Stabilisation of iterated toric fibre products

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Stabilisation of Iterated Toric Fibre Products

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Abstract

In this thesis we will analyse how the toric fibre product of certain ideals behaves when repeatedly applied to a finite number of ideals. Specifically we will investigate bounds on the degrees of generating polynomials of these iterated products. Stabilisation of the toric fibre product is said to occur when there is a uniform bound on these degrees, independent of the number of iterations.

We show that stabilisation occurs when we apply the iterated toric fibre product to the vanishing ideals of the class of affine varieties that are closed under coordinate-wise multiplication. We prove this statement by analysing the inverse (or projective) limit of the varieties that arise from these ideals. This inverse limit can be interpreted as a subset of the variety of infinite-dimensional tensors of rank at most 1. The coordinate ring of this infinite-dimensional tensor space admits certain finiteness properties up to the action of the substitution monoid Subs(\(\mathbb{N}\)). This finiteness results in the existence of a uniform bound on the degree of generators of the iterated toric fibre products, hence in stabilisation.

Among the investigated class of varieties for which the product stabilises are toric varieties. These are varieties that arise as images of monomial maps, and occur frequently in applications. One such application in which toric varieties arise, and a motivating example for this research, is that of graphical models. These are statistical models defined over finite graphs. These graphs can be glued to obtain a bigger model. Following [RS16], we prove that this gluing corresponds to taking the toric fibre product. Hereby we obtain that iterating gluing of graphical models results in toric ideals that are generated in uniformly bounded degree.
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Chapter 1

Introduction

1.1 Motivation

In order to perform certain statistical tests, one needs to be able to draw samples from certain distributions. This cannot always be done efficiently, as already noted in [DS98]. There the notion of a Markov basis is introduced, which can be used to efficiently sample from such distributions using a Markov chain with steps from the Markov basis. They also show that Markov bases are in one-to-one correspondence with generating sets of toric ideals arising from these distributions. This is called the Fundamental theorem of Markov bases in [AHT12], as we will do.

The toric fibre product is a construction of an ideal from two ideals in polynomial algebras, introduced in [Sul07]. Under certain conditions it has been shown that it is possible to explicitly compute generating sets and Gröbner bases of the toric fibre product from generating sets and Gröbner bases of the component ideals.

Our interest lies in graphical models. These are a certain type of statistical model used to model dependence between random variables. These models can be glued together when they have a common set of variables, resulting in a bigger graphical model. Graphical models induce toric ideals, and we will show that the toric ideal of the glueing of two graphical models corresponds to the toric fibre product of the ideals. More specifically, we are interested in families of graphical models obtained by iteratively glueing a finite number of basis models $G_1, \ldots, G_s$. We will investigate bounds on the Markov bases of such models.

An example of such a family is the family of iterated glueings of the graph $K_{3,1}$, the claw graph, glued over the independent set of size 3. This results in the family of graphs $\{K_{3,N} \mid N \geq 1\}$, shown in fig. 1.1 for $N \in \{1, 2, 3, 4, 5\}$. In this case, the Markov degree of binary graphical models over these graphs is at most 12, as shown in [RS14]. This means that the ideals of such graphical models are generated by polynomials (in this case binomials) of degree at most 12.

Figure 1.1: Iterated glueings of the star graph $K_{3,1}$ over the independent set of size 3
1.2 Our contribution

We introduce a class of affine varieties and prove that stabilisation of the iterated toric fibre product occurs when applied to a finite number of these varieties. This class of varieties includes toric varieties, and therefore this implies conjecture 56 of [RS16], which can be restated as:

**Theorem 1.** Given toric ideals $I_1, \ldots, I_s$, there is a uniform bound $C \in \mathbb{N}$ such that for each $a \in \mathbb{N}^s$ the iterated toric fibre product \( \left( \bigtimes_{a_1} A I_1 \right) \times_A \cdots \times_A \left( \bigtimes_{a_s} A I_s \right) \) is generated by binomials of degree at most $C$.

The notation will be introduced later. This theorem will in turn imply that for graphical models $G_1, \ldots, G_s$ compatible over a common subgraph $H$, the family of iterated glueings admits a uniform bound $C$ on the degrees of generators of all ideals.

1.3 Thesis outline

First in chapter 2 we will introduce some abstract results about inverse and direct limits of sets, topological spaces, vector spaces and coordinate rings using category theory. These results will slowly descend to more concrete results that will be used later on. In chapter 3 we discuss the substitution monoid and the space of infinite-dimensional tensors. This space will have a certain finiteness with respect to the action of the substitution monoid. In chapter 4 we recall the definition of the toric fibre product of ideals and introduce a product on the level of varieties that corresponds to the toric fibre product. We also present some results about iterating this product. The main result will be proven in chapter 5 where we show that when applied to a certain class of varieties, the iterated toric fibre product is generated in uniformly bounded degree, i.e. independent of the number of iterations. Finally in chapter 6 we will show how these results imply theorem 1 and finiteness results of graphical models in algebraic statistics.
Chapter 2

Preliminaries

Using category theory we can prove some useful facts concerning direct and inverse limits of vector spaces, and their topologies. Most facts can be found in [Bor94]. First we will briefly recall some definitions, after which we will derive the desired properties of limits of vector spaces and their topologies. The results will start out on a very abstract level and slowly descend to more concrete results about inverse limits of vector spaces.

We denote $\mathbb{Z}$ the set of integers and set $\mathbb{Z}_{\geq a} := \{ n \in \mathbb{Z} \mid n \geq a \}$ for $a \in \mathbb{Z}$. Furthermore we let $\mathbb{N} = \mathbb{Z}_{\geq 1}$ and denote $[n] = \{ 1, \ldots, n \}$ for $n \in \mathbb{N}$.

2.1 Category theory

A category $C$ is a class of objects $\text{Ob}(C)$ and for any two objects $X, Y$ a class of morphisms between them, $\text{Hom}_C(X,Y)$. Morphisms are thought of as generalising functions. For two morphisms $f \in \text{Hom}_C(X,Y)$, $g \in \text{Hom}_C(Y,Z)$ (also denoted $f : X \to Y, g : Y \to Z$) the composition $g \circ f$ is a morphism in $\text{Hom}_C(X,Z)$, and composition of morphisms is associative. Furthermore, for each object $X$ there is a (unique) identity morphism $\text{id}_X : X \to X$ such that $\text{id}_X \circ f = f$ and $g \circ \text{id}_X = g$ for all morphisms $f : Y \to X, g : X \to Z$.

Examples of categories are:

1. Set, the category of sets and functions between them
2. Top, the category of topological spaces and continuous functions between them
3. Vect$_K$, the category of vector spaces over a field $K$ and $K$-linear maps between them
4. TVect$_K$, the category of topological vector spaces over a topological field $K$ and $K$-linear continuous maps between them
5. Var$_K$, the category of affine varieties over $K$ and polynomial maps between them
6. Alg$_K$, the category of $K$-algebras and $K$-algebra homomorphisms between them

We say a morphism $f$ in $\text{Hom}_C(X,Y)$ is an isomorphism if there is a morphism $g$ in $\text{Hom}_C(Y,X)$ such that $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_X$. We then call the objects $X$ and $Y$ isomorphic.

For two categories $I$ and $C$, a covariant functor $F : I \to C$ maps each object $X$ in $I$ to an object $F(X)$ in $C$, and each morphism $f : X \to Y$ in $I$ to a morphism $F(f) : F(X) \to F(Y)$ in $C$ such that:

- $F(\text{id}_X) = \text{id}_{F(X)}$
- $F(f \circ g) = F(f) \circ F(g)$

Dually, a contravariant functor maps objects in $I$ to objects in $C$ and morphisms $f : X \to Y$ in $I$ to morphisms $F(f) : F(Y) \to F(X)$ in $C$ such that:

- $F(\text{id}_X) = \text{id}_{F(Y)}$
- $F(f \circ g) = F(g) \circ F(f)$

Note that a contravariant functor reverses the direction of morphisms and reverses compositions. Typically, when talking about functors we will assume they are covariant, unless we explicitly state they are contravariant.
A small category is a category in which \( \text{Ob}(C) \) and the class of all morphisms \( \text{Hom}_C \) are sets. A locally small category is a category \( C \) in which for all objects \( X \) and \( Y \), \( \text{Hom}_C(X,Y) \) is a set. Examples of locally small categories are \( \text{Set}, \text{Top}, \text{Vect}_\mathbb{K}, \text{TVect}_\mathbb{K}, \text{Var}_\mathbb{K} \) and \( \text{Alg}_\mathbb{K} \). In a locally small category, the \( \text{Hom}_C \) bifunctor \( \text{Hom}_C : C \times C \rightarrow \text{Set} \) is contravariant in the first argument. This means, when fixing an object \( A \) of \( C \), the functor \( \text{Hom}_C(-, A) \) sends an object \( X \) to \( \text{Hom}_C(X,A) \) and a morphism \( g \in \text{Hom}_C(X,Y) \) to a morphism \( \text{Hom}_C(g,A) : \text{Hom}_C(Y,A) \rightarrow \text{Hom}_C(X,A) \) defined as \( \text{Hom}_C(g,A)(f) = f \circ g \).

