Eindhoven University of Technology

MASTER

Modeling and simulation of a laser deflecting system

Goossens, H.J.

Award date:
1995

Disclaimer
This document contains a student thesis (bachelor's or master's), as authored by a student at Eindhoven University of Technology. Student theses are made available in the TU/e repository upon obtaining the required degree. The grade received is not published on the document as presented in the repository. The required complexity or quality of research of student theses may vary by program, and the required minimum study period may vary in duration.

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
Master's Thesis:

Modeling and simulation of a laser deflecting system

H.J. Goossens

Coach : dr. ir. A.A.H. Damen
Supervisor : Prof. dr. ir. P.P.J. van den Bosch
Date : July 1995
Abstract

In this report a model of Differential and Algebraic Equations (DAE) for the dynamics of a rotating laser deflecting system, which is a multibody system, is derived. Classical numerical integration software isn't however able to solve these equations, so the possibility to transform the model into a set of Ordinary Differential Equations (ODE) is investigated. Elimination of the algebraic equations leads to an unmanageable model or yield a stiff numerical problem. This results in a very time consuming simulation. Therefore techniques to solve DAE system directly are examined. First some characteristics of DAE's like their index and structure are explained. It appears that multibody systems always result in an index three DAE. Then the possibilities classical (ODE based) techniques like Backward Differential and Runge-Kutta methods are discussed. Also special techniques for index three DAE's are mentioned: index reduction, Baumgarte stabilization, projection method, stabilizing Lagrange multipliers and state space transformation. The backward difference based code DASSL is used to simulate the rotating system. The results are accurate and obtained much faster than by the above mentioned simulation technique. This makes it possible to compare the behaviour of the nonlinear system to that of the linearized system that is also derived in this report. Only for very small angles the linear model is a good approximation. For large angles there are considerable differences. Furthermore the sensors of the deflecting system are discussed. It appears that these sensors can give rise to unaccuracy of the 3-D position measurement system in which the deflecting system is implemented.
1 Introduction

2 Technical details of the laser deflecting system
   2.1 The mirror's semi sphere
   2.2 The actuators
   2.3 The sensors
      2.3.1 The angle sensors
      2.3.2 The airgap sensor

3 Mechanical description of the mirror
   3.1 Definition of frames
   3.2 Rotational dynamics of the semi sphere
   3.3 Translational dynamics
   3.4 Actuator dynamics
   3.5 Mechanical dynamics of the complete system
   3.6 Euler-Lagrange description

4 Simulation problems
   4.1 Assistant dynamics
   4.2 Solving the DAE system

5 Solving the DAE of a constrained mechanical system
   5.1 Some details on DAE's
      5.1.1 The index
      5.1.2 Structure
      5.1.3 Relation to stiff ODE's
   5.2 Solving DAE's using ODE methods
      5.2.1 BDF methods
      5.2.2 Runge-Kutta methods
   5.3 Special methods for constrained mechanical system DAE's
      5.3.1 Index reduction
      5.3.2 Baumgarte stabilization
      5.3.3 Projection techniques
      5.3.4 Solving the overdetermined system
      5.3.5 Adding extra Lagrange multipliers
      5.3.6 State space transformation
   5.4 Conclusions
Chapter 1

Introduction

The Measurement and Control Group of the Department of Electrical Engineering at Eindhoven University of Technology is doing research on modelling, identification and control of processes. One of these processes is a laser tracking system.

By means of a revolving mirror, a reflector can be followed by a laser beam. In this way the movements of the tool center point (TCP) of a robot can be measured or calibrated. An overview of the tracking system is sketched in figure 1.1.

![Figure 1.1: Schematic view of the laser tracking system](image)

The laser beam is pointed at the rotatable mirror's center and then deflected to the reflector on the robot arm. This reflector is a retroreflector: it returns the laser beam parallel to the incoming beam, independent of the angle of incidence. So if the laser beam doesn't point at the center of the reflector, it is returned parallel to the incoming beam back to the mirror. The reflected laser beam is split in two by a half-way mirror. One part passes the mirror and is received in the interferometer. This device is coupled to the laser and can measure the length the laser beam has travelled. The other part of the laser beam is deflected on a sensor that measures the position of the incoming laser beam in two coordinates. This represents the deviation of the reflected laser beam from the reflectors center. The tracking controller generates new setpoints for the mirror so that the laser beam is pointed at the center of the reflector. The sensors coupled to the rotatable mirror measure the real pointing angles of the mirror. With these two angles and the length of the laser beam the position of the TCP can be calculated.
The tracking system should meet two goals. First it should measure the TCP with high accuracy and second it should follow the reflector even when the robot is moving at a high speed. This is mainly possible because the low mass mirror can rotate with very low friction as it moves on an air bearing.

Both the angles of the mirror and the air bearing's height can be controlled by three actuators. The technical details will follow in the next chapter. For the accuracy of the measurements performed by the tracking system it is very important that the airgap's height has a constant value and that the angles are adjusted accurately. Therefore two controllers are to be designed. The complete system including these extra controllers can be represented by the block scheme of figure 1.2.

To design appropriate controllers, a good mechanical model of the mirror is needed. The differential equations modelling the dynamics of the mirror were partly developed in [32]. A simulation in Simulink based on this model and a linearized model are shown in [31]. Future controller designs could be tested on the now available simulation. Unfortunately the simulation model turns out to be a stiff numerical problem. This will be explained in chapter 4. In practice stiffness means that simulation of a small interval takes a very long time. Because it is very unattractive to use this stiff model for testing controllers, it was decided to search for an other simulation model. This model doesn't consist of a set of Ordinary Differential Equations (ODE), but of a system of both Differential and Algebraic Equations (DAE). From the literature it is known that mechanical systems can be modelled easily by DAE's, but that in general it is very difficult to solve them numerically. The main goal of the project is to investigate if modelling the mirror by a DAE is a good solution to the simulation problems. Some characteristics of differential/algebraic equations will be shown as well as several methods to solve them. Besides a linear model will be derived and it will be investigated by comparing the simulations, if this model is a good approximation of the nonlinear model. Then we can use it as a basis for controller design.
Chapter 2

Technical details of the laser deflecting system

In this chapter the mirror's construction will be shown in detail, so that the mechanical model can be derived in the next chapter. The mirror consists of a steel semi sphere which can rotate on an air bearing. The mirror is driven by three electromagnetic actuators which turn the semi sphere via strings. These very stiff strings are connected to a plastic ring that is placed on top of the mirror's surface. A schematic view is given in figure 2.1.

![Figure 2.1: Schematic view of the laser deflecting system](image)

The plastic ring will be ommitted in further models. Because its mass is very small compared to that of the steel semi sphere, it doesn't play an important role in the behaviour of the mirror. Only the dotted lines drawn between the mirror's center and the connection of the strings will remain in later models of the system.

From figure 2.1 we can see that the actuators can control both the orientation of the mirror and the airgap's height. When all three strings are pulled at the same time, the airgap is reduced. A further description of the actuators will be given in paragraph 2.3. The pretensioning springs prevent the mirror from falling out of its seat.
[17] Bae, D. and Yang, S.
A STABILIZATION METHOD FOR KINEMATIC AND KINETIC CONSTRAINT EQUATIONS

[18] Haug, E. and Yen, J.
GENERALIZED COORDINATE PARTITIONING METHODS FOR NUMERICAL INTEGRATION OF DIFFERENTIAL-ALGEBRAIC EQUATIONS OF DYNAMICS
Berlin: Springer-Verlag, 1991, p. 97-114

[19] Gear, W.
AN INTRODUCTION TO NUMERICAL METHODS FOR ODEs AND DAEs
Berlin: Springer-Verlag, 1991, p. 115-126

[20] Steigerwald, M.
BDF METHODS FOR DAEs IN MULTI-BODY DYNAMICS: SHORTCOMINGS AND IMPROVEMENTS
Berlin: Springer-Verlag, 1991, p. 345-352

[21] Lötstedt, P. and Petzold, L.R.
NUMERICAL SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH ALGEBRAIC CONSTRAINTS I: CONVERGENCE RESULTS FOR BACKWARD DIFFERENTIATION FORMULAS

[22] Lötstedt, P. and Petzold, L.R.
NUMERICAL SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH ALGEBRAIC CONSTRAINTS II: PRACTICAL IMPLICATIONS

[23] Gear, C.W., Leimkuhler, B. and Gupta, G.K.
AUTOMATIC INTEGRATION OF EULER-LAGRANGE EQUATIONS WITH CONSTRAINTS
2.1 The mirror's semi sphere

In this paragraph all important parameters of the mirror's semi sphere will be discussed. An overview is given by the schematic top and side views in figure 2.2. The figure shows the mirror and one of its actuators in equilibrium position.

![Diagram showing the mirror's semi sphere and an actuator in equilibrium position.](image)

**Figure 2.2: Top and side view of the mirror and one actuator**

From the top view can be seen that the actuators are placed around the mirror with a mutual angle of $120^\circ$. The meaning of the other parameters in this figure and their values are described in table 2.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>7.0</td>
<td>mm</td>
<td>Distance from centre of mirror to connection of actuator string. This parameter is equal for each actuator</td>
</tr>
<tr>
<td>$R_m$</td>
<td>5.0</td>
<td>mm</td>
<td>Radius of semi sphere</td>
</tr>
<tr>
<td>$j$</td>
<td>$5.6 \cdot 10^{-8}$</td>
<td>kgm$^2$</td>
<td>Moment of inertia around the centre of the mirror's surface</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>$\pi/12$</td>
<td>rad</td>
<td>Angle between arm of actuator and the mirror surface</td>
</tr>
<tr>
<td>$l$</td>
<td>25.0</td>
<td>mm</td>
<td>Lenght of string connecting the mirror to the actuator</td>
</tr>
<tr>
<td>$d$</td>
<td>-</td>
<td>$\mu$m</td>
<td>Height of air gap between mirror and bearing seat</td>
</tr>
<tr>
<td>$a_i$</td>
<td>-</td>
<td>-</td>
<td>Connection point of the string at the mirror</td>
</tr>
<tr>
<td>$A_i$</td>
<td>-</td>
<td>-</td>
<td>Connection point of the string at the actuator</td>
</tr>
</tbody>
</table>

Notice that the actuator string and the arm are perpendicular in the equilibrium position. When the mirror is rotated, the angle changes. The air bearing of the mirror gives rise to a very low friction. Therefore no friction parameter is included in the model.
2.2 The actuators

The three electromagnetic actuators are based on the principle of a loudspeaker. A coil moves in a magnetic field as a result of a current through the coil. A schematic view of an actuator cross-section is given in figure 2.3.

![Figure 2.3: Cross-section of an actuator](image)

An overview of the parameters used in figure 2.3 is given in table 2.2. The actuator dependent parameters are indicated by a subscript \( i \).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_i )</td>
<td>-</td>
<td>-</td>
<td>Equilibrium position of top of actuator</td>
</tr>
<tr>
<td>( h_i )</td>
<td>-</td>
<td>mm</td>
<td>Movement of actuator relative to offset position ( O_i )</td>
</tr>
<tr>
<td>( C_i )</td>
<td>-</td>
<td>-</td>
<td>Direction of actuator movement (unit vector)</td>
</tr>
<tr>
<td>( m )</td>
<td>9.5 ( \times ) ( 10^{-3} ) kg</td>
<td></td>
<td>Mass of coil</td>
</tr>
<tr>
<td>( F_0 )</td>
<td>0.5 N</td>
<td></td>
<td>Pretensioning force</td>
</tr>
<tr>
<td>( K )</td>
<td>40 N/m</td>
<td></td>
<td>Spring constant</td>
</tr>
<tr>
<td>( F_i )</td>
<td>- N</td>
<td></td>
<td>Tensile force exerted by the string on the actuator</td>
</tr>
<tr>
<td>( I_i )</td>
<td>- A</td>
<td></td>
<td>Current to control actuator</td>
</tr>
<tr>
<td>( V_i )</td>
<td>- V</td>
<td></td>
<td>Voltage to control actuator</td>
</tr>
<tr>
<td>( L )</td>
<td>10(^{-3}) H</td>
<td></td>
<td>Induction of the coil</td>
</tr>
<tr>
<td>( R )</td>
<td>3.5 ( \Omega )</td>
<td></td>
<td>Resistance of the coil</td>
</tr>
<tr>
<td>( A )</td>
<td>3.16 kg( \cdot )m( \cdot )s(^{-2})( \cdot )A(^{-1})</td>
<td></td>
<td>Actuator constant depending on coil radius, number of windings and magnitude of the magnetic field</td>
</tr>
<tr>
<td>( A_i )</td>
<td>-</td>
<td>-</td>
<td>Position of lower end of the string</td>
</tr>
</tbody>
</table>

The permanent magnet produces a homogeneous magnetic field. In this field the coil can move up and down. This movement will be induced by a Lorenzt force which results from a current through the coil. In [31] expressions were derived for the Lorentz force when the actuator is
driven by a current as well as by a voltage:

\[ F_A = -AI \]  
\[ F_A = \frac{-AV - sA^2h}{R + sL} \]  

From figure 2.3 can be derived that the exact position of the top of the actuator \( A_i \) is given by

\[ A_i = h_i C_i + Q_i \]  

The actuators coil is also moving on an air bearing. So the friction associated with this movement is very low. The damping coefficient \( D \) will be included in the model for completeness, but will be presumed zero in simulations. The spring constant \( K \) and actuator constant \( A \) are considered to be equal for each actuator.

