investigate the effect of thermal conductivity. Chapter 9 is concerned with the effects of the Hall term and the $\frac{m_{\text{ion}}}{e^2} \frac{\partial J}{\partial t}$ term in Ohm's law on both the Alfven and the cusp singularity. The conclusions are given in chapter 10.
Chapter 2

Some plasma characteristics

Much of the region from the surface of the sun to the planetary inonospheres is highly if not fully ionized. Ionized gases which are electrically neutral form a plasma. In a plasma, the potential energy of a typical particle due to its nearest neighbor is much smaller than its kinetic energy.

2.1 Debye shielding

In a plasma we have many charged particles flying around at high speeds. Consider a special test particle of charge \( q_T > 0 \) and infinite mass, located at the origin of a three-dimensional coordinate system containing an infinite, uniform plasma. The test charge repels all other ions, and attracts all electrons. Thus, around the test charge the electron density \( n_e \) increases and the ion density decreases. The test charge gathers a shielding cloud that tends to cancel its own charge.

To describe this phenomenon in a quantitative way, consider the Poisson's equation relating the electric potential \( \phi \) to the charge density \( \rho \) due to electrons, ions, and test charge,

\[
\nabla^2 \phi = -\rho = e(n_e - n_i) - q_T \delta(r)
\]

(2.1)

After introducing the test charge, the electrons with temperature \( T_e \) will come to thermal equilibrium with themselves, and the ions with temperature \( T_i \) will do the same. Equilibrium statistical mechanics predicts then

\[
n_e = n_0 \exp\left(\frac{e\phi}{T_e}\right)
\]

(2.2)
and
\[ n_i = n_0 \exp\left(\frac{-e \phi}{T_i}\right) \]  \hspace{1cm} (2.3)

The Boltzmann's constant is absorbed into the temperatures \( T_e \) and \( T_i \). Assuming that \( \frac{e \phi}{T_e} \ll 1 \) and \( \frac{e \phi}{T_i} \ll 1 \), we expand (2.2) and (2.3) and write (2.1) away from \( r = 0 \) as
\[ \nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d \phi}{dr} \right) = n_0 e^2 \phi \left( \frac{1}{T_e} + \frac{1}{T_i} \right) \]  \hspace{1cm} (2.4)

Defining the electron and the ion Debye lengths as
\[ \lambda_{e,i} \equiv \left( \frac{T_{e,i}}{n_0 e^2} \right)^{1/2} \]  \hspace{1cm} (2.5)
equation (2.4) becomes
\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d \phi}{dr} \right) = (\lambda_D)^{-2} \phi \]  \hspace{1cm} (2.6)

where the total Debye length is defined as
\[ \lambda_D^{-2} = \lambda_e^{-2} + \lambda_i^{-2} \]  \hspace{1cm} (2.7)
The solution of (2.6) is given by
\[ \phi = \frac{qT}{r} \exp\left(\frac{-r}{\lambda_D}\right) \]  \hspace{1cm} (2.8)
The potential due to a test charge in a plasma falls off much faster than in vacuum. This phenomenon is known as Debye shielding.

### 2.2 Plasma parameter

In a plasma where each species has density \( n_0 \), the distance between a particle and its nearest neighbor is roughly \( n_0^{-1/3} \). The average potential energy \( \Phi \) of a particle due to its nearest neighbor is, in absolute value,
\[ |\Phi| \sim e^2/r \sim n_0^{-1/3} e^2 \]  \hspace{1cm} (2.9)
The existence of a plasma requires that this potential energy must be much less than the typical particle's kinetic energy
\[ \frac{1}{2} m_s < v^2 > \equiv \frac{3}{2} T_s = \frac{3}{2} m_s v_s^2 \]  \hspace{1cm} (2.10)
where \( m_s \) is the mass of species \( s \), \( < > \) means an average over all particle velocities at a given point in space. \( v_s \) is the thermal speed of species \( s \) defined by
\[ v_s \equiv \left( \frac{T_s}{m_s} \right)^{1/2} \]  \hspace{1cm} (2.11)
The existence of the plasma requires then

\[ n_0^{1/3} e^2 \ll T_\ast \]  
(2.12)

or

\[ n_0^{2/3} \left( \frac{T_\ast}{n_0 e^2} \right) \gg 1 \]  
(2.13)

or

\[ \lambda_s \equiv n_0 \lambda_3^3 \gg 1 \]  
(2.14)

\( \Lambda_s \) is called the plasma parameter of species \( s \). It is just the number of particles of species \( s \) in a box each side of which has length the Debye length.

Note that the Debye length is independent of the magnetic field that is present in a plasma. An other important quantity that is independent of the magnetic field is the plasma frequency. It's given by

\[ \omega_p = \left( \frac{n_s e^2}{\epsilon_0 m_s} \right)^{1/2} \]  
(2.15)

The plasma frequency is the natural frequency of plasma oscillations resulting from charge-density perturbations.

In a uniform magnetic field with no electric field, a charged particle moves in a circle. The frequency of the motion is called the cyclotron frequency and is given by

\[ \Omega_c = \frac{qB}{m} \]  
(2.16)

The radius of the circle is determined by the magnitude of the particle's velocity perpendicular to the magnetic field and the magnitude of the magnetic field and is given by

\[ \rho_c = \frac{v_{\perp}}{\Omega_c} = \frac{m v_{\perp}}{qB} \]  
(2.17)

### 2.3 The fluid description of a plasma

A plasma is a many particle system. To describe its kinetic behaviour, it's necessary to take into account the motions of all the particles. This can be done in an exact way, using for example the Klimontovich equation together with Maxwell's equations. However, we are not interested in the exact solution of all particles in a plasma, but rather in certain average or approximate characteristics. The wave motion we consider in this report can be described by by thinking of the plasma as one fluid. This approach is called magnetohydrodynamics (MHD). Before discussing this approach we first discuss the so-called two fluid model.
2.3.1 The two-fluid model

The first equation of the two fluid model is the continuity equation, which states that the fluid is not being created or destroyed, so that the only way that the fluid density \( n_s(x, t) \) of fluid species \( s \) (electron or ion) can change at a point is by having a net amount of fluid entering or leaving a small spatial volume including that point. The density \( n_s \) is the number of particles of species \( s \) per unit volume. To every element of fluid there corresponds a velocity vector \( v_s(x, t) \) that gives the velocity of the fluid element at the point \( x \) and time \( t \). Mathematically, the continuity equation for fluid species \( s \) is:

\[
\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s v_s) = 0
\]  

(2.18)

The second equation of fluid theory is the force equation, which is simply Newton's second law of motion for a fluid. This can be written for fluid species \( s \) as

\[
n_s m_s \frac{dv_s}{dt} = F_s(x, t)
\]

(2.19)

where

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + v_s \cdot \nabla
\]

(2.20)

On the right side of (2.19) are all forces that act on a fluid element. One such a force is the pressure gradient force. Another force is the gravity, but one also has the Lorentz force that is given by

\[
F_L = q_s n_s (E + v_s \times B)
\]

(2.21)

With these forces (which are per unit volume), equation (2.19) becomes

\[
n_s m_s \frac{dv_s}{dt} + n_s m_s v_s \cdot \nabla v_s = -\nabla P_s + n_s m_s g + q_s n_s E + q_s n_s v_s \times B
\]

(2.22)

The fields \( E(x, t) \) and \( B(x, t) \) are the macroscopic fields. With the given quantities, the total charge density is given by

\[
\rho_c(x, t) = \sum_s q_s n_s(x, t)
\]

(2.23)

While the total current density \( J \) is given by

\[
J(x, t) = \sum_s q_s n_s(x, t) v_s(x, t)
\]

(2.24)

In addition to the the equations above we have the Maxwell equations

\[
\nabla \cdot E = \frac{1}{\varepsilon_0} \rho_c
\]

(2.25)
\[ \nabla \cdot \mathbf{B} = 0 \quad (2.26) \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2.27) \]
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \quad (2.28) \]

The fluid equations provide a complete, but an approximate, description of plasma physics. When the equations above are written for both electrons and ions then we speak about a two-fluid model. When the ion and electron equations are combined then we obtain the one-fluid model, also known as magnetohydrodynamics (MHD).

### 2.3.2 MHD

We are now going to combine the two-fluid equations (ions and electrons) to get one set of equations which will describe the plasma as on single fluid. This single fluid will be characterized by a mass density

\[ \rho_m(x) \equiv m_e n_e(x) + m_i n_i(x) \approx m_i n_i(x) \quad (2.29) \]

a charge density

\[ \rho_c(x) \equiv q_e n_e(x) + q_i n_i(x) = e(n_i - n_e) \quad (2.30) \]

a center of mass fluid flow velocity

\[ \mathbf{v} \equiv \frac{1}{\rho_m} (m_i n_i \mathbf{v}_i + m_e n_e \mathbf{v}_e) \quad (2.31) \]

a current density

\[ \mathbf{J} \equiv q_i n_i \mathbf{v}_i + q_e n_e \mathbf{v}_e \quad (2.32) \]

and a total pressure

\[ P \equiv P_e + P_i \quad (2.33) \]

We wish to derive four equations relating these quantities: a mass conservation equation, a charge conservation equation, a momentum equation, and a generalized Ohm's law.

By multiplying the ion continuity equation by \( m_i \), the electron continuity equation by \( m_e \) and adding, we obtain

\[ \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{v}) = 0 \quad (2.34) \]

which is the mass conservation. The charge continuity follows also from the both continuity equations

\[ \frac{\partial \rho_c}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad (2.35) \]
Consider next the force equation. Regarding \( \mathbf{v} \) and \( \frac{\partial n_s}{\partial t} \) as small quantities, neglecting the products of small quantities, we add (2.22) for electrons and ions to obtain:

\[
\frac{\partial \mathbf{v}}{\partial t} = -\nabla P + \rho_m \mathbf{g} + \rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B} \quad (2.36)
\]

which is the one fluid momentum equation.

Finally it’s desirable to derive an equation for the time derivative of the current, called a generalized Ohm’s law. Multiplying the force equation (2.22) by \( \frac{n_e}{m_e} \), adding the ion version to the electron version, neglecting quadratic terms in the small quantities \( \frac{n_s}{m_s} \) and \( v_s \) we find:

\[
\frac{\partial \mathbf{J}}{\partial t} = -\frac{e}{m_i} \nabla P_i + \frac{e}{m_e} \nabla P_e + (e^2 \frac{n_e}{m_e} + e^2 \frac{n_i}{m_i}) \mathbf{E} + e^2 \frac{n_e}{m_e} \mathbf{v}_e \times \mathbf{B} + \frac{e^2 n_e}{m_e} \mathbf{v}_i \times \mathbf{B} \quad (2.37)
\]

We notice that

\[
\frac{n_e e^2}{m_e} \mathbf{v}_e = -\frac{e}{m_e} \mathbf{J} + \frac{e^2}{m_e m_i} \rho_m \mathbf{v} \quad (2.38)
\]

We use the fact that \( m_e \ll m_i \) to make some simplifications. Furthermore we assume that \( P_e \approx P_i \approx \frac{1}{2} P \) and \( n_i \approx n_e \). We then find

\[
\frac{\partial \mathbf{J}}{\partial t} = -\frac{e}{2m_e} \nabla P + \frac{e^2}{m_e m_i} \rho_m (\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \frac{e}{m_e} \mathbf{J} \times \mathbf{B} \quad (2.39)
\]

For very low frequencies, one can ignore the \( \frac{\partial \mathbf{J}}{\partial t} \) term in the generalized Ohm’s law, whereas for low temperatures the \( \nabla P \) term can be ignored. Furthermore, when the current is small we can neglect the \( \mathbf{J} \times \mathbf{B} \) term (known as the Hall term) compared to the \( \mathbf{v} \times \mathbf{B} \) term. Under all these assumptions, Ohm’s law becomes:

\[
\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \quad (2.40)
\]

In our derivation of the equations above, we didn’t take collisions into account. This implies that the conductivity is assumed to be infinite. In a resistive plasma the Ohm’s law takes the the form [1]

\[
\mathbf{J} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (2.41)
\]

where \( \sigma \) denotes the conductivity of the medium. Under the ideal conditions: infinite conductivity, low frequency condition, no charge imbalances are allowed and we have \( \rho_e = 0 \). We assume furthermore that \( \mathbf{v} \) is a small quantity. The basic equations then become

\[
\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{v}) = 0 \quad (2.42)
\]
\[ \rho_m \frac{\partial v}{\partial t} = -\nabla P + mg + J \times B \quad (2.43) \]

\[ \nabla \times (v \times B) = \frac{\partial B}{\partial t} \quad (2.44) \]

\[ \nabla \times B = \mu_0 J \quad (2.45) \]

The set of equations above consists of 11 unknown quantities, while there are only ten equations. To complete this set of equations, we assume that we deal with adiabatic motions, and that the plasma behaves like an ideal gas. This means that the mass density and the pressure obey the following equation

\[ \frac{d(P\rho_m^{-\gamma})}{dt} = 0 \quad (2.46) \]

\( \gamma \) is the ratio of specific heats.

The last five equations are the MHD-equations. Using these equations we will demonstrate that there are three types of wave motions in a uniform magnetized plasma.
Chapter 3

Wave motions in a uniform plasma

In a uniform magnetized plasma, there are typically three modes of wave motion, driven by different restoring forces. The magnetic tension can drive so-called Alfvén waves. The magnetic pressure together with the plasma pressure produce so-called magneto-acoustic waves.

3.1 Fundamental equations

We consider a uniform plasma that is embedded in a uniform magnetic field. We ignore the gravitational field. The ideal MHD equations are then given by

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{3.1}
\]

\[
\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{J} \times \mathbf{B} \tag{3.2}
\]

\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{3.3}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{v} \times \mathbf{B} \tag{3.4}
\]

\[
\nabla \cdot \mathbf{B} = 0 \tag{3.5}
\]

\[
\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0 \tag{3.6}
\]

The wave motions that result from the MHD model are a direct consequence of the nature of the Lorentz force. One point to notice is that it is directed across the magnetic field, so that any motion or density variation along field lines must be produced by other forces, such as gravity or pressure gradient.
An other point to notice, is that the Lorentz force may be decomposed into a magnetic pressure force and a magnetic tension force.

\[ \mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} \]

\[ = \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla (B^2/(2\mu_0)) \quad (3.7) \]

The first term on the right-hand side of Equation (3.7) is non-zero if \( \mathbf{B} \) varies along the direction of \( \mathbf{B} \); it represents the effect of a tension parallel to \( \mathbf{B} \) of magnitude \( B^2/\mu \) per unit area, which has a resultant effect when the field is curved. The second term in (3.7) represents a scalar pressure force of magnitude \( B^2/(2\mu_0) \) per unit area, the same in all directions. Its component parallel to the magnitude fields cancels with the corresponding tension component, as it must, since the Lorentz force is normal to \( \mathbf{B} \). The Lorentz force therefore has two effects. It acts both to shorten magnetic field lines through the tension force and also to compress the plasma through the pressure term. In the next section we will show, that the tension component give rise to the occurrence of Alfvén waves, while the magnetic pressure gradient term together with the kinetic pressure gradient give rise to the occurrence of slow and fast magneto-acoustic waves.

### 3.2 Alfvén waves and magnetoacoustic waves

One of the effects of the tension \( T \) in an elastic string (of mass density \( \rho_0 \) per unit length) is to permit transverse waves to propagate along the string with speed \( (\frac{E}{\rho})^{1/2} \). So by analogy, it is reasonable to expect the magnetic tension to produce transverse waves that propagate along the magnetic field \( \mathbf{B}_0 \) with speed \( (B_0^2/\rho)^{1/2} \). This is known as the Alfvén speed. On the above intuitive grounds, a purely magnetic wave is expected to exist, driven by the Lorentz force along the magnetic field. The mathematical analysis below supports this.

For more physical insight, we write the MHD equations in the following form

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (3.8) \]

\[ \frac{d\mathbf{v}}{dt} = -\nabla (\rho + \frac{B^2}{2\mu_0}) + \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B}, \quad (3.9) \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{v} \times \mathbf{B}, \quad (3.10) \]

\[ \frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0, \quad (3.11) \]

Starting with these equations we look for linear wave solutions, i.e, wave solutions with small amplitudes. The various quantities in the equations above are
assumed to be a superposition of (i) an state of rest, with a constant magnetic field $B_0$, a constant density $\rho_0$ and a constant pressure $p_0$; and (ii) an unsteady and nonuniform perturbation of mass density $\rho_1$, velocity $\mathbf{v}_1$, pressure $p_1$, and magnetic field $\mathbf{B}_1$. Thus, every quantity can be written as
\[
f(x, t) = f_1 \exp(-i\omega + ik \cdot x) + f_0 \tag{3.12}
\]
with
\[
f_1 \ll f_0 \tag{3.13}
\]
The smallness of $f_1$ allows us to neglect products of perturbations. Substituting (3.12) into the MHD equations above gives the following system of linear algebraic equations
\[
\omega \rho_0 \mathbf{v}_1 = k(p_1 + \frac{B_1 \cdot B_0}{\mu_0}) - \frac{1}{\mu_0} k_{||} \mathbf{B}_1 \cdot B_0, \tag{3.14}
\]
\[
\omega \mathbf{B}_1 = B_0 k \cdot \mathbf{v} - k_{||} B_0 \mathbf{v}_1, \tag{3.15}
\]
\[
\omega p_1 = \gamma p_0 k \cdot \mathbf{v}_1 \tag{3.16}
\]
Note that the continuity equation is decoupled from the other equations. The index $||$ denotes the vector component that is along the equilibrium magnetic field. We calculate the perturbation of the total pressure (kinetic and magnetic pressure). From (3.14) and (3.15) it follows
\[
(p_1 + \frac{B_1 \cdot B_0}{\mu_0}) = \rho_0 \frac{1}{\omega} [(c_s^2 + c_A^2)k \cdot \mathbf{v}_1 - c_A^2 k_{||} \mathbf{v}_{1||}] \tag{3.17}
\]
where
\[
c_s^2 = (\frac{\gamma p_0}{\rho_0})^{1/2}, \tag{3.18}
\]
is the sound velocity and
\[
c_A^2 = (\frac{B_0^2}{\mu_0 \rho_0})^{1/2} \tag{3.19}
\]
is the Alfvén velocity. According to (3.17) the change of the total pressure is determined by the plasma compressions $k \cdot \mathbf{v}_1$ and $k_{||} \cdot \mathbf{v}_{1||}$. From (3.14) and (3.17) we find
\[
\omega^2 \mathbf{v}_1 = k[(c_s^2 + c_A^2)k \cdot \mathbf{v}_1 - c_A^2 k_{||} \mathbf{v}_{1||}] + k_{||}^2 c_A^2 \mathbf{v}_1 - c_A^2 k \cdot \mathbf{v}_1 k_{||} \frac{B_0}{B_0} \tag{3.20}
\]
The projection of $\mathbf{v}_1$ along $B_0$ is
\[
\omega^2 \mathbf{v}_{1||} = k || c_s^2 k \cdot \mathbf{v}_1. \tag{3.21}
\]
So, the motion along the magnetic field is also determined by compression. Using (3.17) we eliminate $v_{||}$ from (3.17). The result is
\[
(\omega^2 - k_{||}^2 c_A^2) \mathbf{v}_1 = k(c_s^2 + c_A^2 - c_A^2 c_s^2 \omega^2)k \cdot \mathbf{v}_1 - k_{||} c_A^2 \frac{B_0}{B_0} k \cdot \mathbf{v}_1 \tag{3.22}
\]
The projection of (3.22) on the the direction perpendicular to \( \mathbf{k} \) and \( \mathbf{B}_0 \) is given by

\[
(\omega^2 - k_{\parallel} c_A^2) \mathbf{v}_1 \cdot (\mathbf{k} \times \mathbf{B}_0) = 0,
\]

(3.23)

The component of (3.22) along \( \mathbf{k} \) is

\[
[\omega^2 - k^2(c_A^2 + c_s^2 - c_A^2 c_s^2 k_{\parallel}^2 / \omega^2)] \mathbf{k} \cdot \mathbf{v}_1 = 0
\]

(3.24)

### 3.2.1 Alfven waves

The dispersion relation for Alfven waves follows from (3.23)

\[
\omega^2 = k_{\parallel}^2 c_A^2
\]

(3.25)

It’s clear from (3.23) and (3.24) that for the Alfven waves under consideration we have

\[
\mathbf{v}_1 \cdot (\mathbf{k} \times \mathbf{B}_0) \neq 0,
\]

(3.26)

and

\[
\mathbf{B}_1 \cdot (\mathbf{k} \times \mathbf{B}_0) \neq 0,
\]

(3.27)

while \( \nabla \cdot \mathbf{v}_1 = 0 \) and \( v_{1\parallel} = 0 \). Thus, compression play no role in the mechanism of Alfven waves. The driving force for the Alfven waves is the magnetic tension alone.

