Order crossovers in inventory management
qualitative and quantitative insights

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Order Crossovers in Inventory Management; Qualitative and Quantitative Insights

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ABSTRACT

Companies in all sort of environments face several kind of uncertainties. The variability in lead time may increase the operating costs of companies, reducing uncertainties is therefore a goal which companies often strive to achieve. If the lead time is variable, the sequence in which orders are placed is not necessarily the order in which they arrive, this is known as order crossovers. Although the effect of order crossovers is known, the variance of the lead time is reduced, knowledge on how to integrate this into the optimal parameter settings of inventory policies lacks. We conducted a study in which data is simulated to research the effect of order crossovers on the optimal parameter settings. Significant real savings can be present between inventory policies incorporating order crossovers and those which do not, depending on the involved parameters. Lead time variance proves to be an important factor in the difference between the optimal parameter values in all four studied inventory policies. It is recommended for practice to investigate the applicability of the assumption on order crossovers, to retrieve the optimal parameter settings appropriately.
MANAGEMENT SUMMARY

PROBLEM
This report studies the effect of order crossovers in inventory management. Orders have crossed if an order arrives at the customer before the arrival of a prior placed order. Variable lead times enable order crossovers. Lead time variability is inherent to production environments and transportation modes, making order crossovers a present concern for companies. Lead times are either deterministic, known beforehand, or stochastic where at the time of ordering the lead time is unknown. A second classification for lead times is either static, no seasonal influences, or dynamic, present seasonal influences. This classification leads to 3 types of order crossovers, represented in Table 1. This thesis focuses on the third type; random crossovers.

<table>
<thead>
<tr>
<th>Type of order crossovers</th>
<th>Lead times</th>
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<tbody>
<tr>
<td>Static</td>
<td>Deterministic</td>
</tr>
<tr>
<td>Dynamic</td>
<td>Dynamic</td>
</tr>
<tr>
<td>No crossovers</td>
<td>Static</td>
</tr>
<tr>
<td>Expected crossovers</td>
<td>Dynamic</td>
</tr>
<tr>
<td>Random crossovers</td>
<td>Static</td>
</tr>
<tr>
<td>General crossovers</td>
<td>Dynamic</td>
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Lead time variability is necessary, but not the only requirement for order crossovers. The time between two order moments should be smaller than the lead time, else order crossovers are not possible.

Even though the originating factors of order crossovers may be known, the exact consequences of order crossovers are less known. Two types of cost savings are present due to order crossovers. First of all apparent savings. Apparent savings are not accounted for in the analysis but occur in practice. Secondly, real savings. Real savings are not accounted for in the analysis without order crossovers, but can be used when taking order crossovers into account by altering the parameters. This leads to the question researched in this report:

**What is the effect of order crossovers on inventory policies' (near-) optimal parameter settings?**

The research focused on four different types of inventory polices. Periodic review with fixed order quantity was the first inventory policy examined. This inventory policy is used in practice if demand is, relatively, stable. Unstable demand is detrimental to the service level. Periodic review with a variable order size is more applicable to handle variable demand due to the dependence of the order size on the observed demand. Periodic review is primarily chosen if costs to keep track of inventory levels are high. If inventory levels are constantly monitored the continuous review inventory policies are in place. If benefits are gathered from a fixed order quantity this is most often chosen. If demand can occur in batches, the variable order size is more appropriate, raising the inventory position to a predetermined level after ordering.
**Analysis**

The research follows the method introduced by Robinson et al. (2001). This method first examines the probability of order crossovers. For all types of inventory policies the lead time distribution is an important contributing factor. The review period is an important factor influencing this probability in periodic review inventory policies. As Figure 1 presents. The x-axis represents the review period and the y-axis the probability of order crossovers.

![Figure 1 Pr(crossover), periodic review inventory policies](image1)

For continuous review inventory policies, the order quantity is an important factor contributing to the probability of order crossovers. A new order is only placed if demand lowers the inventory position to the reorder level. Larger time between orders, due to larger order quantities, equals a lower probability of order crossovers. This is presented in Figure 2, the x-axis is the order size and the y-axis the probability of order crossovers.

![Figure 2 Pr(crossover), continuous review inventory policies](image2)

**Results**

After the probability of order crossovers is calculated, the adjusted lead time variance is incorporated in the method to find the optimal parameters which need to be implemented in the inventory policies. For periodic review inventory policies it is seen that the optimal review period most often decreases. In continuous review inventory policies the order quantity decreases, this makes sure that order crossovers happen more frequently. More important than the adjusted parameters are the real savings which are obtained whilst using these altered parameter settings. Figure 3 shows the impact of the involved parameters on the probability of order crossovers, and thereby the real savings.
RECOMMENDATIONS

Based on the results of the analysis it can be concluded that the assumption regarding order crossovers can make a significant impact. Whether or not it makes a significant impact depends on the involved parameters. It is recommended for companies to thoroughly examine their supply chain to see if it is possible for order crossovers to occur.

If order crossovers are possible, companies should take them into account. Companies can take order crossovers into account by incorporating the shortfall distribution or the adjusted algorithm of Federgruen & Zheng (1991), to find the appropriate parameters involved in the inventory policies.

Several research gaps regarding order crossovers remain. Future research could focus on the presence of seasonality patterns. Bischak et al. (2014) investigated this for a (R, S) inventory policy, but other inventory policies could provide different results. Outdating effects on inventory policies where orders can cross are also still uncertain, changes in arrival sequence could have a detrimental impact on the outdating present in supermarkets.
This report is a result of my graduation project in the Master of Science program Operations Management & Logistics at the school of Industrial Engineering and Innovation Sciences of Eindhoven University of Technology. During this program I developed my professional skills and concluded with this project of 6 months to demonstrate this. Besides demonstrating these learned skills, equally important, this report marks the finish of an amazing student life which started at September 2010.

I would like to thank a number of people who helped me during this thesis project. First of all, Zümbül Atan. Zümbül, you are a great mentor, you brought up the idea of conducting a thesis internally at the TU/e on the topic of order crossovers and immediately got me enthusiastic. Even though we did not see each other often, due to your pregnancy, I could always contact you and you provided me with great support. Thank you for everything and I know for sure you and your family have a wonderful time ahead. Also my second supervisor, Ivo Adan, was important for my project. Ivo, you kept me on track to properly introduce the used terminology. Besides, you helped my thesis considerably regarding notation. Thank you for the meetings we had and the suggestions you provided me.

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Sander Arends
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1. INTRODUCTION

This report represents the study on the influences of order crossovers in several inventory policies. Order crossovers are defined as orders arriving before prior placed orders. This study is the result of the master thesis project conducted at the Eindhoven University of Technology.

The first chapter is organized as an introduction to the rest of the report, and is structured as follows. In the first section we start with a global introduction to inventory management, then we continue with a brief explanation on the main subject of our study, “order crossover”. The next section defines order crossovers and discusses their origin (Section 1.2.). Section 1.3. discusses the problem statement. Based on the proven impact of order crossovers and the current gap in literature, the research question and its sub-questions are formulated (Section 1.4.). Subsequently we describe assumptions of this research and the scope this research can be related to (Section 1.5.). Finally followed by the structure of the report in the last section (Section 1.6.).

1.1. INVENTORY MANAGEMENT

A crucial problem in any manufacturing environment is to control the material flow from suppliers, raw materials, to the end customer, finished products (Axsäter, 2006). Besides controlling the material flow, controlling the total investment in inventory, capital tied to materials, is also of vital importance for companies (Arda & Hennet, 2006). The investment in inventories is often significant and shows a promising potential for improvement according to Axsäter (2006).

Inventory management can not be seen independently of other business functions (Arda & Hennet, 2006). This means that decisions made involving inventory, for example the policy used and its settings, directly influences production and marketing. Due to this interdependence Arda & Hennet (2006) claim that; “The objective of inventory management is therefore to balance conflicting goals”.

In operations strategy, common opposing goals are responsiveness versus efficiency (Wang, Thomas, & Rudi, 2014). If these goals are applied to inventory management, it is possible to state a firm’s responsiveness as few stock outs of a product, a high service level or high inventory levels to immediately respond to a change in demand. This can be achieved by having a large amount of inventory to directly satisfy customer demand. On the other hand it is inefficient to hold these high levels of inventory due to holding cost.

Inventory management research has focused on how to control uncertainties and to effectively balance the opposing goals. Demand uncertainty is apparent to most supply chains, but there are other sources of uncertainty as well. Lead times are almost never constant, possibly causing disruptions or delays for the production department. Due to the presence of several kinds of risks it is often hard to predict the demand and lead times perfectly (Jüttner, 2005). Inventories potentially smoothen these disruptions (Louly, Dolgui, & Hnaien, 2008).
Inventory policies are ultimately concerned with two decisions; the quantity to reorder and its timing (Feeney & Sherbrooke, 1966). The decision when to reorder is either made on a periodical basis or continuously. Continuous review policies constantly monitor their inventory position, based on this inventory position the decision to place an order is made. Periodic review policies can only decide to place an order on certain predetermined time periods. The decision how much to reorder can also differ for each inventory policy. In a fixed quantity model the size of the order does not depend on the observed demand. Orders which have a variable size, and bring the inventory position up-to a predetermined level, are order up to models. Finally, the decision to place an order can depend on the inventory position. If at the order moment the inventory position is beneath a threshold level orders are placed, else no order is placed. Nahmias (2009) states that the combinations of these choices, periodic or continuous review, fixed or variable order quantity and the possible inclusion of a reorder level, are widely known and used in practice.

Inventory control policies are often based on the lead time demand distribution (Robinson, Bradley, & Thomas, 2001), to determine the timing and the optimal quantity of an order. The lead time demand is the demand observed during the lead time of an order. In using this distribution, which is a mixture between the demand and lead time distributions, several assumptions are made. The first assumption made is that all costs (e.g. per unit holding costs) are always the same. However, ordering at multiple suppliers could provide a more reliable supply and/or cheaper and faster deliveries (Arda & Hennet, 2006), due to different cost structures. Secondly the assumption is made that the chosen settings, based on a single period, are optimal if they are applied to all periods. This optimality might be true if the cost structure and demand distribution remain the same, but any alteration could influence the optimal settings. Several other assumptions are made, among others the assumption that orders always arrive in the same sequence. This assumption is questionable. Robinson et al. (2001) already showed that, with variable order quantities and a predetermined optimal review period, benefits are derived from order crossovers. The next section provides more details about order crossovers and its current literature.

1.2. ORDER CROSSOVERS

Before we show that order crossovers impact the optimal inventory policies parameters settings, we first formally define order crossovers. Orders have crossed if an order arrives at the customer before the arrival of a prior placed order. The following statement for order crossovers is in place; “It [order crossovers] occurs when replenishment orders arrive in a sequence that is different than the one in which they were placed.” (Bradley & Robinson, 2005). More general, (Riezebos, 2006) uses the following notation;

\[
O_x = \text{Moment in time of placing order } x \\
L_x = \text{Lead time of order } x \\
R_x = \text{Arrival time of order } x \\
R_x = O_x + L_x
\]

For two orders to cross, Riezebos (2006) shows that the following two conditions must be true;

\[O_A < O_B \text{ and } R_B < R_A\]
Riezebos (2006) proves that lead time variability is necessary, but not the only requirement for order crossovers to occur. If the time between two order moments is larger than the lead time, it is not possible for two orders to cross, as Figure 4 shows.

![Order Crossover Diagram](image)

**Figure 2 Order Crossover**

Riezebos (2006) classifies fluctuations in order lead time as either static or dynamic, and deterministic or stochastic. Deterministic fluctuations are known before any decisions are made whereas stochastic fluctuations relate to uncertainty and can be seen as random variation. Static fluctuations retain from any systematic variation, for example daily patterns. In dynamic fluctuations a dominant systematic pattern is present (Riezebos, 2006). The combination of these two types of fluctuations leads to four situations, presented in Table 2. According to Riezebos (2006), these four types each have a different type of order crossover.

<table>
<thead>
<tr>
<th>Table 2 Type of order crossovers</th>
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<tbody>
<tr>
<td><strong>Lead times</strong></td>
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<tr>
<td></td>
</tr>
<tr>
<td>Stochastic</td>
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<td></td>
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This research focuses on the third type of order crossovers; random crossovers. The models retain from any systematic variation, and lead times are not known before decisions regarding orders are made. Order lead times are inherently stochastic (Xuan, Hao, Jinfeng, & Changrui, 2014) making random crossovers a present concern in inventory management.

How often these order crossovers are present in practice depends on the periodicity of ordering (Hayya, Harrison, & He, 2011). If the period between order moments is large, compared to the lead time, the probability of order crossovers decreases (Song & Zipkin, 1996). A high ordering frequency increases the chance and effects of order crossovers (Hayya et al., 2011). However, when orders are placed extremely close together the effect of order crossovers becomes negligible (Hayya et al., 2011). Orders can be placed extremely close together for example if there are several suppliers for a single product.

The literature study shows that the consequences of order crossovers can have a significant impact on the inventory on hand (Arends, 2015), assuming demand is interchangeable (Liberatore, 1979). Due to order crossovers the effective lead time variance decreases compared to the variance if orders do not cross, maintaining the same mean (Zalkind, 1978), (Hayya, Bagchi, Kim, & Sun, 2008),
Hayya et al. (2008) refer to this phenomenon as the difference between the lead time and the effective lead time. The effective lead time is defined as “the time between placing the \(i^{th}\) order and the time of the receipt of the \(i^{th}\) order”. The index \(i\) is not tagged to a specific order, therefore the receipt of the \(5^{th}\) order can still correspond to the order moment of the \(1^{st}\) order.

Variability may increase the operating costs (Lee & Tang, 1998), independent of the mean lead time (He, Kim, & Hayya, 2005). Orders crossing leads to a higher inventory on hand, or less backorders, compared to a situation where this did not happen. Higher levels of on hand inventory can potentially lead to a higher service level. Higher service level most often correspond with additional costs. These additional costs are most likely incurred due to the higher level of on hand inventory (He, Xu, Keith, & Hayya, 1998), (Robinson, Bradley, & Thomas, 2001). However, order crossovers lead to a decrease in lead time variability, potentially influencing costs linked to the lead time. The following chapters show the difference in costs between inventory policies which account for order crossovers and those which do not. Section 1.3. shows the problem regarding order crossovers, after which the research questions are formulated in the consecutive section.

1.3. **Problem Statement**

As the previous section shows, order crossovers can originate due to several reasons. Companies are potentially aware of these originating factors of order crossovers. If seasonal patterns, company traits or geographical locations (dynamic) influence lead times, or if the lead time variance remains constant (static) this can be known.

Even though the possible causes of order crossovers might be well known, the consequences of order crossovers are less known. Because consequences of order crossovers are less known, inventory managers often do not know how to cope with these consequences to act accordingly. He et al. (1998) found that, compared to analysis which does not take order crossovers into account, two types of cost savings are present;

1. Apparent savings; not accounted for in the analysis but occurs in practice
2. Real savings, not accounted for in the analysis without order crossovers, but are obtained when taking order crossovers into account

The start of extensive research regarding the impact of order crossovers and consequent actions regarding real savings only began several years later, with the article of Robinson et al. (2001). They search for an optimal order up to level if the review period is optimized beforehand. They show that in an (R, S) policy, with a predetermined review period, significant savings can be present. The savings are present even when the probability of order crossovers is small. They also provided proof that the lead time distribution leads to an incorrect estimation of the order up to level. However, it is shown that the optimal review period potentially is lower due to the effects of order crossovers (Bischak, Robb, Silver, & Blackburn, 2014). The predetermined optimal review period might not be optimal assuming order crossovers. Besides the (R, S) policy, other inventory policies might face order crossovers as well and could benefit from their consequences, this area is identified as a research gap which this thesis tries to partially fill.
1.4. Research Questions

Based on the problem statement, we derive the main research question. The problem statement explains that, although the possible causes for order crossovers might be known, the impact of order crossovers is underexposed. Therefore we are interested in the effect of order crossovers, and how to manage their consequences appropriately, which leads to the research question:

**What is the effect of order crossovers on inventory policies’ (near-) optimal parameter settings?**

To answer the main research question we define several sub-questions. Each sub-question aims at a specific inventory policy to see the impact of order crossovers on the optimized parameter settings and the actual costs.

1. What is the effect of order crossovers on the (near-) optimal parameter settings in the (R, Q) policy?
2. What is the effect of order crossovers on the (near-) optimal parameter settings in the (R, S) policy?
3. What is the effect of order crossovers on the (near-) optimal parameter settings in the (s, Q) policy?
4. What is the effect of order crossovers on the (near-) optimal parameter settings in the (s, S) policy?

The chosen inventory policies are widely known and used in practice (Nahmias, 2009). The structure of the sub-questions is based on their perceived complexity. Their intricacy is evaluated first based on the timing of events. Whereas in the periodic review inventory policies timing remains to be constant after a review period is chosen, the timing of orders in continuously review policies is more complicated. After the timing of events, the complexity of order crossovers is evaluated on order sizes. If orders are continuously of the same size, order arrivals have a similar impact upon arriving. On the other hand variable order sizes impact the inventory on hand differently upon arrival. The reorder level is excluded in the sub-questions of the periodic review policies, but a brief analysis is present in Chapter 2 and 3.
1.5. VARIABLES & ASSUMPTIONS

For this master thesis project several assumptions are made, the major assumptions are presented in this section. Specific assumptions which apply to one type of inventory policy, are explained in their corresponding chapter. This section concludes with the definition of used variables.

