MASTER

Interactions in an N finite-dipole system

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Award date:
2015

Link to publication
Interactions in an N finite-dipole system

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R-1848-A
February 2015
master thesis

Interactions in an N finite-dipole system

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internship period;
2-2014 to 2-2015
Abstract

In the present study the interactions between finite-dipoles have been investigated. It is found that two finite-dipoles can have three types of interactions, i.e: scattering, annihilation, or trapping. These interactions are classified into regime diagrams. If the dipoles are placed with point symmetry, the system remains symmetric and therefore the dynamics can be simplified. The simplified dynamics is governed by a so-called "w-function", and the pathlines obtained by numerically solving the w-function can indicate the existence of a stable, unstable or asymptotically stable two-dipole configuration. The w-function is also derived for the symmetric three- and four-dipole configuration. Again, the w-function gives an insight in the existence of stable, unstable, or asymptotically stable dipole configurations. However, the symmetry of a system with \( N \geq 3 \) dipoles is not always conserved. The four-dipole system disintegrates for small disturbances into two two-dipole systems.

Furthermore, we investigated the interaction of N-dipoles in a double-periodic domain. It is found that the interactions in a diluted system consists mostly of two-dipole interactions where the dipoles are scattered or annihilated. Three- and four-dipole interaction are also observed, where a third dipole can modify the annihilation process of two dipoles. A fourth dipole, interacting with the three-dipole system, can make the three-dipole configuration unstable again. It turns out that the annihilation rate scales according to a power law that is given by \( \rho \sim t^{-1} \) when the density \( \rho < 0.05 \). This is in agreement with a model for ballistic annihilation described by the Boltzmann equation.
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1 Introduction

A vortex is a coherent fluid structure containing fluid that is in swirling motion. Vortices are a commonly observed phenomenon in nature and play an important role in many flow situations. They exist for a wide range of length and time scales, e.g. ranging from large eddies in the atmosphere such as tornadoes and hurricanes to small eddies caused by insects as Gerridaes and vinegar flies. Some examples of vortex structures are shown in figure 1.1. Other examples of vortices are smoke rings, whirlpools often seen in the wake of boats and paddles, dust devils, the Great Red Spot in Jupiter’s atmosphere and vortices formed in the wake of airplanes.

![Figure 1.1](image)

**Figure 1.1** – Examples of vortices at different length scales. (a) A vortex in the atmosphere. (b) Dipole vortices created by Gerridaes, visualized with dye. (c) Naruto whirlpool in the ”Naruto Strait” channel in Japan. This vortex is caused by the tides and is so strong that a downdraft appears.

To study the behaviour of vortex structures or fluid motion in general, numerical simulations can be performed, which commonly implies solving the Navier-Stokes equation. Solving the Navier-Stokes equation can be computationally expensive and is for many purposes too detailed. Simplified models can be used to obtain a general understanding of elementary vortex structures or other flow problems. A widely used model is based on an idealised two-dimensional (2D) irrotational flow, i.e. a potential flow. This model uses point vortices and point sources as the building blocks to obtain complex flows. It is a valid simplification for high Reynolds numbers and far from the boundary layers, i.e. in parts of the flow domain where viscous effects play no role. Due to its simplicity it has been an intriguing research field for many decades with applications in various fluid dynamical topics: chaotic advection [21], vortex shedding [22, 23], geophysical applications [24, 25], 2D-turbulence [19, 26, 27], or biological locomotion [7, 15].

In two recent papers finite-dipoles were introduced as a self-propelling mechanism for fish [7, 15]. In [7] the interaction of finite-dipoles was investigated for different
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dipole configurations. The other paper [15], mainly concerned a study of the stability for different fish-school alignments.

The finite-dipole approach is a new concept in the potential flow, and has the remarkable feature that the separation distance \( l \) between the two point vortices is kept constant and is therefore unaffected by other dipoles. This in contrast to the classical point-dipoles where couple exchange can occur [9, 11, 12].

The aim of the present study is to obtain a broader knowledge of the finite-dipoles. Therefore we investigate the interactions (quantitatively) between two-, three-, four- and \( N \) finite-dipole configurations.

The outline of this report is as follows: In chapter 2 we start with the general theory for understanding fluid behaviour, especially vortices. Moreover, some basic point-vortex models are discussed, as well as the more complex finite-dipole model. After this chapter we continue with the results, in which we distinguish three main parts. Part I contains the results of finite-dipoles in an infinite plane. It includes chapter 3 that shortly explains the numerical method that is used in the remaining two chapters of this part. The interaction between two dipoles is the subject of chapter 4. Moreover, regime diagrams are introduced to quantify the interactions between two dipoles. Three- and four-dipole interactions are discussed in chapter 5. Again regime diagrams are made to quantify their interactions, but now only for a symmetric configuration. The stability of different dipole configurations is also discussed.

Part II concerns a study of finite-dipoles in a periodic domain. In chapter 6 we consider the necessary theory to simulate \( N \) finite-dipoles in a periodic plane. The numerical method and simplifications for the governing equations are explained and motivated in chapter 7. The results for \( N \) interacting dipoles are discussed in chapter 8. This chapter includes numerically obtained results of macroscopic system quantities, a model with analogous results from molecular dynamics, the explanation of a renormalization process, and a discussion about the place- and velocity-distribution function.

Finite-dipoles that are bounded by two walls are considered in part III. In chapter 9 we only formulate the governing equation for \( N \) finite-dipoles in a partly bounded domain.

Finally, chapter 10 contains the overall conclusion and recommendations for further research.
2 General theoretical background

In this chapter the elementary equations for understanding fluid behaviour, especially vortices, are discussed. We will start with the derivation of the relevant equations and concepts, and continue with potential flow theory. Then, some simple and some more complex vortex models are explained, which will be useful for understanding the simulations in section I. Moreover, a dimensional analysis is shortly discussed\(^1\).

2.1 Basic equations

The basic equations that govern fluid dynamics are those for the conservation of mass and momentum. Conservation of mass is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0,$$

while this equation can be simplified for an incompressible flow to

$$\nabla \cdot \vec{v} = 0.$$  \hspace{1cm} (2)

The equation for the conservation of momentum, commonly referred to as the Navier-Stokes equation, is given for a incompressible flow by

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} + \vec{g} + \frac{1}{\rho} \vec{f},$$

where \(\vec{v}\) is the velocity vector, \(\rho\) the fluid density, \(p\) the pressure, \(\nu\) the kinematic viscosity, \(\vec{g}\) the gravitational acceleration, and \(\vec{f}\) additional body forces. The left-hand side (LHS) of the Navier-Stokes equation can also be written as the material derivative \(\frac{D\vec{v}}{Dt}\), which represents the total acceleration of a volume element in space and time. The terms on the right-hand side (RHS) represent different volume forces. The Navier-Stokes equation is a set of three coupled second-order non-linear partial differential equations, which can mostly be solved only numerically.

The vorticity \(\vec{\omega}\) is defined as the curl of the velocity field \(\vec{v}\), and is a vector quantity i.e:

$$\vec{\omega} = \nabla \times \vec{v}.$$ \hspace{1cm} (4)

Taking the curl of the Navier-Stokes equation yields the vorticity equation:

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \nabla \vec{v} - \vec{\omega} \nabla \cdot \vec{v} + \frac{\nabla \rho \times \nabla p}{\rho^2} + \nu \nabla^2 \vec{\omega}.$$ \hspace{1cm} (5)

The vorticity equation describes the evolution of the vorticity \(\vec{\omega}\) of a fluid element. The LHS is again the material derivative of the vorticity vector \(\vec{\omega}\). The first term on the RHS describes vortex-tube stretching or tilting of vorticity due to the velocity gradients. The second term describes stretching of vorticity due to flow compressibility. The third

\(^1\) The lecture notes "Fysica van Transportverschijnselen" and "Voortgezette stromingsleer" are used in this as a guideline, see also [5] and [6].
term describes baroclinic effects, i.e. production of vorticity due to the intersection of density and pressure isosurfaces. Finally the last term on the RHS accounts for viscous diffusion.

By making some assumptions about the flow, the equations introduced above can be highly simplified. First of all, we assumed that the fluid was incompressible, i.e. $\rho$ is a constant, which reduces equation (1) to (2). With this assumption, the second term and the third term on the RHS of equation (5) are zero. Secondly, we assume an idealized inviscid fluid, i.e. $\nu = 0$, such that the last term on the RHS of equation (5) is zero. Moreover, we confine ourselves to a two-dimensional flow $\vec{v} = (u, v, 0)$, implying $w = 0$ and $\frac{\partial}{\partial z} = 0$. With this, the first term on the RHS of equation (5) is zero and the vorticity (4) becomes a scalar quantity:

$$\vec{\omega} = (0, 0, \omega_z), \text{ with } \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \quad (6)$$

Finally, we assume a barotropic flow where the iso-density planes coincide with isobaric planes, implying that the third term on the RHS of equation (5) is zero. Then, the vorticity equation reduces to

$$\frac{D\omega}{Dt} = 0. \quad (7)$$

This result expresses that the vorticity is conserved in a two-dimensional, incompressible, barotropic, inviscid flow. Once the vorticity field is known the corresponding velocity field can be calculated using the Biot-Savart law

$$\vec{v}(\vec{r}) = \frac{1}{4\pi} \iint_A \frac{\omega(\vec{r}'\vec{e}_z \times (\vec{r} - \vec{r}')} \cdot d\vec{r}'}{||\vec{r} - \vec{r}'||^3}, \quad (8)$$

which gives the velocity $\vec{v}$ for a two-dimensional case. Here $\vec{r}$ is the vector from the origin to a point of interest, while $\vec{r}'$ is the vector from the origin to a vorticity element.

Finally we introduce the circulation $\Gamma$, which is defined as

$$\Gamma = \oint_C \vec{v} \cdot d\vec{r} \equiv \iint (\nabla \times \vec{v}) \cdot \vec{n} \cdot dA = \iint_A \omega_z \vec{n} \cdot d\vec{A}. \quad (9)$$

The integral is evaluated around a closed contour $C$, $d\vec{r}$ is a line element of that contour. By using the Stokes theorem, one can find the relation between the circulation $\Gamma$ and the flux of vorticity $\vec{\omega}$ of the flow through a surface area $d\vec{A}$ (enclosed by contour $C$). As a result of this relation we can define that the circulation around a closed contour $C$ equals the flux of vorticity through a surface $A$ spanned by $C$.

2.1.1 Stream function and velocity potential

In this section the concept of the stream function and velocity potential will be derived for a two-dimensional flow, for which $\frac{\partial}{\partial z} = 0$. These scalar quantities are useful for the simplifications of the mathematics and the introduction of the complex potential in section 2.1.2.
Stream function: For a two-dimensional incompressible flow, equation (2) can be rewritten as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{10}$$

On the basis of this equation we may introduce a scalar quantity, the stream function $\psi$, which is defined as

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{11}$$

By substitution of the stream function into the vorticity equation (6), we obtain

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega_z. \tag{12}$$

Considering a constant-density and irrotational flow, $\omega_z$ will be zero everywhere except at the locations of point vortices [1]. In that case, the stream function $\psi$ is governed by the Laplace equation,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{13}$$

Velocity potential: In a similar fashion we will define the velocity potential. We consider the flow as irrotational $\nabla \times \vec{v} = 0$, i.e.

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{14}$$

On the basis of this equation we can introduce the velocity potential $\phi$, defined as

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}. \tag{15}$$

Substitution of the velocity potential into the incompressibility equation (2) gives,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{16}$$

Apparently, the velocity potential $\phi$ satisfies a Laplace equation.

2.1.2 Complex potential

The theory that is introduced so far can also be formulated in terms of complex function theory, which has benefits when dealing with singularities in the flow field, moments on 2D objects or when calculating the forces on them. We first introduce the complex coordinate $z$, for which we can write:

$$z = x + iy = r \cos(\theta) + ir \sin(\theta) = r \exp(i\theta), \tag{17}$$
where \( i = \sqrt{-1} \) is the imaginary root. Furthermore \( r = \sqrt{x^2 + y^2} \) is the modulus and \( \theta = \arctan \left( \frac{y}{x} \right) \) is the argument of \( z \) (both plane polar coordinates), while \( x \) and \( y \) are real numbers (Cartesian coordinates), with \( x \) taken along the real axis and \( y \) along the imaginary axis, see also figure 2.1.

The stream function and velocity potential as introduced in section 2.1.1 form an orthogonal set, as expressed by the Cauchy-Riemann equations

\[
\frac{\partial \phi}{\partial x} = u = \frac{\partial \psi}{\partial y}, \quad (18a)
\]

\[
\frac{\partial \phi}{\partial y} = v = -\frac{\partial \psi}{\partial x}. \quad (18b)
\]

Provided that the partial derivatives are continuous (and thus analytic or holomorphic), allows us to define the complex potential \( w \) according to [1, 5, 6]

\[
w(z) = \phi(z) + i\psi(z). \quad (19)
\]

The velocity components \( u \) and \( v \) can be computed with the derivative of the complex potential \( \frac{dw}{dz} \), which can be verified from

\[
\frac{\partial w}{\partial x} = \frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = u - iv, \quad (20a)
\]

\[
\frac{\partial w}{\partial y} = \frac{dw}{dz} = \frac{\partial \phi}{\partial y} + i \frac{\partial \psi}{\partial y} = v + iu. \quad (20b)
\]

Note that the Cauchy-Riemann equations follow directly from these two equations, since we assume a continuous function, and so \( \frac{\partial w}{\partial x} = -i \frac{\partial w}{\partial y} \) holds, leading with the
above derivatives to the orthogonal set. The absolute flow velocity at any point can be calculated with

\[
\frac{dw}{dz} \frac{d\pi}{dz} = \left| \frac{dw}{dz} \right|^2 = u^2 + v^2 = V^2. \tag{21}
\]

These properties will appear to be very useful for calculating velocities, pressures and forces. Singular points are points where \( w \) or \( \frac{dw}{dz} \) are infinite; in these points the orthogonality is lost, and such points are excluded from any integration contour.

### 2.2 Vortex models

#### 2.2.1 Point vortex

The potential vortex is the most simple vortex model. All vorticity is concentrated in one singular point, the vortex centre. The flow is purely azimuthal and singular in the origin. The radial and azimuthal velocity components of this point-vortex flow are given by:

\[
u_r = 0, \quad v_\theta = \frac{\Gamma}{2\pi r}, \tag{22}\]

with \( \Gamma \) the vortex strength and \( r = \sqrt{x^2 + y^2} \) is the radial distance. For this point vortex, one can also write:

\[
\frac{dw}{dz} = u - iv = -iv_\theta e^{-i\theta} = -\frac{i\Gamma}{2\pi z}, \tag{23}\]

in the last step property (17) is used. To find the complex potential we integrate equation (23) with respect to \( z \),

\[
w = -\frac{i\Gamma}{2\pi} \log(z). \tag{24}\]

If the vorticity is not located in the origin but at an arbitrary position \( z_0 \), the complex potential (24) becomes

\[
w = -\frac{i\Gamma}{2\pi} \log(z - z_0). \tag{25}\]

#### 2.2.2 Dipole vortex

Combining two potential vortices of equal, but opposite-signed strengths, separated a distance \( l \) apart, gives a simple model for a dipolar vortex. For a combination of two point vortices with strengths \( \Gamma \) and \(-\Gamma\) located in \( +\frac{l}{2}e^{i\alpha} \) and \(-\frac{l}{2}e^{i\alpha} \), respectively, the complex potential is

\[
w = \frac{i\Gamma}{2\pi} \log(z - \frac{l}{2}e^{i\alpha}) - \frac{i\Gamma}{2\pi} \log(z + \frac{l}{2}e^{i\alpha}), \tag{26}\]

A dipole has a self-propelling speed due to the velocity that each vortex induces in the other. The translational velocity is steady and directed perpendicular to the line
2.2 Vortex models

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connecting their centres. Vortices with strengths of $\Gamma$ and $-\Gamma$ separated a distance $l$ apart will propagate with a velocity of $V = \frac{\Gamma}{2\pi l}$.

When more dipoles are placed in the domain, each vortex also feels a velocity induced by other dipoles, which may lead to a change in the translational velocity, orientation or separation distance.

2.2.3 Finite dipoles

A "finite-dipole" is a point-vortex dipole where the separation distance between the vortices is fixed, imagined as a solid rod that connects the two vortices. For a description of these finite dipoles we follow Tchieu et al. [7]. First we consider N-dipoles, were the $n^{th}$-dipole consists of a positive vortex with strength $+\Gamma_n$ and a negative vortex with strength $-\Gamma_n$, which we will refer to as the left and right vortex, respectively. The position of the left- and right vortex with respect to the dipole centre is given by

$$z_{n,l} = z_n + \frac{i l_n e^{i\alpha_n}}{2}$$ and $$z_{n,r} = z_n - \frac{i l_n e^{i\alpha_n}}{2},$$

and the location of the dipole centre is given by

$$z_n = \frac{z_{n,l} + z_{n,r}}{2}. \quad (28)$$

Due to the multiple dipoles the separation distance $l_n$ would change, but our goal is to keep all separation distances $l_n$ constant. Therefore an additional inter-dipole velocity is needed, such that $\dot{l}_n = 0$. The velocity of the left and right vortices is given by the superposition of the self-induced velocity, the velocity induced by others and the additional velocity term to keep the separation distance constant, i.e.

$$\dot{z}_{n,l} = w_{n,s} + w_{n,o}(z_{n,l}) + i\lambda e^{-i\alpha_n}, \quad (29a)$$
2 GENERAL THEORETICAL BACKGROUND 2.2 Vortex models

\[ \dot{z}_{n,r} = w_{n,s} + w_{n,o}(z_{n,r}) - i\lambda e^{-i\alpha_n}. \]  \hspace{1cm} (29b)

The first term, \( w_{n,s} \), is the self-induced velocity, which is given by

\[ w_{n,s} = \frac{\Gamma_n e^{-i\alpha_n}}{2\pi l_n}, \]  \hspace{1cm} (30)

with \( \frac{\Gamma_n}{2\pi l_n} \) the induced velocity and \( \alpha_n \) the direction. The second term, \( w_{n,o} \), accounts for the induced velocity by all other finite dipoles and is given by,

\[ w_{n,o}(z) = \sum_{j \neq n}^{N} \frac{\Gamma_j}{2\pi l} \left( \frac{1}{z - z_{j,l}} - \frac{1}{z - z_{j,r}} \right). \]  \hspace{1cm} (31)

Finally, the last term is an inter-dipole velocity towards or away from each other, along the line joining the two vortices, and assuring a constant separation distance.

Equation (29a) and (29b) can be rewritten as a system of equations that describes the motion of the dipole centre \( z_n \), and its orientation \( \alpha_n \). Therefore, we substitute equation (29a) and (29b) in the complex conjugated and time derivative equation of the dipole centre (28) to obtain,

\[ \dot{\bar{z}}_n = w_{n,s} + \frac{w_{n,o}(z_{n,l}) + w_{n,o}(z_{n,r})}{2}. \]  \hspace{1cm} (32)

This expression shows that the total velocity of the centre is the addition of the self-induced velocity and the average velocity that each vortex pair "feels" from all other dipoles.

An expression for \( \dot{\bar{z}}_{n,l} \) and \( \dot{\bar{z}}_{n,r} \) can also be obtained from (27). To this end we take again the time derivative and complex conjugate of equation (27) and subtract them, we now find an expression for the change in the separation distance:

\[ \dot{z}_{n,r} - \dot{z}_{n,l} = (\dot{\bar{z}}_n + i\dot{l}_n) e^{-i\alpha_n} = w_{n,o}(z_{n,r}) - w_{n,o}(z_{n,l}) - 2i\lambda_n e^{-i\alpha_n}. \]  \hspace{1cm} (33)

The inter-dipolar velocity, needed to keep the separation distance of the dipole fixed, requires that \( \dot{l}_n = 0 \) and is given by,

\[ \lambda_n = \frac{1}{2} \text{Im} \left[ (w_{n,o}(z_{n,r}) - w_{n,o}(z_{n,l})) e^{i\alpha_n} \right]. \]  \hspace{1cm} (34)

Finally, from equation (33) the change in orientation of the dipole centre can be found, which is a consequence of the interaction with other dipoles,

\[ \dot{\alpha}_n = \frac{\text{Re}[(w_{n,o}(z_{n,r}) - w_{n,o}(z_{n,l})) e^{i\alpha_n}]}{l_n}. \]  \hspace{1cm} (35)

Equations (32) and (35) represent a closed set of 3N equations that governs the motion of N finite dipoles, interacting in an infinite plane.
2.3 Dipole interaction

In this subsection common dipole interactions for the non finite-dipoles are discussed, namely dipole collisions and a dipole-wall interaction.

2.3.1 Dipole collisions

When two equal dipoles, symmetrically-aligned but travelling in opposite directions collide, as shown in figure 2.3a, partner exchange is observed. In partner exchange the original dipoles split into separate vortices, and are then recombined into two new dipoles with exchanged partners. Depending on the angle of collision the newly formed dipoles can be of unequal strength, i.e. the separation distances and so the translational velocity may be different, see for example figure 2.3b.