### 3.2.2 Magneto-acoustic waves

The dispersion relation for magnetoacoustic waves follows from (3.24)

\[
\omega^4 - \omega^2 k^2(c_A^2 + c_s^2) + k_{\parallel}^2 c_A^2 c_s^2 = 0
\]

(3.28)

Obviously, there are two magneto-acoustic waves. *The fast wave* that is determined by the dispersion relation

\[
\omega^2_+ = \frac{1}{2} k^2(c_A^2 + c_s^2) + \frac{1}{2} k^2(c_A^2 + c_s^2)(1 - 4 k_{\parallel}^2 c_A^2 c_s^2 / k^2 (c_A^2 + c_s^2)^2)^{\frac{1}{2}}
\]

(3.29)

and the *slow wave* which is determined by the dispersion relation

\[
\omega^2_- = \frac{1}{2} k^2(c_A^2 + c_s^2) - \frac{1}{2} k^2(c_A^2 + c_s^2)(1 - 4 k_{\parallel}^2 c_A^2 c_s^2 / k^2 (c_A^2 + c_s^2)^2)^{\frac{1}{2}}
\]

(3.30)

From the derivation above it becomes clear that for slow magneto acoustic waves the following holds

\[
k \cdot \mathbf{v}_1 \neq 0,
\]

(3.31)

\[
v_{1\parallel} \neq 0,
\]

(3.32)
\[ B_{\parallel} \neq 0 \]  

while \( \mathbf{v}_1 \cdot (\mathbf{k} \times \mathbf{B}_0) = 0 \) and \( \mathbf{B}_1 \cdot (\mathbf{k} \times \mathbf{B}_0) = 0 \). Notice that Magneto-acoustic waves are dominated by compression of the plasma and the magnetic field. An important limit is when the Alfvén speed is much higher than the sound speed, which implies that the magnetic pressure dominates the thermal pressure. In this limit the dispersion relation for the slow magnetoacoustic wave becomes

\[ \omega^2 \approx k_{\parallel} \frac{c_s^2 c_A^2}{c_A^2 + c_s^2} \]  

(3.34)
Chapter 4

Waves due to stratification: gravity waves

In the previous chapter we discussed the waves that can occur in a uniform plasma. In this chapter we discuss the wave motions that are a result of stratification due to the presence of a gravitational field. We assume that we have no magnetic field. So, Alfvén and magnetoacoustic waves will not appear. The gravitational field is responsible for making the atmosphere stratified. The equilibrium density is a function of height. A stable stratification requires that density decreases with height. A vertically displaced fluid parcel will find itself in the neighbourhood of fluid parcels with a smaller density. Therefore the displaced parcel experiences a force which tends to restore the original equilibrium. The resulting oscillations are the so-called gravity waves [4].

4.1 Gravity waves

Consider a stratified isothermal atmosphere. The equations of concern are the fluid equations

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]  
\quad (4.1)

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \rho g
\]  
\quad (4.2)

\[
\frac{d(\rho \rho^{-\gamma})}{dt} = 0
\]  
\quad (4.3)

Let the isothermal atmosphere be perturbed according to the following scheme:

\[
\rho(x, t) = \rho_0(z) + \rho_1(x, t)
\]  
\quad (4.4)

\[
p(x, t) = p_0(z) + p_1(x, t)
\]  
\quad (4.5)

\[
\mathbf{v}(x, t) = 0 + \mathbf{v}(x, t)
\]  
\quad (4.6)
The static equilibrium is described by

\[ \frac{dp_0}{dz} + \rho g = 0 \] (4.7)

For an ideal gas, the pressure is related to the number density \( n \) and temperature \( T \) through

\[ p = nkT \] (4.8)

Equation (4.7) becomes then

\[ \frac{dp_0}{p} = -\frac{dz}{L} \] (4.9)

where \( L \) is called the scale height and is given by

\[ L = \frac{kT}{mg} \] (4.10)

In (4.10), \( m \) is the mean molecular mass. Integrating (4.9) from the reference height \( z = 0 \) at which \( p_0 = p_{00} \) to an arbitrary height \( z \), we obtain

\[ p_0 = p_{00} \exp\left(-\frac{z}{L}\right) \] (4.11)

The distribution of the density follows from the ideal gas law and is given by

\[ \rho_0 = \rho_{00} \exp\left(-\frac{z}{L}\right) \] (4.12)

with

\[ \rho_{00} = \frac{m}{kT}p_{00} \] (4.13)

Inserting (4.4), (4.5) and (4.6) into (4.1), (4.2) and (4.3) and linearizing the resulting equations, we obtain the following equations:

\[ \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho_0 = 0 \] (4.14)

\[ \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathbf{v} \cdot \mathbf{g} \] (4.15)

\[ \frac{\partial p_1}{\partial t} + \mathbf{v} \cdot \nabla p_0 = c^2 \left( \frac{\partial \rho_1}{\partial t} + \mathbf{v} \cdot \nabla \rho_0 \right) \] (4.16)

We now assume that the perturbed quantities vary like

\[ f_1(x, z, t) = f(z) \exp \left( i(kx - \omega t) \right) \] (4.17)

In such a case, the \( y \)-component of the equation of motion shows that \( v_y = 0 \). The remaining equations (4.14), (4.15) and (4.16) become

\[ -i\omega \frac{\rho_1}{\rho_0} + ikv_x + \frac{\partial v_z}{\partial z} - \frac{1}{L} v_z = 0 \] (4.18)
\[ i \omega v_x = ik \frac{p_1}{\rho_0} \]  
\[ -i \omega v_x = -\frac{\partial (p_1 / \rho_0)}{\partial z} + \frac{1}{L} \frac{p_1}{\rho_0} - g \frac{p_1}{\rho_0} \]  
\[ -i \omega m k T^{-1} \left( \frac{p_1}{\rho_0} - \frac{1}{L} v_z \right) = \gamma [i \omega \left( \frac{p_1}{\rho_0} - \frac{1}{L} v_z \right) ] \]

These four coupled differential equations have constant coefficients in an isothermal atmosphere. This permits us to seek a solution with \( z \) dependence proportional to \( \exp(ikz) \). In Matrix form, the set of equations can be written as

\[ \mathbf{D} \cdot \mathbf{F} = 0 \]  
where \( \mathbf{F} \) is the vector \((\xi_x, \xi_z, v_x, v_z)\). \( \mathbf{D} \) is given by

\[ \mathbf{D} = \begin{pmatrix} -i \omega & 0 & ik & ikz - \frac{1}{L} \\ 0 & ik & -i \omega & 0 \\ g & ikz - \frac{1}{L} & 0 & -i \omega \\ i \omega c^2 & -i \omega & 0 & (\gamma - 1)g \end{pmatrix} \]

The set of homogenous algebraic equations (4.22) has a unique solution when the determinant of the coefficient matrix vanishes, i.e.,

\[ \det(\mathbf{D}) = 0 \]  

The following algebraic equation is then obtained

\[ \omega^4 - \omega^2 c^2 (k_z^2 + k_x^2) + g^2 (\gamma - 1)k_x^2 - i \omega^2 \gamma g k_z = 0 \]  

This equation gives the dispersion relation. It is complex even for the lossless medium. For a forced oscillation \( \omega \) is real. We take \( k \) to be real. This means that \( k_z \) will has to be complex. Let

\[ k_z = k'_z - ik''_z \]

The real part and the imaginary part of (4.25) can be easily separated to give the following two equations.

\[ \omega^4 - \omega^2 c^2 (k_z^2 + k_x^2 - k''_z^2) + g^2 (\gamma - 1)k_x^2 - \omega^2 \gamma g k''_z = 0 \]  
\[ \omega^2 k'_z (2c^2 k''_z - \gamma g) = 0 \]

We consider the case \( k'_z \neq 0 \). So, there is a phase variation along \( z \). The wave associated with this assumption is called an internal gravity wave. From (4.28), \( k''_z \) must be a constant given by

\[ k''_z = \gamma \frac{g}{2c^2} = \frac{1}{2L} \]
The dependence on $z$ of field quantities has the form
\[ \exp\left(\frac{-z}{2L}\right)\exp(ik'_z z) \]  
(4.30)

Inserting (4.29) into (4.27) gives
\[ \frac{k^2}{(1 - \omega_a^2/\omega^2)(1 - \omega_b^2/\omega^2)} + \frac{k^2}{(1 - \omega_b^2/\omega^2)} = \omega/c \]  
(4.31)

where $\omega_a$ is the acoustic cutoff frequency which is given by
\[ \omega_a = \frac{g}{2c} \]  
(4.32)

$\omega_b$ is the Brunt-Vaisla frequency, or the buoyancy frequency. It's given by
\[ \omega_b = (\gamma - 1) \frac{g}{c} \]  
(4.33)

For simplicity the prime on $k_z$ in (4.31) has been ignored so that $k_z$ in (4.31) is real. The dispersion relation (4.31) has two branches:

(i) **Gravity wave branch.** This is the low frequency branch in which $0 < \omega < \omega_b$.

(ii) **Acoustic branch.** In this high frequency branch for which $\omega > \omega_a$, the internal wave can propagate. In the high frequency limit $\omega \gg \omega_a$, we just obtain sound waves.

When $\omega_b < \omega < \omega_a$, the relation (4.31) is contradicted. Therefore, the internal waves cannot exist in this frequency range.

The regions of propagation of the gravity branch and the acoustic branch are distinct in the $\omega k_z$ space as shown by the shaded regions in figure 4.1

![Fig 4.1. Regions of propagation of the gravity branch and the acoustic branch in an isothermal atmosphere](image-url)
Chapter 5

The description of magnetoacoustic-gravity waves with ideal MHD

In this chapter we will consider a stratified compressible atmosphere embedded in a horizontal magnetic field. We assume that this atmosphere is isothermal and perfectly conducting. The effect of a magnetic field on acoustic-gravity waves is to complicate the situation by introducing an extra restoring force and an extra preferred direction in addition to that of gravity.

5.1 Fundamental equations

For the mathematical description we use the ideal MHD equations

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]  \hspace{1cm} (5.1)

\[ \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \rho \mathbf{g} + \mathbf{J} \times \mathbf{B} \]  \hspace{1cm} (5.2)

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \]  \hspace{1cm} (5.3)

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \]  \hspace{1cm} (5.4)

\[ \frac{d(p \rho^{-\gamma})}{dt} = 0 \]  \hspace{1cm} (5.5)

The various quantities above are assumed to be a superposition of an equilibrium state given by index 0, and a harmonic perturbation given by index 1:

\[ f = f_1 + f_0 \]  \hspace{1cm} (5.6)
In order to justify the neglect of the product of two perturbations, it's assumed that their amplitudes are small compared to the equilibrium quantities:

\[ f_1 \ll f_0 \]

We choose the \( z \)-axis along the gravitational acceleration so that

\[ g = -g_\text{ez} \tag{5.7} \]

The equilibrium magnetic field is chosen as

\[ B_0 = B_0 \mathbf{e}_x \tag{5.8} \]

with \( B_0 \) height-independent. We assume that the atmosphere is calm, i.e., \( \mathbf{v}_0 = 0 \). With these conditions, the equilibrium state is given by

\[ -\nabla p_0 + \rho_0 g = 0 \tag{5.9} \]

With the definition of the isothermal sound speed

\[ c^2 = \frac{\gamma p_0}{\rho_0} \tag{5.10} \]

we find

\[ \frac{d\rho_0}{dz} + \frac{1}{L} \rho_0 = 0 \tag{5.11} \]

where

\[ L = c^2 / (\gamma g) \tag{5.12} \]

equation (5.11) yields

\[ \rho_0(z) = \rho_{00} \exp(-z/L) \tag{5.13} \]

Using the definition of the isothermal sound speed we get for the equilibrium pressure \( p_0 \)

\[ p_0(z) = \frac{(c^2/\gamma)}{\rho_{00}} \exp(-z/L) \tag{5.14} \]

With the procedure of linearization and after some algebraic manipulations, equations (5.1) to (5.5) can be written in one closed form

\[
\frac{\partial^2 \mathbf{v}_1}{\partial t^2} = c^2 \nabla \nabla \cdot \mathbf{v}_1 + \nabla (\mathbf{g} \cdot \mathbf{v}_1) + (\gamma - 1) \mathbf{g} \nabla \cdot \mathbf{v}_1 \\
- \frac{1}{\mu_0 \rho_0} (\nabla (\mathbf{B}_0 \cdot \nabla) \mathbf{v}_1 - \mathbf{B}_0 \nabla \cdot \mathbf{v}_1) \\
+ \frac{1}{\mu_0 \rho_0} (\mathbf{B}_0 \cdot \nabla) (\nabla) \mathbf{v}_1 - \mathbf{B}_0 \nabla \cdot \mathbf{v}_1 \tag{5.15}
\]
The vector equation (5.15) can be replaced by its components. By taking its x-component we obtain

\[ \frac{\partial^2 v_{1x}}{\partial t^2} = c^2 \left( \frac{\partial^2 v_{1x}}{\partial x^2} + \frac{\partial^2 v_{1y}}{\partial x \partial y} + \frac{\partial^2 v_{1z}}{\partial x \partial z} \right) - g \frac{\partial v_{1z}}{\partial x} \]  

(5.16)

By taking the y-projection of (5.15) we find

\[ \frac{\partial^2 v_{1y}}{\partial t^2} = c^2 \left( \frac{\partial^2 v_{1x}}{\partial x \partial y} + \frac{\partial^2 v_{1y}}{\partial y^2} + \frac{\partial^2 v_{1z}}{\partial y \partial z} \right) - g \frac{\partial v_{1z}}{\partial y} + a^2(z) \left( \frac{\partial^2 v_{1x}}{\partial y^2} + \frac{\partial^2 v_{1z}}{\partial y \partial z} + \frac{\partial^2 v_{1y}}{\partial z^2} \right) \]

(5.17)

\( a(z) \) is the Alfven velocity which is determined by

\[ a^2(z) = \frac{B_0^2}{\mu_0 \rho_0(z)} = a_0^2 \exp(z/L) \]

(5.18)

Finally the projection of the given vector equation onto the z-axis gives

\[ \frac{\partial^2 v_{1z}}{\partial t^2} = c^2 \left( \frac{\partial^2 v_{1x}}{\partial x \partial z} + \frac{\partial^2 v_{1y}}{\partial x \partial z} + \frac{\partial^2 v_{1z}}{\partial z^2} \right) - g \frac{\partial v_{1z}}{\partial z} + a^2(z) \left( \frac{\partial^2 v_{1x}}{\partial z^2} + \frac{\partial^2 v_{1z}}{\partial x^2} + \frac{\partial^2 v_{1y}}{\partial x \partial z} \right) - g(\gamma - 1) \left( \frac{\partial v_{1x}}{\partial x} + \frac{\partial v_{1y}}{\partial y} + \frac{\partial v_{1z}}{\partial z} \right) \]

(5.19)

A glance at the three components reveals that \( v_{1y} \) is coupled to \( v_{1x} \) and \( v_{1z} \) due to \( \frac{\partial}{\partial y} \). Speaking in physical terms, the velocity component transverse to the plane of gravity and magnetic field is decoupled, when we assume that there is no propagation in the y-direction.

### 5.2 Alfven mode

The component \( v_y \) is due to the assumption \( \frac{\partial}{\partial y} = 0 \) described by the following second order partial differential equation

\[ \frac{\partial^2 v_{1y}}{\partial t^2} = a^2(z) \frac{\partial^2 v_{1y}}{\partial z^2} \]

(5.20)

This is a wave equation describing Alfven waves, propagating along the magnetic field, with wave speed \( a(z) \). Clearly, ignoring the y-dependence of \( v_1 \) implies, that we have separated the Alfven waves that are incorporated in ideal MHD from gravity waves. Since all equilibrium quantities are independent of \( x \), we can fourier transform in this direction and assume harmonic time dependence.

\[ v_1(x, z, t) = v(z, k, \omega) \exp[i(kz - \omega t)] \]

(5.21)
Thus the derivatives $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial z}$ are replaced by algebraic operations, and only derivatives with regard to altitude remain

$$\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = (-i\omega, ik, \frac{d}{dz})$$

(5.22)

Equation (5.20) then becomes

$$(\omega^2 - k^2 a^2(z)v_y = 0$$

(5.23)

In this report we consider $\omega$ and $k$ as given quantities. From (5.23) it then follows that $v_y$ has to be zero everywhere except at the location where $\omega^2 - k^2 a^2$ vanishes. At this location, $v_y$ is undetermined.

5.3 Magnetoacoustic-gravity waves

After Substituting (5.21) into (5.19) and (5.16) and after eliminating $v_x$, we find the following differential equation for $v_z$

$$\beta_1 \frac{d^2 v_z}{dz^2} + \beta_2 \frac{dv_z}{dz} + \beta_3 v_z = 0$$

(5.24)

The coefficients $\beta_1$, $\beta_2$, and $\beta_3$ are given by

$$\beta_1 = \omega^2(c^2 + a^2) - k^2 a^2 c^2$$

(5.25)

$$\beta_2 = -\frac{\omega^2 c^2}{L}$$

(5.26)

$$\beta_3 = k^2 \omega_b^2 c^2 + (k^2 c^2 - \omega^2)(k^2 a^2 - \omega^2)$$

(5.27)

$\omega_b^2$ is the square of the Brunt-Vaisala frequency, which is determined by

$$\omega_b^2 = \frac{(\gamma - 1) g}{\gamma L}$$

Physically, it's the frequency, at which an air parcel oscillates, when we move it from its equilibrium position. The differential equation (5.24) possesses a regular singularity. The location of this singularity is given by the condition

$$\omega^2 = k^2 V_c^2$$

(5.28)

where $V_c$ is determined by

$$V_c^2 = V_c^2(z) = \frac{c^2 a^2}{c^2 + a^2}$$

(5.29)
$V_c$ is called the **cusp speed**, and the resonance determined by (5.28) is called the **cusp resonance**. From (5.28) and (5.29) it follows that the location of this cusp resonance is given by

$$z_c = L \ln \left( \frac{\omega^2}{a_0^2(k^2 - \omega^2/c^2)} \right)$$  \hspace{1cm} (5.30)

In order to have a real valued $z_c$ we assume that $\omega^2 \ll k^2 c^2$.

