Design and optimization of a rotary actuator for a two degree-of-freedom zo-module

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Design and optimization
of a rotary actuator for a
two degree-of-freedom $2\theta$-module

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The department Electrical Engineering
do not accept any responsibility
for the contents of this report

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C Winding factor .................................................. 105
Abstract

In this work an optimized design of a rotary actuator in a zφ-module has been obtained using a multi-physical framework. The number of magnet poles and coils has been selected such to obtain the highest winding factor, which are calculated for slotless machines. The framework contains an accurate 2D semi-analytical magneto static model which offers a fast means to determine the torque capabilities of the slotless permanent magnet actuator. A thermal equivalent circuit of the actuator is presented, which accounts for the forced air cooling inside the airgap. The value of the transfer heat coefficient inside the airgap has been determined with preliminary measurements on a pre-prototype of the actuator.

The rotary actuator has been optimized with the objective to minimize the combined copper losses of the rotary and linear actuator. The thermal constraints, however, are easily met, resulting in an oversized design. Therefore, the actuator also has been optimized with the objective to minimize its volume. Because the rotational stroke of the actuator is limited, a moving-magnet configuration with a single mechanical clearance and a moving-coil configuration with a double mechanical clearance, have been optimized and compared to an initial design of the rotary actuator.

Although the 2D magneto-static model does not account for the edge-effects of the magnets, the reduced magnetic loading in the airgap has been determined with 3D FEM and the flux linkage in the optimized designs of the actuator have been corrected. This effect has been verified by electro-motive force measurements on a pre-prototype. The performance of the optimized designs have been re-evaluated for the magnet edge-effect, together with the temperature dependency of the electrical resistivity of copper.
Acknowledgments

This work is established at the University of Technology Eindhoven in the group of Electromechanics and Power Electronics (EPE) of the department of Electrical Engineering in close cooperation with Wijdeven B.V. I would like to thank my direct supervisors Elena Lomonova and Helm Jansen for their feedback on my work and giving additional insight on the approach to this project.

Furthermore, I want to thank Frank Tacken and Ad Kieboom for given me the opportunity to work on this project at Wijdeven B.V. Many thanks also to the people at Wijdeven for the nice times. Additionally, I want thank Teun Oome, Roy Mennen, Hans Rovers and Koen Meessen for sharing room IM 0.12 and for their help.
## List of symbols

### Symbols

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$m^2$</td>
<td>Cross-sectional area</td>
</tr>
<tr>
<td>$A$</td>
<td>Wb m$^{-1}$</td>
<td>Magnetic vector potential</td>
</tr>
<tr>
<td>$B$</td>
<td>T</td>
<td>Magnetic flux density</td>
</tr>
<tr>
<td>$B_{rem}$</td>
<td>T</td>
<td>Remanent flux density</td>
</tr>
<tr>
<td>$d$</td>
<td></td>
<td>Duty cycle</td>
</tr>
<tr>
<td>$[D]$</td>
<td></td>
<td>Coefficient (denominator)</td>
</tr>
<tr>
<td>$e$</td>
<td>V</td>
<td>Back-EMF</td>
</tr>
<tr>
<td>$e$</td>
<td>K$^{-1}$</td>
<td>Relative increase electrical resistivity</td>
</tr>
<tr>
<td>$E$</td>
<td></td>
<td>Unit vector</td>
</tr>
<tr>
<td>$\vec{E}$</td>
<td>complex</td>
<td>Electric field intensity in phasor representation</td>
</tr>
<tr>
<td>$F$</td>
<td></td>
<td>Coefficient</td>
</tr>
<tr>
<td>$G$</td>
<td></td>
<td>Coefficient</td>
</tr>
<tr>
<td>$H$</td>
<td>A m$^{-1}$</td>
<td>Magnetic field strength</td>
</tr>
<tr>
<td>$i$</td>
<td>A</td>
<td>Current</td>
</tr>
<tr>
<td>$I$</td>
<td>A</td>
<td>Current</td>
</tr>
<tr>
<td>$I$</td>
<td>kg m$^2$</td>
<td>Inertia</td>
</tr>
<tr>
<td>$j$</td>
<td></td>
<td>Unit imaginary part ($j = \sqrt{-1}$)</td>
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<tr>
<td>$J$</td>
<td>A m$^{-2}$</td>
<td>Current density vector</td>
</tr>
<tr>
<td>$k$</td>
<td>W m$^{-1}$ K$^{-1}$</td>
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<tr>
<td>$k_d$</td>
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<td>Distribution factor</td>
</tr>
<tr>
<td>$k_p$</td>
<td></td>
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<tr>
<td>$k_{skew}$</td>
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<tr>
<td>$K_{wn}$</td>
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<td>Distribution factor</td>
</tr>
<tr>
<td>$L$</td>
<td>m</td>
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<tr>
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<td></td>
<td>Number of phases</td>
</tr>
<tr>
<td>$m$</td>
<td>kg m$^{-3}$</td>
<td>Mass density</td>
</tr>
<tr>
<td>$M_0$</td>
<td>A m$^{-1}$</td>
<td>Remanent magnetization vector</td>
</tr>
<tr>
<td>$M_{ij}$</td>
<td>H</td>
<td>Inductance between phase $i$ and $j$</td>
</tr>
<tr>
<td>$M$</td>
<td>kg</td>
<td>Mass</td>
</tr>
<tr>
<td>$n$</td>
<td></td>
<td>Harmonic number</td>
</tr>
<tr>
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<td></td>
<td>Number of turns</td>
</tr>
<tr>
<td>$[N]$</td>
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</tr>
<tr>
<td>Quantity</td>
<td>Unit</td>
<td>Description</td>
</tr>
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<td>----------</td>
<td>------</td>
<td>-------------</td>
</tr>
<tr>
<td>$N_{eph}$</td>
<td>-</td>
<td>Number of coils per phase</td>
</tr>
<tr>
<td>$p$</td>
<td>-</td>
<td>Number of pole pairs</td>
</tr>
<tr>
<td>$q$</td>
<td>-</td>
<td>Coil per slot and per phase</td>
</tr>
<tr>
<td>$q'$</td>
<td>W m$^{-2}$</td>
<td>Heat flux</td>
</tr>
<tr>
<td>$Q$</td>
<td>-</td>
<td>Number of coils</td>
</tr>
<tr>
<td>$P_{cu}$</td>
<td>W</td>
<td>Copper losses</td>
</tr>
<tr>
<td>$r$</td>
<td>m</td>
<td>Radial position</td>
</tr>
<tr>
<td>$R$</td>
<td>Ω</td>
<td>Electrical resistance</td>
</tr>
<tr>
<td>$R$</td>
<td>°C W$^{-1}$</td>
<td>Thermal resistance</td>
</tr>
<tr>
<td>$S$</td>
<td>m$^2$</td>
<td>Surface</td>
</tr>
<tr>
<td>$t$</td>
<td>sec.</td>
<td>Time</td>
</tr>
<tr>
<td>$T$</td>
<td>Nm</td>
<td>Torque</td>
</tr>
<tr>
<td>$T$</td>
<td>°C</td>
<td>Temperature</td>
</tr>
<tr>
<td>$v$</td>
<td>m s$^{-1}$</td>
<td>Velocity</td>
</tr>
<tr>
<td>$V$</td>
<td>m$^3$</td>
<td>Volume</td>
</tr>
<tr>
<td>$X$</td>
<td>-</td>
<td>Coefficient</td>
</tr>
<tr>
<td>$Y$</td>
<td>-</td>
<td>Coefficient</td>
</tr>
<tr>
<td>$z$</td>
<td>m</td>
<td>Axial position</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>-</td>
<td>Magnet pitch to pole pitch ratio</td>
</tr>
<tr>
<td>$\alpha_c$</td>
<td>rad s$^{-2}$</td>
<td>Angular acceleration</td>
</tr>
<tr>
<td>$\alpha_t$</td>
<td>rad</td>
<td>Half the angular span of a coil</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>rad</td>
<td>Half the angular span of a single turn</td>
</tr>
<tr>
<td>$\gamma_i$</td>
<td>rad</td>
<td>Electrical offset angle of coil $i$</td>
</tr>
<tr>
<td>$\Delta\theta$</td>
<td>rad</td>
<td>Relative distance between moving and fixed frame</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>-</td>
<td>Emissivity</td>
</tr>
<tr>
<td>$\theta$</td>
<td>rad</td>
<td>Rotation around $z$ (coils)</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>Wb</td>
<td>Flux linkage</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>T m A$^{-1}$</td>
<td>Permeability of vacuum ($4\pi \times 10^{-7}$)</td>
</tr>
<tr>
<td>$\mu_r$</td>
<td>-</td>
<td>Relative permeability</td>
</tr>
<tr>
<td>$\nu$</td>
<td>-</td>
<td>Spatial frequency</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Ω m$^{-1}$</td>
<td>Electrical resistivity</td>
</tr>
<tr>
<td>$\phi$</td>
<td>rad</td>
<td>Rotation around $z$ (magnets)</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>Wb</td>
<td>Magnetic flux</td>
</tr>
<tr>
<td>$\omega$</td>
<td>rad s$^{-1}$</td>
<td>Angular velocity</td>
</tr>
</tbody>
</table>
Subscripts

- A  Phase A
- B  Phase B
- C  Phase C
- I  In region I
- II In region II
- III In region III
- IV In region IV
- V  In region V

Acronyms

- EMF  Electro-motive force
- FEM  Finite element modeling
- RMS  Root mean square
- TEC  Thermal equivalent circuit
- THD  Total harmonic distortion
Chapter 1

Introduction

1.1 Pick-and-place application

Mecal Applied Mechatronics Veldhoven in corporation with Wijdeven B.V. Oirschot, The Netherlands, is developing a fast, accurate and compact 2 Degree-of-Freedom (2 DoF) module for a pick-and-place application of Ball Grid Array (BGA) components on a Printed Circuit Board (PCB). Attached to a long-stroke robot, the module is responsible for picking up the components from a component feeder and the placing them on a PCB (short stroke), where the long-stroke robot is used for the movement of the component from the feeder over the PCB. In one hour, 10,000 BGA’s need to be placed, so one component every 360 ms. This pick-and-place application is illustrated in Fig. 1.1.

![Figure 1.1: Illustration of the pick-and-place application.](image)

1.2 Rotrans

The 2 DoF module needs to enable rotational and translational motion in order to accurately position and place the components on the PCB and, therefore, is called the \( z\phi \)-module or Rotrans (short for rotation-translation). Wijdeven B.V. already proposed an initial design for the Rotrans, which consists of two actuators positioned above each other along a hollow shaft (used to pick the components with a vacuum). Figure 1.2 shows a 3D overview of this design, where the rotary actuator is placed at the top and the linear actuator is placed at the bottom of the module. The specifications of the \( z\phi \)-module are given in Table 1.1. The
rotary actuator is a slotless permanent magnet actuator and provides ±180° degrees rotation. Because the total stroke of the linear actuator is only 10 mm, a non-commutated short-stroke linear actuator is selected instead of a three phase linear actuator. Whereas the design of the non-commutated short-stroke linear actuator is fixed, the rotary actuator is yet to be optimized for this application.

This work, therefore, concerns the design and optimization of the rotary actuator as part of the zφ-module. First the basic design regarding the rotary actuator is addressed and the test results of a non-optimized pre-prototype are discussed. Prior to the optimization step, an optimum combination of the number of magnet poles and coils is selected. A multi-physical framework of the rotary actuator is discussed and used to obtain an optimized design, which is performed for a configuration with moving magnets and moving coils. Finally, some aspects, such as the magnet edge-effects, of the optimized designs are analyzed. In advance of this graduation work, the performance of the initial design of the rotary actuator (proposed by Wijdeven) was already simulated in FLUX 3D. As a start of this work also measurements were performed on a pre-prototype of this design and these results are given in Appendix A.
1.3 Rotary-Linear actuators

The idea of a combined rotary-linear actuator is not new and nowadays different types of these actuators exist which can be divided into four categories. The first category consists of a linear actuator driving a trolley which holds a rotary actuator. Since the rotary actuator acts as a load for the linear actuator, the dynamical performance is limited [1][2]. In the second category the actuator can only move along a helical path. An example of such actuators is given in [3], which presents an induction motor with helical coils. Rotary and linear motion in these types of actuators are not independent from each other. The third topology consists of two separated actuators along a single shaft, one actuator for each motion, such as two induction motors containing two stators or a combination of two permanent magnet synchronous actuators. Both degrees of freedom can be controlled independently by separate control of the two actuators. In the fourth category both actuators are combined to form one working airgap, which also makes it more difficult to separately control both motions, though only one inverter is needed to power the actuator. In the next subsection an overview is given of rotary-linear actuators of the last two categories in order to compare their performances.

1.3.1 Overview of existing rotary-linear actuators

The switched reluctance motor (SRM) presented in [5] and [4] consists of two stators and one rotor held by two bearings which allow rotational and linear motion, as is shown in Fig. 1.3. Torque is produced by the angular reluctance force experienced by part of the rotor overlapping one of the stators, while linear motion is caused by the difference between the axial forces of the two stators on the rotor. A prototype of a 6/4 SRM is built and tested in [4]. It has a rated torque of 1.5 Nm at 1000 rpm and a maximum thrust force of 30 N over a stroke of 40 mm in a total body size of 100x100x200 mm.

A z-ϕ induction motor combining two stators along a single shaft, designed to press Surfaces Mounted Devices (SMD) against Printed Circuit Boards (PCB) with a force of 10 N for 3 seconds during laser welding, is developed at the University of Twente, The Netherlands [6]. Figure 1.4 illustrates the two stators, one for rotational motion and one for linear motion,
and the low inertia translator made of an aluminum tube. Requirements for the design are that the translator can be positioned at any given position over a stroke of 70 mm and with an accuracy of 10 µm, while it could be rotated over 360 degree to a reference angle within 0.35 mrad accuracy. The final design is capable of producing a force of 15 N and a torque of 60 mNm, resulting in accelerations of 100 m s\(^{-2}\) and 1200 rad s\(^{-2}\). The servo behavior of both actuators was tested and during position regulation a positional error of 25 µm and 0.09 mrad was achieved. With a third order path generator, a z-step of 70 mm was made in 50 ms without overshoot. The total dissipation for the linear actuator is 12.3 W N\(^{-1}\) at a stator frequency of 200 Hz and 1.25 W mNm\(^{-1}\) for the rotary actuator at a frequency of 400 Hz. The total weight of the translator is 0.068 kg and its inertia is estimated at 25×10\(^{-6}\) kg m\(^{-2}\). The outer diameter of the machine is 72 mm although nothing is mentioned about the length of the machine.

The rotary-linear machine in [1], used for integrated circuit wedge wire bonding, contains a three phase servo motor for the rotational axis and a voice coil motor for the linear translation. The six pole servo motor, illustrated in Fig. 1.5, is chosen because the application required rotational accuracy, allowed inexpensive sinusoidal current drives and had a simple and inexpensive to manufacture winding arrangement, while the voice coil motor was chosen since it was easy to manufacture. The actuator needed to be able to accelerate with 30 m s\(^{-2}\) and 200 rad s\(^{-2}\), while achieving a maximum velocity of 0.4 m s\(^{-1}\) and 30 rad s\(^{-1}\). The total load was limited to 500 gr and 150×10\(^{-6}\) kg m\(^{2}\). The actuator should operate over a total stroke of 10 mm and 360 degrees within an accuracy of 8µm and 45mdegrees.

In Fig. 1.6 a rotary-linear induction motor (RLIM) is shown, which is proposed in [7], contains four primaries with independently energized, double-layered ring-windings and a cylindrical conducting secondary made of solid aluminium which is separated from the back iron. By controlling the phase angles of the supply currents in each primary winding, the actuator can operate in any of the four quadrants of the linear/rotary plane. The paper does not focus on designing a actuator according to certain specifications, but rather optimises the rotary force through optimal arrangement of slot currents. Still some specifications can be given. The total length of the four primaries is 196 mm each having 14 slots with 80 windings. The supply frequency is 50 Hz and the excitation current is 2 A. The secondary is 600 mm long and weighs about 1.5 kg. The maximum rotary force possible at rated current is 4 N.
1.3. Rotary-Linear actuators

Figure 1.4: Operating principal of a z-\(\phi\) induction actuator [6].

Figure 1.5: Schematic of the rotary-linear motor with 3-phase servo actuator and voice-coil motor [1].
Figure 1.6: Structure of the rotary-linear induction motor with four primaries [7].

Figure 1.7: Helical motion permanent magnet motor: (a) rotor unit integrated two types of Halbach array and (b) schematic of proposed motor [8].

(radius is not specified) while the linear force is 16 N.

For rotary-linear applications involving arbitrary rotation angles, the above solutions show low force densities and sometimes poor dynamic performances. Besides, the induction machines also produce heat in the secondary. Therefore, several designs with permanent magnets have been proposed. [2] describes a brushless motor including a permanent magnets mover and a ferromagnetic, salient pole stator with 4-phase symmetrical windings, which achieves a decoupled regulation of force and torque. In [8] a motor for helical motion consisting of an exterior polar Halbach quadrupole for the rotational motion and an interior cylindrical Halbach dipole for the linear motion. The stator contains two coil arrays, which interact with the magnetic fields of the two Halbach arrays for both degrees of freedom. This topology is illustrated in Fig. 1.7.

Anorad Corporation has patented a rotary-linear actuator as is shown in Fig. 1.8 [9]. This machine was designed for pick & place applications. The rotary and linear motors are slotted, permanent magnet synchronous motors, which have stationary coils and moving magnets. Significant to this design is the checkerboard magnet array as shown in Fig. 1.8(b). The linear stroke of the machine is 50 mm with a continuous linear force of 2.9 N. With a load of 70 gr it is able to achieve an acceleration of approx. 20 m s\(^{-2}\) and has maximum velocity of 1 m s\(^{-1}\). The rotary actuator can rotate over 360 degree with a maximum angular velocity of 60 rad s\(^{-1}\). Its maximum angular acceleration is 1250 rad s\(^{-2}\).
1.3. Rotary-Linear actuators

Figure 1.8: Rotary-linear actuator patent by Anorad Corporation [9]: (a) cross-section of the actuator and (b) checkerboard magnet array.

Similar checkerboard magnet arrays are used by Psicontrol Mechatronics in Ypre, Belgium and Braincenter in Drachten, The Netherlands, but both topologies are used in combination with a couple of brushless dc-motors placed along the axial length of the actuator. The actuator from Psicontrol was designed for weaving machines and operates over a stroke of 45 mm, while being able to rotate freely. The motor with 54 magnets and 18 coils, produces a force of 40 N at a maximum velocity of 2 m s$^{-1}$ in axial direction and a torque of 1 Nm at 1000 rpm. The rotor weights 131 g and the complete motor is packed in a cylindrical housing measuring 120 mm in axial direction with a diameter of 100 mm.

Braincenter developed its rotary-linear actuator to replace a hydraulic actuator in a razorblade production line. With six dc-motors in axial direction, this machine develops a force of 138 N and a torque of 4.9 Nm, for velocities higher than 0.1 m s$^{-1}$ and 1500 rpm. The actuator is able to make a stroke of 56 mm with an accuracy lower than 2.5 µm in axial direction, although no information is known about the angular stroke.

The checkerboard magnet array shown in Fig. 1.8(b) only has a packing factor of only 50%, making this actuator less efficient compared to a regular motor with similar surface area for the magnets. Filling the unused areas with extra magnet material, though, does not increase the force capabilities. Two separate motors for the $z$ and $\theta$ motion stacked up axially are more efficient, but are less compact than the their checkerboard magnet array counterpart. Anorad Corporation also patented a rotary-linear actuator with two separate actuators along the axis of the motor. An actuator with two different magnetisation patterns is patented in [10] and shown in Figure 1.9. A commercially available custom designed version of this actuator, called the "Cyclone Z-theta positioning module", offers a continuous force of 6 N (peak force is 12N) and a torque of 94 mNm (peak torque is 282 mNm), while it operated over a stroke of 50 mm and its rotational range is unlimited. The actuator is able to accelerate a total weight of 0.42 kg with 20 m s$^{-2}$ to a maximum velocity of 0.875 m s$^{-1}$. The rotary actuator accelerates the moving member (inertia of $85\times10^{-6}$ kg m$^2$) with 6000
Figure 1.9: Rotary-linear actuator patent by Anorad Coorporation [10]: (a) cross-section of the actuator and (b) double magnet array.

rad s^{-2} and has a maximum angular velocity of 60 rad s^{-1}. The rotary-linear actuator designed by Wijdeven B.V. (Rotrans) resembles the design proposed by [10], however it uses a non-commutated short-stroke linear actuator, because it only has to offer a linear stroke of 10 mm. The topology of the Rotrans is based on the third category of $z\phi$-modules. Table 1.2 lists the specification of the initial design of the $z\phi$-module proposed by Wijdeven B.V. together with the other modules which are found in the literature. Comparing the different designs, though, is difficult, since they all apply for different applications and not all specifications are complete.
### Table 1.2: Comparison of different types of rotary-linear actuators.

<table>
<thead>
<tr>
<th>Type motor</th>
<th>Stroke</th>
<th>Stroke Angle</th>
<th>Force</th>
<th>Torque</th>
<th>Acc</th>
<th>Acc Time</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotrans Wijdevel B.V.</td>
<td>10 mm</td>
<td>360°</td>
<td>65 N</td>
<td>1.3 Nm</td>
<td>150 m s⁻²</td>
<td>7700 rad s⁻²</td>
<td>L=105 mm φ=60 mm</td>
</tr>
<tr>
<td>Switched reluctance motor [4]</td>
<td>40 mm</td>
<td>unlim.</td>
<td>30 N</td>
<td>1.5 Nm</td>
<td>-</td>
<td>-</td>
<td>100x100x200 mm</td>
</tr>
<tr>
<td>2 Induction motors [6]</td>
<td>70 mm</td>
<td>unlim.</td>
<td>30 N</td>
<td>1.5 Nm</td>
<td>-</td>
<td>-</td>
<td>L=unknown φ=72 mm</td>
</tr>
<tr>
<td>Voice-coil and servo motor [1]</td>
<td>10 mm</td>
<td>unlim.</td>
<td>&lt;80 N</td>
<td>&lt;0.9 Nm</td>
<td>30 m s⁻²</td>
<td>200 rad s⁻²</td>
<td>unknown</td>
</tr>
<tr>
<td>Induction motor with 4 primaries [7]</td>
<td>&lt;600 mm</td>
<td>unlim.</td>
<td>16 N</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>L=unknown φ=unknown</td>
</tr>
<tr>
<td>PM synchronous motor [9]</td>
<td>50 mm</td>
<td>unlim.</td>
<td>2.9 N</td>
<td>-</td>
<td>20 m s⁻²</td>
<td>1250 rad s⁻²</td>
<td>L=600 mm φ=unknown</td>
</tr>
<tr>
<td>with checkerboard array</td>
<td>45 mm</td>
<td>unlim.</td>
<td>40 N</td>
<td>1 Nm</td>
<td>-</td>
<td>-</td>
<td>unknown</td>
</tr>
<tr>
<td>2 PM synchronous motor [10]</td>
<td>56 mm</td>
<td>unlim.</td>
<td>40 N</td>
<td>1.3 Nm</td>
<td>&gt;20 m s⁻²</td>
<td>-</td>
<td>L=120 mm φ=100 mm</td>
</tr>
<tr>
<td>[11]</td>
<td>50 mm</td>
<td>unlim.</td>
<td>6 N</td>
<td>94 mNm</td>
<td>20 m s⁻²</td>
<td>6000 rad s⁻²</td>
<td>unknown</td>
</tr>
<tr>
<td>25.4 mm</td>
<td>unlim.</td>
<td>40.5 N</td>
<td>0.45 N</td>
<td>0.9 Nm</td>
<td>35 m s⁻²</td>
<td>5060 rad s⁻²</td>
<td>unknown</td>
</tr>
</tbody>
</table>
Chapter 2

Basic design

2.1 Slotless permanent magnet synchronous motor with Halbach array

The basic design of the three-phase AC brushless permanent magnet rotary actuator, which is shown in Fig. 2.1, has a slotless structure in order to provide a low ripple torque. Slotless machines, however, have an large effective airgap and, therefore, a two segmented quasi-Halbach magnet array is used, because it offers high magnetic loading in the airgap. Moreover, such an array offers a sinusoidal flux density and provides self shielding on one side of the array [12]. Concentrated windings are used for their ease of manufacturing, but also have shorter end-windings compared to distributed or lap windings, resulting in lower copper losses. Because the rotary actuator also needs to accommodate the translational motion of the rotor, the coils are elongated so constant torque for any given axial position, is delivered.

![Illustration of a design with (a) a moving magnet configuration and (b) a moving coil configuration.](image)

Figure 2.1: Illustration of a design with (a) a moving magnet configuration and (b) a moving coil configuration.

The same basic design as described above is used for the optimized design of the rotary actuator. Because the rotational angle of the rotary actuator is limited to ±180°, both a moving magnet and a moving coil configuration will be optimized and compared. In the moving
magnet configuration the coils are attached to the back-iron leaving only a single mechanical clearance between the coils and magnets. For moving coils however a double mechanical clearance is required; one additional airgap between the coils and back-iron. Figure 2.1 shows the schematics of the rotary actuator for the moving magnet and moving coil configuration.

Wijdeven B.V. designed a non-optimized pre-prototype of the rotary actuator based on the design consideration as described above, and a pre-prototype of the non-commutated short-stroke linear actuator. Both actuators have been built and tested. Although the design of the linear actuator is out of the scope of this work, its specifications are given in Table 2.1. The design of the rotary actuator is based a configuration with moving coils, however, due to the electrical wiring the magnets were rotated during the tests. Sizes of the rotary actuator are given in Table 5.2 and pictures of the test set-up and coils are shown in Fig. 2.2 and Fig. 2.3, respectively. Each coil in the slotless PM actuator contains 92 turns made of 0.912x0.132 mm square wire. The magnets are made of anisotropic sintered NdFeB, with a remanent flux density of 1.33 T and a relative recoil permeability of 1.1. The core and back-iron are made of steel N398 with a saturation flux density of 1.5 T. Other material properties are given in Table 2.2. The actuator is attached to a Lloyd LF Plus testing machine and the torque is measured with an 50 N load cell. From 2D Finite Element Modeling (FEM), a torque constant of 0.93 Nm A\(^{-1}\) is estimated, however, tests on the pre-prototype show a torque constant of 0.76 Nm A\(^{-1}\). To eliminate the influence of friction, electro-motive force (EMF) measurements are performed, which show a reduced amplitude of 11.7% compared to 2D FEM. This reduction is caused by the lower magnetic loading near the edges of the magnets, as will be discussed in Chapter 6. All test results are given in Appendix A.

\(^1\)The thermal conductivity of the coils is mainly determined by the insulation of the wires.
2.1. Slotless permanent magnet synchronous motor with Halbach array

Figure 2.3: Coil array of the pre-prototype rotary actuator.

Table 2.1: Specifications of the non-commutated short-stroke linear actuator.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{t,lin}$</td>
<td>14 N A$^{-1}$</td>
<td>Force constant</td>
</tr>
<tr>
<td>$R_{coil}$</td>
<td>8.2 Ω</td>
<td>Coil resistance</td>
</tr>
<tr>
<td>$J_{lin}$</td>
<td>100·10$^{-6}$ kg m$^2$</td>
<td>Inertia</td>
</tr>
<tr>
<td>$M_{lin}$</td>
<td>190·10$^{-3}$ kg</td>
<td>Moving mass</td>
</tr>
</tbody>
</table>

Table 2.2: Material properties.

<table>
<thead>
<tr>
<th></th>
<th>$m$ [kg m$^{-3}$]</th>
<th>$k$ [W m$^{-1}$ K$^{-1}$]</th>
<th>$\rho$ [Ω m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>NdFeB</td>
<td>7350</td>
<td>10</td>
<td>-</td>
</tr>
<tr>
<td>N398</td>
<td>7850</td>
<td>73</td>
<td>-</td>
</tr>
<tr>
<td>Copper</td>
<td>8900</td>
<td>$1^1$</td>
<td>1.678e$^{-8}$</td>
</tr>
<tr>
<td>Air</td>
<td>1.067</td>
<td>0.0285</td>
<td>-</td>
</tr>
<tr>
<td>@T=25°C</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

Coil carrier

Coils
Chapter 3

Selection number of magnet poles and coils

3.1 Balanced design

One of the first steps when designing an electric machine is selecting the appropriate combination of the number of magnet poles ($2p$) and coils ($Q$). Making this selection beforehand is difficult, but evaluating every possible combination during the optimization procedure is time consuming. Therefore, the possible combinations of the number of magnet poles and coils have to be limited. In this perspective, the following assumptions are specified:

- concentrated windings are used,
- only balanced three-phase configurations are considered,
- all coils have the same number of turns and have identical dimensions.