### 2.3 The sensors

The mirror system is equipped with several sensors as was indicated in figure 1.2. One sensor measures the rotations of the mirror. The other one measures the translational movements of the semi sphere in it's bearing seat. Both sensor systems will be explained below.

#### 2.3.1 The angle sensors

Compared to the air bearing, a mechanical angle sensor would cause a very high friction. The low friction effects of the air bearing would be canceled out. Therefore a contactless measurement system was developed. It exists of two sets of inductively-coupled coils. The primary coil is attached to the mirror on the plastic ring. The two secondary coils are positioned at the side of the mirror as can be seen in figure 2.4. In the primary coil a magnetic field \( B \) is generated which causes a flux through the secondary coils. This flux depends on the orientation of the mirror, as is shown in figure 2.5. So by measuring the current in the secondary coils induced by the flux, the rotation around the \( X' \) and \( Y' \) axes can be determined.

![Figure 2.4: Top view of the angle sensors and the mirror](image)

The relation between primary and secondary current (respectively \( I_p \) and \( I_s \)) can be considered in the following way. According to the Biot Savart rule, the contribution of a current \( I_p \) through a piece of wire \( dl \) to the magnetic field \( dB \) at a distance \( r \) (see figure 2.6) is

\[ dB = \frac{\mu}{4\pi} \frac{dl \times \hat{r}}{||r||^3} I_p \]  

(2.4)
The current through the secondary coils complies with

\[ I_s = \frac{N_s \phi_m}{L} \]  \hspace{1cm} (2.5)

where \( N_s \) is the number of windings of the secondary coils and \( L \) is the inductance. \( \phi_m \) is the magnetic flux through the coils which can be written as

\[ \phi_m = \int_S B \cdot n_s dS \]  \hspace{1cm} (2.6)

where \( n_s \) is the normal of the secondary coil windings and \( S \) is the surface inside the coil.

Using these formulas we can derive the input-output current ratio as a function of the rotation \( \xi \) around the \( X' \) axis. For this we need some assumptions:

- all coils are assumed to be very flat and the primary coil is positioned exactly in the middle of the secondary coils,
- the mirror can only rotate (vertical movement will induce a current in the secondary coils),
- the distortion of the magnetic field by the metal of the semi sphere is neglected.

With these assumptions the following non-linear expression results.

\[ \frac{I_s}{I_p} = \frac{\mu N_s N_p}{4 \pi L} \int_S \left( \int_{l_i} \frac{d\vec{l}(\xi) \times r(\xi)}{\|r(\xi)\|^3} \right) \cdot n_s dS \]  \hspace{1cm} (2.7)

where \( N_p \) is the number of windings in the primary coil. For the rotation \( \chi \) around the \( Y' \) axis we can use the same formula. Unfortunately we can't evaluate this formula analytically for \( \xi \neq 0 \).
However it can be expected that the current ratio of (2.7) can be approximated by \( \sin(\xi) \). This results from the observation that the magnetic field \( B \) is perpendicular to the mirror plane at a long distance from the primary coil. From the use of the inner produkt of \( B \) and \( \eta_s \) it follows that \( \sin(\xi) \) is a reasonable approximation. More research has to be done to find a more accurate description. Probably (2.7) has to be solved numerically. A precise formula for \( I_s \) as a function of \( \chi \) and \( \xi \) is necessary for accurate positions measurements by the complete laser tracking system. Figure 2.7 shows how the direction of the mirror plain's normal \( \eta \) is constructed from the angles \( \xi \) and \( \chi \). The normal can also be determined by \( \alpha' \) and \( \beta' \), which is the most natural way to characterize the orientation of the mirror.

![Figure 2.7: Measured angles \( \xi \) and \( \chi \) related to normal angles \( \alpha' \) and \( \beta' \)](image)

From this figure we can easily derive that the normal angles \( \alpha' \) and \( \beta' \) can be determined from \( \xi \) and \( \chi \) by

\[
\sin(\alpha') = \frac{\sin(\xi)}{\cos(\chi)} \quad (2.8)
\]
\[
\sin(\beta') = \frac{\sin(\chi)}{\cos(\xi)} \quad (2.9)
\]

### 2.3.2 The airgap sensor

For stabilization of the vertical movement of the mirror in its bearing seat we need to measure the height of the mirror above the seat. Because both mirror and bearing seat are made of metal, this can be done contactless by measuring the electrical capacitance of the airgap between mirror and bearing seat.

For the airgap capacitance \( C_m \) Zorge has derived the following formula in [35].

\[
C_m = -\ln(\cos(\alpha_e)) \frac{\varepsilon_0 \varepsilon_r 2\pi R^2}{d} \quad (2.10)
\]

where \( \alpha_e, R \) and \( d \) are defined as in figure 2.8. The bearing seat is constructed in such a way that \( \alpha_e = \arccos(3/5) \).

In the derivation of (2.10) Zorge has assumed that the horizontal movement of the mirror is neglectable. If this is not the case we get a situation as in figure 2.9, where the mirror has moved horizontally over a distance \( d_h \).

We will now examine the influence of the horizontal displacement \( d_h \) on the capacitance of the airgap. The contribution of a infinitesimal segment of the sphere \( dA \) to \( d \) capacitance is

\[
dC_m = -\frac{\varepsilon_0 \varepsilon_r dA}{d_{\alpha,\beta}} \quad (2.11)
\]
where $\alpha$ and $\beta$ are spherical coordinates defined as in figure 2.10 and $d_{\alpha,\beta}$ is the perpendicular distance between the mirror and the bearing.

Figure 2.10: Definition of variables

This distance $d_{\alpha,\beta}$ can be written as

$$d_{\alpha,\beta} = \|q - p\| - R$$  \hspace{1cm} (2.12)

Relative to origin $O$ and with the $x$ axis defined in the direction of $d_h$, $q$ and $p$ are defined as

$$q = \begin{pmatrix} R \sin(\alpha) \cos(\beta) \\ R \sin(\alpha) \sin(\beta) \\ -R \cos(\alpha) \end{pmatrix} \quad p = \begin{pmatrix} d_h \\ 0 \\ d_v \end{pmatrix}$$  \hspace{1cm} (2.13)
Now (2.12) evaluates to
\[
d_{\alpha, \beta} = R \sqrt{1 - \frac{2d_h}{R} \sin(\alpha) \cos(\beta) + \frac{d_h^2}{R^2} + \frac{d_v^2}{R^2} + 2 \frac{d_v}{R} \cos(\alpha) - R}
\]
(2.14)

Because \(d_h\) and \(d_v\) are very small (order \(10^{-5}\)) we can approximate this result by
\[
d_{\alpha, \beta} \approx \cos(\beta)d_v - \sin(\alpha) \cos(\beta)d_h
\]
(2.15)

Since we can write \(dA\) as \(R^2 \sin(\alpha)d\alpha d\beta\) the capacitance is determined by
\[
C_m = \int_0^{\alpha_c} \int_0^{2\pi} \frac{\varepsilon_\circ \varepsilon_r R^2 \sin(\alpha) d\beta d\alpha}{\cos(\beta)d_v - \sin(\alpha) \cos(\beta)d_h}
\]
(2.16)

Integrating to \(\beta\) leads to
\[
C_m = \int_0^{\alpha_c} \frac{\varepsilon_\circ \varepsilon_r 2\pi R^2 \sin(\alpha) d\alpha}{\sqrt{d_v \cos^2(\alpha) - d_h \sin^2(\alpha)}}
\]
(2.17)

According to the standard primitive
\[
\int \frac{dt}{\sqrt{t^2 - a^2}} = \ln \left( t + \sqrt{t^2 - a^2} \right)
\]
(2.18)

formula (2.17) results in
\[
C_m = \frac{\varepsilon_\circ \varepsilon_r 2\pi R^2}{\sqrt{d_v^2 + d_h^2}} \ln \left( \frac{d_v + \sqrt{d_v^2 + d_h^2}}{\cos(\alpha_c) \sqrt{d_v^2 + d_h^2} + \sqrt{\cos(\alpha_c) \cdot (d_v^2 + d_h^2) - d_h^2}} \right)
\]
(2.19)

In figure 2.11 the value of \(C_m\) is plotted as a function of \(d_h\) and \(d_v\) around the characteristic height \(d_v = 20\mu m\). From the equicapacitive lines we see that the effect of \(d_h\) on \(C_m\) is very low for small horizontal movements. The Taylor series of (2.19) shows that this effect is of second order:
\[
C_m \approx \frac{2\varepsilon \pi R^2}{d_v} \left(- \ln(\cos(\alpha_c)) + \frac{8d_h^3}{9d_v^2} \right)
\]
(2.20)

Since we expect from the theory of airbearings that the horizontal displacements of the mirror are very small, we can conclude that the \(C_m\) is rather convenient for measuring \(d_v\). However, horizontal movements of the mirror cause an uncertainty in the measurement of the airgap heigth. It should be investigated if this uncertainty has serious consequences on the measuring accuracy of the laser tracking system.
Figure 2.11: Capacitance $C_m$ as a function of $d_\nu$ and $d_h$
Chapter 3

Mechanical description of the mirror

The mirror system with its semi sphere and actuators forms a rather complex system. It contains four mutually connected bodies, each with its own dynamics. The system is therefore called a multibody system. In general the dynamics of a complete multibody system can not be derived at once. So we chose to analyse the system in pieces. First the dynamics of the semi sphere and after that the dynamics of the actuators will be investigated. Finally these will be combined to an overall description of the system. In this report only rotational dynamics of the mirror will be investigated. When considering also translational movements of the mirror in its bearing, the model becomes much more complicated. In section 3.3 this will be explained.

3.1 Definition of frames

For convenience of modelling we define several different coordinate frames [37]. First an inertial global frame $\mathcal{g}^0 = [X, Y, Z]$ is defined with its origin in the center of the mirror plane at equilibrium position. The orientation of the global frame in relation to the connection points of the actuator strings is displayed in figure 3.1. In appendix A the vectors $\mathbf{a}_i (i = 1, 2, 3)$ are defined.

In this global frame we can characterize the mirror surface’s unit normal vector $\mathbf{n}$ by its angles $\alpha$ and $\beta$ (see figure 3.2). Remark that we can compute the angles $\alpha$ and $\beta$ by a linear transformation from the measured angles $\alpha'$ and $\beta'$.

To model the dynamics of the mirror a local frame $[x, y, z]$ is defined, with its origin in the centre of the mirror plane, but moving with the mirror. The movements of the mirror can now
be separated into different parts: the translational movement of the origin of the local frame and on the other hand the rotation of the frame. The advantage of such a local frame connected to a body is that every point on the body has a constant position in the local frame. As a matter of fact determining $a_i$ in the local frame becomes very easy. In equilibrium position the local frame is equal to the global frame.

A well known way to describe the orientation of the local frame compared to the global frame is to use three Bryant angles $\psi, \theta$ and $\phi$ which are stored in the vector $\gamma$ [37]. The orientation of the local frame is obtained in the following way.

First the global frame is rotated $\psi$ around the $X$-axis to obtain the frame $[X', Y', Z']$. Next this frame is rotated around the $Y'$-axis by $\theta$, which leads to the frame $[X'', Y'', Z'']$. Finally a rotation around the $Z''$-axis results in the local frame $[x, y, z]$. This transformation is displayed in figure 3.3. It is important that the order of rotations is kept to and that the rotations are all clockwise, seen from the origin. Now for an arbitrary point with coordinates in the local frame $p_L$ the coordinates in the global frame $p_G$ can be calculated by

$$p_G = R_3 p_L \quad \text{and} \quad p_L = R_3^{-1} p_G$$

(3.1)
With the transformation matrix $R_b$ [37]

$$R_b = \begin{pmatrix} 
\cos(\theta) \cos(\phi) & -\cos(\theta) \sin(\phi) & \sin(\theta) \\
\cos(\psi) \sin(\phi) + \sin(\psi) \sin(\theta) \cos(\phi) & \cos(\psi) \cos(\phi) - \sin(\psi) \sin(\theta) \sin(\phi) & -\sin(\psi) \cos(\theta) \\
\sin(\psi) \sin(\phi) - \cos(\psi) \sin(\theta) \cos(\phi) & \sin(\psi) \cos(\phi) + \cos(\psi) \sin(\theta) \sin(\phi) & \cos(\psi) \cos(\theta) 
\end{pmatrix}$$

(3.2)

Notice that we earlier defined the orientation of the mirror by two angles $\alpha$ and $\beta$. The reason to do this is that the normal angles are used to determine the deflection of the laser beam. The rotation the mirror makes around its normal doesn't affect the deflection. For the mechanical behaviour, however, it is important: when the mirror is rotated around its normal, it reacts differently on actuator movements. Therefore three Bryant angles are taken into account in the mirror's model. In the next section we will describe the rotational dynamics of the mirror in terms of Bryant angles. In [31] a relation between normal angles $\alpha, \beta$ and Bryant angles $\psi, \theta$ was found:

$$\alpha = \psi; \quad \beta = \arctan\left(\frac{\tan(\theta)}{\cos(\psi)}\right)$$

(3.3)

$$\psi = \alpha; \quad \theta = \arctan\left(\tan(\beta) \cos(\alpha)\right)$$

(3.4)

So from simulation results in terms of Bryant angles, we can always compute the corresponding normal angles. The singularity of $\tan(\beta)$ and $\tan(\theta)$ at $\pi/2$ isn't a problem, because both angles are physically limited to 1 rad.