The differential equation (5.24) possesses a regular singularity at $z = z_c$. This means that the general solution of this equation will be singular at that point. Before solving this equation exactly, we want to study the behaviour of its solution for $z \to +\infty$ and for $z \to -\infty$. In the case $z \to +\infty$ the coefficients $\beta_1$, $\beta_2$ and $\beta_3$ behave like

$$\beta_1 \to \left( \frac{\omega^2}{c^2} - k^2 \right)$$

$$\beta_2 \to 0,$$  \hspace{1cm} (5.31)

$$\beta_3 \to k^2(k^2 - \omega^2/c^2)$$  \hspace{1cm} (5.32)

Hence for $z \to +\infty$ the differential equation (5.24) looks like

$$\frac{d^2v}{dz^2} - k^2 v = 0$$  \hspace{1cm} (5.34)

The solution of (5.34) is given by

$$M_0 \exp(-kz) + M_1 \exp(kz)$$  \hspace{1cm} (5.35)

For $k \neq 0$, the second term in (5.35) diverges exponentially with altitude, and can be suppressed by setting $M_1 = 0$. In the same way as above, we derive the asymptotic behaviour of the solution of (5.24) for $z \to -\infty$. In this case the coefficients $\beta_1$, $\beta_2$ and $\beta_3$ behave like

$$\beta_1 \to \omega^2/a^2,$$  \hspace{1cm} (5.36)

$$\beta_2 \to -\frac{1}{L} \frac{\omega^2}{a^2},$$  \hspace{1cm} (5.37)

$$\beta_3 \to \frac{1}{a^2} [k^2 \omega^2 - \omega^2(k^2 - \omega^2/c^2)]$$  \hspace{1cm} (5.38)

With these limits, the differential equation (5.24) becomes

$$\frac{d^2v}{dz^2} - \frac{1}{L} \frac{dv}{dz} + \left[ k^2 \left( \frac{\omega^2}{\omega^2} - 1 \right) + \frac{\omega^2}{c^2} \right] v = 0$$  \hspace{1cm} (5.39)

This differential equation has constant coefficients. Its solution is given by

$$N_1 \exp(\lambda_1 z) + N_2 \exp(\lambda_2 z)$$  \hspace{1cm} (5.40)
where $\lambda_1$ and $\lambda_2$ are the two solution of the algebraic equation

$$\lambda^2 - \frac{1}{L}\lambda + \left(k^2\frac{\omega_e^2}{\omega^2} - 1\right) + \frac{\omega^2}{c^2} = 0$$

(5.41)

By solving (5.41) we find

$$\lambda_1 = \frac{1}{2L} + \left(\frac{\omega_e^2}{\omega^2} - 1\right)^{\frac{1}{2}} \sqrt{-k^2 + \frac{1}{c^2}\left(\frac{\omega^2}{\omega_e^2} - \omega^2\right)}$$

(5.42)

$$\lambda_2 = \frac{1}{2L} - \left(\frac{\omega_e^2}{\omega^2} - 1\right)^{\frac{1}{2}} \sqrt{-k^2 + \frac{1}{c^2}\left(\frac{\omega^2}{\omega_e^2} - \omega^2\right)}$$

(5.43)

$\omega_a$ is the cut-off frequency which is determined by

$$\omega_a^2 = \frac{\gamma^2}{4(\gamma - 1)}\omega^2_e = \frac{c^2}{4L^2}$$

(5.44)

In order to have wave solutions, we suppose that $\lambda_1$ and $\lambda_2$ are not real valued. This implies that the expression inside the root has to be negative. This assumption leads us to summarize the previous results as follows:

For $z \to -\infty$ (thus, above the critical level), the wave field $v_z$ is evanescent. For $z \to -\infty$ (thus below the critical level) we find that the wave field $v_z$ consists of upward and downward propagating acoustic-gravity waves (not influenced by the external magnetic field because $a(z)$ vanishes for $z \to -\infty$). Clearly, an upward propagating wave reflects. The question is now, whether this reflection is total. To answer this question it's necessary to calculate the reflection coefficient [19], which is defined as

$$R = \frac{N_2}{N_1}$$

(5.45)

In order to calculate $N_1$ and $N_2$, we have to solve exactly the differential equation (5.24). To do that we use the mathematical theorem known as the Riemann-Papperitz theorem [22][21]. It states, that every differential equation with at most three regular singularities, can be transformed into a hypergeometric differential equation. In our case we have to perform the following transformations: the first is to transform the independent variable $z$ via:

$$\xi = 1 - \xi_0 \exp(-z/L)$$

(5.46)

with

$$\xi_0 = \frac{\omega^2}{a^2_0(k^2 - \omega_e^2/c^2)}$$

(5.47)

Note that by this transformation, the cusp resonance is now located at $\xi = 0$. The second, is a transformation of the dependent variable via:

$$v = \phi \exp(-kz)$$

(5.48)
After applying these transformations we obtain the following differential equation
\[ \xi(\xi-1)\frac{d^2\phi}{d\xi^2} + [\chi - (\alpha + \eta + 1)\xi]\frac{d\phi}{d\xi} - \alpha\beta\phi = 0 \] (5.49)

Indeed, equation (5.49) is the standard form of a hypergeometric differential equation. \(\chi, \alpha\) and \(\eta\) are given by
\[
\begin{align*}
\chi &= 1 \\
\alpha &= \frac{1}{2} + kL + iL, \\
\eta &= \frac{1}{2} + kL - iL,
\end{align*}
\]
with \(l\) determined by
\[ l^2 = \frac{\omega_0^2 - \omega^2}{\omega^2} (k^2 - \frac{\omega_0^2}{c^2} \omega_0^2 - \omega^2) \] (5.53)

The general solution of (5.49) around the point \(\xi = 0\) is given by
\[ \phi = (A_1 + A_2 \ln(\xi))F(\alpha, \eta, 1, \xi) + A_2 \sum_{n=1}^{\infty} C_n \frac{(\alpha)_{n}(\eta)^n}{(n!)^2} \xi^n \] (5.54)

\(A_1\) and \(A_2\) are arbitrary constants of integration. \(C_n\) is given by
\[ C_n = \psi(\alpha + n) + \psi(\eta + n) - \psi(\alpha) - \psi(\eta) - 2\psi(n + 1) + 2\psi(1). \] (5.55)

And by definition the following holds
\[ (\alpha)_n = \alpha(\alpha - 1)(\alpha - 2) \ldots (\alpha - n + 1) \] (5.56)

Note that \(\xi = 1\) is also a regular singular point of the hypergeometric differential equation (5.49). Hence, the given solution is only valid within the interval \(-1 < \xi < 1\). \(F_{\phi}(\xi)\) is the hypergeometric function [21][22] and \(\psi\) is the digamma function [21][22]. To lowest order (5.54) reduces to
\[ \phi = A_1 + A_2 \ln(\xi) \] (5.57)

Indeed, the solution is singular at \(\xi = 0\). As mentioned, the point \(\xi = 1\) is also a regular singular point of equation (5.49). So it's possible to find a solution around it. This solution will be convergent for all \(\xi\) whenever \(|\xi - 1| < 1\), i.e, above the cusp resonance \(z > z_c\). This solution is given by
\[ \phi = D_1(1-\xi)^{-\alpha}F(\alpha, 1-\eta, \alpha-\eta+1; \frac{1}{1-\xi}) + D_2(1-\xi)^{-\eta}F(\eta, 1-\alpha, \eta-\alpha+1; \frac{1}{1-\xi}) \] (5.58)
The point $\xi = \infty$ is also a regular singularity. Thus, it is possible to find a solution of (5.49) around it. It's obvious that this solution will converge for all $\xi$ whenever $|\xi - 1| > 1$, i.e., below the cusp resonance. This solution is given by

$$\phi = E_1 F(\alpha, \eta, \alpha + \eta, 1 - \xi) + E_2(1 - \xi)^{1-\alpha-\eta} \times F(1 - \alpha, 1 - \eta, 2 - \alpha - \eta, 1 - \xi)$$

(5.59)

$A_1, A_2, D_1, D_2, E_1$ and $E_2$ are not arbitrary integration constants, because there has to be a connection between them. For if, there is no connection between them, the differential equation (5.49) would have more than two independent solutions in the region where the convergence area’s overlap. But this is impossible because we deal with a second order differential equation. The connections between $A_1, A_2, D_1, D_2, E_1$, and $E_2$ are determined by the transformation formulas between the hypergeometric functions of the various arguments [21][22].

It follows that

$$A_1 = \frac{\Gamma(\alpha - \eta + 1)}{\Gamma(\alpha)\Gamma(1 - \eta)} \left[2\psi(1) - \psi(\alpha) - \psi(1 - \eta) - \pi i\right] D_1$$

$$+ \frac{\Gamma(\eta - \alpha + 1)}{\Gamma(\eta)\Gamma(1 - \alpha)} \left[2\psi(1) - \psi(\eta) - \psi(1 - \alpha - \pi i)\right] D_2$$

(5.60)

$$A_2 = \frac{-\Gamma(\alpha - \eta + 1)}{\Gamma(\alpha)\Gamma(1 - \eta)} D_1 - \frac{(\eta - \alpha + 1)}{\Gamma(\eta)\Gamma(1 - \alpha)} D_2,$$

(5.61)

$$E_1 = \frac{\Gamma(\alpha - \eta + 1)\Gamma(1 - \alpha - \eta)}{[\Gamma(1 - \eta)]^2} \exp(i\pi \alpha) D_1$$

$$+ \frac{\Gamma(\eta - \alpha + 1)\Gamma(\alpha + \eta - 1)}{[\Gamma(1 - \alpha)]^2} \exp(i\pi \eta) D_2,$$

(5.62)

$$E_2 = \frac{\Gamma(\alpha - \eta + 1)\Gamma(\alpha + \eta - 1)}{[\Gamma(\alpha)]^2} \exp(-i\pi \eta) D_1$$

$$- \frac{-\Gamma(\eta - \alpha + 1)\Gamma(\alpha + \eta - 1)}{[\Gamma(\eta)]^2} \exp(-i\pi \alpha) D_2$$

(5.63)

From (5.58) we find that for $z \to \infty$, that is, for $\xi \to 1$, the asymptotic behaviour of $\phi$ is given by

$$\Phi \sim E_1 + E_2(1 - \xi)^{1-\alpha-\eta},$$

(5.64)

with (5.46) and (5.48) this yields for $\nu_z$

$$\nu_z \sim E_1 \exp(-k\xi) + E_2 \xi^{-2K_1 \xi} \exp(k\xi) \quad z \to \infty$$

(5.65)

Hence, for nonvertical waves ($k \neq 0$) the second term in (5.65) diverges exponentially with altitude, and can be suppressed by setting $E_2 = 0$. Note that
this result is already obtained in (5.35). For \( z \to -\infty \), that is for \( \xi \to -\infty \) we find from (5.59)
\[
\phi \sim D_1 (1 - \xi)^{-\alpha} + D_2 (1 - \xi)^{-\eta}
\]
which means for \( v_z \)
\[
v_z \sim D_1 \xi_0^{-\alpha} \exp\left[\frac{1}{2L} + ilz\right] + D_2 \xi_0^{-\eta} \exp\left[\frac{1}{2L} - ilz\right],
\]
Apparently, confirming the result, we already obtained. Using this result together with (5.45) we see that
\[
\frac{N_2}{N_1} = \frac{D_2}{D_1} \xi_0^{-\eta+\alpha}
\]
By setting \( E_2 = 0 \), and using (5.63) we find
\[
R \equiv \frac{N_2}{N_1} = -\frac{\Gamma(1 + 2ilL)[\Gamma(1/2 + kL - ilL)]^2 \exp(-2\pi ilL)}{\Gamma(1 - 2ilL)[\Gamma(1/2 + kL + ilL)]^2}
\]
or
\[
|R| = \exp(-2\pi ilL),
\]
which yields an absorption coefficient
\[
A = 1 - |R|^2 = 1 - \exp(-4\pi ilL) > 0.
\]

5.4 Concluding Remarks

Clearly, at the cusp resonance absorption takes place, even without including any dissipation mechanism. We can then conclude that an upward propagating acoustic-gravity wave is not totally reflected. This fact, as mentioned in the introduction, leads to discontinuity in the energy flux normal to the critical layer. The physics of what resolves the singularity must be included in order to obtain physically meaningful results. Kamp (1989) [19][20] has analysed the limit of a very small component of the magnetic field parallel to the gravitational acceleration. He showed that by doing so, the singularity is resolved. Furthermore, he showed that through the process of linear mode conversion the wave energy that tunnels to the resonance is converted into another mode that is able to carry off the energy from the resonance.
Chapter 6

Inclusion of resistivity

As we saw in the previous chapter, ideal MHD equations (with horizontal magnetic field) give rise to the occurrence of the cusp resonance. A realistic treatment of this singularity should include for example some dissipation mechanism. In this chapter we investigate the effect of resistivity. We will show, that it resolves the cusp singularity. However, the mathematical price we have to pay for this resolution, is that we now have to deal with a fourth order differential equation. Clearly, new solutions are now allowed. Speaking in physical terms, the inclusion of resistivity gives rise to the occurrence of a new mode.

6.1 Fundamental equations

The starting point is almost the same as in the previous chapter. So let us consider a compressible, isothermal, resistive atmosphere that is stratified due to the presence of a gravitational field. As in the previous chapter, we assume that we deal with an ideal gas. Furthermore we assume that, except resistivity, all other nonidealities (viscosity, thermal conductance, etc) are negligible. The fundamental equations are then given by

\[
\frac{\partial p}{\partial t} + \nabla \cdot (\rho v) = 0 \tag{6.1}
\]

\[
\rho \frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p + \rho g + J \times B, \tag{6.2}
\]

\[
\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \frac{1}{\sigma \mu_0} \nabla^2 B \tag{6.3}
\]

\[
\nabla \times B = \mu_0 J \tag{6.4}
\]

\[
\frac{d(p \rho^{-\gamma})}{dt} = \left( \frac{dp}{dt} + \gamma p \nabla \cdot v \right) = \frac{1}{\sigma} (\gamma - 1) J \cdot J \tag{6.5}
\]
Equation (6.3) states that the magnetic field at a point can be changed by the fluid convection, the first term on the right side of (6.3), or by diffusion due to the second term on the right side of (6.3). When \( v = 0 \), (6.3) becomes

\[
\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}
\]

which is just the standard form of a diffusion equation. For this reason the constant \( 1/ (\sigma \mu_0) \) is called the magnetic diffusivity.

We assume that the various quantities are a superposition of an atmospheric mean state of rest with a constant external magnetic field \( \mathbf{B}_0 \) that is directed along the x-axis and a perturbation. As in the previous chapter we assume also that there is no propagation in the y-direction. Since the equilibrium quantities don't depend on the x-coordinate, we may fourier transform the equations (6.1) to (6.5) (after linearizing them) as follows

\[
f_1(x, z, t) = f(k, w, z) \exp(i(kx - \omega t)).
\]

\( f_1 \) stands for a perturbation of a quantity \( f \).

### 6.2 Alfven mode

After some trivial algebraic manipulations the following differential equation can be derived for \( v_y \)

\[
i \frac{1}{\sigma \mu_0} \frac{\omega}{a^2} \frac{d^2 v_y}{dz^2} - i \frac{2}{L} \frac{\omega}{\sigma \mu_0} \frac{dv_y}{dz} + \left[ k^2 - \frac{\omega^2}{a^2} + O(\epsilon) \right] v_y = 0
\]

It's practical to define a nondimensional parameter \( \epsilon \) as follows:

\[
\epsilon = \frac{1}{\sigma \mu_0} \frac{1}{L^2 \omega}
\]

With this introduction of \( \epsilon \) equation (6.8) becomes

\[
i \epsilon \frac{d^2 v_y}{dz^2} - i \frac{2 \epsilon}{L} \frac{d v_y}{dz} + \frac{1}{L^2} \left( \frac{k^2 a^2 - \omega^2}{a^2} + O(\epsilon) \right) v_y = 0
\]

### 6.2.1 Inner and outer solution

By letting \( \epsilon \to 0 \), we get the ideal case back described by the wave equation (5.20) which is now fourier transformed. We can consider this fourier transformed equation which is given by

\[
\left( k^2 - \frac{\omega^2}{a^2} \right) v_y = 0
\]
as a differential equation of order 0, and which has a regular singularity (the Alfven singularity) in a point $s$ whenever

$$k^2 \alpha^2(s) = \omega^2 \quad (6.12)$$

For given $\omega$ and $k$ (6.11) yields that $v_y$ has to be zero everywhere, except at the resonance, where it may take any value. Again we see, that ideal MHD leads to unphysical results. Looking at the problem in this manner, we can conclude that the inclusion of resistivity resolves the Alfven singularity by raising the order of the differential equation (6.11) by two. We then get the equation (6.10), which is now regular at the Alfven singularity.

We assume that the resistivity plays a role of significance only in a narrow layer around the Alfven resonance. This assumption allows us to use boundary layer theory, which is based on the smallness of the coefficient in front of the highest derivative. In boundary layer theory, we make use of the fact that the solution of for example (6.10) has a different asymptotic expansion in certain distinguished regions. These regions are called the inner region and the outer region. The asymptotics of the solution in these regions are called the inner solution and the outer solution.

The outer region is characterized by the absence of rapid variations in $v_y$. So in this region the derivative, and higher derivatives remain finite, if we let the small parameter $\epsilon$ go to zero. Applying this idea, the outer solution of (6.10) is described by

$$(\omega^2 - k^2 \alpha^2) v_y = 0 \quad (6.13)$$

Thus, the outer solution is trivially given by

$$v_y = 0 \quad (6.14)$$

In order to derive the inner solution, we perform the following transformation

$$\zeta = \zeta_0 \exp\left(\frac{z}{L}\right) - 1 \quad (6.15)$$

with $\zeta_0$ given by

$$\zeta_0 = \frac{k^2 \alpha^2}{\omega^2} \quad (6.16)$$

With this transformation equation (6.10) becomes

$$i\epsilon(\zeta + 1)^2 \frac{d^2 v_y}{d\zeta^2} - i\epsilon(\zeta + 1) \frac{dv_y}{d\zeta} + \zeta v_y = 0 \quad (6.17)$$

In the inner region, that is in the neighbourhood of the Alfven resonance, rapid variations take place if $\epsilon \to 0$. Consequently, the second and the first derivative
are, even to lowest order, not negligible anymore near the resonance. In order to determine the inner solution, a stretching of the variable $\zeta$ is performed according to

$$\zeta = \tau \epsilon^\nu$$

(6.18)

Substitution of this stretching in (6.17) gives for $\epsilon \to 0$, a resulting equation which depends on the value of $\nu$. By applying the method of dominant balance, it's found that the only acceptable limit of $\epsilon \to 0$, is so-called distinguished limit for one value of $\nu$, which is $\nu = 1/3$. The inner differential equation is then given by:

$$\frac{d^2 v_y}{d\tau^2} + \tau v_y = 0$$

(6.19)

Changing the variable $\tau$ according to

$$\tau = \chi \exp(-\frac{i\pi}{6})$$

(6.20)

leads to the equation

$$\frac{d^2 v_y}{d\chi^2} - \chi v_y = 0$$

(6.21)

which is known as the Airy differential equation [4][16][17]. Its solutions are given by the Airy functions:

$$v_y = E_1 \text{Ai}(\chi) + E_2 \text{Bi}(\chi)$$

$$= E_1 \text{Ai}(\exp(\frac{\pi i}{6})\tau) + E_2 \text{Bi}(\exp(\frac{\pi i}{6})\tau)$$

(6.22)

This inner solution has to be matchable to the outer solution. For that, we need its asymptotic behaviour for $\tau \to \pm\infty$. The asymptotics of $\text{Ai}(s)$ and $\text{Bi}(s)$ with $s$ complex are discussed by Bleistein et al [17]. For $\text{Ai}(s)$ the asymptotic expansion is given by:

$$\text{Ai}(s) \sim \frac{s^{-1/4}}{2\sqrt{\pi}} \exp(-\frac{2s^{3/2}}{3}) \quad |s| \to \infty \quad |\arg(s)| < \pi$$

(6.23)

The expansion of $\text{Bi}(s)$ is not bounded for $\tau \to \infty$. Thus, we set $E_2 = 0$. From (6.23) it follows that the asymptotic expansion of the solution of (6.19) is given by

$$v_y \sim E_1 \exp\left(\frac{1}{24}\pi i\right)\tau^{-1/4} \exp\left(-\frac{\sqrt{2}}{3}(1+i)\tau^{3/2}\right) \quad \tau \to \infty$$

(6.24)

$$v_y \sim E_1 \exp\left(-\frac{\pi}{24}i\right)\tau^{-1/4} \exp\left(\frac{\sqrt{2}}{3}(i-1)\tau^{3/2}\right) \quad \tau \to -\infty$$

(6.25)

The expansions (6.24) and (6.25) decrease exponentially for $\tau \to \pm\infty$. This means that the inner solution matches automatically to the outer solution given
by (6.14). The exact solution of equation (6.19) can also be represented by a Laplace integral [18]. We then seek a solution of the form

$$v_y = A \int_C F(s)e^{ist}ds$$

(6.26)

where the contour $C$ and the function $F(s)$ have to be determined. Substituting (6.26) into (6.19) and integrating by parts we get a first order differential equation, which determines $F(s)$. The only bounded solution of (6.26) is given by

$$v_y = A \int_0^\infty e^{-\frac{1}{2}s^3 + is\tau}ds$$

(6.27)

The real part of this solution is

$$Re(v_y) = A \int_0^\infty e^{-\frac{1}{2}s^3} \cos(s\tau)ds = A \int_0^\infty e^{-\frac{1}{2}s^3} \cos(e^{-\frac{1}{2} s} \zeta)ds$$

(6.28)

which is even in $\zeta$. The graph of (6.28) as a function of $\tau$ is shown in figure (6.1). $v_y$ is continuous, and it takes its maximum value at the resonance. We see from figure (6.1) that the region where $v_y \neq 0$ becomes smaller and smaller as we let $\epsilon \to 0$.