2.3.2 Dipole-wall interaction

Viscous forces play no role in the potential flow theory, meaning that the no-slip conditions near boundaries do not hold. However, a solid wall would imply no-penetration, such that the normal velocity is zero. For a point-vortex flow the presence of a straight impermeable wall can be accounted by introducing an image vortex of equal but oppositely-signed strength, this is called the “image principle” [5]. In this model the wall is considered as a mirror, the solid wall only implying a zero normal velocity (slip is allowed). For example a dipolar vortex given by equation (26), which moves towards a wall, will experience velocities induced by its mirror dipole. As a result, the vortex pair will split and the vortices will translate away from each other, see figure 2.4.
2.4 Far field

The velocity field induced by a dipole with the vortices located in \( z_r \) and \( z_l \) is given by

\[
\dot{z} = \frac{\Gamma}{2\pi i} \left( \frac{1}{z - z_l} - \frac{1}{z - z_r} \right). \tag{36}
\]

The velocity induced by a dipole in the far field can be calculated by the Taylor expansion for \( z \to \infty \) of the RHS of equation (36), which gives

\[
\dot{z} \approx \frac{\Gamma}{2\pi i} \left\{ \left( \frac{1}{z} \right) + \frac{z_l}{z^2} + \frac{z_l^2}{z^3} + \cdots \right\} - \left\{ \left( \frac{1}{z} \right) + \frac{z_r}{z^2} + \frac{z_r^2}{z^3} + \cdots \right\},
\]

\[
\dot{z} \approx \frac{\Gamma}{2\pi i} \left( \frac{z_l - z_r}{z^2} + \frac{z_l^2 - z_r^2}{z^3} + \cdots \right) \approx \frac{\Gamma}{2\pi i} \left( \frac{z_l - z_r}{z^2} \right) = \frac{\Gamma}{2\pi i} \frac{il e^{-i\alpha_n}}{z^2}. \tag{37}
\]

In the last step, equation (27) is used for further simplification. The magnitude of this velocity is now given by

\[
|\dot{z}| \approx \frac{\Gamma}{2\pi} \frac{l}{z^2}. \tag{38}
\]

That is, the induced velocity of a dipole on a distance \( z \gg l \) decreases with \( z^{-2} \).

2.5 Dimensionless parameters

The translation velocity of a dipole is given by

\[
v = \frac{\Gamma}{2\pi l}. \tag{39}
\]
The model parameters are the vorticity $\Gamma$ with the dimension [m$^2$s$^{-1}$], and the separation distance $l$ with the dimension [m]. Equation (39) can be non-dimensionalised, therefore we first non-dimensionalize the velocity and length scales in terms of the model parameters:

$$v = v^* \frac{\Gamma}{l}, \quad r = r^* l.$$  \hspace{1cm} (40)

Substituting these into equation (39) gives:

$$\frac{\Gamma}{l} v^* = \frac{\Gamma}{2\pi r^* l} \Rightarrow v^* = \frac{1}{2\pi r^*}.$$  \hspace{1cm} (41)

That is, the dimensionless velocity of the dipole is independent of the vorticity. A change of the vorticity or the separation distance leads to a change in the time scale, i.e. $t = t^* \frac{2\pi l}{\Gamma}$. 

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Part I  
Infinite domain

When finite-dipoles are placed in an infinite domain, each dipole will translate under the influence of; 1) its own translational velocity \(w_{n,s}\), which results in a linear motion, and 2) the velocity induced by all other vortices in the domain, i.e. \(w_{n,o}(z)\). The magnitude of the second one depends mainly on the separation distance of the vortices. When \(\Delta z \leq l\), where \(\Delta z\) is the separation distance between two vortices of different dipoles, the induced velocities by other dipoles can be higher than its own translational velocity, resulting in non-straightforward translation, rotation and interactions of the dipoles. An example is shown in figure 2.5, where five dipoles are initially placed randomly within a domain of size \(2\pi l \times 4\pi l\), and are randomly orientated. The coloured lines represent the paths followed by each dipole centre for a simulation time of \(t = 4\). As can be seen in the figure, all kinds of exotic behaviour are observed. For example, the dipole indicated by the purple path, is bend upwards under the influence of the two dipoles indicated by the green and light blue path, which itself follows a "stable" periodic path. Moreover, the two dipoles at the bottom, indicated with the blue and red lines, circle around each other and around a "centre of mass", that also looks like a stable orbit.

The goal of this section is to explore the different kinds of behaviour dipoles can exhibit, therefore we start with two dipole interactions, which are observed frequently. Then, we continue with three and four dipole interactions.

![Figure 2.5 – Trajectories of five dipoles which are initially placed randomly within a domain with size \(2\pi l \times 4\pi l\) and are a randomly orientation. The coloured lines represent the paths followed by each dipole centre. The hollow circles and the dashed black lines are the initial position, while the filled circles and solid lines are the end positions of the dipoles. The simulation time \(t = 4\).](image-url)
3 Numerical method

The dynamics of the system are obtained by solving equations (32) and (35) which are differential equations for the translation and orientation of a dipole. They form a closed system of $3N$ equations that governs the motion of $N$ finite-dipoles, interacting in an infinite plane. Note that these equations can not be written in a Hamiltonian form due to the additional velocity component that fixes the separation distance, and so a symplectic integrator cannot be used. Therefore this system of ordinary differential equations is solved using the standard package ODE45 in Matlab, with a variable step and explicit time integrator. The relative and absolute tolerances are set to $10^{-6}$ and $10^{-8}$, respectively.

For all simulations and figures obtained, the vortex strength is chosen $\Gamma_n = \Gamma = 2\pi$. The length scales are dimensionless, i.e. $x_{1,2}^* = \frac{x_{1,2}}{\Gamma}$, where the lower subscripts denote for $x$ and $y$, respectively, and the upper subscript is omitted from here on. The dipole has a self-propelling speed of $|w_s| = 1$. Other vortex strengths or separation distances can be chosen, which affects the simulation time.
4 $N = 2$ finite-dipole system

4.1 Dipole-dipole interaction

**Equal direction.** In this subsection examples of numerical results are presented for two dipole interactions. First the observations made by Tchieu et al. [7] are summarized. These authors considered the case of two dipoles, travelling in the same direction, placed initially a horizontal distance $X_0$ and an axis offset $H_0$ apart, see figure 4.1a for a schematic representation. When two dipoles are initially placed in a tandem ($H_0 = 0$) the dipoles remain in an equilibrium, although obtaining a higher translational velocity (figure 4.1b). That is, the two dipoles help each other in travelling forward at a rate faster than their self-induced velocity, which is referred to as ”dipole drafting” [7]. The drafting velocity is given by,

$$\left|\dot{z}\right| = \frac{\Gamma}{2\pi l} \left(1 + \frac{l^2}{X_0^2 + l^2}\right).$$

(42)

As $H_0$ is increased while holding $X_0$ fixed, the dipoles initially assist each other in the forward motion, but then quickly diverge (figure 4.1c). A further increase of $H_0$ leads to a slower rate of divergence. When $H_0$ is fixed and $X_0$ is changed, for example to $H_0 = 4l$ and $X_0 = 2.25l$, the two dipoles move as a whole in an oblique direction with respect their self-propelled velocity (figure 4.1d). Note that in this case the travelling speed is lower than their self-propelled speed, and the orientation angle is unaffected while the overall direction is not. For a further decrease of $H_0$ the dipoles begin to oscillate in what seems close to a periodic trajectory (figure 4.1e). When decreasing $H_0 < 0.83l$ the dipoles will collide: for example, setting the two dipoles initially at $X_0 = 0$ (with $H_0 > l$) they will eventually collide symmetrically and annihilate each other (figure 4.1f).

**Opposite direction.** The research is now further extended by investigating the interaction between two dipoles that are aligned opposite to each other, with a separation distance $X_0$ apart, and an offset $H_0$, see figure 4.2a for a schematic representation. By fixing $X_0 = 500l$ and varying $H_0$, the following three cases can be distinguished: collision, scatter and asymptotical trapping.

For $0 \leq H_0 < H_c$ the dipoles have a translational movement towards each other (figure 4.2b). In the far field, the dipoles are hardly affected by each other. When they get closer to each other in time, they are slightly deflected away from each other making the offset larger. Their self-propelling speed is still dominant, and the dipoles will try to pass each other. Once the distance between the two dipoles gets close to the separation distance $l$, the dipoles get caught in their near field, leading to a strong interaction, which changes their orientation and motion. The dipoles align themselves exactly opposite to each other, forcing the offset $H_0$ to zero and subsequently they annihilate each other. When the initial separation distance is increased but still smaller than $H_c$, the dipoles are trapped in their near field for some longer time before annihilating each other, leading to a circular path around each other (figure 4.2c). An analytical expression for the annihilation is given in the following paragraph.
Figure 4.1 – Interactions between two dipoles for different $H_0$ and $X_0$, with $\alpha_{1,2} = \frac{\pi}{2}$.
(a) A schematic representation of the initial setup. (b) $H_0 = 0$ and $X_0 = 4$. (c) $H_0 = 2$ and $X_0 = 4$. (d) $H_0 = 4$ and $X_0 = 2.25$. (e) $H_0 = 4$ and $X_0 = 1$. (f) $H_0 = 4$ and $X_0 = 0$. 

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4.1 Dipole-dipole interaction

If \( H_0 > H_c \), the dipoles are scattered (figure 4.2d and 4.2e). The dipoles move initially in a similar way as in the case when they get trapped, i.e. in the far field, the dipoles are hardly affected by each other and when they get closer to each other in time, they are slightly deflected away from each other, thus making the offset bigger. Once they are closer to each other they are caught in their near field, leading to a change in their orientation and motion. Since the near field interaction is not strong enough, the dipoles are now not aligned to, but away from each other, causing the dipoles to translate away to infinity. The closer the offset to \( H_c \), the longer they rotate around each other and the more time it takes before they get scattered.

When \( H = H_c \) the dipoles are asymptotically trapped, i.e. the centre of each dipole circles with a constant radius around a centre of mass, located in the origin, and has a constant angular velocity (see figure 4.2f and 4.3). When the location of each dipole centre is described by \( z_n = Re^{\theta(t)} \), this means that \( R, \dot{\theta} = \text{const} \). It turns out that for \( H = H_c \), \( \dot{\theta} = \dot{\alpha} = \sqrt{2} \) and \( R = 0.5 \), which can be interpreted as a solid body rotation. As the asymptotically trapping suggests, this orbit is not stable and after some "solid body" revolutions the dipoles are either annihilated or scattered to infinity. This meta...
In the previous section various interactions between two finite-dipoles where observed, such as scatter, asymptotic trapping, annihilation or periodic motion, depending on the initial conditions. We now introduce a regime diagram (figure 4.4) which classifies the behaviour of two aligned dipoles with opposite orientation, depending on the initial separation distances. The horizontal axis corresponds to $X_0$ and the the vertical axis to $H_0$ (see figure 4.2a). The following types of behaviour, sometimes referred to as ”cases”, can be distinguished:

I) scatter: the two dipoles scatter eventually to infinity, as depicted in figure 4.2d and 4.2e.

II) annihilation: after some interaction the two dipoles finally get aligned opposite to each other, and then cancel each other out, see figures 4.2b and 4.2c.

III) trapping: this is a stable state in which the two dipoles are aligned, such that a tripole-like configuration is formed. The two dipoles circle around a common centre in a closed form, along complicated trajectories.

Figure 4.3 – Asymptotic trapping of two dipoles, the dipoles circle around each other with constant rotational velocity. The centre of each dipole with respect to the centre of mass is described by $z_n = R e^{i\theta(t)}$, the radius and angular velocity are constant (i.e. solid body rotation). With initial conditions $H_0 = 0$ and $X_0 = 1$, $\alpha_1 = \frac{3}{4}\pi$ and $\alpha_2 = \alpha_1 + \pi$. (a) The radius $R = 0.5$ is constant, the orientation velocity of the dipole $\dot{\alpha}_n$ and the rotation speed of the centre around the centre of mass $\dot{\theta}$ is equal; $\dot{\theta} = \dot{\alpha} = \sqrt{2}$. (b) Initial- and final configuration of the dipoles, the path of the centre is given by the green coloured lines for $t = 10$. The dipoles remain in the same configuration, while rotating with respect to the $xy$-axis.

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These cases are also numbered in the figure, the red centre line is a line of symmetry, i.e. similar behaviour is observed for negative values of $H_0$ as for positive. The lines that distinguish area I from II or III represents metastable orbits, where the dipoles circle longer around each other before they eventually "fall" to one of the two sides. The asymptotic case discussed in the previous subsection are the points where the three areas intersect.

Examples of behaviour in regime III are shown in figure 4.5 for different initial conditions. This case is called trapping since the dipoles are caught by each other. Note that the existence of trapped dipoles was already mentioned in a recent paper on finite-dipoles in a double periodic plane [15], where it was called "dipole dancing". This happens when the two vortices of equal sign are closer to each other than to the vortices of the other sign. The two vortices then form a sort of superimposed vortex surrounded by two vortices of opposite sign, thus forming a tripole-type configuration, which circles around the centre. Since the separation distance of the two vortices in the centre is smaller than the separation distance $l$, their induced velocities will be dominant and are responsible for the "small amplitude" periodic motion, with a "small" period. The velocity induced by the inner vortices in the two outer vortices is smaller due to the larger separation distance; these inner vortices are responsible for the large scale motion (rotation of the system) and have a "longer" period, which is clearly visible in figure 4.5b.

The interactions discussed so far are not only confined to finite-dipoles, but they were already observed and well-discussed for non-finite-dipoles, see for example [9, Vincent van Liebergen 20 Master Thesis]
4.2 Classification of interactions

Figure 4.5 – Different examples of trapped dipoles. For all cases $z_1 = 0 + 0i$, $\alpha_1 = 0$ and (a) $z_2 = -0.5 + 1i$, $\alpha_2 = \pi$. (b) $z_2 = 0 + 1.15i$, $\alpha_2 = \pi$. (c) $z_2 = -0.65 + 0.7i$, $\alpha_2 = \pi$. In all configurations the dipoles circle around each other, and have an extra periodic motion (with lower period time and amplitude) due to the higher induced velocity of the inner vortices.

10, 11, 12]. There it was shown that all possible cases of vortex interactions can be characterized by two dimensionless parameters, which are here reformulated in terms of $X_0$ and $H_0$,

$$\delta_h = \frac{l}{|H_0|} \sqrt{\frac{(X_0)^2 + (H_0)^2}{((X_0)^2 + (H_0 + 2l)^2) \left( (X_0)^2 + (H_0 - 2l)^2 \right)}}$$

$$\delta_x = \sqrt{\frac{(X_0)^2 + (H_0 + 2l)^2}{(X_0)^2 + (H_0 - 2l)^2}} \left( \frac{|H_0|^2}{l} \right)$$

With these parameters and the roots $\zeta_1$ and $\zeta_2$ of the polynomial $\zeta^2 - \delta_h \zeta + 1$, four cases can be distinguished, namely;

a) $0 < \delta_h < 2$: direct scatter

b) $\delta_h > 2$ and $\zeta_0 > \zeta_1 > \zeta_2$: exchange scatter

c) $\delta_h > 2$ and $\frac{1}{\delta_h} < \zeta_0 < \zeta_2$: mutual trapping

d) $\delta_h = 2$, $\zeta_0 > 1$: asymptotical trapping.

Although these four cases are specific for non-finite dipoles, similar regimes are observed for the finite-dipoles. The regime diagram is shown in figure 4.6. The regime diagram has a similar shape as the regime diagram for finite dipoles (figure 4.4) but has a less circular shape. The line separating regime b-a and a-c is steeper than the line separating regime II from the I and III for the finite-dipoles. The two points where the three areas intersect (corresponding to regime d) lie on the same location as for the finite-dipoles. Note that exchange scatter is not possible for finite-dipoles and corresponds to annihilation.
4.2 Classification of interactions

Figure 4.6 – Regime diagram for classical point-vortex dipoles, showing the type of interactions: a = direct scatter; b = exchange scatter; c = mutual trapping. The red line indicates a line of symmetry.

Up to now, we only considered two finite-dipoles that are placed opposite to each other and varied the initial separation distance. However, there exist more degrees of freedom that can influence the behaviour of the system. For example, the initial orientation angle of the two dipoles could be changed to $\alpha_1 = \alpha_2$, i.e. they move in the same direction, just as the examples depicted in figure 4.1. The regime diagram for this configuration is shown in figure 4.7a. As can be observed, the regime diagram is different than for the case of the dipoles placed opposite to each other. There are now two regions where two dipoles are trapped: the middle circle and the two left corner regions. Note that for this configuration, trapping should be interpreted in a wider sense: The dipoles can (i) circle around each other as seen before; (ii) they move periodic towards and from each other in a straight path, as in figure 4.1e; (iii) they stay in the same configuration, slowing down each other or increasing the speed as with dipole drafting, corresponding to figure 4.1b and 4.1d. In these cases the dipoles are trapped in the sense that they do neither annihilate each other nor scatter to infinity. The line $H_0 = 0$ corresponds to a stable translational motion of the dipoles where they have a benefit in motion, as observed in figure 4.1b. Area II corresponds again to collision and therefore annihilation of the finite-dipoles, as shown in figure 4.1f.

When the initial orientation angle is changed to $\alpha_1 = \alpha_2 + \frac{\pi}{3}$ or $\alpha_1 = \alpha_2 + \frac{2\pi}{3}$ (see figure 4.8a for a schematic representation), the regime diagram and its symmetry line rotates with half the initial orientation difference between two dipoles $\frac{1}{2}\Delta\alpha$, see...
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4.2 Classification of interactions

Figure 4.7 – Regime diagram for (a) \( \alpha_1 = \alpha_2 \), (b) \( \alpha_1 = \alpha_2 + \frac{2\pi}{3} \) and (c) \( \alpha_1 = \alpha_2 + \frac{2\pi}{3} \), with \( \alpha_1 = 0 \) in all cases. I = scatter, II = annihilation and III = trapping.

Figure 4.7b and 4.7c, respectively. Moreover, the regions with the three possible regimes change. One can see that the second trapping area has disappeared while in the middle trapping area a small region where annihilation (II) occurs shows up. This region grows for larger initial orientation angles and joins the lower annihilation area, which slowly moves upwards for larger orientation angles. For these two cases there are two ways of trapping: again a periodic motion with a net translation in a forward way (figure 4.8c), or circular orbits due to the formation of a tripole as mentioned previously, with no net displacement (figure 4.8b and 4.8d).

Figure 4.8e and 4.8f show an example of annihilation \((z_2 = -0.75 + 0i, \alpha_2 = \frac{2\pi}{3}), which is region II above in the central circle) and scatter (with \( z_2 = -0.77 + 0.4i, \alpha_2 = \frac{2\pi}{3} \), between the two separated regions II).

Figure 4.9 shows the separation distance between two dipole centres on the horizontal plane and the difference in orientation angle on the vertical for three different dipole-dipole configurations, i.e.

\[
\begin{cases}
\Delta z = (z_1 - z_2) \\
\Delta \alpha = (\alpha_1 - \alpha_2)
\end{cases}
\] (45)

The difference in initial conditions of the three examples depicted in the figure are \( \Delta z = 0.5 + i0.8 \), with \( \Delta \alpha = 0 \) for the red line, \( \Delta \alpha = -\frac{1}{3}\pi \) for the blue line, and \( \Delta \alpha = -\pi \) for the green line. From the repeating pattern, it is directly clear that these are trapped dipole configurations, otherwise \( \Delta z \) would be going to infinity (scatter) or zero (annihilation).
As can be seen, the first two (red and blue) move in a three-dimensional phase space, while the green line only evolves in the two-dimensional phase space. This means that for the first two the orientation difference changes as a function of the position \( \Delta \alpha (x_1, y_1, x_2, y_2, t) \), while in the latter case the orientation difference remains constant in time, \( \Delta \alpha (t) = \text{const} = -\pi \), i.e. when two dipoles are placed initially opposite to each other, they remain opposite.

For \( \Delta \alpha \neq \pm \pi \), the orientation difference changes in time between \(-\pi < \Delta \alpha < \pi\), implying that a point of regime diagram 4.7a is interchangeable with a point of regime diagram 4.7b (both in the trapping regime). Consider for example the initial start point of the red line, given by \((\Delta z, \Delta \alpha) = (0.5 + i 0.8, 0)\). As the system evolves, the position and orientation difference between the two dipoles is given by a point on the red line. One of the points that is crossed by the red line is \((\Delta z, \Delta \alpha) \approx (-0.9103 - i 0.1776, \frac{1}{3} \pi)\), i.e. a point in the trapping regime of regime diagram 4.7b. This implies that the initial point lying in the regime diagram of figure 4.7a corresponds to the coordinates \(z = -0.9103 - i 0.1776\) of the regime diagram in figure 4.7b, i.e. the trapped points of the regime diagram correspond to each other and can be mapped.
onto other regime diagrams:

\[ f(z, \Delta \alpha_{\text{initial}}) = \tilde{z} \quad \text{at} \quad \Delta \alpha_{\text{mapped}}, \]

with \( f \) is an unknown mapping function.

We conclude with the observation that the phase space is symmetric around the horizontal plane \( \Delta \alpha = 0 \).