These assumptions imply that the number of coils is a multiple of the number of phase, i.e. $Q = 3j$, where $j$ is an integer. Also the number of coils per pole per phase is less than one, $q = Q/2p/3 < 1$, and $Q \neq 2p$ [13][14][15].

3.2 Winding factor

The selection is further reduced by choosing magnet pole and coil combinations based on the winding factor, $k_w$. The winding factor is a measure for the flux linkage of a certain winding layout and ranges from 0 (no linkage) to 1 (optimal linkage). The torque produced by a three-phase AC brushless PM actuator is related to the flux linkage according to

$$T = \frac{3}{2\omega} I \hat{E} = \frac{3}{2\omega} I k_w d\Lambda_{\text{max}} = \frac{1}{2} I_\text{rms}^2 R_{\text{eq}}$$

where $I$ is the peak phase current, $\hat{E}$ is the peak phase back-EMF, $\omega$ is the angular velocity and $\Lambda_{\text{max}}$ is the maximum possible flux linkage of a single phase. As the expression shows, the torque is directly linked to the winding factor, and it would be desirable to select a combination of magnet poles and coils resulting in the highest possible winding factor. The winding factor can be split into

$$k_w = k_p \cdot k_d \cdot k_{skew},$$

(3.2)
where \( k_p \) is the pitch factor, \( k_d \) is the distribution factor and \( k_{\text{skew}} \) is the skewing factor. Since no skewing is applied, the skewing factor is assumed to be unity. The winding factors can only be found for slotted structures, though ([14][15][16][17]). Therefore, in this section the fundamental winding factor for a slotless permanent magnet machines with concentrated windings is calculated for different combinations of the number of magnet poles and coils. Below, both the pitch and distribution factor will be determined, when only the fundamental harmonic of the magnetic loading due to the magnets is considered.

### 3.2.1 Pitch factor

The pitch factor is a measure for the flux linkage by a single coil. Ideally, the total flux through one magnet pole is linked by all turns in a coil, which is achieved in slotted machines having \( q = 1 \). Where the teeth in a slotted machine provides a low reluctance path for the flux through the coil, letting all turns in the coil link the same amount of flux, in slotless machines however, this low reluctance path is not provided and, hence, not all turns show the same flux linkage.

Therefore, to determine the pitch factor in a slotless machine as is shown in Fig. 3.1, the average flux linkage of a single turn is calculated, by varying the angular span, \( 2\alpha_t \), of a turn at radius \( r \)

\[
\Psi_{av} = \frac{1}{\alpha_c - \beta_o} \int_{-\alpha_c}^{\alpha_c} \Psi_t(\alpha_t) d\alpha_t \\
= \frac{1}{\alpha_c - \beta_o} \int_{-\alpha_c}^{\alpha_c} \int_{-\alpha_t}^{\alpha_t} r L_{\text{act}} B_1(r) \cos(p\phi) d\phi d\alpha_t \\
= \frac{2r L_{\text{act}} B_1(r)}{\alpha_c - \beta_o} \left[ \frac{1}{p^2} (\cos(p\beta_o) - \cos(p\alpha_c)) \right],
\]

(3.3)

where, \( 2\beta_o \) is the opening angle of a coil, \( 2\alpha_c \) is the angular span of a coil, \( L_{\text{act}} \) is the length in axial direction and \( B_1 \) is the amplitude of the fundamental harmonic of the magnetic flux density. With the maximum flux linkage of a single turn being equal to the flux through one magnet pole, \( \Psi_{max} = \frac{2\pi}{p} r L_{\text{act}} B_1(r) \), the pitch factor, \( k_p \), for a slotless machine with concentrated windings becomes

\[
k_p = \frac{\Psi_{av}}{\Psi_{max}} = \frac{1}{p\alpha_c - p\beta_o} \left[ \cos(p\beta_o) - \cos(p\alpha_c) \right].
\]

(3.4)

### 3.2.2 Distribution factor

The distribution factor is a measure of the electrical alignment of all coils in a single phase and can be calculated by writing the back-EMF of a single coil in phasor representation (in per unit)

\[
\overrightarrow{E_{i,pu}} = e^{j(\gamma_i)},
\]

where \( \gamma_i = \frac{2\pi p}{Q} \cdot i \) is the electrical angle offset of coil \( i \). For a sinusoidal flux density, the amplitude of the back-EMF is linked to the peak flux linkage, \( \dot{\Lambda} \), according to

\[
\dot{E} = \left| \frac{d\Lambda}{dt} \right| = \left| \frac{\partial \Lambda}{\partial \phi} \frac{\partial \phi}{dt} \right| = p\omega \dot{\Lambda}.
\]

(3.5)
3.2. Winding factor

Fig. 3.1: Illustration of a slotless actuator for predicting the pitch factor.

Fig. 3.2 shows an example of a phasor representation of the back-EMF for the individual coils and also the resulting phase back-EMF phasors, which are symmetrically displaced by 120° electrical degrees from each other. The minus sign for a phasors means that the windings of the coil are connected in the opposite direction. The phase back-EMF phasors (in per unit) can be calculated according

\[ E_{ph,pu} = \sum_{Q/3} E_{i,pu}. \]

The distribution factor is found by dividing the magnitude of the resulting phase EMF phasor by the number of coils per phase, as given by

\[ k_d = \frac{|E_{ph,pu}|}{Q/3}. \quad (3.6) \]

In [16] a method is presented to obtain the winding layout that gives the highest distribution factor for a given pole and coil count, while retaining a balanced structure. The method is based on the decomposition of the number of coils per pole per slot, \( q \). For a combination of \( 2 \cdot p = 14 \) poles and \( Q = 15 \), as an example, this method is explained in Figure 3.3.

1) \( q \) is written as a fraction which is cancelled down to its lower terms

\[ q = \frac{n}{d} = \frac{5}{14} \]

where \( n \) and \( d \) are integers.

2) A sequence containing \( n = 5 \) times a "1" and \( d - n = 9 \) times a "0" is found where the "1" is distributed as regularly as possible:
Figure 3.2: EMF phasor representation of a three-phase system.

\[ \text{seq} = 10010010010010 \]

3) The sequence is repeated \( \frac{Q}{n} = 3 \) times and compared to a conductor layout for a machine with \( q = 1 \).

4) Conductors of the distributed winding (machine with \( q = 1 \)) corresponding to a "1" in the sequence are kept and form the left conductor of each coil. The right conductor is simply found by taking the return conductor of the left side.

5) The coil layout is described by a vector \( S \) which gives the phase to which every coil is assigned. A minus sign in front of the phase corresponds to a coil which should be wound the opposite direction. Vectors \( S_A, S_B, \) and \( S_C \) give the positions of the coils in the phases. A minus sign is added when the coil is wound the other way around.

### 3.2.3 Analysis of the winding factor

The winding factors for a slotless machine with concentrated windings are calculated for different combinations of the number of magnets poles and coils using expressions (3.2), (3.4) and (3.6). Combinations resulting in the same value of \( q \) have the same winding factor and, therefore, the winding factor as function of the number of coils per magnet pole per phase, \( q \), is shown in Fig. 3.4 when \( \beta_0 = 0 \) and \( \beta_0 = 0.5 \alpha_c \). In Appendix C the most promising pole-coil combinations \( (k_w > 0.65) \) are given. For both cases it can be noticed that machines having \( q = 1/4 \) give the highest winding factor, \( k_w = 0.716 \) and \( k_w = 0.955 \) respectively. These types of machines have \( k_d = 1 \) and therefore show a peak value in the figure. For comparison, slotless machines having \( q = 1 \) (distributed windings) have a winding factor of 0.699 and 0.791, respectively.

Magnet poles and coils combinations resulting in \( q = 1/4 \) are selected for their high winding factor and because they can be obtained with every number of magnet poles which is a multiple
3.2. Winding factor

1) \[ q = \frac{Q}{2p/m} = \frac{\pi}{d} = \frac{5}{14} \rightarrow 5 \times \text{"1"} \]
\[ 9 \times \text{"0"} \]

2) \[ \begin{array}{cccccccccccc}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
\end{array} \]

3) \[ \begin{array}{cccccccccccc}
\end{array} \]

4) \[ \begin{array}{cccccccccccc}
\end{array} \]

\[ S_A = [1 - 2 - 3 - 4 - 5] \]
\[ S_B = [6 - 7 - 8 - 9 - 10] \]
\[ S_C = [11 - 12 - 13 - 14 - 15] \]

Figure 3.3: Determination of the coil layout for a three-phase machine with 14 magnet poles and 15 coils.

Table 3.1: Winding layout for machines with \( q = 1/4 \) and concentrated windings.

<table>
<thead>
<tr>
<th>Winding layout</th>
<th>( [A_+ A_- C_+ C_- B_+ B_- A_+ A_- C_+ C_- B_+ B_- ...] )</th>
</tr>
</thead>
</table>

of four. During optimization the number of magnet poles is varied and the number of coils is determined according to \( Q = 6qp \). Table 3.1 gives the winding layout for a machine with \( q = 1/4 \).
Figure 3.4: Winding factors for slotless machines with concentrated windings versus the number of coils per magnet pole per phase, $q$. 
Chapter 4

Multi-physical framework

To optimize the design of the rotary actuator under certain constraints, a multi-physical model is required. Although finite element analysis is accurate, it is also time consuming and difficult to use in an optimization process. Therefore, in this chapter a multi-physical framework is created and discussed. This framework contains a 2D semi-analytical magneto-static, a thermal and a mechanical model.

4.1 Coordinate systems

The permanent magnet synchronous machine under study here, has a cylindrical design. So, cylindrical coordinates are the obvious choice to model the actuator. Because the actuator consists of a fixed and a moving part, two coordinate systems are required, which are indicated in Fig. 4.1. The first coordinate system \((r, \theta, z)\) is fixed to the coils with the center of the first coil at \(\theta = 0\) and \(z = 0\) in the middle of the coils. The second coordinate system \((r, \phi, z)\) is fixed to the magnets and the center of a radial magnetized magnet is at \(\phi = 0\). To account for the circumferential motion between the magnets and coils, a relative angular displacement between the two coordinate systems \(\Delta \theta\) is introduced, which is defined as \(\Delta \theta = (\theta - \theta_0) - (\phi - \phi_0)\). For simplicity, the offsets \(\theta_0\) and \(\phi_0\) are taken to zero. The angular velocity is defined as \(\omega = \frac{d\Delta \theta}{dt}\).

4.2 Magnetic flux density distribution of Halbach array

4.2.1 General open-circuit magnetic field solution

An analytical model of the open field magnetic field distribution is important to predict the force capabilities, back-EMF waveforms and dynamic performance of a brushless permanent magnet motor. Based on the work of Zhu and Howe in [18] an analytical model is obtained by solving the Laplace/Poisson field equations, without any assumption regarding the relative recoil permeability of the magnets. The solution of the magnetic field is a Fourier series with its coefficients being solved with boundary conditions between different regions in the motor. Different papers give models for the magnet field when a Halbach magnet array is employed, but all have certain assumptions. In [19] and [20] ideal magnetized Halbach arrays are assumed, while [12] also sets the relative permeability to unity. The authors of [21] give a solution to a two-segment Halbach magnetization array, but assume unity
Figure 4.1: Coordinates systems for the slotless permanent magnet actuator.

relative permeability in the magnets and also does not exactly model the magnets with square waveform magnetization. On the other hand, the writers of [22] do not give a solution to the magnetic field distribution.

In this section, therefore, the models obtained from the technique as described in [18] are further extended to account for the two-segmented quasi-Halbach array with straight magnetization. The following assumption are made in the semi-analytical field calculation:

- Two-dimensional fields are assumed in cylindrical coordinates. Therefore, the magnetic field only contains radial and tangential components.
- The actuator is assumed to be infinitely long, so the end effects are neglected in this model.
- The relative permeability of iron is infinite.
- The magnets have a linear demagnetization curve in the 2-quadrant and are fully magnetized in the direction of magnetization.

Since the relative permeability in the iron parts of the actuator are chosen to be infinite, only the solutions for the magnetic field inside the airgap and magnets have to be found. Figure 4.2 shows a schematic of the rotary actuator with the airgap and magnet regions. The figure also shows the leading dimensions used in the magneto-static model.

This model is obtained by first considering Ampères law

\[ \nabla \times \mathbf{H} = \mathbf{J}, \]  

(4.1)

where \( \mathbf{H} \) is the magnetic field strength and \( \mathbf{J} \) is the current density vector. For the airgap region \( \mathbf{B}_1 = \mu_0 \mathbf{H}_1 \), where \( \mathbf{B}_1 \) is the flux density vector and by introducing the vector potential, \( \mathbf{A} \), according to

\[ \mathbf{B} = \nabla \times \mathbf{A}, \]  

(4.2)
for the airgap region \( r_m \leq r \leq r_b \), indicated by subscript I), Eq. (4.1) can be rewritten

\[
\nabla \times \nabla \times \mathbf{A}_I = \nabla (\nabla \cdot \mathbf{A}_I) - \nabla^2 \mathbf{A}_I = \mu_0 \mathbf{J}_I. 
\]

(4.3)

By defining the divergence of the vector potential to be zero \( (\nabla \cdot \mathbf{A}_I) = 0 \) and assuming zero current density, Laplace’s equation is obtained

\[
\nabla^2 \mathbf{A}_I = 0. 
\]

(4.4)

The flux density vector in the magnets \( r_c \leq r \leq r_m \), indicated by subscript II) is described as

\[
\mathbf{B}_{II} = \mu_0 \mu_r \mathbf{H}_{II} + \mu_0 \mathbf{M}_0, 
\]

(4.5)

where \( \mu_r \) is the relative recoil permeability of the magnets and \( \mathbf{M}_0 \) is the remanent magnetization vector.

By taking the curl, using Eqs. (4.1) and (4.2) and also assuming zero current in this region, the Poisson equation is obtained

\[
\nabla^2 \mathbf{A}_{II} = -\mu_0 \nabla \times \mathbf{M}_0. 
\]

(4.6)

Because the magnetic field only has components in radial and tangential direction and depends on \( r \) and \( \phi \), according to Eq. (4.2) the vector potential only has a component in the axial direction and likewise depends on \( r \) and \( \phi \)

\[
\mathbf{A}(r, \phi) = A_z(r, \phi) \mathbf{e}_z. 
\]

(4.7)
The Laplace and Poission equations can therefore be further evaluated

\[
\nabla^2 \mathbf{A}_I = \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A_{zI}}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A_{zI}}{\partial \phi^2} \right) \mathbf{e}_z = 0, \tag{4.8}
\]

\[
\nabla^2 \mathbf{A}_{II} = \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A_{zII}}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A_{zII}}{\partial \phi^2} \right) \mathbf{e}_z = -\mu_0 \nabla \times \mathbf{M}_0. \tag{4.9}
\]

Considering the periodicity of the geometry, the (homogeneous) solution for the vector potential in the airgap can be written as a linear combination of \( \sin(n \phi) \)- and \( \cos(n \phi) \)-terms and \( (r^n \phi) \)- and \( (r^n \phi) \)-terms. By a appropriately choosing the magnetization of the magnets, either the \( \sin(n \phi) \)- or the \( \cos(n \phi) \)-terms can be eliminated. Here the solution can be written as

\[
\mathbf{A}_I(r, \phi) = A_{zI}(r, \phi) \mathbf{e}_z = \sum_{n=1}^{\infty} \left( X_{nI} r^n \phi + Y_{nI} r^{-n} \phi \right) \sin(n \phi) \mathbf{e}_z. \tag{4.10}
\]

For the magnets a particular solution to the curl of the magnetization in the magnets, \( \mathbf{A}_{\text{part}}(r, \phi) \), is added to the homogeneous solution of Eq. (4.10)

\[
\mathbf{A}_{II}(r, \phi) = A_{zII}(r, \phi) \mathbf{e}_z = \sum_{n=1}^{\infty} \left( X_{nII} r^n \phi + Y_{nII} r^{-n} \phi \right) \sin(n \phi) \mathbf{e}_z + \mathbf{A}_{\text{part}}(r, \phi). \tag{4.11}
\]

### 4.2.2 Magnetization of the two-segmented quasi-Halbach array

The remanent magnetization vector \( \mathbf{M}_0 \) of the two-segmented quasi-Halbach array can be analyzed for two cases: curved and straight magnetization (see Fig. 4.3). Although, the curved magnetization pattern either has radial or circumferential components of the magnetization in the different segments of the Halbach array, the straight magnetization shows both radial and tangential components of the magnetization vector in all segments of the quasi-Halbach array.

![Figure 4.3: Magnetization pattern for Halbach magnet array: (a) curved magnetization (b) straight (parallel) magnetization.](image-url)
To illustrate the difference between both cases, the waveforms of the radial and tangential components of the magnetization are illustrated in Fig. 4.4. While for the curved magnetization both components have a square-shaped waveform, the straight waveform shows a more distorted sine-shaped waveform in both components. Especially for low pole counts, straight magnetization approaches ideal Halbach magnetization (containing only sinusoidal waveforms of the magnetization) better. Whether, this gives an advance over curved magnetization is left to be investigated in future work. Because straight magnetization is easier to obtain during fabrication of the magnets, this type of magnetization will be considered. For completeness, also a description for curved magnetization is given.

Both the radial and tangential components of the magnetization vector, $M_r$ and $M_\phi$, respectively, of the two-segmented quasi-Halbach array are described as

\[
\begin{align*}
M_r &= -\frac{B_{rem}}{\mu_0} \cos \left( \phi + \frac{\pi}{p} \right) & -\frac{\pi}{p} \leq \phi \leq -\frac{(2-\alpha)\pi}{2p}, \\
M_\phi &= 0 & -\frac{\pi}{p} \leq \phi \leq -\frac{(2-\alpha)\pi}{2p}, \\
M_r &= 0 & -\frac{(2-\alpha)\pi}{2p} \leq \phi \leq -\frac{\alpha\pi}{2p}, \\
M_\phi &= -\frac{B_{rem}}{\mu_0} \cos \left( \phi + \frac{\pi}{2p} \right) & -\frac{(2-\alpha)\pi}{2p} \leq \phi \leq -\frac{\alpha\pi}{2p}, \\
M_r &= \frac{B_{rem}}{\mu_0} \cos \left( \phi + \frac{\pi}{2p} \right) & -\frac{(2-\alpha)\pi}{2p} \leq \phi \leq -\frac{\alpha\pi}{2p}, \\
M_\phi &= 0 & \frac{\alpha\pi}{2p} \leq \phi \leq \frac{(2-\alpha)\pi}{2p}, \\
M_r &= -\frac{B_{rem}}{\mu_0} \sin \left( \phi - \frac{\pi}{2p} \right) & \frac{\alpha\pi}{2p} \leq \phi \leq \frac{(2-\alpha)\pi}{2p}, \\
M_\phi &= \frac{B_{rem}}{\mu_0} \sin \left( \phi - \frac{\pi}{2p} \right) & \frac{\alpha\pi}{2p} \leq \phi \leq \frac{(2-\alpha)\pi}{2p}, \\
M_r &= -\frac{B_{rem}}{\mu_0} \sin \left( \phi - \pi \right) & \frac{(2-\alpha)\pi}{2p} \leq \phi \leq \frac{\pi}{p}, \\
M_\phi &= \frac{B_{rem}}{\mu_0} \sin \left( \phi - \pi \right) & \frac{(2-\alpha)\pi}{2p} \leq \phi \leq \frac{\pi}{p},
\end{align*}
\]

for curved magnetization, and

\[
\begin{align*}
M_r &= -\frac{B_{rem}}{\mu_0} \cos(\phi + \pi/p) & -\frac{\pi}{p} \leq \phi \leq -\frac{(2-\alpha)\pi}{2p}, \\
M_\phi &= \frac{B_{rem}}{\mu_0} \sin(\phi + \pi/p) & -\frac{\pi}{p} \leq \phi \leq -\frac{(2-\alpha)\pi}{2p}, \\
M_r &= \frac{B_{rem}}{\mu_0} \sin(\phi + \pi/2p) & -\frac{(2-\alpha)\pi}{2p} \leq \phi \leq -\frac{\alpha\pi}{2p}, \\
M_\phi &= -\frac{B_{rem}}{\mu_0} \cos(\phi + \pi/2p) & -\frac{(2-\alpha)\pi}{2p} \leq \phi \leq -\frac{\alpha\pi}{2p}, \\
M_r &= -\frac{B_{rem}}{\mu_0} \sin(\phi - \pi/2p) & \frac{\alpha\pi}{2p} \leq \phi \leq \frac{(2-\alpha)\pi}{2p}, \\
M_\phi &= \frac{B_{rem}}{\mu_0} \cos(\phi - \pi/2p) & \frac{\alpha\pi}{2p} \leq \phi \leq \frac{(2-\alpha)\pi}{2p}, \\
M_r &= -\frac{B_{rem}}{\mu_0} \cos(\phi - \pi/p) & \frac{(2-\alpha)\pi}{2p} \leq \phi \leq \frac{\pi}{p}, \\
M_\phi &= \frac{B_{rem}}{\mu_0} \sin(\phi - \pi/p) & \frac{(2-\alpha)\pi}{2p} \leq \phi \leq \frac{\pi}{p},
\end{align*}
\]

for straight magnetization. Here $B_{rem}$ is the remanent flux density of the magnets and $\alpha = \frac{\phi_p}{\phi_p + \phi_r}$ is the radial magnet arc to pole arc ratio.
The magnetization of the two-segment quasi-Halbach array will be approximated by a Fourier series

\[ M_0 = \sum_{n=1,3,5} M_{nr} \cos(n \pi \alpha) e_r - \sum_{n=1,3,5} M_{n\phi} \sin(n \pi \alpha) e_\phi, \]  

(4.12)

where \( M_{nr} \) and \( M_{n\phi} \) are obtained by applying the Fourier analysis to the magnetization components as given above, resulting in

\[ M_{nr} = \frac{4B_{rem}}{\mu_0 n \pi} \sin \left( \frac{n \pi \alpha}{2} \right), \]

\[ M_{n\phi} = \frac{4B_{rem}}{\mu_0 n \pi} \cos \left( \frac{n \pi \alpha}{2} \right), \]

for curved magnetization and

\[ M_{nr} = \frac{p B_{rem}}{\mu_0 n \pi} \left\{ \frac{1}{np+1} \left[ \sin \left( \frac{2 - \alpha}{2} \frac{\pi n}{2} - \frac{\alpha \pi}{2p} \right) + \cos \left( \frac{2 - \alpha}{2} \frac{\pi n}{2} + \frac{(1 - \alpha) \pi}{2p} \right) \right] + \sin \left( \frac{(np+1) \alpha \pi}{2p} \right) - \cos \left( \frac{(np+1) \alpha \pi}{2p} - \frac{\pi}{2p} \right) \right\}, \]

(4.13)

\[ M_{n\phi} = \frac{p B_{rem}}{\mu_0 n \pi} \left\{ \frac{1}{np+1} \left[ \sin \left( \frac{2 - \alpha}{2} \frac{\pi n}{2} - \frac{\alpha \pi}{2p} \right) + \cos \left( \frac{2 - \alpha}{2} \frac{\pi n}{2} + \frac{(1 - \alpha) \pi}{2p} \right) \right] + \sin \left( \frac{(np+1) \alpha \pi}{2p} \right) - \cos \left( \frac{(np+1) \alpha \pi}{2p} - \frac{\pi}{2p} \right) \right\} - \sin \left( \frac{(np+1) \alpha \pi}{2p} \right) - \cos \left( \frac{(np+1) \alpha \pi}{2p} + \frac{\pi}{2p} \right) \right\}, \]

(4.14)
4.2. Magnetic flux density distribution of Halbach array

for straight magnetization when \( np \neq 1 \).

For the special case when \( p = 1 \) ideal Halbach magnetization is achieved with straight magnetization. For this case the magnetization vector \( \mathbf{M}_0 \) can be described by

\[
\mathbf{M}_0 = \frac{B_{rem}}{\mu_0} \cos(\phi)\mathbf{e}_r - \frac{B_{rem}}{\mu_0} \sin(\phi)\mathbf{e}_\phi. \tag{4.15}
\]

4.2.3 Solution of the open-circuit magnetic field

After having obtained the description of the magnetization vector, the general solution of the open-circuit magnetic field can be further evaluated for the two-segmented quasi-Halbach array. First, the particular solution needs to be found. Second, the solution for the flux density in both regions (magnet and air) is derived from the vector potential. In the third and final step, boundary conditions are used to find the coefficients in the field solutions in order to fully characterize the magnetic field in both regions.