### 3.2 Rotational dynamics of the semi sphere

The dynamics of a rotating object can be derived using Euler’s law

$$\dot{L} = T$$

(3.5)

where $L$ is the angular momentum and $T$ is the resulting torque on the mirror. From this equation we will derive differential equations for the rotational dynamics of the mirror. It's presumed here that the translational movements of the mirror are negligible. In other words: the origin of the local frame doesn’t move.

$L$ can now be written as

$$L = J\omega; \quad \omega = W_b \dot{\gamma}$$

(3.6)

$J$ is the central inertia matrix of the mirror and $\omega$ is the angular velocity. The inertia matrix $J$ was calculated in [32]. It turns out that $J$ is

$$J = jI; \quad j = \frac{1}{5}mR_m^2$$

(3.7)

When applying the local coordinates to Euler's law $\dot{L}$ evaluates to a rather complex expression. This result is derived in [37] and [33]:

$$\dot{L} = j\dot{\omega} = j \{W_b \dot{\gamma} + \Omega_b\} \quad \text{where}$$

$$W_b = \begin{pmatrix} \cos(\theta) \cos(\phi) & \sin(\phi) & 0 \\
-\cos(\theta) \sin(\phi) & \cos(\phi) & 0 \\
\sin(\theta) & 0 & 1 
\end{pmatrix}$$

(3.9)

$$\Omega_b = \begin{pmatrix} 
\cos(\phi) \dot{\phi} \dot{\theta} - \sin(\phi) \cos(\theta) \dot{\phi} \dot{\psi} - \cos(\phi) \sin(\theta) \dot{\theta} \dot{\psi} \\
-\sin(\phi) \dot{\phi} \dot{\theta} - \cos(\phi) \cos(\theta) \dot{\phi} \dot{\psi} + \sin(\phi) \sin(\theta) \dot{\theta} \dot{\psi} \\
\cos(\theta) \dot{\theta} \dot{\psi} 
\end{pmatrix}$$

(3.10)
Notice that $W_b$ is a singular matrix for $\theta = \pi/2$. This isn’t a problem because $\theta$ is limited to 1 rad for constructional reasons. Now we can derive a second order differential equation for rotations of the mirror from (3.8).

$$j\ddot{\gamma} = W_b^{-1}T - W_b^{-1}\Omega_b$$ \hspace{1cm} (3.11)

Which evaluates to

$$j\begin{pmatrix} \ddot{\psi} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\cos(\phi)}{\cos(\theta)} & -\frac{\sin(\phi)}{\cos(\theta)} & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ -\cos(\phi) \tan(\theta) & \sin(\phi) \tan(\theta) & 1 \end{pmatrix} T - \begin{pmatrix} \frac{1}{\cos(\theta)} \dot{\phi} - \tan(\theta) \psi \dot{\theta} \\ -\cos(\theta) \psi \dot{\phi} \\ \tan(\theta) \dot{\phi} + \frac{\sin^2(\theta)}{\cos(\theta)} \dot{\psi} + \cos(\theta) \psi \dot{\psi} \end{pmatrix}$$ \hspace{1cm} (3.12)

The torque $T$ is the result of forces $F_i$ in the actuator strings. We can derive the following expression for the torque in the local frame by calculating the vector product of arm $a_i$ and force $E_i$.

$$\sum T = \sum_{i=1}^{3} a_i \times E_i = \sum_{i=1}^{3} F_i \left( a_i \times \frac{a_i - A_i}{\|a_i - A_i\|_2} \right)$$ \hspace{1cm} (3.13)

The combination of equations (3.12) and (3.13) forms the mechanical description of the mirror’s rotational behavior.

### 3.3 Translational dynamics

As mentioned in the introduction of this chapter, we have only investigated the rotational dynamics of the semi sphere at this moment. In case of translational movement the origins of the local and global frames don’t coincide anymore. So formulas (3.12) and (3.13) can’t be applied in this case.

An additional problem is that we have to model the dynamical behaviour of the airbearing. The forces applied by the bearing on the semi sphere are determined by the momentenious distribution of the air pressure in the bearing. This air pressure depends on the shape of the bearing and especially on the shape of the air inlet. According to Wang [34] it is a difficult job to find the momentenious air pressure.

From experiments [35], we know that the stability of the air gap is within 10 nm in a static situation, while the characteristic gap height is between 10 and 100 $\mu$m. The dynamic behaviour can probably be approximated by a spring model. The spring constant can be determined by the stiffness of the bearing which follows from static load capacity measurements, like those published in [35].

In this report no further attention will be paid to the translational movement of the mirror.

### 3.4 Actuator dynamics

After we have derived the dynamics of the semi sphere we will do the same for the actuators. Here we apply Newton’s law to the actuator displacements $h$

$$m \ddot{h} = F$$ \hspace{1cm} (3.14)

The different forces applied on actuator $i$ are the following.
Mechanical description of the complete system

\[ F_D = -D\dot{h} \quad -\text{Damping force} \]
\[ F_i = -K\dot{h} \quad -\text{Tensile force in string } i \]
\[ F_S = -AI \quad -\text{Actuator force when using current control or} \]
\[ = \frac{-AU - sA^2h}{R + sL} \quad -\text{Actuator force when using voltage control} [31] \]
\[ F_0 = -\text{Pretensioning force due to spring} \]

Since the mass of the coil is very low, we can neglect the gravitational force in relation to the other forces. Substituting the above forces in (3.14) results in the nonlinear differential equation (3.15) for actuator \( i \) (\( i=1, 2, 3 \)). Notice that string force \( F_i \) doesn’t work along the core of the actuator. Therefore the resulting force is calculated by taking the inner product of \( E_i \) and the direction vector \( C_i \).

\[ m\ddot{h}_i = -D\dot{h}_i + E_i \cdot C_i - K\dot{h}_i - F_0 - F_{Ai}(h_i, \dot{h}_i) \quad (3.15) \]

### 3.5 Mechanical description of the complete system

The dynamics of the complete mirror system can be described by the set of differential equations (3.12) and (3.13) and three differential equations of the form (3.15). There are however some other relations that can be observed in the system. The orientation of the mirror and the displacements of the actuators are coupled. The strings that connect the actuators to the mirror are assumed to be inextensible. So the length of the strings is constant. Hence

\[ \| \dot{a}_i - \Delta_i \|_2 = l \quad (3.16) \]

where \( i = 1, 2, 3 \), \( l \) is assumed equal for all strings and \( \dot{a}_i \) and \( \Delta_i \) are expressed in the same coordinate frame. When expressed in global coordinates \( \dot{a}_i = \dot{a}_i(\psi, \theta, \phi) \) and \( \Delta_i = \Delta_i(h_i) \) which follows from the following expression using (2.3) and (3.1)

\[ \| R_ba_{iL} - Q_i + h_iC_i \|_2 = l \quad (3.17) \]

Now the system is described by a total of six nonlinear differential equations and three algebraic equations. From these six differential equations we might conclude that the system has six degrees of freedom. This however is not true, because the three algebraic equations restrict the system in its motion. Only three degrees of freedom remain.

### 3.6 Euler-Lagrange description

A generally used way to describe multibody systems is the Euler-Lagrange form (see for instance [5] and [1]). In the literature this form is also known as the descriptor form. The system is described by expressions of the form (B.1), (B.2) and (B.3).

\[ \dot{\nu} = \ddot{\nu} \quad (3.18) \]
\[ M(\nu)\ddot{\nu} = Q(\nu, p, t) + E(t) + G^T(p)\Lambda \quad (3.19) \]
\[ g(p) = \Omega \quad (3.20) \]

Normally the model in descriptor form isn’t derived in the way we did above. However the resulting model is the same. \( \nu \) is the vector of generalized coordinates, so here \( p = (\psi, \theta, \phi, h_1, h_2, h_3)^T \).
$M$ is the inertia matrix of the complete system. The vector function $Q$ represents the Coriolis and centrifugal forces and $F$ is the vector of external forces that are acting on the system. $\lambda$ is the vector of magnitudes of the constraint forces, that make the solution of the differential equations satisfy the constraint. Except for a constant factor, the constraint forces $\lambda$ and the tensile forces in the mirror system $F_i$ are equal, so $\lambda \in \mathbb{R}^3$. The direction of the constraint forces is determined by the matrix $G^T$, where $G$ is defined as the jacobian of $g(p)$:

$$G(p) = \frac{\partial}{\partial p} g(p)$$

The mirror system is described explicitly in Euler-Lagrange form by the following parameters.

$$M = \begin{pmatrix} jI & 0 \\ 0 & mI \end{pmatrix}$$

$$Q = \begin{pmatrix} -\{\frac{1}{\cos(\theta)} \dot{\phi} - \tan(\theta) \dot{\psi}\} \\ -\{-\cos(\theta) \dot{\psi}\phi\} \\ -\{-\tan(\theta) \dot{\phi} + \frac{\sin(\theta)}{\cos(\theta)} \dot{\psi} - \cos(\theta) \dot{\phi} \psi\} \\ -Dh_1 - Kh_1 \\ -Dh_2 - Kh_2 \\ -Dh_3 - Kh_3 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -F_0 + F_{A1} \\ -F_0 + F_{A2} \\ -F_0 + F_{A3} \end{pmatrix}$$

$$g(p) = \begin{pmatrix} \frac{1}{2}\{\|R_b \mathbf{a}_{1L} - Q_1 + h_1 C_1\|_2^2 - l^2\} \\ \frac{1}{2}\{\|R_b \mathbf{a}_{2L} - Q_2 + h_2 C_2\|_2^2 - l^2\} \\ \frac{1}{2}\{\|R_b \mathbf{a}_{3L} - Q_3 + h_3 C_3\|_2^2 - l^2\} \end{pmatrix}$$

It turns out that $g_i(p)$ can be written as

$$g_i(p) = \frac{1}{2} h_i^2 - h_i l - (Q_i + h_i C_i)^T R_b \mathbf{a}_i + r^2$$

With this description we can write the Jacobian of $g(p)$ as

$$G(p) = \begin{pmatrix} G_{1\psi} & G_{1\theta} & G_{1\phi} & 0 & 0 \\ G_{2\psi} & G_{2\theta} & G_{2\phi} & 0 & 0 \\ G_{3\psi} & G_{3\theta} & G_{3\phi} & 0 & 0 \end{pmatrix}$$

where $G_{1\phi} = -(Q_i + h_i C_i)^T \frac{\partial R_b}{\partial \mathbf{a}_i} \mathbf{a}_i$ and $G_{hi} = h_i - C_i^T R_b \mathbf{a}_i - l$, for $i = 1, 2, 3$ and $\Phi = \psi, \theta, \phi$. 
Chapter 4

Simulation problems

With the previously derived model a simulation should be performed to test controller designs. However, simulating the mirror's dynamics causes some problems. These problems and their solutions will be discussed below.

4.1 Assistent dynamics

In the preceding chapter we have seen that the set of ordinary differential equations (3.12) and (3.15) represents the dynamics of the mirror. Such a model can normally be simulated by any numerical integration package. The numerical simulation software solves the differential equations by calculating the values of the Bryant angles $\gamma$ and actuator positions $h$ from their preceding values, the values of the forces $F_i$ in the strings and the actuator forces $F_A$. The tensions in the strings ($F_i$), however, are unknown. These values have to be computed from the movements of the mirror and the actuators, using the extra equations (3.16). This algebraic problem could be solved by some iterative process during the simulation, but normal simulation packages, like Simulink, aren't able to do that. We can also solve it analytically before the simulation, but that is not a good option either. We would, for example, have to solve $F_i$ from (3.15) and substitute the result in (3.12). Then (3.16) should be used to eliminate $\gamma$ or $h$ from this formula. Since (3.16) is a very nonlinear expression, the model would become very complicated and unmanageable.

One way to avoid these problems of modular systems is given by Iversen [36]. Iversen shows how algebraic problems occur when modelling fluid dynamics in systems build of pipes and vessels. The structure of such systems is shown in figure 4.1. The blocks represent the different modules of the system. Dependencies of modules on other modules are represented by arrows. Thick arrows depict algebraic dependence, thin arrows represent differential dependence. From this figure we see that an algebraic problem occurs during the simulation, because modules 2, 3 and 4 have a mutual algebraic dependence. This is called an "algebraic loop".

In figure 4.2 an extra module is added which breaks the algebraic loop with a differential equation. This is called adding assistent dynamics. In the example of the pipes and vessels, an extra small vessel is added, which acts as a buffer. Now the "algebraic loop" has disappeared and the problem can be solved by any normal numerical simulation package.

Unfortunately some problems remain. First it is difficult to chose an appropriate assistent module. In Iversen's example it is logical to choose an extra vessel, but now the problem remains how to choose the size of the vessel. When the size is chosen too high, an incorrect model will be obtained. When on the other side the size is chosen too low, the model becomes stiff.
For linear systems stiffness is defined in the following way.