**Figure 4.9** – Phase portrait of \((\text{Re}(\Delta z), \text{Im}(\Delta z), \Delta \alpha)\) (eq. 45). The initial differences are \( \Delta z = 0.5 + 0.8 \) with \( \Delta \alpha = 0 \) for the red line, \( \Delta \alpha = -\frac{1}{3}\pi \) for the blue line, and \( \Delta \alpha = -\pi \) for the green line. (a) 3D-view. (b) \( xz \)-plane.
4.3 Annihilation

We now consider two dipoles, their centres are aligned on the x-axis and the orientations are $\alpha_1 = 0$ and $\alpha_2 = \pi$ such that they approach each other and have a purely horizontal velocity component $u$. As the vortices of the respective dipoles approach each other, they cancel one another. An analytical description for $z(t)$ is found by considering the change in propagation speed of the dipole:

$$u = u_{n,s} + u_{n,o}(z).$$  \hspace{1cm} (46)

Due to symmetry, we only consider the induced velocities from $z_{1,l}$ and $z_{2,r}$ in $z_{1,r}$. Note that the vertical velocity components from $z_{2,r}$ and $z_{2,l}$ are suppressed by the additional inter-dipole velocity, and therefore only the horizontal velocity component, $u$, is considered. The horizontal velocity components (see figure 4.10a) are given by

$$u = \frac{\Gamma}{2\pi l} - \frac{\Gamma \cos(\beta)}{2\pi \sqrt{4x^2 + l^2}} = \frac{\Gamma}{2\pi l} - \frac{\Gamma l}{2\pi (4x^2 + l^2)}.$$  \hspace{1cm} (47)

This can be rewritten as

$$\frac{dx}{dt} = \frac{\Gamma}{2\pi l} \frac{4x^2}{4x^2 + l^2}.$$  \hspace{1cm} (48)

Integration gives:

$$x = \frac{-c}{4\pi l} + \frac{\sqrt{(c - \frac{\Gamma l}{2\pi l})^2 + l^2}}{2}, \text{ with } c = -X_0 + \frac{l}{4X_0},$$

Figure 4.10 – (a) Schematic representation of the induced velocities in $z_{1,r}$ when two dipoles are approaching each other. They are initially aligned on the x-axis with $\alpha_1 = 0$ and $\alpha_2 = \pi$ at $z_1 = -z_2 = (-5, 0)$. (b) Numerical (blue) and exact analytical solution (red dots) of the x-position for the right dipole $z_1$ as a function of time for two equal strength dipoles. (c) The position of the dipole centres as a function of time, $z_1$ (blue) $z_2$ (green) and the analytical results (red) obtained for the limits of $x \gg l$ and $x \ll l$.  

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where $X_0$ is the initial separation distance along the real axis. The analytical and numerical solutions are plotted in figure 4.10b, for initial positions $z_1 = (-5, 0)$ and $z_2 = (5, 0)$. Note that in the limit $x \to 0$ and so $x \ll l$ the dipoles will asymptotically approach each other as a function of time, which is easily verified from equation (48).

When we consider two dipoles of different strengths in the same initial configuration as above, equation (47) can be rewritten into

$$\frac{dx}{dt} = \frac{\Gamma_1}{2\pi l} - \frac{\Gamma_2 l}{2\pi (4x^2 + l^2)},$$

for which an exact analytical solution is hard to find. However, in the limit $x \to 0$ and $x \ll l$, it can be seen that the velocity of the dipole is given by a superposition of the two individual vortex strengths. In figure 4.10c the case is shown for two dipoles with strength $\Gamma_1 = 2\Gamma_2$, initial conditions $z_2 = (5, 0) = -z_1$ and orientations $\alpha_1 = 0$ and $\alpha_2 = \pi$. First the two dipoles will approach each other with each individual induced velocity given by $v_n = \frac{\Gamma_n}{2\pi l}$, when the dipoles get closer they reach a transition area, and finally, when the dipoles are in the above limit, the dipoles will propagate with the superimposed velocity $v_1 = \frac{(2\Gamma_1 - \Gamma_2)}{2\pi l}$.

### 4.4 $w$–function

#### 4.4.1 Derivation

Equations (32) and (35) describe the dynamics of the finite-dipole system. When two dipoles are considered, 6 equations should be solved. With the conserved symmetry observed in section 4.2 these equations can be simplified. The orientation difference $\Delta\alpha = \pi$ remains constant in time, so that we can define $\beta = \pi - \alpha_1 = -\alpha_2$, where $\beta$ is a new variable for the orientation. Moreover, when the system is described from a common centre of symmetry, it can be shown that $z_1 = -z_2$. Substituting this in equations (32) and (35), making the equation dimensionless (with $\tau = \frac{\Gamma}{\pi l} t$), and introducing a new complex variable $w$, defined by $w = \frac{2z}{l} \exp(i\beta)$ (not to be confused with the complex potential $W$), the equations can be rewritten to (see Appendix A.1 for the full derivation):

$$\dot{\bar{w}} = \frac{-w^2}{w^2 + 1} - i \bar{w} \dot{\beta},$$

with

$$\dot{\beta} = \text{Im} \left( \frac{1}{w (w^2 + 1)} \right),$$

or

$$\dot{\bar{w}} = \frac{-w^2}{w^2 + 1} - i \bar{w} \text{Im} \left( \frac{1}{w (w^2 + 1)} \right).$$

The variable $w = \frac{2z}{l} \exp(i\beta)$ contains the position (centre) and orientation of one dipole in the complex plane and is itself a complex number, $\dot{\beta} = \text{Im} \left( \frac{1}{w(w^2 + 1)} \right)$ is the angular
velocity of the dipole. The differential equation gives the change in $w$ as a function of time, i.e. the change of position and orientation. Due to symmetry, the original set of six differential equations is reduced to a system of two equations.

Although it is impossible to solve this equation analytically, still valuable information can be obtained from its numerical solution.

### 4.4.2 Interpretation

Figure 4.11 shows the numerically obtained results for equation (52) for different initial conditions, which are indicated with the crosses. The coloured lines, starting from the crosses, indicate the time evolution of $w$ (i.e. pathlines). Initially, fixing $\beta = 0$ gives $w(t) = \frac{2z_1}{l}$, i.e. changing the initial positions of $w$ is directly related to a change of the dipoles centre $z_1$. Describing this in terms of the geometrical variables as represented in figure 4.2a, the following relation is obtained,

$$w(t = 0) = \frac{1}{l} (X_0 + iH_0).$$

Summarizing, the figure shows solutions for different initial positions, given by $w(t = 0) = \frac{1}{l} (X_0 + iH_0)$ and a resolution of 0.1l. The three different colours indicate the different types of behaviour that the dipoles can exhibit: I) scatter; II) annihilation; or III) trapping. Green coloured lines evolve to infinity, like scattered dipoles. Red lines evolve to zero: annihilating dipoles meet each other at the centre of mass, which is at $w = 0$. The blue lines are closed orbits, as with trapped dipoles.

The result is remarkable, since this figure obtained by solving $w$ has a similar shape as the regime diagram of figure 4.4, obtained by analysing the behaviour of dipoles for different initial conditions.

In the next subsection, some interesting aspects of figure 4.11 are discussed.

### 4.4.3 Characteristic features

**Stagnation points.** In section 4.1 it was noticed that two dipoles, placed opposite to each other, can circle around in a solid body rotation. The specific configuration is easily obtained from analysing the $w$-function (see also Appendix A.2). $w$ is given by $w = \frac{2z}{l}e^{i\beta}$, and its derivative by $\dot{w} = \frac{2}{l} \left( \dot{z} + iz\dot{\beta} \right) e^{i\beta}$. Since $z$ is a coordinate in the complex plane, it also can be written as $z = Re^{i\theta}$, with $R$ the magnitude and $\theta$ the argument. Substituting this into $w$ and $\dot{w}$ and taking the complex conjugate gives:

$$\dot{\bar{w}} = \frac{2}{l} \left( \hat{\bar{R}} - i\bar{R} \left( \dot{\bar{\theta}} + \dot{\beta} \right) \right) e^{-i(\beta + \theta)},$$

which should equal equation (52). When equation (94) and (96) are multiplied with $e^{i(\beta + \theta)}$, and since $R, \dot{R}, \dot{\theta}$ and $\dot{\beta}$ are all real valued quantities, it follows that the real part corresponds to $\hat{R}$, while its imaginary part corresponds to $R \left( \dot{\theta} + \dot{\beta} \right)$. For solid
body rotation, \( R \) should be constant, and so \( \dot{R} = 0 \). While the rotational velocity of the dipole centre should equal the velocity of the dipole rotation, i.e,

\[
\dot{\theta} = \dot{\alpha} = -\dot{\beta},
\]

which can only be true when \( \dot{\bar{w}} = 0 \),

\[
\dot{R} = 0 \land \left( \dot{\theta} + \dot{\beta} \right) = 0 \rightarrow \dot{\bar{w}} = 0.
\]

Solutions of equation (52), where \( \text{Re}(\dot{\bar{w}}) = 0 \) and \( \text{Im}(\dot{\bar{w}}) = 0 \), are shown in figure 4.12. The four points of intersection are located at \( w = \pm \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i \) and correspond to solid body rotation. Since the \( w \)-function is symmetric, two of the solutions are caused by the symmetry, the other two correspond to the two points in figure 4.11 where the three different colours (or regimes) intersect, where \( w = \text{const} \).

In terms of \( z_1 \) they correspond to \( z_1 = \frac{i}{2} w = \pm \frac{1}{2\sqrt{2}} \pm \frac{i}{2\sqrt{2}} \) and in Euler form \( z_1 = \pm 0.5e^{\pm \frac{i}{4} \pi} \), see figure 4.13a, or Appendix A.2 for all dipole configurations. The arrows, indicating the self-propelling direction, are shown to clarify the configuration. The circle indicates the constant radius \( R \), while the arrow on the circle indicates the direction of solid body rotation.

The angular velocity can be obtained with equation (51), resulting in \( \dot{\beta} = \pm \frac{1}{\sqrt{2}} \). The angular velocity should be multiplied with a factor 2 for the same dimensionless time as in section 4.1, while \( R \) is in the same form. The results obtained earlier, shown in figure 4.3 and corresponding to asymptotical trapping, are the same as we calculated here analytically.
We conclude with the remark that these points, as was already found numerically, are unstable owing to one of the two positive eigenvalues, leading to exponential growth in time. See Appendix A.3 for the linear stability analyses of these points. The pathlines of $w$ around these points, as shown in figure 4.11, indicate that these points are saddle points, revealing their instability.

**Singularities.** Other interesting findings can be obtained from figure 4.12, namely the singularities of $\dot{w}$ at $w = 0$, $+i$ and $-i$. Some of the corresponding dipole configurations are shown in figure 4.13b and 4.13c. Again, all dipole configurations are depicted in Appendix A.2. As can be seen in the figure, these points are related to dipole configurations where vortices coincide, leading to unphysical induced velocities. Note that these unphysical induced velocities are a result from our formulation, normally vortices that overlap are superimposed. Then, the two cases that are discussed here do correspond to either zero vorticity due to the annihilation, or to the formation of a tripole. The points correspond to the midpoint of the trapping state (blue) and the endpoint of the red lines, respectively, where the dipoles have annihilated each other in figure 4.11.

The singular point at $w = \pm i$ is surrounded by closed lines, indicating that when $w$ would be perturbed, the system stays in that perturbed configuration (a new trapped state). The singularity at $w = 0$ looks like an unstable one, since small perturbations

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Figure 4.13 – Some specific cases of figure 4.12, obtained by using equation (53). See for all configurations Appendix A.2. Some vortices are laying on top of each other, therefore arrows are plotted in the self-propelling direction of each dipole centre to elucidate the configurations. (a) Solid body rotation when \( z_1 = \pm 0.5e^{\pm i\frac{\pi}{4}} \). (b) The dipoles have annihilated each other (laying on top of each other), \( z_1 = 0 \). (c) The opposite vortices coincide with the dipole centres at \( z_1 = \pm \frac{i}{2} \) forming a tripole. For all cases \( \alpha_1 = \pi \), \( z_2 = -z_1 \) and \( \alpha_2 = \alpha_1 + \pi \).

of \( w \) in the negative direction lead to scatter. Note that, for a ”real” finite-dipole system where the symmetry is not conserved, a small perturbations could disrupt the symmetry (which formed the basis of the \( w \)-function), and forcing the dipoles into a trapped state, (see the regime diagrams of figure 4.7a to 4.7c).
5 $N \geq 3$ finite-dipole system

Until now the interaction behaviour of two finite-dipoles was discussed, such as scatter, trapping or annihilation. These interactions are not only confined to a system of two finite-dipoles, but they also occur for systems with $N \geq 3$. Here only the $N = 3$ and $N = 4$ dipole system will be considered, since systems with $N \geq 4$ are unstable and disintegrate into smaller configurations.

5.1 $N = 3$ dipole interactions

Since three dipole interaction turn out to be commonly observed in a gas of many finite-dipoles (see chapter (8)), we now investigate the three dipole system. Its behaviour has already been investigated by Tchieu et al. [7], and a small summary of this work is given here. The configuration as depicted in figure 5.1a was investigated for different values of $H_0$ and $X_0$. When the dipoles are placed in line, with $(H_0, X_0) = (H_0, 0)$ (see figure 5.1b), the two outer dipoles induce a velocity in the negative $x$-direction on the centre dipole, slowing the centre dipole down. The centre dipole also induce a velocity in the negative $x$-direction in the two outer dipoles. Since the velocity component in the negative $x$-direction, induced by the centre dipole is smaller, the outer dipoles are left with a higher induced velocity than the centre dipole. The a-symmetry on the outer two dipoles is also responsible for their change in orientation, causing them to turn inward and putting the system in a fixed equilibrium. This equilibrium is obtained for situations where $X_0 < 0.36l$ and is an attracting state for this regime of initial conditions. As $X_0$ is further increased to $0.36l < X_0 < 1.5l$, the outer dipoles do not overtake the centre one, but align themselves towards each other, and get annihilated, see figure 5.1c. When $X_0$ is even further increased to $1.5l < X_0 < 5.83l$, the outer dipoles initially align themselves towards the centre and then subsequently diverge, see figure 5.1d, this oscillation continues in time with growing amplitude. For $X_0 > 5.83l$ the separation distance between the centre and outer dipoles is so large, that the tendency of the trailing pair to align themselves can not be counteracted by the centre dipole. This results eventually in the annihilation of the trailing pair, the result is a similar situation as shown in figure 5.1c.

Many different initial configurations of a three dipole system lead eventually to one of the discussed end states; equilibrium, annihilation (of two of the three dipoles), or the oscillating behaviour. Of course, one could also imagine that many configurations can lead to scatter, as long as the dipoles are placed initially far from each other or pointing away from each other. As with a two-dipole system, trapping can also occur. Therefore we place the dipole centres equidistantly on a circle of radius $R$, with an initial orientation difference between each dipole of $\pm \frac{2}{3}\pi$. Depending on the initial orientation angle and the radius $R$, trapping can occur, as shown in figure 5.2 for $R = \frac{3}{4}$ and $\alpha_1 = \frac{1}{2}\pi$. In this configuration, the initial symmetry is conserved, leading to the question if it is also possible to find a $w$-function for this configuration, and if so, what we can learn from the solution. This is the subject of the next subsection.
Figure 5.1 – (a) A schematic representation of the three dipole configuration. (b) The dipoles evolve to a static equilibrium with initial conditions \((H_0, X_0) = (4, 0)\). (c) The two outer dipoles align themselves towards each other, annihilating each other, while the middle dipole propagates to infinity, with the initial conditions \((H_0, X_0) = (3, 1)\). (d) Periodic motion of the three dipoles with initial condition \((H_0, X_0) = (4, 2)\).

Figure 5.2 – Trapping configuration of finite-dipoles. The dipoles are placed equidistantly on a circle of radius \(R = \frac{3}{4}\) with \(\alpha_1 = \frac{1}{4}\pi\) and \(\Delta \alpha = \frac{2}{7}\pi\).
5.2 \(w\)-function for \(N = 3\)

The derivation method of the \(w\)-function for a system of three dipoles is identical to that for the two-dipole system. We use the symmetry of the system to simplify equations (32) and (35), see Appendix B.1 for the full derivation. The conserved symmetry for a three dipole system is the orientation difference between each dipole, which is \(\Delta \alpha = \gamma = \frac{2}{3} \pi\), and position \(z_1, z_2 = z_1 \exp(i \gamma), z_3 = z_1 \exp(-i \gamma)\) with respect to the centre of mass. Substituting this into equation (32) and (35), making the equation dimensionless (with \(\tau = \frac{I}{\pi \mu t}\)) and using again the variable \(w = \frac{2z_1}{l} \exp(i \beta)\), we obtain,

\[
\dot{\bar{w}} = \frac{1 + 6w^2 - 3w^4}{1 - 2w^2 - 3w^4} - i \bar{w} \text{Im}\left\{\frac{8w}{1 - 2w^2 - 3w^4}\right\}.
\]

This differential equation gives the change of \(w\) as a function of time, i.e. the change of position and orientation of one particular dipole. Due to symmetry, the initial nine differential equations are reduced to a system of two equations, at the loss of the number of degrees of freedom the system initially had.

5.2.1 Interpretation

Figure 5.3 shows numerically obtained the results from equation (57) for different initial conditions, which are indicated with the crosses. The coloured line starting from the crosses indicates the time evolution of \(\bar{w}\), so they represent path lines. Initially, fixing \(\beta = 0\) gives \(w(t) = \frac{2z_1}{l}\), i.e. changing the initial position of \(w\) is directly related to a change of the dipole centre \(z_1\) (and due to symmetry with \(z_2\), and \(z_3\)). The resolution of the initial conditions is \(0.15l\). The different colours indicate the behaviour that dipoles can exhibit. Note that there is one more possible state, indicated with black lines. Green coloured lines evolve to infinity, like scattered dipoles. Red lines evolve to \(+\frac{1}{\sqrt{3}}\), corresponding to annihilating dipoles. The blue lines represent closed orbits as with trapped dipoles. The black lines, indicating a new state, representing an evolution to a stable and stationary dipole configuration. In the next subsection the different points of figure 5.3 are discussed in some more detail.

5.2.2 Characteristic features

**Stagnation points.** In section 4.4.3 it was shown that solutions where \(\text{Re}(\dot{\bar{w}}) = 0\) and \(\text{Im}(\dot{\bar{w}}) = 0\) indicate stationary solutions, which can be in the form of solid body rotation or annihilation. Figure 5.3 suggests that these states also exist for the three dipole system. To find their location, the contour lines of equation (57) where \(\text{Re}(\dot{\bar{w}}) = 0\) and \(\text{Im}(\dot{\bar{w}}) = 0\) are plotted, see figure 5.4. There are eight intersection points and four singularities (the latter will be treated in the next section). All intersection points are listed in table B.1 and shown in figure B.3 of the Appendix.
\section*{5.2 w-function for $N = 3$}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.3}
\caption{Evolution diagram of $w$ for different initial conditions (cross marks). The coloured lines represent pathlines of $w$ in the complex plane. For green lines $w$ evolves to infinity, for red lines to $\frac{1}{\sqrt{3}}$, corresponding to a singularity, black lines to a stable point at $-\sqrt{1 + \frac{2}{\sqrt{3}}}$ and the blue lines are closed orbits indicating trapping in a stable-closed-orbit.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.4}
\caption{Solutions of $\text{Re}(\dot{w}) = 0$ and $\text{Im}(\dot{w}) = 0$. There are twelve intersection points (from which there are four singularities) for all corresponding dipole configurations, see figure B.3 in the Appendix.}
\end{figure}

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Six of the eight intersection points correspond to solid body rotation, in which there are two main configurations. Four are small variations on the symmetry as depicted in figure 5.5a, while the remaining two are variations on the configuration of figure 5.5b, the arrows indicate the dipole orientation, not their propagation direction. The variations change the direction of solid body rotation. The distances between vortices in the second configuration is much smaller than in the first, leading to higher induced velocities and resulting in a higher angular velocity. Note that, due to the higher induced velocities, this configuration is also less stable. The angular velocity can be calculated with equation (109), and is in agreement with numerical results (which are shown in figure 5.5c). Linear stability analysis (Appendix B.3) shows that all the configurations showing solid body rotation have one positive eigenvalue, making the system unstable in time. These dipole configurations corresponding to saddle-points shown in figure 5.3.

![Diagram](image-url)

**Figure 5.5** – Some specific cases of figure 5.4 corresponding to solid body rotation, (a) for $z_1 \approx 1.5922e^{0.5827i}$, (b) for $z_1 \approx 0.2652e^{i\pi/2}$. (c) Numerical results obtained by solving the finite-dipole system. Depending on the initial configurations the angular velocity is real or negative. Configuration (b) with $(\dot{\theta}_2, \dot{\alpha}_2, R_2)$ is less stable due to higher induced velocities than configuration (a) with $(\dot{\theta}_1, \dot{\alpha}_1, R_1)$, resulting in the deviation for $t > 5$. 
The other two intersection points at $\pm \sqrt{1 + \frac{2}{\sqrt{3}}}$, correspond to two stationary solutions. Only one of the two configurations corresponding with these points is stable. The stable point is recognized in figure 5.3 by the convergent point (where the black lines converge to), while the unstable point at $\sqrt{1 + \frac{2}{\sqrt{3}}}$ is recognized in the divergence of the red and green lines at this point. The dipole configuration corresponding to the stable point is shown in figure 5.6a. The unstable point is shown in the Appendix and corresponds to the same configuration, although with all dipoles rotated over an angle $\pi$, i.e. pointing away from each other.