A familiar example of this phenomenon is from linear algebra, where \( C = \text{Vect}_\mathbb{K} \) and we take \( A = \mathbb{K} \). This is the functor mapping each vector space \( V \) to its dual \( V^* \) and each linear map \( \phi : U \rightarrow W \) to the map \( \phi^* : W^* \rightarrow U^* \) defined as \( \phi^*(f) = f \circ \phi \). Note that in this case the set \( \text{Hom}_C(V, \mathbb{K}) \) has more structure, i.e. it is in fact a vector space, not just a set.

### 2.2 Limits and colimits

In the following we will define what limits and colimits to a functor \( F : I \rightarrow C \) are. Though the definition does not require this, the category \( I \) is often a small category, and the functor can then be thought of as giving a collection of objects \( \{A_i | i \in I\} \), and morphisms \( f_{i,j} \) for certain \( i \) and \( j \), that satisfy certain compatibility constraints.

A **cone** \( (A, \psi) \) to a functor \( F : I \rightarrow C \) is an object \( A \) in \( C \) and a morphism \( \psi_X : A \rightarrow F(X) \) for each object \( X \) in \( I \), such that for every morphism \( f : X \rightarrow Y \) in \( I \) we have \( F(f) \circ \psi_X = \psi_Y \). A **limit** to \( F \) is a cone \( (L, \phi) \) that is universal: For every other cone \( (A, \psi) \) there is a unique morphism \( u : A \rightarrow L \) such that \( \phi_X \circ u = \psi_X \) for all objects \( X \) in \( I \), called a **mediating morphism**. We denote \( (L, \phi) \) as \( \text{lim}_I(F) \), \( \lim_{i \in I}(F) \) or \( \lim_{i \in I}(A_i, f_{i,j}) \) when the functor is given explicitly as objects and morphism \( (A_i, f_{i,j}) \). This results in the commutative diagram in fig. 2.1.

![Figure 2.1: Commutative diagram of a limit](image)

Similarly, a **cocone** \( (A, \psi) \) to \( F \) is an object \( A \) in \( C \) and a morphism \( \psi_X : F(X) \rightarrow A \) for each object \( X \) in \( I \), such that for every morphism \( f : X \rightarrow Y \) in \( I \) we have \( \psi_Y \circ F(f) = \psi_X \), and a **colimit** \( (C, \phi) \) is a universal cocone: For every other cocone \( (A, \psi) \) there is a unique morphism \( u : C \rightarrow A \) such that \( u \circ \phi_X = \psi_X \) for all objects \( X \) in \( I \), again called a **mediating morphism**. We denote \( (C, \phi) \) as \( \text{colim}_I(F) \), \( \text{colim}_{i \in I}(F) \) or \( \text{colim}_{i \in I}(A_i, g_{j,i}) \) when the functor is given as objects and morphism \( (A_i, g_{j,i}) \). This results in the commutative diagram in fig. 2.2.

![Figure 2.2: Commutative diagram of a colimit](image)

Both limits and colimits are unique in a strong sense: For every two (co)limits \( (L, \phi), (M, \psi) \) to \( F \) there is a unique isomorphism \( u : L \rightarrow M \) such that \( \phi_X = \psi_X \circ u \) (resp. \( \phi_X = u \circ \psi_X \)) for all objects \( X \). We can therefore speak of the (co)limit.
The following are examples of limits and colimits:

1. If the class of objects of $I$ is a set and the only morphisms in $I$ are the identity morphisms, the limit is called the product.
   
   (a) In $\textbf{Set}$ the product is the Cartesian product.
   (b) In $\textbf{Vect}_k$ the product is the vector space product.
   (c) In $\textbf{Top}$ the product is the Cartesian product of sets with product topology.

2. Over the same category $I$, the colimit is called the coproduct.
   
   (a) In $\textbf{Set}$ the coproduct is the disjoint union.
   (b) In $\textbf{Vect}_k$ the coproduct is the direct sum.
   (c) In $\textbf{Top}$ the coproduct is the disjoint union of sets with the disjoint union topology.

3. The inverse limit, which we will define in section 2.2.1, is a limit.

4. The direct limit, which we will define in section 2.2.1, is a colimit.

A small (co)limit is the (co)limit of a functor $F : I \rightarrow C$ with $I$ a small category. A category $C$ is called complete (resp. cocomplete) when all small (co)limits exist. The examples $\textbf{Set}$, $\textbf{Top}$, $\textbf{Vect}_k$ and $\textbf{TVect}_k$ are complete and cocomplete. For each of these categories, the $\text{Hom}$-functor, when fixed in the second argument, sends colimits to limits, that is

$$\text{Hom}_C(\text{colim}(A_i, g_{ji}), X) \simeq \lim_{i \in I}(\text{Hom}_C(A_i, X), \text{Hom}_C(g_{ji}, X))$$

More generally, this happens in every category $C$, given that the limit and colimit exist:

$$\text{colim}(\text{Hom}_C(\langle-, X \rangle \circ F)) \simeq \text{Hom}_C(\text{lim}(F), X)$$

When they exist, small limits commute with small limits. That is, given a category $C$, two small categories $I$ and $J$, and a bi-functor $F : I \times J \rightarrow C$, there is a natural isomorphism

$$\lim_{i \in I} \text{lim}_{j \in J} F \simeq \text{lim}_{j \in J} \text{lim}_{i \in I} F$$

Since products and inverse limits are special cases of small limits, they commute whenever they exist.

### 2.2.1 Direct and inverse limits

A directed set $I$ is a partially ordered set such that for each $i, j \in I$ there is a $k \in I$ such that $i, j \leq k$. We can view $I$ as a category with a single morphism $i \rightarrow j$ if and only if $i \leq j$. The transitivity of the partial order means that if $i \leq j \leq k$ then $i \leq k$ and so we have a morphism $i \rightarrow k$. Since there is at most 1 morphism between any two objects, the morphism $i \rightarrow k$ must then be the composition of $i \rightarrow j$ and $j \rightarrow k$.

A functor $F : I \rightarrow C$ gives $(A_i, g_{ji})$ of objects $A_i$ for $i \in I$ and morphisms $g_{ji} : A_i \rightarrow A_j$ for $i \leq j$ such that for $i \leq j \leq k$ we have $g_{ki} = g_{kj} \circ g_{ji}$. We call such a system a direct system. A colimit to the functor is called a direct limit, which we denote $\text{lim}_{i \leq j \in I}(A_i, g_{ji})$. It is an object $A$ with morphisms $n_i : A_i \rightarrow A$ such that $n_i = n_j \circ g_{ji}$ for $i \leq j$. The universal property then states that for any object $B$ with morphisms $m_i : A_i \rightarrow B$ such that $m_i = m_j \circ g_{ji}$ for $i \leq j$, there is a unique morphism $u : A \rightarrow B$ such that $u \circ n_i = m_i$ for all $i \in I$. Though the naming might be somewhat confusing, a direct limit is indeed a colimit.

We can reverse all arrows of the category $I$ to obtain the opposite category $I^{op}$. A functor $F : I^{op} \rightarrow C$ is called an inverse system $(A_i, f_{ij})$ of objects $A_i$ for $i \in I$ and morphisms $f_{ij} : A_j \rightarrow A_i$ for $i \leq j$, such that for $i \leq j \leq k$ we have $f_{i,k} = f_{i,j} \circ f_{j,k}$. A limit to $F$ is called an inverse limit, which we denote $\text{lim}_{i \geq j \in I}(A_i, f_{ij})$. It is an object $A$ with morphisms $n_i : A \rightarrow A_i$ such that $n_i = f_{ij} \circ n_j$ for $i \leq j$. The universal property then states that for any object $B$ with morphisms $m_i : B \rightarrow A_i$ such that $m_i = f_{ij} \circ m_j$ for $i \leq j$, there is a unique morphism $u : B \rightarrow A$ such that $n_i \circ u = m_i$ for all $i \in I$.