**Definition 1** A differential equation $\dot{x} = Ax + b$ is said to be stiff if the eigenvalues $\lambda_i$ of the system comply with

$$Re(\lambda_i) < 0 \quad \text{and} \quad S = \frac{\min|\lambda_i|}{\max|\lambda_i|} \ll 1$$

where $S$ is called the stiffness ratio.

This can be seen as if the system has a part with very fast dynamics and a part with very slow dynamics, as suggested by the relatively small and large eigenvalues of the system. For nonlinear systems we have to use the following, more general definition.

**Definition 2** A numerical integration problem is stiff, if the step length used by a method with a bounded stability region is limited by stability rather than accuracy.

Iversen has found an approach to determine the value of the assistent dynamics automatically from the parameters of the rest of the system. However, this approach is only valid for systems that have the structure of figure 4.1. This is not the case for the mirror system. Therefore Iversens algorithm can’t be applied. Nevertheless assistent dynamics can be put into practice in the mirror system in another way. Our main goal is to remove variable $F_i$ from the equations of motion. If we replace the stiff strings between the actuators and the semi sphere by springs with a high spring constant $C_s$, the algebraic relation is eliminated [31]. Now $F_i$ can be calculated by

$$F_i = C_s(\|\mathbf{s}_i - \Delta_i\|_2 - l) \quad (i = 1, 2, 3)$$

(4.1)

This formula can be substituted in (3.12) and (3.15) in order to eliminate $F_i$. The algebraic equations (3.16) aren’t needed anymore, so a set of differential equations remain which can be simulated by a normal simulation package like Simulink.
In [31] for $C_s = 1.5 \times 10^5$ N/m was chosen. With this value the spring should be a good approximation of the stiff string. But as expected, the numerical solution of the modified system suffers from stiffness now.

According to definition 2 the simulation of the mirror system is a stiff problem, because stability requirements limited the step size to a maximum of $10^{-5}$. This result was derived by solving the system with both a Runge-Kutta method of order 5 and with Gear’s algorithm in Simulink. Unfortunately a simulation of a 1 sec time interval takes approximately one hour on a 80486DX2 processor equipped personal computer with this time step. In other words it isn’t very attractive to use these time consuming simulations for testing controller designs. One solution would be to select another spring constant. This is only possible by a trial and error method. Moreover it is very hard to determine from the simulation results, whether the spring constant is chosen too small. Though it is possible to use a faster computer, it was decided to examine the possibility of simulating the mirror system using other techniques.

### 4.2 Solving the DAE system

In the previous section we have examined methods to convert the mirror system’s model from a set of Differential and Algebraic Equations (DAE) into a set of Ordinary Differential Equations (ODE) in order to simulate the system by classical simulation software. Since DAE’s are very convenient to model many different systems [1], for instance electrical circuits, chemical reactors and constrained mechanical systems, techniques have been developed to solve DAE systems directly. Although it is known that DAE’s are generally harder to simulate numerically than ODE systems, several techniques are applicable for the mirror system. In the next chapter some integration methods for DAE’s will be discussed.
Chapter 5

Solving the DAE of a constrained mechanical system

In general a DAE can be represented by

\[ F(y', y, t) = 0 \] \hspace{2cm} (5.1)

where the Jacobian \( \frac{\partial F}{\partial y'} \) is singular. This singularity causes problems when solving DAE's. Especially in the past fifteen years an increasing amount of research is done to study the problems of DAE's. It has turned out that DAE's can be classified by a parameter called the index. The higher the index of the DAE, the more difficult the integration problem. DAE's that represent constrained mechanical systems, appear to have an index 3. In spite of the problems of this high index, the research has resulted in some methods to solve these index 3 DAE systems. After some characteristics of DAE's are treated, these methods will be discussed in this chapter. Some methods will be explained with help of a simple example. This example is a simplification of the more complicated mirror system.

**Example**
This example concerns the one dimensional system of figure 5.1. Two bodies each of mass \( m \) move on the surface with speed \( v_1 \) and \( v_2 \) as a result of two forces \( F_1 \) and \( F_2 \). The centre of the bodies are indicated by the points \( x_1 \) and \( x_2 \). As in the mirror system, the bodies are coupled by a link which is a solid rod here.

![Figure 5.1: A simple system of moving blocks](image)

The system can now be described in Euler-Lagrange form by

\[ \dot{x}_1 = v_1 \] \hspace{2cm} (5.2)

\[ \dot{x}_2 = v_2 \] \hspace{2cm} (5.3)
Some details on DAE's

5.1 Some details on DAE's

5.1.1 The index

From the literature several definitions of the index of a DAE are known. In [14] for instance, the index is defined as the dimension of the largest block in the block diagonal matrix E in a linear constant-coefficient DAE system of the form

\[ Ey' + Fy = g(t) \]  
\[ (5.9) \]

Since the mirror system leads to a nonlinear variable coefficient DAE, this isn't a useful definition. In this case the following, more general definition is better.

**Definition 3** The index of a DAE is equal to the number of times the DAE, or part of it, has to be differentiated with respect to time to obtain an ODE.

The index according to this definition is called the differential or global index. Because 5.9 may be seen as a local linear approximation of a nonlinear DAE, the index based on this description is also called the local index.

**Example**

Now we can determine the index of the system of figure 5.1. Differentiating the constraint (5.6) results in

\[ \dot{x}_1 - \dot{x}_2 = v_1 - v_2 = 0 \]  
\[ (5.10) \]

Differentiating (5.10) leads to

\[ \ddot{x}_1 - \ddot{x}_2 = \dot{v}_1 - \dot{v}_2 = 0 \]  
\[ (5.11) \]

Now the third differentiation of the constraint combined with the original differential equations leads to the following ODE. This system is called the underlying ODE.

\[ \dot{x}_1 = v_1 \]  
\[ (5.12) \]

\[ \dot{x}_2 = v_2 \]  
\[ (5.13) \]

\[ m\ddot{v}_1 = -F_1 + \lambda \]  
\[ (5.14) \]

\[ m\ddot{v}_2 = F_2 - \lambda \]  
\[ (5.15) \]

\[ \ddot{v}_1 - \ddot{v}_2 = 0 \]  
\[ (5.16) \]
From definition 3 we can conclude that the index of the system is equal to three. The block scheme of figure 5.2 suggests that index three DAE's are indeed difficult to solve. The value of the force \( \lambda \) has to be calculated, such that after two integration steps the constraint is satisfied.

![Figure 5.2: Block scheme of the simple system](image)

The index of the mirror system can be proved to be also three, just as any other mechanical system with constraints on the positions. Because this kind of systems all can be described in Euler-Lagrange form, we could apply definition 3 to the general Euler-Lagrange formulas (B.1), (B.2) and (B.3). For reasons of notational complexity this is omitted here, but it can be imagined that after differentiating the constraint (B.3) three times a differential equation in \( \ddot{v} \) is obtained. This result combined with (B.1) and (B.2) forms an ODE. Hence any constrained mechanical system has index three.

5.1.2 Structure

Not only the index of a DAE is important, but also the structure. The most general form of a DAE is

\[
F(y', y, t) = 0
\]  

(5.17)

This form is called fully implicit. It is the most difficult form. The semi-explicit form is easier to use in numerical proofs. In this form the differential part is separated from the algebraic part, i.e.

\[
\begin{align*}
\dot{y} &= g_1(y, x, t) \\
0 &= g_2(y, x, t)
\end{align*}
\]  

(5.18)

The idea that semi-explicit DAE’s are less complicated than implicit DAE’s is also fed by Gear’s transformation [29]. He demonstrates the relation between implicit index one DAE’s and semi-explicit index two DAE’s. A third important form is the so called Hessenberg or triangular form:

\[
\begin{align*}
0 &= g_1(y_1) \\
\dot{y}_1 &= g_2(y_1, y_2) \\
\dot{y}_2 &= g_3(y_1, y_2, y_3)
\end{align*}
\]  

(5.19)

When \( y_1 = p, y_2 = v \) and \( y_3 = \lambda \) is substituted in (5.19), it can be seen that any constrained mechanical system in Euler-Lagrange description yields a DAE in Hessenberg form. This result will be used later.
5.1.3 Relation to stiff ODE’s

DAE’s can be considered as the limit case for the stiffness ratio $S \to 0$ of stiff differential problems [4]. This can be seen by writing an approximation of (5.18) as

$$\begin{align*}
\dot{y} &= g_1(y, x, t) \\
\epsilon \dot{x} &= g_2(y, x, t)
\end{align*}$$

(5.20)

where $\epsilon > 0$ is a small parameter. When $\epsilon = 0$ we have the original DAE. Writing (5.20) as a normal ODE leads to

$$
\begin{pmatrix}
\dot{y} \\
\dot{x}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
0 & \epsilon^{-1}
\end{pmatrix}
\begin{pmatrix}
g_1(y, x, t) \\
g_2(y, x, t)
\end{pmatrix}
$$

(5.21)

Based on the idea of definition 1 we conclude that large value of $\epsilon^{-1}$ make the ODE stiff. Considering the ODE (5.20) with $\epsilon \to 0$ can help when analysing DAE’s as Knorrenschild demonstrates in [4].

In the case of a DAE we could also think of the system having very fast and very slow dynamics. The differential part of the DAE represents the slow part, while the algebraic part represents infinitely fast dynamics.

5.2 Solving DAE’s using ODE methods

Solving a DAE can be split into two parts. On one hand the differential part has to be solved by numerical integration. Simultaneously the algebraic constraint must be satisfied. It seems obvious to solve the DAE by discrete integration methods like those used for ODE’s. Well known methods are the Backward Differential methods and the Runge-Kutta methods.

5.2.1 BDF methods

The idea behind all linear multistep integration methods is to determine $y_n \equiv y(t_n)$ from

$$F(\dot{y}, y, t) = 0$$

(5.22)

using $k$ values of $y$ at moments before $t_n$. Each of these multistep methods is characterized by the approximation function $\rho(y_n, y_{n-1}, \ldots, y_{n-k})$ for $\dot{y}_n$. Now $y_n$ can be solved from

$$F(\rho(y_n, y_{n-1}, \ldots, y_{n-k}), y_n, t_n) = 0$$

(5.23)

When $\rho$ is a function of $y_n$ the method is implicit and is called a backward differential formula. Otherwise the method is called explicit. Only implicit methods can be applied on DAE’s because of the singularity of the Jacobian $\partial F / \partial \dot{y}$. Also the fact that a DAE can be considered as the limit case of stiff ODE’s suggests that only implicit methods will work, for only these methods are applicable for stiff problems.

A backward differential formula can be characterized by the discretization function

$$\rho(y_n, y_{n-1}, \ldots, y_{n-k}) \equiv \frac{1}{h} \sum_{i=0}^{k} \alpha_i y_{n-i}$$

(5.24)

where $h$ is the timestep. The coefficients $\alpha_i$ depend on the order of the method.

A well known example of BDF is the first order Backward Euler integration formula:

$$\rho \equiv \frac{y_n - y_{n-1}}{h}$$

(5.25)
After substituting $\rho$ in the differential equation the solution of $y_n$ can be obtained. Generally this is done by a Newton iteration process for nonlinear functions $F$.

A lot of research has been done to prove convergence for BDF methods. Some interesting results have been found. One of the first is Petzold's theorem.

**Theorem 1** If $F(\dot{y}, y, t) = 0$ is an index one DAE and is differentiable with respect to $y$ and $\dot{y}$, the solution by a $k$-step BDF method with fixed stepsize $h$ for $k < 7$ converges to order $O(h^k)$ if the initial values are consistent.

With consistent initial values is meant that the constraints as well as their derivatives are satisfied within certain bounds. From this theorem it follows that BDF methods can be used to solve DAE systems under certain circumstances. The practical limitation of $k < 7$, which also holds for ODE's, is not very restrictive, but the constraints on the index, initial values and, especially on the stepsize are. Though there are no general convergence results for higher index DAE's, several people have extended Petzold's result for DAE's in Hessenberg structure. In [25] Brenan and Engquist prove the following theorem.

**Theorem 2** If $F(\dot{y}, y, t) = 0$ is a DAE in Hessenberg structure of index three or lower and is differentiable with respect to $y$ and $\dot{y}$, the solution by a $k$-step BDF with constant stepsize $h$ for $k < 7$ converges globally to order $O(h^k)$ after $k+1$ steps if the initial values are consistent.

The constant stepsize $h$ reduces the simulation speed. A variable stepsize and variable order approach as used in Gear's algorithm for ODE's would accelerate the simulation: when the solution varies slowly, the stepsize can be increased without decreasing the accuracy. Keiper and Gear show in [28] that variable stepsize and variable order BDF methods can be used to solve Hessenberg structure DAE's of index 3 and even 4 with satisfactory accuracy. However it is still necessary that the solution is sufficiently smooth and that the initial values are consistent. Smoothness of the solution is needed because BDF is a multistep method. It uses $k$ preceding values of the solution to compute a new value. If there are discontinuities in the solution, a BDF method is not able to compute the new value correctly. In [20] Steigerwald shows this shortcoming of BDF methods with some examples. Fortunately the mirror system doesn't suffer from any discontinuities.