**Singularities.** Figure 5.4 contains four singularities at $w = \pm i$ and $w = \pm \frac{1}{\sqrt{3}}$. These points correspond to configurations where vortices coincide, leading to unphysical induced velocities as a result of our formulation. Two corresponding dipole configuration with respect to these points are shown in figure 5.6.

Although stability analysis in the current formulation is not possible, the pathlines around these points (figure 5.3) can give us information about these points. At $w = \pm i$, a quadrupole is formed with in the middle a superimposed vortex of strength $3\Gamma$. This singularity is surrounded by elliptic orbits of $w$, i.e. for small perturbations of $w$ (but conserved symmetry) the system stays in a trapped state. At $w = -\frac{1}{\sqrt{3}}$ a divergent singularity is located, for small perturbations around this point $w$ evolve away from this point, while $w = +\frac{1}{\sqrt{3}}$ is a convergent singularity, for small perturbations $w$ evolves again towards this point.

A concluding note, when simulating finite-dipoles, the symmetry of the system could be lost, influencing the stability of the discussed points. In section 5.4 it is shown that the configuration as depicted in figure 5.6a is also linearly stable for perturbations where the symmetry is lost. Moreover, the singularity at $w = \frac{1}{\sqrt{3}}$ is convergent in a 2D phase space, but can be a divergent point if the complete phase space is considered.

---

**Figure 5.6** – Some specific cases corresponding to figure 5.4: (a) The stable configuration with $z_1 = \frac{1}{2} \sqrt{\frac{1}{3}(-3 + 2\sqrt{3})}$. (b,c) Singularities, at (b) $z_1 = +\frac{1}{2\sqrt{3}}$, all vortices coincide so that no the total circulation is zero, while in (c) with $z_1 = +\frac{1}{2}$, a quadrupole is formed with in the middle a vortex of strength $3\Gamma$. In all cases $\alpha_1 = 0$. 

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5.3 \( w \)-function for \( N = 4 \)

For a system of four finite-dipoles we restrict ourselves to the symmetric case, since the system of four dipoles can quickly disintegrate into two smaller systems of only two dipoles. Also a regime diagram is made in terms of \( w \) for this symmetric case.

In figure 5.7a a configuration is shown where the dipoles are placed symmetrically with respect to the common centre of mass. The dipoles interact with each other, leading to a trapped configuration: the dipoles are contracted and expanded to each other while circling around the origin, resulting in the pathlines shown in the figure by the coloured lines. When a small perturbation is put on one of the dipoles, the symmetry is lost and the system disintegrates into the trapping state of two finite-dipoles, which is shown in figure 5.7b.

Although the system seems to be sensitive to small perturbations, it is still possible to derive a function in terms of \( w \):

\[
\dot{w} = w^5 - w^2 + i w \operatorname{Im}\left\{\frac{1 - 5w^2}{w(1 - w^4)}\right\}.
\]  

(58)

This result is obtained for the symmetric case when \( z_1, z_2 = z_1 \exp(i\gamma), z_3 = -z_1 \) and \( z_4 = z_1 \exp(-i\gamma) \), with \( \gamma = \frac{1}{2}\pi \), i.e. all centres are placed equidistantly on a circle, while the orientation difference between the dipoles is \( \Delta \alpha = \frac{1}{2}\pi \). The variable \( w \), given by \( w = \frac{2\nu}{\beta} \exp(i\beta) \) and the same dimensional time is used as before.
Figure 5.8 – Evolution diagram of \( w \) for different initial conditions (cross marks). The coloured lines represent pathlines of \( w \) in the complex plane. For green lines \( w \) evolves to infinity, for red lines to 0, corresponding to a singularity. Magenta lines evolves to 1, which is also a singularity, for black lines a stable point at \( -\sqrt{1 + \frac{2}{\sqrt{3}}} \) and for the blue lines are closed orbits indicating trapping in a stable closed-orbit.

Equation (58) is integrated for different initial conditions, leading to the results as shown in figure 5.8. One more colour is included, indicating one more converging point. The corresponding dipole configuration of the three convergence points of \( w \) are shown in figure 5.9. Figure 5.9a shows a stable configuration. Figures 5.9b and 5.9c show configurations that coincide with singular converging points, with vortices lying on top of each other. Although these singular points look stable in the \( w \)-diagram and may even be so when the symmetry is conserved. Numerical simulations have revealed that simulating four dipoles in this configuration leads to a collapse of the symmetry, and the system evolves into two two-dipole systems. In the next subsection the stability of the point \( w = \sqrt{5} \) is discussed when the symmetry is not conserved.

Figure 5.9 – Some specific cases of figure 5.8 corresponding to (a) the stable configuration \( w = \sqrt{5} \) or \( z_1 = \frac{\sqrt{5}}{2} \), (b,c) the singularities at (b) \( w = 0 \) or \( z_1 = 0 \) and (c) with \( w = 1 \) or \( z_1 = \frac{1}{2} \), respectively. In the last two configurations point vortices coincide, so that the total circulation is zero.
5.4 General stability of equilibrium

The initial configuration as depicted in figure 5.1b ends in a stable configuration where the three dipoles point towards each other and where the vortices are lying on the vertices of a hexagon. The stability of this configuration has been investigated by Tchieu et al. [7] with linear stability analysis. Therefore the system of $3N$ equations is rewritten as a system of real equations in standard state-space form as $\frac{dx}{dt} = F(x)$, with $x = (\text{Re}(z_1), \ldots, \text{Re}(z_N), \text{Im}(z_1), \ldots, \text{Im}(z_N), \alpha_1, \ldots, \alpha_2)^T$ and with $F(x)$ a $3N$ vector with the corresponding equations. The system is linearised about the equilibrium positions, in the form of

$$\frac{d(\delta x)}{dt} = J\delta x,$$

with $\delta x$ representing an infinitesimal perturbation around the equilibrium state and $J = \nabla F$ is the Jacobian matrix evaluated at the equilibrium of interest.

For $N = 3$ the matrix has 9 eigenvalues. Five eigenvalues are zero, related to the five invariants of motion; the invariance in $x-, y$-translations and the in-plane rotation of the initial configuration, the last two invariants are due to reflections about the $x$- and $y$-axes, which do not change the system. The four other eigenvalues are negative and the system is stable for these perturbations, the four modes are denoted to expansion/contraction, shear, opening/closing and tilting modes, see figure 5.10. With the linear superposition of these four stable modes, any deformation can be made, wherefore the system is subsequently stable.

For $N = 4$ the matrix has 12 eigenvalues. Five eigenvalues are still zero, related to the same invariants. Of the remaining eigenvalues, one of the eigenvalues is positive leading to unstable behaviour in time for a straining mode [7]. When a small perturbation is added, the system splits into two systems of two dipoles that collide, see figure 5.10.

Figure 5.10 – Four modes for which the system is linearly stable. Depicted are small perturbations around the equilibrium (dotted line and hollow circles) with respect to the equilibrium (black line and filled circles). The four modes correspond to; (a) expansion or contraction, (b) shear, (c) opening or closing, (d) tilting.
5.4 General stability of equilibrium

Figure 5.11 – Straining mode showing the unstable motion of a four dipole system evolving in time. (a) Initial configuration with a small perturbation. (b,c) The collapse of the system in time.

5.11. It has been proven numerically that the system does not collapse immediately for the other stable nodes, but the system eventually becomes unstable due to numerical noise. For \( N \geq 4 \), all equilibria are linearly unstable, leading to loss of symmetry. Therefore, further derivations of the \( w \)-function are pointless.
Part II

Periodic domain

In the previous part the interaction between finite-dipoles was extensively discussed for different initial conditions. In this part we continue with the interactions of point-vortex dipoles, now looking at a finite-dipole field where many dipoles are initially placed in a domain and left to interact freely.

For the simulation of large systems of point-vortex dipoles periodic boundary conditions (PBCs) are used, which have benefits above simulating the evolution on an infinite domain. An infinite domain has the disadvantage that near the transition region (where the density of dipoles goes to zero) less interactions occur and dipoles can move freely to infinity, see figure 5.12. To suppress these effects the system should be made much larger than the area of interest, resulting in longer computational time and still less accurate results. An other option is to implement PBCs, to approximate a large (infinite) system by using a small part, called a "unit cell". The large system approximated by PBCs, consists of an infinite number of unit cells in the form of a lattice, which are all copies of the original unit cell. During the simulation, only the properties of the original simulation box are computed. The PBCs are often explained as particles that leave a unit cell, for example, on the right-hand side, will enter the unit cell at the left-hand side. Although this also applies to dipoles, one should not neglect the velocity field that each dipole creates, and how even their own image in neighbour cells can be influenced, resulting in changes of orientation and translation.

First, we give the necessary theoretical background to simulate $N$ finite-dipoles in a double periodic domain. Then, the numerical method is explained. Finally, we discuss the results of $N$ finite-dipoles in a double-periodic domain.

![Figure 5.12](image)

**Figure 5.12** – The evolution of finite-dipoles in an infinite domain. The coloured lines represent the path taken by individual dipoles. All dipoles where initially placed random in a square domain near the origin. Most of the dipoles scatter to infinity, trying to fill the entire space, whereas other dipoles annihilate each other.
6 Theory

For the implementation of the PBCs a double-periodic function is needed. This double-periodic function should satisfy \( f(z + 2\omega_1) = f(z) \) and \( f(z + 2\omega_2) = f(z) \), for all values of \( z \) where \( f(z) \) exists. Here, \( 2\omega_1 \) and \( 2\omega_2 \) represent the period size of the unit cell and their ratio should not be purely real. Moreover, a double-periodic complex function which is analytic, except at poles, and has no singularities other than poles in the finite part of the plane is called an elliptic function [3].

The Weierstrass-\( \zeta \) function is such an elliptical function, that is used to calculate the velocity field of point-vortices [13], and is defined as[4]:

\[
\zeta(z; \omega_1, \omega_2) = \frac{1}{z} + \sum_{p,q \neq 0,0} \left[ \frac{1}{z - \Omega_{p,q}} + \frac{1}{\Omega_{p,q}} + \frac{z}{\Omega_{p,q}^2} \right], \tag{60}
\]

with

\[
\Omega_{p,q} = 2p\omega_1 + 2q\omega_2, \quad p, q \in \mathbb{Z} / \{0\}. \tag{61}
\]

In this expression, \( p \) and \( q \) are signed integers, and \( \omega_1 \), and \( \omega_2 \) are the half-periods of the doubly periodic domain. The conjugate velocity field at a point \( z \) in a field of \( N \)-point dipoles with double periodic boundaries is given by [13, 14, 15],

\[
\dot{\bar{z}} = \sum_{n=1}^{N} \frac{\Gamma_n}{2\pi \omega_1} \left[ \zeta(z - z_{n,l}; \omega_1, \omega_2) - \zeta(z - z_{n,r}; \omega_1, \omega_2) + \left( \frac{\pi \omega_1}{\Delta\omega_1} - \frac{\eta_1}{\omega_1} \right) (z_{n,r} - z_{n,l}) - \frac{\pi}{\Delta} (\bar{z}_{n,r} - \bar{z}_{n,l}) \right]. \tag{62}
\]
with $\Delta$ the area of a unit cell, which is given by,

$$\Delta = 2i(\omega_1 \bar{\omega}_2 - \bar{\omega}_1 \omega_2). \quad (63)$$

While $\eta_1 = \zeta(\omega_1; \omega_1, \omega_2)$ is the Weierstrass-$\zeta$ function evaluated at the half period $\omega_1$.

### 6.1 PBCs for finite-dipoles

For finite-dipoles equation (62) should be rewritten such that we obtain $w_{n,s}$ and $w_{n,o}$ which can be plugged into equations (32) and (35). Therefore we follow the procedure as Tsang et al. [15] to find:

$$w_{n,s} = \frac{\Gamma_n}{2\pi i} \left[ \zeta(-i_n e^{i\alpha_n}) - \left( \frac{\pi \omega_1}{\Delta \omega_1} - \frac{\eta_1}{\omega_1} \right) i_n e^{i\alpha_n} - \frac{\pi}{\Delta} i_n e^{-i\alpha_n} \right], \quad (64)$$

$$w_{n,o}(z) = \sum_{j \neq n}^{N} \frac{\Gamma_j}{2\pi i} \left[ \zeta(z - z_{j,l}; \omega_1, \omega_2) - \zeta(z - z_{j,r}; \omega_1, \omega_2) - \left( \frac{\pi \omega_1}{\Delta \omega_1} - \frac{\eta_1}{\omega_1} \right) i_j e^{i\alpha_j} - \frac{\pi}{\Delta} i_j e^{-i\alpha_j} \right]. \quad (65)$$

With these two equations the dynamics of $N$ finite-dipoles in a double periodic domain can be obtained. Note that, the system is still non-Hamiltonian due to the constrained $\dot{l} = 0$. Moreover, for $\omega_1, \omega_2 \to \infty$ equations (64) and (65) reduce back to (30) and (31).

### 6.2 Effects of periodic boundary conditions

We now examine the effects of the PBCs on a dipole. Therefore, we place one dipole into a periodic domain with half-periods $\omega = |\omega_1| = |\omega_2|$. Equation (35) is zero in the absence of other dipoles, implying that the orientation of one dipole remains constant in time, independent from the half-periods. The self-propelling velocity of a finite dipole is given by equation (64), and depends on the variables $w_s(\Gamma, l, \alpha, \omega_1, \omega_2)$. To examine the effect of these variables, we fix the vortex strength $\Gamma$, which only influences the magnitude of $w_s$, and set its value to $\Gamma = 2\pi$, such that the dipole has a self-propelling speed of $w_s = 1$ when $\omega \to \infty$ and $l$ is a unit length. The half-periods are expressed in terms of $l$, i.e. $\omega = Cl$.

The self-propelling speed as a function of the orientation angles is shown in figure 6.2 for different half-periods of the domain. When $\omega = l$ (purple line), the periodic cell is double the size of a finite-dipole. For orientation angles of $\alpha = \frac{\pi}{2}$, the separation distance between a vortex of its closest neighbour (duplication) is equal to the separation distance $l$. The nearest neighbour vortex induces a velocity in the opposite direction of equal magnitude, such that there is no net induced velocity in the self-propelling direction. When the dipole orientation is changed, for example to $\alpha = \left( \frac{n}{2} + \frac{1}{4} \right) \pi$, the distance to a vortex of its nearest duplication is $\sqrt{2}$, in that case the self-propelling velocity is larger than the opposite induced velocities from its nearest neighbours. This leaves the dipole with a net translational velocity, that is much smaller than its...
Figure 6.2 – Graph showing the effect of PBCs on the translation velocity of a dipole, for different half-periods $\omega$, with $|\omega_1| = |\omega_2| = \omega$.

self-propelling speed; this is represented by the purple and red lines, respectively, in figure 6.2.

Increasing the half-periods of the unit cell reduces the influence of the duplications on the dipole in the unit cell, see the blue, green, and cyan lines in figure 6.2. The effect on the self-propelling speed becomes smaller and smaller for the increasing half periods. When the half-period is $\omega = 6l$, the self-propelling speed is almost equal to the speed the dipole has in the unbounded plane.

In all configurations, an orientation of $\alpha = \left(\frac{n}{4} + \frac{1}{4}\right)\pi$ leaves a higher translational velocity than for $\alpha = \frac{n\pi}{2}$, although in both cases the speed is lower than in an unbounded plane. These two configurations correspond to dipoles placed on a diamond lattice or a rectangular lattice, respectively [15]. When the size of the lattice is $\omega < l$, due to its duplications the induced velocities can be larger than its own translational velocity, leading to a translational velocity in the opposite direction than the self-propelling speed.
7 Numerical methods

As in part I, all equations are made dimensionless ((32) and (35)). The vortex strength is chosen $\Gamma_n = \Gamma = 2\pi$. The length scales are dimensionless, i.e. $x_{1,2}^* = \frac{x_{1,2}}{L}$, with the lower subscript for $x$ and $y$, respectively, while the upper subscript is omitted from here on. The dipole has a self-propelling speed of $|w_s| = 1$. Other vortex strengths or separation distances can be chosen, which affects the simulation time.

Since we are working with a periodic cell with dimensionless sides $\omega_1$ and $\omega_2$, the amount of dipoles in a unit cell gives the mean density, defined as

$$\rho = \frac{N}{\Delta} = \frac{N}{L^2}. \quad (66)$$

Here, we limit ourselves to a square domain with $|\omega_1| = |\omega_2| = \omega$, and $\omega_1$ is measured along the $x$-axis, such that $L = 2\omega$.

Each finite-dipole is initially placed randomly in the square domain, i.e. the position of each centre $z_n = x_n + iy_n$ and the orientation $\alpha_n$ is obtained from a uniformly distributed random matrix of dimensions $[3 \times N]$. The first two columns correspond to the position and is a number between 0 and $2\omega$, while the third column (with random numbers between 0 and $2\pi$) represents the orientation angle. For each simulation-run a new random matrix is generated.

7.1 Convergence study

The simulation time strongly depends on; the amount of dipoles, the number of unit cells that is included into equation (60), the solver, and the accuracy used to solve the differential equations. In what follows, we are interested in large systems. In order to reduce computational costs we now examine each individual aspect.

7.1.1 Integration method

A common way to solve a differential equation is the ODE45 package in Matlab, which was also used in part I. It solves the differential equation by taking a time step, estimating the error at this step, and changing the stepsize when the error is larger than the tolerance. Using a variable step ensures that a large stepsize is used for small temporal derivatives and a small stepsize is used for large temporal derivatives. However, since we are integrating a chaotic system, described by $3N$ equations, it is impossible to solve the trajectory of a simulation exactly. Even the smallest computational errors overwhelm numerical trajectories in typical simulations, leading to smaller and smaller variable timesteps taken by the solver and an increase of computational time. Consequently, accuracy is expected only in a statistical sense, based on random initial conditions. In this part (II) we are not interested in the exact dipole trajectories (microscopic quantities), but in system average quantities (macroscopic quantities). This approach is also commonly used in the field of Molecular Dynamics [16]. Therefore, the Runge-Kutta fourth-order solver is used with fixed timesteps, ensuring acceptable computational time.
7 NUMERICAL METHODS

7.1 Convergence study

Figure 7.1 – Graph showing the evolution of the number of active dipoles in the periodic domain for different stepsizes compared to the standard ODE45 solver of Matlab.

Figure 7.2 – Graph showing the path taken by a dipole (same initial configuration for all runs) for different stepsizes and compared to ODE45 in Matlab. The paths are completely different due to the chaotic behaviour of the system, although average quantities of the system are less sensitive to the stepsizes.

Figure 7.1 represents the number of active dipoles in a periodic cell as a function of time for different stepsizes, obtained with the variable step solver (with a relative and absolute tolerance of $10^{-9}$ and $10^{-10}$, respectively). Dipoles become inactive when they get aligned opposite to each other and eventually annihilate each other (as seen in section 4.3).

Each simulation run was started with the same initial conditions. As the figure shows, the decay is not sensitive to the stepsize, although there are some small deviations for lower numbers of active dipoles. However, the path taken by each individual dipole can be completely different, as shown in figure 7.2. Although the dipoles take off at the same point and follow initially the same path, due to the chaotic behaviour of the system the paths start to diverge, and finally become completely different. In spite of the large differences on the individual paths, average system quantities are well defined. Note that the larger deviation for lower active dipole numbers is a result of poor statistics, due to a too small number of dipoles.

7.1.2 Step size

Although the results, in terms of average quantities, seem insensitive to the stepsize of the solver, this stepsize can have a large influence on the computation time and the dipole behaviour in the near field. The computational time scales with $\mathcal{O}(N^2)$, as shown in figure 7.3. Here, the computation time needed to evolve the system one dimensionless time unit, corresponding to the simulations of figure 7.1, is shown as a function of the number of active dipoles. The variable step solver initially needs a large
computational time, and decreases when the number of dipoles diminishes, while all other fixed step solvers scale linearly with the amount of dipoles.

Figure 7.4 shows the path of one individual dipole for different stepsizes. The pathlines are plotted to $t = 8$, for a system initialized with the same initial configuration. With a stepsize of $\frac{1}{120}$, indicated by the blue pathlines, the dipole follows the same main path as with smaller stepsizes, although exact details due to the high induced velocities are lost (the dipole is trapped). With a stepsize of $\frac{1}{120}$, indicated by the green pathline, the main path as well as the details are well maintained, although there are some small deviations on the long term, but these could also be a consequence of the chaotic system. The variable stepsize and fixed stepsizes of $\frac{1}{1000}$ show similar results for the plotted time. However, it can be seen from figure 7.3, that this small stepsize does not yield a better performance, since the details where already captured with $\frac{1}{120}$.

At a first glance this variable step solver seems to be the quickest and most accurate solver. However, we choose here a simulation that has converged with the variable step solver, such that the differences between the solvers could be pointed out. As noted in subsection 7.1.1, the problem with this solver is the sometimes non-converging solution leading on average to longer computational time, especially for higher initial densities. Therefore, we choose the fixed-timestep solver with a timestep of $\frac{1}{240}$, to ensure that small details are well conserved within an acceptable computational time.