![Graph showing $v_y$ as a function of $\tau$ for two different values of $\epsilon$](image)}
6.2.2 WKB approximation

The inner solution obtained by boundary layer theory has only a local validity. Far from the Alfven resonance we can use the WKB method [13][14] to obtain an approximate expression for the solution of (6.10).
Consider the original equation (6.10). We seek a solution in the form of a WKB serie:

\[ v_y = S_0 \exp \frac{1}{\delta} S(z) \]  

(6.29)

\( \delta = \epsilon^\frac{1}{2} \) and \( S(z) \) is given by

\[ S(z) = \int_{z_0}^z \sum_{n=0}^{\infty} \delta^n \phi_n(z') dz' \]  

(6.30)

In our case, we truncate the series given by (6.30) after \( n = 1 \). This means that \( S(z) \) now becomes

\[ S(z) = \int_{z_0}^z (\phi_0 + \delta \phi_1) dz' \]  

(6.31)

Inserting (6.31) into (6.10) and comparing powers of \( \epsilon \) give the following two equations for \( \phi_0 \) and \( \phi_1 \)

\[ \phi_0^2 = -i \frac{1}{L^2 \omega^2} (k^2 a^2 - \omega^2) \]  

(6.32)

\[ \frac{d\phi_0}{dz} + 2\phi_0 (\phi_1 - \frac{1}{L}) = 0 \]  

(6.33)

From (6.33) and (6.32) it then follows that

\[ \phi_1 = \frac{1}{L} \left(1 - \frac{a^2 k^2}{4(k^2 a^2 - \omega^2)}\right) \]  

(6.34)

Note that \( \phi_1 \) is real. The bounded WKB solutions can now be written as

\[ v_y \sim S_0 \exp \left( \int_{z_0}^z \phi_1(z') dz' \right) \times \exp \left( \int_z^\infty -(1 + i) \epsilon^{-\frac{1}{2}} \frac{1}{L \omega} (k^2 a^2 - \omega^2)^{\frac{1}{4}} dz' \right) \quad z > z_a \]  

(6.35)

\[ v_y \sim S_1 \exp \left( \int_{z_0}^z \phi_1(z') dz' \right) \times \exp \left( \int_{z-1}^z (i-1) \epsilon^{-\frac{1}{2}} \frac{1}{L \omega} (k^2 a^2 - \omega^2)^{\frac{1}{4}} dz' \right) \quad z < z_a \]  

(6.36)

where \( z_a \) stands for the location of the Alfven resonance. The WKB solutions (6.35) and (6.36) must be matched to (6.24) and (6.25). In the neighbourhood of the Alfven resonance the WKB approximation takes the form

\[ v_y \sim S_0 \epsilon^{-\frac{1}{4}} \tau^{-\frac{1}{4}} \exp \frac{\sqrt{2}}{3} (1 + i) \epsilon^{-\frac{1}{4}} \tau^\frac{3}{4} \quad \tau > 0 \]  

(6.37)

\[ v_y \sim S_1 \epsilon^{-\frac{1}{4}} |\tau|^{-\frac{1}{4}} \exp \frac{\sqrt{2}}{3} (i-1) \epsilon^{-\frac{1}{4}} |\tau|^\frac{3}{4} \quad \tau < 0 \]  

(6.38)
In order to match (6.37) and (6.38) with (6.24) and (6.25) we require that

\[ S_0 = \epsilon_{\frac{1}{12}} \exp(\frac{1}{24} \pi i) E_1 \quad \text{and} \quad S_1 = \epsilon_{\frac{1}{12}} \exp(-\frac{1}{24} \pi i) E_1 \] (6.39)

From (6.39) it then follows that \(|S_0| = |S_1|\). This principle result indicates that there is total transmission. Note that the detailed behavior of the solution in the neighborhood of the Alfvén resonance can not be obtained from the WKB solution. However after it has passed through the resonance (which is now a turning point) and reaches a region where the WKB solution is again applicable, the solution in this region is predicted by the matching procedure. In chapter 9 we will see that in a lossless case, we have total reflection instead of total transmission.

6.3 Magnetoacoustic-gravity waves

In this section we derive a differential equation for \(v_z\). We saw that in the ideal case, \(v_z\) is described by a second order singular differential equation. By including resistivity we obtain a fourth order regular differential equation. This implies that a new mode is created. The calculation of this new mode is the subject of this section.

After trivial manipulations we can derive the following differential equation for \(v_z\):

\[
\begin{align*}
\frac{d^4v_z}{dz^4} - i 3 \frac{d^3v_z}{dz^3} + i \left( \frac{\omega^2}{c^2} + \frac{3}{L^2} + k^2 \left( \frac{\omega_b^2}{\omega^2} - 2 \right) \right) \frac{d^2v_z}{dz^2} \\
- \frac{1}{L^2 \omega^2} a^2 (\omega^2 c^2 + a^2 - k^2) \frac{dv_z}{dz} + i \frac{1}{L^3} [k^2 (3 - 2 \frac{\omega_b^2}{\omega^2}) - 2 \omega^2 - \frac{1}{L^2} \frac{dv_z}{dz}] \\
+ \frac{1}{L^3} \frac{dv_z}{dz} + i \epsilon \left[ (k^2 - \frac{1}{L^2}) (k^2 (1 - \frac{\omega_b^2}{\omega^2}) - \frac{\omega^2}{c^2}) v_z \\
- a^2 \frac{1}{L^2 \omega^2} [k^2 \frac{\omega_b^2}{a^2} + (k^2 - \frac{\omega^2}{c^2}) (k^2 - \frac{\omega^2}{a^2})] v_z = 0
\end{align*}
\] (6.40)

\(\epsilon\) is defined by (6.9). Note that due to the inclusion of resistivity the equation (6.40) is regular. In addition to magnetoacoustic-gravity waves, (6.40) contains also a new mode which is a result of resistivity. By setting this latter equal to zero, we get our familiar second order singular differential equation back. The equation (6.40) is not so easy to handle, because its coefficients depend on \(z\).

To get some insight in the behaviour of the solution of (6.40) we investigate it for \(z \to \infty\) and \(z \to -\infty\). In the limit \(z \to \infty\), the Alfvén speed becomes very large. In this limit the differential equation (6.40) becomes

\[
\frac{d^2v_z}{dz^2} - k^2 v_z = 0
\] (6.41)
equation (6.41) provides no propagating solutions. A fact which is not surprising, in view of the disappearance of all buoyancy terms like $\omega_b$ and $L$. For $z \to -\infty$ equation (6.40) behaves like

$$i\epsilon \frac{d^4 v_z}{dz^4} - \frac{3}{L^2} i\epsilon \frac{d^3 v_z}{dz^3} + [i\epsilon \left( \frac{\omega^2}{c^2} + \frac{3}{L^2} + k^2 \left( \frac{\omega_b^2}{\omega^2} - 2 \right) \right) - \omega^2] \frac{d^2 v_z}{dz^2}$$

$$+ [i\epsilon \frac{1}{L} \left( k^2 (3 - 2 \frac{\omega_b^2}{\omega^2}) - 2 \frac{\omega^2}{c^2} - \frac{1}{L^2} \right) + \frac{1}{5}] \frac{dv_z}{dz}$$

$$+ i\epsilon \left[ \left( k^2 - \frac{1}{L^2} \right) \left( \frac{1}{k^2 (1 - \frac{\omega_b^2}{\omega^2})} - \frac{\omega^2}{c^2} \right) \right] - \frac{1}{\omega^2 L^2} [k^2 \omega_b^2 - \omega^2 (k^2 - \frac{\omega^2}{c^2})] v_z = 0 \quad (6.42)$$

Hence, in this limit we obtain a fourth order differential equation with constant coefficients. Thus, there exist solutions of the form

$$v_z = A e^{\lambda z} \quad (6.43)$$

with $A$ an arbitrary constant. $\lambda$ is determined by the algebraic equation

$$i\epsilon \lambda^4 - 3 i\epsilon \lambda^3 + [i\epsilon \left( \frac{\omega^2 L^2}{c^2} + 3 + k^2 L^2 \left( \frac{\omega_b^2}{\omega^2} - 2 \right) \right) - 1] \lambda^2$$

$$+ [i\epsilon \left( L^2 k^2 (3 - \frac{2 \omega_b^2}{\omega^2}) - \frac{2 \omega^2}{c^2} L^2 - 1 \right) + 1] \lambda + i\epsilon \left[ \left( L^2 k^2 - 1 \right) \left( \frac{\omega_b^2}{\omega^2} - \omega^2 \right) \right] = 0 \quad (6.44)$$

In order to have wave solutions, the roots of (6.44) have to be complex. We proceed as follows, to prove that's indeed the case. By taking the complex conjugate of (6.44) we find

$$-i\epsilon \lambda^* - 3 i\epsilon \lambda^* + [\lambda \epsilon \left( \frac{\omega^2 L^2}{c^2} + 3 + k^2 L^2 \left( \frac{\omega_b^2}{\omega^2} - 2 \right) \right) - 1] \lambda^*$$

$$+ [-i\epsilon \left( L^2 k^2 (3 - \frac{2 \omega_b^2}{\omega^2}) - \frac{2 \omega^2}{c^2} L^2 - 1 \right) \lambda^* - i\epsilon \left[ \left( L^2 k^2 - 1 \right) \left( \frac{\omega_b^2}{\omega^2} - \omega^2 \right) \right] = 0 \quad (6.45)$$

where $\lambda^*$ denotes the complex conjugate of $\lambda$. Add (6.45) to (6.44) and suppose that $\lambda$ is real valued, then we obtain

$$\lambda^2 - \lambda + L^2 \left[ \frac{k^2 \omega_b^2}{\omega^2} - \left( k^2 - \frac{\omega^2}{c^2} \right) \right] = 0 \quad (6.46)$$

This equation takes the form of equation (5.41) when we substitute

$$\lambda = \frac{\sigma}{L} \quad (6.47)$$
According to chapter 5, equation (5.41) has only complex solutions. But this contradicts our initial assumption that $\lambda$ is real. The conclusion is then: equation (6.44) has no real valued roots. In the same way it can be proved that these roots are also not imaginary. For the differential equation (6.42) this means that it has evanescent wave solutions.

### 6.3.1 Inner and outer solution

The highest derivative in equation (6.40) is multiplied by $\epsilon$ which we will assume small. The smallness of $\epsilon$ allows us to consider the equation (6.40) as a singular perturbation problem. Therefore we can use boundary layer theory to approximate its solution. It’s convenient to perform the following transformations

$$\zeta = 1 - \zeta_0 \exp\left(-\frac{z}{L}\right), \quad \zeta_0 = \frac{\omega^2}{a_0^2 (k^2 - \omega^2/c^2)}$$  \hspace{1cm} (6.48)

and

$$v_z = \Phi \exp(-kz)$$  \hspace{1cm} (6.49)

With these transformations (6.40) becomes

$$i \epsilon (1 - \zeta)^4 \frac{d^4 \Phi}{d\zeta^4} - i \epsilon (1 - \zeta)^3 (9 + 4kL) \frac{d^3 \Phi}{d\zeta^3} + (1 - \zeta)(O(\epsilon) + \zeta) \frac{d^2 \Phi}{d\zeta^2}$$

$$+ [O(\epsilon) + 1 - (\alpha + \beta + 1)\zeta] \frac{d\Phi}{d\zeta} - (\alpha \beta + O(\epsilon)) \Phi = 0$$  \hspace{1cm} (6.50)

where $\alpha$ and $\beta$ are given by

$$\alpha, \beta = \frac{1}{2} + kL \pm il,$$  \hspace{1cm} (6.51)

$l$ is determined by

$$l^2 = \frac{\omega_b^2 - \omega^2}{\omega^2} (k^2 - \frac{\omega_a^2 - \omega^2}{c^2})$$  \hspace{1cm} (6.52)

where $\omega_b$ is the Brunt Vaisala frequency and $\omega_a$ is the cut-off frequency. Equation (6.50) is almost the same as Kamp (1989) [19][20] found when he described magnetoacoustic gravity waves in an atmosphere with a nearly horizontal field. Yet, a fundamental difference has to be noted. In our case, the highest derivative is multiplied by the complex number $i$. This is typical for every loss mechanism. Note that by letting $\epsilon \to 0$, we find the hypergeometric equation which, as we saw in chapter 5, describes the magnetoacoustic gravity waves in the ideal (singular) case. The parameter $\epsilon$ is assumed to be small. The dissipation
process is then assumed to be important only in a narrow layer around the cusp resonance. Sufficiently far from the cusp resonance, the ideal MHD solution is an excellent approximation to the actual solution. As we already noted in the previous chapter this solution is called the outer solution. The area where this solution is valid is called the outer region. This region is characterized by the fact that rapid variations of $\Phi$ are absent. Therefore, in this region, we can approximate equation (6.50) by

$$
\zeta(1 - \zeta) \frac{d^2 \Phi}{d\zeta^2} + [1 - (\alpha + \beta)\zeta] \frac{d\Phi}{d\zeta} - \alpha \beta \Phi = 0 \quad (6.53)
$$

Equation (6.53) has already been discussed in chapter 5. There we saw that in first order we have

$$
\Phi \sim A_1 + A_2 \ln(\zeta) \quad \zeta \to 0 \quad (6.54)
$$

As we can see from (6.54) the outer approximation breaks down near the cusp resonance. Due to the logarithmic term, rapid variations take place. Consequently, the third and fourth derivative in (6.50) are, even to lowest order, not negligible anymore near the resonance. In order to determine the region where these rapid variations appear, we perform (as we did in the previous chapter) a stretching of the $\zeta$ coordinate according to

$$
\zeta = \frac{\tau}{\epsilon^\nu} \quad (6.55)
$$

Substitution of this stretching in (6.50) gives for $\epsilon \to 0$ a resulting equation that depends upon the value of $\nu$. By applying the method of dominant balance it's found that the only acceptable limit for $\epsilon \to 0$ is the so-called distinguished limit for $\nu = 1/3$, confirming the result found in the previous chapter. Substitution of $\nu = 1/3$ yields the following inner differential equation (in the limit $\epsilon \to 0$)

$$
\tau \frac{d^4 \Phi}{d\tau^4} + \frac{d^2 \Phi}{d\tau^2} + \frac{d\Phi}{d\tau} = 0 \quad (6.56)
$$

From a physical point of view, we can deduce the following from equation (6.56): if we neglect the fourth derivative we get a differential equation, which has a logarithmic solution. This enables the matching of the inner solution to the outer solution. The new mode is a result of the balance between the second and the fourth derivative. Equation (6.56) can be solved by the generalized Laplace transform. We then seek a solution of the form

$$
\Phi(\tau) = B_2 \int_C e^{ist} F(s) ds. \quad (6.57)
$$

where the integration contour $C$ and the function $F(s)$ have to be determined. In appendix A it's shown that the only bounded solution is given by

$$
\Phi = B_1 + B_2 \int_0^\infty \frac{1}{s} e^{-s^3/3} (e^{-is\tau} - 1) ds \quad (6.58)
$$
The solution given by (6.58) is regular at the cuspresonance. The real part of the integral is an even function of \( r \) and is given by

\[
R = \int_{0}^{\infty} \frac{1}{s} e^{-\frac{3}{8} s^3 (\cos(3r) - 1)} ds
\] (6.59)

Equation (6.58) gives the quantity \( \Phi \) through the resonance layer; therefore, it only remains to show that this inner solution matches the outer solution obtained earlier. To this end, we will be interested in the behaviour of (6.58) as \( \tau \to \pm \infty \). By the method of steepest descent (appendix B) it can be shown that for \( \tau \to \infty \) the solution (6.58) behaves like

\[
\Phi \sim B_1 + B_2 \left[ -\ln(\tau) - i \frac{1}{2} \pi - \frac{2}{3} \gamma^* - \frac{1}{3} \ln(3) + e^{\frac{3}{8} \pi i} \sqrt{\frac{\pi}{\tau}} e^{-\left(\frac{1}{2} \sqrt{\pi} (\tau^{1+i})\right)} \right]
\] (6.60)

\( \gamma^* \) is here the Euler constant [21]. The asymptotic behaviour of (6.58) for \( \tau \to -\infty \) is given by

\[
\Phi \sim B_1 + B_2 \left[ -\ln(\tau) + i \frac{1}{2} \pi - \frac{2}{3} \gamma^* - \frac{1}{3} \ln(3) + e^{-\frac{3}{8} \pi i} \sqrt{\frac{\pi}{|\tau|}} e^{\left(\frac{1}{2} \sqrt{|\tau|} |\tau^{1+i}|\right)} \right]
\] (6.61)

We identify the subdominant term in (6.60) and (6.61) as the new mode that is generated in the boundary layer. The outer solution of (6.50) is a valid approximation of the exact solution of (6.40) far enough away from the cusp resonance \( \zeta = 0 \). For \( \epsilon^{1/3} \ll \zeta \ll 1 \) and to lowest order, the outer solution is given by

\[
\Phi_{\text{outer}} \sim A_1 + A_2 \ln(\zeta)
\] (6.62)

For the region \( -1 \ll \zeta \ll -\epsilon^{1/3} \), the outer solution behaves like

\[
\Phi_{\text{outer}} \sim A_1 - i\pi A_2 + A_2 \ln(-\zeta)
\] (6.63)

From a physical point of view there has to be an overlap region where both the outer and the inner solution behave in the same way. This observation leads to the following matching procedure [13]:

\[
\lim_{\zeta \downarrow 0} \Phi_{\text{outer}} = \lim_{\tau \to \infty} \Phi_{\text{inner}}, \quad \lim_{\zeta \uparrow 0} \Phi_{\text{outer}} = \lim_{\tau \to -\infty} \Phi_{\text{inner}}
\] (6.64)

From (6.60), (6.62) and (6.64) it follows that \( B_1 \) and \( B_2 \) are given by

\[
B_2 = -A_2
\]

\[
B_1 = A_2 \left( -\frac{1}{2} \pi - \frac{1}{3} \ln(3) + \frac{1}{3} \log(\epsilon) \right) + A_1.
\] (6.65)
In the inner region, we can’t speak about the magnetoacoustic gravity wave and the new mode. In that region we can’t distinguish between the two waves. Only if we are far enough from the cusp resonance, we see from (6.61) and (6.60) that the inner solution is composed of a magnetoacoustic gravity wave and a new mode.

6.3.2 Analytic continuation of the new mode: WKB approximation

Expressions (6.60) and (6.61) are only valid at the "edge" of the resonant layer. Therefore, they have only a local validity. In order to study the behaviour of the new mode in the outer region we will, following Kamp [19], continue it analytically to the outer region. Consider therefore the original differential equation (6.40). We seek a solution in the form of a WKB serie:

$$v_z = S_0 e^{\frac{1}{2} S(z)}.$$  (6.66)

$\delta = \epsilon^{1/2}$ and $S(z)$ is given by

$$S(z) = \int z^2 \sum_{n=0}^{\infty} \delta^n \phi_n(z') dz'.$$  (6.67)

In our case we truncate the series given in (6.67) after n=1. This means that $S(z)$ now becomes

$$S(z) = \int z^2 [\phi_0 + \delta \phi_1] dz'.$$  (6.68)

Inserting (6.68) into (6.40) and comparing powers of $\epsilon$ gives the following two equations.

$$i \phi_0^2 - \frac{1}{L^2} k^3 a^2 \left( \frac{\omega^2 (c^2 + a^2)}{k^2 c^2 a^2} - 1 \right) = 0 $$  (6.69)

$$i(6 \phi_0 \frac{d \phi_0}{dz} + 4 \phi_1 \phi_0^3) - 3i \frac{1}{L} \phi_0^3 - \frac{1}{L^2 \omega^2} a^2 \left( \frac{\omega^2}{V_c^2} - k^2 \right) \left( \frac{d \phi_0}{dz} + 2 \phi_0 \phi_1 \right) + \frac{1}{L^3} \phi_0 = 0$$  (6.70)

From (6.69) it follows that $\phi_0$ vanishes at the cusprecsonance, a fact which we expect because of the break down of the WKB approximation near a turning-point. (by including resistivity the cusp resonance is now a turning point).