### 5.2.2 Runge-Kutta methods

Methods with an acceptable behaviour in the presence of discontinuities are the Runge-Kutta methods. These one-step methods only need one preceding value of the solution to compute the new value. Instead of using preceding values, $M$ values of the solution between the new value and the previous value are computed. With these $M$ points the new value of the solution is computed. The difference between BDF methods and Runge-Kutta methods is demonstrated in figure 5.3. The new value of the solution $y_n$ is computed from $M$ points $k_i$ using a formula of the form

$$y_n = y_{n-1} + h \sum_{i=1}^{M} B_i k_i$$  \hspace{1cm} (5.26)

where the coefficients $B_i$ depend on the method and the intermediate $y$-values $k_i$ are computed by with implicit expression

$$f(t_{n-1} + c_i h, y_{n-1} + h \sum_{j=1}^{M} a_{ij} k_j, k_i) = 0$$  \hspace{1cm} (5.27)
Special methods for constrained mechanical system DAE’s

From the previous paragraph it can be concluded that low index (index one and, to a smaller extend, index two) DAE’s are relatively easy to solve. On the other hand higher index DAE’s cause rather large problems. Because contrained mechanical systems always yield DAE’s with index three, there has been a lot of research on DAE’s for this special case. Since the variable stepsize and variable order method of Gear and Keiper is just shortly known, some different approaches are developped that modify the DAE in order to facilitate the integration problem.

5.3 Special methods for constrained mechanical system DAE’s

From the previous paragraph it can be concluded that low index (index one and, to a smaller extend, index two) DAE’s are relatively easy to solve. On the other hand higher index DAE’s cause rather large problems. Because contrained mechanical systems always yield DAE’s with index three, there has been a lot of research on DAE’s for this special case. Since the variable stepsize and variable order method of Gear and Keiper is just shortly known, some different approaches are developped that modify the DAE in order to facilitate the integration problem.

5.3.1 Index reduction

Problems of higher index DAE’s would be reduced if the index could be decreased. The way to achieve index reduction follows from definition 3 of the global index. When the constraint relations of the DAE are differentiated once, the index is reduced by one. If the constraint of an Euler-Lagrange system is differentiated once we get the following index 2 problem.

\[
\begin{align*}
\ddot{p} & = \nu \\
M(p) \dot{\nu} & = Q(\nu, p, t) + E(t) + G^T(p)\lambda \\
\frac{d}{dt} g(p) & = G(p) \dot{p} = 0 
\end{align*}
\]

(5.28) \hspace{1cm} (5.29) \hspace{1cm} (5.30)

Differentiating again yields the index 1 problem:

\[
\begin{align*}
\ddot{p} & = \nu \\
M(p) \dot{\nu} & = Q(\nu, p, t) + E(t) + G^T(p)\lambda \\
\frac{d}{dt^2} g(p) & = G(p) \ddot{p} + z(p, \dot{p}) = 0 
\end{align*}
\]

(5.31) \hspace{1cm} (5.32) \hspace{1cm} (5.33)

where \( z(p, \dot{p}) = \ddot{p}^T G_p(p) \dot{p} \) if \( g \) is a scalar function. Else \( G_p(p) \) becomes a tensor.

The solutions of these new systems only satisfy the constraints (5.30) or (5.33) on respectively speed and acceleration. The position constraint \( g(p) = 0 \) is not nessecarily satisfied. In practice

Here \( f, c_i \) and \( a_{ij} \) depend on the method. For the same reason as mentioned for BDF methods, only implicit Runge-Kutta methods can be applied for DAE’s. Implicit Runge-Kutta methods give good convergence results for index one and semi-explicit index two DAE’s. Unfortunately no such result has been obtained for systems with higher indexes.
the results are even worse. Differentiating the constraints leads to extra integration constants in the solution. These constants make the solution drift away from the original constraint. In the literature very striking examples are shown of this drift problem (for instance [3] and [5]).

**Example**

We will consider the system of the moving blocks again. Differentiating the constraint leads to a system of which figure 5.4 is the block scheme. The constraint has now changed into $\ddot{x}_1 = \ddot{x}_2$.

![Figure 5.4: Block scheme of the index 1 system](image)

From this scheme it follows that

$$\lambda = \frac{F_2 + F_1}{2} \quad (5.34)$$

which corresponds to the solution we derived in (5.7) and (5.8). On the other hand initial values can be chosen freely. Only if these values are chosen consistently, the original constraint will be satisfied. Moreover the numerical integration blocks will cause aberrations, which can be thought of as state noise. This noise will cause the system to drift away from the original constraint.

To solve the drift problem we have to take the constraint into account again in each timestep of the simulation of the reduced system. This can be done by using:

- "Baumgarte" stabilization,
- projection methods,
- an overdetermined system.

### 5.3.2 Baumgarte stabilization

One way to take the original constraint into account in the reduced system is the so called "Baumgarte" stabilization. Before this stabilization method is discussed, we will elaborate on the Euler-Lagrange description of mechanical systems.

In chapter 4 we have seen that vector $\mathbf{F}$ causes problems during the simulation of the mirror system. The tensile forces in the strings are unknown and have to be computed from the constraints. Normal simulation software can’t solve this algebraic problem, so we can try to modify the model in order to find an explicit expression for the tensile forces. In the Euler-Lagrange description the constraint forces are represented by $\lambda$, which can be solved following a standard method. Differentiating constraint (B.3) twice leads to expression (5.33) which can be written also as

$$G(p)\ddot{v} + \varepsilon(p, v) = 0 \quad (5.35)$$
Combining this with
\[ \dot{\mathbf{u}} = M^{-1}(\mathbf{Q} + \mathbf{F} + G^T\lambda) \] (5.36)
leads to
\[ \lambda = -(GM^{-1}G^T)^{-1}\{GM^{-1}(\mathbf{Q} + \mathbf{F}) + \mathbf{z}\} \] (5.37)
This expression for \( \lambda \) can be substituted in (5.32). Then the constraint force is eliminated from the differential equations and the system can be simulated as a normal ODE.

**Example**
When applying (5.37) to the simple example of the moving blocks, we can compute the constraint force in the rod.
The system of (5.2)-(5.6) is defined by
\[
\begin{align*}
M &= \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \\
G &= \begin{pmatrix} 1 & -1 \end{pmatrix} \\
\mathbf{F} &= \begin{pmatrix} -F_1 \\ F_2 \end{pmatrix} \\
\mathbf{z} &= 0
\end{align*}
\]
Following (5.37), \( \lambda \) evaluates to
\[ \lambda = \frac{F_2 + F_1}{2} \] (5.42)
Substituting (5.42) in the original equations of motion yields
\[
\begin{align*}
m\ddot{x}_1 &= \frac{F_2 - F_1}{2} \\
m\ddot{x}_2 &= \frac{F_2 - F_1}{2}
\end{align*}
\] (5.43) (5.44)
which follows also from combining the two blocks to one body.

This approach can’t be applied unpenalized. The constraint’s second derivative is used to solve \( \lambda \), whereas the original constraint has disappeared from the system. In the previous section we have seen that such a reduced index system suffers from drift problems. The following example shows how bad things can go wrong.

**Example**
Consider the slider-crank system of figure 5.5. From this figure we derive the equations of motion in an Euler-Lagrange form:
\[
\begin{align*}
\dot{\mathbf{w}} &= T + \frac{dg}{d\omega}\lambda \\
\mathbf{h} &= \frac{dg}{dh}\lambda
\end{align*}
\] (5.45) (5.46)
Next we define \( X \) and \( Y \) as
\[
\begin{align*}
X &= d - h - r\cos(\omega) \\
Y &= r\sin(\omega)
\end{align*}
\] (5.47) (5.48)
so that the constraint has the form

\[ g = \sqrt{X^2 + Y^2} - l = 0 \] (5.49)

Now \( \lambda \) is determined according to (5.37) and the resulting ODE is solved with Simulink using Euler’s method with tolerances of \( 10^{-4} \). The parameters of the slider-crank system were \( d=2.0 \) m, \( j=0.05 \) kg·m, \( m=1.0 \) kg, \( r=0.5 \) m, \( l=1.5 \) m and \( T=0.1 \) Nm. The results in figure 5.6 show that the system indeed drifts away from the constraint. After 10 sec the deviation from the constraint is 0.15 m, which is far more than allowable.

To prevent the solution of the DAE to drift away, Baumgarte uses a linear combination of the constraint and its derivatives instead of only the second derivative. A multibody system is then modelled as

\[ \dot{\lambda} = v \] (5.50)

\[ M(p)\ddot{\lambda} = Q(p, v, t) + E(t) + G^T(p)\lambda \] (5.51)

\[ 0 = \ddot{\lambda}(p) + 2\alpha\dot{\lambda}(p) + \beta^2 g(p) \] (5.52)
We have achieved an ODE with differential equations in $p$ and in $g(p)$. Baumgarte suggests to use $\alpha = \beta$, so that $g(p)$ will converge critically damped to zero. Since the mirror system is a nonlinear system, this doesn’t guarantee that the values of vector $p$ converge critically damped to the solution of the DAE. Another problem is to determine if $\alpha$ and $\beta$ should be small or large. The optimal values depend on the dynamics of the simulated system. Perturbances on the constraint with a frequency higher than $\beta$ aren’t suppressed by the stabilization. Good results can be achieved when $\alpha$ and $\beta$ are determined adaptively during the simulation. Further details on Baumgarte stabilization and its implementation, can be found in [16], [17] and [26].

### 5.3.3 Projection techniques

From a geometrical point of view a DAE can be seen as a vectorfield on a manifold. The algebraic part defines the manifold, whereas the differential part defines a direction for each point on the manifold. Vectorfields are well known from ODE techniques. Eich [3], Rheinboldt and Reich have done interesting research on this interpretation of DAE’s. The geometrical approach is visualized in figure 5.7.

\[ \text{Manifold } g(p) = 0 \]

\[ \text{Vectorfield} \]

\[ \text{Projection} \]

Figure 5.7: Geometrical representation of a DAE

Every integration step will result in a solution $\tilde{y}_n$ outside the manifold. A way to regain a solution on the manifold is to project the solution orthogonally back on the manifold by computing

\[ \min_{\tilde{y}_n} \| y_n - \tilde{y}_n \|_2 \text{ subject to } g(y_n) = 0 \quad (5.53) \]

In [3] Eich proves that this projection method applied on BDF integration converges to the correct solution of the DAE for both linear and nonlinear time variant DAE’s. Convergence is even better with projection method than without, because the truncation errors after every integration step are reduced to the part of the error in the manifold. A major advantage of this method is that the solution always satisfies the algebraic constraint. Besides we have the advantage that we can also apply projection to other integration formulas, such as one step Runge-Kutta formulas, in case systems with discontinuities are considered.

**Example**

We will consider the slider-crank system of figure 5.5 again. As we have seen, the constraint can be written as

\[ g(\omega, h) = \sqrt{X(\omega, h)^2 + Y(\omega)^2} - l = 0 \quad (5.54) \]
This can be interpreted as the description of a circle with radius $l$ in the $XY$-plane. The orthogonal projection of the solutions $(\tilde{X}(t), \tilde{Y}(t))$ onto the manifold of (5.54) is very simple now:

$$
X(\omega, h, t) = \frac{l}{\sqrt{\tilde{X}(t)^2 + \tilde{Y}(t)^2}} \tilde{X}(t); \quad Y(\omega, t) = \frac{l}{\sqrt{\tilde{X}(t)^2 + \tilde{Y}(t)^2}} \tilde{Y}(t); 
$$

(5.55)

This explicit formulation of the projected solution can be implemented in the Simulink model of the slider-crank system. The results of a simulation with the same parameters as used in the previous example can be seen in figure 5.8. We see that the violation of the constraint is of order $10^{-16} m$ which is much better than the result of the previous simulation. Unfortunately,

![Simulation results](image)

*Figure 5.8: Results of simulation of the slider-crank system using orthogonal projection for the mirror system we can’t find an explicit formulation such as (5.55) for projection on the manifold.*

### 5.3.4 Solving the overdetermined system

A third way to solve the problems of index reduction, is to consider the following system.

$$
\dot{\hat{p}} = \nu \\
M(p)\dot{\hat{p}} = Q(\nu, \hat{p}, t) + E(t) + G^T(p)\lambda \\
0 = q(p) \\
0 = G(p)\nu \\
0 = G(p)\hat{p} + z(p, \nu)
$$

(5.56) (5.57) (5.58) (5.59) (5.60)

Now the constraint, its first and second derivatives are taken into account. Analytically this system has the same solution as the original system (B.1)-(B.3). It is redundant but not ambiguous. However the dicretized system is overdetermined. It has more equations than variables. The
system can’t be solved by a Newton iteration, so a natural choice would be to use a least square approximation. There are however methods that give better results. When the overdetermined system is modified in a way that it is no longer overdetermined; we can use Newton iteration then. Adding extra variables or removing equations will do the job.