### 7.1.3 Periodic cells

The implementation of the PBCs is obtained with the Weierstrass $\zeta$-function, given by equation (60). The computational time to solve this equation scales linearly with
the number of unit cells $j$ that are included. To minimize the total computational time, we want to minimize $j$ without deteriorating the macroscopic system quantities. The effect of the number of unit cells $j$ that is included into the calculations on the number of active dipoles is plotted in figure 7.5. In each simulation run the same initial configurations were used. The blue line shows that when the dipoles were placed initially in an infinite domain, less interactions occur due to their tendency to move to infinity (see also figure 5.12). When one or more neighbour unit cells are included, the dipoles keep exiting and entering the periodic cell, leading to more interactions and annihilations. The effects from the far field on dipoles are relatively small and have no influence. The divergence between the different lines is caused by the chaotic behaviour, as seen previously, and can be suppressed by taking into account more dipoles. Therefore, in what follows only one neighbour cell ($j = 1$) is included into each direction.

### 7.2 Inactive dipoles

In the previous part we have seen that dipoles can be trapped, scattered or annihilated. The velocity induced by an annihilating dipole pair is given by the quadrupole moment, since the monopole and dipole moment are zero (in the far field), given by:

\[
|\ddot{z}| \approx \frac{\gamma}{2\pi} \frac{2l\delta x}{z^3},
\]

with $l$ the separation distance between the vortices of one dipole, and $\delta x$ the distance between two dipoles that are oriented opposite to each other. The velocity induced by annihilating dipoles reduces quickly with $\frac{1}{z^3}$, while their separation distance reduces...
with $\delta x \propto \frac{1}{t}$ (see section 4.3). Since the influence of an annihilating dipole pair is negligible, we may exclude them from the simulation, therefore the following criteria are used:

$$|z_n - z_j| \leq l\xi_l,$$

$$|\pi - \xi_\alpha| \leq |(\alpha_n - \alpha_j)| \leq (\pi + \xi_\alpha).$$

That is, when the centres of an annihilating dipole pair are within a distance $l\xi_l$ from each other, and the dipoles are aligned opposite to each other within an orientation difference $|\pi \pm \xi_\alpha|$, they will be excluded from the ongoing simulation run.

There is no exact criterion for choosing the values of $\xi_\alpha$ and $\xi_l$. However, a reasonable criterion could be based on the case diagrams discussed in section 4.2, where we have seen that dipoles placed opposite to each other are annihilated (for small separation distances), while dipoles with a slight orientation shift (i.e. $\Delta\alpha \not\equiv \pi$) do not necessarily annihilate each other but are mostly trapped. Therefore, we have choose here $\xi_\alpha = 0.05$ and $\xi_l = 0.15$. 
8 $N$ finite-dipole system

In this section we consider the annihilation rates for a collection of finite-dipoles in a double periodic domain, and investigate the influence of the average system density on these decay rates.

8.1 General observations

First, we discuss the typical behaviour of the dipole density decay which is shown in figure 8.1. Here, the decay is plotted for two systems (cyan and purple line), and an average of several independent simulations (blue line). The initial system density is $\rho = 0.02$ with $N = 100$ randomly placed finite-dipoles. Three typical observations are obtained from the figure, which we discuss in the following three subsections.

8.1.1 Transient effects

Transient effects are related to the initialization of the system. In section 7 we discussed that the dipoles are placed randomly in the domain. It implies that dipoles are hardly ever oriented directly opposite to each other. Therefore, there is always a transition time in which the dipoles must align opposite to each other, and reach the critical annihilation distance. This transition time is visible in figure 8.1, where we have zoomed in on the first few time units.

8.1.2 Variabilities between simulations

The two lines (purple and cyan) in figure 8.1 follow initially the same decay, but start to diverge around $\rho = 0.0075$. At this point there are approximately 30 dipoles left, and the fluctuations become large due to poor statistical quantities. The (sometimes) significant variability between each cycle, can be minimized by averaging

$$ f_a = \langle f \rangle = \frac{1}{N} \sum_{i=1}^{N} f_i. \quad (70) $$

With $N$ the number of independent simulations, and $f$ the considered quantity, for example $\rho$. The averaging of independent simulations is commonly also called ensemble averaging. The average density $\rho_a$, obtained from seven independent simulations, is shown in figure 8.1 (blue line). We see that the fluctuations in the decay are middled out and a smooth curve is obtained. The cause of the fluctuations between each individual simulation (shown within the red circle) will be explained in the next subsection.

Another way to minimize the fluctuations is to increase the number of dipoles. We are able to increase the number of dipoles to a maximum of 500 dipoles, resulting in the dipole decay as shown in figure 8.2. This result is obtained after one simulation run, which gives a much smoother curve comparable to the result of seven averaged simulations each with $N = 100$ dipoles (blue line).
8.1 General observations

8.1.3 Stabilizing effect

In the previous subsection we have seen that there are significant variabilities between each simulation run, which can be averaged out. However, the cause of this variability was unexplained, apart from the note that it is more significant for smaller systems (smaller $N$). The origin of these large fluctuations is clarified with figure 8.3. Here, some snapshots of the system are shown, corresponding to the cyan line in figure 8.1. Figure 8.3a shows the initial configuration at $t = 0$, while figure 8.3b shows the system at $t = 37$. The active dipoles are indicated with red-blue dots (representing positive and negative vorticity, respectively), while inactive dipoles are indicated by the grey coloured dots. There are two circles in the figure that indicate a three-dipole configuration. This configuration emerges when two dipoles are annihilating each other, while a third dipole joins them and disturbs the annihilation process. Now a stable three-dipole configuration is formed, as was discussed in section 5.4, but in a straining mode. Due to the asymmetry of this straining mode, the configuration has a translational velocity. Since this configuration is stable it will be moving in the domain until a fourth (random) dipole crosses its path. This dipole disturb the three-dipole system, and the configuration disintegrates into two two-dipole systems (see the discussion about the instability of a four dipole system in section 5.4). The remaining two-dipole configurations are unstable and become inactive once the dipoles have annihilated each other, see figure 8.3c-d. It causes a sudden big jump in the dipole decay, as shown in figure 8.1. The second indicated three-dipole configurations stays active until $t \approx 530$, revealing their long life time.

Trapped dipole configurations (with two dipoles) can also exhibit a long life time, but only exist in: (i) a dense systems where dipoles overlap and due to the high induced velocities these dipoles can easily put in or escape from a trapped state; (ii) the dipoles
are placed in a trapped state by the random initialization of the system, and remain trapped until a third dipole starts to interact.
8.1 General observations

Figure 8.3 – Snapshots of a dipole system, with initial density $\rho = 0.02$ and $N = 100$. These snapshots correspond to the dipole density decay shown by the cyan line in figure 8.1. Active dipoles are coloured, while inactive dipoles are grey. (a) Initial configuration at $t=0$. (b) The system at $t = 37$, showing that many dipoles are already annihilated. The red circles indicate three-dipole configurations. (c,d,e) Detailed snapshots of the configuration at $t = 72$, 79 and 84, respectively, showing how a 4th dipole (green circle) joins the three-dipole configuration (red circle), and makes the three-dipole configuration unstable, leading to a disintegration into two two-dipole systems. The cyan circles are trapped dipoles.
8.2 Annihilation rate

During the time evolution of the system, dipoles interact with each other leading mostly to scatter or annihilation, which continues until no dipoles are left. Each time that dipoles are annihilated the system density $\rho(t)$ decreases, while the time between each subsequent interaction increases due to the increase of the average dipole distance. To find an analytical expression for the dipole density decay $\rho(t)$, a rational function

$$f_{nm}(t) = \frac{\sum_{i=0}^{n} p_i t^i}{\sum_{i=0}^{m} q_i t^i}, \quad (71)$$

is fitted through the ensemble averaged density. In this expression $p_i$ and $q_i$ are constants, and we sometimes refer to this expression as “rat$nm$”.

<table>
<thead>
<tr>
<th>Constant</th>
<th>Value</th>
<th>(min max)</th>
</tr>
</thead>
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<tr>
<td>$p_1$</td>
<td>0.4462</td>
<td>(0.4453, 0.4471)</td>
</tr>
<tr>
<td>$p_2$</td>
<td>0.5464</td>
<td>(0.4383, 0.6546)</td>
</tr>
<tr>
<td>$p_3$</td>
<td>1.056</td>
<td>(0.5938, 1.518)</td>
</tr>
<tr>
<td>$q_1$</td>
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<td>(17.82, 18.18)</td>
</tr>
<tr>
<td>$q_2$</td>
<td>31.07</td>
<td>(26.12, 36.01)</td>
</tr>
<tr>
<td>$q_3$</td>
<td>52.67</td>
<td>(29.47, 75.88)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.9994</td>
<td></td>
</tr>
</tbody>
</table>

Table 8.1 – Corresponding constants of the rat34 function, obtained with equation (71). The fit is shown in figure 8.4.
8.2 Annihilation rate

The ensemble averaged result of five independent simulations, each started with $\rho_0 = 0.02$ and $N_0 = 500$, is shown in figure 8.4. This figure also shows the deviation between each simulation with the errorbars, and the fit with a rat34 function. This rat34 function gave the best fit in combination with the smallest $R^2$ (the averaged square deviation between the data points and the function). The coefficients corresponding to the rat34 function are given in table 8.1. From this rat34 function a simple rat01 function is expected for large $t$, i.e. $\rho(t) \approx \frac{\rho_0}{q_{1t}}$ after all transient effects have disappeared. The differential equation corresponding to a rat01 function is given by

$$\dot{\rho} \propto k\rho^2. \quad (72)$$

To confirm this relation we differentiate equation (71) for $n = 3$ and $m = 4$ and plot it as $\dot{\rho}(\rho^2)$. The result is shown in figure 8.5. Note that $t = 0$ corresponds to large densities, which relates to the right-hand side of the graph. For $\rho \leq (2.5 \cdot 10^{-4})^{1/2} \approx 0.016$ we clearly observe a linear relation between the decay rate and the system density. The divergence at high densities is a result from the transient effect, discussed in section 8.1.1. For clarity, the derivative of the rat01 function is also plotted in figure 8.5, indicated with the blue line.

We now examine if the differential equation (72) is valid for all numerically obtained results. The data sets are ensemble averages, obtained from five independent simulations. The system density $\rho_0$ is doubled between each ensemble averaged, i.e. $\rho_0 = 0.002; 0.005; 0.01; 0.02; 0.05; 0.1; 0.2; 0.5$, such that a broad density range is obtained. Note that, some of these data sets are obtained with the ODE45 solver (which was possible for lower initial densities). A rational function (equation (71)) is
fitted through the data, and the number of term \( n, m \) is increased until the best fit is obtained \((R^2)\) with a maximum of \( n = 4 \) and \( m = 5 \). The results, after differentiating the fitted function, are collected in figure 8.6, where we have plotted the data on a log-log scale to increase the visibility for lower densities. From the figure its clear that all data sets scale linearly with \( \rho^2 \), except in the limit for high densities, meaning that the relation \( \dot{\rho} \propto k\rho^2 \) is indeed generally valid provided the density is not too high.

The flattening at high densities \((\rho \gtrsim 0.1)\) can have two causes; i) it is related to transient effects, or ii) due to the higher densities dipoles are easily perturbed during the annihilation. The first statement can be checked by running simulations with a higher initial density, such that all transient effects have disappeared once the system has reached the density where currently flattening appears. In the next section we introduce a "renormalization model". This model makes it possible to obtain results with reasonable statistics even when we started at high densities.

8.3 Renormalization

In section 8.1.2 it was noticed that statistical properties become poor for lower dipole-numbers. By increasing the number of dipoles we can increase macroscopic quantities. However, especially at higher densities this is still not sufficient, due to the fast annihilation rate we are soon left with too few dipoles. The number of dipoles can not be increased unlimited, since we are limited by computer memory and time. Therefore, we now introduce a "renormalization" model based on a procedure used for the kinetics of irreversible aggregation, (i.e: \( A_i + A_j = A_{i+j} \)) of point-vortices [19]. In this model, we use the final state of an integration in time (containing few vortices) to create a
new initial state with many more vortices. By integrating over many renormalization cycles we can reach higher accuracy over a larger density regime.

In the renormalization procedure we start with a system prepared with \( N_0 \) dipoles, in a periodic cell with sides of the size \( \omega_0 \), such that a mean density \( \rho_0 = \frac{N_0}{(2\omega_0)^2} \) is obtained. From this state the system evolves and dipoles are annihilated until \( N'_N \) dipoles are left. The evolution of the system from \( N_n \to N'_n \) is called a cycle. At the end of each cycle we renormalize the system by increasing the number of dipoles from \( N'_n \to N_0 \). To remain consistent with the system density \( \rho'_n \) at each cycle, we are forced to increase \( \omega_0 \) (see equation (66)). We choose the following procedure between each cycle: \( 4N'_n = N_0 \) and \( 2\omega_n = \omega_{n+1} \) such that \( \rho'_n = \rho_{n+1} \).

The new position of the \( 3N'_n \) dipoles are related to those of the \( N'_n \) dipoles; they are such positioned that each old dipole is replaced by four new dipoles at \( (z_i + \omega_n (m_x 1 + m_y i)) \) with \( m_{x,y} = 0; 1 \). Some small (random) noise is added to each dipole position \( z_n \) and orientation \( \alpha_n \) to prevent that exact copies are obtained.

We should note that the ratio between the dipole separation distance and the system size \( \delta_n = \frac{l}{\omega_n} = \frac{1}{m\omega_0} \) changes each cycle. We argue however that this ratio is of minor importance since: i) The effects from its own duplications (as a consequence of the PBCs) is negligible if \( \omega \geq 6l \) (see section 6.2). If we start with \( N_0 = 400 \) dipoles and \( \rho_0 = 1 \) the half-periods are \( \omega_0 \geq 10l \). ii) We consider the effects small compared to the velocities induced by the other dipoles, which is conserved since we are copying the dipoles each renormalization.

During each cycle the number of dipoles evolves from \( N = 400 \) to \( N = 100 \), and back to \( N = 400 \) after a renormalization. With this approach transient effects are minimized since the field of dipoles has evolved past its transients. To minimize variabilities between individual cycles we consider again ensemble averaged quantities of independent simulations.

Our goal is to derive \( \dot{\rho} \) such that we can plot \( \dot{\rho}(\rho) \) as in the previous section, but now for a larger density range than the data sets of figure 8.6. We use the averaged result of five independent simulations. It was not possible to fit one function through the entire data set. Therefore the data set is divided into two subsets: one from \( 0.1 \leq \rho \leq 1 \) and one for \( \rho < 0.1 \). The results for both fits are shown in figure 8.7 with the red lines.

In the previous section it was noted that the decay rate flattens for high densities. From figure 8.7 a regime with \( \rho \gtrsim 0.05 \) can be estimated where the decay rate does not scale according to \( \dot{\rho} \sim \rho^2 \). The flattening can not be related to transient effects, since the system is well evolved since the initialization. A comprehensive study of the system gave two possible explanations for this flattening, 1) at high densities the system is dense and the typical vortex distance is smaller than the typical separation distance \( l \) of a dipole. This leads to high induced velocities which easily disturb the annihilation process and therefore the decay rate decreases; 2) in the density regime \( 0.25 < \rho < 0.01 \) many three-dipole configurations are created. Three-dipole configurations are stable and can slow the annihilation rate down (see section 8.1.3). At lower dipole densities the effects of the two discussed possibilities disappear, leading to a decay rate that scales according to \( \dot{\rho} \sim k\rho^2 \). For \( \rho < 0.05 \) the constant is given by \( k \approx 1.863 \).
8.4 Ballistic annihilation

In this section we discuss a model from Molecular Dynamics (MD), that describes "ballistic annihilation". Here, Maxwell molecules or vary hard particles move with constant velocity and annihilate each other when colliding, $A_i + A_j \rightarrow 0$. The model is based on the Boltzmann equation (BE)

$$\frac{\partial P(v, t)}{\partial t} = -P(v, t) \int g \sigma P(w, t) dw,$$

where $P(v, t)$ is the velocity distribution function of a spatially homogeneous gas ($P$ does not depend on the spatial coordinate), and $g = |v - w|$ the relative velocity between two particles. The right-hand side accounts for the collisions where $\sigma$ is a reaction constant. The equation above states that a change of the system in time (LHS) is related to the sum of particles having different velocities times the reaction constant $\sigma$ (RHS). For a detailed discussion see [17] and [18]. The Boltzmann equation as stated above is based on the "molecular chaos assumption", which have allowed us to write the collision term as a momentum-space integral over the product of an one-particle distribution functions [20]. Other important assumptions underlying the BE are [18]:

- The gas is dilute so that the one-body distribution $P(r, v; t)$ describes the state of the gas accurately. Furthermore, the gas is sufficiently dilute that only binary collisions are important.

- Particles move with almost constant velocities between collisions and their trajectories bend only when particles are in a close proximity.

Equation (73) has not been solved and results have to come from numerical simulations, which reveal the following scaling behaviour for the particle concentration $n(t)$:
and the mean velocity $v_{rms}(t)$:

$$v_{rms}(t) = \left[ \frac{1}{n(t)} \int_{-\infty}^{\infty} w^2 P(w, t) \, dw \right]^{1/2} \sim t^{-\beta}. \quad (75)$$

The constants $\alpha$ and $\beta$ are related to each other by $\alpha + \beta = 1$. This can be seen as follows, we cite [17]:

Consider a system of identical particles of fixed radius $r$ at concentration $n$ which move with a velocity of the order of $v_{rms}$. From an elementary mean-free-path argument, the time between collisions is $t \sim \frac{1}{nv_{rms}r^d}$, or $nv_{rms} \propto t^{-1}$. Thus the relation $\alpha + \beta = 1$ should hold for all spatial dimension $d$.

Since the lifetime of particles with velocity $v$ is proportional to $1/v$, faster particles tend to annihilate more quickly, and the typical velocity should decay in time.

For Maxwell molecules, it can be argued that molecules interact at a rate that is independent of the velocity [18]. Therefore, $g$ in equation (73) can be replaced by a constant $u$, and equation (73) now becomes

$$\frac{\partial P(v, t)}{\partial t} = -P(v, t) \sigma u n(t). \quad (76)$$

Integrating this equation over all possible velocities gives:

$$\frac{\partial n(t)}{\partial t} = -u \sigma n^2(t), \quad (77)$$

with the solution

$$n(t) = \frac{n(0)}{1 + u \sigma n(0) t}. \quad (78)$$

Substituting this equation back into equation (76) gives the relation:

$$P(v, t) = \frac{P(v, 0)}{1 + u \sigma n(0) t}. \quad (79)$$

The resulting decay exponents are $\alpha = 1$ and $\beta = 0$, which are universal and trivial in the Maxwell model.

Equation (77) is a more common result (for example in one species annihilation). Although the discussed model is based on molecular dynamics, where particles are seen as spheres and interact due to a potential, there are striking similarities between
the molecular dynamics model and the finite-dipole model. First of all, we have seen that the decay dynamics of finite-dipoles scales with $\rho \sim t^{-1}$, i.e. equation (72), which was found numerically for finite-dipoles at lower densities, has a similar form to that of equation (78). In the following subsections, we discuss more relations between the models.

### 8.4.1 Spatial distribution function

The Boltzmann equation (73) presumes that particles are distributed spatially homogeneous over the domain (then, the convective term $\mathbf{v} \cdot \nabla P$ is zero and the equation simplifies to the given form). Since the dipoles are initially placed randomly in the periodic domain this should be true for $t = 0$, but this is not necessary valid for $t > 0$. Therefore, we now investigate the distribution of finite-dipoles in the domain. For this purpose we use the data obtained with the renormalization process, which ensures that we have reasonable statistics over a wide density range. Each time the dipole number in a cycle reaches $N = 350$, a spatial distribution function $f(x, t)$ is obtained. We choose $N = 350$ because we want as much as possible dipoles. Calculating the spatial distribution function directly after a renormalization (when $N = 400$) would give effectively only the statistics of $N = 100$ finite-dipoles. The results are shown in figure 8.8, where the $x$-axis is normalized by the size of the periodic domain. It is clear that the dipoles are indeed spatially homogeneous distributed over the periodic cell, although the third and fourth graph from the bottom are more peaked. At these densities many three-dipole configurations are observed, which can clarify why these distribution functions are less homogeneous. Once the system reaches lower densities most three-dipole configurations are annihilated leading again to a uniform distribution.

### 8.4.2 Velocity distribution function

Another important assumption made in the Boltzmann equation as stated above, is that particles move with almost constant velocity between collisions and their trajectories are bend only when the particles are in close proximity. Also, the gas should be diluted enough that only binary collisions are important.