After some trivial algebraic manipulations we find from (6.70)

$$5i \phi_0 \frac{d \phi_0}{dz} - 3i \phi_0^2 \frac{1}{L} + 2i \phi_1 \phi_0^2 + \frac{1}{L^3} = 0 $$  (6.71)

Again, we see that the WKB approximation doesn’t hold near the cusprecsonance, because equation (6.71) yields a contradiction whenever $\phi_0$ equals 0. For
\[ L \to \infty, \text{we deal with an homogenous medium. Consequently, we have no resonance and the WKB holds everywhere. Differentiating equation (6.69) with respect to } z \text{ yields} \]

\[ 2\phi_0 \frac{d\phi_0}{dz} = i \frac{a^2}{\omega^2 L^3} (k^2 - \frac{\omega^2}{c^2}) \]  \hspace{1cm} (6.72)

From (6.72), (6.69) and (6.71) it follows that \( \phi_1 \) has to be real valued. The WKB solutions can then be written as

\[ v_z \sim S_0 \exp(\int z^\prime \phi_1(z')dz') \times \exp(\int z^\prime (i-1) e^{-\frac{1}{2} \frac{ka}{L\omega} \left[ \frac{\omega^2(c^2 + a^2)}{c^2 k^2 a^2} - 1 \right]} dz') \] \hspace{1cm} (6.73)

\[ v_z \sim S_1 \exp(\int z^\prime \phi_1(z')dz') \times \exp(-\int z^\prime (i+1) e^{-\frac{1}{2} \frac{ka}{L\omega} \left[ -\frac{\omega^2(c^2 + a^2)}{c^2 a^2 k^2} + 1 \right]} dz') \] \hspace{1cm} (6.74)

The WKB solutions given by (6.73) and (6.74), is only valid in a region far from the cusp resonance. So, for \( z \to -\infty \), WKB is an excellent approximation to the actual solution of (6.40). For \( z \to -\infty \), it follows from 6.69 that \( \phi_0 \) behaves like

\[ \phi_0 \sim -\frac{i}{L^2} \]  \hspace{1cm} (6.75)

Using this result, we conclude that for \( z \to -\infty \) the WKB solution behaves like an evanescent plane wave. A result which we already obtained in the previous section. In the neighbourhood of the resonance, but far enough to guarantee the validity of the WKB approximation, the WKB approximation has to match the expression of the new mode obtained via boundary layer theory. To check this, we investigate the behaviour of (6.73) and (6.74) for \( \zeta \) small enough. It's straightforward to show that this behaviour is given by

\[ v_z \sim S_0 |\zeta|^{-\frac{1}{2}} \exp\left(\frac{2}{3} \zeta^{-\frac{1}{2}} (i-1) |\zeta|^{\sqrt{|\zeta|}} \right) \] \hspace{1cm} (6.76)

\[ v_z \sim S_1 |\zeta|^{-\frac{1}{2}} \exp\left(-\frac{2}{3} \zeta^{-\frac{1}{2}} (i+1) |\zeta|^{\sqrt{|\zeta|}} \right) \] \hspace{1cm} (6.77)

A glance at (6.60) and (6.61) reveal that they match (6.76) and (6.77) if the following holds

\[ S_0 = -\sqrt{\pi} e^{-kL e^{1/4} e^{i \frac{3}{2} \pi}} A_2 \] \hspace{1cm} (6.78)

\[ S_1 = -\sqrt{\pi} e^{-kL e^{1/4} e^{-i \frac{3}{2} \pi}} A_2 \] \hspace{1cm} (6.79)

\textbf{6.3.3 Identification of the new mode}

Obviously, as becomes clear from (6.73), the dispersion relation governing the new mode that is generated is given by

\[ c l^2 = \frac{k^2 a^2}{L^2 \omega^2} \left( \frac{\omega^2(c^2 + a^2)}{k^2 a^2} - 1 \right) \]  \hspace{1cm} (6.80)
where $l$ is the local wavenumber of the mode, which is actually a modified slow magneto-acoustic wave in the limit of a large wavenumber normal to the magnetic field. This can be seen as follows. Consider the dispersion relation for slow magneto-acoustic waves which is discussed in chapter 4.

$$\omega^2 = \frac{1}{2} k^2 (a^2 + c_s^2) - \frac{1}{2} k^2 (a^2 + c_s^2) \left[1 - 4 \frac{k_{||}^2 c_s^2}{k^2 (c_s^2 + c_t^2)^2} \right]^{\frac{1}{2}}$$  \hspace{1cm} (6.81)

From (6.81) we easily deduce that the slow magneto-acoustic wave that propagates nearly perpendicular with respect to the magnetic field is governed by the approximate dispersion relation

$$\omega^2 = \frac{k_{||}^2 c_s^2 a^2}{c_s^2 + a^2}$$  \hspace{1cm} (6.82)

Because of the smallness of $\epsilon$ we can write (6.80) as:

$$\omega^2 = \frac{k^2 a^2 c_s^2}{c_s^2 + a^2} \left(1 + \frac{l^2 L^2 c_s^2}{c_s^2 + a^2} \right)$$  \hspace{1cm} (6.83)

From (6.83) it becomes clear that the mode that is generated near the cusp resonance is a slow magneto-acoustic wave in the limit of a large wavenumber normal to the magnetic field and whose dispersion relation is corrected for resistivity.
Chapter 7

Inclusion of viscosity

In this chapter we consider the effect of viscosity. We will show that it resolves both the Alfven singularity and the cusp singularity, by raising the order of the differential equations in concern. By assuming that viscous effects are only significant in a narrow layer around the resonance, we will show that the inner equations are the same as in the resistive case. Davilla(1987) [23] studied both viscous and resistive effects for compressible MHD perturbations propagating in the inhomogeneous solar atmosphere. He didn’t take gravity into account and considered therefore only Alfven waves.

7.1 Fundamental equations

The starting point is the same as in the previous chapter. So, consider a compressible, isothermal, viscous atmosphere that is stratified due to the presence of a gravitational field. Furthermore, we assume that we deal with an ideal gas. In order to investigate only the effect of viscosity, we ignore all other nonidealities ( resistivity, thermal condutance, etc). The mathematical description is then given by the following equations

\[
\frac{\partial p}{\partial t} + \nabla \cdot (p \mathbf{v}) = 0 \quad (7.1)
\]

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla (p + \frac{B^2}{2\mu_0}) + \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B} + \rho g - \nabla \cdot \Pi \quad (7.2)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \quad (7.3)
\]

\[
\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = -\Pi : \nabla \mathbf{v} \quad (7.4)
\]
\(\Pi\) is the viscosity tensor. If we consider the plasma as a Newton fluid then \(\Pi\) is given by

\[
\Pi_{ij} = -\nu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \frac{2}{3} \delta_{ij} \nu \nabla \cdot v
\]

(7.5)

Where \(\delta_{ij}\) is the Kronecker delta and \(\nu\) is the dynamical viscosity. The term \(\nabla \cdot \Pi\) is then determined by

\[
\nabla \cdot \Pi = -\frac{1}{3} \nu \nabla \nabla \cdot v - \nu \nabla^2 v
\]

(7.6)

In fact this expression for the viscosity tensor is only valid for a unmagnetized plasma and where collisional effects are dominant, i.e., the mean free path is much smaller than a typical length scale \(L\). So, a consistent treatment would also include resistivity. Here, we ignore it, because we want to investigate each process separately. The viscous heating is given by

\[
\rho_0 \frac{\partial v_1}{\partial t} + \nabla \cdot (\rho_0 v_1) = 0
\]

(7.8)

\[
\rho_0 \frac{\partial v_1}{\partial t} = -\nabla (p + \frac{1}{2} \mu_0 B_1 \cdot B_0) + \frac{1}{\mu_0} B_0 \nabla B_1 + \rho_1 g + \frac{1}{3} \nu \nabla (\nabla \cdot v_1) + \nu \nabla^2 v_1
\]

(7.9)

\[
\frac{\partial B_1}{\partial t} = B_0 \cdot \nabla v_1 - B_0 (\nabla \cdot v_1)
\]

(7.10)

\[
\frac{\partial p_1}{\partial t} + v_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot v_1 = 0
\]

(7.11)

Note that the viscous heating term is of higher order, since we consider a calm atmosphere (plasma), with \(v_0 = 0\). After differentiating equation (7.9) partially with respect to \(t\) and after eliminating \(\frac{\partial p_1}{\partial t}\) and \(\frac{\partial B_1}{\partial t}\), we obtain the following vector equation for the perturbation \(v_1\)

\[
\frac{\partial^2 v_1}{\partial t^2} = c^2 \nabla \nabla \cdot v_1 + \nabla (g \cdot v_1) + (\gamma - 1) g \nabla \cdot v_1
\]

\[-(\frac{1}{\mu_0 \rho_0}) (\nabla (B_0 \cdot (B_0 \cdot \nabla) v_1 - B_0 \nabla \cdot v_1))
\]

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\begin{align*}
\left(\frac{1}{\rho_0 \rho_0}\right) (B_0 \cdot \nabla)((B_0 \cdot \nabla)v_1 - B_0 \nabla \cdot v_1) \\
+ \frac{1}{3 \rho_0} \nabla \left( \nabla \cdot \nabla \frac{\partial v_1}{\partial t} \right) + \frac{\nu}{\rho_0} \nabla^2 \left( \frac{\partial v_1}{\partial t} \right)
\end{align*}

(7.12)

The vector equation (7.12) can be replaced by its components. We will assume that \( \frac{\partial}{\partial z} = 0 \). Taking the x-component of (7.12) gives:

\begin{align*}
\frac{\partial^2 v_{1x}}{\partial t^2} = c^2 \left( \frac{\partial^2 v_{1x}}{\partial x^2} + \frac{\partial^2 v_{1z}}{\partial x \partial z} \right) - g \frac{\partial v_{1x}}{\partial x} + \frac{\nu}{\rho_0} \frac{\partial}{\partial t} \left( \frac{4}{3} \frac{\partial^2 v_{1x}}{\partial x^2} + \frac{1}{3} \frac{\partial^2 v_{1z}}{\partial z^2} + \frac{\partial^2 v_{1z}}{\partial x \partial z} \right)
\end{align*}

(7.13)

By taking the y-projection of (7.12) we find

\begin{align*}
\frac{\partial^2 v_{1y}}{\partial t^2} = a^2 (z) \left( \frac{\partial^2 v_{1y}}{\partial z^2} \right) + \frac{\nu}{\rho_0} \frac{\partial}{\partial t} \left( \frac{\partial^2 v_{1y}}{\partial x^2} + \frac{\partial^2 v_{1x}}{\partial z^2} \right)
\end{align*}

(7.14)

\( a \) is the Alfvén velocity which is determined by (5.18). Finally the projection of (7.12) onto the z-axis is given by

\begin{align*}
\frac{\partial^2 v_{1z}}{\partial t^2} = c^2 \left( \frac{\partial^2 v_{1z}}{\partial x^2} + \frac{\partial^2 v_{1z}}{\partial z^2} \right) - g \frac{\partial v_{1z}}{\partial z} + a^2 (z) \left( \frac{\partial^2 v_{1z}}{\partial x^2} + \frac{\partial^2 v_{1z}}{\partial z^2} \right)
- g(\gamma - 1) \left( \frac{\partial v_{1x}}{\partial x} + \frac{\partial v_{1z}}{\partial z} \right) + \frac{\nu}{\rho_0} \frac{\partial}{\partial t} \left( \frac{4}{3} \frac{\partial^2 v_{1z}}{\partial z^2} + \frac{1}{3} \frac{\partial^2 v_{1z}}{\partial z^2} + \frac{\partial^2 v_{1z}}{\partial x \partial z} \right)
\end{align*}

(7.15)

The assumption \( \frac{\partial}{\partial z} = 0 \) decouples the Alfvén wave from the magneto-acoustic-gravity wave, that is incorporated in the equations for \( v_{1x} \) and \( v_{1z} \). In the following section we will investigate the viscous Alfvén mode.

### 7.2 Alfvén mode

Since all equilibrium quantities don’t depend on the \( x \)-coordinate, we may Fourier transform our equations in this direction. Thus, we assume that a perturbation behaves like

\( f_1 = f(k, \omega, z) \exp \left( ikx - \omega t \right) \)

(7.16)

In the \( y \)-direction the motion is determined by

\( \frac{d^2 v_y}{dz^2} + \frac{i}{\delta} \frac{1}{L^2 \omega^2} (\omega^2 - k^2 a^2 + O(\delta)) v_y = 0 \)

(7.17)

Where \( \delta \) is defined by

\( \delta = \frac{\mu}{L^2 \omega} \)

(7.18)

\( \mu \) is the kinematical viscosity, which is given by

\( \mu = \frac{\nu}{\rho_0} \)

(7.19)
7.2.1 An approximate solution to (7.17)

In the subsequent sections we take $\mu$ to be constant [24]. The differential equation (7.17) can be put into a system of two simultaneous first-order equations by defining a column vector $\mathbf{x}$

$$
\mathbf{x} = \begin{pmatrix} x_1(z) \\ x_2(z) \end{pmatrix} = \begin{pmatrix} v_x \\ \frac{dv_x}{dz} \end{pmatrix}
$$

Equation (7.17) can be written as

$$
\frac{dx}{dz} = \mathbf{A}(z) \cdot \mathbf{x}
$$

where the matrix $\mathbf{A}$ is given by

$$
\mathbf{A}(z) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{L^2}q^2(z) & 0 \end{pmatrix}
$$

$q^2(z)$ is given by

$$
q^2(z) = \frac{i}{\delta} \left( \frac{\omega^2 - k^2 a^2}{\omega^2} + O(\delta) \right)
$$

We now change the dependent variable $\mathbf{x}$ by the transformation

$$
\mathbf{x} = \Phi(z) \cdot \mathbf{y}
$$

where $\mathbf{y}$ is the new dependent vector and $\Phi(z)$ is the transformation matrix yet to be determined. Substituting (7.24) into (7.21), we have

$$
\frac{d\Phi}{dz} \cdot \mathbf{y} + \Phi \cdot \frac{dy}{dz} = \mathbf{A} \cdot \Phi \cdot \mathbf{y}
$$

Multiplying through by $\Phi^{-1}$, provided $\Phi$ is nonsingular, we obtain

$$
\frac{dy}{dz} = (\Phi^{-1} \cdot \mathbf{A} \cdot \Phi) \cdot \mathbf{y} - \Phi^{-1} \cdot \frac{d\Phi}{dz} \cdot \mathbf{y}
$$

The idea is to introduce a transform $\Phi$ such that the matrix

$$
\mathbf{D} = \Phi^{-1} \cdot \mathbf{A} \cdot \Phi
$$

becomes diagonal. If this is achieved, (7.25) becomes two uncoupled equations when the second term on the right-hand side is neglected. Solutions can then be obtained by direct integration. From matrix theory, we know, that the diagonalization of the matrix $\mathbf{A}$ is achieved by choosing $\Phi$ such that the columns in the matrix $\Phi$ are the eigenvectors of the eigenvalue problem

$$
\mathbf{A} \cdot \mathbf{v} = \lambda \mathbf{v}
$$
For given by (7.22), the eigenvalues are obtained easily to be

\[
\lambda_1 = -i \frac{1}{L} q
\]
\[
\lambda_2 = i \frac{1}{L} q
\]

(7.29) (7.30)

The corresponding eigenvectors are

\[
v_1 = \begin{pmatrix} 1 \\ -i \frac{1}{L} q \end{pmatrix}
\]

(7.31)

and

\[
v_2 = \begin{pmatrix} 1 \\ i \frac{1}{L} q \end{pmatrix}
\]

(7.32)

Therefore the transform matrix has the form

\[
\Phi = \begin{pmatrix} 1 & 1 \\ -i \frac{1}{L} q & i \frac{1}{L} q \end{pmatrix}
\]

(7.33)

substituting (7.33) into (7.27), we obtain the diagonal matrix D:

\[
\Phi^{-1} \cdot A \cdot \Phi = \begin{pmatrix} -i \frac{1}{L} q & 0 \\ 0 & i \frac{1}{L} q \end{pmatrix}
\]

(7.34)

where

\[
\Phi^{-1} = \frac{L}{2i q} \begin{pmatrix} i \frac{1}{L} q & -1 \\ i \frac{1}{L} q & 1 \end{pmatrix}
\]

(7.35)

The transformed equation (7.26) now has the form

\[
\frac{dy}{dz} = \frac{d}{dz} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -i \frac{1}{L} q & 0 \\ 0 & i \frac{1}{L} q \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \frac{1}{4 q^2} \frac{dq^2}{dz} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\]

(7.36)

Equation (7.36) is still a set of two uncoupled equations. But under the condition that the off-diagonal terms on the right-hand side of (7.36) can be neglected, we obtain two uncoupled equations for \(y_1\) and \(y_2\). The exact condition to neglect the off-diagonal terms is given by K.C. Yeh et al (1972) [4]. In our case this condition yields

\[
\left| \frac{3 L^2}{4} \frac{1}{q^3} \left( \frac{dq}{dz} \right)^2 - \frac{L^2}{2q^2} \frac{d^2 q}{dz^2} \right| \ll 1
\]

(7.37)

By assuming \(\delta\) small enough, the condition (7.37) is satisfied, if \(q\) is not too close to zero. We have then the following two uncoupled differential equations for \(y_1\) and \(y_2\):

\[
\frac{dy_1}{dz} = \left[ -i \frac{1}{L} q - \frac{1}{2q \frac{dq}{dz}} \right] y_1
\]

(7.38)

\[
\frac{dy_2}{dz} = \left[ i \frac{1}{L} q - \frac{1}{2q \frac{dq}{dz}} \right] y_2
\]

(7.39)
The solutions of (7.38) and (7.39) can be obtained immediately by integration:

\[
y_1 = c_1 q^{-\frac{1}{2}} \exp -i \left( \frac{1}{L} \int^z q(s)ds \right)
\]

(7.40)

\[
y_2 = c_2 q^{-\frac{1}{2}} \exp i \left( \frac{1}{L} \int^z q(s)ds \right)
\]

(7.41)

By using (7.20) and (7.24) we obtain the following bounded solution:

\[
v_y \sim C_1 (1 - \frac{k^2 a^2}{\omega^2})^{\frac{1}{4}} \exp(i - 1) \frac{\delta^{-\frac{1}{2}}}{L} \int^z (1 - \frac{k^2 a^2}{\omega^2})^{\frac{1}{2}} dz' \quad z < z_a \quad (7.43)
\]

\[
v_y \sim C_2 (1 - \frac{k^2 a^2}{\omega^2})^{\frac{1}{4}} \exp -(i + 1) \frac{\delta^{-\frac{1}{2}}}{L} \int^z (1 - \frac{k^2 a^2}{\omega^2})^{\frac{1}{2}} dz' \quad z > z_a \quad (7.44)
\]

### 7.2.2 The behaviour near the Alfven resonance

At the Alfven resonance \( q \) becomes equal to zero. The given solution doesn’t hold any more. To investigate the behaviour of the solution of (7.17) near the resonance, we notice that the location of the singularity is a turning point of the equation (7.17). As known, the study of the behaviour of the solution of a second differential equation near its turning point consists of expanding the function \( q(z) \) about this point. In this expansion all higher order terms except the linear term are discarded. Using this procedure the equation (7.23) becomes

\[
\frac{d^2 v_y}{dz^2} - \frac{i L^2}{2} \delta (z - z_a) v_y = 0
\]

(7.45)

\( z_a \) is the location of the Alfven resonance. It’s determined by the condition

\[
\omega^2 = a_0^2 k^2 \exp(\frac{z_a}{L})
\]

(7.46)

The equation (7.45) can be simplified by performing the transformation

\[
z - z_a = \frac{\delta^{\frac{1}{2}}}{L} \exp(\frac{-1}{6\pi} i) w
\]

(7.47)

(7.45) then becomes

\[
\frac{d^2 v_y}{dz^2} - w v_y = 0
\]

(7.48)

This is the Airy equation. We already met this equation in the resistive case. Clearly: provided that resistivity and viscosity are small enough, we obtain the same differential equation describing the behaviour of the resistive and viscous Alfven wave near the Alfven resonance. Actually this is a direct result of boundary layer theory. Consider an arbitrary dissipation mechanism. This mechanism
raises the order of the ideal singular equation. Let us assume that in the ideal case we deal with an equation of order 0 (as in the case of the Alfven wave). The inclusion of any dissipation mechanism, will result in a differential equation of order 2. In general, this equation will have the form

\[
i \epsilon \frac{d^2 f}{dz^2} + g(z) \epsilon \frac{df}{dz} + h(z)(\omega^2 - k^2 a^2) v_y = 0 \tag{7.49}
\]

\( \epsilon \) represents the dissipation mechanism under consideration, and is assumed to be small. Now, the only distinguished limit is that for which \( \Delta = O(\epsilon^{1/2}) \), where \( \Delta \) is the thickness of the boundary layer. This implies that the inner equation is given by the Airy equation

\[
\frac{d^2 f}{d\tau} - \tau f = 0 \tag{7.50}
\]

where \( \tau = 0 \) is the location of the singularity. The conclusion above means, that all the results we obtained in the resistive case are valid in the viscous case.