As an example we consider a system of directed subsets of a certain set:
Theorem 2. Let $M$ be a set and for each $i \in I$ let $M_i$ be a subset of $M$ such that for $i \leq j, M_i \subseteq M_j$. Denote $g_{j,i} : M_i \to M_j$ the inclusion map $m \mapsto m$. Then $\lim_{\to j} (M_i, g_{j,i})$ is $\left( \bigcup_{i \in I} M_i, q_i \right)$ with maps $q_i$ being the inclusion maps.

Proof: Define $q_i : M_i \to \bigcup_{i \in I} M_i$ as $m \mapsto m$. Then obviously $q_i = q_j \circ g_{j,i}$ for $i \leq j$, and therefore $(\bigcup_{i \in I} M_i, q_i)$ is a cocone. Suppose $(N, r_i)$ is another cocone. For each $m \in \bigcup_{i \in I} M_i$ there is an $i$ such that $m \in M_i$, and then for all such $i$ the value $r_i(m)$ is the same because each $g_{j,i}$ is the inclusion map and $r_i = r_j \circ g_{j,i}$. So we define $u : \bigcup_{i \in I} M_i \to N$ as $u(m) = r_i(m)$ for $i$ such that $m \in M_i$. Then obviously $u \circ q_i = r_i$, and $u$ is unique, because for any other mediating morphism $w$ we have $u(m) = u(q_i(m)) = r_i(m) = w(q_i(m)) = w(m)$ for $m \in M_i$. So $u = w$. Therefore $(\bigcup_{i \in I} M_i, q_i)$ is the direct limit.

We will as much as possible reinterpret direct systems as such a directed subset system and use the above to obtain the direct limit.

For a directed set $I$, a subset $I'$ is cofinal if for every $i \in I$ there is an $i' \in I'$ such that $i \leq i'$. If $(A_i, f_{i,j}, I)$ is an inverse system (explicitly stating over which index set) with inverse limit $(A, \phi_i)$, then by the universal property there is a unique morphism $v : A \to A$ such that $\phi_i = \phi_i \circ v$ for all $i \in I$. By definition the identity morphism satisfies this, so this morphism $v$ must be the identity. Now if $I'$ is a cofinal subset of $I$, then $(A_i, f_{i,j}, I')$ is also an inverse system. Suppose it has limit $(A', \psi_i)$. Then $(A, \phi_i)$ is also a cone to $(A_i, f_{i,j}, I')$, so we have a unique mediating morphism $u : A \to A'$. Now the other way around, for $i \in I$, choose $j_i \in I'$ such that $i \leq j_i$, then $(A_i, f_{i,j}, I')$ is a cone to $(A_i, f_{i,j_i}, I)$, and thus we have a unique mediating morphism $w : A' \to A$. It is easy to show that $u \circ w$ and $w \circ u$ are the identity on $A'$ and $A$ respectively, because $\phi_i = \phi_i \circ (w \circ u)$ and $\psi_j = \psi_j \circ (u \circ w)$ for all $i \in I$ and $j \in I'$. Therefore $A$ and $A'$ are isomorphic. Therefore we can always take inverse limits over a cofinal index subset and obtain the same inverse limit. A similar argument shows this for direct limits over cofinal index subsets.

## 2.3 A double system

Suppose $C$ is a complete and cocomplete category and we have a simultaneous direct and inverse system $(A_i, g_{j,i}, f_{i,j})$ with $g_{j,i} : A_i \to A_j, f_{i,j} : A_j \to A_i$ and $f_{i,j} \circ g_{j,i} = \text{id}_{A_i}$ for $i \leq j$, over a directed set $I$. Then we can create left- or right-inverses for most maps that arise from the inverse and direct limits. Suppose $(A, p_i)$ is the inverse limit, and $(A', t_i)$ the direct limit.

First, fix an element $m \in I$. Then for each $i \in I$ define $h_{i,m} : A_m \to A_i$ as $h_{i,m} = f_{i,k} \circ g_{k,m}$ for some $k \geq m, i$. Note that by the properties of $f_{i,j}$ and $g_{j,i}$, the definition of $h_{i,m}$ does not depend on the choice of $k$, as long as $k \geq m, i$, which always exists. Then we obviously have for $i \leq j$, $h_{i,m} = f_{j,i} \circ h_{j,m}$. Therefore by the universal property of the inverse limit we obtain a unique map $r_m : A_m \to A$ for each $m \in I$ such that for all $i \in I$ we have $p_i \circ r_m = h_{i,m}$. In particular we have the following three identities:

1. $p_m \circ r_m = \text{id}_{A_m}$
2. $p_i \circ r_m = f_{i,m}$, for $i \leq m$
3. $p_j \circ r_m = g_{j,m}$, for $m \leq j$

That is, $r_m$ is a right-inverse of $p_m$ for each $m$ and we can then see that for $i \leq m$ and $j \in I$:

$$p_j \circ r_m \circ g_{m,i} = h_{j,m} \circ g_{m,i} = f_{j,k} \circ g_{k,m} \circ g_{m,i} = f_{j,k} \circ g_{k,i} = h_{j,i} = p_j \circ r_i$$

Note that a morphism into an inverse limit is uniquely determined by the composition with all its projection morphisms $p_j$ by the universal property. Therefore indeed $r_m \circ g_{m,i} = r_i$ and so by the universal property of the direct limit, we get a unique morphism $v : A' \to A$ such that for all $i \in I$ we have $r_i = v \circ t_i$. Furthermore, we can define $u_i = p_i \circ v$, and see that we then have $u_i \circ t_i = \text{id}_{A_i}$ for all $i \in I$, and for $i \leq j$ we have $f_{i,j} \circ u_j = u_i$. All this can be summarised in the diagrams in fig. 2.3, where dashed arrows are sections (right-inverses) of the morphisms in the other direction of the other diagram.
2.3.1 Double systems in Set, Vect, Top, TVect

Now suppose that $C$ is Set, Top, Vect$^K$ or TVect$^K$. In these cases the direct limit of a double system can be embedded in the direct limit. To do so, we will construct the inverse limit. First we will show what the direct limit is in Set:

Let $D = \bigcup_{i \in I} A_i \times \{i\}$, the disjoint union of the $A_i$. Define an equivalence relation as $(a, i) \sim (b, j)$ if and only if there exists a $k \geq i, j$ such that $g_{k,i}(a) = g_{k,j}(b)$. That is, two pairs are the same if they eventually become the same in the direct system. Denote $A'' = D / \sim$, and $[a, i]$ the equivalence class of $(a, i)$. Quite obviously each $A_i$ injects in $A''$ as $t_i : A_i \to A''$ with $a \mapsto [a, i]$, and then $t_i = t_j \circ g_{j,i}$. Using the fact that each element of $a \in A''$ is of the form $t_i(a')$ for some $i \in I$, $a' \in A_i$, we can easily show that $A''$ is indeed the direct limit.

Now if the $A_i$ are vector spaces we define vector space operations on $A''$ as:

1. $[a, i] + [b, j] = [g_{k,i}(a) + g_{k,j}(b), k]$ for some $k \geq i, j$
2. $\lambda [a, i] = [\lambda a, i]$

The addition operation is independent of the choice of $k$. Furthermore these operations are well-defined, and indeed satisfy all axioms of a vector space. We can see that the morphisms $t_i$ are then linear maps. Therefore $(A'', t_i)$ is a cocone to the direct system of vector spaces in Vect$^K$, and in fact it is the direct limit.

If the $A_i$ are topological spaces we can endow $A''$ with the final topology, in which a set $U \subseteq A''$ is open if and only if $t_i^{-1}(U)$ is open in $A_i$ for each $i \in I$. Then obviously each $t_i$ is continuous and therefore $(A'', t_i)$ is a cocone in Top, which is again the direct limit.

Finally, if the $A_i$ are topological vector spaces, we can define both of the above structures, the vector space operations and the topology on $A''$. In this case the vector space operations are continuous and therefore $(A'', t_i)$ is a cocone in TVect$^K$. It is again the direct limit.

Therefore in these cases we can actually say something about the structure of the underlying set: It is a quotient of the disjoint union, and in fact it is the union of the images of the morphisms $t_i$. That is, for each $a \in A''$ there is an $i \in I$ and $a' \in A_i$ such that $a = t_i(a')$. Therefore, for each $a \in A''$ there is an $i \in I$ such that $a = t_i(u_i(a))$. Using this fact we can see:

**Proposition 1.** The mediating morphism $v : A' \to A$ is injective.

**Proof:** If $v(a) = v(b)$ then $u_i(a) = p_i(v(a)) = p_i(v(b)) = u_i(b)$ for all $i \in I$, meaning $(t_i \circ u_i)(a) = (t_i \circ u_i)(b)$ for all $i \in I$. Now let $k_1$ and $k_2$ such that $(t_{k_1} \circ u_{k_1})(a) = a$ and $(t_{k_2} \circ u_{k_2})(b) = b$ and let $k \geq k_1, k_2$. Then we have $a = t_k(u_k(a)) = t_k(u_k(b)) = b$. Therefore $v$ is injective.  