1.5.1. DISTRIBUTIONS

Two distributions are important for examining the impact of order crossovers; the lead time and demand distribution. For the periodic review policies, lead times are measured in periods, are non-negative and integer valued. The Poisson distribution is used, this complies to non-negative and integer values constraint. Lead times are non-negative in continuous review policies, simulated by the Gamma distribution. Besides, lead time is independent and identically distributed. Furthermore, lead times are assumed to be static, stochastic thus the type of order crossovers is random, seasonal patterns are therefore excluded. As Bischak et al. (2014) show, correlated lead times decrease the probability of order crossover and therefore also the consequences of order crossovers are scaled down. Demand size is assumed to be Poisson distributed, it is non-negative and integer valued. The demand distribution is independent and identically distributed. The Poisson distribution is often used in practice. In companies information about the shape of the distribution is often incomplete. The Poisson distribution only requires one parameter to be estimated and is a reasonable approximation of reality (Ramaekers & Janssens, 2008). In continuous review policies, the demand is simulated by the arrival rate distribution following an exponential distribution, often used to simulate the interarrival times of a Poisson distribution (Glasserman & Wang, 1998). Demand and lead time distributions are jointly independent over time, meaning the lead time does not depend on the demand, or vice versa.

1.5.2. ORDER OF EVENTS

Before analyzing inventory policies, we first establish the order in which events happen. We assume that the first thing that happens, is the arrival of orders of which the lead time has passed. Stochastic lead times result in the possibility of zero, one or more orders arriving in one single period. Following the arrival of orders, demand is observed. After observing demand an order can be placed at the supplier. The order size in policies with a variable order quantity depend on this observed demand. With constant order quantities, the observed demand thus has no influence on the order quantity. Observed demand determines the time of order placement in continuous review policies. The order arrives immediately if the lead time is 0, however the order quantity does not change, even when this instant delivery is observed. A lead time of 0 is possible in practice, for instance if the truck delivering products carries additional inventory which it can supply directly. After orders are placed, the demand for that period is fulfilled. Finally costs are incurred which are briefly discussed in the next section. To summarize, the order of events;

1. Possible arrival of outstanding orders
2. Demand $D_t$ observed
3. Order of size $Q_t$ placed
4. Demand $D_t$ realized
5. Costs incurred based on ending/average inventory levels
1.5.3. Costs

We analyzed four basic inventory policies and how order crossovers impact the (near-) optimal parameter settings and, more importantly, its corresponding costs. The costs assumed to be present and considered in the model are the ordering costs, holding costs and penalty costs. Ordering costs are incurred every time an order is placed, regardless of the order size. Periodic review inventory policies incur holding and penalty costs at the end of each period, for the positive (on-hand) and negative (backordered) inventory respectively. Continuous review inventory policies incur holding and backorder costs based on their average positive (on-hand) and negative (backordered) inventory during a period. The case of lost demand is not investigated, but it can be hypothesized that the penalty costs are increased, for instance to the level of the selling price, to represent the case of lost demand.

1.5.4. Variables

Table 3 defines the variables used throughout this report. These variables are required to calculate the probability of order crossovers and to calculate the costs corresponding to the inventory policy.

<table>
<thead>
<tr>
<th>Table 3 Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lead time probability mass function</td>
</tr>
<tr>
<td>Lead time cumulative distribution function</td>
</tr>
<tr>
<td>Stochastic, stationary demand in period $t$</td>
</tr>
<tr>
<td>Stochastic, stationary lead time at time $t$</td>
</tr>
<tr>
<td>Demand, $d$, probability mass function during review period, $R$</td>
</tr>
<tr>
<td>Demand cumulative distribution function during review period</td>
</tr>
<tr>
<td>Set of order moments (periodic review)</td>
</tr>
<tr>
<td>Number of order moments time $b$ to time $d$</td>
</tr>
<tr>
<td>Size of order placed at time $t$</td>
</tr>
<tr>
<td>Reorder level</td>
</tr>
<tr>
<td>Order up to level</td>
</tr>
<tr>
<td>Service level</td>
</tr>
<tr>
<td>Per-unit holding costs</td>
</tr>
<tr>
<td>Per-unit shortage costs</td>
</tr>
<tr>
<td>Fixed ordering costs</td>
</tr>
<tr>
<td>Review period</td>
</tr>
<tr>
<td>Mean demand</td>
</tr>
<tr>
<td>Standard deviation demand</td>
</tr>
<tr>
<td>Mean lead time</td>
</tr>
<tr>
<td>Standard deviation lead time</td>
</tr>
</tbody>
</table>
To answer the research question, a structure for the research project is chosen. The chapters which follow all have the same structure. Robinson et al. (2001) already investigated the consequences of order crossovers in an \((R,S)\) inventory policy, with \(R\) set to a fixed level. We follow the same structure as they did.

Chapters 2 and 3 analyze the periodic review inventory policy, respectively with a fixed and variable order quantity. First, each chapter examines the practical use of the inventory policies to see when they are used. Secondly, the probability of order crossovers is examined for the inventory policies. The first thing to know when analyzing the influence of order crossovers, evidently has to be the fact that order crossovers can occur. If order crossovers can occur their impact on the effective lead time demand is investigated. Afterwards methods to find new optimized settings for the inventory policy are provided. To see if any significant differences are present in the costs using the optimized settings, either assuming order crossovers can occur or not, a simulation studies is conducted. These simulations allow us to see whether the new method yields a significant improved solution. Finally, a sensitivity analysis is conducted. This sensitivity analyses shows how the parameters impact the probability and consequences of order crossovers.

Chapter 4 focuses on the continuous review policy with a fixed order quantity. The same approach is followed as in the \((R,S)\) policy. Moreover an heuristic providing the optimal values for the reorder level and order quantity is examined and adjusted. Afterwards the shortfall distribution and adjusted heuristic are both checked in a simulation, to see if the retrieved optimal values are an improvement compared to traditional methods. Chapter 4 concludes with a sensitivity analysis to check the impact of the involved parameters on the real savings. The case of continuous review with a variable order size is examined in Chapter 5, following a more qualitative approach. Finally after the impact of order crossovers is shown in all four inventory policies, Chapter 6 concludes and provides recommendations for both practice and further research.
2. PERIODIC REVIEW, FIXED ORDER QUANTITY

The previous chapter explained the general problem of order crossovers in inventory management. This chapter focuses specifically on the periodic review inventory policy with fixed order quantity. In this policy an order is placed every $R$ periods, with a fixed order of the size of an average’s review period demand.

The rest of this chapter is organized as follows. Section 2.1. explains the likelihood of this inventory policy in practice and when this policy is most likely adopted. In Section 2.2. we discuss the probability of order crossovers in periodic review inventory policies. Section 2.3. first shows that apparent savings are present. Section 2.3. continues by showing the real savings caused by order crossovers and describes the influences of changes in the parameter values. The final Section, 2.4., concludes and answers the first sub-question.

2.1. POLICY APPLICABILITY

The $(R, Q)$ inventory policy is a straightforward policy to understand and use in practice. Every $R$ periods an order, always of quantity, $Q$, is placed. This fixed order quantity is equal to the mean demand during the review period. This is intuitively concluded by the reasoning that every period the faced demand is the average period's demand. If the order quantity is larger than the mean demand during the review period, inventory on hand steadily but continuously increases. Whilst if the order quantity is lower compared to the mean demand during the review period, the inventory level eventually is below 0 and all demand is backordered.

In a simulation this intuition is tested, by changing the order quantity to $Q = (\mu_D + k) \times R$, where $k; \{-3, -2, -1, 0, 1, 2, 3\}$. The situation $k = 0$ represents an order quantity equal to the mean demand during the review period. An example, for a mean demand of 10 and variation 0, and review periods 1 up to 5, is presented in Table 4. As Table 4 shows, the optimal order quantity indeed is equal to the mean demand per period times the length of the review period. If the order quantity is set equal to 49 with a mean demand in the review period of 50 the amount of backorders at the end of simulation (with period length 10.000) is 1919. This is logical since every 5 periods, 1 unit is short. Differentiating from the $Q = \mu_D \times R$ is not optimal. It is assumed that this optimal order quantity, $Q^* = \mu_D \times R$, holds for every value of the mean and variance of demand.

<table>
<thead>
<tr>
<th>$R(Q^*)$</th>
<th>$k$</th>
<th>-3</th>
<th>-2</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (10)</td>
<td>€134.603</td>
<td>€89.690</td>
<td>€44.773</td>
<td>€73</td>
<td>€5.059</td>
<td>€10.047</td>
<td>€15.036</td>
</tr>
<tr>
<td>2 (20)</td>
<td>€134.541</td>
<td>€89.631</td>
<td>€44.712</td>
<td>€57</td>
<td>€5.039</td>
<td>€10.028</td>
<td>€15.018</td>
</tr>
<tr>
<td>3 (30)</td>
<td>€134.501</td>
<td>€89.583</td>
<td>€44.659</td>
<td>€56</td>
<td>€5.036</td>
<td>€10.027</td>
<td>€15.017</td>
</tr>
<tr>
<td>4 (40)</td>
<td>€134.470</td>
<td>€89.548</td>
<td>€44.620</td>
<td>€59</td>
<td>€5.037</td>
<td>€10.028</td>
<td>€15.019</td>
</tr>
<tr>
<td>5 (50)</td>
<td>€134.428</td>
<td>€89.509</td>
<td>€44.573</td>
<td>€61</td>
<td>€5.041</td>
<td>€10.032</td>
<td>€15.022</td>
</tr>
</tbody>
</table>
In an (R, Q) inventory policy an inventory manager can therefore only decide on the length of the review period, and the starting inventory. The review period influences the order quantity. The starting inventory is determined based on the probability that an order arrives with a certainty to reach the predetermined service level. This starting inventory can be seen as a certain number of periods of demand which must be in the inventory level when an order is placed.

The applicability of a fixed order size in practice is disputable. The reason why the practicality of the (R, Q) policy is questionable derives from the fact that there is a lack of control on the inventory level after the actual demand is observed. The demand is observed prior to placing an order. In the (R, Q) policy the size of the order is independent to the observed demand. In the (R, S) policy the demand influences the order size and in the (s, Q) policy the demand triggers the order moments (Hillier & Lieberman, 2010). In the (R, Q) policy such a relationship is non-existent. This non-existing relationship ensures that the (R, Q) policy is only used if demand is constant or has a very small variance.

Due to the assumption that demand is independent over time, the chance of demand being over (or under) the average period’s demand is the same in each period, regardless the previously observed demand. In these periods of time the inventory position drops and rises dramatically. These noteworthy fluctuations have a detrimental impact on the service level. We can demonstrate this with an example. If we assume the demand is Poisson(10) distributed and simulate this five times for 100 periods we get, with different seed values, respectively 963, 986, 1026, 977 and 1003 for the observed demand. These simulated demand quantities have a significant impact on the inventory position of all the future periods. Therefore an essential assumption for the (R, Q) inventory policy is made that the demand is constant.

Because the (R, Q) inventory policy’s order moment or order size is independent of the observed demand the starting inventory is crucial to reach the required service level. Traditionally, when assumed that orders can not cross, the lead time distribution is used in order to calculate this required starting inventory. This level of inventory on hand is required to cover most of the demand during the lead time. For example, let’s say we have a lead time which is Poisson(10) distributed. If we want to have a service level (r) of 90% then we must find the height of the starting inventory whereof the cumulative distribution of the lead time is closest to, and at least, 90%. For a review period of 1 and lead time of Poisson(10) this starting inventory is $14 \times \mu_D$.

However, due to order crossovers the lead time variance changes and is adjusted to an effective lead time, still with the same mean. The adjusted variance is $VAR(L > l) = \sum_{i=0}^{\infty} F_i (1 - F_i)$ for a review period of 1 (Robinson et al. (2001)). If we compute this variance and then calculate the starting inventory which satisfies $r > 0.9$, we retrieve the value of $13 \times \mu_D$. These two values represent how much products must be in the inventory position in order to reach the predetermined service level.
The effective lead time variance, as calculated by Robinson et al. (2001), assumes the review period is 1. To calculate the effective lead time variance for other review periods, a simulation is used. In the simulation the effective lead times are calculated by subtracting the order moment of the $i^{th}$ order, from the arrival moment of the $i^{th}$ order. The mean of this effective lead time remains equal to the mean of the simulated lead time, however the variance decreases. These formula’s do not take demand into account. Due to the assumption of constant demand, the starting inventory is deducted by multiplying the number of periods with the constant demand.

The next section determines the probability of order crossovers in the (R, Q) inventory policy.

### 2.2. Probability of Order Crossovers

Before we say something about the impact of order crossovers, first it is important to know how likely it is that these events occur. Numerous options exist that enable order crossovers when the lead time is stochastic. A simple example is presented in Table 5. Order 1 is placed at time 0 and order 2 is placed at time 1. Lead times are uniformly distributed \{1, 2, 3, 4\}. As mentioned in Section 1.2., order crossovers are defined as; “an order which is placed later arrives before the previously placed order”. Arriving at the same time is not order crossing. Table 5 shows the arrival times \{Order$_1$, Order$_2$\}, where the events of order crossover are highlighted. If only these two orders are taken into account, the probability of crossovers is 3/16.

<table>
<thead>
<tr>
<th>Lead time Order 1/Lead time Order 2</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 {1,2}</td>
<td>(2,2)</td>
<td>{3,2}</td>
<td>{4,2}</td>
<td></td>
</tr>
<tr>
<td>2 {1,3}</td>
<td>(2,3)</td>
<td>{3,3}</td>
<td>{4,3}</td>
<td></td>
</tr>
<tr>
<td>3 {1,4}</td>
<td>(2,4)</td>
<td>{3,4}</td>
<td>{4,4}</td>
<td></td>
</tr>
<tr>
<td>4 {1,5}</td>
<td>(2,5)</td>
<td>{3,5}</td>
<td>{4,5}</td>
<td></td>
</tr>
</tbody>
</table>

Robinson et al. (2001) show that, if lead times are independent and identically distributed, the probability any given order with a lead time of $l$ did not cross the $k^{th}$ order is $F_{l+k}$. However, this formula assumes that every period an order is placed. If the review period is no longer 1, not every period an order is placed but only every $R^{th}$ period. For instance if the review period is 2, no order is placed in periods 1, 3, and so on. This impacts the ordering, holding and penalty costs incurred, as well as the probability of order crossovers.

The orders placed in the $R^{th}$ prior period remain to have the same probability of no order crossover. The chance of no order crossover when no order is placed is equal to 1, simply because there is no order to be crossed. If we want to know what the probability is that an order with lead time $l$ does not cross the $k^{th}$ order we must include the parameter $R$. The $k^{th}$ order is placed at time $k * R$. In formula;

$$\text{Pr(No crossing order k)} = F_{l+kR}$$
However, we are not interested in the probability of an order not crossing another order, the chance of an order crossover happening is more interesting. Obviously for these two orders the chance of an order crossing is equal to $1 - F_{l+kR}$. Instead of comparing two single orders it is more important to know if any orders cross. To calculate the probability of an order crossing any order, the artesian product of the chances of no crossovers is retrieved, to retrieve all possibilities of order crossovers. This is multiplied with the probability of that lead time, for all the lead times, and subtracted from 1, in formula;

$$ \Pr(Order \, crossovers) = 1 - \sum_{l=0}^{\infty} f_{l} \prod_{k=1}^{\infty} F_{l+kR} $$

Figure 5 is a visual representation of the influence of the review period on the crossover probability. On the horizontal axis the review period is represented and on the vertical axis the chance of order crossovers is provided. The lead times are Poisson distributed, respectively 2, 4 and 8 as mean and variance. Probability of order crossovers is independent of the demand distribution in the (R, Q) policy.

As Figure 5 shows, the probability of order crossovers decreases steadily whilst the review period increases. This is intuitively explained by the fact that in a larger review period, more periods exist where no orders are placed and the orders which are placed are further apart. If the review period equals 1, the order placed 2nd requires a lead time which is at least 2 periods shorter compared to the order placed first. If the review period is 2, the 2nd order already requires a lead time which is at least 3 periods shorter compared to the first order placed. With a lead time distribution which remains the same, this probability obviously decreases. The effect of a higher review period looks similar for all lengths of lead times, a longer review period decreases the chance of order crossover.

A higher mean and variance of the lead time, increases the probability of order crossover as well. Increased lead times lead to the expectation of more order moments before the order arrives. Each of these placed orders before the order arrives has a probability to cross other orders.
To see if the assumption of the lead time distribution influences the probability of order crossovers, the Poisson distribution is compared with the two-point distribution. The values are presented in Appendix A, Table 18. The probabilities of the two-point distribution and Poisson distribution are compared with the probabilities found in Figure 5 and presented in Appendix A, Figure 15, 16 & 17. As the graphs show, the length of the review period has a bigger impact on the probability of order crossovers in the two-point distribution compared to the Poisson distribution. This increased impact is explained by the fact that the Poisson distribution has probabilities across all integer values of the lead time, whereas the two-point distribution has not.

A concise example; a lead time with mean and variance 2, the two-point distribution can be $f_1 = \frac{2}{3}, f_4 = \frac{1}{3}$. According to the two-point distribution it is not possible for a review period of 3 for orders to cross. The absence of this possibility makes sense because an order placed in period 0 arrives, at its latest, in period 4. The next order is only placed in period 3 which arrives earliest in period 4, so no order crossing is possible. In a Poisson distribution it is possible, with a probability of 0,05, that the lead time is larger than 4 and a probability of 0,13 that the lead time is 0, so orders can cross in this situation.