### 7.3 Magnetoacoustic-gravity mode

In this section we will analyse the differential equation describing the magnetoacoustic-gravity mode. From our experience with the resistive case, we expect a fourth order differential equation. Indeed, after fourier transforming (7.13) and (7.15) according to (7.16) and after eliminating \( v_1x \) from these equations we obtain:

\[
i \delta \frac{d^4 v_{1z}}{dz^4} + \frac{1}{L_i} i \delta \frac{c^2 + 3a^2}{c^2 + a^2} \frac{d^3 v_{1z}}{dz^3} + \frac{1}{L^2} \left( \frac{k^2 c^2 a^2}{c^2 + a^2} - 1 + O(\delta) \right) \frac{d^2 v_{1z}}{dz^2} + \frac{1}{L^3} \left( \frac{c^2}{c^2 + a^2} + O(\delta) \right) \frac{dv_{1z}}{dz}
\]

\[- \frac{1}{L^4} \left( \frac{k^2 c^2 a^2}{c^2 + a^2} \right) + L^2 \frac{(k^2 c^2 - \omega^2)(k^2 a^2 - \omega^2)}{\omega^2(c^2 + a^2)} + O(\delta) v_{1z} = 0 \tag{7.51}
\]

where \( \delta \) is defined by (7.18). As in the resistive case, equation (7.51) is not so easy to handle. To approximate it, we will use boundary layer theory. Performing the familiar transformations

\[
\zeta = 1 - \zeta_0 \exp\left(-\frac{z}{L}\right) \tag{7.52}
\]

\[
v_z = \phi \exp\left(-kz\right) \tag{7.53}
\]

Equation (7.51) becomes

\[
i \delta (1 - \zeta)^4 \frac{d^4 \phi}{dz^4} - i \delta (1 - \zeta)^3 (6 + 4kL - \frac{c^2}{c^2} (1 - \zeta) + 3a_0^2) \frac{d^3 \phi}{dz^3} + \frac{(1 - \zeta)(O(\delta) + \zeta)^2 d^2 \phi}{d\zeta^2} + (O(\delta) + 1 - \zeta(\alpha + \beta + 1)) \frac{d\phi}{d\zeta} + (\alpha \beta + O(\delta)) \phi = 0 \tag{7.54}
\]

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where $\alpha$ and $\beta$ are given by

$$
(\alpha, \beta) = \frac{1}{2} + kL \pm ilL 
$$

(7.55)

$l$ is determined by

$$
l^2 = \frac{\omega_b^2 - \omega^2}{\omega^2} (k^2 - \frac{\omega^2 \omega_a^2 - \omega^2}{c^2 \omega_b^2 - \omega^2})
$$

(7.56)

where $\omega_b$ is the Brunt Vaisala frequency and $\omega_a$ is the cut-off frequency. Note that by letting $\delta \to 0$, we obtain the familiar hypergeometric equation, which describes magnetoacoustic-gravity waves in the ideal singular case. The smallness of $\delta$ allows us to use boundary layer theory. The viscous effects are then assumed to be important only in a narrow layer around the cusp resonance. Sufficiently far from the cusp level the ideal MHD solution is an excellent approximation to the actual solution. Therefore in the outer region, we can approximate equation (7.54) by

$$
d^2\phi d\zeta^2 + (1 - (\alpha + \beta)\zeta) \frac{d\phi}{d\zeta} - \alpha\beta\phi = 0
$$

(7.57)

Equation (7.57) has already been discussed in chapter 5. There we saw that in first order its solution behaves like

$$
\phi \sim A_1 + A_2 \ln(\zeta)
$$

(7.58)

Due to the logarithmic term, rapid variations take place. Consequently, the third and fourth derivative in (7.54) are, even to lowest order, not negligible anymore. In order to determine the region where these rapid variations take place, we perform (as in the resistive case) a stretching of the $\zeta$ coordinate according to

$$
\zeta = \frac{\tau}{\delta^\sigma}
$$

(7.59)

Substitution of this stretching in (7.54) gives for $\delta \to 0$ a resulting equation that depends upon the value $\sigma$. By applying the method of dominant balance it's found that the only acceptable limit for $\delta \to 0$ is the distinguished limit for $\sigma = \frac{1}{3}$. We can then conclude: provided that viscous and resistive effects are represented by a small parameter $\delta$ then the boundary layer where these effects are important has a thickness of $O(\delta^{\frac{1}{3}})$. Substitution of $\sigma = \frac{1}{3}$ yields the following inner differential equation

$$
\frac{i d^4\phi}{d\tau^4} + \tau \frac{d^3\phi}{d\tau^2} + \frac{d\phi}{d\tau} = 0
$$

(7.60)

Another important conclusion: resistive and viscous effects are described by the same inner equation. Clearly, to resolve a singularity, nature doesn't make any difference between resistivity and viscosity (if they are small enough). This conclusion allows us to use the results we obtained in the resistive case. The
new viscous mode that's generated in the resonant layer is then given by the subdominant term of (6.64). In the same way as we did in the previous chapter it can be shown that this new mode is actually a slow magneto-acoustic wave in the limit of a large wave number normal to the magnetic field and whose dispersion relation is corrected for viscous effects.
Chapter 8

Thermal conductivity

Up to now, we have assumed that all oscillations were adiabatic. Most authors who consider the subject of the heating of the solar atmosphere restrict themselves mainly to adiabatic oscillations. Babaev et al [25] noticed however that the waves in the solar atmosphere are highly nonadiabatic. Hence, it becomes necessary for the theory of waves in the solar atmosphere to take into account nonadiabatic effects. Babaev et al [25] considered radiative heat exchange which obeys Newton cooling law. They considered an atmosphere which is embedded in a magnetic field with an arbitrary direction. They derived a fourth order differential equation for magneto-acoustic-gravity waves which they solved analytically in terms of Meijers' hypergeometric G-functions. In this chapter we will take thermal conductivity into account. The energy equation then becomes

\[
\frac{d(p\rho^{-\gamma})}{dt} = (\gamma - 1)\kappa \nabla^2 T \quad (8.1)
\]

where \( \kappa \) is the coefficient of thermal conductivity.

8.1 Fundamental equations

We consider a compressible, isothermal, perfectly conducting atmosphere that is stratified due to the presence of a gravitational field. Taking only thermal conductance into account we obtain the following equations

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{v}\rho = 0 \quad (8.2)
\]

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla (p + \frac{B^2}{2\mu_0}) + \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B} + \rho \mathbf{g} \quad (8.3)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \quad (8.4)
\]
\[ \frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = \kappa (\gamma - 1) \nabla^2 T \quad (8.5) \]

\[ T = \frac{1}{(\gamma - 1)c_v \rho^2} p \quad (8.6) \]

As usual, we will assume that the various quantities above, are a superposition of an atmospheric mean state of rest with a constant magnetic field \( \mathbf{B}_0 \) that is directed along the \( x \)-axis and a perturbation, which we will denote by \( f_i \) where \( f \) stands for an arbitrary quantity from the fundamental equations. After linearizing we obtain

\[
\frac{\partial p_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}) = 0 \quad (8.7)
\]

\[
\frac{\partial \mathbf{v}_1}{\partial t} = -\nabla (p_1 + \frac{1}{\mu_0} \mathbf{B}_1 \cdot \mathbf{B}_0) + \frac{1}{\mu_0} \mathbf{B}_0 \nabla \mathbf{B}_1 + \rho_1 \mathbf{g} \quad (8.8)
\]

\[
\frac{\partial \mathbf{B}_1}{\partial t} = \mathbf{B}_0 \cdot \nabla \mathbf{v}_1 - \mathbf{B}_0 (\nabla \cdot \mathbf{v}_1) \quad (8.9)
\]

\[
\frac{\partial p_1}{\partial t} + \mathbf{v}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \mathbf{v}_1 = (\gamma - 1) \kappa \nabla^2 T_1 \quad (8.10)
\]

\[
\frac{T_1}{T_0} = \frac{p_1}{p_0} \frac{\rho_1}{\rho_0} \quad (8.11)
\]

where \( T_0 \) is given by

\[
T_0 = \frac{c^2}{\gamma (\gamma - 1)c_v} \quad (8.12)
\]

Note that the Alfven mode is unaffected by the inclusion of thermal conductivity. After eliminating \( p_1, p_0 \) and \( \mathbf{B}_1 \) we obtain the following equations relating \( T_1 \) and \( \mathbf{v}_1 \)

\[
\mathbf{v}_1 \cdot \nabla p_0 + \gamma \nabla \cdot \mathbf{v}_1 + \frac{\partial}{\partial t} \left( \frac{T_1}{T_0} \right) - \frac{1}{\rho_0} \nabla \cdot (\rho_0 \mathbf{v}_1) = \frac{(\gamma - 1)}{\rho_0} \kappa \nabla^2 T_1 \quad (8.13)
\]

\[
-\frac{1}{\mu_0 \rho_0} \mathbf{B}_0 \cdot \nabla [\mathbf{B}_0 \cdot \nabla \mathbf{v}_1 - \mathbf{B}_0 (\nabla \cdot \mathbf{v}_1)] - g \frac{1}{\rho_0} \nabla \cdot (\rho_0 \mathbf{v}_1) \quad (8.14)
\]

If we ignore the velocity perturbation in equation (8.13), we will just obtain the diffusion equation for the temperature. In order to decouple the Alfven mode from the magnetoacoustic-gravity mode we assume, as usual, that there is no propagation in the \( y \)-direction.
8.1.1 The isothermal cusp resonance

The perturbations are assumed to behave like

\[ f_1 = f(z, k, \omega) \exp(ikx - \omega t) \tag{8.15} \]

After Fourier transforming (8.13) and (8.14) according to (8.15), we can eliminate \( v_z \). We then obtain the following equations relating \( v_z \) and \( \frac{T_1}{T_0} \):

\[ (\gamma - 1)(1 - A) \frac{dv_z}{dz} + \frac{\gamma - 1}{L} v_z = i\omega(1 - (\gamma - 1)A + O(\epsilon)) \frac{T_1}{T_0} - \epsilon \omega L^2 \frac{d^2}{dz^2} \frac{T_1}{T_0} \tag{8.16} \]

\[ (a^2 + \frac{c^2}{\gamma}(1 - A)) \frac{d^2 v_z}{dz^2} + \left( \frac{a}{L}(A - 1) + \omega ^2 \right) \frac{dv_z}{dz} = i\omega \frac{c^2}{\gamma}(A - 1) \frac{dT_1}{dz} \frac{T_1}{T_0} + A g(\frac{T_1}{T_0}) \tag{8.17} \]

where \( A \) and \( \epsilon \) are given by

\[ A = \frac{k^2 c^2 / (\gamma)}{k^2 c^2 / \gamma - \omega^2} \tag{8.18} \]

\[ \epsilon = (\gamma - 1) \frac{\kappa T_0}{\omega P_0 L^2} \tag{8.19} \]

Notice that the effect of conductivity is not explicitly present in equation (8.17). The reason is obvious: unlike viscosity and resistivity, thermal conductivity is only apparent in the energy equation. The equations (8.16) and (8.17) can be combined into a system of four first ordinary differential equations

\[ F \cdot \frac{dw}{dz} = D \cdot w = \begin{pmatrix} 0 & 1 & 0 & 0 \\ D_{21} & 0 & D_{23} & D_{24} \\ 0 & 0 & 0 & 1 \\ D_{41} & D_{42} & D_{43} & 0 \end{pmatrix} \cdot w \tag{8.20} \]

Where

\[ w = \begin{pmatrix} v_z \\ \frac{dv_z}{dz} \\ \frac{d^2 v_z}{dz^2} \\ f_1 \end{pmatrix} \tag{8.21} \]

\( F \) is a diagonal matrix with as non-zero elements

\[ F_{11} = 1 \]

\[ F_{22} = a^2 + \frac{c^2}{\gamma}(1 - A) \tag{8.22} \]

\[ F_{33} = 1 \]

\[ F_{44} = -\epsilon \omega L^2 \tag{8.23} \]
The elements of the matrix \( D \) are,

\[
\begin{align*}
D_{21} &= -\left( \frac{g}{L} (A - 1) + \omega^2 \right) \\
D_{23} &= Agi\omega \\
D_{24} &= \frac{c^2}{\gamma} i\omega (A - 1) \\
D_{41} &= \frac{1}{L} (\gamma - 1) \\
D_{42} &= (\gamma - 1)(1 - A) \\
D_{43} &= -i\omega (A - 1) + 1 + O(\epsilon)
\end{align*}
\] (8.24)

The system of equations (8.20) has singular solutions if \( \det(F) = 0 \) for some point. From (8.22) it's obvious that this is the case when

\[
a^2 + \frac{c^2}{\gamma} (1 - A) = 0
\] (8.25)

or if

\[
\omega^2 = k^2 \frac{c^2}{\gamma^2} + a^2
\] (8.26)

Clearly, the inclusion of thermal conductivity, resolves the cusp resonance, but another isothermal cusp resonance occurs. This fact was already seen by D. Herman et al [26]. They stated that by including conductivity the cusp continuum is replaced by the isothermal continuum.

8.1.2 Inner equation

In order to compare the effect of thermal conductivity with viscosity and resistivity we derive one single differential equation for \( v_z \). By eliminating \( \frac{T_1}{L} \) from the system (8.16) and (8.17) we obtain

\[
\begin{align*}
-i\omega \frac{\gamma - 1}{\gamma} \kappa^* \left( \frac{c^2 + \gamma a^2}{c^2 a^2} - \frac{k^2}{\omega^2} \right) \frac{d^4 v_z}{dz^4} \\
-3i\omega \frac{\gamma - 1}{\gamma} \kappa^* \left( \frac{\gamma}{c^2 L} - \frac{k^2}{\omega^2 L} \right) \frac{d^3 v_z}{dz^3} \\
+\left[ \omega^2 \frac{c^2 + a^2}{c^2 a^2} - k^2 + O(\kappa^*) \right] \frac{d^2 v_z}{dz^2} - \left( \frac{1}{L} \frac{\omega^2}{a^2} + O(\kappa^*) \right) \frac{dv_z}{dz} \\
+\left[ k^2 \omega^2 \frac{1}{a^2} + (k^2 - \frac{\omega^2}{a^2})(k^2 - \frac{\omega^2}{a^2}) + O(\kappa^*) \right] v_z = 0
\end{align*}
\] (8.27)

Equation (8.27) confirms the occurrence of the isothermal cusp resonance. By applying the transformations (7.52) and (7.53), equation (8.27) becomes

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\[ \begin{align*}
&i\delta(1 - \frac{\Lambda^2}{\gamma - 1}(1 - \zeta)^2) \frac{d^4\Phi}{d\zeta^4} \\
&-i\delta(1 - \zeta^2)\left[ \frac{(3 + 4kL)\gamma - (6 + 4kL)}{\gamma - 1} - (6 + 4kL)\frac{\Lambda^2}{\gamma - 1}(1 - \zeta)^2 \right] \frac{d^3\Phi}{d\zeta^3} \\
&+\Lambda^2[\{\zeta(1 - \zeta) + O(\delta)\}] \frac{d^2\Phi}{dz^2} + (1 - 2(1 + kL)\zeta + O(\delta)) \frac{d\Phi}{d\zeta} \\
&-\left( \frac{\omega_b}{\omega} \right)^2 k^2 L^2 + kL + \frac{\omega^2}{4(\omega_a)^2} + O(\delta)\Phi = 0 \quad (8.28)
\end{align*} \]

with

\[ \delta = \frac{(\gamma - 1)^2}{\gamma} \frac{\kappa}{\omega \rho_0 RL^2} \quad (8.29) \]

and

\[ \Lambda^2 = \frac{k^2 \omega}{\omega^2} - 1 > 0 \quad (8.30) \]

\( \delta \) is assumed to be small and constant. This allows us to use boundary layer theory. Following the same procedure as we did in the previous chapter, we find that the inner equation is given by

\[ i\frac{d^4\Phi}{d\tau^4} + \Lambda^2(\zeta \frac{d^2\Phi}{d\tau^2} + \frac{d\Phi}{d\tau}) = 0 \quad (8.31) \]

where \( \tau \) is determined by

\[ \tau = \delta^{-\frac{1}{2}} \zeta \quad (8.32) \]

(8.31) is the same kind of equation that we met earlier in the case of viscosity and resistivity. In the same way as in the resistive case it can easily be shown that the new mode that is generated is a modified slow magneto-acoustic wave.
Chapter 9

The inclusion of the Hall term in Ohm's law

The MHD equations result from the two fluid model. One of these equations is the Ohm's law. In fact, this law is an equation for the time derivative of the current and can be derived form the momentum equations for electrons and ions. By neglecting collisions and assuming that electrons and ions have the same pressure, Ohm's law is given by [2][27]

$$E + v \times B = \frac{m_i m_e}{e^2 \rho} \frac{\partial J}{\partial t} + \frac{1}{2e \rho} m_i J \times B + \frac{1}{2e} m_i \frac{dv}{dt} \quad (9.1)$$

The $J \times B$ term is known as the Hall term. In The following section we will investigate the effect of this Hall term on MHD singularities. A mathematical analysis will show that this term resolves the cusp singularity, by raising the order of the ideal singular equation, and thus, by introducing a new mode.

9.1 The Hall term

As usual we consider a compressible, isothermal, perfectly conducting atmosphere that is stratified due to the presence of a gravitational field. Viscosity and all other nonidealities are ignored. Taking the Hall term into account the fundamental equations become

$$\rho \frac{\partial v}{\partial t} + \nabla \cdot \rho v = 0 \quad (9.2)$$

$$\rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla (p + \frac{B^2}{2 \mu_0}) + \frac{1}{\mu_0} B \cdot \nabla B + \rho g \quad (9.3)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times v \times \mathbf{B} - \frac{m_i}{2e \rho \mu_0} [\nabla \times (\mathbf{B} \cdot \nabla \mathbf{B}) + \frac{\nabla \rho}{\rho} \times (\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2} \nabla B^2)] \quad (9.4)$$
As usual, we will assume that the various quantities in the fundamental equations are a superposition of an atmospheric mean state of rest with a constant magnetic field $B_0$ that is directed along the $x$-axis and a perturbation, which we will denote by $f_1$ where $f$ stands for a quantity from the fundamental equations. After linearizing these equation we obtain

\begin{equation}
\frac{\partial p_1}{\partial t} + \nabla \cdot (\rho_0 v_1) = 0
\end{equation}

\begin{equation}
\rho_0 \frac{\partial v_1}{\partial t} = -\nabla (p_1 + \frac{1}{\mu_0} B_1 \cdot B_0) + \frac{1}{\mu_0} B_0 \nabla B_1 + \rho_1 g
\end{equation}

\begin{equation}
\frac{\partial B_1}{\partial t} = B_0 \cdot \nabla v_1 - B_0 (\nabla \cdot v_1)
\end{equation}

\begin{equation}
- \frac{mi}{2e \rho \mu_0} [\nabla \times (B_0 \cdot \nabla B_1) + \nabla \rho_0 \times (B_0 \cdot \nabla B_1 - \frac{1}{\mu_0} \nabla B_1 \cdot B_0)]
\end{equation}

\begin{equation}
\frac{\partial p_1}{\partial t} + v_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot v_1 = 0
\end{equation}

It’s straightforward to show that the Hall term couples the $v_{1y}$ component to $v_{1x}$ and $v_{1z}$ even with our familiar assumption $\frac{\partial f}{\partial y} = 0$. By taking for example the $x$-component of the Ohm’s law we obtain

\begin{equation}
\frac{\partial B_{1x}}{\partial t} = -B_0 [\frac{\partial v_{1y}}{\partial y} + \frac{\partial v_{1z}}{\partial z}]
\end{equation}

\begin{equation}
+ \frac{a^2}{2\Omega_i} [\frac{\partial^2 B_{1z}}{\partial x \partial y} - \frac{\partial^2 B_{1y}}{\partial x \partial z} + \frac{1}{L} (\frac{\partial B_{1y}}{\partial y} - \frac{\partial B_{1y}}{\partial x})]
\end{equation}

where $\Omega_i$ is the ion cyclotron frequency which is given by

\begin{equation}
\Omega_i = \frac{eB_0}{m_i}
\end{equation}

So, it’s obvious that by assuming $\frac{\partial f}{\partial y} = 0$, we still keep the $y$-component of the perturbation, i.e., $B_{1y}$. Thus, by including the Hall term Alfvén waves mix with the magnetoacoustic-gravity waves. It’s then impossible to separate one from the other. In order to simplify the algebra, we set $\frac{\partial f}{\partial y} = 0$. We assume now that a perturbation behaves like

\begin{equation}
f_1 = f(k, w, z) \exp i(kx - \omega t)
\end{equation}

After fourier transforming the equations (9.6) to (9.9) according to (9.12) we can derive a fourth differential equation for $v_z$. This equation is given by

\begin{equation}
\frac{d^4 v_z}{dz^4} + \frac{3\omega^2 - k^2 a^2}{\omega^2 - k^2 a^2} \frac{d^3 v_z}{dz^3}
\end{equation}
where $f$ is defined as

$$
\frac{\omega^2 - k^2a^2}{a^2}\left[\frac{\omega^2(a^2 + c^2)}{k^2a^2c^2} - 1 + O(\epsilon)\right] + \frac{1}{L}(O(\epsilon) - \frac{\omega^2}{k^2a^2})\frac{dv_z}{dz} + k^2(\omega + \frac{\omega_0}{k^2a^2} + (1 - \frac{\omega^2}{k^2c^2})(1 - \frac{\omega^2}{k^2a^2})v_z) = 0
$$

(9.13)

where $\epsilon$ is defined as

$$
\epsilon = \frac{\omega^2}{\Omega_c^2}
$$

(9.14)

Note that the highest derivative is not multiplied by a complex number as in the case of a loss mechanism. The differential equation (9.13) is not easy to handle. To approximate it, we will assume that $\epsilon$ is small, so that we can use boundary layer theory. Actually, the smallness of $\epsilon$ is a condition for the validity of the MHD model.