Roughly speaking, both statements could be verified from the velocity distribution function. A single dipole has, in the absence of other dipoles, a well defined self-propelling speed: $|\mathbf{w}_s| = 1$. Therefore, in a diluted gas we expect a narrow peak in the velocity distribution function around this self-propelling speed. Again we used the data obtained with the renormalization process at each time that $N = 350$ finite-dipoles are left. The results are shown in figure 8.9. In the beginning dipoles overlap due to the high density ($\rho \approx 0.88$), which results in high induced velocities and a broad peak in the velocity distribution. At each subsequent renormalization the system density decreases. Therefore, we expect a more and more narrow peak in the velocity distribution. This becomes visible after the second renormalizations (red line), where most dipoles already move with their self-propelling velocity. The skewness of the distribution in the direction of $0$ indicates that many dipoles are annihilating each other, and the existence of three-dipole configurations that have a translational speed de-
Figure 8.8 – Graph showing the spatial distribution function $P(x,t)$, at each cycle that the system reaches $N = 350$ finite dipoles. From the bottom to the top the densities are $\rho = \frac{7}{8}; \frac{7}{8\pi}; \frac{7}{8\pi^2}; \frac{7}{8\pi^3}; \frac{7}{8\pi^4}$. The $x$-axis is normalized, by dividing by the size of each period of the domain, i.e. $\bar{x} = \frac{x}{2\pi n}$. The blue colour represents the spatial distribution function of the $x$-position of finite-dipoles, the green colour represents the spatial distribution function of the $y$-position of finite-dipoles.
8.4 Ballistic annihilation

Figure 8.9 – Graph showing the velocity distribution function $P(v,t)$ at each cycle for which the system reaches $N = 350$ finite-dipoles. The density at each cycle is given in the legend. Initially, the system is dense and high velocities are induced, resulting in a broad distribution function. Once the system becomes more diluted, dipoles tend to translate with their self-propelling speed, leading to a peak around $v = 1$.

Depending on the strain mode (which is generally smaller than the self-propelling speed). Between the second and third renormalization most three-dipole configurations are annihilated. The peak becomes more narrow, indicating that most dipoles translate with their intrinsic velocity.

Note that the velocity distribution function at $t = 0$ has a high-velocity tail. This high-velocity tail is also shown in figure 8.10, which is obtained for $N = 10000$ finite-dipoles. Here the velocity difference is considered $w = |v - \langle v \rangle|$, between the velocity of each dipole centre and the averaged velocity $\langle v \rangle$. The high-velocity tail scales in good approximation with a power law $P(w) \sim w^{-3}$, shown by the green line. This seems to be consistent with observations coming from simulations of point-vortices [29].

8.4.3 Velocity dependence

The dipole density decay for finite-dipoles scales with $\rho \sim t^{-1}$ at lower densities. From the relation $\alpha + \beta = 1$, which is valid for a system that can be described by the
Boltzmann equation (73), a constant root-mean-square (RMS) velocity is expected, according to: \( v_{\text{rms}} \sim t^0 \).

Figure 8.11 shows \( v_{\text{rms}} \), as obtained by averaging the same five simulations as the data set used for figure 8.7. For \( 0 < t < 50 \), the graph shows a strong decrease of \( v_{\text{rms}} \). In this time interval the system density decreases from \( \rho = 1 \) to \( \rho \approx 0.05 \), where the density does not scale with \( 1/t \). From \( 50 < t < 200 \) there are still relatively high fluctuations in \( v_{\text{rms}} \), which indicates that there are still many dipoles interacting. For \( t > 200 \), the system is diluted and most dipoles translate with their self-propelling velocity, i.e. \( v_{\text{rms}} \) now becomes (almost) a constant.

### 8.4.4 Concluding remarks

In the three previous subsections we have seen that a deluted gas of finite-dipoles seems to have the necessary conditions to be described by the Boltzmann equation (73). The dipoles are spatially homogeneously distributed and have a constant velocity between the collisions. However, one important condition is not yet verified, i.e. whether the dipoles are uncorrelated such that the collision term can be written as a momentum-space integral over the product of one-particle distribution functions [20].
Figure 8.11 – Graph showing the RMS velocity $v_{rms}$ as a function of time, from five averaged data sets, corresponding to the decay rate in figure 8.4. Initially the dipoles experience high induced velocities due to the high density of the system. The system becomes quickly deluted.
Part III

Bounded domain
9 Theory

9.1 Vortices bounded by two walls

When the image principle is applied on vortices which are bounded by two walls, oriented opposite to each other, things get more complicated since in that case there are infinitely many mirror images. The velocity a vortex experiences due to its own mirrors is given by

\[ \dot{\bar{z}}_n = -
\frac{i \Gamma_n}{2\pi} \frac{\pi}{2D} \cot \left( \frac{\pi A_n}{D} \right), \tag{80} \]

see Appendix D for the full derivation. In (80) \( D \) is the separation distance between the two solid walls and is a purely imaginary quantity when the walls are oriented on the real axis. \( A_n \) is the vertical distance from the vortex to the upper wall, i.e. \( A_n = i \text{Im} (D - z_n) \) and is also an imaginary quantity. When more vortices are considered not only their direct influence onto each other should be considered, but also the mirrors of these vortices (note that mirrors can only influence the real vortex and not other mirrors). Their influence is given by,

\[ \dot{\bar{z}}_n = \sum_{j \neq n} \frac{i \Gamma_j}{2\pi} \frac{\pi}{2D} \left\{ \cot \left( \frac{\pi (\lambda_j - 2A_j)}{2D} \right) - \cot \left( \frac{\pi \lambda_j}{2D} \right) \right\}, \tag{81} \]

with \( \lambda_j = (z_n - z_j) \) the separation distance between two dipoles, and \( A_j = i \text{Im} (D - z_j) \) the distance from the \( j^{th} \) vortex to the upper site of the wall.

9.2 Finite-dipoles bounded by walls

Equations (80) and (81) govern the motion of vortices on a domain bounded by two walls oriented parallel to the real axis. Now these equations will be rewritten for the finite-dipoles, in the form of their self-propelling speed \( w_{n,s} \) and the velocities \( w_{n,o} \), induced by others. The velocities induced by others are not only due to the real vortices, but also due to their mirrors and its own mirrors, i.e. \( w_{n,o}(z) = w_{n,ma}(z) + w_{n,mo}(z) \) (the orientation of a real dipole can change under the influence of its own mirror). The equations (32) and (35) governing the velocity and orientation of a finite-dipole are

\[ \dot{\bar{z}}_n = w_{n,s} + \frac{(w_{n,o}(z_{n,l}) + w_{n,o}(z_{n,r}))}{2}, \tag{82} \]

and

\[ \dot{\alpha}_n = \text{Re} \left[ \left( w_{n,o}(z_{n,r}) - w_{n,o}(z_{n,l}) \right) e^{i\alpha_n} \right] \frac{1}{l_n}. \tag{83} \]

The self-propelling speed of the dipole is still given by equation (30), i.e.

\[ w_{n,s} = \frac{\Gamma_n e^{-i\alpha_n}}{2\pi l_n}, \]

while \( w_{n,o}(z) \) is given by its own mirrors,
Figure 9.1 – Schematic representation of finite-dipoles bounded by two solid walls. The impenetrability of fluid through the wall is obtained by placing mirrors of the dipoles, and since the walls are placed opposite to each other, there are infinitely many mirrors.

\[
\begin{align*}
    w_{n,ms}(z) &= -\frac{i\Gamma_n}{2\pi} \frac{\pi}{2D} \left\{ -\cot \left( \frac{\pi ((z - z_{n,l}) - 2A(z_{n,l}))}{2D} \right) \right. \\
    &\quad \left. + \cot \left( \frac{\pi ((z - z_{n,r}) - 2A(z_{n,r}))}{2D} \right) \right. \\
    &\quad \left. - \cot \left( \frac{\pi ((z - z_{n,l}) - (z - z_{n,l}))}{2D} \right) \right. \\
    &\quad \left. + \frac{2D}{\pi} \frac{1}{(z - z_{n,r}) - (z - z_{n,l})} \right\} \\
    \text{where } D &\text{ is the distance between the two solid walls, which is a purely imaginary quantity (due to its orientation on the real axis), } \\
    A(z) &= i \text{Im}(D - z) \text{ is the distance from the vortex to the upper wall and } (z - z_n) = \lambda_n \text{ was already substituted into the equation but represents the distance between two vortices, see figure 9.1 for a schematic representation. The velocity due to other dipoles and their mirrors in } z_n \text{ is given by} \\
    w_{n,mo}(z) &= \sum_{j \neq n}^{N} -\frac{i\Gamma_n}{2\pi} \frac{\pi}{2D} \left\{ \cot \left( \frac{\pi ((z - z_{j,l}) - 2A_j(z))}{2D} \right) \right. \\
    &\quad - \cot \left( \frac{\pi ((z - z_{j,r}) - 2A_j(z))}{2D} \right) \right\}, \\
    \text{where the } n\text{-th dipole, which is considered, should be excluded from the summation. } \\
    A_j(z) &\text{ represents the vertical distance between the } j\text{-th vortex to the wall.}
\end{align*}
\]
10 Conclusion and recommendations

10.1 Conclusion

A finite-dipole consists of two point-vortices of equal strength, having opposite vorticity, and placed a separation distance $l$ apart. In the finite-dipole formulation the separation distance $l$ is fixed, which is in contrast to the classical point-vortex dipole where the separation distance can change under the influence of other dipoles. In the present study we investigated the interaction between such finite-dipoles.

First we considered two-, three, and four-dipole interactions in an infinite plane. For a two-dipole system three interaction types can be identified, which depend on the initial configuration, i.e: scattering, annihilation, or trapping. The interactions were classified in a regime diagram, at different regime diagrams where made for different orientation angles between the two dipoles. The symmetry is conserved for dipoles that where placed point symmetric. Therefore, the dynamics could be simplified and reformulated in terms of the so called $w-$function. This function shows the existence of stable, unstable or asymptotically-stable dipole configuration.

The $w$-function was also derived for a three- and four-dipole system. These systems were less stable and did not always conserve the initial symmetry. The three-dipole system has a configuration which is linearly stable, i.e. after a small perturbation the dipoles re-obtain their symmetric configuration. The four-dipole system is mostly unstable, leading to the disintegration of the four-dipole system into two two-dipole systems.

For the simulation of $N$ dipoles periodic boundary conditions were used. The dynamics of a system of $N$ dipoles can be divided into two-, three-, and four-dipole interactions. Dipoles that interact are mostly scattered or annihilated. By excluding annihilating dipoles from the periodic domain, decay dynamics was obtained. It was found that the system density scales according to $\rho \sim t^{-1}$ when $\rho < 0.05$. In this regime the mean velocity stays constant in time $v_{rms} = 1$. The decay dynamics seems insensitive for the stepsize of the solver, while individual dipole paths are sensitive to the stepsize and computational noise (chaotic system). A renormalization model was introduced to find reasonable statistics over a wide density regime. Each time that $1/4$ of the initial number of dipoles is left, the system is re-prepared to the initial number of dipoles while the system retains the previous density. The spatial- and velocity-distribution where determined from the renormalization data. The finite-dipoles where distributed homogeneous over the periodic domain, especially for lower densities. The velocity distributions is flat for high densities (high induced velocities), while it becomes narrow and peaked around the self-propelling velocity for lower system densities. The decay dynamics suggests that the Boltzmann equation maybe used to model the statistics of a gas of interacting finite-dipoles providing the density is sufficiently low.
10.2 Recommendations

Some recommendations can be formulated in addition to the present study of finite-dipoles:

- For the two-dipole system is would be interesting to see how passive tracer particles are advected by the velocity field that is induced by the two trapped dipoles, and if it can form a model for a chaotic mixer.

- For a further research to the N-dipole system it can be interesting to program the system in a faster computer code, such that the number of dipoles can be increased and better statistical system quantities are obtained (macroscopic quantities).

- For a diluted system, the dipoles translate with their self-propelling speed. Therefore the system may be simplified, and only the effects of dipoles in close approximation are encountered. With this the number of calculations at each timestep can be reduced and therefore the computational time. It would be interesting to see if this has an influence on the macroscopic system quantities.

- A more comprehensive study on the effects of the critical annihilation distance on the annihilation rate and other macroscopic system quantities would be derirable.

- For a dense systems ($\rho = 1$), the tail of the velocity distribution function scales with $P(w) \sim \frac{a}{w^2}$, which is consistent with observations comming from simulations of point-vortices [29]. More research is needed to confirm and clarify these observations.
References


REFERENCES


Appendix
A. The \( w \)-function for \( N = 2 \)

A.1 Derivation of the \( w \)-function

The self-induced velocity of a dipole is given by equation (30). Due to the symmetry of the system, for dipoles which are placed opposite to each other, the following relations follow; \( \alpha_1 = \pi - \beta \) with \( \beta \) a variable, \( \alpha_2 = -\beta \) and \( z_1 = -z_2 \) (see figure A.1). The equation for its self-induced velocity can now be rewritten to

\[
\dot{z}_1 = \frac{\Gamma e^{-i\alpha_1}}{2\pi l} = -\frac{\Gamma e^{i\beta}}{2\pi l}, \quad (86)
\]

\[
\dot{z}_2 = \frac{\Gamma e^{-i\alpha_2}}{2\pi l} = \frac{\Gamma e^{i\beta}}{2\pi l} = -\dot{z}_1. \quad (87)
\]

In each vortex a velocity is induced by other dipoles in the plane and is given by equation (31). For \( N = 2 \) this equation can be simplified to

\[
\dot{z}_{1,o}(z) = \frac{\Gamma}{2\pi i} \left( \frac{1}{z - z_{2,l}} - \frac{1}{z - z_{2,r}} \right). \quad (88)
\]

The total velocity consists of the self-induced velocity and the velocity induced by others (equation (32)), i.e.
\[ \dot{\bar{z}}_1 = \dot{z}_{1,\alpha} + \frac{\dot{z}_{1,\alpha}(z_{1,l}) + \dot{z}_{1,\alpha}(z_{1,r})}{2}. \] (89)

\[ \dot{\bar{z}}_1 = -\frac{\Gamma e^{+i\beta}}{2\pi l} + \frac{1}{2 \cdot 2 \pi i} \left( \frac{1}{z_{1,l}-z_{2,l}} - \frac{1}{z_{1,l}-z_{2,r}} + \frac{1}{z_{1,r}-z_{2,l}} - \frac{1}{z_{1,r}-z_{2,r}} \right). \] (90)

In the last step equations (86) and (88) are used. The positions of the left and right vortices are given by equation (27), which can be simplified with the observed symmetry to

\( z_{1,r} = z_1 + \frac{il e^{-i\beta}}{2}, \)
\( z_{2,r} = z_2 - \frac{il e^{-i\beta}}{2} = -z_1 - \frac{il e^{-i\beta}}{2} = -z_{1,r} \)
\( z_{1,l} = z_1 - \frac{il e^{-i\beta}}{2}, \)
\( z_{2,l} = z_2 + \frac{il e^{-i\beta}}{2} = -z_1 + \frac{il e^{-i\beta}}{2} = -z_{1,l} \)

Substituting these into equation (90) and simplifying:

\[ \dot{\bar{z}}_1 = -\frac{\Gamma e^{+i\beta}}{2\pi l} + \frac{1}{2 \cdot 2 \pi i} \left( \frac{1}{z_{1,l}-z_{2,l}} - \frac{1}{z_{1,l}-z_{2,r}} + \frac{1}{z_{1,r}-z_{2,l}} - \frac{1}{z_{1,r}-z_{2,r}} \right). \]

\[ \dot{\bar{w}}_1 = \frac{-\Gamma e^{i\beta}}{2\pi l} + \frac{1}{4 \pi i} \frac{i le^{-i\beta}}{z_1^2 + \frac{1}{4} l^2 e^{-2i\beta}} \]

\[ \dot{\bar{w}}_1 = \frac{-\Gamma e^{i\beta}}{2\pi l} + \frac{1}{2 \pi l} \left( \frac{1}{z_1^2 + \frac{1}{4} l^2 e^{-2i\beta}} \right) \] (91)

We now introduce a variable \( w \), defined by \( w = \frac{1}{2} z e^{i\beta} \), and contains the position (centre) and orientation of one dipole in the complex plane and which is itself a complex number. This variable \( w \) should not be confused with the complex potential. Substituting this together with the dimensionless time (\( \tau = \frac{\Gamma}{\pi l} t \)) into equation (91) leads to,

\[ \frac{d}{d\tau} \left( \bar{w} e^{i\beta} \right) = \frac{-w^2}{w^2 + 1} e^{i\beta}, \]

\[ \dot{\bar{w}} + i \bar{w} \dot{\beta} = \frac{-w^2}{w^2 + 1} \] (92)

This differential equation gives the evolution of \( w \) in the complex plane in time. The equation contains the variable \( \dot{\beta} \) which is given by equation (35) and can be simplified, using the same simplifications and substitutions as above, into the following form:
A.2 Stationary solutions

\[ \dot{\alpha}_1 = -\beta = \frac{1}{l} \text{Re} \left\{ \frac{-\Gamma}{2\pi i} \frac{p^2 e^{-i\beta} e^{-i2\beta}}{z_1 (4z_1^2 + l^2 e^{-2i\beta})} \right\} \]

Now using again the variable \( w \) and the dimensionless time \( \tau \) one to obtains:

\[ -\frac{\Gamma}{\pi l^2} \frac{d\beta}{d\tau} = \frac{1}{l} \text{Re} \left\{ \frac{-\Gamma}{2\pi i} \frac{l^2 e^{-i\beta} e^{-i2\beta}}{z_1 (4z_1^2 e^{-2i\beta} + 1)} \right\} = \frac{-\Gamma}{\pi l^2} \text{Re} \left\{ \frac{1}{i} \frac{1}{w (w^2 + 1)} \right\} \]

\[ \dot{\beta} = -\text{Im} \left\{ \frac{1}{w (w^2 + 1)} \right\} \]  \hspace{1cm} (93)

This is a differential equation that describes the evolution of the orientation angle in time as a function of \( w(z,\beta,t) \). Combining equation (92) and (93) leads to:

\[ \dot{w} = \frac{-w^2}{w^2 + 1} - i \, \bar{w} \, \text{Im} \left\{ \frac{1}{w (w^2 + 1)} \right\} \] \hspace{1cm} (94)

A.2 Stationary solutions

Numerical simulations have revealed that two dipoles can circle in an orbit around each other with a constant radius and angular velocity, which is equal to the rotation speed of the dipole centre around the origin. Therefore this configuration rotates as a solid body rotation. The angular velocity \(-\dot{\alpha} = \dot{\beta} = \dot{\theta} = \text{const}\), while the position of the dipole centre is given by \( z = R e^{i\theta} \), with \( R \) the radius of the circular orbit and \( \theta \) its argument. Looking for an analytical solution of these orbits, we use the \( w \)-function, take the time derivative, and substitute the above relations. This leads to:

\[ \dot{w} = \frac{2}{l} \left( \bar{\dot{z}} + i \bar{\dot{z}} \beta \right) e^{i\beta} = \frac{2}{l} \left( \left( \dot{R} + iR\dot{\theta} \right) e^{i\theta} + iR\dot{\beta} e^{i\theta} \right) e^{i\beta} = 0. \]

With this and equation (92), the following relation can be obtained,

\[ \dot{w} = \frac{2}{l} \left( \dot{R} + iR \left( \dot{\theta} + \dot{\beta} \right) \right) e^{i(\beta+\theta)}, \] \hspace{1cm} (95)

while its complex conjugated is,

\[ \dot{\bar{w}} = \frac{2}{l} \left( \bar{\dot{R}} - i\bar{R} \left( \dot{\theta} + \dot{\beta} \right) \right) e^{-i(\beta+\theta)}, \] \hspace{1cm} (96)

which is equal to equation (94). When equation (94) and (96) are multiplied with \( e^{i(\beta+\theta)} \), and since \( R, \dot{R}, \dot{\theta} \) and \( \dot{\beta} \) are all real-valued quantities, it follows that the real part corresponds to \( \dot{R} \), while its imaginary part corresponds to \( R \left( \dot{\theta} + \dot{\beta} \right) \). For solid body rotation, \( R \) should be constant, and so \( \dot{R} = 0 \). The rotational velocity of the dipole centre should be equal the velocity of the dipole rotation, i.e,

\[ \dot{\theta} = \dot{\alpha} = -\dot{\beta}, \] \hspace{1cm} (97)
Figure A.2 – Graph showing the solutions of \( \text{Re}(\dot{\bar{w}}) = 0 \) and \( \text{Im}(\dot{\bar{w}}) = 0 \). The lines cross at \( w = \pm \frac{1}{\sqrt{2}} \pm i \frac{1}{\sqrt{2}} \), and singularities are located at \( w = 0, +i \) and \(-i\).

\[
\dot{R} = 0 \wedge (\dot{\theta} + \dot{\beta}) = 0 \rightarrow \dot{\bar{w}} = 0. \tag{98}
\]

Summarizing, for solid body rotation the real and imaginary parts of \( \dot{\bar{w}} \), given by equation (94) should be zero. Solutions of \( \text{Re}(\dot{\bar{w}}) = 0 \) and \( \text{Im}(\dot{\bar{w}}) = 0 \) are shown in figure A.2. The intersection points are located at \( w = \pm \frac{1}{\sqrt{2}} \pm i \frac{1}{\sqrt{2}} \): two are due to the symmetry of the system, the other two correspond to the two points in figure 4.11 where the three different colours (or regimes) intersect, at exact these points \( w = \text{const} \).

In terms of \( z_1 \) they correspond to \( z_1 = \frac{1}{2} \bar{w} = \pm \frac{l}{2\sqrt{2}} \pm i \frac{l}{3\sqrt{2}} \), or in Euler form to \( z_1 = \pm 0.5e^{\pm i \frac{\pi}{4}} \), see figures A.3a-d for these dipole configurations. The arrows, indicating the self-propelling direction, are shown to clarify the configuration. The circle indicates the constant radius \( R \), while the arrow on the circle indicates the direction of solid body rotation.