The particular solution to Eq. (4.6) can be written in the form

\[
A_{\text{part}}(r, \phi) = \sum_{n=1,3,5,\ldots}^\infty G_n r \sin(n p \phi) \mathbf{e}_z. \tag{4.16}
\]

By substituting this solution and the description of magnetization into Eq. (4.9), \( G_n \) is obtained as

\[
G_n = \mu_0 n p M_{nr} - M_{n\phi}. \tag{4.17}
\]

This partial solution is not valid for the case when \( np=1 \). In this special case, let \( r = e^t \), so Eq. (4.6) becomes

\[
\frac{\partial^2 A_{z,\text{part}}(r, \phi)}{\partial t^2} + \frac{\partial^2 A_{z,\text{part}}(r, \phi)}{\partial \phi^2} = -\mu_0 (M_{1r} - M_{1\phi}) e^t \sin(\phi). \tag{4.18}
\]

By assuming

\[
A_{\text{part}}(r, \phi) = G_1 t e^t \sin(n p \phi) \mathbf{e}_z, \tag{4.19}
\]

and substituting this solution into Eq. (4.18), the constant \( G_1 \) is obtained as

\[
G_1 = -\mu_0 \frac{M_{1r} - M_{1\phi}}{2}. \tag{4.20}
\]

The final solution for the partial solution when \( np=1 \) as function of \( r \) becomes

\[
A_{\text{part}}(r, \phi) = G_1 t ln(r) \sin(n p \phi) \mathbf{e}_z. \tag{4.21}
\]

With \( \mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r} \frac{\partial A_z}{\partial \phi} \mathbf{e}_r - \frac{\partial A_z}{\partial r} \mathbf{e}_\phi \) the radial and tangential components of the flux
density in the airgap and magnets for \( np \neq 1 \) become

\[
B_{rI}(r, \phi) = \sum_{n=1,3,5,\ldots}^{\infty} (npX_{nI}r^{np-1} + npY_{nI}r^{-np-1}) \cos(np\phi),
\]

\[
B_{\phi I}(r, \phi) = -\sum_{n=1,3,5,\ldots}^{\infty} (npX_{nI}r^{np-1} - npY_{nI}r^{-np-1}) \sin(np\phi),
\]

\[
B_{rII}(r, \phi) = \sum_{n=1,3,5,\ldots}^{\infty} (npX_{nII}r^{np-1} + npY_{nII}r^{-np-1} + npG_n) \cos(np\phi),
\]

\[
B_{\phi II}(r, \phi) = -\sum_{n=1,3,5,\ldots}^{\infty} (npX_{nII}r^{np-1} - npY_{nII}r^{-np-1} + G_n) \sin(np\phi).
\]

For \( np = 1 \), the radial and tangential components of the flux density in the airgap and magnets become

\[
B_{rI}(r, \phi) = (X_{II} + X_{III}r^{-2}) \cos(\phi),
\]

\[
B_{\phi I}(r, \phi) = -(X_{II} - X_{III}r^{-2}) \sin(\phi),
\]

\[
B_{rII}(r, \phi) = (X_{III} + Y_{III}r^{-2} + G_1\ln(r)) \cos(\phi),
\]

\[
B_{\phi II}(r, \phi) = -(X_{III} - Y_{III}r^{-2} + G_1\ln(r)) \sin(\phi).
\]

The constants \( X_{nI}, Y_{nI}, X_{nII}, Y_{nII}, X_{II}, X_{III}, Y_{III} \) and \( Y_{III} \) are found by satisfying the following boundary conditions:

\[
H_{\phi I}|_{r=r_b} = 0, \quad (4.30)
\]

\[
H_{\phi II}|_{r=r_c} = 0, \quad (4.31)
\]

\[
B_{rI}|_{r=r_m} = B_{rII}|_{r=r_m}, \quad (4.32)
\]

\[
H_{\phi I}|_{r=r_m} = H_{\phi II}|_{r=r_m}, \quad (4.33)
\]

In Appendix B.1 the boundary conditions are further evaluated by substituting the solutions for the flux density given by Eqs. (4.22) to (4.29), into Eqs. (4.30) to (4.33). For numerical calculations the final solutions can be written as
4.2. Magnetic flux density distribution of Halbach array

\[ B_{rI}(r, \phi) = \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{np[N_{nI}]}{(n^2p^2 - 1)[D_n]} \left[ \left( \frac{r}{r_b} \right)^{np-1} \left( \frac{r_m}{r_b} \right)^{np+1} + \left( \frac{r_m}{r} \right)^{np+1} \right] \cos(np\phi), \]  
(4.34)

\[ B_{\phi I}(r, \phi) = -\sum_{n=1, 3, 5, \ldots}^{\infty} \frac{np[N_{nI}]}{(n^2p^2 - 1)[D_n]} \left[ \left( \frac{r}{r_b} \right)^{np-1} \left( \frac{r_m}{r_b} \right)^{np+1} - \left( \frac{r_m}{r} \right)^{np+1} \right] \sin(np\phi), \]  
(4.35)

\[ B_{rII}(r, \phi) = \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{np([N_{nIIr}] + [N_{nII\phi}])}{(n^2p^2 - 1)[D_n]} \left[ \left( \frac{r}{r_m} \right)^{np-1} + \left( \frac{r_c}{r} \right)^{np+1} \left( \frac{r_c}{r_m} \right)^{np-1} \right] \cos(np\phi) \]
\[ + \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{\mu_0 np(M_{nr} - npM_{n\phi})}{n^2p^2 - 1} \left( \frac{r_c}{r} \right)^{np+1} \cos(np\phi) \]
\[ + \sum_{n=1, 3, 5, \ldots}^{\infty} npG_n \cos(np\phi), \]  
(4.36)

\[ B_{\phi II}(r, \phi) = -\sum_{n=1, 3, 5, \ldots}^{\infty} \frac{np([N_{nIIr}] + [N_{nII\phi}])}{(n^2p^2 - 1)[D_n]} \left[ \left( \frac{r}{r_m} \right)^{np-1} - \left( \frac{r_c}{r} \right)^{np+1} \left( \frac{r_c}{r_m} \right)^{np-1} \right] \sin(np\phi) \]
\[ + \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{\mu_0 np(M_{nr} - npM_{n\phi})}{n^2p^2 - 1} \left( \frac{r_c}{r} \right)^{np+1} \sin(np\phi) \]
\[ - \sum_{n=1, 3, 5, \ldots}^{\infty} G_n \sin(np\phi), \]  
(4.37)

where,

\[ [N_{nI}] = \frac{\mu_0}{\mu_r} \left[ (M_{nr} + M_{n\phi})(np - 1) + 2(M_{nr} - npM_{n\phi}) \left( \frac{r_c}{r_m} \right)^{np+1} \right] - (M_{nr} - M_{n\phi})(np + 1) \left( \frac{r_c}{r_m} \right)^{2np}, \]  
(4.38)

\[ [N_{nIIr}] = \frac{\mu_0}{\mu_r} M_{nr} \left[ (\mu_r np - 1) \left( \frac{r_m}{r_b} \right)^{2np} - (\mu_r np + 1) \right. \]
\[ -(\mu_r - 1) \left( \frac{r_c}{r_m} \right)^{np+1} + (\mu_r + 1) \left( \frac{r_c}{r_m} \right)^{np+1} \left( \frac{r_m}{r_b} \right)^{2np} \],  
(4.39)

\[ [N_{nII\phi}] = \frac{\mu_0}{\mu_r} M_{n\phi} \left[ (np - \mu_r) \left( \frac{r_m}{r_b} \right)^{2np} + (np + \mu_r) \right. \]
\[ + np(\mu_r - 1) \left( \frac{r_c}{r_m} \right)^{np+1} - np(\mu_r + 1) \left( \frac{r_c}{r_m} \right)^{np+1} \left( \frac{r_m}{r_b} \right)^{2np} \],  
(4.40)

\[ [D_n] = \left( \frac{\mu_r + 1}{\mu_r} \right) \left( 1 - \left( \frac{r_c}{r_m} \right)^{2np} \right) \]
\[ - \left( \frac{\mu_r - 1}{\mu_r} \right) \left( \left( \frac{r_m}{r_b} \right)^{2np} - \left( \frac{r_c}{r_m} \right)^{2np} \right), \]  
(4.41)
for \( np \neq 1 \) and

\[
B_{rI}(r, \phi) = \frac{[N_{II}]}{[D_1]} \left[ 1 + \left( \frac{r_b}{r} \right)^2 \right] \cos(\phi),
\]

\[
B_{\phi I}(r, \phi) = -\frac{[N_{II}]}{[D_1]} \left[ 1 - \left( \frac{r_b}{r} \right)^2 \right] \sin(\phi),
\]

\[
B_{rII}(r, \phi) = \frac{([N_{IIIc}]-[N_{IIIm}])}{[D_1]} \left[ 1 + \left( \frac{r_c}{r} \right)^2 \right] \cos(\phi)
+ G_1 \left( \ln \left( \frac{r}{r_m} \right) + \left( \frac{r_c}{r} \right)^2 \ln \left( \frac{r_c}{r_m} \right) - 1 \right) \cos(\phi)
- \mu_0 M_1 \phi \cos(\phi),
\]

\[
B_{\phi II}(r, \phi) = -\frac{([N_{IIIc}]-[N_{IIIm}])}{[D_1]} \left[ 1 - \left( \frac{r_c}{r} \right)^2 \right] \sin(\phi)
+ G_1 \left( -\ln \left( \frac{r}{r_m} \right) + \left( \frac{r_c}{r} \right)^2 \ln \left( \frac{r_c}{r_m} \right) \right) \sin(\phi)
- \mu_0 M_1 \phi \left( \frac{r_c}{r} \right)^2 \sin(\phi),
\]

where,

\[
[N_{II}] = \frac{\mu_0}{2\mu_r} \left( (M_{1r} + M_{1\phi}) \left( \frac{r_m}{r_b} \right)^2 - \left( \frac{r_c}{r_b} \right)^2 \right)
+ 2(M_{1r} - M_{1\phi}) \left( \frac{r_c}{r_b} \right)^2 \ln \left( \frac{r_m}{r_c} \right),
\]

\[
[N_{IIIc}] = G_1 \left[ \ln \left( \frac{r_c}{r_m} \right) \left( \left( \frac{\mu_r + 1}{\mu_r} \right) \left( \frac{r_c}{r_b} \right)^2 - \left( \frac{\mu_r - 1}{\mu_r} \right) \left( \frac{r_c}{r_m} \right)^2 \right)
+ 1 - \left( \frac{r_m}{r_b} \right)^2 \right],
\]

\[
[N_{IIIm}] = \mu_0 M_{1\phi} \left[ 1 - \left( \frac{r_m}{r_b} \right)^2 \right],
\]

\[
[D_1] = \left( \frac{\mu_c + 1}{\mu_r} \right) \left( 1 - \left( \frac{r_c}{r_b} \right)^2 \right)
\]
\[
- \left( \frac{\mu_r - 1}{\mu_r} \right) \left( \left( \frac{r_m}{r_b} \right)^2 - \left( \frac{r_c}{r_m} \right)^2 \right),
\]

for \( np = 1 \).

The solutions for the flux density are implemented in MATLAB and for the dimension of the initial design of the rotary actuator proposed by Wijdevn B.V. (given in Appendix A), the results for the airgap flux density \( (r = r_g = (r_b + r_m)/2) \) were compared with FEM. Figure 4.5 shows excellent agreement between the analytical and FE-model.

For a fixed airgap length and stator radius \( (r_b=27.5 \text{ mm} \text{ and } r_m=25.2 \text{ mm}) \) the influence of the core radius to magnet radius ratio, \( r_c/r_m \), on the fundamental harmonic of the airgap...
4.2. Magnetic flux density distribution of Halbach array

flux density for different number of magnet poles is shown in Fig. 4.6a. For low values of \( r_c/r_m \), i.e. large length of the magnets relative to the rotor radius, the airgap flux density decreases significantly for higher pole counts. It can also be observed, that the magnitude of the flux density in the airgap remains constant for low values of, \( r_c/r_m \).

More interesting to see is the influence of the variable magnet length versus the airgap length for fixed shaft and stator radii. In Fig. 4.6b the fundamental harmonic of the airgap flux density is shown as function of the magnet radius to back-iron radius ratio, \( r_m/r_b \), when \( r_b=27.5 \text{ mm}, r_c=19.5 \text{ mm} \) and \( \alpha=2/3 \). Clearly, the airgap flux density increases for larger values of the magnet length and decreasing airgap. For \( r_m/r_b<0.95 \), a lower magnet pole count gives a higher magnetic loading. Such values of \( r_m/r_b \) would be relevant for slotless machines, since they generally have a large airgap. In slotless machines, the benefit of a larger magnetic loading with increasing magnet length and smaller airgap is compromised by a lower electrical loading, as less windings can be placed in a smaller airgap. Slotted machines have a small airgap so \( r_m/r_b \) approaches unity. In this case a larger pole count would increase the magnetic loading.

The influence of variable magnet arc to pole arc ratio, \( \alpha \), for fixed dimensions of the motor is shown in Fig. 4.6c and d. The latter figure shows a minimum value of the total harmonic distortion (THD) for all pole counts when \( \alpha=0.5 \). Here, the third harmonic of the flux density is zero, while the fifth harmonic shows its peak value. At the same time, for higher magnet pole counts the magnitude of the first harmonic shows its peak value, as can be seen in Fig. 4.6c. Higher pole counts have a lower THD, so a more sinusoidal flux density can be expected, which in turn gives lower torque ripple. However, machines having a fractional number of the number of coils per slot per phase (\( q<1 \)) filter higher harmonics in the back-EMF. To illustrate this, the THD as function of the magnet arc to pole arc ratio again is shown in Fig. 4.7, but now every harmonic of the flux density is corrected with the winding factor. Basically, this figure shows the total harmonic distortion of the back-EMF.
Figure 4.6: Influence of design parameters on the magnetic loading in the airgap: (a) airgap flux density for variable core radius to magnet radius ratio, $r_c/r_m$ ($r_m=25.2$ mm, $r_b=27.5$ mm and $\alpha=2/3$), (b) airgap flux density for variable magnet radius to back-iron radius ratio, $r_m/r_b$ ($r_c=19.5$ mm, $r_b=27.5$ mm and $\alpha=2/3$), (c) airgap flux density for variable radial magnet arc to magnet pole arc ratio, $\alpha$ ($r_c=19.5$ mm, $r_m=25.2$ mm and $r_b=27.5$ mm), (d) total harmonic distortion of airgap flux density for variable $\alpha$. 

Multi-physical framework
4.3 Torque calculation

A general method of calculating the torque produced by an actuator is using Maxwell stress tensor [23]. In the case of a slotless structure, using the Lorentz force is a more convenient way to determine the torque. Lorentz force is a force acting on a steady current imparted by an external B-field. Figure 4.8 shows the permanent magnet actuators including the coils and the torque on a current distribution $J$ inside the coils volume, $V$, is calculated according to

$$ T = \int_V r \times (J \times B_{\text{ext}})dV = \int_V r \times (J \times B_1) dV, $$

(4.50)

where $r$ is the vector to the point about which the torque is computed and $B_1$ is the magnetic flux density inside the airgap region. Considering only current density in the axial direction in the airgap region and radial and tangential components of the magnetic flux density, the torque only has a component about the $z$-axis

$$ T_z = \int_V J_z B_1 r_dV. $$

(4.51)

For a coil having $N$ conductors, all carrying a current $I$, the current density $J_z$ can be expressed as

$$ J_z = \frac{NI}{A_{\text{bundle}}}. $$

(4.52)

$A_{\text{bundle}}$ is the cross-sectional area of one bundle of a coil given by

$$ A_{\text{bundle}} = \frac{1}{2}(r_{co}^2 - r_{co}^2)(\alpha_c - \beta_o), $$

(4.53)

where $\alpha_c$ is the total coil span and $\beta_o$ is the coil opening angle.
Given the current density, the torque is only calculated along the active length of the magnets, $L_{act}$. For a single coil $i$, with offset angle $\theta_i$, the integral of Eq. (4.51) needs to be evaluated over both bundles of the coil, giving

$$T_{z,i} = J_z \left( \int_{V_+} B_{r1}rdv - \int_{V_-} B_{r1}rdv \right),$$

$$= J_z L_{act} \left( \int_{A_+} B_{r1}rdv - \int_{A_-} B_{r1}rdv \right),$$

$$= J_z L_{act} \left( \int_{R_{ci}}^{R_{co}} \int_{-\alpha_c}^{-\beta_o} B_{r1}(r, \theta - \theta_i)r^2d\theta dr - \int_{R_{ci}}^{R_{co}} \int_{\beta_o}^{\alpha_c} B_{r1}(r, \theta - \theta_i)r^2d\theta dr \right),$$ (4.54)

where $A_+$ and $A_-$ are the cross-sectional areas of both bundles.

By using the identity

$$\sin(np(\alpha_c - \Delta \theta + \theta_i)) - \sin(np(\beta_o - \Delta \theta + \theta_i))$$
$$- \sin(np(-\beta_o - \Delta \theta + \theta_i)) + \sin(np(-\alpha_c - \Delta \theta + \theta_i))$$
$$= 2(\cos(np\beta_o) - \cos(np\alpha_c)) \sin(np(\Delta \theta - \theta_i)),$$

the torque on coil $i$ can be expressed as

$$T_{z,i} = - \sum_{n=1,3,5,}^{\infty} J_z,i L_{act} B_{n,e} R_{n,e} \sin(np(\Delta \theta - \theta_i)).$$ (4.55)
4.3. Torque calculation

For \( np \neq 1 \)

\[
B_{n,e} = \frac{np|N_{nI}|}{(n^2p^2 - 1)|D_n|}, \quad (4.56)
\]

\[
R_{n,e} = \frac{2}{np} \left[ \cos(np\beta_o) - \cos(n\alpha_{e}) \right] \left[ \frac{1}{np + 2} \left( \frac{R_m}{R_s} \right)^{np+1} \left( \frac{R_c}{R_s} \right)^{np-1} \left( \frac{R_{ci}}{R_{ei}} \right)^{np-1} \left( \frac{R_{ci}}{R_{ci}} \right)^{np+1} \right] , \quad (4.57)
\]

where \([N_{nI}]\) and \([D_n]\) are given by Eqs. (4.38) and (4.41), respectively. For \( np=1 \)

\[
B_{n,e} = \frac{|N_{1I}|}{|D_1|}, \quad (4.58)
\]

\[
R_{n,e} = \frac{2}{3} \left[ \cos(\beta_o) - \cos(\alpha_e) \right] \left[ R_{c}^3 - R_{c_t}^3 + 3R_s^2R_{c} - 3R_s^2R_{c_t} \right] , \quad (4.59)
\]

where \([N_{1I}]\) and \(D_1\) are given by Eqs. (4.46) and (4.49), respectively.

For a 3-phase machine with wye-connection, the phase current densities can be written as

\[
J_A = J \sin(p\Delta\theta), \quad (4.60)
\]

\[
J_B = J \sin(p\Delta\theta - \frac{2\pi}{3}), \quad (4.61)
\]

\[
J_C = J \sin(p\Delta\theta + \frac{2\pi}{3}). \quad (4.62)
\]

The total torque of the slotless permanent magnet actuator can then be calculated according to

\[
T_{\text{Lorentz}} = T_A + T_B + T_C
\]

\[
= - \sum_{ixC_A} \sum_{n=1,3,5,..}^{\infty} L_{act}B_{n,e}R_{n,e,J} \sin(p\Delta\theta) \sin(np(\Delta\theta - \theta_i))
\]

\[
- \sum_{ixC_B} \sum_{n=1,3,5,..}^{\infty} L_{act}B_{n,e}R_{n,e,J} \sin(p\Delta\theta - \frac{2\pi}{3}) \sin(np(\Delta\theta - \theta_i))
\]

\[
- \sum_{ixC_C} \sum_{n=1,3,5,..}^{\infty} L_{act}B_{n,e}R_{n,e,J} \sin(p\Delta\theta + \frac{2\pi}{3}) \sin(np(\Delta\theta - \theta_i)). \quad (4.63)
\]

For a combination of the number of magnet poles and coils resulting in \( q=1/4 \), all coils in a phase are electrically aligned. The common (mechanical) angle offset \( \theta_i \) for each phase is

\[
\theta_A = 0,
\]

\[
\theta_B = -\frac{\pi}{6},
\]

\[
\theta_C = \frac{\pi}{6}.
\]
Considering only the first harmonic of the magnetic field the total torque becomes

\[
T_{\text{Lorentz}} = -\frac{Q}{3} L_{\text{act}} B_{1,e} R_{1,e} J \left( \sin^2(p\Delta\theta) + \sin^2(p\Delta\theta - \frac{2\pi}{3}) + \sin^2(p\Delta\theta + \frac{2\pi}{3}) \right),
\]

\[
= \frac{Q}{2} L_{\text{act}} B_{1,e} R_{1,e} J. \tag{4.64}
\]

Figure 4.9 shows the torque for the initial design determined according to Eq. (4.63). The analytical model shows excellent agreement with FE-simulation.

### 4.4 Stationary and moving circuits

An electro-mechanical devices converts electrical energy to mechanical energy or vice versa. Such a device can be split into three subsystems: electrical, magnetic and the mechanical subsystem. The interconnection from the magnetic to the electrical subsystem follows from the quasi-static field theory and can be obtained for stationary and moving circuits. The objective of this section is to show that for this application both stationary coils and moving coils result in the same electrical behavior when the same conditions are considered. These conditions are that the relative (rotational) motion of the stationary member to the moving member is the same in both cases and all dimensions are equal. In the following subsections the electrical behavior for stationary and moving circuits in a cylindrical actuator will be analyzed for a single coil placed in a sinusoidal magnetic field with amplitude \( \tilde{B} \), where the flux density (only radial component) is given by

\[
B_{\text{ext}} = \tilde{B} \cos(p(\phi))e_r \tag{4.65}
\]

Figure 4.10 shows the single coil placed above the magnets. An electrical circuit consisting a voltage source \( V_s \) and a resistance \( R \), is attached to the coil.
4.4. Stationary and moving circuits

4.4.1 Stationary coil

To analyze the electrical circuit for a quasi-static field, the following integral along a contour $C$ needs to be evaluated [23]

$$
\oint_C \mathbf{E} \cdot d\mathbf{l} = - \oint_C \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l} - \oint_C \nabla \varphi \cdot d\mathbf{l}.
$$

(4.66)

For the single coil problem as is shown in Fig. 4.10 this integral becomes

$$
V_s = i(R + R_{\text{coil}}) + M \frac{di}{dt} + \omega \frac{\partial \Lambda(\Delta \theta)}{\partial \Delta \theta},
$$

(4.67)

where $R_{\text{coil}}$ and $M$ are the resistance and the inductance of the coil, respectively. The last term on the right hand side of the equation is the voltage induced by a time varying flux linked by the coil, $\Lambda$, and is referred to as the electro-motive force or back-EMF. The flux linkage of a coil in a moving magnetic field, $\mathbf{B}$, is determined according to

$$
\Lambda = \int_{\text{coil}} \mathbf{B} \cdot d\mathbf{S}.
$$

(4.68)

When a magnetically linear system is considered, $\mathbf{B}$ can be written as a superposition of two fields: $\mathbf{B}_i$ is due to the current $i$ in the coil and $\mathbf{B}_{\text{ext}}$ is the field produced by the magnets. Only $\mathbf{B}_{\text{ext}}$ depends on $\Delta \theta$, so for now it suffices to evaluate Eq. (4.68) as follows

$$
\Lambda = \int_{\text{coil}} \mathbf{B}_{\text{ext}} \cdot d\mathbf{S} = \int_0^{L_{\text{act}}} \int_{-\alpha_t}^{\alpha_t} \hat{B} \cos(p\phi) r \, d\phi \, dl,
$$

(4.69)

$$
= \frac{2}{p} r L_{\text{act}} \hat{B} \sin(p\alpha_t) \cos(p\Delta \theta),
$$

(4.70)

where $r$ is the radius of the coil and $L_{\text{act}}$ the active axial length of the magnets. The back-EMF, therefore, becomes

$$
e = \omega \frac{\partial \Lambda(\Delta \theta)}{\partial \Delta \theta} = -2\omega r L_{\text{act}} \hat{B} \sin(p\alpha_t) \sin(p\Delta \theta).
$$

(4.71)
Similarly, the inductance can be obtained as

\[
M = \frac{\partial \Lambda(i)}{\partial i} = \frac{\partial}{\partial i} \int_{\text{coil}} (B_1(i) + B_{\text{ext}}(\Delta \theta)) \cdot dS = \frac{\partial}{\partial i} \int_{\text{coil}} B_1(i) \cdot dS. \tag{4.72}
\]

This integral will not be evaluated for now, but the equation shows that the inductance only depends on the field produced by the coil itself.

### 4.4.2 Moving coil

For a coil moving through a static (time constant) magnetic field, the electrical equation becomes [23]

\[
V_s = i(R + R_{\text{coil}}) + M \frac{di}{dt} - \int_{\text{coil}} [(\omega \times r) \times B_{\text{ext}}] \cdot dl. \tag{4.73}
\]

Again, the last term is referred to as the back-EMF. With the rotation fixed to the z-axis, \( \omega = \omega e_z \) and \( r = r e_r \), where \( r \) is the distance of the conductors of the coil perpendicular to the z-axis. The line integral is evaluated only along the active length of the magnets (\( B_{\text{ext}} \neq 0 \)) and can be split for the two conductors of the coil:

1. The first conductor is placed at \( \theta = -\alpha_t \) and the integral is in positive z-direction with \( dl = dlez \), so

\[
- \int_{0}^{L_{\text{act}}} [(\omega e_z \times r e_r) \times \hat{B} \cos(p(-\alpha_t - \Delta \theta)) e_r] \cdot dlez = \omega r L_{\text{act}} \hat{B} \cos(p(\alpha_t + \Delta \theta)). \tag{4.74}
\]

2. The second conductor is placed at \( \theta = \alpha_t \) and the integral is in negative z-direction with \( dl = dlez \), so

\[
- \int_{0}^{L_{\text{act}}} [(\omega e_z \times r e_r) \times \hat{B} \cos(p(\alpha_t - \Delta \theta)) e_r] \cdot dlez = -\omega r L_{\text{act}} \hat{B} \cos(p(\alpha_t - \Delta \theta)). \tag{4.75}
\]

Summing both results gives

\[
e = -2\omega r L_{\text{act}} \hat{B} \sin(p\alpha_c) \sin(p\Delta \theta), \tag{4.76}
\]

which is the same result for the back-EMF obtained for a stationary coil. The inductance for moving coils is defined as [23]

\[
M = \frac{\partial \Lambda_{\text{coil}}(i)}{\partial i} = \frac{\partial}{\partial i} \int_{\text{coil}} B_1(i) \cdot dS, \tag{4.77}
\]

which is similar to the inductance for the stationary coil.

It can be concluded that the electrical equations for both stationary and moving coils are the same (if the resistance of the coil is constant). Therefore, the method for calculating the back-EMF and inductance for the case of fixed coils and moving magnets, can be used for the case where the coils are moved.
Back-EMF calculation

Predicting the back-EMF of the slotless machine is less straightforward than for slotted machine, because not all windings in a coil in a slotless machine have the same flux linkage. Although the total back-EMF can be determined by calculating the back-EMF for every single winding in a coil, when designing an electric machine the number of turns in the coil might still be unknown. Therefore, in this section the average back-EMF of a turn in a coil will be determined. Once the total number of turns is known, the total back-EMF of the actuator can be determined multiplying the average back-EMF with the number of turns. The structure as is shown Fig. 4.11 is used in the analysis.

Consider a single turn with a cross-sectional area $dA$ running somewhere in coil $i$ (with offset angle $\theta_i$) at radius $r$ and spanning $2\alpha_i$, which is shown in Fig. 4.11. The flux linked by this turn can be expressed according to

$$
\Psi_{t,dA}(\alpha_i, r) = \int_{S_i} B_1 dS = L_{act} \int_{-\alpha_i}^{\alpha_i} B_{rI}(r, \theta + \theta_i) rd\theta,
$$

where $L_{act}$ is the active length of the actuator and $B_{rI}$ is the radial component of the open field flux density in the airgap. Using Eqs. (4.34) and (4.42) and evaluating this integral gives

$$
\Psi_{t,dA}(\alpha_i, r) = \sum_{n=1,3,5,...}^{\infty} \frac{2L_{act}[N_{gI}]}{(n^2p^2 - 1)[D_n]} \left[ \frac{r}{R_s} \right]^{np-1} \left( \frac{R_m}{R_s} \right)^{np+1} + \left( \frac{R_m}{r} \right)^{np+1} \cdot r \sin(np\alpha_i) \cos(np(\Delta\theta - \theta_i)),
$$

Figure 4.11: Schematic of the slotless permanent magnet motor including the coils.
for $np \neq 1$ and

$$\Psi_{t,dA}(\alpha_t, r) = 2L_{act} \frac{[N_{1l}]}{[D_1]} \left[ 1 + \left( \frac{R_s}{r} \right)^2 \right] \sin(\alpha_t) \cos(\Delta \theta - \theta_i), \quad (4.80)$$

for $np = 1$.

By assuming that the coil contains a large number of turns like the one described above, the total flux linkage of this coil becomes

$$\Psi = \int_{S_{bundle}} \Psi_{t,dA}(\alpha_t, r) dA = \int_{r_{ci}}^{r_{co}} \int_{\beta_b}^{\alpha_c} \Psi_{t,dA}(\alpha_t, r) r d\alpha_t dr. \quad (4.81)$$

For $np \neq 1$ the total flux linkage becomes

$$\Psi = \sum_{n=1,3,5,..}^{\infty} \frac{2L_{act}}{np} \frac{[N_{nI}]}{(n^2 p^2 - 1) [D_n]} \left[ \cos(np\beta_o) - \cos(np\alpha_c) \right]$$

$$\cos(np(l::(} - (}i)) \right] \cos(n^{\cdot} \alpha_c). \quad (4.82)$$

For $np = 1$ the total flux linkage becomes

$$\Psi = \frac{2L_{act}}{3} \frac{[N_{1l}]}{[D_1]} \left[ r_{co}^3 - r_{ci}^3 + 3r_{co}^2 r_{co} - 3r_{ci}^2 r_{ci} \right]$$

$$\cdot \left[ \cos(np\beta_o) - \cos(np\alpha_c) \right] \cos(\Delta \theta - \theta_i). \quad (4.83)$$

The average flux linkage per turn can be calculated as

$$\Psi_{av} = \frac{\Psi}{A_{bundle}}, \quad (4.84)$$

where $A_{bundle}$ is the cross-sectional area of one bundle of the coil given by Eq. 4.53.

The average back-EMF for a turn in coil $i$ can now be expressed as

$$e_{av,i} = -\frac{d\Psi_{av}}{dt} = -\frac{\partial \Psi_{av}}{\partial t}, \quad (4.85)$$

where $\omega$ is the mechanical angular velocity of the moving member. By using Eq. (4.84) the average back-EMF can be further evaluated

$$e_{av,i} = \sum_{n=1,3,5,..}^{\infty} \omega \Psi_n \sin(np(\Delta \theta - \theta_i)), \quad (4.86)$$

where

$$\Psi_n = \frac{B_{n,c} R_{n,e} L_{st}}{A_{bundle}}. \quad (4.87)$$
4.6. Armature field

Figure 4.12: Phase back-EMF for the initial design of the slotless PM actuator with $\omega = 90$ rad s$^{-1}$.

is the amplitude of the flux linkage. For $n_p \neq 1$, $B_{n,p}$ and $R_{n,p}$ are given by Eqs. (4.56) and (4.57), respectively and for $n_p = 1$ they are given by Eqs. (4.58) and (4.59).