By performing the familiar transformations

$$
\zeta = 1 - \zeta_0 \exp(-\frac{z}{L}), \quad \zeta_0 = \frac{\omega^2}{a_0^2(k^2 - \frac{\omega^2}{c^2})}
$$

(9.15)

$$
v_z = \phi \exp(-kz)
$$

(9.16)

equation (9.13) becomes

$$
\epsilon(1 - \zeta)^4 \frac{d^4 \phi}{d\zeta^4} + \epsilon(1 - \zeta)^3[-(6 + 4kL) + \frac{3\omega^2(1 - \zeta) - k^2a_0^2}{1 - \zeta - k^2a_0^2} \frac{d^3 \phi}{d\zeta^3} + \Lambda^2(1 + \zeta(\frac{k^2c^2}{\omega^2} - 1))[(\zeta + O(\epsilon))(1 - \zeta)) \frac{d^2 \phi}{d\zeta^2} + (1 - (\alpha + \beta + 1)\zeta + O(\epsilon)) \frac{d\phi}{d\zeta} - \alpha\beta\phi] = 0
$$

(9.17)

where $\alpha$ and $\beta$ are given by

$$
(\alpha, \beta) = \frac{1}{2} + kL \pm ilL
$$

(9.18)

$\Lambda^2$ is determined by

$$
\Lambda^2 = \frac{L^2\omega^2}{c^2}(1 - \frac{\omega^2}{k^2c^2})
$$

(9.19)

and $l$ is as we know given by

$$
l^2 = \frac{\omega_b^2 - \omega^2}{\omega^2}(k^2 - \frac{\omega^2}{c^2}\omega_a^2 - \frac{\omega^2}{\omega_b^2})
$$

(9.20)

where $\omega_b$ is the Brunt Vaisala frequency and $\omega_a$ is the cut-off frequency. The smallness of $\epsilon$ allows us to use boundary layer theory to approximate the solution of equation (9.17). Far from the cusp resonance the ideal MHD solution is an
excellent approximation to the actual solution. Therefore in the outer region, we can approximate equation (9.17) by
\[
\zeta(1 - \zeta) \frac{d^2 \phi}{d\zeta^2} + [1 - (\alpha + \beta)\zeta] \frac{d\phi}{d\zeta} - \alpha\beta\phi = 0
\] (9.21)
Equation (9.21) has already been discussed in chapter 5. There we saw that in first order its solution behaves like
\[
\phi \sim A_1 + A_2 \ln(\zeta) \quad \zeta \to 0
\] (9.22)
By performing a stretching of the $\zeta$ coordinate according to
\[
\zeta = \frac{r}{ce}
\] (9.23)
we can see that for the distinguished limit we have $\sigma = \frac{1}{2}$. The inner equation is then given by
\[
\frac{d^4 \phi}{d\tau^4} + \Lambda^2 (\tau \frac{d^2 \phi}{d\tau^2} + \frac{d\phi}{d\tau}) = 0
\] (9.24)
Again, we obtain the same kind of equation that we met earlier. This indicates that the new mode will be a modified slow magneto-acoustic wave.

9.1.1 The solution of equation (9.24)
The differential equation (9.24) can be solved by the Laplace method, as we did in the case of resistivity, but it can also be solved directly. Perform therefore the transformation
\[
\tau = -\Lambda^{-\frac{3}{2}} x
\] (9.25)
with this transformation (9.24) becomes
\[
\frac{d^4 \phi}{dx^4} - \frac{d}{dx}(x \frac{d\phi}{dx}) = 0
\] (9.26)
After integrating (9.26) and setting $\frac{d\phi}{dx} = \psi(x)$ we find
\[
\frac{d^2 \psi}{dx^2} - x\psi = C
\] (9.27)
where $C$ is an integration constant. The solution of (9.27) is given by [16][28]
\[
\psi(x) = C_1\text{Ai}(x) + C_2\text{Bi}(x) + C_3\text{Gi}(x)
\] (9.28)
The function $\text{Bi}(x)$ is not bounded for large $x$, so we set $C_2$ equal to zero. Going back to the original variable, the physical solution of (9.26) is then given by
\[
\phi(\tau) = D_1 \int_0^\Lambda \text{Ai}(-x) dx + D_3 \int_0^\Lambda \text{Gi}(-x) dx + D_4
\] (9.29)
In order to match the inner solution (9.29) to the outer solution, we need its asymptotic behaviour for $\tau \to \pm \infty$. This is explicitly given in [28]. For $\tau \to \infty$ the behaviour of (9.29) is given by

$$\phi(\tau) \sim D_4 + D_1\left[\frac{2}{3} - \pi^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} \cos\left(\frac{2}{3} \tau^{-\frac{1}{2}} \Lambda + \frac{\pi}{4}\right)\right]$$

$$-D_3\left[\frac{2\gamma^* + \ln(3)}{3\pi} + \frac{1}{\pi} \ln(\Lambda^{-\frac{3}{2}} \tau) - \pi^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \sin\left(\frac{2}{3} \tau^{-\frac{1}{2}} \Lambda + \frac{\pi}{4}\right)\right]$$

(9.30)

For $\tau \to -\infty$ the asymptotic behaviour is

$$\phi(\tau) \sim D_4 - \frac{1}{3} D_1 - D_3\left[\frac{2\gamma^* + \ln(3)}{3\pi} + \frac{1}{\pi} \ln(-\tau^{\frac{3}{2}})\right]$$

(9.31)

The outer solution is a valid approximation of the exact solution far away from the cusp resonance $\zeta = 0$. For the region $0 < \epsilon^3 \ll \zeta \ll 1$, and to lowest order, the outer solution is given by

$$\phi \sim A_1 + A_2 \ln(\tau) + \frac{1}{3} A_2 \ln(\epsilon)$$

(9.32)

For the region $-1 \ll \zeta \ll \epsilon^\frac{1}{3}$, the outer solution is to lowest order given by

$$\phi \sim A_1 - i\pi A_2 + A_2 \ln(-\tau) + \frac{1}{3} A_2 \ln(\epsilon)$$

(9.33)

The matching procedure has been given in chapter 6. We write it down again

$$\lim_{\zeta \to 0} \phi^{\text{outer}} = \lim_{\tau \to \infty} \phi^{\text{inner}}$$

$$\lim_{\zeta \to 0} \phi^{\text{outer}} = \lim_{\tau \to -\infty} \phi^{\text{inner}}$$

(9.34)

Using this matching procedure we can deduce from (9.30),(9.31), (9.32) and (9.33) that

$$iD_1 = D_3 = -\pi A_2$$

(9.35)

and

$$D_4 = A_1 + A_2\left[\frac{2}{3} \pi i + \frac{1}{3} \ln(\epsilon) - \frac{2\gamma^* + \ln(3)}{3} - \frac{2}{3} \ln(\Lambda)\right]$$

(9.36)

For $\tau \to \infty$ we have then

$$D_4 + D_1\pi^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} \exp\left(-i\left(\frac{2}{3} \tau^{-\frac{1}{2}} \Lambda + \frac{\pi}{4}\right)\right)$$

$$+ D_1\left[\frac{2}{3} - \frac{2\gamma^* + \ln(3)}{3\pi} - \frac{i}{\pi} \ln(\Lambda^{-\frac{3}{2}} \tau)\right]$$

(9.37)

The subdominant term in (9.37) is the new mode that is generated in the resonant layer by the process of mode conversion. It carries off the acoustic gravity wave energy that, coming from $z = -\infty$, partially tunnels toward the cusp resonance. Notice that the generated mode is propagating only above the cusp resonance.
9.1.2 Analytic continuation of the new mode

The mode given in expression (9.37) has only a local validity. In order to understand it, we continue it analytically to the outer region. Consider therefore the original differential equation (9.13). We seek a solution in the form of a WKB serie:

\[ u_\star = S_0 \exp \left( \frac{1}{\delta} S(z) \right) \]  

(9.38)

where \( \delta = \epsilon \frac{1}{2} \). \( S(z) \) is given by

\[ S(z) = \int^z \sum_{n=0}^{\infty} \delta^n \phi_n (z') \, dz' \]  

(9.39)

In our case we truncate the series (9.39) after \( n = 1 \). This means that \( S(z) \) now becomes

\[ S(z) = \int^z [\phi_0 + \delta \phi_1] \, dz' \]  

(9.40)

By inserting (9.38) into (9.13) and comparing powers of \( \epsilon \), we obtain the following two equations for \( \phi_0 \) and \( \phi_1 \)

\[ \phi_0^2 = - \frac{\omega^2 - k^2 a^2}{a^2} \left( \frac{\omega^2 (a^2 + c^2)}{k^2 a^2 c^2} - 1 \right) \]  

(9.41)

\[ \frac{(6 \phi_0^2 \frac{d \phi_0}{dz} + 4 \phi_1 \phi_0^3)}{\omega^2 - k^2 a^2} \left( \frac{\omega^2 (a^2 + c^2)}{k^2 a^2 c^2} - 1 \right) \left( \frac{d \phi_0}{dz} + 2 \phi_0 \phi_1 \right) \]

\[ - \frac{1}{L} \frac{\omega^2 (a^2 - k^2 a^2)}{k^2 a^4} \phi_0 = 0 \]  

(9.42)

From (9.41) it follows that we have propagating solutions only if

\[ \frac{\omega^2 - k^2 a^2}{a^2} \left( \frac{\omega^2 (a^2 + c^2)}{k^2 a^2 c^2} - 1 \right) > 0 \]  

(9.43)

In the previous section we noticed that the new mode is only propagating above the cusp resonance. However, from (9.43) it follows that this mode propagates also below the Alfven resonance. This doesn’t contradict our previous analysis because the mode obtained by boundary layer theory has only a local validity.

In the region between the Alfven and cusp resonance condition (9.43) is not satisfied, which implies that the mode is not propagating in that region. A result which is consistent with boundary layer theory.

Thus, actually the problem is more complicated, because we have two resonances. A complete the treatment should also include the investigation of the
solutions of (9.13) near the Alfvén resonance. However, in this chapter we consider only the region above the cusprecance. From (9.41) it follows that \( \phi_0 \frac{d\phi_0}{dz} \) is real. By using (9.42) it then follows that \( \phi_1 \) has to be real valued. The WKB solution can then be written as

\[
v_z \sim \exp \int^z \phi_1 dz' \times [S_0 \exp \int^z i\epsilon^{-\frac{1}{2}} \text{Im}(\phi_0) dz' + S_1 \exp \int^z -i\epsilon^{-\frac{1}{2}} \text{Im}(\phi_0) dz']
\]

where \( \text{Im}(\phi_0) \) is given by

\[
\sqrt{\frac{\omega^2 - k^2a^2}{a^2} \left( \frac{\omega^2(a^2 + c^2)}{k^2a^2c^2} - 1 \right)}
\]

In order to match (9.44) with (9.37) we must require that \( S_0 = 0 \). \( S_1 \) can be determined by investigating the behaviour of (9.44) near the cusprecance. After then we can match it with the new mode in expression (9.37). Trivial algebraic manipulations reveal that in the neighbourhood of the cusprecance we have

\[
\phi_0 \sim -\frac{1}{E^2} \Lambda^2 \zeta
\]

\[
\phi_0 \frac{d\phi_0}{dz} \sim -\frac{1}{2E^2} \Lambda^2 (1 + O(\zeta))
\]

From (9.42) it follows that for \( \zeta \to 0 \), \( \phi_1 \) behaves like

\[
\phi_1 \sim (-\frac{3}{4L} \zeta^{-1} + O(1))
\]

With these approximations, the WKB solution is given by

\[
v_z = S_1 \zeta^{-\frac{3}{4}} \exp(-i\frac{2}{3} \zeta^{\frac{3}{2}} \Lambda)
\]

From (9.16) and (9.15) it follows that

\[
\Phi = S_1 \zeta^k \zeta^{-\frac{3}{4}} \exp(-i\frac{2}{3} \zeta^{3} 2\Lambda)
\]

In order to match the mode given in (9.37) with (9.50) we require

\[
S_1 = D_1 e^{\frac{i}{4}} \zeta_0^{-k} \pi^{-\frac{3}{4}} \Lambda^{-\frac{3}{4}} \exp(-i\frac{\pi}{4})
\]

Finally, we obtain the following analytical continuation of the generated mode in the outer region

\[
v_z = D_1 e^{\frac{i}{4}} \zeta_0^{-k} \pi^{-\frac{3}{4}} \Lambda^{-\frac{3}{4}} \exp(\int^z \phi_1 dz') \times \exp(-i(\int^z e^{\frac{i}{2}} \text{Im}(\phi_0) dz'))
\]
9.1.3 Identification of the mode that is generated near the cusp resonance

Obviously, as becomes clear from (9.52), the dispersion relation governing the generated mode is given by

$$\epsilon l^2 = \frac{\omega^2 - k^2 a^2}{a^2} \left( \frac{\omega^2 (a^2 + c^2)}{a^2 c^2} - 1 \right)$$  \hspace{1cm} (9.53)

where $l$ is the local wavenumber of the mode, which is actually a modified slow magneto-acoustic wave in the limit of a large wavenumber normal to the magnetic field. This can be seen as follows.

Consider the dispersion relation for slow magneto-acoustic waves which is discussed in chapter 4.

$$\omega^2 = \frac{1}{2} k^2 (a^2 + c^2) - \frac{1}{2} k^2 (a^2 + c^2) \left[ 1 - \frac{k_0^2}{k^2} \left( \frac{c^2}{a^2} + \frac{c^2}{c^2} \right)^2 \right]$$  \hspace{1cm} (9.54)

From (9.54) we easily deduce that the slow magneto-acoustic wave that propagates nearly perpendicular with respect to the magnetic field is governed by the approximate dispersion relation

$$\omega^2 = \frac{k_0^2 c^2 a^2}{c^2 + a^2}$$  \hspace{1cm} (9.55)

Near the cusp resonance, we approximate the relation (9.53) by

$$\epsilon l^2 = - \frac{k^2 a^2}{a^2 + c^2} \left( \frac{\omega^2 (a^2 + c^2)}{a^2 c^2} - 1 \right)$$  \hspace{1cm} (9.56)

which can be written as

$$\omega^2 = (k^2 \frac{a^2 c^2}{a^2 + c^2} - \epsilon c^2 l^2)$$  \hspace{1cm} (9.57)

From (9.57) it becomes obvious that the mode that is generated near the cusp resonance is a slow magneto-acoustic wave in the limit of a large wavenumber normal to the magnetic field field and which is modified with the term $-\epsilon c^2 l^2$.

9.2 The term $\frac{m_i m_e}{e^2} \frac{\partial \mathbf{J}}{\partial t}$

We now take the term $\frac{m_i m_e}{e^2} \frac{\partial \mathbf{J}}{\partial t}$ in (9.1) into account. The equations of concern are then given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$  \hspace{1cm} (9.58)
As it becomes clear from equation (9.60), Alfvén waves decouple from the magnetoo acoustic gravity waves if there is no propagation in the y-direction. As usual we will assume that the various quantities in the equations above are a superposition of an atmospheric mean state of rest with a constant magnetic field \( \mathbf{B} \) along the x-axis and a perturbation which we will denote by \( \epsilon_1 \) where \( \epsilon \) stands for a quantity from the equations above. After linearizing we obtain the following system of equations

\[
\rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla(p + \frac{B^2}{2\mu_0}) + \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B} + \rho g \tag{9.59}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{v} \times \mathbf{B}
\]

\[
-\frac{m_i m_e}{\epsilon^2} \frac{1}{\rho} \frac{\partial}{\partial t} (\nabla^2 \mathbf{B} + \frac{1}{L^2} [\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}] e_y - (\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial x}) e_x) \tag{9.60}
\]

\[
\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0 \tag{9.61}
\]

9.2.1 Alfvén mode

We assume that a perturbation behaves like

\[
f_1 = f(kx, \omega, z) \exp (ikx - \omega t) \tag{9.66}
\]

Furthermore we set \( \frac{\partial f}{\partial y} = 0 \), and we fourier transform the equations (9.62) to (9.65) according to (9.66). After some trivial algebraic manipulations, we can then derive the following differential equation for \( v_y \)

\[
\epsilon \frac{d^2 v_y}{dz^2} + \frac{1}{L} \frac{dv_y}{dz} + k^2 [1 - \frac{\omega^2}{a^2 k^2} + O(\epsilon)] v_y = 0 \tag{9.67}
\]

with

\[
\epsilon = \frac{\omega_i}{\omega_e} \tag{9.68}
\]

\( \omega_i \) and \( \omega_e \) are the ion and the electron gyrofrequency respectively.
9.2.2 Inner equation

By including the term $\frac{m_v m_x x A}{\varepsilon}$ we obtain a second order differential equation describing the motion in the $y$-direction. We approximate the solution of (9.67) by means of boundary layer theory. We make therefore use of the smallness of $\varepsilon$. After transforming (9.67) according to

$$\xi = -\xi_0 \exp\left(\frac{z}{L}\right) + 1$$

(9.69)

where

$$\xi_0 = \frac{k^2 a_0^2}{\omega^2}$$

(9.70)

we obtain the following differential equation for $v_y$

$$\varepsilon(\xi - 1)^2 \frac{d^2 v_y}{d\xi^2} + 2\varepsilon(\xi - 1)^2 \frac{dv_y}{d\xi} - k^2 L^2 (1 + O(\varepsilon)) v_y = 0$$

(9.71)

By applying boundary layer theory, it's easily seen that the inner equation is given by

$$\frac{d^2 v_y}{d\zeta^2} - k^2 L^2 \zeta v_y = 0$$

(9.72)

where $\zeta$ is given by

$$\zeta = \varepsilon^{-\frac{1}{4}} \xi$$

(9.73)

As we already saw in previous chapters, the physical solution of (9.72) is given by the Airy function

$$v_y = C_0 \text{Ai}(\zeta(kL)^{\frac{3}{2}})$$

(9.74)

The asymptotic behaviour of $v_y$ for $\zeta \to \infty$ is given by [17]

$$v_y \sim \frac{1}{2\sqrt{\pi}} (kL)^{-\frac{1}{4}} \zeta^{-\frac{1}{4}} \exp\left(-\frac{2}{3} k L \zeta^{\frac{3}{2}}\right)$$

(9.75)

For $\zeta \to -\infty$ the asymptotic behaviour is given by

$$v_y \sim \frac{1}{\sqrt{\pi}} (kL)^{-\frac{1}{4}} \zeta^{-\frac{1}{4}} \left[\sin\left(\frac{2}{3} k L \zeta^{\frac{3}{2}}\right) - \cos\left(\frac{2}{3} k L \zeta^{\frac{3}{2}}\right)\right] = \sqrt{\frac{2}{\pi}} (kL)^{-\frac{1}{4}} \zeta^{-\frac{1}{4}} \sin\left(\frac{2}{3} k L \zeta^{\frac{3}{2}}\right)$$

(9.76)

For a region far from the resonance we use the WKB method to approximate the solution of (9.67). Following the procedure of the previous section, we can derive that the WKB solution of (9.67) is given by

$$v_y \sim \exp\left(\int^{z} \phi_1 dz'\right) \times \exp\left(\int^{z} \varepsilon^{-\frac{1}{2}} \phi_0 dz'\right)$$

(9.77)
where

\[ \phi_0 = \pm \sqrt{-k^2(1 - \frac{\omega^2}{a^2k^2})} \]  
(9.78)

\[ \phi_1 = -\frac{1}{2} \frac{1}{L}(\frac{1}{a^2k^2} - \frac{\omega^2}{\omega^2}) \]  
(9.79)

The asymptotic solutions (9.75) and (9.76) must be matched to the approximate WKB solutions which actually represent incoming and outgoing waves. In the neighbourhood of \( \zeta = 0 \) and for \( \zeta < 0 \) the WKB solution is given by

\[ v_y = e^{-\frac{i}{\sqrt{2}}|\zeta|^{\frac{1}{2}}}[B_1 \exp i \left(\frac{2}{3}kL|\zeta|^{\frac{3}{2}}\right) + B_2 \exp \left(-i\frac{2}{3}kL|\zeta|^{\frac{3}{2}}\right)] \]  
(9.80)

while for \( \zeta > 0 \) and in the neighbourhood of \( \zeta = 0 \) the WKB solution is given by

\[ v_y = e^{-\frac{i}{\sqrt{2}}|\zeta|^{\frac{1}{2}}}[C_1 \exp \left(\frac{2}{3}kL|\zeta|^{\frac{3}{2}}\right) + C_2 \exp \left(-\frac{2}{3}kL|\zeta|^{\frac{3}{2}}\right)] \]  
(9.81)

From (9.81) and (9.75) it follows that we must require that \( C_1 = 0 \). Furthermore, from (9.80) and (9.76) it follows that \( B_1 = B_2 \). This principal result indicates that there is total reflection, since the amplitudes of the incoming and outgoing waves are the same.