2.3. ANALYSIS

This section first describes the simulation used. Section 2.3.2. analyzes two cases where the assumption is made that orders can or can not cross. The apparent and real savings are calculated for both these cases. Finally the sensitivity of included parameters (Section 2.3.3.) and reorder level (Section 2.3.4.) is discussed.

2.3.1. SIMULATION

In the conducted simulation where order crossovers are allowed, independent lead times are simulated. If order crossovers do not occur however, lead times are first generated for the entire length of the simulation and afterwards sorted from small to large. Sorting the lead times results in a situation where order crossovers are impossible to happen. However, the lead times are no longer independent. The distribution remains the same as in the situation of order crossovers.

Even though lead times are sorted in simulating the case of no order crossover, the total costs when the probability of order crossovers is 0 are not significantly different. When the lead times are sorted the lead times are no longer independent, but this sorting does not make a difference if order crossovers are impossible to happen. Sorting the lead times from small to large is therefore an accurate representation of the case where order crossovers do not happen.

Because the order quantity depends on the length of the review period the only parameter which needs to be optimized is the review period. The optimal review period is found by calculating the costs when $R = 1$, and comparing these costs when $R = 2$. If the costs decreased, the review period is increased by 1 again. When the costs start to increase the previous review length is optimal since the cost function is convex (Flynn, 2008). The way costs are calculated is explained in Section 1.5.3.
The simulation consists of the following parameters which are tested for their effects on order crossovers and thereby the total costs; review period, required service level and lead time. Besides, the demand, fixed ordering costs, holding costs and penalty costs obviously also influence the costs and these influences are observed too. It is required to start with inventory, since the order quantity or order moment does not depend on the inventory position. Section 2.1. shows how this inventory at the start is determined. The number of periods simulated is at least 100,000. If, after these 100,000 periods, including another period influences total costs, this period is added. Simulation stops if the average cost per period is stable.

2.3.2. Results

Appendix A, Table 19, provides the parameter settings from which the results in Table 6 & 7 are retrieved. Table 6 and 7 provide results for different review periods and their involved costs.

Table 6 Costs with and without order crossovers, Lead time Poisson(2)

<table>
<thead>
<tr>
<th>R Poisson(2)</th>
<th>Assuming Order Crossovers Total costs</th>
<th>Assuming no Order Crossovers Total costs</th>
<th>Apparent savings</th>
</tr>
</thead>
<tbody>
<tr>
<td>R = 1, Q = 10</td>
<td>€63,32</td>
<td>€76,84</td>
<td>€13,52</td>
</tr>
<tr>
<td>R = 2, Q = 20</td>
<td>€45,30</td>
<td>€53,62</td>
<td>€8,32</td>
</tr>
<tr>
<td>R* = 3, Q* = 30</td>
<td>€43,01</td>
<td>€46,38</td>
<td>€3,37</td>
</tr>
<tr>
<td>R** = 4, Q** = 40</td>
<td>€43,49</td>
<td>€44,94</td>
<td>€1,45</td>
</tr>
<tr>
<td>R = 5, Q = 50</td>
<td>€45,55</td>
<td>€46,03</td>
<td>€0,48</td>
</tr>
</tbody>
</table>

Table 7 Costs with and without order crossovers, Lead time Poisson(10)

<table>
<thead>
<tr>
<th>R Poisson(10)</th>
<th>Assuming Order Crossovers Total costs</th>
<th>Assuming no Order Crossovers Total costs</th>
<th>Apparent savings</th>
</tr>
</thead>
<tbody>
<tr>
<td>R = 1, Q = 10</td>
<td>€73,33</td>
<td>€106,87</td>
<td>€33,54</td>
</tr>
<tr>
<td>R = 2, Q = 20</td>
<td>€57,69</td>
<td>€83,18</td>
<td>€25,49</td>
</tr>
<tr>
<td>R* = 3, Q* = 30</td>
<td>€56,85</td>
<td>€75,81</td>
<td>€18,96</td>
</tr>
<tr>
<td>R* = 4, Q* = 40</td>
<td>€59,16</td>
<td>€73,02</td>
<td>€13,86</td>
</tr>
<tr>
<td>R** = 5, Q** = 50</td>
<td>€63,01</td>
<td>€72,58</td>
<td>€9,57</td>
</tr>
<tr>
<td>R = 6, Q = 60</td>
<td>€65,72</td>
<td>€72,64</td>
<td>€6,92</td>
</tr>
<tr>
<td>R = 7, Q = 70</td>
<td>€68,72</td>
<td>€73,76</td>
<td>€5,04</td>
</tr>
</tbody>
</table>

The actual perceived cost is in practice the decisive factor on which the optimal values of an inventory policy are based. Tables 6 & 7 show the effect of the chosen settings on the costs and apparent savings. A ‘∗’ represents the optimal value, for R and Q, assuming order crossovers. The optimal values assuming no order crossovers are represented by ‘∗∗’. Both simulations always reach their minimum required service level, which is here set to 90%. Assuming no order crossovers whilst they actually happen lead to the situation where the service level requirement is exceeded.

As these two tables present, order crossovers outperform the simulation with no order crossovers cost wise, due to apparent savings. Apparent savings are present in practice but not accounted for in the analysis. Apparent savings are calculated if the costs of the same row of assuming order
crossovers are subtracted from the costs of assuming no order crossovers. These apparent savings are reduced if the review period is increased. An increase in review period causes order crossovers to be less likely. As expected, the cost difference for a lead time distribution with larger mean and variance is bigger, this is due to the fact that order crossovers happen more frequently.

We are however not searching for apparent savings, but for real savings. Real savings are present if the optimized parameters when assuming order crossovers are different to the case when no order crossovers are assumed. This answers the question if it is appropriate to use the assumption of no order crossovers or if they have to be taken into account. To answer this question, we compare the total costs retrieved when the optimal parameter values of assuming no order crossovers are used in a setting where orders can cross.

If we first examine Table 6, we see that the optimal review period and order quantity when assuming orders can not cross are \( R = 4 \) and \( Q = 40 \). Assuming orders can cross results in optimal \( R = 3 \) and \( Q = 30 \). If the total costs are examined the difference between these observed costs, which are the real savings, are only €0.48 (€43.49 - €43.01) or just 1.1%. When studying Table 7, the difference between the optimal review period and order quantity is larger (\( R = 3 \) and \( Q = 30 \) vs. \( R = 5 \) and \( Q = 50 \)). The real savings observed here are €6.16 (€63.01 - €56.85), 10.83% of the total costs. Based on these results we conclude that the appropriateness of the assumption of no order crossovers depends on the parameter settings.

2.3.3. **Sensitivity**

Table 8 corresponds to situations where only one parameter is altered, represented in the first column. The 2nd and 3rd column represent the optimal parameter settings, when order crossovers are assumed or not assumed to occur. The final two columns present the savings (real and total).

<table>
<thead>
<tr>
<th>Alteration setting</th>
<th>Optimal parameters assuming order crossovers</th>
<th>Optimal parameters assuming no order crossovers</th>
<th>Total savings</th>
<th>Real savings</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h = \text{€}0.50 )</td>
<td>( R^* = 4, Q^* = 40 )</td>
<td>( R^* = 8, Q^* = 80 )</td>
<td>€7.00 (14.65%)</td>
<td>€5.08 (11.07%)</td>
</tr>
<tr>
<td>( h = \text{€}2 )</td>
<td>( R^* = 2, Q^* = 30 )</td>
<td>( R^* = 3, Q^* = 30 )</td>
<td>€34.39 (29.21%)</td>
<td>€4.21 (4.79%)</td>
</tr>
<tr>
<td>( p = \text{€}4 )</td>
<td>( R^* = 3, Q^* = 30 )</td>
<td>( R^* = 5, Q^* = 50 )</td>
<td>€9.62 (15.65%)</td>
<td>€4.32 (7.69%)</td>
</tr>
<tr>
<td>( p = \text{€}12 )</td>
<td>( R^* = 2, Q^* = 20 )</td>
<td>( R^* = 4, Q^* = 40 )</td>
<td>€18.78 (24.07%)</td>
<td>€1.94 (3.17%)</td>
</tr>
<tr>
<td>( A = \text{€}0 )</td>
<td>( R^* = 1, Q^* = 10 )</td>
<td>( R^* = 1, Q^* = 10 )</td>
<td>€34.48 (59.69%)</td>
<td>€0.00 (0%)</td>
</tr>
<tr>
<td>( A = \text{€}100 )</td>
<td>( R^* = 4, Q^* = 40 )</td>
<td>( R^* = 7, Q^* = 70 )</td>
<td>€9.62 (11.87%)</td>
<td>€5.23 (6.82%)</td>
</tr>
<tr>
<td>( \mu_D = 5 )</td>
<td>( R^* = 4, Q^* = 40 )</td>
<td>( R^* = 7, Q^* = 35 )</td>
<td>€4.66 (11.50%)</td>
<td>€1.70 (4.53%)</td>
</tr>
<tr>
<td>( \mu_D = 20 )</td>
<td>( R^* = 2, Q^* = 20 )</td>
<td>( R^* = 4, Q^* = 80 )</td>
<td>€44.65 (33.30%)</td>
<td>€16.11 (15.26%)</td>
</tr>
<tr>
<td>( r = 0.8 )</td>
<td>( R^* = 4, Q^* = 40 )</td>
<td>( R^* = 5, Q^* = 50 )</td>
<td>€18.02 (22.89%)</td>
<td>€3.76 (5.83%)</td>
</tr>
<tr>
<td>( r = 0.95 )</td>
<td>( R^* = 3, Q^* = 30 )</td>
<td>( R^* = 6, Q^* = 60 )</td>
<td>€14.13 (18.96%)</td>
<td>€9.33 (13.31%)</td>
</tr>
</tbody>
</table>
As presented in Table 8, the real and total savings depends on the chosen parameters. Significant savings can be obtained if the parameters are optimized when assuming orders can cross. To draw conclusions about the sensitivity of the assumption, we handle the parameters separately;

- Per unit holding costs; An increase in holding costs decreases the optimal review period.
- Per unit backordering costs; An increase in backordering costs decreases the optimal review period.
- Fixed order costs; An increase in fixed ordering costs increases the optimal review period.
- Mean demand; If the demand increases, but remains to be constant, the optimal review period decreases. This is due to the fact that the share of the fixed ordering costs compared to the costs corresponding with holding and backordering decreases. If demand is made stochastic, the (R, Q) inventory policy is no longer stable and provides significant high inventory levels or number of backorders.
- Review period; A higher review period leads to a decrease in the present apparent savings. The real savings are also lower if the review period increases, because the probability of order crossovers reduces. If the review period is chosen such that the maximum lead time is always within the review period the assumption of crossovers has no impact.
- Service level; Higher required service levels require more inventory. A larger impact is present when assuming orders can not cross, which results in larger real cost savings.
- Variance lead time; Lead time variance contributes significantly to the probability of order crossovers as Section 2.2. shows. Higher variance of the lead time increases the probability of order crossovers and therefore the real savings increase.
- Mean lead time; A higher average of the lead time contributes to an inventory position at ordering which must be increased, the effect is the same on both policies which are and are not incorporating order crossovers (He, Kim, & Hayya, 2005).
- Order quantity; Larger order quantity increases the review period, thereby the real savings decrease.

2.3.4. Reorder Level

To increase the influence on the inventory position in a periodical review inventory policy with fixed order size, a reorder level can be incorporated. This is the (R, s, Q) inventory policy. This inventory policy monitors the inventory position every $R$ periods in order to make a replenishment decision (Janssen, Heuts, & de Kok, 1998). When the inventory position falls below the reorder level $s$, an order with fixed order quantity $Q$ is ordered. This order ensures that the inventory position is between $s$ and $s + Q$. When a reorder level is included in this inventory policy the probability of order crossovers decreases. This decrease is caused since not in every review period an order needs to be placed, but only if in that period the inventory position is below $s$.

If we arbitrarily set a reorder point it is concluded that, due to the assumption of constant demand, this reorder point is reached with certainty in predetermined periods. This inclusion of the reorder level therefore can be viewed as another way of setting a review period. Due to the assumption of constant demand we do not elaborate further on this addition to the inventory policy.
2.4. Conclusion

First we discussed the likelihood of the \((R, Q)\) policy in practice. This inventory policy is often chosen if demand is, relatively, stable over time. Afterwards we examined the probability of order crossovers. We saw that the review period and variance of the order lead time are important factors influencing this probability. Apparent savings are present if the assumption is made that order can not cross, while in reality they can. Besides apparent savings, real savings could be present if the parameters are optimized assuming that order crossovers can happen.

We saw that order crossovers do not always significantly impact the real cost savings. A low probability of order crossovers explains the absence of real cost savings. If real cost savings are present, we analyzed the relationships of the parameters involved and the real cost savings, this is presented in Figure 6. A ‘+‘-sign represents a positive relationship, for instance if the order costs rise the review period also increases. A ‘-‘-sign represents a negative relation, an increase in mean demand leads to a decrease of the review period.

<table>
<thead>
<tr>
<th>Backorder costs</th>
<th>Reorder level</th>
<th>Service level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean demand</td>
<td>Review period</td>
<td>Probability crossovers</td>
</tr>
<tr>
<td>Holding costs</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Order quantity</td>
<td>Mean lead time</td>
<td>Real savings</td>
</tr>
<tr>
<td>Order costs</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

![Figure 4 Concluding figure (R, Q)](image)

It is now possible to answer the first research question; What is the effect of order crossovers on the (near-) optimal parameter settings in the \((R, Q)\) policy?

In the case of a periodic review, fixed order quantity it is seen that if the probability of order crossovers is large, the real savings can be significant. Assuming the wrong assumption leads to \(R\) and \(Q\) values which are not optimal. Optimal review period and order quantity, are lower in cases where orders can cross.

The appropriateness of the assumption depends on the parameter settings involved. The parameter settings which influence the real savings are investigated and summarized in Figure 6. Some of these parameters influence the review period, which on its turn influences the probability of order crossovers, which is the driver for the real savings.

The next chapter relaxes the constraint of fixed order quantities and constant demand.
3. Periodic Review, Variable Order Quantity

Chapter 2 discussed the consequences of order crossovers in a periodic review inventory policy with fixed order quantities. The constraint of ordering in fixed quantities is relaxed in this chapter to handle the variable order quantity. This policy remains to order every $R$ periods, but now the order size equals the observed demand during the review period. The order quantity restores the inventory position to a predetermined order up to level, $S$, after ordering.

The rest of this chapter is structured as follows. First, Section 3.1. discusses the (R, S) practicality and current literature. Section 3.2. shows that the probability of order crossovers potentially differs from the (R, Q) policy. In Section 3.3. the method of Robinson et al. (2001) is followed in order to calculate the shortfall distribution. The shortfall distribution helps to calculate the appropriate order up to level. Afterwards Section 3.4. compares total costs of the retrieved order up to levels with methods not incorporating order crossovers. Section 3.4. also analyzes the sensitivity of the assumption on the parameters involved. Finally Section 3.5. concludes this chapter.

3.1. Policy Applicability

First we investigate the applicability of the (R, S) policy in practice. Afterwards existing literature is briefly examined. Section 3.1.2. shows the altered assumptions in this inventory policy.

Comparable to the (R, Q) policy, the (R, S) policy places orders every period. This is used in practice to make it easier for an inventory manager. The inventory manager only has to check once every review period to see if an order needs to be placed. Continuous review policies require the manager to constantly watch the inventory position. The orders are of a size $Q_t$. The order quantity, $Q_t$, equals the demand during the review period, $\sum_{x=t-R+1}^{t} D_x$, given the fact that $t$ is one of the order moments, $O_t$. With a review period equal to 1, this order quantity equals the demand at time $t$, $D_t$.

This order quantity ensures that the inventory position equals $S$, the order up to level, after each order placement. Due to this variable order quantity, this inventory policy can be used to handle variable demand as well. The order up to level is chosen such that there is a balance between the holding and backorder costs, whilst subject to a predetermined service level. The inventory manager therefore has to decide on the review period and order up to level.

3.1.1. Existing Literature

Robinson et al. (2001) investigated the impact of order crossovers in an (R, S) inventory policy. Their paper presents an exact iterative algorithm to compute the number of outstanding orders and formulae for the shortfall distribution. The shortfall distribution optimizes the order up to level whilst assuming orders can cross. They find that, even if the chance of order crossovers is small, total cost savings can be significant.

Robinson et al. (2001) state in their article; “...there is no explicit fixed order cost, although the period length may have been set to reflect the costs of monitoring the inventory level and placing an order”. Order crossovers can influence the optimal period length (Bischak et al. 2014). The article of
Robinson et al. (2001) is unable to see the impact of different settings for the review period. The formulas presented for the shortfall distribution in Section 3.3. incorporate fixed ordering costs. By incorporating fixed ordering costs the optimal review period is determined, instead of determining $R$ beforehand.

The goal of this chapter is twofold, first check and elaborate on the formulas found by Robinson et al. (2001). These formulas provide optimal order up to levels based on the assumption orders can cross. The optimal order up to levels are needed to calculate the optimal review period, and corresponding optimal costs. Second, compare these retrieved costs with the costs when the order up to level and review period are optimized assuming orders can not cross.