The angular velocity corresponding with these configurations is directly given by equation (93), resulting in \( \dot{\beta} = \pm \frac{1}{\sqrt{2}} \). Since the dimensionless time differs a factor two with dimensionless time used in the simulations (as in section 4.1), this result must be multiplied with a factor 2 for the same dimensionless time. We conclude that the results obtained earlier, shown in figure 4.3 corresponding to asymptotical trapping, are the same as we calculated here analytically.
**Singularities.** Figure A.2 reveals a few other interesting points, namely the singularities of $\dot{w}$ at $w = 0, +i$ and $-i$. The corresponding dipole configurations are depicted in figure A.3e,f,g. These points are related to configurations where vortices coincide, leading to non-physical induced velocities. In figure 4.11 these points correspond to the endpoint of the red pathlines at $w = 0$, where the dipoles annihilated each other, and the midpoint of the trapping state (blue) $w = \pm i$.

![Diagram](image)

**Figure A.3** – Some specific cases of figure A.2. (a,b,c,d) Solid body rotation when $z_1 = \pm 0.5 e^{\pm i \frac{\pi}{4}}$. The circle represents $R$, and the arrow on it the rotation direction. (e) The dipoles have annihilated each other (lying on top of each other), $z_1 = 0$. (f,g) The opposite vortices coincide with the dipole centres at $z_1 = \pm i$, forming a tripole. For all cases $\alpha_1 = \pi$, $z_2 = -z_1$ and $\alpha_2 = \alpha_1 + \pi$. 
A.3 Linear stability analysis

Using linear stability analysis, we want to examine if the orbits holding to equation (98) are stable. For this purpose equation (94) is linearised:

\[
\hat{w} + i \omega \text{Im} \left\{ \frac{1}{w(w^2 + 1)} \right\} = -\frac{w_0^2}{w^2 + 1}. \tag{99}
\]

We write \( w = w_0 + \delta w \), in which \( w_0 \) is the main position and \( \delta w \) is a perturbation to this position, and this is substituted in the equation above. The terms are separately linearised and higher order terms are neglected:

\[
\begin{align*}
\frac{1}{w} &= \frac{1}{w_0 + \delta w} = \frac{1}{w_0 \left(1 + \frac{\delta w}{w_0}\right)} \\
\frac{1}{w^2 + 1} &= \frac{1}{(w_0 + \delta w)^2 + 1} = \frac{1}{1 + w_0^2 + \frac{2w_0 \delta w}{1 + w_0^2} + \mathcal{O}(\delta w^2)} \\
(w_0 + \delta w)^2 &\approx w_0^2 + 2w_0 \delta w + \mathcal{O}(\delta w^2)
\end{align*}
\]

These are substituted into equation (99) and multiplied out, higher order terms are again neglected and the equation is further simplified:

\[
\begin{align*}
\hat{w}_0 + \delta w + i \left( \hat{w}_0 + \delta w \right) \text{Im} \left\{ \frac{1}{w_0} \left(1 - \frac{\delta w}{w_0}\right) \right\} &= -\left( w_0^2 + 2w_0 \delta w \right) \frac{1}{1 + w_0^2} \left(1 - \frac{2w_0}{1 + w_0^2} \delta w\right)
\end{align*}
\]

\[
\delta \hat{w} + i \left( \hat{w}_0 + \delta w \right) \text{Im} \left\{ \frac{1}{w_0} \left(1 - \frac{1 + 3w_0^2}{w_0^2} \delta w\right) \right\} = -\left( \frac{w_0^2}{1 + w_0^2} \left(1 + \frac{2}{w_0 (1 + w_0^2)} \delta w\right) \right)
\]

After subtracting the zeroth order equation, an equation is obtained for the first order perturbations:

\[
\begin{align*}
\delta \hat{w} + i \delta \hat{w} \text{Im} \{ A \delta w \} + i \delta \hat{w} \text{Im} \{ B \} = C \delta w,
\end{align*}
\]

with the constants \( A = -\frac{(1 + 3w_0^2)}{w_0^2(1 + w_0^2)} \), \( B = \frac{1}{w_0(1 + w_0^2)} \) and \( C = \frac{-2w_0}{(1 + w_0^2)} \), which only depend on the initial position. Since \( \delta w \) is a point in the complex plane, i.e. \( \delta w = \delta x + i \delta y \), we can rewrite this equation into:

\[
\begin{align*}
\delta \hat{x} - i \delta \hat{y} + i \hat{w}_0 \{ A (\delta x + i \delta y) \} + i (\delta \hat{x} - i \delta \hat{y}) \text{Im} \{ B \} = C (\delta x + i \delta y),
\end{align*}
\]

\[
\begin{align*}
\delta \hat{x} - i \delta \hat{y} + i \hat{w}_0 \{ \text{Re} \{ A \} \delta y + \text{Im} \{ A \} \delta x \} + i \text{Im} \{ B \} \delta x + \text{Im} \{ B \} \delta y = C \delta x + iC \delta y.
\end{align*}
\]
A.3 Linear stability analysis

which consists of a real part and an imaginary part. For the real part we write:

\[ \dot{\delta x} = \text{Im}\left\{ \bar{w}_0 \right\} (\text{Re}\{A\} \delta y + \text{Im}\{A\} \delta x) - \text{Im}\{B\} \delta y + \text{Re}\{C\} \delta x - \text{Im}\{C\} \delta y, \]

\[ \dot{\delta x} = (\text{Im}\left\{ \bar{w}_0 \right\} \text{Im}\{A\} + \text{Re}\{C\}) \delta x + (\text{Im}\left\{ \bar{w}_0 \right\} \text{Re}\{A\} - \text{Im}\{B\} - \text{Im}\{C\}) \delta y, \]

\[ \delta x = M_{11} \delta x + M_{12} \delta y, \]

with \( M_{11} = (\text{Im}\left\{ \bar{w}_0 \right\} \text{Im}\{A\} + \text{Re}\{C\}) \), and \( M_{12} = (\text{Im}\left\{ \bar{w}_0 \right\} \text{Re}\{A\} - \text{Im}\{B\} - \text{Im}\{C\}) \).

For the imaginary part we obtain:

\[ i \dot{\delta y} = i (\text{Re}\left\{ \bar{w}_0 \right\} (\text{Re}\{A\} \delta y + \text{Im}\{A\} \delta x) + \text{Im}\{B\} \delta x - \text{Im}\{C\} \delta x - \text{Re}\{C\} \delta y), \]

\[ i \dot{\delta y} = i ((\text{Re}\left\{ \bar{w}_0 \right\} \text{Im}\{A\} + \text{Im}\{B\} - \text{Im}\{C\}) \delta x + (\text{Re}\left\{ \bar{w}_0 \right\} \text{Re}\{A\} - \text{Re}\{C\}) \delta y), \]

\[ i \dot{\delta y} = i (M_{21} \delta x + M_{22} \delta y), \]

with \( M_{21} = (\text{Re}\left\{ \bar{w}_0 \right\} \text{Im}\{A\} + \text{Im}\{B\} - \text{Im}\{C\}) \), and \( M_{22} = (\text{Re}\left\{ \bar{w}_0 \right\} \text{Re}\{A\} - \text{Re}\{C\}) \).

These two equations can be put into a matrix form,

\[
\begin{bmatrix}
\dot{\delta x} \\
\dot{\delta y}
\end{bmatrix} =
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\begin{bmatrix}
\delta x \\
\delta y
\end{bmatrix}.
\] (101)

The two eigenvalues of this matrix are the roots of the equation:

\[ \lambda^2 - \lambda (M_{11} + M_{22}) + (M_{11}M_{22} - M_{12}M_{21}) = 0. \]

Eigenvalues with \( \text{Re}\{\lambda_{1,2}\} > 0 \) will eventually grow to infinity in time due to the disturbance of the form \( e^{\lambda t} \), which leads to unstable behaviour.

The eigenvalues of the stationary solution as discussed in the previous section, corresponding to solid body rotation, are \( \lambda_{1,2} = -1 \pm \sqrt{17} \), since one of the eigenvalues is positive, the orbit is unstable.
B.1 Derivation of the \( \dot{w} \)-function

The self-induced velocity of the dipole is given by equation (30). Using the properties \( \alpha_1 = \alpha = -\beta \), \( \alpha_2 = \alpha_1 + \gamma \), and \( \alpha_3 = \alpha_1 - \gamma \), with \( z_1 = Re^{i\lambda} \), \( z_2 = z_1e^{i\gamma} \) and \( z_3 = z_1e^{-i\gamma} \) (see figure B.1), this equation can be rewritten as:

\[
\dot{z}_1 = \frac{\Gamma e^{-ia_1}}{2\pi l} = \frac{\Gamma e^{-ia}}{2\pi l}.
\]  

(102)

The velocity in each vortex induced by the other dipoles is given by equation (31), and for \( N = 3 \) this equation becomes:

\[
\ddot{z}_{1,o}(z) = \frac{\Gamma}{2\pi i} \left( \frac{1}{z - z_{2,l}} - \frac{1}{z - z_{2,r}} + \frac{1}{z - z_{3,l}} - \frac{1}{z - z_{3,r}} \right).
\]  

(103)

The total velocity consists of the self-induced velocity plus the velocity induced by the other dipoles (equation (32)), i.e.:

\[
\ddot{z}_1 = \dot{z}_{1,s} + \frac{\dot{z}_{1,o}(z_{1,l}) + \dot{z}_{1,o}(z_{1,r})}{2},
\]  

(104)

or:
\[ \dot{z}_1 = \frac{\Gamma e^{-i\alpha}}{2\pi l} + \frac{1}{2\pi i} \frac{1}{\gamma} \left\{ \frac{1}{z_{1,l} - z_{2,l}} - \frac{1}{z_{1,l} - z_{2,r}} + \frac{1}{z_{1,r} - z_{2,l}} - \frac{1}{z_{1,r} - z_{2,r}} \right\} + \frac{1}{\gamma} \left( \frac{1}{z_{1,l} - z_{3,l}} - \frac{1}{z_{1,l} - z_{3,r}} + \frac{1}{z_{1,r} - z_{3,l}} - \frac{1}{z_{1,r} - z_{3,r}} \right) \right\}. \] (105)

In the last step equations (102) and (103) are used. The position of the left and right vortices are given by equation (27), and can be rewritten due to the symmetry properties to:

\[ z_{1,r} = z_1 + \frac{i\alpha}{2} \gamma, \quad z_{1,l} = z_1 - \frac{i\alpha}{2} \gamma, \]
\[ z_{2,r} = z_2 + \frac{i\alpha}{2} \gamma, \quad z_{2,l} = z_1 e^{i\gamma}, \]
\[ z_{3,r} = z_3 + \frac{i\alpha}{2} \gamma, \quad z_{3,l} = z_1 e^{-i\gamma}. \]

Substituting these into equation (105) and simplifying leads to:

\[ \dot{z}_1 = \frac{\Gamma e^{-i\alpha}}{2\pi l} + \frac{1}{2\pi i} \frac{1}{\gamma} \left( \frac{il64z_2^4 e^{i(\alpha+\gamma)}}{(l^2 e^{2\alpha} + 4z_1^2)(l^2 e^{2\alpha} + e^{i(\alpha+\gamma)}) + 2\alpha l - 2i\alpha z_1 e^{i\gamma}} \right) \] (105),

\[ \dot{z}_1 = \frac{\Gamma e^{-i\alpha}}{2\pi l} + \frac{1}{2\pi i} \frac{1}{\gamma} \left( \frac{il64z_2^4 e^{i(\alpha+\gamma)}}{(l^2 e^{2\alpha} + 4z_1^2)(l^2 e^{2\alpha} + e^{i(\alpha+\gamma)}) + 2\alpha l - 2i\alpha z_1 e^{i\gamma}} \right) \] (105),

\[ \dot{z}_1 = \frac{\Gamma e^{-i\alpha}}{2\pi l} + \frac{1}{2\pi i} \frac{1}{\gamma} \left( \frac{il64z_2^4 e^{i(\alpha+\gamma)}}{(l^2 e^{2\alpha} + 4z_1^2)(l^2 e^{2\alpha} + e^{i(\alpha+\gamma)}) + 2\alpha l - 2i\alpha z_1 e^{i\gamma}} \right) \] (105).

Since \( \gamma = \frac{2\pi}{3} \), one obtains:

\[ \dot{z}_1 = \frac{\Gamma}{2\pi l} \left( \frac{l^4 e^{4\alpha} + 24l^2 z_1 e^{2\alpha} - 48z_1^4}{l^4 e^{4\alpha} - 8l^2 z_1^2 e^{2\alpha} - 48z_1^4} e^{-i\alpha} \right). \] (106)

We now introduce again the variable \( w \), defined as \( w = \frac{2\alpha}{l} e^{-i\gamma} = \frac{2\alpha}{l} e^{-i\alpha} \), containing the position (centre) and orientation of one particular dipole in the complex plane and which is itself a complex number. This is substituted together with the dimensional time \( \tau = \frac{l}{\pi \rho} \) into equation , to obtain:

\[ \frac{d}{d\tau} (\hat{w}e^{-i\alpha}) = e^{-i\alpha} \left( 1 + 6w^2 - 3w^4 \right) \]

\[ \hat{w} + i\hat{w} \hat{\beta} = \frac{1}{1 - 2w^2 - 3w^4} \] (107).

This is a differential equation which describes the time evolution of \( w \) in the complex plane. The equation contains the variable \( \hat{\beta} \), which is given by equation (35), using the same simplifications and substitutions as above, this equation can be rewritten into the following form:
\[ \dot{\alpha}_1 = -\dot{\beta} = \frac{1}{l} \Re \left\{ \left( \dot{z}_{1,o}(z_{1,r}) - \dot{z}_{1,o}(z_{1,l}) \right) e^{i\alpha} \right\}, \]

\[ \dot{\alpha}_1 = -\dot{\beta} = \frac{1}{l} \Re \left\{ \frac{\Gamma}{2\pi i} \left( \frac{-32l^2 z_1 e^{i(2\alpha+\gamma)}}{(l^2 e^{2\alpha} + 4z_1^2)(l^2 e^{2\alpha} (1 + e^{i\gamma})^2 + 4z_1^2 (1 - e^{i\gamma})^2)} \right) e^{i\alpha} \right\}, \]

\[ \dot{\alpha}_1 = -\dot{\beta} = \frac{1}{l} \Re \left\{ \frac{\Gamma}{2\pi i} \left( \frac{-32l^2 z_1 e^{i(2\alpha+\gamma)}}{(l^2 e^{2\alpha})^2 (1 + \frac{4z_1^2}{\tau e^{i\gamma}}) \left( (1 + e^{i\gamma})^2 + \frac{4z_1^2}{\tau e^{i\gamma}} (1 - e^{i\gamma})^2 \right)} \right) e^{i\alpha} \right\}. \] (108)

Now using again the variable \( w \) and the dimensional time \( \tau \), we obtain:

\[ -\frac{\Gamma}{\pi l^2} \frac{d\beta}{d\tau} = \frac{\Gamma}{\pi l^2} \Re \left\{ -i \left( \frac{-8we^{i\gamma}}{(1 + w^2) ((1 + e^{i\gamma})^2 + w^2 (1 - e^{i\gamma})^2)} \right) \right\}, \]

\[ \dot{\beta} = -\text{Im} \left\{ \frac{8w}{(1 + w^2) (1 - 3w^2)} \right\}. \] (109)

This is a differential equation which describes the evolution of the orientation angle as a function of \( w(z, \beta) \). Combining equation (107) and (109) leads to:

\[ \dot{\dot{w}} = \frac{1 + 6w^2 - 3w^4}{1 - 2w^2 - 3w^4} - i \dot{w} \text{Im} \left\{ \frac{8w}{1 - 2w^2 - 3w^4} \right\}. \] (110)

### B.2 Stationary solution

We now search for a stationary solution, i.e. for which \( \dot{\beta} = \dot{\dot{w}} = 0 \). Equations (107) and (109) then simplify to:

\[ 0 = \dot{\dot{w}} = \frac{1 + 6w^2 - 3w^4}{1 - 2w^2 - 3w^4}, \] (111)

and

\[ 0 = \dot{\beta} = -\text{Im} \left\{ \frac{8w}{(1 + w^2) (1 - 3w^2)} \right\}. \] (112)

The first equation is satisfied when the numerator is zero, i.e. solutions of \( 1 + 6w^2 - 3w^4 = 0 \). The possible solutions are:

\[ w_1 = -i \sqrt[3]{\frac{1}{3}(-3 + 2\sqrt{3})}, \quad w_2 = i \sqrt[3]{\frac{1}{3}(-3 + 2\sqrt{3})}, \quad w_3 = -\sqrt[3]{\frac{1}{3}(3 + 2\sqrt{3})}, \quad w_4 = \sqrt[3]{\frac{1}{3}(3 + 2\sqrt{3})}. \]
B.3 Linear stability analysis

From the second equation we find $\dot{\beta} = 0$, if $w$ is real or zero. This leaves only $w_3$ and $w_4$ as solutions. The stability analysis in the next section shows that only $w_3$ is stable.

Another possibility for stationary solutions is again the solid body rotation. With the same assumptions as made in section A.2 an expression for $\dot{\bar{w}}$ is obtained:

$$\dot{\bar{w}} = \frac{2}{l} \left( \bar{\dot{R}} - i \bar{R} \left( \dot{\bar{R}} + \dot{\bar{\beta}} \right) \right) e^{-i(\beta + \theta)}, \quad (113)$$

which should now equal be identical to equation (110). When equation (110) and (113) are multiplied with $e^{i(\beta + \theta)}$ and since $\dot{R}, \ddot{R}, \dot{\theta}$ and $\dot{\beta}$ are all real valued quantities, it follows that the real part corresponds to $\ddot{R}$, while its imaginary part corresponds to $R \left( \dot{\theta} + \dot{\beta} \right)$. For solid body rotation, $R$ is constant, and $\ddot{R} = 0$, while the rotational velocity of the dipole centre should equal the velocity of the dipole rotation, i.e. $\dot{\theta} = \dot{\alpha} = -\dot{\beta}$, from which it follows that

$$\ddot{R} = 0 \land \left( \dot{\theta} + \dot{\beta} \right) = 0 \rightarrow \dot{\bar{w}} = 0. \quad (114)$$

Solutions of $\text{Re}(\dot{\bar{w}}) = 0$ and $\text{Im}(\dot{\bar{w}}) = 0$ are shown in figure B.2. The points of intersection are listed in table B.1. (a,b,c,d,e) and (f) correspond to solid body rotations, at exactly these points $w = \text{const}$. In terms of $z_1$ they correspond to $z(t = 0) = \frac{1}{2} (X_0 + i H_0) = \frac{1}{2} \left( \text{Re} \{ w \} + i \text{Im} \{ w \} \right)$, see figures B.3a-f for these dipole configurations. The arrows indicate the dipole orientation, and are only shown to clarify the configurations. The circle indicates the constant radius $R$, while the arrow on the circle the direction of solid body rotation. The angular velocity corresponding with these configurations is directly given by equation (109), resulting in $\dot{\beta} \approx \pm 0.5707$ or $\dot{\beta} \approx \pm -1.8849$. The dimensionless time differs a factor two with the dimensionless time used in the simulations (as in section 4.1); therefore, the result must be be multiplied with 2 for the same dimensionless time. We conclude that this analytically obtained result corresponds to the one found numerically, as shown in figure 4.3.

At the points $w = \pm i$ and $w = \pm i \sqrt{\frac{1}{7}(-3 + 2\sqrt{3})}$, the $w$-function has singularities and is not well defined. Due to superposition, these points correspond to (i) no vorticity or dipoles, or (ii) a quadrupole with in the middle a vortex of strength $\pm 3\Gamma$ surrounded by three vortices of opposite strength $\mp \Gamma$, equally separated.