### 9.3 Magnetoacoustic-gravity mode

After fourier transforming equations (9.58) to (9.61) according to (9.66), we can derive a fourth order differential equation for \( v_z \). This equation is given by

\[ \varepsilon \frac{d^4v_z}{dz^4} - \frac{3}{L} \frac{d^3v_z}{dz^3} + k^2 \left(\frac{\omega^2(c^2 + a^2)}{k^2c^2a^2} - 1 + O(\varepsilon)\right) \frac{d^2v_z}{dz^2} 
+ \frac{k^2}{L} \left(O(\varepsilon) - \frac{\omega^2}{k^2a^2}\right) \frac{dv_z}{dz} + k^4 \left(O(\varepsilon) + \frac{\omega^2}{k^2c^2} + (1 - \frac{\omega^2}{k^2a^2})(1 - \frac{\omega^2}{k^2c^2})\right)v_z = 0 \]  
(9.82)

where \( \varepsilon \) is given by (9.68). By performing the familiar transformations

\[ \zeta = 1 - \zeta_0 \exp(-\frac{z}{L}) \]  
(9.83)

\[ v_z = \phi(-kz) \]  
(9.84)

equation (9.82) becomes

\[ \varepsilon(\zeta - 1) \frac{d^4\phi}{d\zeta^4} - \varepsilon(1 - \zeta)^3(9 + 4kL) \frac{d^3\phi}{d\zeta^3} - \Gamma^2(\zeta + O(\varepsilon))(1 - \zeta) \frac{d^2\phi}{d\zeta^2} 
+ (1 - (\alpha + \beta + 1)\zeta + O(\varepsilon))\frac{d\phi}{d\zeta} - \alpha\beta\phi = 0 \]  
(9.85)
where

\[ \Gamma^2 = k^2 L^2 \left(1 - \frac{\omega^2}{k^2 c^2}\right) \]  

(9.86)

\( \alpha \) and \( \beta \) are given by (9.18) and (9.20). We use the smallness of \( \varepsilon \) to approximate the solution of (9.85). In the outer region, we can approximate equation (9.85) by

\[ \zeta (1 - \zeta) \frac{d^2 \phi}{d\zeta^2} + [1 - (\alpha + \beta) \zeta] \frac{d\phi}{d\zeta} - \alpha \beta \phi = 0 \]  

(9.87)

This equation has already been discussed in chapter 3. There we saw that in first order its solution behaves like

\[ \phi \sim A_1 + A_2 \ln(\zeta) \quad \zeta \to 0 \]  

(9.88)

By performing a stretching of the \( \zeta \) coordinate according to

\[ \zeta = \frac{r}{\varepsilon^\nu} \]  

(9.89)

it is easily seen that for the distinguished limit we have \( \nu = \frac{1}{3} \). The inner equation is then given by

\[ \frac{d^4 \phi}{d\tau^4} - \Gamma^2 (\frac{d^2 \phi}{d\tau^2} + \frac{d\phi}{d\tau}) = 0 \]  

(9.90)

By performing the transformation

\[ \tau = -\Gamma^{\frac{3}{2}} x \]  

(9.91)

equation (9.90) becomes

\[ \frac{d^4 \phi}{dx^4} - \frac{d}{dx} \left( x \frac{d\phi}{dx} \right) = 0 \]  

(9.92)

In the previous section we saw that the bounded solution of (9.90) is given by

\[ \phi(x) = D_1 \int_0^{\frac{3}{2}} \text{Ai}(x)dx + D_3 \int_0^{\frac{3}{2}} \text{Gi}(x)dx + D_4 \]  

(9.93)

In order to match the inner solution (9.93) with the outer solution, we need the asymptotic behaviour of (9.93) as \( \tau \to \pm \infty \). According to the previous section the asymptotic behaviour is given by

\[ \phi(\tau) = D_4 + D_4 \left[ \frac{2}{3} - \pi^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} \cos \left( \frac{2}{3} \Gamma^{\frac{3}{2}} + \frac{\pi}{4} \right) \right] - D_3 \left[ \frac{2\gamma + \ln(3)}{3\pi} + \frac{1}{\pi} \ln(\Gamma^{\frac{3}{2}} |\tau|) - \pi^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} |\tau|^{-\frac{3}{2}} \sin \left( \frac{2}{3} \Gamma |\tau|^{\frac{3}{2}} + \frac{\pi}{4} \right) \right] \quad \tau \to -\infty \]  

(9.94)

(9.95)
\[ \phi(\tau) = D_4 - \frac{1}{3} D_1 - D_3 \left( \frac{2\tau^* + \ln(3)}{3\pi} + \frac{1}{\pi} \ln(\tau \Gamma^3) \right) \quad \tau \to \infty \quad (9.96) \]

Using the matching procedure given by (9.34) we can easily deduce that for \( \tau \to -\infty \) we have

\[ \phi(\tau) = D_4 - D_3 i \pi^{-\frac{1}{2}} |\tau|^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} \exp(i \frac{2}{3} \tau^2 \Gamma + \frac{\pi}{4}) \]

\[ + D_3 \left( \frac{2}{3} i - \frac{2\tau^* + \ln(3)}{3\pi} + \frac{1}{\pi} \ln(\tau \Gamma^3) \right) \quad (9.97) \]

The subdominant term in (9.97) is the new mode that is generated in the resonant layer by the process of mode conversion. In the next section, we will see that it actually represents a modified slow magnetoacoustic wave in the limit of a large wavenumber normal to the magnetic field.

### 9.3.1 Identification of the new mode

Using the WKB method that we applied in the previous chapter, it can easily be shown that the analytic continuation of the new mode that is generated in the resonant layer is given by

\[ v(z) \sim \exp \left( \int_0^z \phi_1(z') \, dz' \right) \times \exp \left( i \int_0^z \left| \phi_0(z') \right| \, dz' \right) \quad (9.98) \]

where \( \phi_0 \) and \( \phi_1 \) are determined by

\[ \phi_0^2 = -k^2 \left( \frac{\omega^2 (c^2 + a^2)}{k^2 a^2 c^2} - 1 \right) \quad (9.99) \]

\[ \frac{6\phi_0}{dz} + 4\phi_1 \phi_0 = \frac{3}{L} \phi_0^3 - \frac{\omega^2}{La} \phi_0 \]

\[ + k^2 \left( \frac{\omega^2 (c^2 + a^2)}{k^2 a^2 c^2} - 1 \right) \left( \frac{d\phi_0}{dz} + 2\phi_1 \phi_0 \right) = 0 \quad (9.100) \]

From (9.99) it is quite obvious that the new mode propagate only below the cusp resonance. A result which is already obtained by boundary layer theory. Furthermore, it follows from (9.99) and (9.100) that \( \phi_1 \) is real valued.

Obviously as it becomes clear from (9.98) the dispersion relation for governing the new mode is given by

\[ \epsilon l^2 = k^2 \left( \frac{\omega^2 (c^2 + a^2)}{k^2 a^2 c^2} - 1 \right) \quad (9.101) \]

where \( l \) is the local wavenumber of the mode. Expression (9.101) can be written as

\[ \omega^2 = \frac{c^2 a^2}{c^2 + a^2} \epsilon l^2 + \frac{c^2 a^2 k^2}{c^2 + a^2} \quad (9.102) \]
from (9.102) it follows that the mode that is generated by means of mode conversion is actually a slow magneto-acoustic wave in the limit of a large wavenumber normal to the magnetic field and whose dispersion relation is modified with the term $cl^2 \frac{c^2 - a^2}{c^2 + a^2}$. 
Chapter 10

Summary and Conclusions

In an atmosphere that is permeated with a magnetic field, there are typically four modes of wave motions, driven by different restoring forces. The mag­netic tension can drive so-called Alfven waves. The magnetic pressure, the plasma pressure and gravity can act separately and generate compressional Alfven waves, sound waves and gravity waves, respectively; but, when acting together, these three forces produce so-called magnetoacoustic-gravity modes.

In chapter 5 it’s shown that in an atmosphere permeated with a horizontal magnetic field (in the z-direction) ideal MHD equations can possess two singu­larities:

1-The cusp singularity:

This singularity occurs when we try to describe magnetoacoustic-gravity waves with ideal MHD. Its location is determined by the condition $c^2a^2k^2/(c^2 + a^2) = \omega^2$. $a$ is the Alfven velocity, and $c$ is the sound speed. At this singularity the z-component of the velocity perturbation possesses a logarithmic singularity. An incoming magnetoacoustic-gravity wave with a given frequency $\omega$ and a given wavenumber $k$ within the so-called cusp continuum will be reflected. Part of the energy of the wave will be absorbed and the energy fluxes of the incoming and outgoing wave will be different. This means that absorption takes place, even without including any dissipation mechanism. Energy is therefore accumulated ad infinitum.

A realistic treatment of the cusp singularity should include for example some dissipation mechanism. In the present report, we investiagted the effect of re­sistivity, viscosity and thermal conductance on the cusp resonance. We also included some terms in a generalized Ohm’s law. Each of these nonideal terms is investigated separtely.
In the ideal case, magnetoacoustic-gravity waves are described by a second order singular differential equation. By including one of these nonideal MHD terms, it's shown, that the order of the differential equation is raised by two. Furthermore, the obtained fourth-order differential equations are regular. So our first conclusion is: the cusp resonance is resolved by including one of these nonideal terms. The mathematical price that we have to pay for the resolution of the cusp resonance, is that we now has to deal with fourth-order differential equations, which are very complicated. Instead of trying to solve them exactly, we use singular perturbation techniques, or, more specifically, boundary layer theory, to approximate their solutions in some asymptotic sense. One important condition for the application of this technique is the assumption that the nonideal terms under consideration, are only important in a narrow layer (so-called inner region) around the cusp resonance. Outside this narrow layer, i.e., within the so-called outer region, magnetoacoustic-gravity waves are accurately described by the equations of ideal MHD.

The fact that we obtain a fourth-order differential equation means that new solutions are introduced and that mode conversion takes place. In the inner region the complicated fourth order differential equations are approximated by so-called inner equations. In this report it's shown that in all cases we obtain the same inner equation. These leads us to conclude that the new mode that is generated near the cusp resonance is the same for all the nonideal terms that we included. Indeed, by using the WKB method, it's demonstrated that mode conversion takes place near the cusp resonance and that the new mode is always a modified slow magneto-acoustic wave.

2-Alfvén singularity

We encounter this singularity, when ideal MHD is used to describe Alfvén waves in a stratified medium permeated with a horizontal field. The location of this singularity is determined by the condition $\omega^2 - k^2a^2 = 0$. Because of the geometry that is chosen in this report, the Alfvén wave is in the ideal case described by a zeroth order "differential equation". Except thermal conductivity, all the nonideal terms that we included resolve the Alfvén singularity by raising the order of the original differential equation by two. Using the WKB approximation, we found that that the solutions of the obtained second-order differential equation can be interpreted as Alfvén waves modified by the nonideal term in concern. The fact that thermal conductivity doesn't resolve the Alfvén singularity, is not so surprising, since the energy equation is not involved in the mechanism of Alfvén waves. These waves are a result of magnetic tension alone.
Appendix A

The solution of equation (6.56)

We will solve the differential equation

\[ \frac{d^4 y}{dx^4} + 2 \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0 \quad (A.1) \]

by means of the generalized Laplace method. We then seek a solution of the form

\[ y(x) = \int_C f(t) \exp(x t) dt \quad (A.2) \]

With the contour C and the function f(t) yet to be determined. Filling (A.2) into (A.1) yields

\[ [f(t)t^2 \exp(x t)dt]_C - \int_C t[t \frac{df}{dt} + f(t)(1 - it^3)] \exp(x t)dt = 0 \quad (A.3) \]

Here, the first term is to be interpreted as the difference of the values of f(t)t^2 exp(x t) at the endpoints of the contour C (or the difference of the limiting values when these endpoints are at infinity). In order to satisfy (A.1), we first require that f(t) satisfy the differential equation

\[ t \frac{df}{dt} + f(t)(1 - it^3) = 0 \quad (A.4) \]

Once f(t) is determined, the next step is to select C so that the endpoints contributions are zero as well. The solution of equation (A.4) is given by

\[ f(t) = \frac{\exp\left(\frac{1}{2}it^3\right)}{t} \quad (A.5) \]
Hence we must choose $C$ so that
\[
[t \exp\left(i \frac{1}{3} t^3 + xt\right)]_C \tag{A.6}
\]
vanishes at its points. We choose the contour to begin in 0 and to end at infinity. It's easily seen that for all $x$, we may choose $C$ to be any infinite contour which starts at zero and ends at infinity in one of sectors given in figure A.1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{The four possible contours each representing a independent solution for the equation A.1.}
\end{figure}

We have then the following four solutions
\[
y_1(x) = c_1 \int_0^{\infty + i 0} \frac{exp\left(\frac{1}{3} i t^3 \left(\exp(x t) - 1\right)\right)}{t} dt \tag{A.7}
\]
\[
y_2(x) = c_2 \int_0^{\infty - i 0} \frac{exp\left(-\frac{1}{3} i t^3 \left(1 - \exp(-x t)\right)\right)}{t} dt \tag{A.8}
\]
\[
y_3 = c_3 \int_0^{\infty} \frac{exp\left(-\frac{1}{3} i t^3 \left(\exp(-x t) - 1\right)\right)}{t} dt \tag{A.9}
\]
\[
y_4 = c_4 \int \frac{exp\left(-\frac{1}{3} i t^3 + x t\right)}{t} dt = c_4 \text{Res}\left(\frac{exp\left(-\frac{1}{3} i t^3 + x t\right)}{t}\right) = c_4 \tag{A.10}
\]
c_1, c_2, c_3 and $c_4$ are integration constants. It is obvious that for $x \to \pm \infty$ the only bounded solution is given by
\[
y_1(x) = c_1 \int_0^{\infty + i 0} \frac{exp\left(\frac{1}{3} i t^3 \left(\exp(x t) - 1\right)\right)}{t} dt + c_4 \tag{A.11}
\]
Appendix B

The asymptotic behaviour of (A.11)

In order to investigate the asymptotic behaviour of (A.11) for \(x \to \infty\), we introduce the following integral

\[
I_\delta(x) = \frac{1}{t^\delta} \int_0^\infty \exp\left(-\frac{1}{3}t^3\right) \left(\exp(-ixt) - 1\right) dt
\]  

The desired behaviour is then obtained via

\[
I(x) = \lim_{\delta \to 1} I_\delta
\]

The asymptotic behaviour of (B.1) can be investigated by means of so-called steepest descent method. The details of this technique are discussed in [17]. It is a method which can be used to study the asymptotic behaviour of integrals of the form

\[
S(\lambda) = \int_C g(z) \exp(\lambda w(z)) dz
\]

Here \(C\) is a fixed contour in the complex \(z\) plane, while \(g(z)\) and \(w(z)\) are analytic functions in some region \(D\) that includes \(C\). It’s also allowed that \(g\) and \(w\) have singularities at the endpoints of \(C\) as long as \(S(\lambda)\) is convergent.

In order to apply the steepest descent method to our case, we write (B.1) as

\[
I_\delta = \int_0^\infty \frac{\exp\left(-\frac{1}{3}t^3 - ixt\right)}{t^\delta} dt - \int_0^\infty \frac{\exp\left(-\frac{1}{3}t^3\right)}{t^\delta} dt
\]

The last integral in B.4 can be calculated exactly. It can easily be shown that

\[
M = \int_0^\infty \frac{\exp\left(-\frac{1}{3}t^3\right)}{t^\delta} dt = 3^{-\frac{1}{3}\delta} \Gamma\left(1 - \frac{1}{3}(\delta + 2)\right)
\]
For the study of the asymptotic behaviour of

\[ K(x) = \int_0^\infty \frac{\exp(-\frac{1}{2}t^3 - ixt)}{t^\delta} \, dt \]  

we put this integral in the form (B.3) by means of the transformation \( t = \sqrt{x}s \), we then obtain

\[ k(x) = (\sqrt{x})^{1-\delta} \int_0^\infty \frac{\exp(\Lambda(-\frac{1}{3}s^3 - is))}{s^\delta} \, ds \quad \Lambda = x\sqrt{x} \]  

(B.7)

The saddle points of \( w(s) = -\frac{1}{3}s^3 - is \) are given by

\[ s = \pm \exp(-\frac{1}{4}\pi i) \]  

(B.8)

The saddle point \(-\exp(\frac{1}{4}\pi i)\) is not admissible [17], so that the contribution to the asymptotic behaviour comes only from \( \exp(-\frac{1}{4}\pi i) \). From [17], it follows that this contribution is given by

\[ K_1 = \exp\left(\frac{3}{8}\pi i\right)\sqrt{\left(\frac{\pi}{\Lambda}\right)} \exp\left(-\frac{2}{3}\Lambda(1-i)\right) \quad x \to \infty \]  

(B.9)

Up to now, we have calculated the contribution of the saddle point \( \exp(-\frac{1}{4}\pi i) \). We must also take into account the contribution of the endpoint \( s = 0 \) to the asymptotic behaviour of (B.7). This is done by using so-called Watson's lemma[17]. Using this lemma it is straightforward to show that the contribution from point 0 is given by

\[ K_2 = \frac{\Gamma(1-\delta)}{x^{1-\delta} - \frac{1}{3} \frac{\Gamma(4-\delta)}{x^{4-\delta}}} + O(x^{-7+\delta}) \]  

(B.10)

The asymptotic integral of \( I(x) \) is now given by

\[ I(x) = \lim_{\delta \to 1}(K_1 + K_2 - M) \]  

(B.11)

It can be shown by straightforward calculation that this limit is given by

\[ I(x) = -\ln(x) - \frac{1}{2} \pi i - \frac{2}{3} \gamma^* - \frac{1}{3} \ln(3) + \exp\left(\frac{3}{8}\pi\right)\sqrt{\left(\frac{\pi}{\Lambda}\right)} \exp\left(-\frac{2}{3}\Lambda(1-i)\right) \quad x \to \infty \]  

(B.12)

The asymptotic behaviour for \( x \to -\infty \) is the complex conjugate of (B.12).
References