The total phase back-emf is obtained by summing the back-EMF of all coils in the phase, each having $N$ turns, can be calculated as

$$e_{ph} = \frac{Q}{3} \sum_{i} N e_{av,i}$$

Figure 4.12 shows the phase back-EMFs for the slotless permanent magnet actuator with the dimensions of the initial design, containing 92 turns per coil and $\omega = 90$ rad s$^{-1}$. The results show good agreement with FEM.

4.6 Armature field

To calculated the inductance of the concentrated windings inside the slotless PM actuator and to determine the demagnetization of the magnets, the magnetic field, or armature field, due to the current in the coils needs to be determined. To find a solution for the armature field, first a description of the current density for a certain winding layout needs to found. Using the same technique which is used to find the magnetic field due to the magnets, which is described in [24], the description for the armature is found.

4.6.1 Current density distribution

The current density inside the airgap of the slotless PM actuator with a winding layout for $q = 1/4$, is shown in Fig. 4.13. The current density distribution in axial direction can be described by

$$J_{dist} = \{J_A(\Delta \theta)K_{wA}(\theta) + J_B(\Delta \theta)K_{wB}(\theta) + J_C(\Delta \theta)K_{wC}(\theta)\} e_z.$$ (4.88)
where $J_A$, $J_B$ and $J_C$ are the current densities for the different phases and are given by Eqs. (4.60), (4.61) and (4.62). $K_{wA}$, $K_{wB}$ and $K_{wC}$ are the winding distribution functions for the phases, respectively. For a machine with $q=1/4$ the winding distribution functions can be represented by Fourier series

\[
K_{wA}(\theta) = \sum_{n=1}^{\infty} K_{wn} \sin(n \theta),
\]

\[
K_{wB}(\theta) = \sum_{n=1}^{\infty} K_{wn} \sin(n \left(\theta - \frac{2\pi}{Q}\right)),
\]

\[
K_{wC}(\theta) = \sum_{n=1}^{\infty} K_{wn} \sin(n \left(\theta + \frac{2\pi}{Q}\right)),
\]

where

\[
K_{wn} = \frac{2}{\pi n} \left[\cos(n \pi/Q) - \cos(n \beta_0)\right].
\] (4.89)

Here, $\nu = \frac{m}{Q}$ is the spatial frequency, where $m=3$ is the number of phases.

### 4.6.2 General solution of the armature field

When predicting the armature field, the same assumptions as for predicting the magnetic field due to the magnets, are made. However, now no magnet material is assumed and the space will be replaced by air. This will not result in a significant error in the calculation of the magnetic field, since the armature field is only in the order of a couple of percent of the field produced by the magnets [25]. In case of the moving coil configuration, for simplicity, the coils are assumed to be attached to the back-iron.

The inductance needs to be determined for the part of the coils which is next to back-iron and the part which is in mid-air. The last part consist of straight windings and end-windings.

Again, a solution for the vector potential, which is introduced by Eq. 4.2, needs to be found. However, now three regions will be considered as is illustrated in Fig. 4.15. Region III, ranging from $r_{ci} \leq r \leq r_b$, contains the current density carried by the coils and has permeability equal to free space, $\mu_0$. Region IV is air region and for the part of the coil next to the back-iron it ranges from $r_c \leq r \leq r_{ci}$ and in mid-air it ranges from $0 \leq r \leq r_{ci}$. Region V is also a air region and ranges from $r_b \leq r \leq \infty$. 

![Figure 4.13: Current density distribution.](image)

\[\text{Figure 4.13: Current density distribution.}\]
4.6. Armature field

Figure 4.14: Side view of the structure of the rotary actuator for calculation of the armature field.

Figure 4.15: Schematic of the slotless permanent magnet motor without the magnet material.
The Poisson equation for the coil region and the Laplace equations for the air regions are given as

\[
\nabla^2 \mathbf{A}_{\text{III}} = -\mu_0 \mathbf{J}_{\text{dist}}, \tag{4.90}
\]

\[
\nabla^2 \mathbf{A}_{\text{IV}} = 0, \tag{4.91}
\]

\[
\nabla^2 \mathbf{A}_{\text{V}} = 0. \tag{4.92}
\]

The vector potential is described as function of \( r \) and \( \theta \) and because of the three phase current distribution, the following (homogeneous) solution for the vector potential in region IV and V is proposed

\[
\mathbf{A}_{\text{IV}}(r, \theta) = \sum_{n=1}^{\infty} \left( X_{AnIV} r^{\nu} + Y_{AnIV} r^{-\nu} \right) \sin(\nu \theta) \mathbf{e}_z \\
+ \sum_{n=1}^{\infty} \left( X_{BnIV} r^{\nu} + Y_{BnIV} r^{-\nu} \right) \sin(\nu(\theta + \frac{2\pi}{Q})) \mathbf{e}_z \\
+ \sum_{n=1}^{\infty} \left( X_{CnIV} r^{\nu} + Y_{CnIV} r^{-\nu} \right) \sin(\nu(\theta - \frac{2\pi}{Q})) \mathbf{e}_z, \tag{4.93}
\]

\[
\mathbf{A}_{\text{V}}(r, \theta) = \sum_{n=1}^{\infty} \left( X_{AnV} r^{\nu} + Y_{AnV} r^{-\nu} \right) \sin(\nu \theta) \mathbf{e}_z \\
+ \sum_{n=1}^{\infty} \left( X_{BnV} r^{\nu} + Y_{BnV} r^{-\nu} \right) \sin(\nu(\theta + \frac{2\pi}{Q})) \mathbf{e}_z \\
+ \sum_{n=1}^{\infty} \left( X_{CnV} r^{\nu} + Y_{CnV} r^{-\nu} \right) \sin(\nu(\theta - \frac{2\pi}{Q})) \mathbf{e}_z. \tag{4.94}
\]

The solution for the region III is

\[
\mathbf{A}_{\text{III}}(r, \theta) = \sum_{n=1}^{\infty} \left( X_{AnIII} r^{\nu} + Y_{AnIII} r^{-\nu} \right) \sin(\nu \theta) \mathbf{e}_z \\
+ \sum_{n=1}^{\infty} \left( X_{BnIII} r^{\nu} + Y_{BnIII} r^{-\nu} \right) \sin(\nu(\theta + \frac{2\pi}{Q})) \mathbf{e}_z \\
+ \sum_{n=1}^{\infty} \left( X_{CnIII} r^{\nu} + Y_{CnIII} r^{-\nu} \right) \sin(\nu(\theta - \frac{2\pi}{Q})) \mathbf{e}_z \\
+ \mathbf{A}_{\text{part}}, \tag{4.95}
\]

The particular solution to equation (4.90) can be written in the form

\[
\mathbf{A}_{\text{part}}(r, \theta) = \sum_{n=1,3,5,\ldots}^{\infty} F_{An} r^{2} \sin(\nu \theta) \mathbf{e}_z \\
+ \sum_{n=1,3,5,\ldots}^{\infty} F_{Bn} r^{2} \sin(\nu(\theta + \frac{2\pi}{Q})) \mathbf{e}_z \\
+ \sum_{n=1,3,5,\ldots}^{\infty} F_{Cn} r^{2} \sin(\nu(\theta - \frac{2\pi}{Q})) \mathbf{e}_z. \tag{4.96}
\]
By substituting this solution and the description of current density distribution given by Eq. (4.88) into Eq. (4.90), constants $F_{An}$, $F_{Bn}$ and $F_{Cn}$ are obtained

\[ F_{An} = \mu_0 \frac{J_A K_{wn}}{(\nu^2 - 4)}, \]  
\[ F_{Bn} = \mu_0 \frac{J_B K_{wn}}{(\nu^2 - 4)}, \]  
\[ F_{Cn} = \mu_0 \frac{J_C K_{wn}}{(\nu^2 - 4)}. \]

This particular solution is not valid for the case when $\nu = 2$. For this special case, let $r = e^t$, so Eq. (4.90) becomes

\[ \frac{\partial^2}{\partial t^2} A_{\text{part}}(r, \theta) + \frac{\partial^2}{\partial \theta^2} A_{\text{part}}(r, \theta) = -\mu_0 e^{2t} J_{\text{dist}}. \]  

The particular solution can therefore be written in the form

\[ A_{\text{part}}(r, \theta) = F_{A2} r^2 \ln(r) \sin(2\theta)e_z + F_{B2} r^2 \ln(r) \sin\left(2\theta + \frac{2\pi}{Q}\right)e_z + F_{C2} r^2 \ln(r) \sin\left(2\theta - \frac{2\pi}{Q}\right)e_z. \]

Constants $F_{A2}$, $F_{B2}$ and $F_{C2}$ are obtained by substituting this solution into Eq. (4.100)

\[ F_{A2} = -\mu_0 \frac{J_A K_{wn}}{4}, \]  
\[ F_{B2} = -\mu_0 \frac{J_B K_{wn}}{4}, \]  
\[ F_{C2} = -\mu_0 \frac{J_C K_{wn}}{4}. \]

The total particular solution for $\nu = 2$ then becomes

\[ A_{\text{part}}(r, \theta) = F_{A2} r^2 \ln(r) \sin(2\theta)e_z + F_{B2} r^2 \ln(r) \sin\left(2\theta + \frac{2\pi}{Q}\right)e_z + F_{C2} r^2 \ln(r) \sin\left(2\theta - \frac{2\pi}{Q}\right)e_z. \]

Since the solution for the vector potential contains separate solutions to the three different phases which all have the same form, only one general solution to a single phase will be further considered. The solution for all three phases are obtained simply by shifting the general function with the appropriate angle and using the right constant for the particular solution. Reminding Eq. (4.2), the radial and tangential components of the flux density in region III,
IV and V become

\[
B_{rIII}(r, \theta) = \sum_{n=1,3,5}^{\infty} (\nu X_{nIII} r^{\nu-1} + \nu Y_{nIII} r^{\nu-1} + \nu F_n r) \cos(\nu \theta), \tag{4.106}
\]

\[
B_{\thetaIII}(r, \theta) = -\sum_{n=1,3,5}^{\infty} (\nu X_{nIII} r^{\nu-1} - \nu Y_{nIII} r^{\nu-1} + 2F_n r) \sin(\nu \theta), \tag{4.107}
\]

\[
B_{rIV}(r, \theta) = \sum_{n=1,3,5}^{\infty} (\nu X_{nIV} r^{\nu-1} + \nu Y_{nIV} r^{-\nu-1}) \cos(\nu \theta), \tag{4.108}
\]

\[
B_{\thetaIV}(r, \theta) = -\sum_{n=1,3,5}^{\infty} (\nu X_{nIV} r^{\nu-1} - \nu Y_{nIV} r^{-\nu-1}) \sin(\nu \theta), \tag{4.109}
\]

\[
B_{rV}(r, \theta) = \sum_{n=1,3,5}^{\infty} (\nu X_{nV} r^{\nu-1} + \nu Y_{nV} r^{-\nu-1}) \cos(\nu \theta), \tag{4.110}
\]

\[
B_{\thetaV}(r, \theta) = -\sum_{n=1,3,5}^{\infty} (\nu X_{nV} r^{\nu-1} - \nu Y_{nV} r^{-\nu-1}) \sin(\nu \theta), \tag{4.111}
\]

for \(\nu \neq 2\) and

\[
B_{rIII}(r, \theta) = (2X_{2III} r + 2Y_{2III} r^{-3} + 2F_2 r \ln(r)) \cos(2\theta), \tag{4.112}
\]

\[
B_{\thetaIII}(r, \theta) = -(2X_{2III} r - 2Y_{2III} r^{-3} + 2F_2 r \ln(r) + F_2 r) \sin(2\theta), \tag{4.113}
\]

\[
B_{rIV}(r, \theta) = (2X_{2IV} r + 2Y_{2IV} r^{-3}) \cos(2\theta), \tag{4.114}
\]

\[
B_{\thetaIV}(r, \theta) = -(2X_{2IV} r - 2Y_{2IV} r^{-3}) \sin(2\theta), \tag{4.115}
\]

\[
B_{rV}(r, \theta) = (2X_{2V} r + 2Y_{2V} r^{-3}) \cos(2\theta), \tag{4.116}
\]

\[
B_{\thetaV}(r, \theta) = -(2X_{2V} r - 2Y_{2V} r^{-3}) \sin(2\theta). \tag{4.117}
\]

for \(\nu = 2\).

The coefficients in the solutions of the magnetic fields need to be found for the part of the coils which is next to the back-iron and for the part which is in mid-air. This leads to different expressions for the coefficients, which will be obtained in the following two subsections.

### 4.6.3 Armature field for coils next to the back-iron

For the part of the coils which are next to the back-iron only the coefficients for region III and IV need to be obtained. These coefficients \((X_{nIII}, Y_{nIII}, X_{nIV}, Y_{nIV}, X_{2III}, Y_{2III}, X_{2IV}, Y_{2IV})\) are obtained by satisfying the field solutions to the following boundary conditions

\[
H_{\thetaIII}|_{r=r_h} = 0, \tag{4.118}
\]

\[
H_{\thetaIV}|_{r=r_c} = 0, \tag{4.119}
\]

\[
B_{rIII}|_{r=r_c} = B_{rIV}|_{r=r_c}, \tag{4.120}
\]

\[
H_{\thetaIII}|_{r=r_c} = H_{\thetaIV}|_{r=r_c}. \tag{4.121}
\]

In Appendix B.2 the boundary conditions are further evaluated by substituting the solutions for the flux density given by Eqs. (4.106) to (4.115) into Eqs. (4.118) to (4.121). For numerical
4.6. Armature field

calculations the total field solutions become

\[
B_{rIII}(r, \theta) = \sum_{n=1}^{\infty} F_n r c_i \frac{[N_{nI}]}{2[D_n]} \left[ \left( \frac{r}{r_c} \right)^{\nu - 1} \left( \frac{r_c}{r_c} \right)^{\nu + 1} + \left( \frac{r_c}{r} \right)^{\nu + 1} \left( \frac{r_b}{r_c} \right)^{2\nu} \right] \cos(\nu \theta)
\]

\[
+ \sum_{n=1}^{\infty} 2F_n r_b \left( \frac{r_b}{r} \right)^{\nu + 1} \cos(\nu \theta)
\]

\[
+ \sum_{n=1}^{\infty} \nu F_n r \cos(\nu \theta), \quad (4.122)
\]

\[
B_{\theta III}(r, \theta) = - \sum_{n=1}^{\infty} F_n r c_i \frac{[N_{nI}]}{2[D_n]} \left[ \left( \frac{r}{r_c} \right)^{\nu - 1} \left( \frac{r_c}{r_c} \right)^{\nu + 1} - \left( \frac{r_c}{r} \right)^{\nu + 1} \left( \frac{r_b}{r_c} \right)^{2\nu} \right] \sin(\nu \theta)
\]

\[
+ \sum_{n=1}^{\infty} 2F_n r_b \left( \frac{r_b}{r} \right)^{\nu + 1} \sin(\nu \theta)
\]

\[
- \sum_{n=1}^{\infty} \nu F_n r \sin(\nu \theta), \quad (4.123)
\]

\[
B_{rIV}(r, \theta) = \sum_{n=1}^{\infty} F_n r c_i \frac{[N_{nII}]}{2[D_n]} \left[ \left( \frac{r}{r_c} \right)^{\nu - 1} \left( \frac{r_c}{r_c} \right)^{\nu + 1} + \left( \frac{r_c}{r} \right)^{\nu + 1} \right] \cos(\nu \theta), \quad (4.124)
\]

\[
B_{\theta IV}(r, \theta) = - \sum_{n=1}^{\infty} F_n r c_i \frac{[N_{nII}]}{2[D_n]} \left[ \left( \frac{r}{r_c} \right)^{\nu - 1} \left( \frac{r_c}{r_c} \right)^{\nu + 1} - \left( \frac{r_c}{r} \right)^{\nu + 1} \right] \sin(\nu \theta), \quad (4.125)
\]

where,

\[
[N_{nI}] = (\nu - 2) - (\nu + 2)\left( \frac{r_c}{r_c} \right)^{2\nu} + 4 \left( \frac{r_b}{r_c} \right)^{\nu + 2},
\]

\[
[N_{nII}] = (\nu - 2) - (\nu + 2)\left( \frac{r_b}{r_c} \right)^{2\nu} + 4 \left( \frac{r_b}{r_c} \right)^{\nu + 2},
\]

\[
[D_n] = 1 - \left( \frac{r_b}{r_c} \right)^{2\nu},
\]
for $\nu \neq 2$ and

\[
B_{rIII}(r, \theta) = F_{2rb} \frac{[N_{2I}]}{2[D_2]} \left[ \left( \frac{r}{r_b} \right) + \left( \frac{r_b}{r} \right)^3 \right] \cos(2\theta) + F_{2rb} \left( \frac{r_b}{r} \right)^3 (2\ln(r_b) + 1) \cos(2\theta) + 2F_{2rln}(r) \cos(2\theta),
\]

\[
B_{\theta III}(r, \theta) = -F_{2rb} \frac{[N_{2I}]}{2[D_2]} \left[ \left( \frac{r}{r_b} \right) - \left( \frac{r_b}{r} \right)^3 \right] \sin(2\theta) + 2F_{2rl}(r) \cos(2\theta),
\]

\[
B_{rIV}(r, \theta) = F_{2rc} \frac{[N_{2II}]}{2[D_2]} \left[ \left( \frac{r}{r_c} \right) + \left( \frac{r_c}{r} \right)^3 \right] \cos(2\theta),
\]

\[
B_{\theta IV}(r, \theta) = -F_{2rc} \frac{[N_{2II}]}{2[D_2]} \left[ \left( \frac{r}{r_c} \right) - \left( \frac{r_c}{r} \right)^3 \right] \sin(2\theta),
\]

where,

\[
[N_{2I}] = \left( \frac{r_b}{r_c} \right)^4 (2 + 4\ln(r_b)) - \left( \frac{r_c}{r_b} \right)^4 (1 + 4\ln(r_c)),
\]

\[
[N_{2II}] = \left( \frac{r_b}{r_c} \right)^4 (1 - 4\ln \left( \frac{r_c}{r_b} \right)) - \left( \frac{r_c}{r_b} \right)^4 ,
\]

\[
[D_2] = 1 - \left( \frac{r_b}{r_c} \right)^4 ,
\]

for $\nu = 2$.

### 4.6.4 Armature field for coils in mid-air

For the part of the coils in mid-air only the coefficients for all regions (III, IV and V) need to be obtained. These coefficients ($X_{nIII}, Y_{nIII}, X_{nIV}, Y_{nIV}, X_{nV}, Y_{nV}, X_{2III}, Y_{2III}, X_{2IV}, Y_{2IV}$) are obtained by satisfying the field solutions to the following boundary conditions

\[
A_{\theta IV} \big|_{r=0} = 0, \quad (4.130)
\]

\[
A_{\theta V} \big|_{r=\infty} = 0, \quad (4.131)
\]

\[
B_{rIII} \big|_{r=r_{ci}} = B_{rIV} \big|_{r=r_{ci}}, \quad (4.132)
\]

\[
H_{\theta III} \big|_{r=r_{ci}} = H_{\theta IV} \big|_{r=r_{ci}}, \quad (4.133)
\]

\[
B_{rIII} \big|_{r=r_{b}} = B_{rV} \big|_{r=r_{b}}, \quad (4.134)
\]

\[
H_{\theta III} \big|_{r=r_{b}} = H_{\theta V} \big|_{r=r_{b}}. \quad (4.135)
\]
For the total field solutions become

\[ B_{rIII}(r, \theta) = \sum_{n=1}^{\infty} \frac{1}{2} F_n \left[ - (\nu + 2) r_b \left( \frac{r}{r_b} \right)^{\nu-1} - (\nu - 2) r_{ci} \left( \frac{r_{ci}}{r} \right)^{\nu+1} + 2\nu r \cos(\nu \theta) \right], \quad (4.136) \]

\[ B_{\thetaIII}(r, \theta) = \sum_{n=1}^{\infty} \frac{1}{2} F_n \left[ (\nu + 2) r_b \left( \frac{r}{r_b} \right)^{\nu-1} - (\nu - 2) r_{ci} \left( \frac{r_{ci}}{r} \right)^{\nu+1} - 4r \sin(\nu \theta) \right], \quad (4.137) \]

\[ B_{rIV}(r, \theta) = \sum_{n=1}^{\infty} \frac{1}{2} (\nu + 2) F_n \left[ r_{ci} \left( \frac{r}{r_{ci}} \right)^{\nu-1} - r_b \left( \frac{r}{r_b} \right)^{\nu-1} \right] \cos(\nu \theta), \quad (4.138) \]

\[ B_{\thetaIV}(r, \theta) = -\sum_{n=1}^{\infty} \frac{1}{2} (\nu + 2) F_n \left[ r_{ci} \left( \frac{r}{r_{ci}} \right)^{\nu-1} - r_b \left( \frac{r}{r_b} \right)^{\nu-1} \right] \sin(\nu \theta), \quad (4.139) \]

\[ B_{rV}(r, \theta) = \sum_{n=1}^{\infty} \frac{1}{2} (\nu - 2) F_n \left[ r_{ci} \left( \frac{r}{r_{ci}} \right)^{\nu+1} - r_b \left( \frac{r}{r_b} \right)^{\nu+1} \right] \cos(\nu \theta), \quad (4.140) \]

\[ B_{\thetaV}(r, \theta) = -\sum_{n=1}^{\infty} \frac{1}{2} (\nu - 2) F_n \left[ r_{ci} \left( \frac{r}{r_{ci}} \right)^{\nu+1} - r_b \left( \frac{r}{r_b} \right)^{\nu+1} \right] \sin(\nu \theta), \quad (4.141) \]

for \( \nu \neq 2 \) and

\[ B_{rIII}(r, \theta) = \frac{1}{2} F_2 \left[ r_{ci}^4 r^{-3} - r + 4 r \ln \left( \frac{r}{r_b} \right) \right] \cos(2\theta), \quad (4.142) \]

\[ B_{\thetaIII}(r, \theta) = -\frac{1}{2} F_2 \left[ r - r_{ci}^4 r^{-3} + 4 r \ln \left( \frac{r}{r_b} \right) \right] \sin(2\theta), \quad (4.143) \]

\[ B_{rIV}(r, \theta) = 2 F_2 r \ln \left( \frac{r_{ci}}{r_b} \right) \cos(2\theta), \quad (4.144) \]

\[ B_{\thetaIV}(r, \theta) = -2 F_2 r \ln \left( \frac{r_{ci}}{r_b} \right) \sin(2\theta), \quad (4.145) \]

\[ B_{rV}(r, \theta) = \frac{1}{2} F_2 r^{-3} \left( r_{ci}^4 - r_b^4 \right) \cos(2\theta), \quad (4.146) \]

\[ B_{\thetaV}(r, \theta) = \frac{1}{2} F_2 r^{-3} \left( r_{ci}^4 - r_b^4 \right) \sin(2\theta), \quad (4.147) \]

for \( \nu = 2 \).

### 4.6.5 Total solution armature field

Now the armature field has been solved for a single phase when the coils are next to the back-iron and in mid-air, the armature field due to all phase can be obtained by
Figure 4.16: Armature reaction field in the coils predicted analytically and with FEM for a peak current density of 5 A/mm² and Δθ=0.

The radial and tangential components of the total armature reaction field in the different regions can then be obtained as

\[
\begin{align*}
B_{rIII}(r, \theta) &= B_{rIII}(r, \theta) |_{F_n=F_A^n} \quad \quad \quad B_{θIII}(r, \theta) &= B_{θIII}(r, \theta) |_{F_n=F_A^n}, \\
B_{rIV}(r, \theta) &= B_{rIV}(r, \theta) |_{F_n=F_A^n} \quad \quad \quad B_{θIV}(r, \theta) &= B_{θIV}(r, \theta) |_{F_n=F_A^n}, \\
B_{rV}(r, \theta) &= B_{rV}(r, \theta) |_{F_n=F_A^n} \quad \quad \quad B_{θV}(r, \theta) &= B_{θV}(r, \theta) |_{F_n=F_A^n}, \\
B_{BrIII}(r, \theta) &= B_{rIII}(r, \theta + \frac{2\pi}{Q}) |_{F_n=F_B^n} \quad \quad \quad B_{θrIII}(r, \theta) &= B_{θrIII}(r, \theta + \frac{2\pi}{Q}) |_{F_n=F_B^n}, \\
B_{BrIV}(r, \theta) &= B_{rIV}(r, \theta + \frac{2\pi}{Q}) |_{F_n=F_B^n} \quad \quad \quad B_{θrIV}(r, \theta) &= B_{θrIV}(r, \theta + \frac{2\pi}{Q}) |_{F_n=F_B^n}, \\
B_{BrV}(r, \theta) &= B_{rV}(r, \theta + \frac{2\pi}{Q}) |_{F_n=F_B^n} \quad \quad \quad B_{θrV}(r, \theta) &= B_{θrV}(r, \theta + \frac{2\pi}{Q}) |_{F_n=F_B^n}, \\
B_{CrIII}(r, \theta) &= B_{rIII}(r, \theta - \frac{2\pi}{Q}) |_{F_n=F_C^n} \quad \quad \quad B_{θrIII}(r, \theta) &= B_{θrIII}(r, \theta - \frac{2\pi}{Q}) |_{F_n=F_C^n}, \\
B_{CrIV}(r, \theta) &= B_{rIV}(r, \theta - \frac{2\pi}{Q}) |_{F_n=F_C^n} \quad \quad \quad B_{θrIV}(r, \theta) &= B_{θrIV}(r, \theta - \frac{2\pi}{Q}) |_{F_n=F_C^n}, \\
B_{CrV}(r, \theta) &= B_{rV}(r, \theta - \frac{2\pi}{Q}) |_{F_n=F_C^n} \quad \quad \quad B_{θrV}(r, \theta) &= B_{θrV}(r, \theta - \frac{2\pi}{Q}) |_{F_n=F_C^n}.
\end{align*}
\]  

Figure 4.16 shows excellent agreement between the flux density in the coils \((r = \frac{R_c + R_s}{2})\) predicted analytically and predicted with FEM. Here the initial design, given in Appendix A, is used for comparison and the peak current density is set to 5 A/mm².
4.7 Inductance calculation

The prediction of the self- and mutual-induction of an electric motor is important for the match between the motor and servo-amplifier. Especially for slotless actuator, like the one discussed in this work, an accurate prediction is important as slotless machine inherent low values of the inductance. In this section self- and mutual inductances a predicted for the particular design of the slotless PM actuator as discussed in this work and will be split into (see Fig. 4.17)

\[ M_{self} = M_{fe} + M_{st} + M_{end}, \]

where, \( M_{fe} \) is the inductance of the part of the coil next to the back-iron, \( M_{st} \) is the inductance of the straight part of the coil in mid air and \( M_{end} \) is the end-winding inductance.

4.7.1 Self inductance

Energy stored in the magnetic field of an coil carrying a current \( I \) can be expressed according to

\[ W_m = \frac{1}{2} \int_{V} \mathbf{A} \cdot \mathbf{J} dV. \]

(4.155)

For a linear system, both \( \mathbf{A} \) and \( \mathbf{J} \) are proportional to the current \( I \) and, therefore, Eq. (4.155) can be written as

\[ W_m = \frac{1}{2} M I^2, \]

(4.156)

where \( M \) is called the self-inductance of the coil. The phase self-inductance, \( M_{ph} \), can be obtained by rewriting Eq. (4.156) and substituting the magnetic energy from Eq. (4.155) [23]

\[ M = \frac{1}{I_{ph}^2} \int_{V_{ph}} A_{ph} \cdot J_{ph} dV, \]

(4.157)

where \( V_{ph} \) is the volume containing the phase current density. Only the \( z \)-components of the current density and vector potential are assumed and by evaluating the integral from
Eq. (4.157) over both bundles of each coil in phase A, B and C (see Figure 4.15), the phase self-inductances for a machine with \( q = 1/4 \) coil per magnet pole per phase, becomes

\[
M_A = \frac{N_{cph}}{I_A^2} \left[ \int_{V_{A+}} A_{III}(r, \theta) \cdot J_A(r, \theta) \, dv + \int_{V_{A-}} A_{III}(r, \theta) \cdot J_A(r, \theta) \, dv \right],
\]

\[
M_B = \frac{N_{cph}}{I_B^2} \left[ \int_{V_{B+}} A_{BIII}(r, \theta) \cdot J_B(r, \theta) \, dv + \int_{V_{B-}} A_{BIII}(r, \theta) \cdot J_B(r, \theta) \, dv \right],
\]

\[
M_C = \frac{N_{cph}}{I_C^2} \left[ \int_{V_{C+}} A_{CIII}(r, \theta) \cdot J_C(r, \theta) \, dv + \int_{V_{C-}} A_{CIII}(r, \theta) \cdot J_C(r, \theta) \, dv \right],
\]

where \( M_{fe} \) is the axial length of the part of the coil which is considered, \( N_{cph} = Q/3 \) is the number of coils per phase and \( I_A, I_B \) and \( I_C \) are the phase currents. Because of the balanced design of the motor, the self-inductances of all phases are the same. So, for now only the phase inductance of phase A will be determined, for separate parts of the coil.