5.3.5 Adding extra Lagrange multipliers

Two extra variables should be added to the system (5.56)-(5.60) in the following way.

\[
\begin{align*}
\dot{p} &= v + G^T \mu + [u^T G_p(p)]^T \eta \\
M(p) \ddot{p} &= Q(u, p, t) + E(t) + G^T(p) \Lambda + G^T(p) \eta \\
0 &= g(p) \\
0 &= G(p) v \\
0 &= G(p) \dot{v} + \zeta(p, v)
\end{align*}
\]

(5.61) (5.62) (5.63) (5.64) (5.65)

This system can be seen as the index 1 system (5.56), (5.57) and (5.60) coupled to the constraints (5.58) and (5.59) via the extra variables \( \mu \) and \( \eta \). Analytically this system (5.61)-(5.65) gives the same solution as the overdetermined DAE (5.56)-(5.60) when \( \mu \) and \( \eta \) are zero. In [15] and [27] Führer and Leimkuhler show that the discretized versions of both systems give also the same integration result for the variables of interest \( p, v \) and \( \lambda \). They can be solved accurately, satisfying the constraint, whereas the uninteresting variables \( \mu \) and \( \eta \) may differ from zero.

This method of coupling constraints via extra Lagrange multipliers can also be applied to the index 2 system (5.56), (5.57) and (5.59). This results in the system

\[
\begin{align*}
\dot{p} &= v + G^T \mu \\
M(p) \ddot{p} &= Q(u, p, t) + E(t) + G^T(p) \Lambda \\
0 &= g(p) \\
0 &= G(p) v
\end{align*}
\]

(5.66) (5.67) (5.68) (5.69)

This description gives results that are comparable to that of the previous system. Hence it is difficult to determine which of both formulations can be applied best.

5.3.6 State space transformation

Previously we have seen that the number of generalized coordinates of a constrained mechanical system isn’t equal to the number of degrees of freedom of the system. The number of constraints determines the real number of degrees of freedom. With this in mind, we can partition the generalized coordinates as

\[
p = \begin{pmatrix} u \\ w \end{pmatrix}
\]

(5.70)

where \( u \) is the set of dependent variables and \( w \) is the set of independent variables. The set of variables \( u \) is chosen in such a way that \( G_u^2 = \partial g / \partial u \) is a regular matrix. This implies that we can write \( u \) as a function of \( w \):

\[
u = h(w, t)
\]

(5.71)

This yields

\[
\dot{u} = \frac{\partial h}{\partial w} \dot{w} = h'(w, \dot{w}) \quad \text{and}
\]

\[
\ddot{u} = h''(w, \dot{w}, \ddot{w})
\]

(5.72) (5.73)
Since the differential equations can be written as
\[
\begin{pmatrix}
M^{uu} & M^{uw} \\
M^{wu} & M^{ww}
\end{pmatrix}
\begin{pmatrix}
\ddot{w} \\
\dot{w}
\end{pmatrix} = \begin{pmatrix}
Q^u + E^u \\
Q^w + E^w
\end{pmatrix} + \begin{pmatrix}
G^{T}_{\dot{u}} \\
G^{T}_{\dot{w}}
\end{pmatrix} \lambda
\]  
(5.74)

and \(G^{T}_{\dot{u}}\) is regular, \(\lambda\) can be solved:
\[
\lambda = \left( G^{T}_{\dot{u}} \right)^{-1} \left( M^{uu} \ddot{w} + M^{uw} \dot{w} - Q^u - E^u \right)
\]  
(5.75)

When this result, together with (5.71), (5.72) and (5.73), is substituted in the original set of differential equations, the overdetermined system is reduced to a set of ordinary differential equations in \(w\).

**Example**

In the system of the moving blocks we can make \(x_1\) the independent variable and \(x_2\) the dependent variable. The functions \(h, h'\) and \(h''\) are now defined as
\[
\begin{align*}
x_2 &= h(x_2) = x_1 + 1 \\
\dot{x}_2 &= h'(x_2) = \dot{x}_1 \\
\ddot{x}_2 &= h''(x_2) = \ddot{x}_1
\end{align*}
\]  
(5.76) (5.77) (5.78)

and \(\lambda\) can be written as
\[
\lambda = -(m\ddot{x}_2 - F_2)
\]  
(5.79)

With this we can rewrite the original differential equations as the following differential equations in the independent variables \(x_1\) and \(v_1\).
\[
\begin{align*}
\dot{x}_1 &= v_1 \\
\dot{m}\dot{v}_1 &= \frac{F_2 - F_1}{2}
\end{align*}
\]  
(5.80) (5.81)

This leads to the analytical solution of (5.7) and (5.8). Position \(x_2\) can now be computed using (5.76).

Although this method yields accurate simulation results, as is shown in [18], it has some disadvantages. In complex systems the choice of dependent and independent coordinates is not trivial. Moreover it is numerically expensive to compute the functions \(h, h'\) and \(h''\) at every integration step. In general this is necessary because most practical systems have nonlinear constraints which can't be solved analytically for the dependent variables, as we have mentioned in chapter 4 for the mirror system.

### 5.4 Conclusions

Though we have seen that simulating DAE systems is not as simple as simulating ODE systems, we can conclude that there are some good methods to simulate the mirror system. There are no algorithms known for solving general higher index DAE's, but because of its Hessenberg structure we can solve DAE's of constrained mechanical systems. We can solve it directly by applying a BDF or projection method. On the other hand we can use indirect methods, of
which stabilising Lagrange methods will give the best results. However, it is not obvious which one of these three approaches should be applied for simulation of the mirror system. Because of availability of computer code and convenience of implementation (no modification of the DAE needed), it was decided to use a variable stepsize, variable order BDF method. Nevertheless it should be said that the projection method is very appealing, as it seems to require very few calculations per integration step, whereas it always satisfies the constraint. Unfortunately no computer code for this was available.
Chapter 6

Simulations of the mirror system using DASSL

For simulation of the nonlinear model of the mirror system a BDF based computer code DASSL (Differential Algebraic System SoLver) was available. Originally this code was written to solve index one DAE's. After some modifications the package can also solve index three DAE's according to theorem 2. First the algorithm behind DASSL and the modification for index three DAE's will be explained. Next some simulation results will be shown.

6.1 The DAE solver DASSL

DASSL was written for solving DAE's of the form

\[ F(\dot{y}, y, t) = 0 \]  \hspace{1cm} (6.1)

The underlying idea for solving a DAE of this form is to replace the derivative by a backward difference approximation, for example Euler's first order approximation:

\[ F\left(\frac{y_n - y_{n-1}}{h}, y_n, t_n\right) = 0 \]  \hspace{1cm} (6.2)

as was explained in section 5.2.1. From this equation the solution \( y_n \) at time \( t_n \) is solved by using Newton's iteration method:

\[ y_n^{m+1} = y_n^m - \Gamma^{-1} F\left(\frac{y_n^m - y_{n-1}}{h}, y_n^m, t_n\right) \]  \hspace{1cm} (6.3)

where

\[ \Gamma = \left(h^{-1} \frac{\partial F}{\partial \dot{y}} + \frac{\partial F}{\partial y}\right) \]  \hspace{1cm} (6.4)

is called the iteration matrix and \( m \) is the iteration index. The algorithms in DASSL are an extension of this approach. But instead of using the first order approximation as in (6.2), it uses difference approximations of order one to five. Each timestep the algorithm choses an approriate order and stepsize \( h \), based on the previous solution. After a solution of the Newton iteration is found, a local truncation error is estimated. If this error is larger than the tolerances specified by the user, the stepsize is reduced and/or the order is changed and the step is attempted again. The description of the DAE problem is implemented in routine 'RES3' in the form \( \text{DELTA} = F(\text{YPRIME}, Y, T) \) according to 6.1. The differential equations for a controller for the mirror system can be added in this routine. Further details about the implementation of DASSL are found in [24]. Practical implications of using DASSL are explained in the source code.
6.2 Modifications to DASSL

When index three problems are solved with the original version of DASSL, the code doesn't produce very good results. For problems other than index one, DASSL should be modified. In [22] Petzold and Lötstedt show that the iteration matrix $\Gamma$ of a DAE problem in the general form of (B.1), (B.2) and (B.3) leads to roundoff errors in $p$, $\nu$ and $\lambda$ proportional to $p/h$, $p/h^2$ and $p/h^3$. This can be improved by scaling the constraints in (B.3) by $1/h$. Now the roundoff errors of the Newton iteration are proportional to $p$, $p/h$ and $p/h^2$. This scaling should be implemented in DASSL by the user. Stepsize $h$ is passed to routine 'RES3', so that the constraints can be scaled by $1/h$. Although the errors in $\nu$ and $\lambda$ become considerably smaller than $p/h^2$ and $p/h^3$ (as in the unscaled problem), we still have to be careful when using these variables in error tests in the code. The unreliability of $\nu$ and $\lambda$ isn't a very big problem, since we aren't very much interested in these variables. But they are used in the test for terminating the Newton iteration and for testing the local truncation error. Because of the unreliability we could exclude $\nu$ and $\lambda$ from the tests, but here it was chosen to scale the error estimates of $\nu$ and $\lambda$ by respectively $h$ and $h^2$ for both tests.

6.3 Simulation results

Simulations of the current controlled and the voltage controlled mirror system have been performed with the modified version of DASSL. The behaviour of the mirror after a step on each of the actuators is shown in the next section. For these simulations DASSL was set to compute the Newton iteration matrix automatically. Absolute and relative tolerances were chosen equal for all variables. All simulations were performed in some seconds, which is much faster than the simulation method used in [31]. As mentioned before, the latter needed one hour to compute the same solutions.

6.3.1 Simulation of the voltage controlled system

In figure 6.1 the behaviour of the mirror system in terms of Bryant angles after a step of 0.15 V on each of the actuators can be seen. These results correspond to the results in [31]. We can observe that a step on actuator 1 affects only $\theta$, whereas steps on the other actuators affect all angles.

Besides the Bryant angles, the values of $\bar{h}$, $g(\bar{v})$, $E_A$ and $\lambda$ are computed. In figure 6.2 the behaviour of these variables after a step of 0.15 V on actuator 3 is shown. Because the constraint forces $\lambda$ are the tensile forces in the strings (except for a constant factor), they must be permanently negative. Only in that case, the simulation is valid. From figure 6.2 we conclude that this is true. We conclude that the violation of the constraints is of order $10^{-9}$, which shows that DASSL satisfies the constraint rather accurately. This accuracy could be improved by setting lower tolerances, but this results in high frequency oscillations in the solution. For the above results the absolute and relative tolerances of the local truncation errors were chosen $10^{-3}$. The small differences between $h_1$ and $h_2$ respectively $\lambda_1$ and $\lambda_2$ in figure 6.2 are probably a result of the accumulation of many errors of the order $10^{-3}$. By choosing different tolerance values for the different variables we may be able to improve the accuracy of the simulation results without oscillations. Introducing small damping parameters in the model may also prevent high frequency oscillations.
6.3.2 Simulation of the current controlled system

In figure 6.3, 6.4 and 6.5 the behaviour of the mirror in terms of $\psi$, $\theta$ and $\phi$ after a step of 5 mA on respectively actuator 1, 2 and 3 is shown. Since the system is undamped for current control, the system exhibits a stable oscillation. The higher frequency oscillations are a numerical problem. This may be solved as mentioned in the previous section. For these simulations tolerances of $10^{-4}$ were used.
Simulation results

Figure 6.3: Orientation the mirror system after a step of 5 mA on actuator 1

Figure 6.4: Orientation the mirror system after a step of 5 mA on actuator 2

Figure 6.5: Orientation the mirror system after a step of 5 mA on actuator 3
Chapter 7

The linearized model

Since most control theory is based on linear models, we have to find a linear model for the mirror system. Therefore the DAE is linearized and the result is described in ODE form. Next we will look at the differences between the linear and the nonlinear system.

7.1 The linear DAE model

The DAE model of section 3.6 is linearized for small values of the Bryant angles by taking the first order Taylor approximations $\dot{Q}$ and $\ddot{q}$ of $Q$ and $\ddot{q}$:

$$
\dot{Q} = \begin{pmatrix}
0 \\
0 \\
-D\dot{h}_1 - K\dot{h}_1 \\
-D\dot{h}_2 - K\dot{h}_2 \\
-D\dot{h}_3 - K\dot{h}_3 \\
\end{pmatrix}
$$

(7.1)

$$
\ddot{q} = \begin{pmatrix}
-h_1 + r\theta \\
-h_2 - \frac{1}{2}r\theta - \frac{1}{2}\sqrt{3}r\psi \\
-h_3 - \frac{1}{2}r\theta + \frac{1}{2}\sqrt{3}r\psi \\
\end{pmatrix}
$$

(7.2)

The model is completed by computing the Jacobian $\tilde{G}$ of $\ddot{q}$:

$$
\tilde{G} = \begin{pmatrix}
0 & r & 0 & -1 & 0 & 0 \\
-\frac{1}{2}r\sqrt{3} & -\frac{1}{2}r & 0 & 0 & -1 & 0 \\
\frac{1}{2}\sqrt{3} & -\frac{1}{2}r & 0 & 0 & 0 & -1 \\
\end{pmatrix}
$$

(7.3)

Since $\tilde{G}^T\lambda$ doesn’t depend on any of the generalized coordinates, it is linear in $\lambda$. Now we have derived a linear DAE model of the mirror system in Euler-Lagrange form. Physically the above constraints can be interpreted as if the orientation of the tensile forces $F_i$ are constant (equal to that in the equilibrium position). This means that the mirror isn’t excitated to turn around its normal, i.e. angle $\phi$ is zero in the linear model.