### 3.1.2. Assumptions

Relaxing the restriction of a fixed order quantity makes this inventory policy more applicable for practice. Data about observed demand is directly incorporated in the placed orders. This information feed allows to relax the assumption of constant demand. Due to the order size being variable, any demand variations are included in order size. The service level does not necessarily suffer when the demand is larger than the mean demand for several periods in a row. The service level is dramatically impacted if this happens in the (R, Q) inventory policy. Often variable demand is faced in practice, therefore this inventory policy is more likely to be adopted.

Relaxing the constant demand assumption allows to relax the assumption that orders are always placed at a review period. A reason for this assumption could be that an inventory manager spends time to check the inventory position, and updates his documents accordingly, costing time and thereby incurring costs. However, due to the increase in computer presence the inventory does not need to be physically checked. Combined with the fact that there is no need for a truck to make a trip to deliver goods, this assumption is invalid and changed. The assumption is changed to; *No order is placed if demand is observed to be zero*.

Section 3.2. shows the impact of variable demand, and placing no orders if demand during the review period is zero, on the probability of order crossovers.

### 3.2. Probability of Order Crossovers

This section calculates the probability of order crossovers in the (R, S) inventory policy, incorporating the altered assumptions.

The assumption that orders are only placed if there is demand during the review period, ensures a different probability of order crossovers for (R, Q) and (R, S) inventory policies. We defined $g_d^R$ as the probability of demand $d$ during $R$. The probability of 0 demand during the review period is $g_0^R$. Incorporating this in the formula for probability of order crossovers, it now looks like this;

$$
\Pr(\text{Order crossovers}) = 1 - \sum_{l=0}^{\infty} f_l \sum_{k=1}^{\infty} \left[ F_{l+kR}^R + g_0^R (1 - F_{l+kR}) \right]
$$
The formula takes the artesian product of the chances that orders do not cross, or of orders that are not placed. These are multiplied with the probability that this specific lead time occurs and subtracted from 1. This provides the probability of order crossovers.

The probability that demand equals zero, and thus no order is placed, is lower for larger review periods. Figure 5 of Section 2.2. is adjusted to show how the demand distribution influences the chance of order crossovers. Figure 7 shows the probabilities of order crossovers with different parameters for the demand distribution. The x-axis shows the review period and the vertical axis shows the probability of order crossover. Lead time is Poisson distributed, with mean and variance 2, 4, or 8. Demand is either Poisson(10) or Poisson(2) distributed.

If demand is Poisson(10) distributed, Figure 5 ((R, Q) inventory policy) and Figure 7 ((R, S) inventory policy) illustrate an identical probability of order crossovers. The probability of 0 demand, even when the review period is 1, is small; $4.5 \times 10^{-5}$. This small probability is the reason for the similar probability of order crossovers.

If demand is Poisson(2) distributed, the chance of 0 demand, with $R = 1$, is 0.135. Figure 7 shows that the probability of order crossovers is different compared to a demand distribution which is Poisson(10) distributed. This difference is directly linked to the larger probability of 0 demand. However, if the review period increases the probability of zero demand decreases, since the demand in the entire review period needs to be zero.

![Probability Order Crossover, order every R if d >0](image)

*Figure 5 Probability order crossover*

Section 2.2. already showed that the probability of order crossovers differs slightly for different lead time distributions. If orders are only placed if demand is observed to be non-negative, the impact of different lead time distributions is comparable. The impact of a two-point distribution instead of a Poisson distribution for the lead time is presented in Appendix B, Table 20, if orders are only placed if demand occurred.
The demand distribution only influences the probability of order crossovers if the chance of 0 demand is altered. A probability of zero demand which equals 0, has a similar probability of order crossovers as (R, Q) inventory policies.

3.3. **SHORTFALL**

To calculate the optimal reorder level, Robinson et al. (2001) first define the shortfall variable, \(SF_t\). This variable denotes the inventory shortfall after all order arrivals at time \(t\), but before demand of period \(t\) is observed. \(SF_t\) is equal to the total amount of inventory ordered in or before period \(t\), but which has not arrived yet. Because holding and penalty costs are incurred based on ending inventory levels, it is required to define the shortfall at the end of the period. The shortfall at the end of the period is defined as; \(SF_t^e = SF_t + D_t\). The shortfall distribution enables us to provide a more accurate result of the expected costs. The shortfall distribution has some similarities compared to the effective lead time distribution introduced in Chapter 2. The shortfall distribution however also incorporates the demand.

First we present formulae to calculate the shortfall distribution (Section 3.3.1.). After the calculations, Section 3.3.2. approximates this value by means of a normal distribution.

3.3.1. **CALCULATED SHORTFALL**

The shortfall depends on the number of orders which are still outstanding. To calculate the probability mass function of the shortfall at the end of the period, Robinson et al. (2001) first provide an iterative algorithm to calculate these outstanding orders. This iterative algorithm calculates the probability that \(l\) of the \(m\) most recently placed orders are still outstanding after orders are placed in the current period. They define this as \(g_l|_m\). "Define \(g_l|_m = \lim_{m \to \infty} \{g_l|_m\}\) to be the probability that \(l\) orders are outstanding after the order has been placed in a period.". This value is used to calculate the shortfall. By taking the limit as \(m \to \infty\), \(g_l\) is iteratively calculated with the following formulas (Robinson et al. 2001);

\[
\begin{align*}
g_0|_0 & \equiv 1 \\
g_0|m+1 & = g_0|m \ast F_m & \quad m \geq 0 \\
g_m+1|m+1 & = g_m|m \ast (1 - F_m) & \quad m \geq 0 \\
g_k|m+1 & = g_k|m \ast F_m + g_{k-1}|m \ast (1 - F_m) & \quad 1 \leq k \leq m
\end{align*}
\]

The probability of outstanding order depends on the probability of outstanding orders in the previous period combined with the cumulative lead time probability of the placed order. The demand is not incorporated in these formulae. As Section 3.2. shows, if the probability of zero demand is negligible, every order moment an order is placed. Appendix B, Figure 18 graphically depicts the limits of outstanding orders when the review period is 1 and the lead time is Poisson(3) distributed.
The formulae of Robinson et al. (2001) are no longer valid when the review period is increased, because not in every period an order is placed. For example if \( R = 2 \); at period 2 there can be at most 1 outstanding order, whilst the presented formulae assume 2 orders can be outstanding. The probability of \( g_{1|m} \) is therefore altered to incorporate that not every period necessarily places an order. For a review period of 2, the formulas are altered as follows:

\[
\begin{align*}
g_{0|0} &= 1 \\
g_{0|m+1} &= F_m & \text{for } m < R \\
g_{0|m+1} &= g_{0|m-1} * F_m & \text{for } m \geq R \\
g_{l|m+1} &= 1 - F_m & \text{for } m < R, l \leq O_0^m \\
g_{l|m+1} &= g_{l|m-1} * F_m + g_{l-1|m-1} * (1 - F_m) & \text{for } m \geq R, l \leq O_0^m \\
\end{align*}
\]

for all equations; \( m \geq 0, l \geq 1 \)

The restriction \( l \leq O_0^m \) is included. This makes sure that the number of orders, \( l \), is at most the number of periods in which orders can be placed, \((m/R = O_0^m)\).

For a review period larger than 1, the number of outstanding orders no longer converges to one limit. There are \( R \) limits, for each possible day in the review period. For instance if the review period is 2, there is a limit present for all possibilities of orders outstanding, at each order moment and at each period which is not an order moment (the two distinct types of periods).

With a review period of 3, there is a limit at the order moment, the period after the order moment and the 2\textsuperscript{nd} period after the order moment (period prior to order moment). Figure 8 presents the limit of outstanding orders if \( R = 2 \). A review period of 3 is presented in Appendix B, Figure 19. The lead time is in both cases Poisson(3) distributed.

![Limit of Outstanding Orders, R=2](image)

Figure 6 Outstanding orders, lead time Poisson(3), Review period 2
If the retrieved limits at a review period of 1 and 2 are compared, it is readily seen that the probabilities of a high number of outstanding orders decreases. This decrease aligns with the fact that the probability of order crossovers is lower at higher review periods.

With the probability of number of outstanding orders, Robinson et al. (2001) provide the following formula for the probability mass function of the inventory shortfall; \( \Pr(SF^e = sf^e) = \sum_{i=0}^{\infty} g_i \Pr\{D^{(i+1)} = sf^e\} \). This function sums the multiplied chances of \( i \) outstanding orders with the probability that demand of these \( i \) orders + 1 equals \( sf^e \). The shortfall at the end of the period is required because costs are incurred at the end of a period. This function is valid if every period an order is placed. With larger review periods, less orders are placed and the chances of \( i \) outstanding orders were adapted. Because each order has got, in the case of \( R = 2 \), 2 limits, found at \( O_i \) and \( O_i - 1 \) (the order moment and the moment prior to the order moment). The formula for the inventory shortfall is altered to incorporate that as well;

\[
\Pr(SF^e = sf^e) = \left( \sum_{i=0}^{\infty} \Pr(D^{(i+1)} = sf^e) * g_j\mid R - i \right)
\]

\[
j = \left\lfloor \frac{i}{R} \right\rfloor
\]

\( \forall i \in \{1, 2 \ldots R\} \)

In the formula \( k \) represents a large enough number to retrieve a limit for the number of orders outstanding. For a review period of 2, the total cumulative probability now accumulates to 2 instead of 1. In case of a general review period it accumulates to \( R \). To make sure the cumulative probability mass function correctly accumulates to 1 several options exist. The lead time values could be multiplied with the conditional chance of this lead time occurring. However, multiplying with the conditional chances does not have the desirable result. The total cumulative probability does approach 1, but not exactly reach 1. The probabilities are divided by \( R \) instead of the conditional chance. This method is used before and Wensing & Kuhn (2015) show this is approximately correct. The chance of being in a period is uniform distributed, hence dividing by \( R \) is appropriate. Comparing this to the simulation, the costs are around 5% different then the costs when this function is used. This difference is comparable to the difference between the costs calculated for a review period of 1 in simulation versus the formulas provided by Robinson et al. (2001).

### 3.3.2. Estimated Shortfall

To avoid tedious calculations of the shortfall distribution, a common heuristic is to use an approximation for this distribution. A frequent used approximation is a normal distribution with the same mean and variance (Robinson et al., 2001). Robinson et al. (2001) provide formulas to calculate the mean and variance of the shortfall distribution, under the assumption that \( D \) and \( L \) are jointly independent. Standard formulas for the mean and variance of the shortfall conclude \( E(SF) = \mu_D \sum_{k=0}^{\infty} (1 - F_k) = \mu_D \mu_L = E(LTD) \) respectively \( Var(SF) = \mu_L \sigma_D^2 + \mu_D^2 \sum_{i=0}^{\infty} F_i(1 - F_i) \), Robinson et al. (2001).
Using the formulas above and comparing their results to the function retrieved in Section 3.3.1, provide similar results. This estimation is therefore considered correct. The comparison of the probability mass function and the cumulative distribution function is presented in Figure 20 & 21, Appendix B.

The formulas Robison et al. (2001) provide are not yet applicable for a review period not equal to 1. Accounting for the review period in the formulas estimating the mean and variance of the shortfall distribution, these formulas are;

\[ E(SF) = R\mu_D \sum_{k=0}^{\infty} (1 - F_k) = R\mu_D \mu_L \]

\[ Var(SF) = \mu_L * (R * \sigma_D^2) + \mu_D^2 R \sum_{i=0}^{\infty} F_i (1 - F_i) \]

Comparing these approximated values to the retrieved values of the calculation, described in Section 3.3.2. For a review period of 2, Appendix B, Figure 22 & 23 are retrieved as an example (both the p.m.f. and c.d.f.). For different setting for demand, lead time or the review period results are comparable. The cumulative distribution function again has a maximum difference with the calculated values of 1.5%. These formulas are tested for other review periods and the differences compared to the calculated shortfall are all comparable to the difference for a review period of 1. Therefore the approximation of the shortfall is considered valid.

3.4. ANALYSIS

This section first searches for the optimal order up to levels assuming orders can and cannot cross. The next section explains the cost structure and shows how the optimal review period is found. The optimal parameters are implemented in a simulation which is described in Section 3.4.3. The consecutive paragraphs show the results of the simulation, comparing the optimal values and their corresponding costs in both situations. Section 3.4.5. discusses the impact of parameters, Section 3.4.6. briefly describes the impact of a reorder level.

3.4.1. ORDER UP TO LEVELS

Traditionally the lead time demand (ltd) distribution is often used to calculate the order up to level. Approximating this distribution with a normal distribution, the mean equals \( \mu_{TD} = \mu_L * \mu_D \), the variance is equal to \( \sigma_{TD}^2 = \mu_L * (\sigma_D^2 * R) + \mu_D^2 * \sigma_L^2 \), Robinson et al. (2001). The review period only influences the demand variance during the lead time.

Section 3.3. presented the shortfall distribution as an improvement for the ltd distribution. Appendix B, Figure 24 presents the differences in order up to levels between these distributions. The approximation uses the normal distribution, whereas the calculated values present the actual computed values. The shortfall distribution has a lower order up to level compared to the ltd distribution. Order crossovers cause the variance to decrease and therefore the same service level is reached with a lower order up to level. The parameters are presented in Appendix B, Table 21.
As Robinson et al. (2001) mentioned, the difference in order up to level $S$, is not per se valuable in practice. The differences in costs when these values are actually used are meaningful. Therefore these values are compared when they are used in the simulation. First, the following section discusses these involved costs.

### 3.4.2. Cost Structure

The cost structure for the $(R, S)$ policy is similar to the $(R, Q)$ policy, incorporating holding, backordering and fixed ordering costs. The expected costs for a review period of 1 looks as follows; $A + h \sum_{s_f < S} (S - s_f^e) \cdot \Pr\{SF^e = s_f^e\} + p \sum_{s_f > S} (s_f^e - S) \cdot \Pr\{SF^e = s_f^e\}$ and is solved as a standard newsvendor problem with discrete demand. The expected costs per period should incorporate the fixed ordering costs $A$, based on the length of the review period. This is represented in the following formula;

$$\frac{A}{R} + h \sum_{s_f < S} (S - s_f^e) \cdot \Pr\{SF^e = s_f^e\} + p \sum_{s_f > S} (s_f^e - S) \cdot \Pr\{SF^e = s_f^e\}$$

For $(R, S)$ systems it is possible to find the optimal review period, and its corresponding optimal costs, by using heuristics provided by Flynn (2008). These heuristics calculate the optimal order up to levels, each subject to the same service level, for each review period. Afterwards the review period with the lowest total costs is selected. This is possible because the cost function is convex (Flynn, 2008). Appendix B, Figure 25, shows an example of this convexity.

If we use the shortfall distribution as a base and compare it with the $ltd$ distribution ($R = 2$), Figure 9 is retrieved. The costs of the normal approximation of the shortfall distribution compared to the calculated shortfall show many similarities. Comparing Figure 9, with the figure Robinson et al. (2001) retrieve for a review period of 1, it is seen that the difference between the shortfall and the $ltd$ distribution slightly decreases if the review period increases. This decrease is caused by a lower probability of order crossovers if the review period increases.

![Relative costs](image)

*Figure 7 Total costs for different service levels*
3.4.3. Simulation

The simulation for the (R, S) inventory policy is comparable to the (R, Q) policy. Different is that demand is now also Poisson distributed. Most importantly the order quantity is no longer fixed but variable, it depends on the observed demand during the review period. Similar to the (R, Q) policy, the same parameters are included (review period, required service level, lead time, fixed ordering costs, holding costs, penalty costs and demand).

A part of the simulation output, used to validate the formulas found for the determination of the order up to point and the total costs, is represented in Appendix B, Table 22. There are some slight differences between the costs simulated and the calculated costs using the adapted formulas of Robinson et al. (2001). These differences can arise due to the fact that the orders which potentially cross each other are not necessarily of the same size, but are distributed $R \ast \text{Poisson}\sim(D)$. This is not accounted for in the formulas but in the simulation this case can occur.

It is not required to start with inventory, since the order quantity depends on the inventory position. After the first order the inventory position reaches the order up to level. A warm-up period is incorporated, which makes sure that the first order has arrived to start with inventory on hand. The number of periods simulated is at least 100.000. After these 100.000 periods the inclusion of an additional period is examined. If this additional period influences total costs this period are added, simulation only stops if the average cost per period is stable.

3.4.4. Results

Appendix B Table Tables 23 and Table 10 presented below, show the retrieved costs and service level per review period. These costs and service levels are retrieved if the order up to level is optimized, either assuming orders can cross or that they can not cross. Appendix B, Table 23 shows that the assumption regarding order crossovers does not influence the optimal settings. The optimal order up to level and the optimal review period are the same in both cases. With these parameters no real savings are obtained if order crossovers are taken into account. Optimal values assuming no order crossovers are marked with a `*`, optimal values assuming order crossovers are marked with a `**`.