If $A_i$ are topological spaces we can endow $A$ with the initial topology, for which a subset $U \subseteq A$ is open if and only if it is of the form $\bigcup_{i \in I} p_i^{-1}(U_i)$ for $U_i \subseteq A_i$ open for each $i \in I$. This is the unique topology that makes $A$ the inverse limit of topological spaces.

**Proposition 2.** If the objects $A_i$ are topological spaces (and thus $f_{i,j}, g_{i,j}, p_i, r_i, t_i$ and $u_i$ are continuous functions), then $v(A')$ is dense in $A$.

**Proof:** To show that $v(A')$ is dense, let $U = \bigcup_{i \in I} p_i^{-1}(U_i)$ be a non-empty open subset of $A$. Then $U_i \neq \emptyset$ for some $i \in I$ because each $p_i$ is surjective. Thus let $U_i$ be non-empty and $x \in U_i$. Then $p_i(r_i(x)) = x \in U_i$ so $r_i(x) \in p_i^{-1}(U_i)$. But also $r_i(x) = v(t_i(x))$ and $t_i(x) \in A'$ so $r_i(x) \in v(A')$. Therefore $r_i(x) \in U \cap v(A') \neq \emptyset$, meaning $v(A')$ is dense in $A$.  

![Figure 2.3: Commutative diagrams of a double system](image-url)
From now on, when we have a double system in \textbf{Set}, \textbf{Vect}_\mathbb{K}, \textbf{Top} or \textbf{T Vect}_\mathbb{K}, we directly identify the direct limit as a subset of the inverse limit (dense when applicable). The morphism \( v \) becomes the inclusion map and we can then identify the maps \( u_i \) and \( p_i \), and the maps \( t_i \) and \( r_i \) because \( u_i = p_i \circ v \) and \( r_i = v \circ t_i \). We will only use the more general maps \( p_i \) and \( r_i \).

2.4 Vector spaces over sets

Fix a field \( \mathbb{K} \) and let \( I \) be a directed set. If a vector space \( U \) is the dual of a vector space \( W \) we say it has coordinate ring \( S(W) \), the symmetric algebra of \( W \), and denote it \( \mathbb{K}[U] \). When choosing a basis \( \{ w_b | b \in B \} \) of \( W \), we can find an isomorphism \( \mathbb{K}[U] \simeq \mathbb{K}[x_b | b \in B] \) with the algebra generated by (algebraically independent) functionals \( \{ x_b | b \in B \} \) defined as

\[
x_b \left( \sum_{b \in B} c_b w_b \right) = c_b
\]

The isomorphism is defined as \( w_b \mapsto x_b \), and extended linearly and multiplicatively, where multiplication in \( S(W) \) is the (symmetric) tensor product. Then \( W \simeq \bigoplus_{b \in B} \mathbb{K} \) and so we can see that:

\[
U = \text{Hom}_{\text{Vect}_\mathbb{K}}(W, \mathbb{K}) \simeq \text{Hom}_{\text{Vect}_\mathbb{K}}(\bigoplus_{b \in B} \mathbb{K}, \mathbb{K}) \simeq \prod_{b \in B} \text{Hom}_{\text{Vect}_\mathbb{K}}(\mathbb{K}, \mathbb{K}) \simeq \prod_{b \in B} \mathbb{K} \simeq \text{Hom}_{\text{Set}}(B, \mathbb{K})
\]

The isomorphism taking the direct sum out as a product comes from the fact that the direct sum is a colimit in the category of vector spaces, and the product is the corresponding limit.

On the other hand if \( U \simeq \text{Hom}_{\text{Set}}(B, \mathbb{K}) \simeq \prod_{b \in B} \mathbb{K} \) for some set \( B \) we get by the above that \( U \simeq \text{Hom}_{\text{Vect}_\mathbb{K}}(\bigoplus_{b \in B} \mathbb{K}, \mathbb{K}) \) and so \( \mathbb{K}[U] \simeq \mathbb{K}[x_b | b \in B] \).

Vector spaces

Suppose for each \( i \in I \) we have a set \( A_i \), and for \( i \leq j \) maps \( \alpha_{j,i} : A_j \to A_i \) that form a direct system in \textbf{Set}. Let \((A, \gamma_i)\) be the direct limit of the direct system. Then by point-wise addition and multiplication by scalars of \( \mathbb{K} \), we can make \( \text{Hom}_{\text{Set}}(A_i, \mathbb{K}) \) into a vector space, call it \( V_i \). Then \( V_i \simeq \prod_{a \in A_i} \mathbb{K} \). From the \( \text{Hom}_{\text{Set}} \)-bifunctor we get linear maps \( f_{i,j} : V_j \to V_i \) defined as \( f_{i,j} = \text{Hom}_{\text{Set}}(\alpha_{j,i}, \mathbb{K}) \). When we view elements \( v \in V_i \) as functions \( A_i \to \mathbb{K} \), then the morphism \( f_{i,j} \) evaluated at \( v \) is \( v \circ \alpha_{j,i} \), which boils down to:

\[
f_{i,j}(v)(a) = (v \circ \alpha_{j,i})(a) = v(\alpha_{j,i}(a))
\]

When viewing \( v \) again as a tuple \((v_a)_{a \in A_i}\) of elements of \( \mathbb{K} \) we then have \( f_{i,j}(v) = (v_{\alpha_{j,i}(a)})_{a \in A_i} \). These maps \( f_{i,j} \) form an inverse system because \( \text{Hom}_{\text{C}} \) is contravariant in the first argument, so for \( i \leq j \leq k \):

\[
f_{i,k} = \text{Hom}_{\text{Set}}(\alpha_{k,i}, \mathbb{K}) = \text{Hom}_{\text{Set}}(\alpha_{k,j} \circ \alpha_{j,i}, \mathbb{K}) = \text{Hom}_{\text{Set}}(\alpha_{j,i}, \mathbb{K}) \circ \text{Hom}_{\text{Set}}(\alpha_{k,j}, \mathbb{K}) = f_{i,j} \circ f_{j,k}
\]

Then because the \( \text{Hom} \)-functor takes colimits to limits, we have:

\[
\lim_{i \in I}(V_i, f_{i,j}) = \lim_{i \in I}(\text{Hom}_{\text{Set}}(A_i, \mathbb{K}), \text{Hom}_{\text{Set}}(\alpha_{j,i}, \mathbb{K})) = \text{Hom}_{\text{Set}}(\lim_{i \in I}(A_i, \alpha_{j,i}), \mathbb{K}) = (\text{Hom}_{\text{Set}}(A, \mathbb{K}), \text{Hom}_{\text{Set}}(\gamma_i, \mathbb{K}))
\]

Denote this inverse limit as \((V, p_i)\). Then we see that \( V \simeq \text{Hom}_{\text{Set}}(A, \mathbb{K}) \simeq \prod_{a \in A} \mathbb{K} \) and the morphisms \( p_i \) are given as:

\[
p_i = \text{Hom}_{\text{Set}}(\gamma_i, \mathbb{K}) \quad \text{and therefore} \quad p_i((v_a)_{a \in A}) = (v_{\gamma_i(a)})_{a \in A_i}
\]

From this we get the two commutative diagrams in fig. 2.4 and fig. 2.5.
Coordinate ring of inverse limit

We can take the coordinate ring of each $V_i$, given as $\mathbb{K}[V_i] \simeq \mathbb{K}[x_a^i | a \in A_i]$, where $x_a^i$ is the linear functional mapping $(v_b)_{b \in A_i}$ to $v_a$. The induced morphisms $f_{i,j}^* : \mathbb{K}[V_i] \to \mathbb{K}[V_j]$ are defined as $f_{i,j}^*(h) = h \circ f_{i,j}$. Then we can see that $f_{i,j}^*(x_a^i) = x_{\alpha_{i,j}(a)}^j$:

$$f_{i,j}^*(x_a^i)((v_b)_{b \in A_j}) = (x_a^i \circ f_{i,j})((v_b)_{b \in A_j}) = x_a^j((v_{\alpha_{i,j}(b)})_{b \in A_j}) = v_{\alpha_{i,j}(a)} = x_{\alpha_{i,j}(a)}^j((v_b)_{b \in A_j})$$