Notice that from (3.3) and (3.4) it follows that for small angles normal angles $\alpha$ and $\beta$ are equal to the Bryant angles $\psi$ and $\theta$. 
### 7.2 The linear model in ODE form

The linear DAE system of the previous section can be transformed to an ODE using the method of section 5.3.6. The sets of dependent and independent coordinates is chosen as follows.

\[ \mathbf{u} = (h_1, h_2, h_3)^T \]  \hspace{1cm} (7.4)

\[ \mathbf{w} = (\psi, \theta, \phi)^T \]  \hspace{1cm} (7.5)

From (7.2) it follows that the function \( h \) in \( \mathbf{u} = h(\mathbf{w}) \) is defined by \( \tilde{G}_{w}w \). The functions \( h' \) and \( h'' \) are defined as \( \tilde{G}_{w}w' \) and \( \tilde{G}_{w}w'' \).

We can solve \( \lambda \) from

\[ m\ddot{\mathbf{u}} = Q^u + E^u_A + \tilde{G}_u^T \lambda \]  \hspace{1cm} (7.6)

because \( \tilde{G}_u^T = -I \) is a regular matrix. After substituting \( Q^u = -D\mathbf{u} - K\mathbf{u} \) this results in

\[ \lambda = m\ddot{\mathbf{u}} + D\dot{\mathbf{u}} + K\mathbf{u} + E^u_A \]  \hspace{1cm} (7.7)

This gives rise to the following differential equation in \( \mathbf{w} \).

\[ (jI + mG_w^T G_w)\ddot{\mathbf{w}} = DG_w^T \dot{\mathbf{w}} + KG_w^T \mathbf{w} + G^T e_A(\mathbf{w}, \dot{\mathbf{w}}) \]  \hspace{1cm} (7.8)

The actuator forces are different for voltage and current control, so we get different models.

#### 7.2.1 The linear model for current control

For current control (7.8) leads to the following transfer functions

\[ \psi = \frac{Ar\sqrt{3}(I_2 - I_3)}{N_c} \]  \hspace{1cm} (7.9)

\[ \theta = \frac{Ar(-2I_1 + I_2 + I_3)}{N_c} \]  \hspace{1cm} (7.10)

\[ \phi = 0, \text{ where} \]  \hspace{1cm} (7.11)

\[ N_c = (2j + 3r^2m)s^2 + 3r^2Ds + 3r^2K \]  \hspace{1cm} (7.12)

If matrixes \( B \) and \( \hat{B} \) are defined as

\[ B = \begin{pmatrix} 0 & 1 & -1 \\ -2 & 1 & 1 \end{pmatrix} \]  \hspace{1cm} (7.13)

\[ \hat{B} = \begin{pmatrix} 0 & -1/3 \\ 1/2 & 1/6 \\ -1/2 & 1/6 \end{pmatrix} \]  \hspace{1cm} (7.14)

such that

\[ \begin{pmatrix} I_{\psi} \\ I_{\theta} \end{pmatrix} = B \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \]  \hspace{1cm} and

\[ \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \hat{B} \begin{pmatrix} I_{\psi} \\ I_{\theta} \end{pmatrix} \]  \hspace{1cm} (7.15)

(7.16)
we can write (7.9) and (7.10) as

\[ \frac{\psi}{I_\psi} = \frac{Ar\sqrt{3}}{N_c} = H_{\psi,I} \]  

(7.17)

\[ \frac{\theta}{I_\theta} = \frac{Ar}{N_c} = H_{\theta,I} \]  

(7.18)

When all numerical values, including \( D=0 \), are substituted in (7.12) this gives rise to two conjugate complex poles at \( \pm 62.43j \). The corresponding resonance frequency is 9.93 Hz. It is obvious that we get two complex poles, because we didn’t introduce any damping in the system. In [31] two extra poles at \( \pm 1.458 \cdot 10^4 j \) were found. These poles are the result of substituting the actuator strings by stiff springs (as was explained in chapter 4). Since poles are the eigenvalues of the system we can conclude that the maximum eigenvalue is much larger than the minimum eigenvalue. According to definition 1, the simulation of the system with the extra springs is indeed a stiff problem.

The Bode plots of \( H_{\psi,I} \) and \( H_{\theta,I} \) are drawn in figure 7.1. We can clearly see the resonance frequency at 9.93 Hz.

![Bode plots of H_{\psi,I} and H_{\theta,I}](image)

**Figure 7.1:** Bode plots of \( H_{\psi,I} \) and \( H_{\theta,I} \)

### 7.2.2 The linear model for voltage control

For voltage control we get the following transfer functions.

\[ \psi = \frac{Ar\sqrt{3}(V_2 - V_3)}{N_v} \]  

(7.20)

\[ \theta = \frac{Ar(-2V_1 + V_2 + V_3)}{N_v} \]  

(7.21)

\[ \phi = 0, \text{ where} \]  

(7.22)

\[ N_v = L(2j + 3r^2m)s^3 + (3r^2DL + 3r^2mR + 2jR)s^2 + 3r^2(DR + KL + A^2)s + 3r^2KR \]  

(7.23)
Using $B$ and $\tilde{B}$ as in the previous section yields functions

\[
\frac{\psi}{V_{\psi}} = \frac{Ar\sqrt{3}}{N_o} = H_{\psi,v} \tag{7.24}
\]
\[
\frac{\theta}{V_{\theta}} = \frac{Ar}{N_o} = H_{\theta,v} \tag{7.25}
\]

These third order functions turn out to have poles at -14.74, -289.6 and -3196 when numerical values are substituted. The -3dB points are at 2.34, 46.1 and 509 Hz. Like for current control two more poles were found in [31] due to the extra springs. Since we have introduced damping because of the voltage control we have a damped system here. The negative poles point out that the system is stable, as was expected from experiments.

The Bode plots are drawn in figure 7.2.

![Bode plots of $H_{\psi,v}$ and $H_{\theta,v}$](image.png)

Figure 7.2: Bode plots of $H_{\psi,v}$ and $H_{\theta,v}$

### 7.3 Simulations of the linear model

The linear system was simulated in DAE form using DASSL. Several experiments were carried out. The results of step functions on $I_{\psi}, I_{\theta}, V_{\psi}$ and $V_{\theta}$ were computed and compared to the result of such a step function on the nonlinear system. Inputs were chosen $I_{\psi}, I_{\theta} = 50$ mA for current control and $V_{\psi}, V_{\theta} = 0.15$ V for voltage control. For comparable results the pretensioning force $F_0$ had to be chosen 0. In the next section we will discuss this further. The results of the linear system are drawn in solid lines, the results of the nonlinear system are drawn in dashed lines.
Simulations of the linear model

Figure 7.3: Simulation results of step on $I_\psi$

Figure 7.4: Simulation results of step on $I_\theta$

In these figures we see that for current control the outputs have an oscillation frequency of 9.9 Hz, as we expected from the linear model. For steps on $I_\theta$ the differences between the linear and the nonlinear model are neglectable. For steps on $I_\psi$ there is a considerable difference. First there is the difference in the output for angle $\psi$. The nonlinear model shows an oscillation with non constant amplitude. Besides the behaviour of the angles $\theta$ and $\phi$ isn’t small like in the linear model. This phenomenon also occurs for voltage control, though less extreme.
7.4 Model errors of the linear system

The differences between the linear and the nonlinear system can be seen as model errors in the linear model. Controllers for the mirror system can be developed based on the linear model, but so that they have a reasonable behaviour despite the model errors. Examples are $H_{\infty}$ and $\mu$ controllers. For $\mu$ controllers the model errors have to be structured in a certain way. Here we will show some important errors and their influence. A further study on model errors and how to use them in a robust controller design, should be the subject for future research.
7.4.1 Observability and controllability of $\phi$

In the linear model the rotation around the mirror’s normal $\phi$ isn’t excited. Moreover this angle isn’t measured. It is neither observable nor controllable. From simulations of the nonlinear system, however, we have seen that this rotation can have a considerable magnitude. This could cause oscillations or other undesirable behaviour. Future simulations will have to show if the uncontrollability is a serious problem.

7.4.2 Influence of the pretensioning forces

In the previous section we have taken the pretensioning force of the actuator springs $F_0 = 0$, whereas the real value is estimated at 0.5 N. From the linear model we should expect that this force has no influence on the behaviour of the system, because it doesn’t occur in the transfer functions (7.9)-(7.23). However, from simulations of the nonlinear system it appears that $F_0$ has a considerable effect on the output, as can be seen in figures 7.7 and 7.8. For different values of $F_0$ the solutions of $\psi$ and $\theta$ were computed. It appears that the final values of $\psi$ and $\theta$ are much larger for $F_0 = 0.5$ N, than we should expect from figures 7.5 and 7.6. Besides the influence of $F_0$ is nonlinear as can be seen in the second plots. The range of $F_0$ from 0.3 N to 0.7 N corresponds to a difference in length of the springs of 1 cm. The pretensioning force is used to adjust the static airgap height in the bearing, so the value of 0.5 N may be altered. With the right plots in figures 7.7 and 7.8 the corresponding final values can be estimated.

This behaviour can be explained by looking at the geometrical properties of the mirror system. In the linear model it was assumed that the orientation of the forces $F_i$ didn’t change from the equilibrium orientation. No torque results from the pretensioning forces $F_0$. In the real system there is a resulting torque as a consequence of these forces, which makes the mirror rotate more than we expected from the linear model.

![Figure 7.7: Simulation results of step on $V_\psi$ with different values of $F_0$](image)
7.4.3 Linearity of final values

Finally we will investigate the behaviour of the final values of $\psi$ and $\theta$, when the nonlinear system is excited with step function of different magnitude on the inputs $V_\psi$ and $V_\theta$. In figure 7.9 the behaviour of the solutions of $\theta$ for $V_\theta$ in the range from 0.05 V to 0.25 V. $F_0$ was chosen 0.5 N during this simulation. We see that $V_\theta$ has a linear effect on $\theta$. The solution for $\psi$ was smaller than $10^{-12}$ and can be neglected. When we look at the results of steps on $V_\psi$ in figure 7.10, we see that the effect of $V_\psi$ on $\psi$ is also linear. However, the (side) effect on $\theta$ is nonlinear. It isn’t obvious how to take this into account as a model error in the linear model.

Figure 7.8: Simulation results of step on $V_\theta$ with different values of $F_0$

Figure 7.9: Simulation results of steps with different magnitude on $V_\theta$
Model errors of the linear system

Figure 7.10: Simulation results of steps with different magnitude on $V_\psi$. 
Chapter 8

Conclusions

Since the mirror system is a multibody system, we can model its rotational dynamics conveniently by a DAE. This DAE can’t be solved by a classic ODE solver. Transforming the DAE straightforwardly to an ODE results in an unmanageable model. By replacing the strings between the actuators and the mirror by springs, we can also obtain an ODE model of the system. But this model yields a stiff numerical problem, resulting in a very time consuming simulation. Therefore methods to solve DAE directly were investigated. Although DAE’s, and especially higher index DAE’s, are generally more difficult to solve numerically than ODE’s, we can conclude that there are several interesting methods that can solve the mirror systems DAE (and DAE’s of any multibody system): BDF and projection methods and stabilizing techniques like Baumgarte or with extra Lagrange multipliers.

The computer code DASSL, implementing a variable stepsize and variable order BDF method, is now available. With this code any multibody system can be simulated. We used this package to simulate the mirror system with satisfactory results. The results corresponded to those of the method in [31], but were obtained much faster.

On the other hand we tried to derive a linear model that is a good approximation of the mirror system. This linear model will be used to design controllers for mirror. However, from the simulations we have to conclude that the linear model derived in this report by linearizing the original model (for small rotations) around the equilibrium point, doesn’t satisfy. Especially the influence of the pretensioning forces, side effects of actuator movements and the uncontrollability of the rotation around the mirrors normal vector, cause large differences between the linear and the nonlinear model.

Finally it appears that extra research has to be done on the accuracy of the position measurement carried out by the complete laser tracking system. We have seen that the measurement of the vertical position of the mirror is cursed with an uncertainty because of the horizontal movement of the mirror. Moreover we have seen that the formula to determine the orientation of the mirror from the currents in the anglesensors is very complex. This makes it difficult to obtain an accurate measurement of the orientation of the mirror.
Chapter 9

Recommendations

In further research mirror system, special attention should be paid to the following items. The nonlinear DAE model should be extended for horizontal and vertical movement of the mirror. This has effect not only on the behaviour of the mirror, but also on the orientation measurement. Since the flux through the secondary coils of the angle sensor changes, when the mirror moves vertically, this disturbs the angle measurements. When extending the model, the Bryant angles should be eliminated. The use of normal angles \( \alpha \) and \( \beta \) together with a rotation around the normal \( \phi \), will not lead to more complicated expressions, whereas we don't have to convert the Bryant angles to normal angles anymore.

Moreover an extensive study should be made to the overall accuracy of the laser tracking system. The system should determine positions with an accuracy of order \( 10^{-5} \) m, but there are several effects in the system that give rise to a lower accuracy. Next to those we have mentioned before, there is the precision of the electronic equipment which is used in the different sensorsystems.