<table>
<thead>
<tr>
<th>Review period</th>
<th>Assuming Order Crossovers</th>
<th>Assuming no Order Crossovers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S</td>
<td>Total costs</td>
</tr>
<tr>
<td>1</td>
<td>122</td>
<td>€76,21</td>
</tr>
<tr>
<td>2</td>
<td>133</td>
<td>€63,09</td>
</tr>
<tr>
<td>3*</td>
<td>142*</td>
<td>€62,24</td>
</tr>
<tr>
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<td>152</td>
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</tr>
<tr>
<td>5**</td>
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<td>€66,78</td>
</tr>
<tr>
<td>6</td>
<td>169</td>
<td>€69,73</td>
</tr>
<tr>
<td>7</td>
<td>177</td>
<td>€74,45</td>
</tr>
</tbody>
</table>

Table 9 Example Lead time Poisson(10) Demand Poisson(10)
In Table 9 there are differences in the optimal order up to level and the corresponding optimal review period. As Table 9 shows, the optimal review period assuming no order crossovers is 5, with a corresponding optimal order up to level of 166. The corresponding costs are €76,50. If the assumption is made that orders do cross the optimal review period is 3, and the corresponding order up to level 142. The optimal review period can be different due to the effect of crossovers already shown by Bischak et al. (2014). Total costs are in this case €62,24. The total cost savings in this situation is €14,26 (€76,50 - €62,24), 18,65%.

However, we are not searching for total cost savings, but for real cost savings. If the optimal parameters assuming no order crossovers are implemented in the simulation where orders can cross, total costs of €67,79 are retrieved. Apparent savings of €8,71 (€76,50 – €67,79) are present. The real savings, obtained when the optimal parameters assuming order crossovers are used, are €5,55 (€67,79 - €62,24), 8,19%. The next section alters the involved parameters to investigate their influences.

### 3.4.5. Sensitivity

Any alteration made in the parameters regarding costs, service level or demand influences the optimal settings. Table 10 corresponds to situations where only one parameter is altered, represented in the first column. The 2nd and 3rd column present the optimal parameter settings to reach the required service level. The final two columns present the savings. Total savings is the difference between the total costs of assuming no order crossovers when orders can not cross and total costs assuming order crossovers if they can cross. Real savings are obtained if the optimal parameter settings assuming no order crossovers are implemented in the simulation where orders can cross, minus the costs of the optimized parameters assuming orders can cross.

<table>
<thead>
<tr>
<th>Alteration setting</th>
<th>L<del>Poisson(10) D</del>Poisson(10)</th>
<th>Optimal parameters assuming order crossovers</th>
<th>Optimal parameters assuming no order crossovers</th>
<th>Total savings</th>
<th>Real savings</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h=€0,50$</td>
<td>$R^* = 4, S^* = 152$</td>
<td>$R^* = 6, S^* = 173$</td>
<td>€7,47 (14,51%)</td>
<td>€1,14 (2,48%)</td>
<td></td>
</tr>
<tr>
<td>$h = €2,-$</td>
<td>$R^* = 2, S^* = 133$</td>
<td>$R^* = 4, S^* = 160$</td>
<td>€31,94 (25,57%)</td>
<td>€19,00 (16,96%)</td>
<td></td>
</tr>
<tr>
<td>$p = €4,-$</td>
<td>$R^* = 3, S^* = 142$</td>
<td>$R^* = 5, S^* = 166$</td>
<td>€11,39 (17,03%)</td>
<td>€7,72 (12,41%)</td>
<td></td>
</tr>
<tr>
<td>$p = €12$</td>
<td>$R^* = 3, S^* = 142$</td>
<td>$R^* = 5, S^* = 166$</td>
<td>€16,39 (19,90%)</td>
<td>€4,42 (6,28%)</td>
<td></td>
</tr>
<tr>
<td>$A = €0,-$</td>
<td>$R^* = 1, S^* = 122$</td>
<td>$R^* = 1, S^* = 142$</td>
<td>€30,85 (50,60%)</td>
<td>€13,03 (30,20%)</td>
<td></td>
</tr>
<tr>
<td>$A = €100$</td>
<td>$R^* = 4, S^* = 152$</td>
<td>$R^* = 6, S^* = 173$</td>
<td>€8,65 (10,16%)</td>
<td>€2,22 (2,82%)</td>
<td></td>
</tr>
<tr>
<td>$\mu_D = \sigma_D^2 = 2$</td>
<td>$R^* = 7, S^* = 37$</td>
<td>$R^* = 10, S^* = 42$</td>
<td>€0,28 (1,20%)</td>
<td>€0,28 (1,20%)</td>
<td></td>
</tr>
<tr>
<td>$\mu_D = \sigma_D^2 = 20$</td>
<td>$R^* = 2, S^* = 260$</td>
<td>$R^* = 4, S^* = 316$</td>
<td>€41,21 (29,84%)</td>
<td>€18,07 (15,72%)</td>
<td></td>
</tr>
<tr>
<td>$r = 0,8$</td>
<td>$R^* = 3, S^* = 130$</td>
<td>$R^* = 5, S^* = 149$</td>
<td>€17,21 (20,43%)</td>
<td>€4,02 (5,66%)</td>
<td></td>
</tr>
<tr>
<td>$r = 0,95$</td>
<td>$R^* = 3, S^* = 152$</td>
<td>$R^* = 5, S^* = 181$</td>
<td>€15,63 (19,40%)</td>
<td>€10,32 (13,71%)</td>
<td></td>
</tr>
</tbody>
</table>

Table 10 shows that the real and total savings depend on the involved parameters. Significant savings might be present if the parameters are optimized when assuming orders can cross. We handle the variables one by one in order to draw conclusions;
- Per unit holding costs; An increase in holding costs increases the real savings. Assuming no order crossovers overestimates the order up to level, causing additional inventory stock.

- Per unit backordering costs; An increase in backordering costs decreases the real savings. Assuming no order crossovers overestimates the order up to level, causing less units to be backordered compared to the assumption of order crossovers. An increase in the penalty costs therefore increases cost when assuming no order crossovers less.

- Review period; A higher review period leads to a decrease in the real savings. With a higher review period the probability of order crossovers reduces. If the review period is chosen such that the maximum lead time is always within the same review period the assumption of crossovers has no impact.

- Fixed ordering costs; An increase in fixed ordering costs increases the optimal value for the review period. An increase of the review period, decreases the real savings present.

- Demand; If the demand increases the optimal review period decreases, increasing real savings. This is due to the fact that the share of the fixed ordering costs compared to the costs corresponding with holding and backordering decreases. The variance of the demand influences the probability of order crossovers slightly, as Section 3.2. showed.

- Service level; Higher required service levels require higher order up to levels. This effect is larger if the assumption is made that orders can not cross. The overestimation of the order up to level increases, thereby increasing the real cost savings.

- Variance lead time; The lead time variance contributes significantly to the probability of order crossovers as Section 3.2. shows. Higher variance of the lead time increases the probability of order crossovers, therefore the real savings increase.

- Mean lead time; A higher average of the lead time contributes to an order up to level which must be increased. The effect is the same on both policies which are and are not incorporating order crossovers, real cost savings remain the same if only the mean lead time is increased. He et al. (2005) already showed that the mean lead time can be independent of the operating costs.

3.4.6. Reorder level

Incorporating a reorder level in the (R, S) policy influences the probability of order crossovers. Orders are no longer placed if the demand is smaller than the difference between the order up to level and the reorder level during the review period; \( d_{(S-s)}^R \). What is expected for this situation is that the real savings decrease. This decrease in real savings is expected due to the fact that the effective lead time variance, and thereby the effective lead time demand variance, is adjusted since order crossovers are less likely to happen.

If we arbitrarily set the reorder level in such a manner that \( s = S - (\mu_p \times R) \), we can already see that the service level slightly drops. The average costs on the other hand remain roughly the same, this is a result of a small increase in the penalty costs, but this is offset by the minor decrease in the ordering costs. The impact of this is larger in the case of bigger reviewing times, since an order moment less is significantly less beneficial. However, the probability that the demand during the review period is 0, or \( S - s \), if the review period is increased, decreases.
3.5. CONCLUSION

This chapter started with a brief recapitulation of existing literature and the likelihood of the (R, S) policy in practice. Robinson et al. (2001) showed that order crossovers can lead to significant total savings. However, unlike Robinson et al. (2001) the review period is optimized in this chapter after order crossovers are incorporated. Section 3.2. showed that, similar to (R, Q) policies, the lead time and review period are important factors influencing the probability of order crossover. Furthermore the probability of 0 demand is also slightly contributing to this probability.

We saw that order crossovers do not always significantly impact the real cost savings. If order crossovers are unlikely to happen, traditional methods yield accurate results for the optimal parameter settings. If order crossovers are likely to happen however, the parameter settings do substantially differ. Figure 10 shows the impact of the involved parameters and the real savings. A ‘+’-sign represents a positive relationship, for instance if the demand variance increases the probability of order crossovers also (slightly) increases. A ‘-’-sign represents a negative relationship, an increase in reorder level leads to a decrease of the probability of order crossovers.

![Figure 8 Concluding figure (R, S)](image)

It is now possible to answer the first research question; What is the effect of order crossovers on the (near-) optimal parameter settings in the (R, S) policy?

If the probability of order crossovers is large, significant savings are present in the (R, S) policy, the appropriateness of the assumption depends on the involved parameters. Influences of parameters on the probability of order crossovers, and thereby the real savings present, are presented in Figure 10. The same required service level but with lower costs is reached, while the order up to level is decreased. Bischak et al. (2014) showed that the review period can change due to order crossovers. Optimal review periods presented in Section 3.3.4. and 3.3.5. are lower in the case of assuming crossovers compared to assuming no order crossovers.

The next chapter orders in fixed quantities, similar to Chapter 2, but periodic review is substituted for continuous reviewing.
4. CONTINUOUS REVIEW, FIXED ORDER QUANTITY

The previous two chapters examined the impact of order crossovers in two periodic review policies. This chapter focuses on the continuous review inventory policy with a fixed order quantity, the \((s, Q)\) inventory policy. In a \((s, Q)\) inventory policy an order is placed directly if the inventory position falls to the reorder level, \(s\). This order is always of a fixed size, \(Q\). This policy benefits from the simplicity of always the same order size, which the \((R, S)\) policy lacks, as well as control over the inventory position, which is absent in the \((R, Q)\) policy.

The rest of this chapter is organized as follows, Section 4.1. examines when this inventory policy is used. Section 4.2. determines the probability of order crossovers. Section 4.3., comparable to the \((R, S)\) inventory policy, calculates the shortfall distribution in order to calculate appropriate reorder level. Afterwards, Section 4.4. investigates the method of Federgruen & Zheng (1991) to find the optimal \(s\) and \(Q\) levels. Afterwards an improvement for their heuristic is suggested. Section 4.5. discusses the simulation used and the retrieved optimal parameters. More important, the optimal costs and the sensitivity of the used variables is also discussed in Section 4.5. Section 4.6. concludes this chapter with a brief summary.

4.1. POLICY APPLICABILITY

Equivalent to the \((R, Q)\) policy, the \((s, Q)\) policy always orders in a fixed order quantity. From a cost perspective the fixed order quantity is logical. Boxes/pallets on which the goods are transported are always of fixed optimized sizes and the truck actually transporting the goods is fully utilized as spare capacity is allocated to other products or companies. Different as the \((R, Q)\) policy, this policy constantly monitors the inventory position. Constant monitoring is enabled by technology innovations, such as barcodes and intelligent shelves.

Due to the continuous information feed on the inventory position it is possible to directly place an order if demand occurs which drops the inventory position to the reorder level. An assumption which is made in this inventory policy is that only single demand occurs. No customer can place an order of a batch of products. This assumption makes sure that the inventory position after ordering is equal to \(s + Q\), this can be thought of as the order up to level. This assumption is relaxed in the continuous review, variable order quantity inventory policy.

4.2. PROBABILITY CROSSOVER

In an \((s, Q)\) inventory policy an order is placed immediately if the inventory position reaches \(s\), an order of size \(Q\) is placed at that time. If the inventory position reaches \(s\) for the second time another order is placed and a possibility exists that orders crossover. Orders have crossed if the lead time of the first order exceeds the required time for the inventory position to reach \(s\) again plus the lead time of the second order. In formulas; \(l_{\text{order }1} > (l_{\text{order }2} + \frac{Q}{\mu_D})\).
The time between the 1\textsuperscript{st} and the \(k\textsuperscript{th}\) order is not \(k\) anymore as it was in the periodic review policy. For the \(k\textsuperscript{th}\) order it takes \(k \frac{Q}{\mu_D}\) before that order is placed. Demand of \(k \times Q\) must occur for the inventory position to reach the reorder level, before the \(k\textsuperscript{th}\) order is placed. This time and the lead time of the \(k\textsuperscript{th}\) order need to be smaller than the lead time of order 0 for the \(k\textsuperscript{th}\) order to arrive sooner than the order placed at time 0. In formulas; \(l_{\text{order } 0} > l_{\text{order } k} + k \frac{Q}{\mu_D}\) instead of \(F_{l+kR}\) as the probability of order 1 crossing order \(k\), in an \((R, S)\) inventory policy, the probability that any given order with a lead time of \(l\) did not cross the previous \(k\textsuperscript{th}\) order in an \((s, Q)\) inventory policy, has a probability of; \(F_{l+kQ \over \mu_D}\).

The probability that it did not strictly cross any previously placed orders is \(\prod_{k=1}^{\infty} F_{l+kQ \over \mu_D}\). The probability of order crossover is;

\[
1 - \sum_{l=0}^{\infty} f_l \prod_{k=1}^{\infty} F_{l+kQ \over \mu_D}
\]

The formula takes the artesian product of the chances that orders do not cross. These are multiplied with the probability that this specific lead time occurs and subtracted from 1. This provides the probability of order crossovers.

![Probability Order Crossovers](image)

Figure 9 Probability order crossover \((s, Q)\)

Figure 11 shows 4 examples of the influence of order quantity on the probability of order crossovers. On the horizontal axis the order quantity is represented and on the vertical axis the chance of order crossovers is provided. As Figure 11 shows, the probability of order crossovers decreases if the order quantity increases. This is intuitively explained by the fact that the time between order placements increases if the order quantity increased. The time between order moments is also larger if the demand has a lower mean. The order quantity must first be consumed before a new order is placed, for a higher \(Q\) or lower mean demand this takes longer. As Hayya et al.
(2011) showed, the periodicity of ordering influences the probability of order crossovers. As the order quantity increases this frequency of ordering decreases. Larger lead times result in more order moments before the arrival time of orders, each which have a probability to cross another order.

Appendix C, Figure 26, shows the probability in an \((s, Q)\) inventory policy if the lead time is Poisson(3) distributed. No smooth line is present in this graph since the lead time values are always integer valued. Due to the fact that the lead time is an integer value, an average time of 0,9 between orders (if the order quantity is 9 for example) needs a lead time value which is 1 smaller compared to an average time of 1 between orders. Whilst if the average time between order moments is 1, the lead time needs to be 2 periods shorter than the prior lead time. The bigger drops at order quantity 10 and 20 are explained by the fact that they are a multiple of the mean daily demand.

If we set the order quantity equal to the mean daily demand, so \(\frac{Q}{A} = 1\), and Poisson lead times we see that the probability of order crossovers, in a \((s, Q)\) inventory policy, equals the probability of order crossover when a review period of 1 is assumed. If we change the order quantity to \(N\) times the mean daily demand, the probability of order crossover is equal to the probability of order crossover with a review period of \(N\).

### 4.3. Shortfall

Following the procedure of Robinson et al. (2001), after calculating the probability of order crossovers it is required to define the probability of outstanding orders. With this probability it is possible to calculate and estimate the shortfall distribution, which in turn helps to calculate the optimal \(s + Q\) value.

#### 4.3.1. Calculated Shortfall

Robinson et al. (2001) assumed that every \(R\) periods an order was placed. As mentioned earlier, not every \(R\) periods but every \(\frac{Q}{A}\) periods an order is placed. If we replace this in the probability of outstanding orders we retrieve the following formula’s;

\[
g_{0|0} = 1
\]

\[
g_{0|m+1} = g_{0|m} * \frac{F_{mq}}{\mu D} \quad \text{for } m \geq 0
\]

\[
g_{m+1|m+1} = g_{m|m} * \left(1 - \frac{F_{mq}}{\mu D}\right) \quad \text{for } m \geq 0
\]

\[
g_{k|m+1} = g_{k|m} * \frac{F_{mq}}{\mu D} + g_{k-1|m} * \left(1 - \frac{F_{mq}}{\mu D}\right) \quad \text{for } 1 \leq k \leq m
\]

The probability of outstanding order thus depends on the probability of outstanding orders in the previous period as well as the cumulative probability of the lead time of the order placed. If this is graphically depicted Figure 12 is retrieved.
The graph represents the limit of outstanding orders when the lead time is Gamma(3) distributed, Demand is Poisson(10) distributed and the order quantity, Q, is 10. These formulae are still valid when the order quantity is altered. With the formula’s for the number of orders outstanding and the probability of order crossover it is possible to determine the chance of the shortfall and thereby determine the optimal value for \( s + Q \).

The shortfall depends on the number of outstanding orders and the size of them. Therefore the probability of the shortfall can be defined as follows;

\[
Pr\{S^e = s^e\} = \sum_{d=0}^{\infty} Pr\{D = d\} \times g_{d/Q}^{\infty}
\]

However, the total cumulative probability now accumulates to \( Q \) instead of 1. In order to make sure that the total cumulative probability function correctly accumulates to 1, several options are possible. First of all, the solutions can be divided by \( Q \). The second approach could be to multiply the solutions with the conditional chance of \( Prob(D = d) \) given the demand range which is necessary for the number of orders. For instance for an order quantity of 10, 2 orders can only be outstanding if demand is 20 to 29. If demand is lower, only 1 order is placed, and if demand is larger at least 3 orders are placed. However, there is only a slight difference between the two options. The graph based on the conditional chance is very pointed. This pointed shape is explained by the fact that for example in the region 35-39, the chance of a demand of 35 given a demand between 35-39 is much larger compared to the conditional chance for a value of 39. This graph looks like a normal distribution and for the following paragraph a goal is to find a formula to approximate this distribution, as can be seen in 27 & 28 in Appendix C.