**Self-inductance of the part of the coil next to the back-iron**

Reminding the vector potential of phase A by the armature reaction field

\[
A_{III} = \sum_{n=1}^{\infty} F_n \left\{ \frac{r_{ci}}{r_c} 2\nu \left[ \frac{(\nu - 2) - (\nu + 2)}{2} \left( \frac{r_c}{r_{ci}} \right)^{2\nu} + 4 \left( \frac{r_{ci}}{r_c} \right)^{\nu+2} \right] \right\} \left( \frac{r}{r_c} \right)^{\nu} \left( \frac{r_{ci}}{r_c} \right)^{\nu} \left( \frac{r_{b}}{r_c} \right)^{2\nu} \sin(\nu \theta)
\]

\[
+ \sum_{n=1}^{\infty} \frac{2}{F_n r_b^2} \left( \frac{r_b}{r} \right)^{\nu} \sin(\nu \theta)
\]

\[
+ \sum_{n=1,3,5,}^{\infty} \nu F_n r_b^2 \sin(\nu \theta).
\]

For simplicity, replace the term between \{ .. \} by \( R_c \). Since the integrals over the radius \( r \) and the angle \( \theta \) are independent, they can be separated resulting in five different integrals to be
4.7 Inductance calculation

solved

\[ I_{r1} = \int_{r_{ci}}^{r_b} \left[ \left( \frac{r}{r_c} \right)^\nu \left( \frac{r_{ci}}{r_c} \right)^\nu + \left( \frac{r_{ci}}{r} \right)^\nu \left( \frac{r_b}{r_c} \right)^{2\nu} \right] rdr, \quad (4.161) \]

\[ = \left[ \frac{1}{\nu + 2} \left( \frac{r_{ci}}{r_c} \right)^\nu \left( \frac{r_b}{r_c} \right)^{\nu} - \frac{r_{ci}}{r_c} \right] - \frac{1}{\nu - 2} \left( \frac{r_b}{r_c} \right)^{2\nu} \left( \frac{r_{ci}}{r_b} \right)^{\nu} \left( \frac{r_{ci}}{r_c} \right)^{\nu}. \]

\[ I_{r2} = \int_{r_{ci}}^{r_b} \left( \frac{r_b}{r} \right)^\nu rdr, \]

\[ = \frac{1}{\nu - 2} \left[ \left( \frac{r_b}{r_c} \right)^\nu \left( \frac{r_{ci}}{r_c} \right)^{\nu} \right], \quad (4.162) \]

\[ I_{r3} = \int_{r_{ci}}^{r_b} r^3 dr = \frac{1}{4} \left[ r_b^4 - r_{ci}^4 \right], \quad (4.163) \]

\[ I_{\theta 1} = \int_{-\alpha_c}^{-\beta_o} \sin(\nu \theta) d\theta - \int_{\alpha_c}^{\beta_o} \sin(\nu \theta) d\theta = \frac{2}{\nu} \left[ \cos(\nu \pi) - \cos(\nu \beta_o) \right]. \quad (4.164) \]

Using the expression for the vector potential (Eq. (4.2)), the integrals from Eq. (4.158) become

\[ M_A = N_{cph} L_{fe} \sum_{n=1}^{N} \frac{F_n}{I_A^2} \left[ R_e I_{r1} - \frac{2}{\nu} r_b^2 I_{r2} + I_{r3} \right] \cdot I_{\theta 1}. \]

By filling in the expressions for the coefficient \( F_{An} \) and current density \( J_A \), given by Eqs. (4.97) and (4.60), respectively, the self-inductance for a phase becomes

\[ M_{ph} = \mu_0 L_{fe} N_{cph} N^2 \sum_{n=1}^{\infty} \frac{K_{wn}}{\nu^2 - 4} \left[ R_e I_{r1} - \frac{2}{\nu} r_b^2 I_{r2} + I_{r3} \right] \cdot I_{\theta 1}, \quad (4.165) \]

reminding

\[ R_e = r_{ci}^2 \left[ \frac{((\nu - 2) - (\nu + 2) \left( \frac{r_{ci}}{r_c} \right)^{2\nu} + 4 \left( \frac{r_{ci}}{r_b} \right)^{2\nu} + 2r_{ci}^2}{2\nu (1 - \left( \frac{r_{ci}}{r_b} \right)^{2\nu}} \right]. \quad (4.166) \]

**Self-inductance of the part of the coil in mid-air**

Reminding the vector potential of phase A by the armature reaction field

\[ A_{AIII} = \sum_{n=1}^{\infty} \frac{1}{2\nu} F_n \left[ -(\nu + 2) r_b^2 \left( \frac{r}{r_b} \right)^\nu - (\nu - 2) r_{ci}^2 \left( \frac{r_{ci}}{r} \right)^\nu + 2r_{ci}^2 \right] \sin(\nu \theta). \quad (4.167) \]

Because the integrals over the radius \( r \) and the angle \( \theta \) are independent, they can be separated resulting in only one integral left to be solved

\[ I_{r4} = \int_{r_{ci}}^{r_b} \left[ -(\nu + 2) r_b^2 \left( \frac{r}{r_b} \right)^\nu - (\nu - 2) r_{ci}^2 \left( \frac{r_{ci}}{r} \right)^\nu + 2r_{ci}^2 \right] rdr, \]

\[ = -r_b^{\nu+2} - r_{ci}^{\nu+2} - r_{ci}^{\nu+2} \left( r_{b}^{\nu+2} - r_{ci}^{\nu+2} \right) + \frac{1}{2} \nu \left( r_b^4 - r_{ci}^4 \right). \quad (4.168) \]

The phase self-inductance for this part of the coil becomes

\[ M_{ph} = \mu_0 L_{st} \frac{N_{cph} N^2}{A_{bundle}} \sum_{n=1}^{\infty} \frac{K_{wn}}{2\nu (\nu^2 - 4)} I_{r4} I_{\theta 1}. \quad (4.169) \]
4.7.2 Mutual inductance

The inductance considered in the previous section, is calculated for an isolated set of coils. However, other sets (different phases) can be magnetically coupled with one other, giving rise to the mutual inductance. For instance, a current through the coils of phase A generates a flux through the coils of phase B. The mutual inductance from phase \( i \) to \( j \) in a linear system can, therefore, be expressed as

\[
M_{i,j} = \frac{1}{I_i I_j} \int_{V_j} A_1 \cdot J_2 dV.
\]

Similar to the self inductance, the volume integral needs to be evaluated over both conductors of a coil. The mutual inductances \( M_{AB}, M_{BC} \) and \( M_{CA} \) are calculated according to

\[
M_{AB} = \frac{N_{cph}}{I_{AIB}} \left[ \int_{V_{B+}} A_{AII} \cdot J_{B} dV + \int_{V_{B-}} A_{AIII} \cdot J_{B} dV \right],
\]

\[
= \frac{N_{cph} L_{fe}}{I_{AIB}} \left[ \int_{r_{ci}}^{r_{b}} \int_{-\alpha_c - \frac{2\pi}{3}}^{-\beta_0 - \frac{2\pi}{3}} A_{AII}(r, \theta) J_{B r} r d\theta dr - \int_{r_{ci}}^{r_{b}} \int_{-\beta_0 + \frac{2\pi}{3}}^{-\alpha_c + \frac{2\pi}{3}} A_{AIII}(r, \theta) J_{B r} r d\theta dr \right],
\]

\[
M_{BC} = \frac{N_{cph}}{I_{BIC}} \left[ \int_{V_{C+}} A_{BII} \cdot J_{C} dV + \int_{V_{C-}} A_{BIII} \cdot J_{C} dV \right],
\]

\[
= \frac{N_{cph} L_{fe}}{I_{BIC}} \left[ \int_{r_{ci}}^{r_{b}} \int_{-\alpha_c + \frac{2\pi}{3}}^{-\beta_0 - \frac{2\pi}{3}} A_{BII}(r, \theta) J_{C r} r d\theta dr - \int_{r_{ci}}^{r_{b}} \int_{-\beta_0 + \frac{2\pi}{3}}^{-\alpha_c + \frac{2\pi}{3}} A_{BIII}(r, \theta) J_{C r} r d\theta dr \right],
\]

\[
M_{CA} = \frac{N_{cph}}{I_{CIA}} \left[ \int_{V_{A+}} A_{CII} \cdot J_{A} dV + \int_{V_{A-}} A_{CIII} \cdot J_{A} dV \right],
\]

\[
= \frac{N_{cph} L_{fe}}{I_{CIA}} \left[ \int_{r_{ci}}^{r_{b}} \int_{-\alpha_c}^{-\beta_0} A_{CII}(r, \theta) J_{A r} r d\theta dr - \int_{r_{ci}}^{r_{b}} \int_{-\beta_0}^{-\alpha_c} A_{CIII}(r, \theta) J_{A r} r d\theta dr \right].
\]

It can be shown that

\[
M_{BA} = M_{AB}, \quad M_{CB} = M_{BC}, \quad M_{CA} = M_{CA}.
\]

**Mutual inductance of the part of the coil next to the back-iron**

The integrals for the mutual inductances are in the same form as for the self inductance, so the solutions are given directly as

\[
M_{AB} = M_{BA} = \mu_0 L_{fe} \frac{N_{cph} N^2}{A_{bundle}} \sum_{n=1}^{\infty} \frac{K_{wn}}{(\nu^2 - 4)} \left[ \frac{1}{\nu} R_e I_{r} - \frac{2}{\nu^2} r_b^2 I_{r} + \frac{1}{\nu} I_3 \right] \cdot I_{\theta 2},
\]

\[
M_{BC} = M_{CB} = \mu_0 L_{fe} \frac{N_{cph} N^2}{A_{bundle}} \sum_{n=1}^{\infty} \frac{K_{wn}}{(\nu^2 - 4)} \left[ \frac{1}{\nu} R_e I_{r} - \frac{2}{\nu^2} r_b^2 I_{r} + \frac{1}{\nu} I_3 \right] \cdot I_{\theta 3},
\]

\[
M_{CA} = M_{AC} = \mu_0 L_{fe} \frac{N_{cph} N^2}{A_{bundle}} \sum_{n=1}^{\infty} \frac{K_{wn}}{(\nu^2 - 4)} \left[ \frac{1}{\nu} R_e I_{r} - \frac{2}{\nu^2} r_b^2 I_{r} + \frac{1}{\nu} I_3 \right] \cdot I_{\theta 4}.
\]
4.7. Inductance calculation

where

\[ I_{\theta 2} = \int_{-\alpha - \beta_0 + \frac{2\pi}{Q}}^{\alpha - \beta_0 + \frac{2\pi}{Q}} \sin(\nu\theta) d\theta - \int_{-\alpha - \beta_0 + \frac{2\pi}{Q}}^{\alpha - \beta_0 + \frac{2\pi}{Q}} \sin(\nu\theta) d\theta, \]

\[ = \frac{2}{\nu} \cos\left(\frac{\nu}{Q}\right) \left[ \cos\left(\frac{2\pi}{Q}\right) - \cos(\nu\beta_0) \right], \quad (4.176) \]

\[ I_{\theta 3} = \int_{-\alpha - \beta_0 + \frac{2\pi}{Q}}^{\alpha - \beta_0 + \frac{2\pi}{Q}} \sin\left(\nu\theta + \frac{2\pi}{Q}\right) d\theta - \int_{-\alpha - \beta_0 + \frac{2\pi}{Q}}^{\alpha - \beta_0 + \frac{2\pi}{Q}} \sin\left(\nu\theta + \frac{2\pi}{Q}\right) d\theta, \]

\[ = \frac{2}{\nu} \cos\left(\frac{\nu}{Q}\right) \left[ \cos\left(\frac{2\pi}{Q}\right) - \cos(\nu\beta_0) \right], \quad (4.177) \]

\[ I_{\theta 4} = \int_{-\alpha + \beta_0 - \frac{2\pi}{Q}}^{\alpha + \beta_0 - \frac{2\pi}{Q}} \sin\left(\nu\theta - \frac{2\pi}{Q}\right) d\theta - \int_{-\alpha + \beta_0 - \frac{2\pi}{Q}}^{\alpha + \beta_0 - \frac{2\pi}{Q}} \sin\left(\nu\theta - \frac{2\pi}{Q}\right) d\theta, \]

\[ = \frac{2}{\nu} \cos\left(\frac{\nu}{Q}\right) \left[ \cos\left(\frac{2\pi}{Q}\right) - \cos(\nu\beta_0) \right]. \quad (4.178) \]

Self-inductance of the part of the coil in mid-air

The mutual inductance of this part of the coil is given by

\[ M_{AB} = M_{BA} = \mu_0 L_\text{st} \frac{N_{\text{ph}} N^2}{A_{\text{bundle}}} \sum_{n=1}^{\infty} \frac{K_{wn}}{2\nu(\nu^2 - 4)} I_{\nu 4} I_{\theta 2}, \quad (4.179) \]

\[ M_{BC} = M_{CB} = \mu_0 L_\text{st} \frac{N_{\text{ph}} N^2}{A_{\text{bundle}}} \sum_{n=1}^{\infty} \frac{K_{wn}}{2\nu(\nu^2 - 4)} I_{\nu 4} I_{\theta 3}, \quad (4.180) \]

\[ M_{CA} = M_{AC} = \mu_0 L_\text{st} \frac{N_{\text{ph}} N^2}{A_{\text{bundle}}} \sum_{n=1}^{\infty} \frac{K_{wn}}{2\nu(\nu^2 - 4)} I_{\nu 4} I_{\theta 4}, \quad (4.181) \]

4.7.3 End-winding inductance

To calculate the inductances of the end-windings of the coils, the technique used to calculate the inductances of the straight parts, is difficult to use, because 3D field descriptions are required. The end-windings of the coils are, therefore, considered as one half of a disc coil and the inductance for a disc coil is obtained from [26]. Figure 4.18 shows a schematic of a disc coil with its leading dimensions. The variables \( a, b \) and \( c \) are calculated according to

\[ a = \frac{1}{4}(\alpha_c - \beta_0)(r_b - r_c), \quad (4.182) \]

\[ b = r_b - r_c, \quad (4.183) \]

\[ c = \frac{1}{2}(\alpha_c - \beta_0)(r_b + r_c). \quad (4.184) \]

The inductance of a disc coil is given by

\[ M_{\text{disc}} = 0.1N^2 aP' \times 10^{-6}, \quad (4.185) \]

where \( N \) is the number of turns. \( P' \) is the product of \( P \) and \( F \), which can be found in [26] depending the ratios \( b/c \) and \( c/2a \).
The total inductances of the end-windings in a phase can be calculated as
\[ M_{\text{end}} = 0.1N_{\text{cpb}}N^2aP' \times 10^{-6}, \]  
(4.186)

The mutual inductances due to the end-windings are obtained by determining the ratio of the mutual inductance to the self-inductance for the straight parts of the coils and multiplying this ratio with the end-winding inductance from Eq. 4.186.

### 4.7.4 Total inductance initial design

To check the expression obtained for the calculation of the self- and mutual-inductances and for the active part of the initial design of the rotary actuator the inductances are calculated, resulting in
\[ M_{ij,\text{fem}} = \begin{bmatrix} M_A & M_{AB} & M_{AC} \\ M_{BA} & M_B & M_{BC} \\ M_{CA} & M_{CB} & M_C \end{bmatrix} = \begin{bmatrix} 507 & -131 & -131 \\ -131 & 507 & -131 \\ -131 & -131 & 507 \end{bmatrix} \mu H. \]  
(4.187)

For comparison, a FE-model was created without the magnetic material. A sinusoidal current was fed to the different phases in separate simulations and the voltage over the terminals of all the phases were predicted. The inductances were calculated by using the electrical equation of the voltage over a inductance \( j \) (having no resistance) due to a varying current in phase \( i \)
\[ V_j = M_{ij} \frac{dI_j}{dt} \rightarrow M_{ij} = \frac{V_j}{\frac{d}{dt}I_j}. \]  
(4.188)

For a current with an amplitude of 1 A and frequency of 200 Hz the results were exactly the same as predicted
\[ M_{ij,\text{fem}} = \begin{bmatrix} M_A & M_{AB} & M_{AC} \\ M_{BA} & M_B & M_{BC} \\ M_{CA} & M_{CB} & M_C \end{bmatrix} = \begin{bmatrix} 816 & -199 & -199 \\ -199 & 816 & -199 \\ -199 & -199 & 816 \end{bmatrix} \mu H. \]  
(4.189)

Finally the total self- and mutual inductances according to Eq. 4.154, are calculated for the initial design, which results in
\[ M_{ij,\text{tot}} = \begin{bmatrix} M_A & M_{AB} & M_{AC} \\ M_{BA} & M_B & M_{BC} \\ M_{CA} & M_{CB} & M_C \end{bmatrix} = \begin{bmatrix} 816 & -199 & -199 \\ -199 & 816 & -199 \\ -199 & -199 & 816 \end{bmatrix} \mu H, \]  
(4.190)
which only shows an error of 3% compared to the phase self-inductances measured on the pre-prototype of the initial design ($M_{\text{meas}}=840 \mu\text{H}$).

### 4.8 Resistance Calculation

To predict the ohmic losses in the coils, the resistance needs to be calculated. The average value of a single turn will be considered, so the total resistance can be obtained by multiplying this average resistance with the number of turns, while not exactly knowing how every single turn is laid inside the coil. The geometry of a single coil is shown Fig. 4.19. The resistance consists of the end-winding resistance and the resistance in the straight part of the coil. The average resistance per turn in these two parts of the coil are:

\[
R_{\text{st},av} = \frac{\rho_{\text{cu}}2L_{\text{st}}}{A_w} \tag{4.191}
\]

\[
R_{\text{end},av} = \rho_{\text{cu}}(\alpha_c + \beta_i)\frac{\pi(r_{\text{co}} + r_{\text{ci}})}{2A_w} \tag{4.192}
\]

where $\rho_{\text{cu}}$ is the electrical resistivity of copper, $L_{\text{st}}$ is the length of the straight part of the coil and $A_w$ is the cross-sectional area of the wire. For a square wire $A_w = w_w t_w$, where $w_w$ is the width and $t_w$ the thickness of the wire. The total resistance for a single coil can be obtained by

\[
R_{\text{coil}} = N(R_{\text{st},av} + R_{\text{end},av}) \tag{4.193}
\]

The resistance of a single in the initial design of the rotary actuator was measured and predicted. The Table 4.1 shows both results.

![Illustration of a single coil.](image)

<table>
<thead>
<tr>
<th>$R_{\text{coil}}$ ($\Omega$)</th>
<th>Predicted</th>
<th>Measured</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.01</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

### 4.9 Mechanical model

The required torque to accelerate the moving member of the rotary actuator is directly linked to the inertia, which consists of the load inertia and rotor inertia. Although the load inertia is fixed by the specifications, the rotor inertia can be determined for a certain design of the actuator. Not only the inertia is of concern, also the total moving mass of the rotary actuator is important to known, since the linear actuator should be able to achieve its specified acceleration. Because the concepts with moving magnets and moving coils are under study here, the inertia and mass for different parts of the actuator need to be calculated.
4.9.1 Moving magnet

In the concept of moving magnet the magnet array and core iron are directly attached to the shaft (the shaft is considered to be load) as is shown in Figure 2.1. Therefore the inertia and mass are calculated by

\[ I_{MM} = I_{core} + I_{magnet}, \]
\[ M_{MM} = M_{core} + M_{magnet}, \]

where

\[ I_{core} = \frac{\pi}{2} m_c L_{act}(r_c^4 - r_i^4), \]
\[ I_{magnet} = \frac{\pi}{2} m_m L_{act}(r_m^4 - r_c^4), \]
\[ M_{core} = \frac{\pi m_c L_{act}(r_c^2 - r_i^2)}, \]
\[ M_{magnet} = \frac{\pi m_m L_{act}(r_m^2 - r_c^2)}. \]

Here, \( m_c \) and \( m_m \) are the mass densities for the core steel and the magnet material.

4.9.2 Moving coil

In this concept not only the mass and inertia of the coils are of concern, but also the coil carrier needs to be taken into account. However, the design of the coil carrier is unknown at this point and for now its mass and inertia are estimated from the initial design and added to the load.

The total inertia and moving mass for the concept with moving coil can therefore be calculated by

\[ I_{MC} = I_{coil} = I_{st} + I_{end}, \]
\[ M_{MC} = M_{coil} = M_{st} + M_{end}, \]

where

\[ I_{st} = \frac{1}{2} Q m_w L_{st}(\alpha_c - \beta_o)(r_{co}^4 - r_{ct}^4), \]
\[ I_{end} = \frac{\pi}{5} Q m_w (\alpha_c^2 - \beta_o^2)(r_{co}^5 - r_{ct}^5), \]
\[ M_{st} = Q m_w L_{st}(\alpha_c - \beta_o)(r_{co}^2 - r_{ct}^2), \]
\[ M_{end} = \frac{\pi}{4} Q m_w (\alpha_c^2 - \beta_o^2)(r_{co} - r_{ct})(r_{co} + r_{ct})^2, \]
\[ L_{st} = L_{act} + z_{stroke}. \]

Here, \( m_w \) is the mass density of the windings. Normally, copper windings are used because of their low electrical resistivity, though aluminum windings could be considered as an alternative, because of the low mass density. When calculating the inertia and mass of the coils, the packing factor of the wire should be taken into account, because the volume of the coil is not completely filled with copper. For square wire, like the wire used in the initial design of the rotary actuator, the packing factor is around 90%, but it can be as low as 50% or even lower for a round wire.
4.10 Thermal model

Analysis of thermal behavior of the electro-mechanical device is crucial to guarantee a safe operation of the machine under certain operating conditions. Irreversible damage on the insulation material of the coil windings occur for high coil temperatures and magnets can be irreversibly demagnetized. Additionally, the ohmic losses in the wires increase due to the increasing electrical resistivity of copper and model parameters change, such as the decrease of the remanent flux density of the magnets.

Although exact prior determination of the thermal behavior is difficult, an estimate of the temperature in certain parts of the machine under worst-case conditions is essential when designing an electronic device. Various methods exist to analyze the thermal behavior of an electrical machine. Here, a thermal equivalent circuit (TEC) of the slotless PM actuator will be used, because it offers a fast means for determining the temperature distribution of the actuator during optimization. Such a TEC is an analogy of an electric circuit with the heat being analogous to current and the temperature analogous to the voltage. Prior knowledge of the heat flow is required to lump the electrical device into thermal resistances, thermal capacitances and thermal sources. While the transient thermal behavior of the machine can be modeled with TEC, here only the steady-state temperatures in different parts of the machine will be calculated. Transient modeling might be useful to investigate thermal stresses when the machine is temporarily overloaded.

When calculating the thermal resistances and heat sources, different types of heat transfer need to be considered [27].

- **Conduction**
  Conduction can be viewed as the transfer of energy from the more energetic to the less energetic particles of a substance due to interactions between the particles. The heat transfer due to conduction can be quantified by \( q'_k \) as the heat flux in W m\(^{-2}\). Using the rate equation, this heat flux can be calculated as function of the temperature \( T \):

  \[
  q'_k = -k \nabla T,
  \]

  where \( k \) is the thermal conductivity of the material, \( \nabla \) the three-dimensional Nabla operator.

- **Convection**
  The convection heat transfer mode is comprised of two mechanisms: heat transfer due to random molecular motion (diffusion) and heat transfer by bulk motion of a fluid. Regardless of the particular nature of the convection heat transfer process, the appropriate rate equation is of the form

  \[
  q'_h = h(T_s - T_{sur}),
  \]

  where \( q'_h \), the convective heat flux, is proportional to the difference between the surface and fluid temperatures, \( T_s \) and \( T_{sur} \), respectively. The constant \( h \), in W m\(^{-2}\) K\(^{-1}\), is the convection heat transfer coefficient. It depends on the conditions in the boundary layer, which are influenced by surface geometry, the nature of the fluid and thermodynamic and transport properties.
Figure 4.20: Radial heat flow inside slotless permanent magnet actuator with moving coils.

- **Radiation**

  Thermal radiation is energy emitted by matter that is at a finite temperature. The net rate of radiation heat transfer from the surface, expressed per unit area of the surface, is

  \[ q_E = \varepsilon \sigma_{SB} (T_s^4 - T_{sur}^4), \]

  where \( \varepsilon \) is the emissivity of a surface ranging from 0 to 1, \( \sigma_{SB} = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4} \) the Boltzmann constant, \( T_s \) the surface temperature and \( T_{sur} \) the surrounding temperature.

Taking only the active part of the actuator (axial length of magnets), the geometry of the actuator can be split into cylinders made of different materials. Because of the periodicity around the z-axis, only the axial and radial heat flow are of interest. To further simplify the thermal model, only radial heat flow will be considered. In Fig. 4.20 the heat flow is shown for the concept with moving coils. In the case of moving magnets, the back iron is directly attached to the coils. Assuming only heat production inside the coils (eddy currents losses in the back iron will be ignored), heat flows through to the sides of the airgap, where part of it is conducted over the airgap to the magnets and back-iron and the rest of the heat flow is carried away due to convection, resulting from forced air cooling in the airgap.

### 4.10.1 Components of the TEC

**Heat sources**

The ohmic losses inside the coils are introduced as a heat source \( P_{cu} \) and for the active part of the actuator is calculated according to

\[
P_{cu} = \frac{1}{V_{coils}} \rho_{cu} J_{rms}^2 dV, \]

\[
= \frac{Q \rho_{cu} J_{rms}^2 L_{act}}{r_{co}^2 - r_{ci}^2},
\]

where \( \rho_{cu} \) is the electrical resistivity of copper and \( J_{rms} \) is the rms current density inside the coils.
Conductive thermal resistance

For every single cylindrical component of the actuator, the conductive thermal resistance is split into two resistances, with the center node giving the average temperature of that component. Components with heat generation have an additional resistance, as is shown in Fig. 4.21. Without heat generation, $T_{mid}$ gives the average temperature of the component. Otherwise, $T_{av}$ gives the average temperature, while $T_{mid}$, in this case, would give an average temperature which is too high. Therefore, $R_3$ is a negative resistance. Heat generation is introduced as a current through this resistance [28]. The thermal resistances are calculated according to

\[
R_1 = \frac{1}{4\alpha c k L_{act}} \left[ 1 - \frac{2r_2^2ln(r_1/r_2)}{(r_1^2 - r_2^2)} \right],
\]

\[
R_2 = \frac{1}{4\alpha c k L_{act}} \left[ \frac{2r_1^2ln(r_1/r_2)}{(r_1^2 - r_2^2)} - 1 \right],
\]

\[
R_3 = \frac{-1}{8\alpha c (r_1^2 - r_2^2)k L_{act}} \left[ r_1^2 + r_2^2 - \frac{4r_1^2r_2^2ln(r_1/r_2)}{(r_1^2 - r_2^2)} \right],
\]

where $r_2$ and $r_1$ are the inside and outside radii of the cylindrical component, respectively, and $T_2$ and $T_1$ are the corresponding temperatures. Here, $L_{act}$ is the axial length of the component and $k$ is the thermal resistivity of the material.

Convective thermal resistance

The convective heat transfer between the contact area $A$ and a cooling fluid (air in this case), inside and outside the actuator are modeled by a the convective resistance

\[
R_h = \frac{1}{hA}.
\]

The heat transfer functions for the exterior and interior surfaces of the actuator need to be found. These values may be obtained from the dimensionless Nusselt number, which are mostly determined empirically. The Nusselt number is defined as

\[
Nu = \frac{hl}{k},
\]

where $l$ is the length of the convective boundary layer.