Finally attention should be paid to control and identification theory based on DAE systems. In [30] is explained how to design LQG controllers for linear DAE systems. It appears that many results of the linear control theory for state space models can be applied also on linear DAE's. In this report we could transform our linear DAE model to an ODE very easily, but this isn't true for all models. Therefore control and identification theory is a very interesting subject.
Bibliography

[1] Mathis, W.
*Recent Developments in Numerical Integration of Differential Equations*

[2] Campbell, S.L. and Moore, E.
*Progress on a General Numerical Method for Nonlinear Higher Index DAEs II*

*Convergence Results for a Coordinate Projection Method Applied to Mechanical Systems with Algebraic Constraints*
SIAM Journal on Numerical Analysis, Vol. 30 (1993), No. 5, p. 1467-1482

*Differential/Algebraic Equations as Stiff Ordinary Differential Equations*
SIAM Journal on Numerical Analysis, Vol. 29 (1992), No. 6, p. 1694-1715

[5] Dam, A.A. ten
*Stable Numerical Integration of Dynamical Systems Subject to Equality State-Space Constraints*

[6] Petzold, L.R.
*Recent Developments in the Numerical Solution of Differential/Algebraic Systems*

*On Modelling and Differential/Algebraic Systems*

*A New Technique for Solving High-Index Differential-Algebraic Equations Using Dummy Derivatives*
*PROJECTED COLLOCATION FOR HIGER-ORDER HIGHER-INDEX DIFFERENTIAL-ALGEBRAIC EQUATIONS*  

[10] Campbell, S.L.  
*DESCRIPTOR SYSTEMS IN THE 1990s*  

*NUMERICAL METHODS FOR DIFFERENTIAL ALGEBRAIC EQUATIONS*  

[12] Barrlund, A.  
*CONSTRAINED LEAST SQUARES METHODS FOR LINEAR TIMEVARYING DAE SYSTEMS*  

[13] Reich, S.  
*ON A GEOMETRICAL INTERPRETATION OF DIFFERENTIAL-ALGEBRAIC EQUATIONS*  

[14] Petzold, L.R.  
*METHOD AND SOFTWARE FOR DIFFERENTIAL-ALGEBRAIC SYSTEMS*  
Berlin: Springer-Verlag, 1991, p. 127-140

[15] Führer, C. and Leimkuhler, B.  
*A NEW CLASS OF GENERALIZED INVERSES FOR THE SOLUTION OF DISCRETIZED EULER-LAGRANGE EQUATIONS*  
Berlin: Springer-Verlag, 1991, p. 143-154

[16] Ostermeyer, G.  
*ON BAUMGARTE STABILIZATION FOR DIFFERENTIAL ALGEBRAIC EQUATIONS*  
Berlin: Springer-Verlag, 1991, p. 193-208
[24] Petzold, L.R.  
A DESCRIPTION OF DASSL: A DIFFERENTIAL/ALGEBRAIC SYSTEM SOLVER  
In: IMACS Transactions on Scientific Computation, Ed. by R.S. Stepleman, Montreal,  
August 1982  
Amsterdam: North Holland, 1983, Vol. 1, p. 65-68

[25] Brenan, K.E. and Engquist, B.E.  
BACKWARD DIFFERENTIATION APPROXIMATION OF NONLINEAR DIFFERENTIAL/ALGEBRAIC SYSTEMS  

[26] Baumgarte, J.  
STABILIZATION OF CONSTRAINTS AND INTEGRALS OF MOTION IN DYNAMICAL SYSTEMS  

[27] Führer, C. and Leimkuhler, B.  
FORMULATION AND NUMERICAL SOLUTION OF THE EQUATIONS OF CONSTRAINED MECHANICAL MOTION  

[28] Gear, C.W. and Keiper, G.  
THE ANALISYS OF GENERALIZED BDF METHODS APPLIED TO HESSENBERG FORM DAEs  

[29] Gear, C.W.  
DIFFERENTIAL-ALGEBRAIC EQUATION INDEX TRANSFORMATIONS  

CONTROL ANALYSIS AND SYNTHESIS OF LINEAR MECHANICAL DESCRIPTOR SYSTEMS  
In: Advanced Multibody Systems, Ed. by W. Schielen,  
Kluwer, 1993, p. 463-468

[31] Slots, T.D.  
MODELLING AND IDENTIFICATION OF A LASER DEFLECTING SYSTEM  
Measurement and Control Group, Faculty of Electrical Engineering, Afstudeerverslag, Eindhoven: University of Technology, 1994

[32] Jong, M. de  
ANALYSE VAN EEN 3-D LASERAFBUIGSYSTEEM  
Fundamental Mechanics Group, Faculty of Mechanical Engineering, Stageverslag, Eindhoven: University of Technology, 1986

[33] Sol, E.J.  
KINEMATICS AND DYNAMICS OF MULTIBODY SYSTEMS,  
a systematic approach to systems with arbitrary connections  
Eindhoven: Technische Universiteit Eindhoven, 1983  
Doctoral Dissertation
[34] Wang, J.
_DESIGN OF GAS BEARING SYSTEMS FOR PRECISION APPLICATIONS_
Eindhoven: Technische Universiteit Eindhoven, 1993
Doctoral Dissertation

[35] Zorge, R.A.
_MEASUREMENTS ON A MINIATURE SPHERICAL AIR BEARING BY USING ITS ELECTRICAL GAP CAPACITANCE_
Measurements, Vol. 11 (1993), p. 159-172

[36] Iversen, T.
_MODULAR TECHNIQUES IN DYNAMIC PROCESS SIMULATION_
Trondheim: University of Trondheim, Norwegian Institute of Technology, 1990
Doctoral Dissertation

[37] Sauren, A.
_MULTIBODY DYNAMICA_
Eindhoven: Technische Universiteit Eindhoven, 1986
Dictaatnr. 4659
Appendix A

Definitions of vectors

In figure A.1 the definition of the global frame in relation to the mirror and the angle sensor coils and the laser source is shown. The positive $X$ axis is defined in the direction of the deflected laserbeam. From this definition the coordinates of the vectors $a_i$, $C_i$ and $O_i$ follow. In table A.1 they are given in respectively the local and the global frame. $O_i$ is computed with the geometrical relation $O_i = a_{iL} - lC_i$, which is valid in equilibrium.

![Figure A.1: Top view of mirror system with global frame](image)

Table A.1: Definition of characteristic vectors in the mirror system

<table>
<thead>
<tr>
<th>Vector</th>
<th>Actuator 1</th>
<th>Actuator 2</th>
<th>Actuator 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{iL}$ (local)</td>
<td>$\begin{pmatrix} -r \cos(\epsilon) \ 0 \ r \sin(\epsilon) \end{pmatrix}$</td>
<td>$\begin{pmatrix} \frac{1}{2}r \cos(\epsilon) \ -\frac{1}{2}\sqrt{3}r \cos(\epsilon) \ r \sin(\epsilon) \end{pmatrix}$</td>
<td>$\begin{pmatrix} \frac{1}{2}r \cos(\epsilon) \ \frac{1}{2}\sqrt{3}r \cos(\epsilon) \ r \sin(\epsilon) \end{pmatrix}$</td>
</tr>
<tr>
<td>$C_i$ (global)</td>
<td>$\begin{pmatrix} \sin(\epsilon) \ 0 \ \cos(\epsilon) \end{pmatrix}$</td>
<td>$\begin{pmatrix} -\frac{1}{2}\sin(\epsilon) \ \frac{1}{2}\sqrt{3}\sin(\epsilon) \ \cos(\epsilon) \end{pmatrix}$</td>
<td>$\begin{pmatrix} -\frac{1}{2}\sin(\epsilon) \ -\frac{1}{2}\sqrt{3}\sin(\epsilon) \ \cos(\epsilon) \end{pmatrix}$</td>
</tr>
<tr>
<td>$O_i$ (global)</td>
<td>$\begin{pmatrix} -r \cos(\epsilon) - l \sin(\epsilon) \ 0 \ r \sin(\epsilon) - l \cos(\epsilon) \end{pmatrix}$</td>
<td>$\begin{pmatrix} \frac{1}{2}{r \cos(\epsilon) - l \sin(\epsilon)} \ -\frac{1}{2}\sqrt{3}{r \cos(\epsilon) + l \sin(\epsilon)} \ r \sin(\epsilon) - l \cos(\epsilon) \end{pmatrix}$</td>
<td>$\begin{pmatrix} \frac{1}{2}{r \cos(\epsilon) + l \sin(\epsilon)} \ \frac{1}{2}\sqrt{3}{r \cos(\epsilon) + l \sin(\epsilon)} \ r \sin(\epsilon) - l \cos(\epsilon) \end{pmatrix}$</td>
</tr>
</tbody>
</table>
Appendix B

Euler-Lagrange description of the mirror system

The mirror system is described by

\[ \begin{align*}
\mathbf{u} &= \dot{\mathbf{p}} \\
M(p)\ddot{p} &= Q(\mathbf{u}, p, t) + E(t) + G^T(p)\Delta \\
g(p) &= 0
\end{align*} \]

where

\[ \mathbf{p} = \begin{pmatrix} \psi \\ \theta \\ \phi \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} \]

\[ M = \begin{pmatrix} jI & 0 \\ 0 & mI \end{pmatrix} \]

\[ Q = \begin{pmatrix} -\left(\frac{1}{\cos(\theta)}\dot{\phi}\dot{\theta} - \tan(\theta)\dot{\psi}\dot{\theta}\right) \\ -\left(-\cos(\theta)\dot{\psi}\dot{\phi}\right) \\ -\left(-\tan(\theta)\dot{\phi}\dot{\theta} + \frac{\sin^2(\theta)}{\cos(\theta)}\dot{\psi} + \cos(\theta)\dot{\psi}\right) \\ -D\dot{h}_1 - Kh_1 \\ -D\dot{h}_2 - Kh_2 \\ -D\dot{h}_3 - Kh_3 \end{pmatrix} \]

\[ \mathbf{F} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -F_0 + F_{A1} \\ -F_0 + F_{A2} \\ -F_0 + F_{A3} \end{pmatrix} \]

\[ g(p) = \begin{pmatrix} \frac{1}{2}\|R_8a_{1L} - Q_1 + h_1C_1\|_2^2 - l^2 \\ \frac{1}{2}\|R_8a_{2L} - Q_2 + h_2C_2\|_2^2 - l^2 \\ \frac{1}{2}\|R_8a_{3L} - Q_3 + h_3C_3\|_2^2 - l^2 \end{pmatrix} \]
Now $g_i(p)$ be written as

$$g_i(p) = \frac{1}{2} h_i^2 - h_i l - (Q_i + h_i C_i)^T R \delta \theta_i + r^2 \quad (B.9)$$

This yields

$$G(p) = \begin{pmatrix} G_{1\psi} & G_{1\theta} & G_{1\phi} & G_{h1} & 0 & 0 \\ G_{2\psi} & G_{2\theta} & G_{2\phi} & 0 & G_{h2} \\ G_{3\psi} & G_{3\theta} & G_{3\phi} & 0 & 0 & G_{h3} \end{pmatrix} \quad (B.10)$$

where

$G_{1\psi} = -(Q_i + h_i C_i)^T \frac{\partial R}{\partial \psi} \delta \psi_i$,

$G_{1\theta} = -(Q_i + h_i C_i)^T \frac{\partial R}{\partial \theta} \delta \theta_i$,

$G_{1\phi} = -(Q_i + h_i C_i)^T \frac{\partial R}{\partial \phi} \delta \phi_i$ and

$G_{h1} = h_i - C_i^T R \delta \theta_i - l$. 

Appendix C

List of symbols

\( \mathbf{n} \)   Mirror surface normal vector
\( e^0 = [X, Y, Z] \)   Global frame
\([x, y, z]\)   Local frame
\([X', Y', Z]\)   Global frame through the secondary coils of angle sensor
\( \alpha, \beta \)   Mirror normal vector angles defined in \( [X, Y, Z] \)
\( \alpha', \beta' \)   Mirror normal vector angles defined in \( [X', Y', Z] \)
\( \xi, \chi \)   Angles measured by angle sensor
\( \gamma = (\psi, \theta, \phi)^T \)   Vector of Bryant angles
\( r \)   Distance from centre of mirror to connection of actuator string
\( R_m \)   Radius of semi sphere
\( j \)   Moment of inertia around the centre of the mirror’s surface
\( \epsilon \)   Angle between arm of actuator and the mirror surface
\( l \)   Length of string connecting the mirror to the actuator
\( d, d_v \)   Height of air gap between mirror and bearing seat
\( d_h \)   Horizontal movement of mirror in the bearing seat
\( g_i \)   Connection point of the string at the mirror
\( A_i \)   Connection point of the string at the actuator
\( Q_i \)   Equilibrium position of top of actuator
\( h_i \)   Movement of actuator relative to offset position \( Q_i \)
\( C_i \)   Direction of actuator movement (unit vector)
\( m \)   Mass of coil
\( F_0 \)   Pretensioning force
\( K \)   Spring constant of actuator spring
\( D \)   Damping in the actuator
\( F_i \)   Tensile force exerted by the string on the actuator
\( F_A \)   Actuator force
\( I_i \)   Current to control actuator
\( V_i \)   Voltage to control actuator
\( L \)   Induction of the coil
\( R \)   Resistance of the coil
\( A \)   Actuator constant depending on coil radius, number of windings and magnitude of the magnetic field