For the remainder of this chapter we use the option to divide by \( Q \), this proves to be an accurate approximation.
4.3.2. Estimated Shortfall

As mentioned the graph of the shortfall distribution looks like a normal distribution. Therefore a normal distribution with the same mean and variance is used to approximate this shape, as Robinson et al. (2001) claim is common to do. If we take the approximation of Robinson et al. (2001) as a starting point it is easy to see that the mean still coincides with the mean that they have calculated; \( E(SF) = \mu_D \sum_{k=0}^{\infty} (1 - F_k) = \mu_D \mu_L = E(LTD) \).

However, the variance of the shortfall does no longer coincide with the variance Robinson et al. (2001) calculated. This discrepancy is due to the fact that not every period an order is placed but the periodicity depends on the order quantity. Incorporating this in the approximation of the variance of the shortfall the following formula is retrieved;

\[
Var(SF) = \left( \frac{Q}{\mu_D} \right) \mu_L \sigma_D^2 + \left( \frac{Q}{\mu_D} \right) \mu_D^2 \sum_{l=0}^{\infty} F_l (1 - F_l)
\]

Comparing these found formulas with the calculated shortfall show that the comparison is accurate. Both the probability mass function and the cumulative distribution function are compared. The differences between the approximation and the calculated shortfall, for the cumulative distribution function, from a service level of 80% upwards, differ at most 4%. Before this service level the difference is at most 12%. However, due to the fact that most service levels are chosen around the region of 0.9, this approximation is considered a correct approximation.

4.4. Traditional Algorithm

To see if the approximated shortfall is an improvement compared to traditional methods for an \((s, Q)\) inventory policy, the optimization algorithm of Federgruen & Zheng (1991) is used as a comparison. Their article; "An efficient algorithm for computing an optimal \((s, Q)\) policy in continuous review stochastic inventory system" (Federgruen & Zheng, 1991) searches for the optimal reorder level, \(s\), and order quantity, \(Q\), and thereby the optimal costs. No service level constraint is included in their optimization algorithm. The algorithm is investigated and adjusted to incorporate order crossovers. This adjusted algorithm also does not require a predetermined service level to be met.

The costs of an \((s, Q)\) inventory policy are described as follows in Federgruen & Zheng (1991);

\[
C(s, Q) = \frac{K \lambda}{Q} + h \sum_{j=0}^{\infty} \text{Prob}[IL(\infty) = j] + p \sum_{j=-\infty}^{1} (-j)\text{Prob}[IL(\infty) = j] = \\
\frac{K \lambda}{Q} + \frac{1}{Q} \sum_{y=r+1}^{r+Q} \left\{ h \sum_{i=0}^{y} (y-i)p_i + p \sum_{i=y+1}^{\infty} (i-y)p_i \right\}
\]

The latter term is rewritten to; \( G(y) = h \sum_{j=0}^{y} (y-j)p_j + p \sum_{j=y+1}^{\infty} (j-y)p_j = (h + p) \sum_{j=0}^{y-1} p_j + p(\lambda L - y) \). Where \( p_j = \text{Prob}[LD(\infty) = j] \) and LD represents the demand during the lead time.
Their algorithm is the following;

\[
\begin{align*}
\text{Step 0. Calculate } & \ G(0) \text{ and } \Delta G(0); L := 0; \\
\text{while } & \Delta G(L) < 0 \text{ do} \\
& \text{begin } L := L + 1, \text{ evaluate } \Delta G(L), G(L + 1) := G(L) + \Delta G(L) \\
& \text{end} \\
S := & \ k + G(L), Q := 1, C^* := S, s := L - 1, \\
R := & \ L + 1; \\
\text{Step 1. Repeat} \\
& \text{begin if } G(s) \leq G(R) \\
& \quad \text{then if } C^* \leq G(s) \\
& \quad \quad \text{then stop.} \\
& \quad \text{else begin } S := S + G(s), s := s - 1, \\
& \quad \quad \text{if } r < 0, \text{ evaluate } \Delta G(s) \text{ and } G(s) := G(s + 1) - \Delta G(s), \\
& \quad \quad \text{end;} \\
& \quad \text{else if } C^* \leq G(R) \\
& \quad \quad \text{then stop.} \\
& \quad \text{else begin } S := S + G(R), \text{ evaluate } G(R), G(R + 1) := G(R) + \Delta G(R), R := R + 1 \\
& \quad \quad \text{end;} \\
& \quad Q := Q + 1, C^* := S/Q \\
& \text{end.}
\end{align*}
\]

The idea behind the algorithm is as follows. First, step 0 makes the trade-off between penalty and holding costs which leads to an optimal value for the reorder level, \( s \). The ordering costs are added to the corresponding expected holding and penalty costs which present the total costs, represented as \( S \). Because the ordering quantity is first set to 1, the costs per product, \( C^* \), are equal to \( S \).

Step 1 checks if the penalty costs or holding costs are cheaper to include when the ordering quantity is increased. Afterwards it is checked whether this addition (either penalty or holding costs) is beneficial for the optimal costs per product. If it is beneficial to increase the order quantity is increased and the optimal costs are adjusted accordingly. Step 1 repeats itself until the increase in order quantity is not beneficial anymore. If the order quantity is no longer increased, the optimal order quantity and reorder level are found and the algorithm is finished (Federgruen & Zheng, 1991).

**4.4.1. Dependent Variance**

As Zalkind (1978), Hayya et al. (2008) and Hayya & Harrison (2009) show, the lead time variance is lower in situations where orders cross. Due to the fact that the variance of the effective lead time is lowered, the variance of the effective lead time demand also changes. The crossing of orders does not impact the holding and penalty costs structure though, since orders are of the same size and are therefore interchangeable. However each order quantity has a different effect on the effective lead time demand variance. This difference is caused by the order frequency. If the order quantity is smaller the reorder level is reached faster and the probability of order crossovers is increased. This increase in the frequency of order crossovers results in a decrease in the effective lead time demand variance (Hayya et al. 2011).
This dependent effective lead time demand variance is the difficulty of order crossovers in a \((s, Q)\) inventory policy. During a simulation this effective lead time demand variance, and its mean, are simulated. The simulation first simulates the demand, and if the inventory position drops to \(s\) an order of size \(Q\) is placed. The lead time for this order is simulated, after which the arrival time is calculated. Subtracting arrival times from the order moments provides the effective lead time, of which the variance is calculated. Table 11 represents a few values to see how it actually is affected.

<table>
<thead>
<tr>
<th>Order quantity</th>
<th>VAR(ELTD) (L \sim \text{Gamma}(3), D \sim \text{Poisson}(10))</th>
<th>VAR(ELTD) (L \sim \text{Gamma}(10), D \sim \text{Poisson}(10))</th>
<th>VAR(ELTD) (L \sim \text{Gamma}(10), D \sim \text{Poisson}(5))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15,11</td>
<td>27,53</td>
<td>12,74</td>
</tr>
<tr>
<td>2</td>
<td>21,44</td>
<td>45,57</td>
<td>20,61</td>
</tr>
<tr>
<td>5</td>
<td>49,98</td>
<td>96,49</td>
<td>47,56</td>
</tr>
<tr>
<td>10</td>
<td>85,93</td>
<td>189,35</td>
<td>82,77</td>
</tr>
<tr>
<td>15</td>
<td>121,38</td>
<td>268,51</td>
<td>118,81</td>
</tr>
<tr>
<td>20</td>
<td>164,03</td>
<td>314,81</td>
<td>141,85</td>
</tr>
<tr>
<td>100</td>
<td>312,72</td>
<td>958,63</td>
<td>255,25</td>
</tr>
<tr>
<td>(VAR(LTD))</td>
<td>330</td>
<td>1100</td>
<td>300</td>
</tr>
</tbody>
</table>

To use this information, the \(p_j\) value should be updated each time after the order quantity is altered. This is dangerous to do however, since changing the variance of the distribution of the effective demand during the lead time influences the costs and might alter the convexity of \(C(s, Q)\). For example the expected costs which are used to approximate the best order level at an order quantity of 1 is not similar to the expected costs of order quantities which are higher.

### 4.4.2. Improved Algorithm

When the used algorithm is examined, it shows that the costs for the increment of the order quantity \((G(R)\) or \(G(s)\)) are added to the current costs, \(S\), to calculate the optimal costs, \(C^*\). However, when the variance of the effective lead time demand is updated, these costs continuously change. For example the costs of \(G(35)\) (with \(L \sim \text{Gamma}(3)\) and \(D \sim \text{Poisson}(10)\)) for \(Q = 1\) are €2,36 whereas for \(Q = 2\) they are €4,05. Adding the incremental costs therefore results in incorrect total costs. The total costs have to be calculated after each increase in order quantity. In updating these costs the optimal cost for that order size is reached when the \(Q\) most optimal values for \(y\) are added. These are consecutive since the cost function is convex.

Incorporating these changes in the algorithm of Federgruen and Zheng (1991), results in an adjusted algorithm;

**Step 0. Calculate** \(G(0)\) and \(\Delta G(0)\); \(L \leftarrow 0;\)

**while** \(\Delta G(L) < 0\) **do**

**begin** \(L \leftarrow L + 1;\) **evaluate** \(\Delta G(L), G(L + 1) \leftarrow G(L) + \Delta G(L)\)

**end**

\(S := \kappa + G(L), Q := 1, C^* := S, s := L - 1,\)

\(R := L + 1, y := 0;\)

**while** \(y < R\) **do**

36
begin y = 0 evaluate $\Delta G(y), G(y + 1) := G(y) + \Delta G(y)$
end

Step 1. Repeat
begin if $G(s) \leq G(R)$
then if $C^* \leq G(s)$
then stop.
else begin $S := k + \sum_{y=s+1}^{R-1} G(y)$, $s:=s-1$,
end;
else if $C^* \leq G(R)$
then stop.
else begin $S := k + \sum_{y=s+1}^{R-1} G(y)$, $R = R + 1$,
end;
end
while $y < R$
do
begin evaluate $\Delta G(y), G(y + 1) := G(y) + \Delta G(y)$
end
end.

The next section analyzes the two cases where the assumption is made that orders can or cannot cross. It provides results for the traditional algorithm of Federgruen & Zheng (1991), using the lead time distribution and the effective lead time distribution. Section 4.5. also shows the optimal values and costs for the adjusted algorithm and the shortfall distribution presented in Section 4.3.

4.5. ANALYSIS

This section compares the costs retrieved using the optimal parameters assuming order crossover versus the costs when assumed orders can not cross. The lead time demand is used in the algorithm of Federgruen & Zheng (1991) for the case of no order crossovers. If order crossovers are assumed four cases are checked; the lead time demand distribution, the shortfall distribution, the effective lead time demand in the original algorithm and in the adjusted algorithm. First the simulation is explained in the next section, after which Section 4.5.2. discusses the optimal parameters. More important than the optimal parameters, are the involved costs, Section 4.5.3. Section 4.5.4. discusses the sensitivity of the results on the involved parameters.

4.5.1. SIMULATION

This section briefly describes the simulation for the $(s, Q)$ inventory policy. The demand is still Poisson distributed, comparable to the periodic review policies. However, the timing of the demand is important in the $(s, Q)$ inventory policy. The timing of demand is simulated by the interarrival times. If the demand is Poisson(4) distributed, the time between arrivals is, on average 0.25. This interarrival time is simulated as an exponential distribution. As mentioned in Section 1.5.1. the lead time is now Gamma distributed.
The starting inventory is set equal to the reorder level. This makes sure that no multiples of the order quantity are ordered. The number of periods simulated is at least 100,000. After these 100,000 periods the impact of an additional period is examined. If this additional period influences total costs this period is added. Simulation only stops if average costs per period are stable.

From a single run some rows are represented in Appendix C, Table 25, this provides a clearer picture of the used simulation.

4.5.2. Optimal Parameters

The mean and variance are traditionally approximated by making use of the lead time demand distribution; \( \mu_{LTD} = \mu_D \ast \mu_L \) and \( \sigma_{LTD}^2 = \mu_L \ast \sigma_D^2 + \mu_L^2 \ast \sigma_L^2 \). Implementing this for an example with parameters as in Table 26 in Appendix C, with demand Poisson \( \sim 10 \) and lead time Gamma \( \sim 3 \), where \( \mu_{LTD} = 30 \sigma_{LTD}^2 = 330 \). The iterative algorithm of Federgruen and Zheng (1991) concludes that the optimal value of \( Q \) is 19 and the optimal reorder value of \( s \) is 45.

Incorporating the dependent effective lead time demand variance in the traditional algorithm of Federgruen & Zheng (1991), leads to a new order quantity and reorder level of respectively \( Q = 12 \) and \( s = 35 \) (\( d \sim \text{Poisson}(10), l \sim \text{Gamma}(3) \)).

The lower reorder level is explained by the fact that the variance of the effective lead time demand at lower order quantities is much smaller due to order crossovers occurring more often at lower order quantities. At a lower reorder level the certainty of having no stock-out is comparable to the higher reorder level at the same order quantity when the traditional lead time demand is used. The difference in order quantity is explained by the fact that the expected costs at the lower order quantities is significantly lower in the situation of the effective lead time demand (with changing variability) compared to the traditional lead time demand distribution. These lower costs result in a steeper slope downhill for the average costs per product. This steep slope has as a consequence that the tipping point of a beneficial increase of the order size is reached sooner.

The dependent effective lead time demand variance in the improved algorithm provides values of \( Q = 16 \) and \( s = 38 \). To compare the retrieved approximation of the shortfall distribution the acquired service level found by the algorithm of Federgruen & Zheng (1991) is implemented as a service level constraint. This makes it possible to compare the two models based on the same service level constraint. The shortfall distribution, where the required service level and order quantity is equal to the original algorithm with the lead time demand distribution has \( s = 43 \). This is summarized in Table 12. Table 13 provides a second example of these four methods to calculate the reorder level and corresponding order quantity.

<table>
<thead>
<tr>
<th>Table 12 Example adjusted algorithm ((s, Q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand<del>Poisson(10) Lead time</del>Gamma(3)</td>
</tr>
<tr>
<td>Reorder level</td>
</tr>
<tr>
<td>Order quantity</td>
</tr>
</tbody>
</table>
Table 13 Example 2 adjusted algorithm (s, Q)

<table>
<thead>
<tr>
<th>Demand~Poisson(10)</th>
<th>Lead time~Gamma(10)</th>
<th>Lead time demand</th>
<th>Shortfall distribution</th>
<th>Effective lead time demand</th>
<th>Effective lead time demand adj. algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reorder level</td>
<td>133</td>
<td>126</td>
<td>106</td>
<td>114</td>
<td></td>
</tr>
<tr>
<td>Order quantity</td>
<td>23</td>
<td>23</td>
<td>9</td>
<td>19</td>
<td></td>
</tr>
</tbody>
</table>

The adjusted algorithm however requires significantly more time to reach the optimal solution, due to the constant updating of the effective lead time demand variance. A possible way to reduce the computation time is to use the order quantity retrieved using the lead time demand. This found order quantity has an effective lead time demand variance. This variance should be used to calculate the reorder level which is optimal for that specific order quantity. The computational time significantly decreases. Section 4.5.3. shows results for this 5th method as well.

4.5.3. Results

The retrieved values of s and Q are not per se valuable in practice, but the actual cost difference in a situation where orders cross is. Therefore these values are used in the simulation described in Section 4.5.1. and the costs and service level are compared. Table 14 & 15 presents these values.