The outside surfaces at inner and outer radii are considered as vertical cylindrical planes with free convection and for this case an expression for the Nusselt number can be found in [29]
The Rayleigh number is calculated as \( Ra = Gr Pr \), where \( Gr \) and \( Pr \) are the Grashof and Prandtl number, respectively. When calculating the heat transfer coefficient, the length of the actuator should be taken for \( l \). The heat transfer function, \( h_{end} \), at both radii is estimated to be 10 W m\(^{-2}\) K\(^{-1}\).

Inside the airgap, the Nusselt numbers for convection due to circumferential velocity of one surface can be found in [28]. The convective heat transfer between two rotating smooth cylinders is given by Taylor and the modified expressions, due to the effect of slots, for the Nusselt number are [28]

\[
\begin{align*}
Nu &= 2.2 \quad \text{for} \quad Ta < 41 \\
Nu &= 0.23 Ta^{0.63} Pr^{0.27} \quad \text{for} \quad 41 < Ta < 100
\end{align*}
\]

The Taylor number is found with \( Ta = 2Re \frac{l}{r^*} \), where \( r^* = r_{s,i}/r_{s,o} \) is the ration between the radii of the inner and outer cylindrical surface and the Reynolds number is obtained by \( Re = U \frac{r_{s,o} - r_{s,i}}{\nu} \) with \( U \) is the circumferential velocity of the rotating surface and \( \nu \) is the kinematic viscosity of air. In this case \( l \) should be taken as the length of the airgap \((r_{s,o} - r_{s,i})\).

In this study, however, forced air cooling inside the airgap is assumed. In [30] the Nusselt numbers inside an concentric annulus with forced air cooling are determined for different cases. In this work, though, developed and laminar air flow inside is annulus is assumed. Due to the difficult nature of the air flow in the airgap, the heat transfer coefficient \( h_{gap} \) is left open for now and a minimum value for safe operation of the actuator will be selected later.

**Radiative thermal resistance**

The radiative heat flow depends on the temperature by the power four. However, to account for the radiation in the TEC, for small temperature differences the heat flow per unit area can be linearized and expressed as

\[
q_\epsilon = h_\epsilon (T_1 - T_2),
\]

where \( h_\epsilon \) is the radiation heat transfer coefficient, which can be approximated by \( h_\epsilon \approx 6\epsilon \) [31]. For low temperatures, the radiation inside the airgap of the actuator is relatively small compared to the conductive heat flow, however at the surfaces in contact with the surrounding air, radiation can not be omitted. The radiation heat transfer coefficient is added to the heat transfer coefficient for the natural convection, \( h_{end} \).

**4.10.2 Solution of the TEC**

Figure 4.22 shows the TEC for the concept with moving coils. For moving magnets, the thermal resistances \( R_{1g2}, R_{2g2}, R_{h,co} \) can simply be omitted. Resistance \( R_{h,rb} \), however is remained to account for the convective heat transfer inside the coil opening.

To solve the thermal network for a given heat dissipation inside the coils, for each node the heat flow needs to be balanced. Analogous to an electrical network, Kirchoffs current law
4.10. Thermal model

Figure 4.22: Thermal Equivalent Circuit of the radial heat flow inside the slotless pm actuator with moving coils.

can be applied to a node \( i \) (steady state)

\[
P_i = \sum_j \frac{1}{R_{ij}}(T_i - T_j) + \frac{1}{R_{ch,i}}(T_i - T_0),
\]  

(4.214)

where \( P_i \) is the heat generation injected to node \( i \), \( R_{ij} \) is the thermal resistance between node \( i \) and \( j \), \( T_i \) and \( T_j \) are the nodal temperatures of \( i \) and \( j \), respectively and \( R_{ch,i} \) is the convective thermal resistance of node \( i \) to the surrounding temperature, \( T_0 \).

All nodal equations can be written in matrix form

\[
[P] = [Y][T],
\]  

(4.215)

where \([Y]\) is a square matrix with all conductances. The nodal temperatures can thus be solved by

\[
[T] = [Y]^{-1}[P].
\]  

(4.216)

Using the dimensions of the initial design of the slotless actuator and the material properties from Table 2.2, the temperatures inside the coils and magnets as function of the heat \( h_{gap} \) for both the moving coil and moving magnet configuration are shown in Fig. 4.23. As the figure shows, the temperature drops significantly when \( h_{end} \) is increased from 5 to 30 W m\(^{-2}\) K\(^{-1}\). Although high values for the heat transfer function can be achieved by forced air cooling [32], \( h_{gap} \) is set to a conservative value of 15 W m\(^{-2}\) K\(^{-1}\) during the optimization procedure. This value, however, was also estimated from preliminary measurements on the pre-prototype with an air flow of 25 L min\(^{-1}\) through the airgap.

The temperature distribution inside the initial design of the actuator with \( h_{end}=10 \) W m\(^{-2}\) K\(^{-1}\), \( h_{gap}=15 \) W m\(^{-2}\) K\(^{-1}\) and a heat production of 0.5 W in the active part of the actuator, is simulated using the TEC and the results are given in Table 4.2. For comparison a thermal simulation is done in FEM and the temperature distribution is shown in Fig. 4.24. It can be concluded that the TEC gives an good approximate of the temperature distribution inside the slotless PM actuator.
Multi-physical framework

Figure 4.23: Influence of the airgap heat transfer coefficient on the coil and magnet temperatures when 0.5 W of heat is produced in the coils.

Figure 4.24: Temperature distribution calculated with FEM.

Table 4.2: Simulated temperature distribution using TEC for $P_{cu}=0.5$ W.

<table>
<thead>
<tr>
<th>Component</th>
<th>$T$ [°C]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core</td>
<td>48.94</td>
</tr>
<tr>
<td>Magnet</td>
<td>49.02</td>
</tr>
<tr>
<td>Airgap 1</td>
<td>51.46</td>
</tr>
<tr>
<td>Coil</td>
<td>54.18</td>
</tr>
<tr>
<td>Airgap 2</td>
<td>50.96</td>
</tr>
<tr>
<td>Back-iron</td>
<td>48.25</td>
</tr>
</tbody>
</table>
Chapter 5

Design optimization

The multi-physical framework is implemented in MATLAB and used in the optimization procedure, which is performed with sequential quadratic programming (SQP). Because no specifications on the electrical behavior are given, only the magnetic, thermal and mechanical models considered in the optimization procedure. In this section the optimization objective, design constraints and results are discussed.

5.1 Optimization objective

To create an efficient design of the zφ-module the rotary actuator is optimized with the objective to minimize the copper losses inside the rotary and linear actuator combined. These losses are calculated for a third order motion profiles as is shown in Fig. 5.1 and Fig. 5.2. By taking the losses inside the linear actuator into account, the moving mass of the rotary actuator is limited. The objective function can be written as

\[ f(\alpha, \beta_0, r_i, l_c, l_m, l_w, l_b, L_{act}) = P_{cu,rot} + P_{cu,lin} \]  \hspace{1cm} (5.1)  

where \( \alpha = \frac{\phi}{\phi_{q} + \phi_{c}}, \beta_0, r_i, l_c, l_m, l_w, l_b \) and \( L_{act} \) are the optimization variables, which are indicated in Fig. 5.3.

The initial design of the rotary actuator already showed that the specifications given in Table 1.1 are easily met within the volume constraints, making this design oversized for the application. Therefore, the design is also optimized with the objective to minimize the volume of the rotary actuator. The objective function can expressed as

\[ f(\alpha, \beta_0, r_i, l_c, l_m, l_w, l_b, L_{act}) = \pi r_o^2 L, \]  \hspace{1cm} (5.2)  

where \( r_o \) is the outer radius of the actuator and \( L \) is the total length of the coils.
Figure 5.1: Third order motion profile for the rotary actuator.

Figure 5.2: Third order motion profile for the linear actuator.
Figure 5.3: Schematic of the rotary actuator with the optimization variables.
5.2 Optimization procedure

To determine the copper losses of both the linear and rotary actuator during a third order motion profile, first the rms torque and force are calculated according to

\[
T_{\text{rms}} = \alpha_{\phi,\text{rms}}(I_{\text{rot}} + I_{\text{lin}}),
\]

\[
= \sqrt{\left( \frac{1}{3}d_\phi\alpha_\phi \right)(I_{\text{rot}} + I_{\text{lin}})}, \tag{5.3}
\]

\[
F_{\text{rms}} = \alpha_{z,\text{rms}}(M_{\text{rot}} + M_{\text{lin}}),
\]

\[
= \sqrt{\left( \frac{1}{3}d_z\alpha_z \right)(M_{\text{rot}} + M_{\text{lin}})}. \tag{5.4}
\]

where \(d_\phi, \alpha_\phi, d_z, \alpha_z\) are the duty cycles and peak accelerations for the rotary and linear actuator, respectively. For the rotary actuator the required current density to achieve this rms torque, is determined by

\[
J_{\text{rms}} = \frac{T_{\text{rms}}}{\sqrt{2k_{t,\text{rot}}}}, \tag{5.5}
\]

where \(k_{t,\text{rot}}\) is the torque constant of the rotary actuator in Nm m² A⁻¹ and is determined using the torque model from Section 4.3, when sinusoidal currents are assumed. The total copper losses inside the actuator are then determined according to

\[
P_{\text{cu,rot}} = P_{\text{cu,st}} + P_{\text{cu,end}}, \tag{5.6}
\]

where

\[
P_{\text{cu,st}} = \frac{\pi Q_{\text{cu}} I_{\text{rms}}^2}{4} (L_{\text{act}} + z_{\text{stroke}})(\alpha_c - \beta_\alpha)(r_{co}^2 - r_{ci}^2), \tag{5.7}
\]

\[
P_{\text{cu,end}} = \frac{\pi Q_{\text{cu}} I_{\text{rms}}^2}{4} (\alpha_c^2 - \beta_\alpha^2)(r_{co} - r_{ci})(r_{co} + r_{ci})^2. \tag{5.8}
\]

The copper losses inside the linear actuator are obtained by determining the required current according to

\[
I_{\text{rms}} = \frac{F_{\text{rms}}}{k_{t,\text{lin}}}, \tag{5.9}
\]

where \(k_{t,\text{lin}}\) is the force constant of the linear actuator in N A⁻¹ and is given in Table 2.1. The losses are calculated according to

\[
P_{\text{cu,lin}} = I_{\text{rms}}^2 R_{\text{lin}}, \tag{5.10}
\]

where \(R_{\text{lin}}\) is the coil resistance of the linear coil and is given in Table 2.1.

5.3 Constraints

The design of the rotary actuator is subject to several constraints, which can be divided in magnetic, volume and thermal constraints. Table 5.1 lists all the constraints.

**Magnetic constraints**

The semi-analytical magneto-static model is based on the assumption that the core and back-iron are infinitely permeable. This model gives a good approximation of the flux density
5.3. Constraints

Table 5.1: List of all constraints for the rotary actuator.

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{\text{core}} &lt; 1.4 \text{T}$</td>
<td>Saturation in core</td>
</tr>
<tr>
<td>$B_{\text{back}} = 1.4 \text{T}$</td>
<td>Saturation in back-iron</td>
</tr>
<tr>
<td>$r_i &gt; 18 \text{mm}$</td>
<td>Minimum inner radius</td>
</tr>
<tr>
<td>$r_o &lt; 30 \text{mm}$</td>
<td>Maximum outer radius</td>
</tr>
<tr>
<td>$L &lt; 40 \text{mm}$</td>
<td>Maximum length actuator</td>
</tr>
<tr>
<td>$T_w &lt; 100 \degree \text{C}$</td>
<td>Maximum coil temperature</td>
</tr>
<tr>
<td>$T_m &lt; 60 \degree \text{C}$</td>
<td>Maximum magnet temperature</td>
</tr>
</tbody>
</table>

Distribution inside the airgap when steel with a high relative permeability is used and, therefore, saturation of the steel has to be avoided. The flux density inside the core and back-iron are calculated according to

\[
\begin{align*}
B_{\text{core}} &= \frac{1}{A_{\text{core}}} \int_0^{\pi/p} r_c L_{\text{act}} B_{r11}(r = r_c) d\phi, \quad (5.11) \\
B_{\text{back}} &= \frac{1}{A_{\text{back}}} \int_0^{\pi/p} r_b L_{\text{act}} B_{r11}(r = r_b) d\phi, \quad (5.12)
\end{align*}
\]

where $A_{\text{core}}$ and $A_{\text{back}}$ are the cross-sectional areas in the $rz$-plane of the core and back-iron. To limit the radial length of the both parts, the peak flux density inside them is fixed to 1.4T. However, for manufacturing reasons, this length has a minimum of 0.5 mm.

**Volume constraints**

Due to the size of the shaft, there is a minimum inequality constraint for the inner radius of the actuator. The outer radius has a maximum inequality constraint and for the moving magnet and moving coil configuration, the outer radius is determined according to

\[
\begin{align*}
  r_o &= r_i + l_c + l_m + l_{cl} + l_w + l_b, \\
  r_o &= r_i + l_c + l_m + 2l_{cl} + l_w + l_b,
\end{align*}
\]

respectively, where $l_{cl} = 0.2 \text{ mm}$ is the clearance airgap.

The total length of the actuator is constrained by the available height inside $z\phi$-module minus the height of the linear actuator and is equal to the length of the coils

\[
L = L_{\text{coil}} = L_{\text{act}} + z_{\text{stroke}} + 2L_{\text{end}}
\]

where $L_{\text{end}}$ is the length of the end-windings and $z_{\text{stroke}}$ is the linear stroke. This length decreases with increasing number of coils, leaving more space available to increasing the active length of the rotary actuator.

**Thermal constraints**

To prevent damage of the winding insulation, possible irreversible demagnetization of the magnets or substantial loss of performance due to a lower value of the intrinsic magnetization, the coil and magnet temperatures, which are predicted using the model presented in Section 4.10, are constrained to 100 °C and 60 °C, respectively.
5.4 Optimization results

The optimization is performed for the moving magnet configuration and moving coil configuration and Fig. 5.4 shows the minimized copper losses for different magnet counts. Both configurations have a minimum at $p=14$ and the sizes and specifications of these designs are given in Table 5.2, which also gives the design of the pre-prototype with moving coils. The optimized design with moving coils shows a significant reduction of the copper losses (48%) compared to the initial design. The moving magnet configuration, though, also shows reduced copper losses (24%). Both designs are constrained by the volume.

For the moving magnet configuration the optimization procedure tends to increase the inner radius of the rotary actuator, while occupying the maximum outer volume. For the moving coil configuration the radial length of the windings, $l_w$, is minimized and a lower bound is required. This bound is set to the width of a single wire which is used in the pre-prototype (0.912 mm). The design has long magnets because their inertia and mass are not of any influence on the performance of the rotary and linear actuator. However, the radial length can be decreased, without reducing the airgap flux density significantly. This influence will be addressed in Chapter 6.

In Fig. 5.5, the results for optimization with the objective to minimize the volume of the rotary actuator are shown. The figure shows a minimum for $p=18$ and $p=16$ for the moving coil and moving magnet configurations, respectively. In Table 5.3 the specifications of both configurations are given. Again, a lower bound is set to the length of the windings. Both designs occupy only half the volume of the initial design. The minimization of the volume, however, is limited by the magnet temperature.
Table 5.2: Initial and optimized design with minimized copper losses (MC = moving coil, MM = Moving magnet).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Initial MC</th>
<th>Opt. MC</th>
<th>Opt. MM</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>8</td>
<td>14</td>
<td>14</td>
<td>Number of magnet pole pairs</td>
</tr>
<tr>
<td>$Q$</td>
<td>12</td>
<td>21</td>
<td>21</td>
<td>Number of coils</td>
</tr>
<tr>
<td>$r_i$ [mm]</td>
<td>18.0</td>
<td>18.0</td>
<td>24.8</td>
<td>Inner radius</td>
</tr>
<tr>
<td>$r_o$ [mm]</td>
<td>30.0</td>
<td>30.0</td>
<td>30.0</td>
<td>Outer radius</td>
</tr>
<tr>
<td>$l_c$ [mm]</td>
<td>1.5</td>
<td>0.5</td>
<td>0.5</td>
<td>Radial core length</td>
</tr>
<tr>
<td>$l_m$ [mm]</td>
<td>5.7</td>
<td>8.6</td>
<td>2.1</td>
<td>Radial magnet length</td>
</tr>
<tr>
<td>$l_w$ [mm]</td>
<td>1.9</td>
<td>0.9</td>
<td>1.1</td>
<td>Radial length coils</td>
</tr>
<tr>
<td>$l_b$ [mm]</td>
<td>2.5</td>
<td>1.6</td>
<td>1.2</td>
<td>Radial length back-iron</td>
</tr>
<tr>
<td>$L_{act}$ [mm]</td>
<td>15.0</td>
<td>18.7</td>
<td>18.6</td>
<td>Active length actuator</td>
</tr>
<tr>
<td>$L$ [mm]</td>
<td>37</td>
<td>37</td>
<td>37</td>
<td>Total axial length rotary actuator</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.67</td>
<td>0.5</td>
<td>0.7</td>
<td>Radial magnet arc to total magnet pole arc ratio</td>
</tr>
<tr>
<td>$\beta_o$ [degree]</td>
<td>1.7</td>
<td>3.4</td>
<td>1.5</td>
<td>Coil opening angle</td>
</tr>
<tr>
<td>$V$ [m$^3$]</td>
<td>$104.6\times10^{-6}$</td>
<td>$104.6\times10^{-6}$</td>
<td>$104.6\times10^{-6}$</td>
<td>Actuator volume</td>
</tr>
<tr>
<td>$P_{cu,tot}$ [W]</td>
<td>13.04</td>
<td>6.84</td>
<td>9.96</td>
<td>Total copper losses</td>
</tr>
<tr>
<td>$P_{cu,rot}$ [W]</td>
<td>7.49</td>
<td>3.40</td>
<td>5.63</td>
<td>Rotary copper losses</td>
</tr>
<tr>
<td>$P_{cu,lin}$ [W]</td>
<td>5.55</td>
<td>3.44</td>
<td>4.33</td>
<td>Linear copper losses</td>
</tr>
<tr>
<td>$T_{coil}$ [°C]</td>
<td>36.9</td>
<td>28.2</td>
<td>35.2</td>
<td>Coil temperature</td>
</tr>
<tr>
<td>$T_{magnet}$ [°C]</td>
<td>34.6</td>
<td>27.1</td>
<td>32.9</td>
<td>Magnet temperature</td>
</tr>
<tr>
<td>$J_{tot}$ [kg mm$^2$]</td>
<td>142</td>
<td>103</td>
<td>119</td>
<td>Total inertia</td>
</tr>
<tr>
<td>$M_{tot}$ [g]</td>
<td>282</td>
<td>222</td>
<td>249</td>
<td>Total moving mass</td>
</tr>
</tbody>
</table>

Figure 5.5: Minimized volume for different numbers of the magnet pole pairs.
Table 5.3: Initial and optimized design with minimized volume (MC = moving coil, MM = Moving magnet).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Initial MC</th>
<th>Opt. MC</th>
<th>Opt. MM</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>8</td>
<td>18</td>
<td>16</td>
<td>Number of magnet pole pairs</td>
</tr>
<tr>
<td>$Q$</td>
<td>12</td>
<td>27</td>
<td>24</td>
<td>Number of coils</td>
</tr>
<tr>
<td>$r_t$ [mm]</td>
<td>18.0</td>
<td>18.0</td>
<td>18.0</td>
<td>Inner radius</td>
</tr>
<tr>
<td>$r_o$ [mm]</td>
<td>30.0</td>
<td>22.8</td>
<td>22.7</td>
<td>Outer radius</td>
</tr>
<tr>
<td>$l_c$ [mm]</td>
<td>1.5</td>
<td>0.5</td>
<td>0.5</td>
<td>Radial core length</td>
</tr>
<tr>
<td>$l_m$ [mm]</td>
<td>5.7</td>
<td>2.4</td>
<td>2.3</td>
<td>Radial magnet length</td>
</tr>
<tr>
<td>$l_w$ [mm]</td>
<td>1.9</td>
<td>0.9</td>
<td>0.9</td>
<td>Radial length coils</td>
</tr>
<tr>
<td>$l_b$ [mm]</td>
<td>2.5</td>
<td>0.6</td>
<td>0.8</td>
<td>Radial length back-iron</td>
</tr>
<tr>
<td>$L_{act}$ [mm]</td>
<td>15.0</td>
<td>15.5</td>
<td>15.8</td>
<td>Active length actuator</td>
</tr>
<tr>
<td>$L$ [mm]</td>
<td>37</td>
<td>30.5</td>
<td>31.4</td>
<td>Total axial length rotary actuator</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.67</td>
<td>0.57</td>
<td>0.59</td>
<td>Radial magnet arc to total magnet pole arc ratio</td>
</tr>
<tr>
<td>$\beta_0$ [degree]</td>
<td>1.70</td>
<td>1.26</td>
<td>1.21</td>
<td>Coil opening angle</td>
</tr>
<tr>
<td>$V$ [m$^3$]</td>
<td>104.6x10$^{-6}$</td>
<td>49.9x10$^{-6}$</td>
<td>50.9x10$^{-6}$</td>
<td>Actuator volume</td>
</tr>
<tr>
<td>$P_{cu,\text{tot}}$ [W]</td>
<td>13.04</td>
<td>17.27</td>
<td>15.05</td>
<td>Total copper losses</td>
</tr>
<tr>
<td>$P_{cu,\text{rot}}$ [W]</td>
<td>7.49</td>
<td>13.99</td>
<td>11.35</td>
<td>Rotary copper losses</td>
</tr>
<tr>
<td>$P_{cu,\text{lin}}$ [W]</td>
<td>5.55</td>
<td>3.28</td>
<td>3.70</td>
<td>Linear copper losses</td>
</tr>
<tr>
<td>$T_{coil}$ [°C]</td>
<td>36.9</td>
<td>66.4</td>
<td>66.9</td>
<td>Coil temperature</td>
</tr>
<tr>
<td>$T_{magnet}$ [°C]</td>
<td>34.6</td>
<td>60</td>
<td>60</td>
<td>Magnet temperature</td>
</tr>
<tr>
<td>$J_{tot}$ [kg mm$^2$]</td>
<td>142</td>
<td>91</td>
<td>94</td>
<td>Total inertia</td>
</tr>
<tr>
<td>$M_{tot}$ [g]</td>
<td>282</td>
<td>217</td>
<td>230</td>
<td>Total moving mass</td>
</tr>
</tbody>
</table>
Chapter 6

Discussion of the optimized designs

In the multi-physical framework several assumptions are made in order to simplify the optimization procedure. Some assumptions, however, can have a large influence on the performance of the rotary actuator and can, therefore, not be neglected. In this chapter, the optimized designs are discussed and re-evaluated.

The optimized design with moving coils uses a large magnet volume, which, however, can be decreased as is shown in Fig. 6.1. The figure depicts the airgap flux density as function of the radial length of the magnets, \( l_m \), while the outer radius and airgap length are kept constant. By changing \( l_m \) from 8.6 mm to 5.4 mm, the magnet volume decreases by 32.7\%, while the airgap flux density is only reduced by 3.2\%.

As is already mentioned in , the reduced amplitude of the EMF measurements compared to 2D FEM simulations, can be explained by the edge-effects of the magnets. For the design of the pre-prototype, the peak airgap flux density along the axial length of the magnets is simulated with 3D FEM, and Fig. 6.2 shows a comparison to the flux density predicted with 2D FEM. A reduction of 10.3\% of the flux linkage by the coils and, hence, a similar reduction of the EMF, is predicted from this analysis. Likewise, the magnet edge-effects are predicted for the optimized designs and the results are given in Table 6.1.

Another parameter of influence on the performance of the rotary actuator is the increasing resistivity of copper with higher temperatures. Not only will the ohmic losses increase, also the temperature inside the actuator will be higher. The resistivity as function of the temperature can be expressed as

\[
\rho_{cu}(T) = \rho_{cu,T_0}(1 + e(T - T_0)),
\]

where \( \rho_{cu,T_0} \) is the resistivity at reference temperature, \( T_0 \) and \( e = 0.00396 \text{ K}^{-1} \) is introduced as the relative increase of the resistivity per Kelvin [33]. This effect is included in the TEC by adding a negative resistance next to the heat source. The temperature dependency of

<table>
<thead>
<tr>
<th>Design</th>
<th>Objective function</th>
<th>Reduced magnetic loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Wijdeven</td>
<td>( P_{cu} )</td>
<td>10.3 %</td>
</tr>
<tr>
<td>Moving coil</td>
<td>( P_{cu} )</td>
<td>6.3 %</td>
</tr>
<tr>
<td>Moving magnet</td>
<td>( P_{cu} )</td>
<td>4.7 %</td>
</tr>
<tr>
<td>Moving coil</td>
<td>( V )</td>
<td>5.4 %</td>
</tr>
<tr>
<td>Moving magnet</td>
<td>( V )</td>
<td>5.1 %</td>
</tr>
</tbody>
</table>
Discussion of the optimized designs

Figure 6.1: Influence of the magnet length on the airgap flux density.

the resistivity is not taken into account for in the optimization, because certain resulted in a temperature runaway. This, however, can be avoided by limiting the design range of the variation parameters. Also eddy currents, induced in the core and back-iron, have an influence on the thermal behavior of the rotary actuator and give rise to the losses.

The copper losses in the optimized designs are recalculated while the reduced magnetic loading and the increasing resistivity are accounted for. In Tables 6.2 and 6.3 the re-evaluated values are given, with the old values indicated between brackets.

With the objective to minimize the copper losses, the optimized design with moving coils shows a higher reduction of the copper losses (51%) compared to the initial design, than the moving-magnet configuration (30%). This is due to the fact that the moving-coil configuration has a lower inertia and moving mass. Both optimized design, however use the same amount of volume as the initial design and the thermal constraints are easily met.