We define functionals $x_a^\infty : V \to \mathbb{K}$ as $x_a^\infty((v_b)_{b \in A}) = v_a$ and claim the direct system $(\mathbb{K}[V_i], f_{i,j}^*)$ has the following direct limit:

$$\mathbb{K}[x^\infty] := \mathbb{K}[x_a^\infty | a \in A] \quad \text{with} \quad q_i : \mathbb{K}[V_i] \to \mathbb{K}[x^\infty] \quad \text{defined as} \quad q_i(x_a^i) = x_{\gamma_i(a)}^\infty$$

Since we already saw that $f_{i,j}^*(x_a^i) = x_{\alpha_{i,j}(a)}^j$, we can see that $q_i = q_j \circ f_{i,j}^*$. Therefore indeed this is a cocone. Suppose $(Y, y_i)$ is another cocone. Define $u : \mathbb{K}[x^\infty] \to Y$ as $x_a^\infty \mapsto y_i(x_a^i)$ where $a = \gamma_i(a')$. For any two such choices $(i, a'), (j, b')$ we have $y_i(x_a^i) = y_j(x_b^j)$, and thus this value is independent of the specific pair $(i, a')$. Then obviously $u(q_i(x_a^i)) = y_i(x_a^i)$. Therefore we have a well-defined mediating morphism $u : \mathbb{K}[x^\infty] \to Y$. To see that $u$ is unique, suppose that $w$ is also a mediating morphism. Let $x_a^\infty \in \mathbb{K}[x^\infty]$ and $a = \gamma_i(a')$. Then

$$w(x_a^\infty) = w(q_i(x_a^i)) = u(q_i(x_a^i)) = u(x_a^\infty)$$

Since $u$ and $w$ agree on generators of $\mathbb{K}[x^\infty]$ we see that indeed $u$ and $w$ are the same morphism.

Note that we have a characterisation of $q_i$ as $p_i^*$:

$$q_i(x_a^i)((v_b)_{b \in A}) = x_{\gamma_i(a)}^\infty((v_b)_{b \in A}) = v_{\gamma_i(a)} = x_a^i(p_i((v_b)_{b \in A}))$$

We note that $\mathbb{K}[x^\infty] \simeq S(\bigoplus_{a \in A} \mathbb{K})$ and $V \simeq \prod_{a \in A} \mathbb{K} \simeq (\bigoplus_{a \in A} \mathbb{K})^*$, so $\mathbb{K}[V] \simeq \mathbb{K}[x^\infty]$. Therefore the coordinate ring of $V$ is $\mathbb{K}[x^\infty]$, and from now on we denote $\mathbb{K}[V] = \mathbb{K}[x^\infty]$. Since this algebra is the direct limit of the coordinate rings of the vector spaces $V_i$ we see:

$$\lim_{i \in I} \left( \mathbb{K}[V_i], f_{i,j}^* \right) \simeq \mathbb{K} \left[ \lim_{i \in I} (\mathbb{K}[V_i], f_{i,j}) \right]$$

This results in the commutative diagram in fig. 2.6
2.4.1 Topology on the inverse limit

When a vector space $U$ admits a coordinate ring, we can endow it with the Zariski topology, for which the closed subsets are exactly $V(J) := \{ v \in U \mid \forall f \in J : f(v) = 0 \}$ for $J \subseteq \mathbb{K}[U]$. If desired, we can take $J$ to be an ideal, because $J$ and the ideal generated by $J$ give the same zero-locus. In the case of finite-dimensional vector spaces this is the familiar Zariski topology. Since the inverse limit $V$ constructed above admits a coordinate ring $\mathbb{K}[V]$, we can endow it with the Zariski topology. Furthermore, since each $V_i$ has a coordinate ring, they also admit the Zariski topology, and we can thereby also endow $V$ with the initial topology, for which a set is closed exactly when it is of the form $\bigcap_{i \in I} p_i^{-1}(C_i)$ with each $C_i \subset V_i$ closed. We claim these two topologies are the same.

**Theorem 3.** The Zariski topology and the initial topology on $V$ are the same.

**Proof:** Recall that $q_i(f) = f \circ p_i$.

**Zariski-closed $\Rightarrow$ Closed in the initial topology:** Let $C = V(J)$ be a Zariski-closed subset of $V$ with $J$ an ideal. Define for $i \in I$, $J_i = q_i^{-1}(J) \subseteq \mathbb{K}[V_i]$. Then $J_i$ is an ideal, because $q_i$ is a $\mathbb{K}$-algebra homomorphism, and thus pulls back ideals to ideals. Note that we then have $J = \bigcup_{i \in I} q_i(J_i)$. We claim $C = \bigcap_{i \in I} p_i^{-1}(V(J_i))$:

$$v \in C \iff \forall i \in I, f \in J_i : f(v) = 0$$

Thus indeed $C$ is also Zariski-closed.

**Closed in the initial topology $\Rightarrow$ Zariski-closed:** If $C$ is closed in the initial topology, then $C = \bigcap_{i \in I} p_i^{-1}(V(J_i))$ for ideals $J_i \subseteq \mathbb{K}[V_i]$. Let $G = \bigcup_{i \in I} q_i(J_i) \subseteq \mathbb{K}[V]$. Then we have:

$$v \in C \iff \forall i \in I, f \in J_i : f(p_i(v)) = 0$$

Thus a set is Zariski-closed if and only if it is closed in the initial topology. Therefore the topologies coincide.

We have seen that the coordinate ring of the inverse limit is the direct limit of the coordinate rings. But now we can see that the (unique) topology that makes $V$ the inverse limit is actually the Zariski topology coming from this coordinate ring.
2.4.2 Double system of vector spaces

Suppose that the maps  have left-inverses  such that  for . Then we have a double system of vector spaces  

The maps  induce maps  as \( g_{ij} : V_i \to V_j \) as \( g_{ij} = \text{Hom}_\text{Set}(\beta_{ij}, k) \). Then we can see that  if and  if . These maps induce maps  for which we have  if and  if . The maps  also induce maps  for . Then we can define  as \( \text{K}[V] \to \text{K}[V] \) for which we have  and thus  and see that  and .

Now the direct limit  of the system  is dense in . If for each  we have a subset  such that they form a direct and inverse system  (i.e.  and ) (i.e.  or  and ) then we call  a subsystem of . For such a subsystem the direct limit  is dense in the inverse limit . Furthermore,  is the subset  and  is the subset of consisting of elements  such that  for all . Applying to the Zariski topology; any polynomial in \( \text{K}[V] \) that vanishes on all of  vanishes on all of .

Now we note the following two propositions:

**Proposition 3.** If  is generated by a set of polynomials , then  is  for and the ideal generated by the lifts of the elements of .

**Proof:** If  then  for certain . So for  we have  and thus  is an ideal, consisting of elements consisting of elements . Then we have a subset  such that  and see that  and .

Thus  and since  for we have  if and only if for all  we have  if .

**Claim  \( s_i(G) \subseteq  \):** Let  for some . But  uses only finitely many variables, so for some \( f \in I \) and  we have  by taking a  and letting  we can see that  so  by . So let  then  for some  and  if . Therefore  for some  . Therefore 

Thus indeed  for .

**Claim  \( I(X_i) \subseteq  \):** If  then  for all . So  meaning , with  for all . Then 

Therefore indeed  for .

Therefore  for .

We will use the results of this chapter throughout the remaining chapters of this thesis to quickly obtain direct and inverse limits, and coordinate rings of inverse limits.
Chapter 3

The substitution monoid and infinite-dimensional tensors

One of the tools we will use is that of taking direct and inverse limits, which in our application will turn out to be interpretable as the space of infinite-dimensional tensors. These tensors will have an action of the Substitution monoid \( \text{Subs}(\mathbb{N}) \). We look at the analogue of rank-1 tensors in this space, which will be a Zariski-closed subset. This variety will be closed under the action and its coordinate ring will be \( \text{Subs}(\mathbb{N}) \)-Noetherian, i.e. the ideal of any subvariety stable under the action of \( \text{Subs}(\mathbb{N}) \) is generated by the orbits of finitely many polynomials. We will prove these facts in this chapter. We let \( \mathbb{K} \) be an algebraically closed field.