B.3 Linear stability analysis

Using linear stability analysis we wish to examine if the solutions found in the previous section are stable. For that purpose equation (110) is linearised:

$$\dot{\bar{w}} = \frac{1 + 6 w^2 - 3 w^4}{(1 + w^2)(1 - 3 w^2)} - i \bar{w} \text{Im} \left\{ \frac{8w}{(1 + w^2)(1 - 3w^2)} \right\}. \quad (115)$$

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Again, we write \( w = w_0 + \delta w \), in which \( w_0 \) is the main position and \( \delta w \) is a perturbation to this position, and this expression is substituted in the equation above. The terms are again linearised and higher order terms are neglected:

\[
\begin{align*}
\dot{w} &= \dot{w}_0 + \delta \dot{w} \approx \delta \dot{w} \\
\frac{1}{w^2+1} = \frac{1}{(w_0 + \delta w)^2+1} &\approx \frac{1}{1+w_0^2} \left( 1 - \frac{2w_0 \delta w}{1+w_0^2} + O(\delta w^2) \right) \\
\frac{1}{1-3w^2} = \frac{1}{(1-3(w_0 + \delta w)^2)} &\approx \frac{1}{1-3w_0^2} \left( 1 + \frac{6w_0 \delta w}{1-3w_0^2} + O(\delta w^2) \right) \\
(w_0 + \delta w)^2 &\approx w_0^2 + 2w_0 \delta w + O(\delta w^2) \\
(w_0 + \delta w)^4 &\approx w_0^4 + 4w_0^3 \delta w + O(\delta w^2)
\end{align*}
\]

These are substituted into equation (115) and multiplied out, higher order terms are again neglected and the equation is further simplified:

\[
\begin{align*}
\delta \dot{w} &= -i \left( \bar{w}_0 + \delta \bar{w} \right) \text{Im} \left\{ 8(w_0 + \delta w) \frac{1}{1+w_0^2} \left( 1 + \frac{6w_0}{1-3w_0^2} \delta w \right) \left( 1 - \frac{2w_0}{1+w_0^2} \delta w \right) \right\} \\
&\quad + (1 + 6(\bar{w}_0^2 + 2w_0 \delta w) - 3(w_0^4 + 4w_0^3 \delta w)) \frac{1}{1+w_0^2} \left( 1 + \frac{6w_0}{1-3w_0^2} \delta w \right) \left( 1 - \frac{2w_0}{1+w_0^2} \delta w \right),
\end{align*}
\]

\[
\begin{align*}
\delta \ddot{w} &= -i \left( \bar{w}_0 + \delta \bar{w} \right) \text{Im} \left\{ \frac{8(w_0 + \delta w)}{(1+6w_0^2)(1-3w_0^2)} \left( 1 - \left( \frac{2w_0}{1+w_0^2} - \frac{6w_0}{1-3w_0^2} \right) \delta w \right) \right\} \\
&\quad + \frac{(1+6w_0^2-3w_0^4+12w_0^2)(1+w_0^2)}{(1+w_0^2)(1-3w_0^2)} \left( 1 - \left( \frac{2w_0}{1+w_0^2} - \frac{6w_0}{1-3w_0^2} \right) \delta w \right).
\end{align*}
\]
B.3 Linear stability analysis

Figure B.3 – Specific three-dipole configurations corresponding to figure B.2. (a,b,c,d) Solid body rotation with $z_1 = \pm 0.7961e^{\pm 0.5827i}$. (e,f) Solid body rotation when $z_1 = \pm 0.1326e^{i\frac{\pi}{2}}$. The circle represents $R$, and the arrow the rotation direction. (g, h) The three vortices of equal sign lying over each other, giving a vortex of strength $3\Gamma$ in the middle surrounded by three vortices of opposite sign, forming a quadrupole. $z_1 = \pm \frac{1}{2}$.

(i) Stationary and stable configuration with $z_1 = -\frac{1}{2}\sqrt[3]{\frac{1}{3}(-3 + 2\sqrt{3})}$. (j) Stationary but unstable configuration, with $z_1 = \frac{1}{2}\sqrt[3]{\frac{1}{3}(-3 + 2\sqrt{3})}$. (k,l) The dipoles have annihilated each other (lying on top of each other) with $z_1 = \pm \frac{1}{2\sqrt{3}}$. $\alpha_1 = 0$. 

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\[ \delta \dot{w} = -i \left( \bar{w}_0 + \delta \bar{w} \right) \Im \left\{ \frac{8w_0}{(1+w_0^2)(1-3w_0^2)} + \frac{8w_0}{(1+w_0^2)(1-3w_0^2)} \right\} \delta w \]
\[ + \left( \frac{12w_0(1-w_0^2)}{(1+w_0^2)(1-3w_0^2)} + \frac{4w_0(1+3w_0^2)}{(1+w_0^2)(1-3w_0^2)} \right) \delta w \]

The first expression on right hand side of equation (116) can be rewritten to a part containing the zeroth order equation and a part that only confines the first order perturbation:

\[ \begin{align*}
1^{\text{th}} \text{ term of RHS } &\rightarrow = -i \left( \bar{w}_0 + \delta \bar{w} \right) \Im \left\{ \frac{8w_0}{(1+w_0^2)(1-3w_0^2)} + \frac{8w_0}{(1+w_0^2)(1-3w_0^2)} \right\} \\
&+ \bar{w}_0 \Im \left\{ \frac{8w_0}{(1+w_0^2)(1-3w_0^2)} \left( 1 + \frac{4w_0(1+3w_0^2)}{(1+w_0^2)(1-3w_0^2)} \right) \delta w \right\} \\
&+ \delta \bar{w} \Im \left\{ \frac{8w_0}{(1+w_0^2)(1-3w_0^2)} \right\}
\end{align*} \]

The second expression on right hand side is also rewritten to a part of the differential equation at the stationary point and a perturbation:

\[ \begin{align*}
2^{\text{nd}} \text{ term of RHS } &\rightarrow = \frac{12w_0(1-w_0^2)}{(1+w_0^2)(1-3w_0^2)} + \frac{1+6w_0^2-3w_0^4}{(1+w_0^2)(1-3w_0^2)} + \frac{4w_0(1+3w_0^2)}{(1+w_0^2)(1-3w_0^2)} \delta w
\end{align*} \]

Since the zeroth order equation is for the stationary case equal to zero, i.e. \( \dot{w}(w_0) = 0 \), it can be removed from equation (116), so that we are left with the following equation for the first order perturbation:

\[ \delta \dot{w} = -i \left( \bar{w}_0 \Im \left\{ \frac{8w_0}{(1+w_0^2)(1-3w_0^2)} \left( 1 + \frac{4w_0(1+3w_0^2)}{(1+w_0^2)(1-3w_0^2)} \right) \delta w \right\} \right) + \delta \bar{w} \Im \left\{ \frac{8w_0}{(1+w_0^2)(1-3w_0^2)} \right\} \]

Or in simplified form, in terms of the constants \( A, B \) and \( C \):

\[ \delta \dot{w} = C \delta w - i \left( \bar{w}_0 \Im \{ A \delta w \} + \delta \bar{w} \Im \{ B \} \right), \]

with:

\[ \begin{align*}
A &= \frac{8}{(1+w_0^2)(1-3w_0^2)} \left( 1 + \frac{4w_0(1+3w_0^2)}{(1+w_0^2)(1-3w_0^2)} \right) = \frac{8(1+2w_0^2+9w_0^4)}{(1+w_0^2)(1-3w_0^2)}, \\
B &= \frac{1+6w_0^2-3w_0^4}{(1+w_0^2)(1-3w_0^2)} \left( 1 + \frac{4w_0(1+3w_0^2)}{(1+w_0^2)(1-3w_0^2)} \right) = \frac{16w_0(1+3w_0^2)}{(1+w_0^2)(1-3w_0^2)}, \\
C &= \frac{8w_0}{(1+w_0^2)(1-3w_0^2)} = B \frac{2(1+3w_0^2)}{(1+w_0^2)(1-3w_0^2)}.
\end{align*} \]

These three constants only depend on the stationary solution \( w_0 \). What we are left with is an equation of the same form as equation (100). Therefore, from here on exactly the same procedure is followed as in section A.3, resulting in:

\[ \delta \dot{x} = M_{11} \delta x + M_{12} \delta y, \]
and,

\[ i \delta \dot{y} = i (M_{21} \delta x + M_{22} \delta y) , \]

with

\[
\begin{align*}
M_{11} &= (\text{Im} \{ \bar{w}_0 \} \text{Im} \{ A \} + \text{Re} \{ C \}) \\
M_{12} &= (\text{Im} \{ \bar{w}_0 \} \text{Re} \{ A \} - \text{Im} \{ B \} - \text{Im} \{ C \}) \\
M_{21} &= (\text{Re} \{ \bar{w}_0 \} \text{Im} \{ A \} + \text{Im} \{ B \} - \text{Im} \{ C \}) \\
M_{22} &= (\text{Re} \{ \bar{w}_0 \} \text{Re} \{ A \} - \text{Re} \{ C \})
\end{align*}
\]

These two equations can be put into a matrix form

\[
\begin{bmatrix} \delta \dot{x} \\ \delta \dot{y} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} .
\]

(119)

The two eigenvalues from this matrix are the roots of the equation: \( \lambda^2 - \lambda (M_{11} + M_{22}) + (M_{11}M_{22} - M_{12}M_{21}) = 0 \). Eigenvalues with \( \text{Re} \{ \lambda_{1,2} \} > 0 \) will eventually grow to infinity due to the perturbation of the form \( e^{\lambda t} \), which leads to unstable behaviour. The eigenvalues of all stationary points, where this linear stability analysis is valid, are listed in table (B.1), the letters corresponds to the points indicated in figure B.2 and B.3.

Table B.1 – Eigenvalues of equation (119) for the different solutions indicated in figure (B.2).

<table>
<thead>
<tr>
<th>point</th>
<th>Coordinate</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>Behaviour</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>+1.3295 + i 0.8761</td>
<td>1.0387</td>
<td>-0.9076</td>
<td>unstable</td>
</tr>
<tr>
<td>b</td>
<td>-1.3295 + i 0.8761</td>
<td>-1.0387</td>
<td>0.9076</td>
<td>unstable</td>
</tr>
<tr>
<td>c</td>
<td>-1.3295 - i 0.8761</td>
<td>-1.0387</td>
<td>0.9076</td>
<td>unstable</td>
</tr>
<tr>
<td>d</td>
<td>+1.3295 - i 0.8761</td>
<td>1.0387</td>
<td>-0.9076</td>
<td>unstable</td>
</tr>
<tr>
<td>e</td>
<td>-i 0.2652</td>
<td>3.2068</td>
<td>-3.2068</td>
<td>unstable</td>
</tr>
<tr>
<td>f</td>
<td>i 0.2652</td>
<td>3.2068</td>
<td>-3.2068</td>
<td>unstable</td>
</tr>
<tr>
<td>g</td>
<td>-i</td>
<td>~</td>
<td>~</td>
<td>singularity</td>
</tr>
<tr>
<td>h</td>
<td>i</td>
<td>~</td>
<td>~</td>
<td>singularity</td>
</tr>
<tr>
<td>i</td>
<td>(-\sqrt{\frac{3}{2}}(3 + 2\sqrt{3}))</td>
<td>-1.1800</td>
<td>-0.6813</td>
<td>stable</td>
</tr>
<tr>
<td>j</td>
<td>(\sqrt{\frac{1}{2}}(3 + 2\sqrt{3}))</td>
<td>0.6813</td>
<td>1.1800</td>
<td>unstable</td>
</tr>
<tr>
<td>k</td>
<td>(-\frac{1}{\sqrt{3}})</td>
<td>~</td>
<td>~</td>
<td>singularity</td>
</tr>
<tr>
<td>l</td>
<td>(\frac{1}{\sqrt{3}})</td>
<td>~</td>
<td>~</td>
<td>singularity</td>
</tr>
</tbody>
</table>
C  The \( w \)-function for \( N = 4 \)

C.1 Derivation of the \( w \)-function

The self-induced velocity of the dipole is given by equation (30). Using the properties \( \alpha_1 = \alpha = -\beta \), \( \alpha_2 = \alpha_1 + \gamma \), \( \alpha_3 = -\alpha_1 \), \( \alpha_4 = \alpha_1 - \gamma \), and with \( z_1 = \text{Re}^{i\lambda} \), \( z_2 = \text{Re}^{i(\lambda+\gamma)} \), \( z_3 = -z_1 \) and \( z_4 = z_1 e^{-i\gamma} \), this equation can be rewritten as:

\[
\dot{z}_1 = \frac{\Gamma e^{-i\alpha_1}}{2\pi l} = \frac{\Gamma e^{i\beta}}{2\pi l}.
\]  (120)

The velocity in each vortex induced by the other dipoles is given by equation (31). For \( N = 4 \) this equation reads:

\[
w_{1,o}(z) = \sum_{j=2}^{4} \frac{\Gamma_j}{2\pi l} \left( \frac{1}{z - z_{j,l}} - \frac{1}{z - z_{j,r}} \right).
\]  (121)

The total velocity consists of the self-induced velocity and the velocity induced by other vortices (equation (32)), i.e.:

\[
\dot{\bar{z}}_1 = \dot{z}_{1,s} + \frac{\dot{z}_{1,o}(z_{1,l}) + \dot{z}_{1,o}(z_{1,r})}{2}.
\]

New using equations (120) and (121) and expressing all \( z_{j,l,r} \) in terms of \( z_{1,l-r} \),

\[
\begin{align*}
z_{1,r} &= z_1 + \frac{i le^{-i\beta}}{2}, \\
z_{2,r} &= z_{1,r} e^{i\gamma}, \\
z_{3,r} &= -z_{1,r}, \\
z_{4,r} &= z_{1,r} e^{-i\gamma}, \\
z_{1,l} &= z_1 - \frac{i le^{-i\beta}}{2}, \\
z_{2,l} &= z_{1,l} e^{i\gamma}, \\
z_{3,l} &= -z_{1,l}, \\
z_{4,l} &= z_{1,l} e^{-i\gamma},
\end{align*}
\]

one obtains

\[
\dot{\bar{z}}_1 = \frac{\Gamma}{\pi} 10 z_1 e^{i\alpha} - 8 z_1^4 e^{-i\alpha}.
\]  (122)

Again using the variable \( w = \frac{2z_1 e^{i\beta}}{\pi} = \frac{2z_1 e^{-i\alpha_1}}{\pi} \), which combines the position (centre) and orientation of one particular dipole in the complex plane, and the dimensional time \((\tau = \frac{T}{\pi l} t)\), equation (122) can be simplified to,

\[
\dot{w} + i \dot{w} \beta = w^5 \frac{5 - w^2}{1 - w^4}
\]  (123)

This is a differential equation which describes the time evolution of \( w \) in the complex plane. The equation contains the variable \( \dot{\beta} \) which is given by equation (35) and which can be found using the same simplifications and substitutions as above.
C.2 Stationary solution

Solutions of equation (125) for \( \text{Re}(\dot{w}) = 0 \) and \( \text{Im}(\dot{w}) = 0 \) are shown in figure C.1, only the upper half is plotted. The points (a,b,c,d) correspond to solid body rotations, at these points \( w = \text{const} \) corresponding to a saddle point. In terms of \( z_1 \) they correspond to \( z(t = 0) = \frac{w}{2} \), see figures C.2a-d for these dipole configurations. The arrows indicate the dipole orientation, and are only shown to clarify the configurations. The circle indicates the constant radius \( R \). The angular velocity corresponding with these configurations is directly given by equation (124). Note that the dimensionless time differs a factor two with the dimensionless time used in the simulations; therefore, the result must be be multiplied with 2 for the same dimensionless time.

Figures C.2e and C.2i correspond to \( w = \pm \sqrt{5} \): these are both stationary solutions, but only the first one is stable in terms of \( w \). In section 5.4 it has been pointed out that also the "stable" configuration can collapse for straining modes, i.e. when the symmetry (which forms the basis of the \( w \)-function) is broken. The four singularities correspond to coordinates where vortices coincide. At \( w = i \) one vortex of strength...
4$\Gamma$ in the middle is surrounded by four vortices of opposite strength, while for C.2f-h, vortices of opposite sign lie on top of each other, so that the total net vorticity (or circulation) is zero.
Figure C.2 – Specific cases of figure B.2. (a,b) Solid body rotation when $z_1 \approx \pm 1.0944e^{i\frac{1}{6}\pi}$, the circle represents $R$. (c,d) Solid body rotation when $z_1 \approx \pm 0.2284e^{i\frac{1}{6}\pi}$, the circle represents $R$. (e) Stable and stationary configuration at $z_1 = -\frac{\sqrt{5}}{2}$. (f, g, h) Configurations where vortices of opposite sign are lying on top of each other such that the total net vorticity is zero, at $z_1 = -\frac{1}{2}, 0, \frac{1}{2}$, respectively. (i) Stationary configuration that is unstable at $z_1 = \frac{\sqrt{5}}{2}$. (j) Formation of a vortex with strength $4\Gamma$ in the middle surrounded by four vortices of opposite vorticity, with $z_1 = \pm \frac{i}{2}$. For all cases $\alpha_1 = 0$. 

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In this section an analytical solution\(^2\) is obtained for a vortex that is bounded by two parallel walls. In potential flow theory a wall can be seen as a mirror, to obtain a zero velocity component perpendicular to the wall (i.e. the wall is impenetrable). Since we consider two parallel walls, there are infinitely many mirrors (see figure D.1).

The velocity induced in vortex \(z_n\), is given by its mirrors plus all other vortices in the channel and their mirrors, giving

\[
\dot{z}_n = \sum_{j=1}^{N} -\frac{i\Gamma_j}{2\pi} \left\{ \left( \frac{-1}{z_n - (z_j + 2A_j)} + \frac{1}{z_n - (z_j + 2D)} - \frac{1}{z_n - (z_j + 2(D + A_j))} + \ldots \right) \right. \\
\left. \left( \frac{-1}{z_n - (z_j - 2(D - A_j))} + \frac{1}{z_n - (z_j - 2D)} - \frac{1}{z_n - (z_j - 2(2D - A_j))} \right) + \ldots \right\} + \sum_{j\neq n}^{N} \frac{-i\Gamma_j}{2\pi} \frac{1}{z_n - z_j}.
\]

This equation can be rewritten into the following form:

\[
\dot{z}_n = \sum_{j=1}^{N} -\frac{i\Gamma_j}{2\pi} \left\{ \left( \sum_{m=0}^{\infty} \frac{-1}{z_n - (z_j + 2(mD + A_j))} \right) + \left( \sum_{m=1}^{\infty} \frac{-1}{z_n - (z_j - 2(mD - A_j))} \right) + \sum_{m=1}^{\infty} \left( \frac{1}{z_n - (z_j - 2mD)} + \frac{1}{z_n - (z_j + 2mD)} \right) \right\} + \sum_{j\neq n}^{N} \frac{-i\Gamma_j}{2\pi} \frac{1}{z_n - z_j}.
\]

Now we put the first two terms into the same summation, therefore we should compensate for \(m = 0\) in the second term, i.e.

\[
\dot{z}_n = \sum_{j=1}^{N} -\frac{i\Gamma_j}{2\pi} \left\{ \sum_{m=0}^{\infty} \left( \frac{-1}{z_n - (z_j + 2(mD + A_j))} + \frac{-1}{z_n - (z_j - 2(mD - A_j))} \right) + \frac{1}{z_n - (z_j + 2A_j)} + \sum_{m=1}^{\infty} \left( \frac{1}{z_n - (z_j - 2mD)} + \frac{1}{z_n - (z_j + 2mD)} \right) \right\} + \sum_{j\neq n}^{N} \frac{-i\Gamma_j}{2\pi} \frac{1}{z_n - z_j}.
\]

This equation can be simplified if we distinguish two cases:

Case (1) if \(n = j\),

\[
\dot{z}_n = -\frac{i\Gamma_n}{2\pi} \left\{ \left( \sum_{m=0}^{\infty} \frac{-1}{2(mD + A_n)} + \frac{1}{2(mD - A_n)} \right) - \frac{1}{2A_n} - \sum_{m=1}^{\infty} \left( \frac{1}{2mD - 2D} - \frac{1}{2mD} \right) \right\} - \frac{-i\Gamma_n}{2\pi} \left\{ \left( \sum_{m=0}^{\infty} \frac{A_n}{(mD)^2 - A_n^2} \right) - \frac{1}{2A_n} \right\}.
\]

\(^2\)With thanks to J.A. van Hooft

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A summation of this type converges for $m \to \infty$ to $\cot(x)$, and the equation then simplifies to

$$\dot{\mathbf{z}}_n = -i \frac{\Gamma_n \pi}{2\pi} \cot\left(\frac{\pi A_n}{D}\right).$$

(126)

Case (2) if $n \neq j$ and $(z_n - z_j) = \lambda_j$

$$\dot{\mathbf{z}}_n = \sum_{j \neq n} -i \frac{\Gamma_j}{2\pi} \left\{ \sum_{m=0}^{\infty} \left( \frac{-1}{\lambda_j - 2(mD + A_j)} + \frac{-1}{\lambda_j - 2(mD - A_j)} \right) + \frac{1}{\lambda_j - 2A_j} \right\},$$

which we can rewrite to

$$\dot{\mathbf{z}}_n = \sum_{j \neq n} -i \frac{\Gamma_j}{2\pi} \left\{ \sum_{m=0}^{\infty} \left( \frac{-2(\lambda_j - 2A_j)}{(\lambda_j - 2A_j)^2 - (2mD)^2} \right) + \frac{1}{\lambda_j - 2A_j} + \sum_{m=0}^{\infty} \left( \frac{2\lambda_j}{\lambda_j^2 - (2mD)^2} \right) - \frac{1}{\lambda_j} \right\}.$$

Again, both summations converge to $\cot(x)$ for $m \to \infty$, so that this equation is simplified to

$$\dot{\mathbf{z}}_n = \sum_{j \neq n} i \frac{\Gamma_j}{2\pi} \frac{\pi}{2D} \left\{ \cot\left(\frac{\pi (\lambda_j - 2A_j)}{2D}\right) - \cot\left(\frac{\pi \lambda_j}{2D}\right) \right\}.\quad (127)$$

Equation (126) gives the induced velocities in vortex $n$ due to its own mirrors, while equation (127) gives the induced velocities of vortex $j$ and its mirrors into vortex $n$. 
Figure D.1 – Schematic representation of two vortices ($n$ and $j$) bounded by two walls, located at the horizontal axis a distance $D$ apart. Only a few mirrors are shown. Red indicate positive vorticity and blue negative. $z_{n,j}$ gives the location of the vortex in the complex plane.