Table 14 Example different optimization methods

<table>
<thead>
<tr>
<th>Demand~Poisson(10)</th>
<th>Optimal parameters (s, Q)</th>
<th>Service level</th>
<th>Costs</th>
<th>Real savings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lead time demand</td>
<td>(45, 19)</td>
<td>97,19%</td>
<td>€54,40</td>
<td>-</td>
</tr>
<tr>
<td>Shortfall distribution</td>
<td>(43, 19)</td>
<td>95,57%</td>
<td>€54,35</td>
<td>€0,05 (0,00%)</td>
</tr>
<tr>
<td>Effective lead time demand</td>
<td>(35, 12)</td>
<td>87,26%</td>
<td>€62,35</td>
<td>-€7,95 (-14,61%)</td>
</tr>
<tr>
<td>Effective lead time demand adj. algorithm</td>
<td>(38,16)</td>
<td>92,28%</td>
<td>€54,03</td>
<td>€0,37 (0,68%)</td>
</tr>
<tr>
<td>Adj. algorithm, reduced computational time</td>
<td>(38, 19)</td>
<td>91,23%</td>
<td>€52,79</td>
<td>€1,61 (2,96%)</td>
</tr>
</tbody>
</table>

Table 14 shows that the shortfall distribution leads to real savings of only €0,05. However, if the effective lead time demand distribution is used in the existing algorithm of Federgruen & Zheng (1991) the total costs increase. The lower order quantity results in more order moments, therefore a larger fixed ordering cost is present. Section 4.4.1. already expected this, since each order quantity has got a different effective lead time demand variance. This change is incorporated in the adjusted algorithm. However, Table 14 shows that this improved algorithm provides no significant real savings. The absence of significant real savings is explained by the relative small lead time. If orders crossed, the time until the crossed order arrives is relatively small, decreasing the impact of the crossed order. If the method is chosen to first find the optimal order quantity, after which the optimal reorder level is determined the final row is retrieved. It is seen that the real cost savings are largest in this case. This is counterintuitive, because it is hypothesized that the benefits due to order crossovers are larger if the probability of these crossovers is larger. However, the part of the fixed ordering costs is large in this situation and the probability of order crossovers low. Setting the reorder level based on the optimal order quantity found earlier therefore yields appropriate results in this case.
Table 15 Example 2 different optimization methods

<table>
<thead>
<tr>
<th>Demand~Poisson(10)</th>
<th>Optimal parameters (s, Q)</th>
<th>Service level</th>
<th>Costs</th>
<th>Real savings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lead time demand</td>
<td>(133, 23)</td>
<td>98,84%</td>
<td>€70,30</td>
<td>-</td>
</tr>
<tr>
<td>Shortfall distribution</td>
<td>(126, 23)</td>
<td>97,36%</td>
<td>€66,33</td>
<td>€3,67 (5,22%)</td>
</tr>
<tr>
<td>Effective lead time demand</td>
<td>(106,9)</td>
<td>85,23%</td>
<td>€80,82</td>
<td>-€10,02 (-14,25%)</td>
</tr>
<tr>
<td>Effective lead time demand adj. algorithm</td>
<td>(114, 19)</td>
<td>92,11%</td>
<td>€59,78</td>
<td>€10,52 (14,96%)</td>
</tr>
<tr>
<td>Adj. algorithm, reduced computational time</td>
<td>(115, 23)</td>
<td>92,56%</td>
<td>€60,78</td>
<td>€9,52 (13,54%)</td>
</tr>
</tbody>
</table>

Table 15 also presents the real savings present, but in this setting the lead time distribution has a mean and variance of 10. It is seen that the shortfall distribution provides 5,22% real savings. The effective lead time demand in the existing algorithm still has a worse result compared to the algorithm using the lead time demand. However, if we examine the impact of the adjusted algorithm a significant real saving is present (14,96%). The adjusted algorithm where the order quantity is predetermined based on the $ltd$ distribution, still provides real savings. The real savings however now no longer outperform the effective lead time demand.

Appendix C, Table 27, presents four other examples where the results are similar; the reorder level is lowered and the order quantity is also lowered or remains the same, comparing the algorithm with the adjusted algorithm. Similar results are obtained in these four cases.

4.5.4. Sensitivity

Any alteration regarding the used parameters results in different optimal settings of the order quantity and reorder level. Table 16 corresponds to situations where only one of the involved parameters is altered, presented in the first column. The second and third column present the optimal settings using the original algorithm or the adjusted algorithm respectively. The last column presents the real savings. The optimal parameter values are retrieved if the parameters are optimized using the adjusted algorithm using the effective lead time demand variance, assuming order crossovers, represented by $s^*$ and $Q^*$. Assuming no order crossovers the optimal parameter values are retrieved using the algorithm of Federgruen & Zheng (1991), represented by $s^{**}$ and $Q^{**}$. Both these values are implemented in the case where orders can cross to find the corresponding costs and real savings.

Table 16 Sensitivity (s, Q) inventory policy

<table>
<thead>
<tr>
<th>Alteration setting (L<del>Gamma(10), D</del>Poisson(10))</th>
<th>Assuming order crossovers (s*, Q*)</th>
<th>Costs</th>
<th>Service level</th>
<th>Assuming no order crossovers (s**, Q**)</th>
<th>Costs</th>
<th>Service level</th>
</tr>
</thead>
<tbody>
<tr>
<td>h = €0,50</td>
<td>(121, 23)</td>
<td>€45,27</td>
<td>95,35%</td>
<td>(142, 27)</td>
<td>€49,65</td>
<td>99,49%</td>
</tr>
<tr>
<td>h = €2,-</td>
<td>(108, 15)</td>
<td>€85,96</td>
<td>85,19%</td>
<td>(121, 19)</td>
<td>€98,21</td>
<td>96,49%</td>
</tr>
<tr>
<td>p = €4,-</td>
<td>(106, 21)</td>
<td>€52,53</td>
<td>83,94%</td>
<td>(116, 25)</td>
<td>€54,99</td>
<td>92,34%</td>
</tr>
<tr>
<td>p = €12,-</td>
<td>(116, 18)</td>
<td>€64,10</td>
<td>94,95%</td>
<td>(137, 22)</td>
<td>€76,16</td>
<td>99,16%</td>
</tr>
</tbody>
</table>
Table 16 shows that the real savings fluctuate, depending on the involved variables. Significant savings can be present using the adjusted algorithm rather than the traditional algorithm of Federgruen & Zheng (1991). The variables are discussed one by one in order to discuss their impact:

- Per unit holding costs; Increasing the holding cost, increases the real savings. The reorder level is lower when order crossovers are assumed, causing lower inventory levels. For example in Table 16 the maximum inventory position assuming order crossovers is 2 higher than the minimum inventory position assuming no order crossovers. If holding inventory becomes more expensive, the savings regarding inventory holding costs significantly increase.

- Per unit backordering costs; An increase in backordering costs decreases the order quantity. Lower order quantities have a larger probability of order crossovers, as Section 4.2. showed. Due to order crossovers occurring more often, the real savings increase.

- Fixed ordering costs; Increase in fixed ordering costs results in an order quantity which increases. Larger order quantities, with the same demand distribution, lead to a chance of order crossovers which is lower (Hayya et al. 2011). Due to the lower probability of order crossovers, the real cost savings also decrease.

- Demand; Higher demand rates lead to order quantities which are larger, but the reorder level is reached more often. As Hayya et al. (2011) showed, a larger ordering periodicity leads to a probability of order crossovers which is increased. Due to order crossovers occurring more often, the real savings also significantly increase.

- Mean lead time; An alteration in the mean lead time does not influence the probability of order crossovers or the total involved costs (He et al. 2005).

- Variance lead time; If the lead time variance decreases the probability of order crossovers drops significantly as well. This decrease of order crossovers causes the real savings to decrease if the lead time variance decreases.
4.6. CONCLUSION

In this chapter we started with the applicability of the (s, Q) policy, which is often used to benefit from the fixed order quantity but remain in control over the inventory position. Furthermore we discussed the probability of order crossovers and saw that the order quantity is an important contributing factor. Finally we introduced a method which can determine the optimal s + Q value where the order quantity is already determined. We compared this novel approximation with the algorithm of Federgruen & Zheng (1991), and adjusted this algorithm to incorporate order crossovers accordingly.

Comparable to the previous two chapters, the parameters have similar influences on the probability of order crossovers and thereby the impact of the assumption on order crossovers. The relationships and their directional influence are provided in Figure 13.

![Figure 11 Concluding figure (s, Q)](image)

This chapter enables us to answer the third research question; What is the effect of order crossovers on the (near-) optimal parameter settings in the (s, Q) policy?

Comparing the algorithm of Federgruen & Zheng (1991) with the adjusted algorithm it is shown that the reorder level can be lowered due to orders crossing. The order quantity can also be slightly different compared to the quantity retrieved by the original algorithm. With a lower order quantity the probability of order crossovers is increased. The improved algorithm requires a significant longer computational time, by taking the order quantity of the traditional algorithm and retrieving the optimal reorder level for this quantity proves to be a small improvement as well. The adjusted algorithm does not always provide improved costs, this depends on the involved parameters. If order crossovers do not happen often, the adjusted algorithm provides similar results as the original algorithm. Therefore the conclusion can be drawn that the appropriateness of the assumption on order crossovers depends on the involved parameters.

The next chapter no longer orders in fixed quantities, but a variable order size is considered.
5. Continuous Review, Variable Order Quantity

The previous chapter examined the impact of order crossovers in a continuous review inventory policy with a fixed order quantity. This chapter relaxes the assumption of fixed order quantity. The \((s, S)\) inventory policy is an inventory policy which places an order if the inventory position falls to or below \(s\). The order has a size such that the inventory position reaches \(S\) after an order is placed. This is a more flexible policy than the \((s, Q)\) inventory policy since the order quantity can differ at each order moment. However, the benefits of the economical order quantity are not present in this policy.

The rest of this chapter is organized as follows, Section 5.1. discusses the applicability of this policy and the differences with the \((s, Q)\) inventory policy. Afterwards Section 5.2. determines the probability of order crossovers. The next sections investigate and discuss the impact of alterations in the lead time mean (Section 5.3.) and variance (Section 5.4.) potentially caused by order crossovers. Section 5.5. concludes this chapter with a brief summary.

5.1. Policy Applicability

The difference between an \((s, S)\) inventory policy and a \((s, Q)\) inventory policy is primarily that demand occurs in batches in the \((s, S)\) inventory policy. A common assumption in the \((s, Q)\) inventory policy is that demand occurs with one unit at a time. Due to the relaxation of this assumption it is possible for the inventory position to fall below \(s\), whereas if only single demand occurs this is impossible. The inventory position is at its lowest \(s\) in an \((s, Q)\) inventory policy because orders are placed immediately after the reorder level is reached.

The effect of lead time uncertainty is studied by (Song J. S., 1994), where demand occurs following a compound Poisson process. This process means that "the demand epochs occur according to a Poisson process, and at each such epoch there is a demand for a batch of random size". The replenishment orders of the system follow a stochastic lead time. Two assumptions are made here, first of all lead times are exogenous, meaning that the order quantity has no influence of the order lead time. This assumption can be justified by the fact that, for example, the supplier’s operation is relatively large compared to the order size. A relative small order means that the order size contributes little to the supplier’s total workload, determining the lead times of orders. The second assumption which is made is sequential deliveries. Sequential deliveries means that orders are always being processed and arriving in the same order. This assumption is justified by thinking that the supplier’s production and distribution system is operating sequentially on a single line, thus orders can never cross. However, by assuming this the order lead times are no longer independent. This assumption is relaxed to see the effects of order crossovers. Relaxing this assumption is justified by the fact that the supplier can operate on multiple lines or different transportation modes can be, randomly, assigned to the orders. Several more reasons for order crossovers are previously described in Section 1.2.
Song (1994) assesses the lead time effects by comparing two models where only the specifications of the lead time parameters are different. Representing two potential companies operating in the same industry, identical in structure and function, but using different suppliers. They show that the optimal base stock level of system 1 is higher when the lead time in system 1 is stochastically larger. Furthermore they show that a more variable lead time always results in higher optimal average costs.

5.2. Probability of Order Crossover

As mentioned previously, orders are placed in a (s, S) inventory policy if the inventory position falls to or below s. Order crossovers are defined as “...when replenishment orders arrive in a sequence that us different than the one in which they were placed.” In the (s, S) inventory policy there are three types of uncertainties to take into account.

First of all the timestamps on which a demand occurs, or the demand epoch rate as mentioned by Song (1994). Secondly, when demand occurs the size of the order is variable as well which has a probability mass function. This probability mass function is defined as φ(Z) (batch-size random variable, discrete in value), Zt belonging to the demand at time t. Combining these two results in the quantity and timing of the orders results in how often the reorder level is reached. As described by Hayya et al. (2011) the ordering periodicity is of importance for the probability of order crossovers. Besides the ordering periodicity also the combination of the lead time and its variance are required for orders to cross (e.g. the maximum lead time should at least be larger than the minimum time between the orders). The lead time of order i is defined as Li. Combining this information provides a condition for orders to cross when the lead time demand plus the lead time of an order k is smaller than the lead time of order k - 1. If we define Ok as the order moment of order k we can provide the following formula for order crossover.

\[ L_{k-1} \geq L_{k+\frac{S-s}{\mu_D}} \text{ if } \text{Prob} \left[ \sum_{t=0}^{O_k-O_{k-1}} D_t Z_t \geq (S-s) \right] \]

If the probability that the demand epochs and demand batches are at least the difference between S and s, an order will be placed. This will on average take \( \frac{S-s}{\mu_D} \).

However, this only represents the probability of order crossover with the previous order. If we want to calculate the chance of an order crossing any previously placed order the formula is altered in the following way;

\[ 1 - \prod_{k=1}^{\infty} L_{k+\frac{S-s}{\mu_D}} \text{ if } \text{Prob} \left[ \sum_{t=0}^{O_k-O_0} D_t Z_t \geq k(S-s) \right] \]
The chance of no order crossover is calculated with the following formula. The chance that the lead time of the \( k \)th order plus the time it takes to place this \( k \)th order, represents probability these orders cross.

\[
F_{t+k \left( \frac{s-s}{\mu_D} \right)}
\]

The time between orders now depends on how often the inventory position will fall to or below \( s \). This formula overestimates the probability of order crossover, since it is possible for the inventory position to fall far below the reorder level \( s \) and thereby a relatively large order is placed. However, this formula looks to the average demand (epochs and batch size). The part which is considered to fall below \( s \) is considered, in the average case, to be part of the next time when the reorder level could be reached. The accuracy of the formula therefore will deteriorate if the variability in order size if relatively large compared to the difference between \( S \) and \( s \).

The formula presented above only provides the chance of crossovers for 2 orders. If we want to find the chance that it did not cross any order we have to take the artesians probability; \( \prod_{k=1}^{\infty} F_{t+k \left( \frac{s-s}{\mu_D} \right)} \). If we continue to expand this to the chance of a crossover we retrieve the following formula;

\[
1 - \sum_{l=0}^{\infty} f_l \prod_{k=1}^{\infty} F_{t+k \left( \frac{s-s}{\mu_D} \right)}
\]

The formula above takes the artesian product of the chances of no order crossovers. This artesian product is multiplied by the chances of the incorporated lead time occurring. This is subtracted from 1 to retrieve the probability of order crossover.

Order crossovers occur more often if the difference between the order up to level and the reorder level are closer together. This is explained by the fact that the order periodicity increases if this difference is lower, Hayya et al. (2011) showed that ordering periodicity influences the occurrence of order crossovers.

### 5.3. Effect of Altered Mean Lead Time

This subsection studies the effects of uncertainties regarding the mean of the lead time, with an unaltered variance of the lead time. The result is intuitive, a larger lead time results in a larger lead time demand, but a larger lead time does not result in a different probability of order crossover if the variance of the lead time remains unaltered. A larger lead time demand requires a higher base-stock level to compensate the higher possibility of stock out. As Section 4.5. shows, order crossovers lead to situations in which the base stock can be lowered compared to a situation if order crossovers are not incorporated. The effect of a higher base stock level on the average cost behavior are more elaborate. A larger lead time does not necessarily have to result in a higher average cost, since it is variability rather than the mean lead time which influence the operating costs (Lee & Tang, 1998), (He et al., 2005).
5.4. **Effect of Altered Variance Lead Time**

In order to study these effects of lead time variability on the optimal policy, Song (1994) used two lead time distributions with the same mean but with a different density spread. Song (1994) states that an immediate, and important, fact of a noisier lead time demand always leads to higher costs, for any fixed base stock policy. Thereby the optimal cost are also always higher. This higher cost is derived from the fact that the lead time demand is more variable if the lead time variance is increased. Order crossovers lead to situations in which the variability of the lead time decreases but the mean remains the same (Zalkind, 1978). The effective lead time variance differs for each ordering periodicity (Hayya et al., 2011), the periodicity is influenced by the difference between the reorder level and order up to level. In the algorithm of Zheng & Federgruen (1991) this effective lead time variance needs to be updated each time after different settings are chosen for \( s \) and \( S \).

5.5. **Conclusion**

In this chapter first the applicability of order crossovers was examined. This policy includes the assumption that demand can be of different sizes. Variability on the size of the demand, or at the rate that demand occurs influence the probability of order crossovers by the fact that order moments can be closer or further apart. Combined with the lead time variability the probability of order crossovers was calculated in section 5.2. The previous section discussed the effects of the mean and variance of the lead time on this policy are discussed. Due to orders crossing the effective lead time variance is decreased and thereby lower costs can be obtained. The effects of the parameters are shown in Figure 14 below.

![Flowchart](image.png)

**Figure 12 Concluding figure (s, S)**

It is now possible to answer the fourth, and final, sub question; What is the effect of order crossovers on the (near-) optimal parameter settings in the (s, S) policy?

The (s, S) policy includes three important types of uncertainty; lead time, demand epoch and demand size. Due to these differences in possible demand sizes the order sizes are different as well, reducing the impact of order crossovers. A crossed order can either be smaller and therefore still backorder costs or incurred, or larger which shifts the holding costs upwards.
6. CONCLUSION

This final chapter concludes the main findings of this master thesis project which studied the effects of order crossovers on the optimal inventory setting. The introduction explained inventory management as a global concept, after which the idea of order crossovers was introduced. It was already shown that the variance of the lead time is reduced due to order crossovers, this is referred to as the effective lead time. The influence of this effective lead time on optimal inventory policy settings are observed as a gap in the literature, which this thesis tries to, partially, fill.

First of all the (R, Q) policy was investigated. A common assumption for this policy is constant demand, else the inventory position is very unstable. The review period is observed as the biggest impact on the probability of order crossovers. For small review periods it is more important to use the appropriate effective lead time distribution, since order crossovers are more likely to occur. Due to this lowered effective lead time variance the inventory position of this policy at its start can be lower and still reach the required service level. Reaching this required service level with a lower inventory leads to lower total costs.