3.1 The substitution monoid

In [DK14] the monoid \( \text{Subs}(\mathbb{N}) \) is defined, which is an associative monoid with a natural left-action on infinite strings over an alphabet with a distinguished element \( 0 \). Here we recall its definition and action on infinite strings. \( \text{Subs}(\mathbb{N}) \) is the monoid of infinite sequences \( \sigma = (\sigma_1, \sigma_2, \ldots) \) of finite pairwise disjoint subsets of \( \mathbb{N} \). Multiplication of \( \sigma, \pi \in \text{Subs}(\mathbb{N}) \) is defined as:

\[
(\sigma \pi)_i = \bigcup_{j \in \pi_i} \sigma_j
\]

The monoid has identity element \( (\{1\}, \{2\}, \{3\}, \ldots) \). We denote \( \text{im} \sigma = \bigcup_{i \in \mathbb{N}} \sigma_i \), the union of all its elements. Fix an alphabet \( \mathcal{A} \). If \( \mathcal{A} \) has a distinguished element \( 0 \), the left-action of \( \text{Subs}(\mathbb{N}) \) on \( \mathcal{A}^\mathbb{N} \), the set of infinite strings over \( \mathcal{A} \), is defined as:

\[
(\sigma s)_i = \begin{cases} 
0 & \text{if } i \notin \text{im} \sigma \\
_s m & \text{if } i \in \sigma_m
\end{cases} \quad \text{for } s \in \mathcal{A}^\mathbb{N}
\]

Note that the subset \( \mathcal{A}_0^\mathbb{N} \) of strings for which only finitely many elements are not \( 0 \) is stable under this action. We note two important submonoids. First we have \( \text{Subs}_<(\mathbb{N}) \) of sequences \( \sigma \) in which all \( \sigma_i \) are non-empty, and \( \max \sigma_1 < \max \sigma_2 < \max \sigma_3 < \cdots \). Within this monoid there is the submonoid \( \text{Subs}^s_<(\mathbb{N}) \) of sequences \( \sigma \) such that \( \text{im} \sigma = \mathbb{N} \). The latter is the monoid we will use most often, because of the following fact:

**Theorem 4.** Let \( \mathcal{A} \) be an alphabet with a distinguished element \( 0 \). If \( s \in \mathcal{A}_0^\mathbb{N}, \sigma \in \text{Subs}_<(\mathbb{N}) \), then there exists a \( \pi \in \text{Subs}^s_<(\mathbb{N}) \) such that \( \sigma s = \pi s \).

**Proof:** Let \( l_1, \ldots, l_n \in \mathbb{N} \) be such that exactly \( s_{l_i} \neq 0 \), and set \( \mathbb{N} \setminus \{l_1, \ldots, l_n\} = \{m_1 < m_2 < \cdots\} \). Then set:

\[
\pi_k = \begin{cases} 
\sigma_{l_i} & \text{if } k = l_i \text{ for some } i \in [n] \\
\sigma_{m_1} \cup \{l \in \mathbb{N} \mid l \notin \text{im} \sigma, l < \max \sigma_{m_1} \} & \text{if } k = m_1 \\
\sigma_{m_i} \cup \{l \in \mathbb{N} \mid l \notin \text{im} \sigma, \max \sigma_{m_{i-1}} < l < \max \sigma_{m_i} \} & \text{if } k = m_i \text{ for some } i > 1
\end{cases}
\]

Then for \( i \in \mathbb{N} \), we have:
1. \( \pi_i \) is finite
2. \( \sigma_i \subseteq \pi_i \), and so \( \pi_i \neq \emptyset \)
3. max \( \pi_i = \max \sigma_i \) and so max \( \pi_1 < \max \pi_2 < \max \pi_3 < \cdots \)
4. \( \forall j : \pi_i \cap \pi_j = \emptyset \)

Now if \( l \notin \operatorname{im}(\sigma) \), then either \( l < \max \sigma_1 \) in which case \( l \in \pi_1 \), or there is some \( k > 1 \) such that \( \max \sigma_{k-1} < l < \max \sigma_k \), meaning that by definition \( l \in \pi_k \) then. So indeed \( \pi \in \text{Subs}_\leq(\mathbb{N}) \).

To show \( \sigma s = \pi s \), let \( i \in \mathbb{N} \). If \( i \in \operatorname{im}(\sigma) \), with \( i \in \pi_k \subseteq \pi_k \) then \( (\sigma s)_i = s_k = (\pi s)_i \). If \( i \notin \operatorname{im}(\sigma) \) then \( (\sigma s)_i = 0 \). Since \( i \notin \operatorname{im}(\sigma) \), \( i \in \pi_{m_j} \) for some \( j \), because \( \pi_{lk} = \sigma_{lk} \) for all \( k \in [n] \). Then since \( s_{m_j} = 0 \), we have \( (\sigma s)_i = 0 = s_{m_j} = (\pi s)_i \).

This result shows that when using infinite strings with only finitely many elements not \( 0 \), we lose no information by using the smaller monoid \( \text{Subs}_\leq^\infty(\mathbb{N}) \). The idea being that when \( i \notin \operatorname{im}(\sigma) \) then \( (\sigma s)_i = 0 \) and so we can put \( i \) in \( \sigma_k \) for sufficiently large \( k \) (such that \( s_k = 0 \) and max \( \sigma_k > i \)) without altering the value of \( \sigma s \) or changing the sequence max \( \sigma_1 < \max \sigma_2 < \ldots \).

Furthermore, this monoid has a more natural action on any set of infinite strings over \( A \), without having to select a distinguished element, and the subset \( A_{\text{fin}}^\infty := \{ s \in A^\infty \mid \exists k \in \mathbb{N} : \forall n \geq k : s_n = s_k \} \) of strings that are eventually constant is stable under this action.

### 3.1.1 Sequences over a finite alphabet

For our purposes it will suffice to consider sequences over finite alphabets. So for \( d, a \in \mathbb{N} \) denote

\[
\begin{align*}
\text{seq}_d(d) &= \{(s_1, \ldots, s_d) \mid \forall k \in [a] : s_k \in [d]\} \\
\text{seq}_\infty(d) &= \{(s_1, s_2, s_3, \ldots) \mid \forall k \in \mathbb{N} : s_k \in [d]\} \\
\text{seq}_{\text{fin}}^\infty(d) &= \{s \in \text{seq}_\infty(d) \mid \exists k \in \mathbb{N} : \forall n \geq k : s_k = s_n\}
\end{align*}
\]

Then for \( s \in \text{seq}_{\text{fin}}^\infty(d) \) we define \( s_\infty \) as the stabilising element, i.e. the element \( s_\infty = \lim_{n \to \infty} s_n \). For \( \sigma \in \text{Subs}_\leq^\infty(\mathbb{N}) \) we then have \( (\sigma s)_\infty = s_\infty \).

Consider the following maps for \( a \leq b \) in \( \mathbb{N} \):

\[
\alpha_{b,a} : \text{seq}_a(d) \to \text{seq}_b(d) \quad \text{defined as} \quad \forall l \in [b] : (\alpha_{b,a}(s))_l = s_{\min(a,l)}
\]

These maps correspond to repeating the last element. Then \( \alpha_{c,b} \circ \alpha_{b,a} = \alpha_{c,a} \) for \( a \leq b \leq c \), so we have a direct system \( (\text{seq}_a(d), \alpha_{b,a}) \). The direct limit is \( \text{seq}_{\text{fin}}^\infty(d) \) with morphisms

\[
\gamma_a : \text{seq}_a(d) \to \text{seq}_{\text{fin}}^\infty(d) \quad \text{defined as} \quad \forall l \in \mathbb{N} : (\gamma_a(s))_l = s_{\min(a,l)}
\]

To see this, note that \( \text{seq}_a(d) \) is in bijection with the following subset of \( \text{seq}_{\text{fin}}^\infty(d) \):

\[
\left\{ s \in \text{seq}_{\text{fin}}^\infty(d) \mid \forall k \geq a : s_k = s_a \right\}
\]

Under these identifications, the maps \( \alpha_{b,a} \) are inclusions and the system forms a directed subset system, and the union of these sets is exactly \( \text{seq}_{\text{fin}}^\infty(d) \). Therefore by theorem 2 it is the direct limit.

The maps \( \alpha_{b,a} \) have left-inverses, mapping a sequence to its restriction to the first \( a \) elements:

\[
\beta_{a,b} : \text{seq}_b(d) \to \text{seq}_a(d) \quad \text{defined as} \quad \forall l \in [a] : (\beta_{a,b}(s))_l = s_l
\]

These maps induce the following map:

\[
\delta_a : \text{seq}_{\text{fin}}^\infty(d) \to \text{seq}_a(d) \quad \text{defined as} \quad \forall l \in [a] : (\delta_a(s))_l = s_l
\]

They map an infinite sequence \( s \) to its restriction to the first \( a \) elements. Now these maps satisfy \( \beta_{a,b} \circ \delta_b = \delta_a \) and \( \delta_a \circ \gamma_a = \text{id}_{\text{seq}_a(d)} \).