For optimal design with minimized volume, both the moving-magnet and moving-coil configuration occupy 51% and 52%, respectively, less volume than the initial design. On the other hand, the copper losses have increased by 25% and 47%, respectively. Also the temperatures have increased and exceed the thermal constraint for the magnets. In this case, the moving-magnet configuration is preferred because it has lower copper losses.
Table 6.2: Re-evaluated values for the initial and optimized design with minimized copper losses (MC = moving coil, MM = Moving magnet). Old values are indicated between brackets.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Initial MC</th>
<th>Opt. MC</th>
<th>Opt. MM</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>8</td>
<td>14</td>
<td>14</td>
<td>Number of magnet pole pairs</td>
</tr>
<tr>
<td>$Q$</td>
<td>12</td>
<td>21</td>
<td>21</td>
<td>Number of coils</td>
</tr>
<tr>
<td>$r_i$ [mm]</td>
<td>18.0</td>
<td>21.2</td>
<td>24.8</td>
<td>Inner radius</td>
</tr>
<tr>
<td>$r_o$ [mm]</td>
<td>(18.0)</td>
<td>30.0</td>
<td>30.0</td>
<td>Outer radius</td>
</tr>
<tr>
<td>$l_c$ [mm]</td>
<td>1.5</td>
<td>0.5</td>
<td>0.5</td>
<td>Radial core length</td>
</tr>
<tr>
<td>$l_m$ [mm]</td>
<td>5.7</td>
<td>5.4</td>
<td>2.1</td>
<td>Radial magnet length</td>
</tr>
<tr>
<td>$l_w$ [mm]</td>
<td>1.9</td>
<td>0.9</td>
<td>1.1</td>
<td>Radial length coils</td>
</tr>
<tr>
<td>$l_b$ [mm]</td>
<td>2.5</td>
<td>1.6</td>
<td>1.2</td>
<td>Radial length back-iron</td>
</tr>
<tr>
<td>$L_{act}$ [mm]</td>
<td>15.0</td>
<td>18.7</td>
<td>18.6</td>
<td>Active length actuator</td>
</tr>
<tr>
<td>$L$ [mm]</td>
<td>37</td>
<td>37</td>
<td>37</td>
<td>Total axial length rotary actuator</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.67</td>
<td>0.5</td>
<td>0.7</td>
<td>Radial magnet arc to radial magnet pole arc ratio</td>
</tr>
<tr>
<td>$\beta_o$ [degree]</td>
<td>1.7</td>
<td>3.4</td>
<td>1.5</td>
<td>Coil opening angle</td>
</tr>
<tr>
<td>$V$ [m$^3$]</td>
<td>104.6 × 10^{-6}</td>
<td>104.6 × 10^{-6}</td>
<td>104.6 × 10^{-6}</td>
<td>Actuator volume</td>
</tr>
<tr>
<td>$P_{cu,tot}$ [W]</td>
<td>16.44</td>
<td>8.50</td>
<td>12.45</td>
<td>Total copper losses</td>
</tr>
<tr>
<td></td>
<td>(13.04)</td>
<td>(6.84)</td>
<td>(9.96)</td>
<td></td>
</tr>
<tr>
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<td>10.90</td>
<td>5.06</td>
<td>8.11</td>
<td>Rotary copper losses</td>
</tr>
<tr>
<td></td>
<td>(7.49)</td>
<td>(3.40)</td>
<td>(5.63)</td>
<td></td>
</tr>
<tr>
<td>$P_{cu,lin}$ [W]</td>
<td>5.55</td>
<td>3.44</td>
<td>4.33</td>
<td>Linear copper losses</td>
</tr>
<tr>
<td>$T_{coil}$ [°C]</td>
<td>42.9</td>
<td>31.2</td>
<td>40.5</td>
<td>Coil temperature</td>
</tr>
<tr>
<td></td>
<td>(36.9)</td>
<td>(28.2)</td>
<td>(35.2)</td>
<td></td>
</tr>
<tr>
<td>$T_{magnet}$ [°C]</td>
<td>39.7</td>
<td>28.9</td>
<td>35.2</td>
<td>Magnet temperature</td>
</tr>
<tr>
<td></td>
<td>(34.6)</td>
<td>(27.1)</td>
<td>(32.9)</td>
<td></td>
</tr>
<tr>
<td>$J_{tot}$ [kg mm$^2$]</td>
<td>142</td>
<td>103</td>
<td>119</td>
<td>Total inertia</td>
</tr>
<tr>
<td>$M_{tot}$ [g]</td>
<td>282</td>
<td>222</td>
<td>249</td>
<td>Total moving mass</td>
</tr>
</tbody>
</table>
Table 6.3: Re-evaluated values for the initial and optimized design with minimized volume (MC = moving coil, MM = Moving magnet). Old values are indicated between brackets.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Initial MC</th>
<th>Opt. MC</th>
<th>Opt. MM</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>8</td>
<td>18</td>
<td>16</td>
<td>Number of magnet pole pairs</td>
</tr>
<tr>
<td>$Q$</td>
<td>12</td>
<td>27</td>
<td>24</td>
<td>Number of coils</td>
</tr>
<tr>
<td>$r_i$ [mm]</td>
<td>18.0</td>
<td>18.0</td>
<td>18.0</td>
<td>Inner radius</td>
</tr>
<tr>
<td>$r_o$ [mm]</td>
<td>30.0</td>
<td>22.8</td>
<td>22.7</td>
<td>Outer radius</td>
</tr>
<tr>
<td>$l_c$ [mm]</td>
<td>1.5</td>
<td>0.5</td>
<td>0.5</td>
<td>Radial core length</td>
</tr>
<tr>
<td>$l_m$ [mm]</td>
<td>5.7</td>
<td>2.4</td>
<td>2.3</td>
<td>Radial magnet length</td>
</tr>
<tr>
<td>$l_w$ [mm]</td>
<td>1.9</td>
<td>0.9</td>
<td>0.9</td>
<td>Radial length coils</td>
</tr>
<tr>
<td>$l_b$ [mm]</td>
<td>2.5</td>
<td>0.6</td>
<td>0.8</td>
<td>Radial length back-iron</td>
</tr>
<tr>
<td>$L_{act}$ [mm]</td>
<td>15.0</td>
<td>15.5</td>
<td>15.8</td>
<td>Active length actuator</td>
</tr>
<tr>
<td>$L$ [mm]</td>
<td>37</td>
<td>30.5</td>
<td>31.4</td>
<td>Total axial length rotary actuator</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.67</td>
<td>0.57</td>
<td>0.59</td>
<td>Radial magnet arc to total magnet pole arc ratio</td>
</tr>
<tr>
<td>$\beta_o$ [degree]</td>
<td>1.70</td>
<td>1.26</td>
<td>1.21</td>
<td>Coil opening angle</td>
</tr>
<tr>
<td>$V$ [m³]</td>
<td>104.6×10⁻⁶</td>
<td>49.9×10⁻⁶</td>
<td>50.9×10⁻⁶</td>
<td>Actuator volume</td>
</tr>
<tr>
<td>$P_{cu,\text{tot}}$ [W]</td>
<td>16.44</td>
<td>24.2</td>
<td>20.58</td>
<td>Total copper losses</td>
</tr>
<tr>
<td>$P_{cu,\text{rot}}$ [W]</td>
<td>(13.04)</td>
<td>(17.27)</td>
<td>(15.05)</td>
<td>Rotary copper losses</td>
</tr>
<tr>
<td>$P_{cu,\text{lin}}$ [W]</td>
<td>10.90</td>
<td>20.92</td>
<td>16.87</td>
<td>Linear copper losses</td>
</tr>
<tr>
<td>$T_{\text{coil}}$ [°C]</td>
<td>42.9</td>
<td>85.3</td>
<td>85.6</td>
<td>Coil temperature</td>
</tr>
<tr>
<td>$T_{\text{magnet}}$ [°C]</td>
<td>(36.9)</td>
<td>(66.4)</td>
<td>(66.9)</td>
<td>Magnet temperature</td>
</tr>
<tr>
<td>$J_{\text{tot}}$ [kg mm²]</td>
<td>142</td>
<td>91</td>
<td>94</td>
<td>Total inertia</td>
</tr>
<tr>
<td>$M_{\text{tot}}$ [g]</td>
<td>282</td>
<td>217</td>
<td>230</td>
<td>Total moving mass</td>
</tr>
</tbody>
</table>
Figure 6.2: 2D and 3D FEM comparison of flux density along the axial length of the permanent magnet.
Discussion of the optimized designs
Chapter 7

Conclusions and recommendations

7.1 Conclusions

• In this thesis a new design has been proposed for a rotary actuator as part of a $z\phi$-module, because the initial design proposed by Wijdeven B.V. was not optimized for the pick-and-place application.

• Combinations of the number of magnet poles and coils, which are analyzed during optimization, have been selected based upon the winding factor. The winding factor has been calculated for slotless machines and combinations resulting in 1/4 coil per magnet pole per phase have the highest winding factor.

• A fast multi-physical framework has been created containing a
  - 2D semi-analytical magneto-static model of the slotless permanent magnet actuator with a two-segmented Halbach magnet array and straight magnetization,
  - thermal equivalent circuit to model the radial heat flow inside and the forced convection due to air cooling inside the airgap,
  - simplified mechanical model, which couples the linear and rotary actuator inside the $z\phi$-module.

• The design of the rotary actuator has been optimized using the multi-physical framework for a moving coil and moving magnet configuration with two objectives:
  - The combined copper losses of the rotary and linear actuator have been minimized so a efficient design of the $z\phi$-module is obtained and an optimal balanced is created between the moving mass and inertia.
  - The volume of the rotary has been minimized in order to show that the same specifications are obtained within a smaller design.

• The performance of the optimized models have been re-evaluated for:
  - Magnet edge-effects. The magneto-static model does not account for the finite length of the magnets, because is requires 3D modeling. Due to this effect the actual flux linkage of the actuator is reduced. This reduction has been analyzed with 3D FEM and the same reduction of flux linkage has been verified by EMF-measurements on a pre-prototype of the initial design.
Conclusions and recommendations

- Temperature dependency of the electrical resistivity of copper. This effect was not taken into account during optimization because it results in thermal runaway. With higher temperatures, however, a higher electrical resistivity increases the copper losses.

- The re-evaluated designs of the rotary actuator have been compared to the initial design proposed by Wijdeven B.V.

  - For the minimized copper losses the moving coil configuration results in the lowest losses, which are 51% less compared to the initial design.

  - For minimized volume both configurations only occupy 50% of the initial design volume. The copper losses, however, have increased (upto 47% for the moving coil configuration). Also the maximum magnet temperature has been exceeded and a different magnet grade should therefore be selected.

7.2 Recommendations

- A model of the magnet edge-effects should be created and the thermal runaway due to the temperature dependency of the electrical resistivity needs to be solved. This way, both effects can be accounted for during the optimization step.

- The effect of eddy-currents in the back-iron and magnets on the performance of the rotary actuator should be investigated.

- The thermal model needs to be verified by extensive measurements on a prototype.

- The mechanical strength of the coil array needs to be determined.
Bibliography


Appendix A

Initial Design

The first designs for the 2-DOF actuator were proposed by Wijdeven B.V. and is called the Rotrans (Rotatie-Translatie (dutch)). These designs consist of a voice-coil actuator at the bottom of the machine for the linear motion, as can be seen in Fig. A.1. A slotless synchronous permanent magnet machine at the top of the 2-DOF machine, is used for the rotational motion.

Particular to this machine are the moving windings/coils instead of moving magnets. This choice was made, as it was believed that eddy currents in the back-iron would be minimized, which cause additional (undesired) torque and losses. A configuration with moving coils limits the motion of the machine, because of the electrical wiring to the servo-amplifier. Some degree of freedom was achieved by forming wires, made of phosphor bronze, into a helical spring.

![Diagram of the first version of the 2-DOF machine.](image)

Figure A.1: 3D impression of the first version of the 2-DOF machine.

Initially the rotary actuator contained 12 magnet poles and 18 coils, but after redesign a combination of 16 magnet poles and 12 coils was chosen. A quasi-Halbach magnet array with two segments per pole was selected, since it offers a higher magnetic loading compared to a radial magnetized array and produces a more sinusoidal flux density in the airgap. The
Table A.1: Leading dimensions for the rotary actuator.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Description</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>Number of pole pairs</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>$Q$</td>
<td>Number of coils</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>$r_i$</td>
<td>Inner radius actuator</td>
<td>18</td>
<td>mm</td>
</tr>
<tr>
<td>$r_c$</td>
<td>Outer radius core</td>
<td>19.5</td>
<td>mm</td>
</tr>
<tr>
<td>$r_m$</td>
<td>Outer radius magnets</td>
<td>25.2</td>
<td>mm</td>
</tr>
<tr>
<td>$r_s$</td>
<td>Inner radius back iron</td>
<td>27.5</td>
<td>mm</td>
</tr>
<tr>
<td>$r_o$</td>
<td>Outer radius actuator</td>
<td>30</td>
<td>mm</td>
</tr>
<tr>
<td>$l_{cl}$</td>
<td>Length clearance airgap</td>
<td>0.2</td>
<td>mm</td>
</tr>
<tr>
<td>$l_{gap}$</td>
<td>Length airgap</td>
<td>2.3</td>
<td>mm</td>
</tr>
<tr>
<td>$l_{coil}$</td>
<td>Thickness coil</td>
<td>1.9</td>
<td>mm</td>
</tr>
<tr>
<td>$\phi_p$</td>
<td>Angular pitch pointer magnet</td>
<td>15</td>
<td>degree</td>
</tr>
<tr>
<td>$\phi_c$</td>
<td>Angular pitch concentrator magnet</td>
<td>7.5</td>
<td>degree</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Radial magnet arc to pole arc ratio</td>
<td>2/3</td>
<td></td>
</tr>
<tr>
<td>$L_{act}$</td>
<td>Stack height magnets</td>
<td>15</td>
<td>mm</td>
</tr>
</tbody>
</table>

ratio between the angular pitch of the radial and angular magnetised magnets was 2:1. Stack height of the magnets is 15 mm, while the length of the coils is set to 37 mm, with the straight part of the coils being 23 mm long. For a stroke of 10 mm this opening is too short, giving loss of torque when the magnets are close to the ends of the windings. Additionally an extra force is created in the axial direction, since is an attracting or repelling Lorentz force acts on the current conducting windings.

The coils contain 92 turns of 0.9x0.12 mm square copper wire. The three phases of the machine are wye-connected. Total resistance of one coil was estimated to be 1.17 $\Omega$. Magnets are anisotropic sintered neodymium, with a remanence flux density of 1.33 T and a relative permeability of 1.1. The back iron and inner core were made of N398 steel, having a saturation flux density around 1.5 T. Figure A.3 shows the BH-curve for the magnets and steel.

Figure A.2 and Table A.1 show the dimensions of the initial design proposed by Wijdeven B.V.

A.1 Specifications and requirements

Together with the voice-coil actuator, the rotary actuator has following specifications:

Table A.2: Specifications of the rotrans.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Load mass</td>
<td>0.5</td>
<td>kg</td>
</tr>
<tr>
<td>Load inertia</td>
<td>$1.66\times10^{-4}$</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>Friction force</td>
<td>unknown</td>
<td>N</td>
</tr>
<tr>
<td>Friction torque</td>
<td>unknown</td>
<td>Nm</td>
</tr>
</tbody>
</table>

Since air bearings will be used, the friction force and torque are assumed to be very small and, therefore, are neglected. Maximum torque is required when maximum acceleration of
A.1. Specifications and requirements

Figure A.2: Simplified drawing of the rotary actuator with the leading design parameters.

Figure A.3: B-H curve for (a) the magnets at $T=20 \degree C$ and (b) steel.
deceleration of the actuator is required and is calculated as follows:

\[ T_{\text{max}} = I \alpha + T_{\text{friction}}, \quad (A.1) \]

where \( T_{\text{max}} \) is the peak torque, \( I \) the load inertia, \( \alpha \) the angular acceleration and \( T_{\text{friction}} \) the friction torque. Without any friction, the required torque is equal to the desired acceleration.

To calculate the rms torque over one cycle of rotation the profile of the acceleration as is shown in Fig. 5.1 is used. One cycle consist of a triangular acceleration waveform for 70 ms and zero acceleration for 110 ms. The rms torque is calculated as

\[ T_{\text{rms}} = \sqrt{\frac{1}{\tau} \int_0^\tau T^2 dt} = \sqrt{\frac{1}{t_{\text{triangle}}} \int_{t_{\text{triangle}}}^\tau T^2 dt} = \sqrt{\frac{t_{\text{triangle}}}{\tau} T_{\text{rms,triangle}}^2} = \sqrt{d(T_{\text{max}}) \frac{2}{\sqrt{3}}} \quad (A.2) \]

where \( T_{\text{rms}} \) is the rms torque during one cycle, \( \tau \) the period of one cycle, \( t_{\text{triangle}} \) the time in which the actuator accelerates/decelerates, \( T_{\text{rms,triangle}} \) is the rms value of the torque with a triangular profile and \( d \phi \) is the duty cycle. Table A.3 the peak and rms torque are given for a load inertia of \( 1.66 \times 10^{-4} \) kg m\(^2\). For control reasons a dynamical factor of 1.5 is used to give the required torque performance of the rotary actuator.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{\text{max}} )</td>
<td>0.85</td>
<td>Nm</td>
</tr>
<tr>
<td>( T_{\text{rms}} )</td>
<td>0.31</td>
<td>Nm</td>
</tr>
<tr>
<td>( T_{\text{max,1.5}} )</td>
<td>1.3</td>
<td>Nm</td>
</tr>
<tr>
<td>( T_{\text{rms,1.5}} )</td>
<td>0.47</td>
<td>Nm</td>
</tr>
</tbody>
</table>

A.2 FEM modelling of rotary actuator

In the very first design of the machine, the core behind the magnets and the back iron were connected via a circular U-shape core made also made of steel N398. A 3D FE-model, shown in Fig. A.4, was used to simulate whether flux would also flow through this U-shape part (rz-plane), rather than only through the magnets, back iron and core (rθ-plane). Figure A.4 shows the a section of the rotary motor. To the left the iron parts and magnets are shown, coils are simulated as non-meshed coils and are situated in the airgap. To the right the flux density across the iron parts are shown. The simulations showed that the U-shaped iron frame had no contribution to the flux flow, so this was eventually made of aluminium in order to reduce the mass and inertia of the machine.
To simulate the performance of the rotary actuator, a 2D FE-model was used, because the 3D model was time consuming. Having a model depth equal to the stack height of the magnets, the 2D model does not take the drop-off in the magnetic field near the edges of the magnets into account. End-windings of the coils are neglected, resulting in a lower induction seen by the inverter, which can have an effect on transient modelling with a driving circuit. Only a quarter of the machine is simulated, as it saves calculation time. Some simulation results, like the electromagnetic torque, therefore, are multiplied by 4 in order to get the correct results. The dimensions used for the 2D simulations are given in Table A.1. Figure A.5 shows the flux lines and flux density distribution in the actuator.
The quasi-Halbach magnet array gives a radial magnetic field in the airgap with an amplitude of the first harmonic of 0.97 T and 0.1 T for the third component, as is shown in Fig. A.6.

For constant torque production, the currents are commuted. The phase currents are determined as

\[ I_a = I \cos(p\theta + \theta_{\text{comm}}), \]
\[ I_b = I \cos(p\theta + \theta_{\text{comm}} + 2\pi/3), \]
\[ I_c = I \cos(p\theta + \theta_{\text{comm}} - 2\pi/3), \]

where \( I \) is the peak current, \( p \) is the number of pole pairs, \( \theta \) is the angular position and \( \theta_{\text{comm}} \) is the commutation offset angle. For \( I=1.4 \) A and \( \theta_{\text{comm}}=36 \) degree, the torque and phase currents are shown in Fig. A.7. Here the actuator produces a constant torque of 0.47 Nm, as is specified.
A.3. Tests

Figure A.8 shows the back-EMF for all three phases at maximum velocity of 900 rpm. The peak EMF is approximately 21 V.

![Figure A.8: Phase back-emf for the rotary actuator for a velocity of 900 rpm.](image)

A.3 Tests

To test the rotary actuator, a prototype was build by Wijdeven. A special housing was designed to link the core and magnets with the back iron. Instead of moving the coils, the magnets were moved, which have a larger inertia. The air bearing was replaced by a sintered bronze bearing, which slides over a steel surface. The Elmo Cornet AC-Powered digital servo drive with PWM switching frequency of 22 kHz and a current loop bandwidth of 2.5 kHz was selected to drive the actuator. Voltage output of the inverter is variable between ±60 and 270 V and the amplitude for a continuous sinusoidal current is 9 A. The Heidenhain ROD1020 encoder with 3600 lines (14400 for quadrature signal) and TTL interface is used for measuring the angular position, velocity and acceleration of the actuator.

Self-inductances of the coils were calculated by FE-modeling, although this does not take the end-windings into account. Table A.4 gives the predicted and measured inductance and resistance of a single coil. Since the inverter switches between two phases, it sees 8 coils in total (4 per phase). The total voltage over the coils can be expressed as

$$u(t) = \sum (R_{coil}i(t) + L \frac{di(t)}{dt} + \frac{dA}{dt}),$$

(A.3)

where $R$ is the resistance of one coil, $i(t)$ the current, $L$ the inductance, $\frac{dA}{dt}$ the voltage over the coil induced by time varying flux linkage. Taking the maximum value of the all three terms on the right side of the equation, the peak voltage over two phase becomes (assuming only self inductance)

$$U_{max} = 8R_{coil}I + 8p\omega L_{self}I + \sqrt{3}U_{emf},$$

(A.4)

where the values for $R_{coil}$ and $L_{self}$ are given in Table A.4, $I=1.4$ A, $\omega_c=90$ rad s$^{-1}$ and $U_{emf}$ given in the previous section, the maximum line-line voltage is 80 V. It is highly unlikely that the voltage becomes this larger, since all terms on the right hand side of Eq. A.3 are
92 Initial Design

sinusoidal which are out of phase. Though, the inverter output voltage is set to 100 V, so it is capable of driving the actuator.

As the inverter switches +V over the coils for one half of the switching period and -V during the other half, the magnitude of the ripple current through is:

$$I_{\text{ripple}} = \frac{V}{2f_{\text{switch}}L_{\text{phase-phase}}} \quad (A.5)$$

Set at an output voltage of ±100 V, the ripple current produced by the inverter is 1.4 A, which is quite significant. Extra inductance in series with the actuator reduces the ripple.

Table A.4: Predicted and measured inductance and resistance of one coil.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Predicted</th>
<th>Measured</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>1.17</td>
<td>1.0</td>
<td>Ω</td>
</tr>
<tr>
<td>L</td>
<td>0.14</td>
<td>0.21</td>
<td>mH</td>
</tr>
</tbody>
</table>

During assembly of the coils in the coil carrier, the insulation of the wires was damaged, resulting in short circuit between coils via the coil carrier. After a second try the same problem occurred with a couple of coils. In order to perform tests, one coil per phase was removed from the electrical circuit (still placed in carrier), so 3 coils per phase were left. For test performed below the required torque was scaled by 75%.

A.3.1 Test 1: Pull test at rms torque

The actuator was attached to a Lloyd LF Plus testing machine with a load cell of 50 N. A wire attached to the load cell was wound around the actuator at a radius of 32.5 mm. The Elmo software was set torque command and the peak current was set to 1.8 A in order to produce the specified rms torque of 0.35 Nm (0.47 for 12 coils). This pull test was performed with a air flow of 20, 40 and 80 liters of air per minute through the actuator. Tables A.5, A.6 and A.7 give the measured torque (torque=force*radius) (with friction torque) and the temperature of the coils. The torque constant (for 12 coils) of the machine is calculated according to

$$k_T = \frac{T}{I_{\text{rms}}} = \sqrt{2} \frac{T}{I} = \sqrt{2} \frac{0.47}{1.8} = 0.37 \, [Nm \, A_{\text{rms}}^{-1}] \quad (A.6)$$

Table A.5: Pull test at rms torque with air flow of 20 liter min⁻¹.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>11.0</td>
<td>0.3575</td>
<td>22</td>
</tr>
<tr>
<td>10</td>
<td>11.0</td>
<td>0.3575</td>
<td>26</td>
</tr>
<tr>
<td>20</td>
<td>11.0</td>
<td>0.3575</td>
<td>31</td>
</tr>
<tr>
<td>30</td>
<td>11.0</td>
<td>0.3575</td>
<td>35</td>
</tr>
<tr>
<td>40</td>
<td>11.0</td>
<td>0.3575</td>
<td>37</td>
</tr>
<tr>
<td>50</td>
<td>11.0</td>
<td>0.3575</td>
<td>38</td>
</tr>
<tr>
<td>60</td>
<td>11.0</td>
<td>0.3575</td>
<td>40</td>
</tr>
<tr>
<td>70</td>
<td>11.0</td>
<td>0.3575</td>
<td>41</td>
</tr>
<tr>
<td>80</td>
<td>11.0</td>
<td>0.3575</td>
<td>42</td>
</tr>
<tr>
<td>90</td>
<td>11.0</td>
<td>0.3575</td>
<td>43</td>
</tr>
<tr>
<td>100</td>
<td>11.0</td>
<td>0.3575</td>
<td>43</td>
</tr>
<tr>
<td>300</td>
<td>11.0</td>
<td>0.3575</td>
<td>45</td>
</tr>
</tbody>
</table>
Table A.6: Pull test at rms torque with air flow of 40 liter min$^{-1}$.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>11.0</td>
<td>0.3575</td>
<td>22</td>
</tr>
<tr>
<td>10</td>
<td>11.0</td>
<td>0.3575</td>
<td>25</td>
</tr>
<tr>
<td>20</td>
<td>11.0</td>
<td>0.3575</td>
<td>29</td>
</tr>
<tr>
<td>30</td>
<td>11.0</td>
<td>0.3575</td>
<td>31</td>
</tr>
<tr>
<td>40</td>
<td>11.0</td>
<td>0.3575</td>
<td>32</td>
</tr>
<tr>
<td>50</td>
<td>11.0</td>
<td>0.3575</td>
<td>33</td>
</tr>
<tr>
<td>60</td>
<td>11.0</td>
<td>0.3575</td>
<td>34</td>
</tr>
<tr>
<td>70</td>
<td>11.0</td>
<td>0.3575</td>
<td>35</td>
</tr>
<tr>
<td>80</td>
<td>11.0</td>
<td>0.3575</td>
<td>35</td>
</tr>
<tr>
<td>90</td>
<td>11.0</td>
<td>0.3575</td>
<td>36</td>
</tr>
<tr>
<td>100</td>
<td>11.0</td>
<td>0.3575</td>
<td>36</td>
</tr>
<tr>
<td>300</td>
<td>10.9</td>
<td>0.3543</td>
<td>38</td>
</tr>
</tbody>
</table>

Table A.7: Pull test at rms torque with air flow of 80 liter min$^{-1}$.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10.9</td>
<td>0.3543</td>
<td>23</td>
</tr>
<tr>
<td>10</td>
<td>10.9</td>
<td>0.3543</td>
<td>26</td>
</tr>
<tr>
<td>20</td>
<td>10.9</td>
<td>0.3543</td>
<td>29</td>
</tr>
<tr>
<td>30</td>
<td>10.9</td>
<td>0.3543</td>
<td>30</td>
</tr>
<tr>
<td>40</td>
<td>10.9</td>
<td>0.3543</td>
<td>31</td>
</tr>
<tr>
<td>50</td>
<td>10.9</td>
<td>0.3543</td>
<td>32</td>
</tr>
<tr>
<td>60</td>
<td>10.9</td>
<td>0.3543</td>
<td>32</td>
</tr>
<tr>
<td>300</td>
<td>10.9</td>
<td>0.3543</td>
<td>32</td>
</tr>
</tbody>
</table>

A.3.2 Test 2: Slow rotation at rms torque

During this test the actuator is rotated 360 degrees while the a sinusoidal current with an amplitude of 1.8 A was fed to the actuator in order to produce a rms torque of 0.35 Nm (0.47 Nm for 12 coils). Force is measured while the actuator is turned both ways in one minute. The test machine winds or unwinds the wire around the actuator by moving down or up resp, with a velocity of 208 mm min$^{-1}$. The difference between average torque measured in both tests divided by two, is the estimated (coulomb) friction torque. The air flow is set to 20 liter min$^{-1}$. Table A.8 gives the peak, average and minimum force and torque for both tests. The friction force/torque was estimated to be 0.45 N / 0.015 Nm.

Table A.8: Slow rotation test at rms torque with air flow of 20 liter min$^{-1}$.

<table>
<thead>
<tr>
<th></th>
<th>+360°(unwind)</th>
<th>-360°(wind)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force</td>
<td>Torque [Nm]</td>
<td>Force [N]</td>
</tr>
<tr>
<td>Max</td>
<td>12.05</td>
<td>0.40</td>
</tr>
<tr>
<td>Avg</td>
<td>11.7</td>
<td>0.38</td>
</tr>
<tr>
<td>Min</td>
<td>11.25</td>
<td>0.37</td>
</tr>
</tbody>
</table>
A.3.3 Test 3: Pull test at peak torque

The peak current was increased to 5 A in the torque command mode of the Elmo test software in order to produce the peak torque of 0.98 Nm (1.3 Nm for 12 coils). To avoid overheating of the motor the air flow was set to 80 liter min$^{-1}$ and the peak current was fed only momentarily. The results for 3, 4 and 5 A are shown in Table A.9.

<table>
<thead>
<tr>
<th>$I$</th>
<th>Force [N]</th>
<th>Torque [Nm]</th>
<th>Duration peak current [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>18.3</td>
<td>0.595</td>
<td>0.30</td>
</tr>
<tr>
<td>4.0</td>
<td>24.35</td>
<td>0.79</td>
<td>0.40</td>
</tr>
<tr>
<td>5.0</td>
<td>30.8</td>
<td>1.0</td>
<td>0.27</td>
</tr>
</tbody>
</table>

A.3.4 Test 4: Free acceleration at peak torque

The rotary actuator is accelerated with peak torque to a maximum velocity of 860 rpm, while the peak current was set to 5 A in velocity mode. Figure A.10 shows the velocity command and actual velocity of the actuator and the current command and actual current through phase A. Between 0 and 0.2 s the actuator accelerates with a constant current command of 5 A. Its acceleration during this period was estimated to be constant at 5300 rad s$^{-2}$. With a load inertia of $200 \times 10^{-6}$ kg m$^{-2}$, this prototype actuator produces an estimated torque of 1.06 Nm.
A.3. Tests

A.3.5 Test 6: Slow rotation at rms torque with 12 coils

The same test from Test 2 is performed again but now with all coils (12) working. Table A.10 gives the results from this test and shows that the actuator produces specified rms torque.

Table A.10: Slow rotation test at rms torque with air flow of 20l/min with all 12 coils working.