If the fixed order quantity restriction is released, the (R, S) inventory policy is observed. This inventory policy can handle variable demand since the inventory position reaches a predetermined level at order moments. The probability of 0 demand influences the probability of order crossovers slightly as well. The variable demand can be incorporated in the shortfall distribution to approximate the optimal order up to level to reach the required inventory position. Similar to the (R, Q) policy order crossovers lead to a lower variance and thereby lower order up to levels and/or review periods. This leads to a situation where also lower costs can be assured.

The effects of order crossovers in continuous review inventory policies are examined in Chapter 4 and 5. It is seen that the order quantity influences the probability of order crossovers in the (s, Q) inventory policy. In the (s, S) inventory policy the difference between the reorder level and the order up to level primarily influences to probability of order crossovers. It is seen that the reorder level in the (s, Q) policy is lowered, due to the lower effective lead time demand variance. The order quantity will most likely decrease as well, however some cases might lead to similar order quantities being optimal. If the demand sizes are variable as well the effects between the two models are moderated, due to the order quantities no longer being the same size but also variable.

Based on this master thesis a set of recommendations is made, both for practice and academics. First the applications in practice are discussed, after which the future research areas for academia are explored.
6.1. **Recommendations**

Chapter 2, 3, 4 & 5 show that the optimal parameter settings for four basic inventory policies should be altered to incorporate order crossovers, and are compared with the optimal settings in situations which do not incorporate order crossovers. It is shown that implementing these retrieved parameter values retrieve the optimal costs, and possible significant real savings, as close as possible to the required service level. Therefore the first recommendation which is made for organizations is that the assumption whether orders can cross needs to be verified.

The reduced costs due to order crossovers comes at a cost of increased computation time. If the complexity of the inventory policy increases the time to find the optimal parameters also increases. Therefore it is recommended to first calculate the probability of order crossovers. Even though Bradley & Robinson (2005) show that effects of crossovers can be large if its probability is small, the effects are still moderated if the probability hereof is relatively small.

Finally, the benefit of crossovers is that the variance of the effective lead time demand is reduced. Variance is primarily influencing the inventory costs, rather than its mean (Lee & Tang, 1998 & He et al., 2001). Therefore companies should find other ways of reducing the variance of the lead time demand next to the possibility of order crossovers.

6.2. **Future Research**

This master thesis project investigated the area of order crossovers in more detail than existing literature, but several gaps remain. The following subsections will provide some of these gaps and briefly discuss the potential impact order crossovers can have.

6.2.1. **Uncertainty**

Yield is another type of uncertainty which can influence the optimal parameters in inventory policies. If part of an order is of unacceptable quality or if the supplier is sometimes unable to deliver products this should be included, this is called yield. Order crossovers could have an influence on the optimal parameter settings where yield is included.

The demand uncertainty faced by companies can be influenced by seasonality or include a week pattern. However, the described models in Chapter 2-5 either assume constant demand or demand which is identically distributed over time (either in singles or batches). Seasonality potentially has a significant impact on the influence on crossovers, we demonstrate this with an example. We place a small order on a Tuesday and a large order on a Friday and expect the order to arrive on Tuesday respectively Friday, with a peak demand on Friday. If the order placed on a Friday arrives at Tuesday already, and the Tuesday order only arrives on the Friday then the difference in size between the two orders is stored and incurs additional holding costs.

The lead time of orders might be influenced by the lead time of the previously placed orders, this is called autocorrelation. It is shown that positive autocorrelation reduces the probability of order crossovers (Bischak et al., 2014). However, this is only shown in a periodic review, order up to
policy. It can be hypothesized that autocorrelation has a similar impact on other inventory policies. The possibility of autocorrelation being negative, for instance lead times of the 2nd order becomes smaller if the lead time of the 1st order is observed to be large, has not yet been accounted for in literature as far as we know.

6.2.2. Emergency Orders

After an emergency order is placed the lead time of this order is known and can be implemented in the decision which is made on how much and when to order. For instance if the past three orders have a lead time which is significantly larger than the expected lead time it can be expected that the predetermined service level is not be reached in these instances due to the long lead times. Emergency orders are ordered after a regular order is made (Chiang & Gutierrez, 1998) and arrive before the regular order arrives (Tagaras & Vlachos, 2001). Emergency orders are classified as expected order crossovers (Riezebos, 2006). Combining emergency orders in inventory policies which allow for random crossovers could decrease the effective lead time demand variance even more. The combination of these two types of order crossovers is at the moment of writing inexistent to our knowledge.

6.2.3. Outdating

In retail there are numerous products which have a limited shelf life, for instance apples, bread and sushi. If the expiration date is assumed to have a time which is a constant factor higher than the order moment (Expiration date = Order moment + Constant) it might be required for the retail store to change their replenishment policy. In traditional replenishment policies often first-in-first-out is assumed, but if orders can cross the order which arrives first has a expiration date which is later than the order which arrived second. The benefits of order crossovers might be reduced in this case, due to the fact that if a FIFO policy is followed the products in the order which is crossed could expire and therefore additional outdating costs needs to be included in the cost estimation.

6.2.4. Assembly Systems

Every component must be on site before assembly can start (Louly et al., 2008), therefore the longest variance of a component matters. Louly et al. (2008) provide two, nearly similar, possible options for firms to take this uncertainty into account; either safety stock or incorporate safety time in the component lead time. Safety time is subjective however. The uncertainty of the lead time of all components is shown to be a large influence on costs, inventory costs being a major contributor (Gupta & Brennan, 1995). Due to order crossing the problem of multi-period inventory control is becoming more complex. The possibilities of crossovers at each of the components make this problem highly relevant in the assembly systems. Due to the dependence on multiple components, the effect of order crossovers can be hypothesized to be less beneficial; if orders have crossed at 3 out of the four components and reduced the lead time, but the fourth component is not ready only additional inventory costs are incurred.
6.2.5. Additional Waiting Time

Instead of orders crossing, it can be assumed that the orders are not allowed to cross but additional waiting time is incurred for the order which otherwise would have crossed. An example is presented in Table 17, which recalculates the effective order lead time. This could for example be the case if two trains have to deliver the product to a customer, but the second train can not overtake the first and therefore has the same arrival time as the first customer. This can also be done on purpose, because the first order incurs higher penalty costs while waiting. This assumption represents situations which can occur in practice and is therefore worthwhile to investigate. Even though order crossovers are still ignored in this situation, assuming that additional waiting time is incurred could resemble similar properties. The variance of the effective order lead time (in the example 1,167) might still be lower than the variance of the order lead time (3,6). However, due to the required waiting time the mean has shifted upwards (from 3,4 to 4,5). As shown by, He et al. (2005) and Lee & Tang (1998), it is primarily the variance rather than the mean which influences inventory costs.

Table 17 Additional Waiting Time

<table>
<thead>
<tr>
<th>Order moment</th>
<th>Order lead time (Uniform 1-6)</th>
<th>Order arrival time</th>
<th>Effective order lead time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>12</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>14</td>
<td>6</td>
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<td>9</td>
<td>4</td>
<td>14</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>14</td>
<td>4</td>
</tr>
</tbody>
</table>
BIBLIOGRAPHY


APPENDICES

A. CHAPTER 2

Table 18 Probability Order Crossovers, 2-Point distribution

<table>
<thead>
<tr>
<th>Review period</th>
<th>( f_1 = \frac{2}{3}, f_4 = \frac{1}{3} )</th>
<th>( f_2 = \frac{1}{2}, f_6 = \frac{1}{2} )</th>
<th>( f_4 = \frac{1}{3}, f_{10} = \frac{2}{3} )</th>
<th>( f_6 = \frac{2}{3}, f_{12} = \frac{1}{3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3704</td>
<td>0.43</td>
<td>0.34</td>
<td>0.57</td>
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<tr>
<td>2</td>
<td>0.222</td>
<td>0.25</td>
<td>0.27</td>
<td>0.34</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.25</td>
<td>0.19</td>
<td>0.19</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0.19</td>
<td>0.19</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0.19</td>
<td>0.19</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 13 Probability Order Crossover, mean and variance = 2

![Probability Order Crossover](image-url)
Figure 14 Probability Order Crossover, mean and variance = 4

Table 19 Settings simulation Table 6 and 7

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand</td>
<td>10</td>
</tr>
<tr>
<td>Fixed ordering costs</td>
<td>€50,-</td>
</tr>
<tr>
<td>Per unit holding costs</td>
<td>€1,-</td>
</tr>
<tr>
<td>Per unit backorder costs</td>
<td>€9,-</td>
</tr>
<tr>
<td>Required service level</td>
<td>90%</td>
</tr>
</tbody>
</table>
B. Chapter 3

Table 20 Probability Order Crossovers, 2-Point distribution, Order if Demand > 0

<table>
<thead>
<tr>
<th>Review period</th>
<th>( f_1 = \frac{2}{3}, f_4 = \frac{1}{3} )</th>
<th>( f_2 = \frac{1}{2}, f_6 = \frac{1}{2} )</th>
<th>( f_4 = \frac{1}{3}, f_{10} = \frac{2}{3} )</th>
<th>( f_6 = \frac{2}{3}, f_{12} = \frac{1}{3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.329</td>
<td>0.409</td>
<td>0.31</td>
<td>0.54</td>
</tr>
<tr>
<td>2</td>
<td>0.218</td>
<td>0.245</td>
<td>0.25</td>
<td>0.32</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.245</td>
<td>0.19</td>
<td>0.19</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0.19</td>
<td>0.19</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0.19</td>
<td>0.19</td>
</tr>
</tbody>
</table>

Figure 16 Outstanding orders, lead time Poisson(3), Review period 1

Figure 17 Outstanding orders, lead time Poisson(3), Review period 3
Figure 18 P.M.F. Shortfall Demand ~Poisson(10), Lead time ~Poisson(3), Review Period 1

Figure 19 C.D.F. Shortfall Demand ~Poisson(10), Lead time ~Poisson(3), Review Period 1
Figure 20. PMF Shortfall Demand ~Poisson(10), Lead time ~Poisson(3), Review Period 2

Figure 21. CDF Shortfall Demand ~Poisson(10), Lead time ~Poisson(3), Review Period 2
Figure 22 Order up to levels for different service levels

Figure 23 Costs belonging to (R, S) model
Table 21 Parameter settings Section 3.4.1.

<table>
<thead>
<tr>
<th>R</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>h</td>
<td>€ 15,65</td>
<td>€ 20,84</td>
<td>€ 25,00</td>
<td>€ 29,08</td>
</tr>
<tr>
<td>p</td>
<td>€ 5,89</td>
<td>€ 7,60</td>
<td>€ 8,98</td>
<td>€ 9,69</td>
</tr>
<tr>
<td>A</td>
<td>€ 25,00</td>
<td>€ 12,50</td>
<td>€ 8,33</td>
<td>€ 6,25</td>
</tr>
<tr>
<td>Total</td>
<td>€ 46,54</td>
<td>€ 40,94</td>
<td>€ 42,31</td>
<td>€ 45,02</td>
</tr>
<tr>
<td>S</td>
<td>45</td>
<td>55</td>
<td>64</td>
<td>73</td>
</tr>
</tbody>
</table>

The first column represents the period and the 2nd column the inventory after orders have arrived but before demand is fulfilled. The 3rd column shows the demand which is Poisson~(10) distributed, and the fourth column then shows the inventory after demand, this column is required for the determination of the holding and penalty costs. The final two columns show the lead time (Poisson~(3) distributed) and the period in which the order arrives.

Total periods simulated are 10,000 and a warm-up period 100 periods has been chosen in order for the system to reach a stable state at period 0. Due to this warm-up period the inventory after order arrivals are in period 2, for example, 35 since an order of size 10 was placed in period -1.

Table 22 Example simulation (R, S) inventory policy

<table>
<thead>
<tr>
<th>Period</th>
<th>Inventory after order arrivals</th>
<th>Demand</th>
<th>Inventory after demand</th>
<th>Order lead time</th>
<th>Order arrival time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>35</td>
<td>8</td>
<td>27</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>35</td>
<td>10</td>
<td>25</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>35</td>
<td>7</td>
<td>28</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>28</td>
<td>7</td>
<td>21</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>38</td>
<td>14</td>
<td>24</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>24</td>
<td>11</td>
<td>13</td>
<td>3</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 23 Example Lead time Poisson(2) Demand Poisson(2)

<table>
<thead>
<tr>
<th>L<del>Poisson(2), D</del>Poisson(2) Review period</th>
<th>Assuming Order Crossovers</th>
<th>Assuming no Order Crossovers</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>Total costs</td>
<td>S</td>
</tr>
<tr>
<td>------------------------</td>
<td>-----------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>€55,28</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>€31,51</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>€24,45</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
<td>€21,01</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>€19,38</td>
</tr>
<tr>
<td>6</td>
<td>16</td>
<td>€18,46</td>
</tr>
<tr>
<td>7</td>
<td>18</td>
<td>€18,10</td>
</tr>
</tbody>
</table>

Table 24 Parameter Settings Table 9 and 23

<table>
<thead>
<tr>
<th>Demand</th>
<th>Poisson(10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed ordering costs</td>
<td>€50,-</td>
</tr>
<tr>
<td>Per unit holding costs</td>
<td>€1,-</td>
</tr>
<tr>
<td>Per unit backorder costs</td>
<td>€9,-</td>
</tr>
<tr>
<td>Required service level</td>
<td>90%</td>
</tr>
</tbody>
</table>
Figure 24 Probability order crossover, Poisson distributed lead time

Figure 25 PDF Shortfall
In the example of Table 25, the demand is Poisson(10) distributed, lead time is Gamma(10) distributed. The reorder level is 114 and the order quantity is 19.

**Table 25 Example Simulation (s, Q) inventory policy**

<table>
<thead>
<tr>
<th>Period</th>
<th>Inventory after order arrivals</th>
<th>Demand</th>
<th>Inventory after demand</th>
<th>Inventory position</th>
<th>Order lead time</th>
<th>Order arrival time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>115</td>
<td>0</td>
<td>115</td>
<td>115</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.05</td>
<td>115</td>
<td>1</td>
<td>114</td>
<td>133</td>
<td>7,23</td>
<td>7,28</td>
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<tr>
<td>0.15</td>
<td>114</td>
<td>1</td>
<td>113</td>
<td>132</td>
<td>-</td>
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</tr>
<tr>
<td>...</td>
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<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>7.23</td>
<td>46</td>
<td>1</td>
<td>45</td>
<td>121</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7.28</td>
<td>64</td>
<td>0</td>
<td>64</td>
<td>121</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7.37</td>
<td>64</td>
<td>1</td>
<td>63</td>
<td>120</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7.44</td>
<td>63</td>
<td>1</td>
<td>63</td>
<td>119</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

**Table 26 Parameter settings Section 4.4.**

<table>
<thead>
<tr>
<th>Demand</th>
<th>Poisson(10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lead time</td>
<td>Gamma(4)</td>
</tr>
<tr>
<td>Ordering costs</td>
<td>€50,-</td>
</tr>
<tr>
<td>Holding costs</td>
<td>€1,-</td>
</tr>
<tr>
<td>Penalty costs</td>
<td>€9,-</td>
</tr>
</tbody>
</table>
Table 27 Comparison order crossovers vs. no order crossovers

<table>
<thead>
<tr>
<th>Demand, Poisson</th>
<th>Lead time, Gamma</th>
<th>Order crossovers</th>
<th>No order crossovers</th>
<th>Real savings</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Optimal (s, Q)</td>
<td>Costs, Service level</td>
<td>Optimal (s, Q)</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>(19, 14)</td>
<td>€25,99, 98,64%</td>
<td>(21, 15)</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>(55, 19)</td>
<td>€53,78, 95,41%</td>
<td>(58,20)</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>(116,19)</td>
<td>€58,14, 92,08%</td>
<td>(130,23)</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>(46,15)</td>
<td>€36,54, 99,91%</td>
<td>(51,17)</td>
</tr>
</tbody>
</table>
D. CHAPTER 5

Existing algorithm \((s, S)\) inventory policy.

In order to be able to compare the use of the effective lead time demand rather than the lead time demand an existing algorithm to find optimal \((s, S)\) policies and its values has been used. The algorithm is the following (Zheng & Federgruen, 1991)

\[
\begin{align*}
\text{Step 0.} & \quad s := y^*; \\
& \quad S_0 := y^*; \\
& \quad \text{Repeat } s := s - 1 \text{ until } c(s, S_0) \leq G(s); \\
& \quad s_0 := s; \ c^0 := c(s_0, S_0); \ S^0 := S_0; \ S := S^0 + 1; \\
\end{align*}
\]

\[
\begin{align*}
\text{Step 1. While } G(S) & \leq c^0 \text{ do} \\
& \quad \text{begin If } c(s, S) < c^0 \\
& \quad \quad \text{then begin } S^0 := S. \\
& \quad \quad \text{While } c(s, S^0) \leq G(s+1) \text{ do } s := s + 1; \\
& \quad \quad c^0 = c(s, S^0); \\
& \quad \quad \text{end; } \\
& \quad S := S + 1; \\
& \quad \text{end.}
\end{align*}
\]

The algorithm is relatively straightforward, in step 0 the algorithm is initialized by entering with an initial order up to level, which has \(y^*\) as an arbitrary minimum of the costs function. Then an optimal reorder level is found which corresponds to this order up to level, by making a loop over the reorder level, constantly decreasing it by 1 until the cost are equal or lower than the arbitrary minimum.

Step 1 then continues by searching the smallest increase of \(S\) which improves the total costs, after which a new optimal reorder level is determined. This new \(s\) is searched for by adjusting the former \(s\) value upwards with quantity of 1. This step repeats itself.