Note that none of these maps actually depend on the value \( d \), and we will therefore omit it when using these maps.

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3.2 Tensor spaces and the inverse limit

Let \( n \in \mathbb{N} \), and \( V = \mathbb{K}^n \). On \( V \) we define a bilinear associative commutative Hadamard-product as:

\[
(v_1, \ldots, v_n) \cdot (w_1, \ldots, w_n) := (v_1 w_1, \ldots, v_n w_n)
\]

This product has identity element \( e = (1, \ldots, 1) \). Define the vector spaces \( T_a(V) = \bigotimes_{k=1}^b V \) for \( a \in \mathbb{N} \) and morphisms for \( a \leq b \):

\[
f_{a,b} : T_b(V) \rightarrow T_a(V) \text{ defined as } \quad v_1 \otimes \cdots \otimes v_b \mapsto v_1 \otimes \cdots \otimes v_{a-1} \otimes \left( \bigotimes_{k=a}^b v_k \right)
\]

Then we have \( f_{a,b} \circ f_{b,c} = f_{a,c} \) for \( a \leq b \leq c \), and thus we have an inverse system \( (T_a(V), f_{a,b}) \). Let \( (T_\infty(V), p_a) \) be the inverse limit of this system. Note that we can identify \( T_a(V) \) as the set of tuples of elements of \( \mathbb{K} \) indexed by \( \text{seq}_{\mathbb{N}}(n) \), and we can see that \( f_{a,b}((v_i)_{i \in \text{seq}_{\mathbb{N}}(n)}) = (v_{a,b}(i))_{i \in \text{seq}_{\mathbb{N}}(n)} \) under this identification, because the product is coordinate-wise. Then we have \( \mathbb{K}[T_a(V)] \cong \mathbb{K}[x_i | i \in \text{seq}_{\mathbb{N}}(n)] \) and the inverse limit \( (T_\infty(V), p_\infty) \) is given as \( \mathbb{K}^{\text{seq}^{fin}_{\infty}(n)} \) with morphisms

\[
p_\infty((t_i)_{i \in \text{seq}^{fin}_{\infty}(n)}) = (t_{\infty}(i))_{i \in \text{seq}_{\mathbb{N}}(n)}
\]

Under this identification, \( T_\infty(V) \) has a natural right-action of \( \text{Subs}^{+}_{\infty}(\mathbb{N}) \) as \( (t\sigma)_i = t\sigma_i \). By the results on coordinate rings in section 2.4, the space \( T_\infty(V) \) has the coordinate ring

\[
\mathbb{K}[T_\infty(V)] := \mathbb{K}[x_i | i \in \text{seq}^{fin}_{\infty}(n)]
\]

which is the direct limit of the coordinate rings \( \mathbb{K}[T_a(V)] \) with morphisms \( q_a : \mathbb{K}[T_a(V)] \rightarrow \mathbb{K}[T_\infty(V)] \) given as

\[
q_a(x_i) = x_{\gamma a(i)}
\]

The coordinate ring \( \mathbb{K}[T_\infty(V)] \) has a \( \text{Subs}^{+}_{\infty}(\mathbb{N}) \) action defined as \( \sigma x_i = x_{\sigma i} \), extended by multiplicativity and linearity. This action satisfies the property that for any \( t \in T_\infty(V) \), \( f \in \mathbb{K}[T_\infty(V)] \) we have \( (\sigma f)(t) = f(t\sigma) \). Furthermore, for \( f \in \mathbb{K}[T_a(V)] \) we have \( q_a(f) = f \circ p_a \).

We note that the action of \( \text{Subs}^{+}_{\infty}(\mathbb{N}) \) on \( T_\infty(V) \) is by continuous maps:

**Theorem 5.** The map \( f_{\sigma} : T_\infty(V) \rightarrow T_\infty(V) \) defined as \( t \mapsto t \cdot \sigma \) is continuous.

**Proof:** Let \( V(J) \subseteq T_\infty(V) \) be a closed set. Define \( J' = \{ \sigma \cdot f | f \in J \} \). We claim \( f_{\sigma}^{-1}(V(J)) = V(J') \):

\[
x \in f_{\sigma}^{-1}(V(J)) \iff f_{\sigma}(x) \in V(J)
\]

\[
\iff x \cdot \sigma \in V(J)
\]

\[
\iff \forall f \in J : \quad f(x \cdot \sigma) = 0
\]

\[
\iff \forall f \in J : \quad (\sigma f)(x) = 0
\]

\[
\iff \forall g \in J' : \quad g(x) = 0
\]

\[
\iff x \in V(J')
\]

Therefore the pre-image of a closed set is a closed set, so \( f_{\sigma} \) is continuous.

Therefore if a set \( X \) is stable under the action, so is its closure.

Furthermore, since the maps \( g_{b,a} \) have left-inverses \( \beta_{a,b} \) we also get the following maps:

\[
g_{b,a} : T_a(V) \rightarrow T_b(V) \text{ defined as } \quad t \mapsto (t_{\beta_{b,a}(i)})_{i \in \text{seq}_{\mathbb{N}}(b)}
\]

\[
r_a : T_a(V) \rightarrow T_\infty(V) \text{ defined as } \quad t \mapsto (t_{\delta a(i)})_{i \in \text{seq}^{fin}_{\infty}(n)}
\]

\[
s_a : \mathbb{K}[T_\infty(V)] \rightarrow \mathbb{K}[T_a(V)] \text{ defined as } \quad x_i \mapsto x_{\delta a(i)}
\]

Then in fact \( g_{b,a}(v_1 \otimes \cdots \otimes v_a) = v_1 \otimes \cdots \otimes v_a \otimes e \otimes \cdots \otimes e \), and these maps satisfy:

\[
f_{a,b} \circ g_{b,a} = \text{id}_{T_a(V)}
\]

\[
r_b \circ g_{b,a} = r_a
\]

\[
p_a \circ r_a = \text{id}_{T_a(V)}
\]

\[
s_a \circ q_a = \text{id}_{\mathbb{K}[T_a(V)]}
\]
3.3 Infinite-dimensional rank-1 tensors

Within each $T_a(V)$ we have the subset $T_a^{\leq 1}(V)$ of rank at most 1 tensors (pure tensors), which is a Zariski-closed subset. These subsets are mapped to each other under the maps $f_{a,b}$ and $g_{b,a}$. Therefore $(T_a^{\leq 1}(V), g_{b,a}, f_{a,b})$ is a subsystem of $(T_a(V), g_{b,a}, f_{a,b})$. The inverse limit is exactly the subset of elements $t$ of $T_\infty(V)$ such that $p_a(t) \in T_a^{\leq 1}(V)$ for all $a \in \mathbb{N}$. We denote it $T_a^{\leq 1}(V)$, and note that its coordinate ring is the direct limit of the coordinate rings $\mathbb{K}[T_a^{\leq 1}(V)]$ under the maps $f_{a,b}^*$, defined as $h \mapsto h \circ f_{a,b}$. We will describe the coordinate rings $\mathbb{K}[T_a^{\leq 1}(V)]$ and consequently describe $\mathbb{K}[T_\infty^{\leq 1}(V)]$.

We will use the characterisation of the ideal $I(T_a^{\leq 1}(V))$ as given in [Grottmann], formulated slightly differently. A bipartition of a set $S$ is a pair of subsets $(I, J)$ such that $I \cup J = S$, $I \cap J = \emptyset$. For a bipartition $(I, J)$ of $[a]$ and two tuples $i \in [n]^I$, $j \in [n]^J$, denote $i \oplus j$ the string $s \in \text{seq}_a(n)$ such that $s_l = i_l$ if $l \in I$ and $s_l = j_l$ if $l \in J$. We say $i_1, i_2, j_1, j_2$ satisfy $(*)_a$ if there is a bipartition $(I, J)$ of $[a]$ such that $i_1, i_2 \in [n]^I$, $j_1, j_2 \in [n]^J$. Then the pure tensors have a vanishing ideal that is characterised as:

$$I \left( T_a^{\leq 1}(V) \right) = \langle x_{i_1 \oplus j_1} x_{i_2 \oplus j_2} - x_{i_1 \oplus j_2} x_{i_2 \oplus j_1} \mid i_1, i_2, j_1, j_2 \text{ satisfy } (*_a) \rangle$$

For $a \in \mathbb{N}$ denote

$$M_{a \times n} = \left\{ M \in \mathbb{Z}_{\geq 0}^{a \times n} \mid \exists d \in \mathbb{Z}_{\geq 0} : \forall k \in [a] : \sum_{i=1}^n M_{k,i} = d \right\}$$