<table>
<thead>
<tr>
<th></th>
<th>+360°(unwind)</th>
<th>-360°(wind)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max</td>
<td>16.2</td>
<td>0.54</td>
</tr>
<tr>
<td>Avg</td>
<td>15.8</td>
<td>0.52</td>
</tr>
<tr>
<td>Min</td>
<td>15.45</td>
<td>0.51</td>
</tr>
</tbody>
</table>

A.3.6 Back-EMF measurements

The back-EMF of the rotary actuator are measured for a angular velocity of 90 rad s\(^{-1}\). The actuator is driven by an external motor and the induced voltages were measured using a oscilloscope. Because of the lack of a velocity controller, the velocity is adjusted manually by measuring the frequency of the induced voltages. For a machine with 8 pole pairs (p) the electrical frequency and a mechanical angular velocity of \(\omega=90\) rad s\(^{-1}\) becomes

\[
f_e = p \frac{\omega_m}{2\pi} = 8 \times \frac{90}{2\pi} \approx 115Hz.
\]

In Table A.11 the measured phase back-EMF and phase-phase back-EMF are given. Because it was difficult to control the velocity of the actuator precisely, the actual frequencies are also given in the table.

Table A.11: Measured phase back-EMF and phase-phase back-EMF and actual frequency.

<table>
<thead>
<tr>
<th></th>
<th>(f_e)</th>
<th>(V_{pp})</th>
<th>(V_{rms})</th>
</tr>
</thead>
<tbody>
<tr>
<td>phase-phase back-EMF</td>
<td>115.1</td>
<td>61.6</td>
<td>21.6</td>
</tr>
<tr>
<td>phase back-EMF</td>
<td>115.3</td>
<td>35.4</td>
<td>12.4</td>
</tr>
</tbody>
</table>
Figure A.3.6 shows screen drops from the oscilloscope for the measured phase back-EMF and phase-phase back-EMF. As the figure shows, the three-phase system is balanced (all graphs are displaced by 120 electrical degrees from one other).

Figure A.11: Measured back-EMF on a three-phase slotless permanent magnet actuator: (a) phase-phase back-EMF, (b) phase back-EMF.
Appendix B

Coefficient derivation

B.1 Open field coefficients

B.1.1 Coefficients for $np \neq 1$

The boundary conditions (4.30)-(4.33) must hold for every harmonic, so the summation sign is eliminated for ease of reading.

• The first boundary condition (4.30) gives

$$H_{\theta I}|_{r=r_c} = \frac{1}{\mu_0} B_{\theta I}(r, \theta)|_{r=r_c} = 0,$$

$$np X_{nl} r_c^{np} - np Y_{nl} r_c^{-np} = 0,$$

$$Y_{nl} = X_{nl} r_c^{2np}. \quad (B.1)$$

• The second boundary condition (4.31) gives

$$H_{\theta II}|_{r=r_b} = \frac{1}{\mu_0 \mu_r} (B_{\theta II}(r, \theta) - \mu_0 M_\theta)|_{r=r_b} = 0,$$

$$np X_{nl} r_b^{np} - np Y_{nl} r_b^{-np} + G_n - \mu_0 M_{n\theta} = 0,$$

$$Y_{nl} = X_{nl} r_b^{2np} + \frac{1}{np} G_n r_b^{np+1} - \frac{\mu_0}{np} M_{n\theta} r_b^{-np+1}. \quad (B.2)$$

• The third boundary condition (4.32) gives

$$B_{r I}|_{r=r_m} = B_{r II}|_{r=r_m},$$

$$X_{nl} r_m^{np} + Y_{nl} r_m^{-np} = X_{nl} r_m^{np-1} + Y_{nl} r_m^{-np-1} + G_n.$$

By substituting Eqs. (B.1) and (B.2) into the above equation, an expression for $X_{nl}$ is obtained as

$$X_{nl} = \frac{X_{nl}(r_m^{np} + r_b^{2np} r_m^{-np}) + G_n(1 + \frac{1}{np} r_b^{np+1} r_m^{-np}) - \mu_0 M_{n\theta} (\frac{1}{np} r_b^{np+1} r_m^{-np})}{r_m^{np} + r_b^{2np} r_m^{-np}}. \quad (B.3)$$
The fourth boundary condition (4.33) gives

\[ H_{\theta I}|_{r=r_m} = \frac{1}{\mu_0} B_{\theta I}|_{r=r_m} = \frac{1}{\mu_0 \mu_r} (B_{\theta III} - \mu_0 M_\theta)|_{r=r_m} = H_{\theta II}|_{r=r_m}, \]

\[ np X_{nII} r_m^{n-1} - np Y_{nII} r_m^{n-1} = \frac{1}{\mu_r} (np X_{nII} r_m^{n-1} - np Y_{nII} r_m^{n-1} + G_n - \mu_0 M_{n\theta}). \]

By substituting Eqs. (B.1), (B.2) and (B.3) into the above equation and separating the variables \( X_{nII}, G_n \) and \( M_{n\theta} \) the following equation is obtained

\[ np X_{nII} \left[ \frac{1}{\mu_r} \left( r_m^{n-1} - r_b^{2n} r_m^{n-1} \right) \right] \left( r_m^{n-1} + r_c^{2n} r_m^{n-1} \right) \]

\[ = G_n \left[ \left( r_m^{n-1} - r_c^{2n} r_m^{n-1} \right) (np + r_b^{np} r_m^{np-1}) \right] \]

\[ - \frac{1}{\mu_r} (1 - r_b^{np} r_m^{np-1}) \left( r_m^{n-1} + r_c^{2n} r_m^{n-1} \right) \]

\[ - \mu_0 M_{n\theta} \left[ (r_m^{n-1} - r_b^{2n} r_m^{n-1}) (r_b^{np} r_m^{np-1}) \right] \]

\[ - \frac{1}{\mu_r} \left( 1 - r_b^{np} r_m^{np-1} \right) \left( r_m^{n-1} + r_c^{2n} r_m^{n-1} \right). \]

By further extending and rewriting this equation, the following expression for \( X_{nII} \) is obtained

\[ X_{nII} = \frac{G_n [N_{nIIc}] + \mu_0 M_{n\theta} [N_{nIIm}]}{np r_m^{n-1} [D_n]}, \]

where

\[ [N_{nIIc}] = (np - \frac{1}{\mu_r}) r_m^{2n} + \left( 1 + \frac{1}{\mu_r} \right) r_m^{n-1} - (np + \frac{1}{\mu_r}) r_m^{2n} \]

\[ - (1 - \frac{1}{\mu_r}) r_c^{2n} \left( \frac{r_b}{r_m} \right)^{n+1}, \]

(B.5)

\[ [N_{nIIm}] = \frac{1}{\mu_r} r_m^{2n} + \frac{1}{\mu_r} r_c^{n-1} - (1 + \frac{1}{\mu_r}) r_m^{n-1} \]

\[ + (1 - \frac{1}{\mu_r}) r_c^{2n} \left( \frac{r_b}{r_m} \right)^{n+1}, \]

(B.6)

\[ [D_n] = \left( \frac{\mu_r + 1}{\mu_r} \right) \left( r_c^{2n} - r_b^{2n} \right) \]

\[ - \left( \frac{\mu_r - 1}{\mu_r} \right) \left( r_m^{2n} - r_c^{2n} \left( \frac{r_b}{r_m} \right)^{2n} \right). \]

(B.7)

Substituting Eq. (B.4) into (B.3) gives the following expression for \( X_{nI} \)

\[ X_{nI} = \frac{\mu_0 [N_{nI}]}{\mu_r \left( n^2 p^2 - 1 \right) r_m^{n-1} [D_n]} \]

where

\[ [N_{nI}] = (M_{nr} + M_{n\theta})(np + 1) r_m^{2n} + 2(M_{nr} - np M_{n\theta}) r_m^{n-1} r_b^{np+1} \]

\[ - (M_{nr} - M_{n\theta})(np + 1) r_b^{2n}. \]
B.1. Open field coefficients

The total solution for the flux density in the airgap and magnet region becomes

\[ B_{rI}(r, \theta) = \sum_{n=1,3,5,\ldots}^{\infty} npX_{nI}(r^{n_{p}-1} + r^{-n_{p}+1} r_{c}^{2n_{p}}) \cos(n_{p}\theta), \quad (B.10) \]

\[ B_{\theta I}(r, \theta) = -\sum_{n=1,3,5,\ldots}^{\infty} npX_{nI}(r^{n_{p}-1} - r^{-n_{p}+1} r_{c}^{2n_{p}}) \sin(n_{p}\theta), \quad (B.11) \]

\[ B_{rII}(r, \theta) = \sum_{n=1,3,5,\ldots}^{\infty} npX_{nII}(r^{n_{p}-1} + r^{-n_{p}-1} r_{b}^{2n_{p}}) \cos(n_{p}\theta) + \sum_{n=1,3,5,\ldots}^{\infty} (G_{n} - \mu_{0}M_{n\theta}) r^{-n_{p}-1} r_{b}^{2n_{p}+1} \cos(n_{p}\theta) \]
\[ + \sum_{n=1,3,5,\ldots}^{\infty} npG_{n} \cos(n_{p}\theta), \quad (B.12) \]

\[ B_{\theta II}(r, \theta) = -\sum_{n=1,3,5,\ldots}^{\infty} npX_{nII}(r^{n_{p}-1} - r^{-n_{p}-1} r_{b}^{2n_{p}}) \sin(n_{p}\theta) + \sum_{n=1,3,5,\ldots}^{\infty} (G_{n} - \mu_{0}M_{n\theta}) r^{-n_{p}-1} r_{b}^{2n_{p}+1} \sin(n_{p}\theta) \]
\[ - \sum_{n=1,3,5,\ldots}^{\infty} G_{n} \sin(n_{p}\theta). \quad (B.13) \]

B.1.2 Coefficients for \( np = 1 \)

- The first boundary condition (4.30) gives
  \[ H_{\theta I}|_{r=r_{c}} = \frac{1}{\mu_{0}} B_{\theta I}(r, \theta)|_{r=r_{c}} = 0, \]
  \[ X_{1I} - Y_{1I} r_{c}^{-2} = 0, \]
  \[ Y_{1I} = X_{1I} r_{c}^{2}. \quad (B.14) \]

- The second boundary condition (4.31) gives
  \[ H_{\theta II}|_{r=r_{b}} = \frac{1}{\mu_{0}\mu_{e}} (B_{\theta II}(r, \theta) - \mu_{0}M_{\theta})|_{r=r_{b}} = 0, \]
  \[ X_{1II} - Y_{1II} r_{b}^{-2} + G_{1}(ln(r_{b}) + 1) - \mu_{0}M_{1\theta} = 0, \]
  \[ Y_{1II} = X_{1II} r_{b}^{2} + G_{1} r_{b}^{2}(ln(r_{b}) + 1) - \mu_{0}M_{1\theta} r_{b}^{2}. \quad (B.15) \]

- The third boundary condition (4.32) gives
  \[ B_{rI}|_{r=r_{m}} = B_{rII}|_{r=r_{m}}, \]
  \[ X_{1I} + Y_{1I} r_{m}^{-2} = X_{1II} + Y_{1II} r_{m}^{-2} + G_{1} ln(r_{m}). \]
By substituting Eqs. (B.14) and (B.15) into the above equation, an expression for $X_{nI}$ is obtained as

$$X_{nI} = \frac{X_{II}[1 + r_b^2 r_m^{-2}] + G_1[r_b^2 r_m^{-2}(\ln(r_b) + 1) + \ln(r_m)] - \mu_0 M_{n\theta} r_b^2 r_m^{-2}}{1 + r_c^2 r_m^{-2}}.$$  \hspace{1cm} (B.16)

The fourth boundary condition (4.33) gives:

$$H_{\theta I}|_{r=r_m} = \frac{1}{\mu_0} B_{\theta I}|_{r=r_m} = \frac{1}{\mu_0 \mu_r} (B_{\theta II} - \mu_0 M_{\theta}|_{r=r_m} = H_{\theta II}|_{r=r_m},$$

$$X_{II} - Y_{II} r_m^{-2} = \frac{1}{\mu_r} (X_{II} - Y_{II} r_m^{-2} + G_1(\ln(r_m) + 1) - \mu_0 M_{\theta}).$$

By substituting Eqs. (B.14), (B.15) and (B.16) into the above equation and separating the variables $X_{II}$, $G_1$ and $M_{\theta}$ the following equation is obtained

$$X_{II} = \left[ \frac{1}{\mu_r} (1 - r_b^2 r_m^{-2})(1 + r_c^2 r_m^{-2}) ight.$$

$$\left. - (1 + r_b^2 r_m^{-2})(1 - r_c^2 r_m^{-2}) \right]$$

$$= G_1[(r_b^2 r_m^{-2}l_0(r_b) + r_b^2 r_m^{-2} + \ln(r_m)) (1 - r_c^2 r_m^{-2})]$$

$$- \left[ \frac{1}{\mu_r} (1 + \ln(r_m) - r_b^2 r_m^{-2}ln(r_b) - r_b^2 r_m^{-2}) (1 + r_c^2 r_m^{-2}) \right]$$

$$- \frac{\mu_0 M_{\theta}}{(1 - r_b^2 r_m^{-2})r_b^2 r_m^{-2}}$$

$$- \frac{1}{\mu_r} (1 - r_b^2 r_m^{-2})(1 + r_c^2 r_m^{-2})].$$

By further extending and rewriting this equation the following expression for $X_{II}$ is obtained

$$X_{II} = G_1[N_{IIc}] + \mu_0 M_{\theta}[N_{IIm}]$$

$$\left[ \frac{D_1}{} \right],$$

where

$$[N_{IIc}] = \left( \frac{\mu_r + 1}{\mu_r} \right) [r_b^2 \ln(r_b) + r_b - r_c \ln(r_m)]$$

$$- \left( \frac{\mu_r - 1}{\mu_r} \right) [r_b^2 r_m^{-2}ln(r_b) + r_b^2 r_m^{-2} - r_m^2ln(r_m)]$$

$$- \frac{1}{\mu_r} (r_m^2 + r_c^2),$$

$$[N_{IIm}] = \left( \frac{\mu_r + 1}{\mu_r} \right) r_b^2 - \left( \frac{\mu_r - 1}{\mu_r} \right) r_b^2 r_c^2 r_m^{-2} - \frac{1}{\mu_r} (r_m^2 + r_c^2),$$

$$[D_1] = \left( \frac{\mu_r + 1}{\mu_r} \right) [r_b^2 - r_b^2]$$

$$- \left( \frac{\mu_r - 1}{\mu_r} \right) [r_m^2 - r_c^2(r_b/r_m)^2].$$

Substituting Eq. (B.22) into (B.16) gives the following expression for $X_{II}$

$$X_{II} = \frac{[N_{II}]}{[D_1]}.$$  \hspace{1cm} (B.26)
B.2. Armature field coefficients

where

\[ [N_{1I}] = \frac{\mu_0}{2\mu_r} \left[ (M_{1r} + M_{1\theta})(r_m^2 - r_b^2) + 2(M_{1r} - M_{1\theta})r_b^2 \ln \left( \frac{r_m}{R_r} \right) \right]. \]  (B.27)

The total solution for the flux density in the airgap and magnet region becomes

\[ B_{c1}(r, \theta) = X_{1I} [1 + r^{-2} r_c^2] \cos(\theta), \]  (B.28)
\[ B_{c2}(r, \theta) = -X_{1I} [1 - r^{-2} r_c^2] \sin(\theta), \]  (B.29)
\[ B_{rII}(r, \theta) = X_{1I} [1 + r^{-2} r_b^2] \cos(\theta). \]
\[ + G_1 [\ln(r) + r^{-2} r_b^2 \ln(r_b) + r^{-2} r_b^2] \cos(\theta) \]
\[ - \mu_0 M_{1\theta} r^{-2} r_b^2 \cos(\theta), \]  (B.30)
\[ B_{\theta II}(r, \theta) = -X_{1I} [1 - r^{-2} r_c^2] \sin(\theta) \]
\[ + G_1 [r^{-2} r_b^2 \ln(r_b) + r^{-2} r_b^2 - ln(r) - 1] \sin(\theta) \]
\[ - \mu_0 M_{1\theta} r^{-2} r_b^2 \sin(\theta). \]  (B.31)

B.2 Armature field coefficients

B.2.1 Coefficients for \( \nu \neq 2 \)

The boundary conditions (4.118)-(4.121) must hold for every harmonic, so the summation sign is eliminated for ease of reading.

- The first boundary condition (4.118) gives

\[ H_{\theta I}|_{r=r_c} = \frac{1}{\mu_0} B_{\theta I}(r, \theta)|_{r=r_c} = 0, \]
\[ \nu X_{nIII} r_c^{\nu-1} - \nu Y_{nIII} r_c^{\nu-1} + 2 F_n r_c = 0, \]
\[ Y_{nIII} = X_{nIII} r_c^{2\nu} + \frac{2}{\nu} F_n r_c^{\nu+2}. \]  (B.32)

- The second boundary condition (4.119) gives

\[ H_{\theta II}|_{r=r_b} = \frac{1}{\mu_0} (B_{\theta II}(r, \theta)|_{r=r_b} = 0, \]
\[ \nu X_{nIV} r_b^{\nu-1} - \nu Y_{nIV} r_b^{\nu-1} = 0, \]
\[ Y_{nIV} = X_{nIV} r_b^{2\nu}. \]  (B.33)

- The third boundary condition (4.120) gives

\[ B_{r I}|_{r=r_{ci}} = B_{r II}|_{r=r_{ci}}, \]
\[ X_{nIII} r_{ci}^{\nu-1} + Y_{nIII} r_{ci}^{\nu-1} + F_n r_{ci} = X_{nIV} r_{ci}^{\nu-1} + Y_{nIV} r_{ci}^{\nu-1}. \]

By substituting Eqs. (B.32) and (B.33) into the above equation, an expression for \( X_{nIII} \) is obtained as

\[ X_{nIII} = \frac{X_{nIV} (r_{ci}^{\nu-1} + r_b^{2\nu} r_{ci}^{\nu-1}) - F_n (r_{ci} + \frac{2}{\nu} r_{ci}^{\nu+2} + r_b^{2\nu} r_{ci}^{\nu-1})}{r_{ci}^{\nu-1} + r_b^{2\nu} r_{ci}^{\nu-1}}. \]  (B.34)
The fourth boundary condition (4.121) gives

\[ H_{\theta I}|_{r=r_{ci}} = \frac{1}{\mu_0} B_{\theta I}|_{r=r_{ci}} = \frac{1}{\mu_0} (B_{\theta I}|_{r=r_{ci}} = H_{\theta II}|_{r=r_{ci}}, \]

\[ \nu X_{nIV} r_{ci}^{\nu-1} - \nu Y_{nIV} r_{ci}^{\nu-1} + 2F_\nu r_{ci} = \nu X_{nIV} r_{ci}^{\nu-1} - \nu Y_{nIV} r_{ci}^{\nu-1}. \]

By substituting Eqs. (B.32), (B.33) and (B.34) into the above equation and separating the variables \( X_{nIV} \) and \( F_\nu \) the following equation is obtained:

\[ X_{nIV} [(r_c^{\nu-1} + r_b^{2\nu} - r_{ci}^{\nu-1})(r_{ci}^{\nu-1} - r_c^{2\nu} - r_{ci}^{\nu-1})
\]

\[-(r_{ci}^{\nu-1} - r_b^{2\nu} - r_{ci}^{\nu-1})(r_{ci}^{\nu-1} + r_c^{2\nu} - r_{ci}^{\nu-1})]

\[ = F_\nu [(r_{ci} + \frac{2}{\nu} r_c^{\nu+2} - r_{ci}^{\nu-1})(r_{ci}^{\nu-1} - r_c^{2\nu} - r_{ci}^{\nu-1})
\]

\[-(r_{ci} - \frac{2}{\nu} r_c^{\nu+2} - r_{ci}^{\nu-1})(r_{ci}^{\nu-1} + r_c^{2\nu} - r_{ci}^{\nu-1})]. \]

By further extending and rewriting this equation the following expression for \( X_{nIV} \) is obtained

\[ X_{nIV} = \frac{F_\nu [(\nu - 2)r_c^{\nu+2} + 4r_c^{\nu+2} - (\nu + 2)\nu^2 r_c^{\nu-2}]}{2\nu[r_b^{2\nu} - r_c^{2\nu}]} . \]

(B.35)

Substituting Eq. (B.35) into (B.34) gives the following expression for \( X_{nIII} \)

\[ X_{nIII} = \frac{F_\nu [(\nu - 2)r_c^{\nu+2} + 4r_c^{\nu+2} - (\nu + 2)\nu^2 r_c^{\nu-2}]}{2\nu[r_b^{2\nu} - r_c^{2\nu}]} . \]

(B.36)

The total solution for the flux density in the coil and airgap region becomes

\[ B_{rI}(r, \theta) = \sum_{n=1}^{\infty} \nu X_{nIII}(r_{ci}^{\nu-1} - r_{ci}^{\nu-1} - r_{ci}^{\nu-1}) \cos(n\theta) \]

\[ + \sum_{n=1}^{\infty} \frac{2}{\nu} F_\nu r_{ci}^{\nu-1} \nu^{\nu+2} \cos(n\theta) \]

\[ + \sum_{n=1}^{\infty} \nu F_\nu r_{ci} \cos(n\theta), \]

(B.37)

\[ B_{\theta I}(r, \theta) = -\sum_{n=1}^{\infty} \nu X_{nIII}(r_{ci}^{\nu-1} - r_{ci}^{\nu-1} - r_{ci}^{\nu-1}) \sin(n\theta) \]

\[ + \sum_{n=1}^{\infty} \frac{2}{\nu} F_\nu r_{ci}^{\nu-1} \nu^{\nu+2} \sin(n\theta) \]

\[ - \sum_{n=1}^{\infty} 2F_\nu r_{ci} \sin(n\theta), \]

(B.38)

\[ B_{rII}(r, \theta) = \sum_{n=1}^{\infty} \nu X_{nIV}(r_{ci}^{\nu-1} + r_{ci}^{\nu-1} - r_{ci}^{\nu-1}) \cos(n\theta), \]

(B.39)

\[ B_{\theta II}(r, \theta) = -\sum_{n=1}^{\infty} \nu X_{nIV}(r_{ci}^{\nu-1} - r_{ci}^{\nu-1} - r_{ci}^{\nu-1}) \sin(n\theta). \]

(B.40)
B.2.2 Coefficients for $\nu = 2$

The boundary conditions (4.118)-(4.121) must hold for every harmonic, so the summation sign is eliminated for ease of reading.

- The first boundary condition (4.118) gives
  \[ H_{01}|_{r=r_c} = \frac{1}{\mu_0} B_{01}(r, \theta)|_{r=r_c} = 0, \]
  \[ 2X_{2II} r_c - 2Y_{2II} r_c^{-3} + 2F_2 r_c \ln(r_c) + F_2 r_c = 0, \]
  \[ Y_{2II} = X_{2II} r_c^4 + F_2 r_c^4 \ln(r_c) + \frac{1}{2} F_2 r_c^4. \]  
  \( \text{(B.41)} \)

- The second boundary condition (4.119) gives
  \[ H_{0II}|_{r=r_b} = \frac{1}{\mu_0} (B_{0II}(r, \theta)|_{r=r_b} = 0, \]
  \[ 2X_{2IV} r_b - 2Y_{2IV} r_b^{-3} = 0, \]
  \[ Y_{2IV} = X_{2IV} r_b^4. \]  
  \( \text{(B.42)} \)

- The third boundary condition (4.120) gives
  \[ B_{rI}|_{r=r_{ci}} = B_{rII}|_{r=r_{ci}}, \]
  \[ X_{2III} r_{ci} + Y_{2III} r_{ci}^{-3} + 2F_2 r_{ci} \ln(r_{ci}) = X_{2IV} r_{ci} + Y_{2IV} r_{ci}^{-3}. \]

By substituting Eqs. (B.41) and (B.42) into the above equation, an expression for $X_{nIII}$ is obtained as
\[ X_{2III} = \frac{X_{2IV}(1 + r_{b}^4 r_{ci}^{-4}) - F_2(\ln(r_{ci}) + r_{c}^4 r_{ci}^{-4} \ln(r_c) + \frac{1}{2} r_{c}^4 r_{ci}^{-4})}{1 + r_{c}^4 r_{ci}^{-4}}. \]  
  \( \text{(B.43)} \)

- The fourth boundary condition (4.121) gives
  \[ H_{0I}|_{r=r_{ci}} = \frac{1}{\mu_0} B_{0I}|_{r=r_{ci}} = \frac{1}{\mu_0} (B_{0II}|_{r=r_{ci}} = H_{0II}|_{r=r_{ci}}, \]
  \[ 2X_{2III} r_{ci} - 2Y_{2III} r_{ci}^{-3} + 2F_2 r_{ci} \ln(r_{ci}) + F_2 r_{ci} = 2X_{2IV} r_{ci} - 2Y_{2IV} r_{ci}^{-3}. \]

By substituting Eqs. (B.41), (B.42) and (B.43) into the above equation and separating the variables $X_{2IV}$ and $F_2$ the following equation is obtained
\[ X_{2IV}[(r_{ci} - r_{b}^4 r_{ci}^{-3})(1 + r_{c}^4 r_{ci}^{-4}) - (r_{ci} - r_{c}^4 r_{ci}^{-3})(1 + r_{b}^4 r_{ci}^{-4})]
= F_2[(r_{ci} \ln(r_{ci}) + \frac{1}{2} r_{ci} - r_{c}^4 r_{ci}^{-3} \ln(r_c) - \frac{1}{2} r_{c}^4 r_{ci}^{-3})(1 - r_{c}^4 r_{ci}^{-4})
- (\ln(r_{ci}) + r_{c}^4 r_{ci}^{-4} \ln(r_c) + \frac{1}{2} r_{c}^4 r_{ci}^{-4})(r_{ci} + r_{c}^4 r_{ci}^{-3})]. \]
By further extending and rewriting this equation the following expression for $X_{2IV}$ is obtained

$$X_{2IV} = \frac{F_2[r_c^4 - r_{ca}^4 - r_c^4 \ln\left(\frac{r_{ca}}{r_c}\right)]}{4[r_b^4 - r_c^4]}.$$

(B.44)

Substituting Eq. (B.44) into (B.43) gives the following expression for $X_{2III}$

$$X_{2III} = \frac{F_2[2r_c^4 - r_{ca}^4 - r_b^4 + 4r_c^4 \ln(r_c) - 4r_b^4 \ln(r_b)]}{4[r_b^4 - r_c^4]}.$$

(B.45)

The total solution for the flux density in the coil and airgap region becomes

$$B_{rI}(r, \theta) = 2X_{2III}(r + r^{-3}r_c^4) \cos(2\theta) + F_2 r^{-3}r_c^4 (2\ln(r_c) + 1) \cos(2\theta) + 2F_2 r \ln(r) \cos(2\theta),$$

(B.46)

$$B_{\theta I}(r, \theta) = -2X_{2III}(r - r^{-3}r_c^4) \sin(2\theta) + F_2 r^{-3}r_c^4 (2\ln(r_c) + 1) \sin(2\theta) - F_2 r (2\ln(r) + 1) \sin(2\theta),$$

(B.47)

$$B_{rII}(r, \theta) = 2X_{2IV}(r + r^{-3}r_b^4) \cos(2\theta),$$

(B.48)

$$B_{\theta II}(r, \theta) = -2X_{2IV}(r - r^{-3}r_b^4) \sin(2\theta).$$

(B.49)
Appendix C

Winding factor
### Table C.1: Winding factors for slotless machine with concentrated windings ($\beta_o=0$).

| Q/2p | 4   | 6   | 8   | 10  | 12  | 14  | 16  | 18  | 20  | 22  | 24  | 26  | 28  | 30  | 32  | 34  | 36  | 38  | 40  | 42  |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 6    |     | 0.716 | 0.716 | 0.694 | 0.667 | 0.716 | 0.716 | 0.713 | 0.659 |     |     |     |     |     |     |     |     |     |     |
| 9    |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 12   | 0.699 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 15   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 18   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 21   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 24   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 27   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 30   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 33   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 36   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 39   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 42   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 60   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |

### Table C.2: Winding factors for slotless machine with concentrated windings ($\beta_o=0.5\alpha_c$).

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Winding Factor