The set of $a \times n$-matrices with constant row sum. These form an additive monoid. Define $\mathbb{K}[y^{M_{a \times n}}] := \mathbb{K}[y^M \mid M \in M_{a \times n}] \subseteq \mathbb{K}[y_k,i \mid k \in [a], i \in [n]]$, the sub-algebra with exponents exactly the matrices of $M_{a \times n}$. We give the following characterisation of $\mathbb{K}[T_a^{\leq 1}(V)]$:

**Theorem 6.** For all $a \in \mathbb{N}$, $\mathbb{K}[T_a^{\leq 1}(V)] \cong \mathbb{K}[y^{M_{a \times n}}]$.

**Proof:** Define $\phi_a : \mathbb{K}[T_a(V)] \to \mathbb{K}[y_k,i \mid k \in [a], i \in [n]]$ as

$$\phi_a(x_i) = \prod_{k=1}^a y_{k,i_k}$$

This is a monomial map and therefore it has a kernel generated by binomials. We claim $\ker \phi_a = I(T_a^{\leq 1}(V))$:

**Claim** $\ker \phi_a \subseteq I(T_a^{\leq 1}(V))$: We only need to check that all binomial generators of $\ker \phi_a$ are in $I(T_a^{\leq 1}(V))$, so let $x^A - x^B$ be such a binomial, with $A, B$ finite multisets in $\text{seq}_a(n)$, and let $t = v_1 \otimes \cdots \otimes v_a \in T_a^{\leq 1}(V)$. Then

$$\phi_a(x^A) = \prod_{i \in A} \phi_a(x_i) = \prod_{i \in A} \prod_{k=1}^a y_{k,i_k}$$

and similarly for $B$. Since $t$ is rank 1, we have for each $i \in \text{seq}_a(n)$: $t_i = \prod_{k=1}^a (v_k)_{i_k}$. Therefore we know that:

$$x^A(t) = \prod_{i \in A} \prod_{k=1}^a (v_k)_{i_k}$$

$$= \left( \prod_{i \in A} \prod_{k=1}^a y_{k,i_k} \right) \begin{pmatrix} v_1 & \vdots & v_a \end{pmatrix} \quad \text{(the polynomial evaluated in)}$$

$$\left( \text{the } a \times n \text{ matrix } (v_1 \cdots v_a)^T \right)$$

$$= \phi_a(x^A)$$

Since $\phi(x^A - x^B) = 0$, we can replace $A$ by $B$ and work the other way around to obtain $x^A(t) = x^B(t)$, meaning $x^A - x^B \in I(T_a^{\leq 1}(V))$. 

\[\square\]
Claim \( \text{I}(T_{a}^{\leq 1}(V)) \subseteq \ker \phi_{a}: \) Again we only need to check for binomial generators. Suppose \((I, J)\) is a bipartition of \([a]\) and \(i \in [n]^{I}, j \in [n]^{J}. \) Then:

\[
\phi_{a}(x_{i} \oplus j) = \prod_{k=1}^{a} y_{k,(i \oplus j)_{k}} = \prod_{k \in I} y_{k,i_{k}} \prod_{k \in J} y_{k,j_{k}}
\]

Therefore if \(i_{1}, i_{2}, j_{1}, j_{2} \) satisfy \((\ast_{a})\) we can easily see that \(\phi_{a}(x_{i_{1} \oplus j_{1}}, x_{i_{2} \oplus j_{2}}) = \phi_{a}(x_{i_{1} \oplus j_{2}}, x_{i_{2} \oplus j_{1}})\). Therefore all binomial generators of \( \text{I}(T_{a}^{\leq 1}(V)) \) are in \( \ker \phi_{a}, \) so \( \text{I}(T_{a}^{\leq 1}(V)) \subseteq \ker \phi_{a}. \)

Since \( \ker \phi_{a} = \text{I}(T_{a}^{\leq 1}(V)) \) we can see by the first isomorphism theorem that \( \mathbb{K}[T_{a}^{\leq 1}(V)] \cong \text{im} \phi_{a}. \) Now we claim that \( \text{im} \phi_{a} = \mathbb{K}[y^{M_{a \times n}}]. \)

Claim \( \text{im} \phi_{a} \subseteq \mathbb{K}[y^{M_{a \times n}}]: \) For \( j \in \text{seq}_{a}(n) \) define \( M^{j} \) as \( M_{k,j_{k}} = 1 \) and 0 elsewhere. Then obviously \( M^{j} \) has constant row sum 1 and thus \( M^{j} \in M_{a \times n}. \) Furthermore, \( \phi_{a}(x_{j}) = y^{M^{j}}. \) Therefore, since each generator is mapped into \( \mathbb{K}[y^{M_{a \times n}}], \) we get \( \text{im} \phi_{a} \subseteq \mathbb{K}[y^{M_{a \times n}}]. \)

Claim \( \mathbb{K}[y^{M_{a \times n}}] \subseteq \text{im} \phi_{a}: \) Suppose \( M \in M_{a \times n}. \) We use induction on the constant row sum \( d \) of \( M. \) If \( d = 0 \) then \( M \) is completely 0, so \( y^{M} = 1 = \phi(1). \) If \( d > 0, \) let for each \( k \in [a], \) \( i_{k} \) be such that \( M_{k,i_{k}} > 0. \) Then define

\[
M'_{k,i} = \begin{cases} 
M_{k,i} - 1 & \text{if } i = i_{k} \\
M_{k,i} & \text{if } i \neq i_{k} 
\end{cases}
\]

Then \( M' \) has constant row sum \( d - 1 \) and thus \( M' \in M_{a \times n}. \) By induction, there is a monomial \( x^{j} \) such that \( \phi_{a}(x^{j}) = y^{M'}. \) Furthermore, if we define \( j \in \text{seq}_{a}(n) \) as \( j_{k} = i_{k} \) then \( \phi_{a}(x_{j}x^{j}) = y^{M}. \) Therefore, \( \mathbb{K}[y^{M_{a \times n}}] \subseteq \text{im} \phi_{a}. \)

Therefore by the first isomorphism theorem we can see that indeed \( \mathbb{K}[T_{a}^{\leq 1}(V)] \cong \mathbb{K}[y^{M_{a \times n}}]. \) \( \blacksquare \)

Notice that the proof above actually shows that each of the monoids \( M_{a \times n} \) is finitely generated by the matrices \( \{M^{j} \mid j \in \text{seq}_{a}(n)\}. \)

Define the monoid \( x_{a \times n} = \{\chi^{M} \mid M \in M_{a \times n}\} \) with \( \chi^{M}\chi^{N} = \chi^{M+N}. \) This monoid is the multiplicative variant of the additive monoid \( M_{a \times n}. \) Then we can see that \( \mathbb{K}[y^{M_{a \times n}}] \cong \mathbb{K}[\chi_{a \times n}] \) by the obvious map \( y^{M} \mapsto \chi^{M} \) extended by linearity and multiplicativity.

The maps \( f^{a}_{a,b} \) map \( x_{i} \in \mathbb{K}[T_{a}(V)] \) to \( x_{a_{b}(i)} \in \mathbb{K}[T_{b}(V)]. \) Under the map \( \phi_{b} \) this is then mapped to \( \chi^{M} \) with \( M_{b,\alpha_{b}(i)} = 1 \) and 0 elsewhere. Since \( \alpha_{b} \) repeats the last element of \( i (b-a) \) times, the corresponding map \( M_{a \times n} \to M_{b \times n} \) sends \( M \) to the matrix \( M \) augmented with \( (b-a) \) rows, all equal to the \( a^{'} \)th (last) row of \( M. \) With similar arguments as in section 3.1.1 the direct limit of these maps of matrices is:

\[
M_{a \times n} := \left\{ M \in \mathbb{Z}_{\geq 0}^{n \times n} \mid \exists d \in \mathbb{Z}_{\geq 0} : \forall k \in \mathbb{N} : \sum_{i=1}^{n} M_{k,i} = d, \exists k_{0} \in \mathbb{N} : \forall k \geq k_{0} : M_{k} = M_{k_{0}} \right\}
\]

with morphisms mapping a finite matrix to its infinite augmentation by repeating the last row ad infinitum. This results in maps \( x_{a \times n} \to x_{b \times n} \) and thereby \( \mathbb{K}[\chi_{a \times n}] \to \mathbb{K}[\chi_{b \times n}] \) that correspond to the maps \( f^{a}_{a,b}. \) Its direct limit is the algebra \( \mathbb{K}[\chi_{a \times n}] := \mathbb{K}[\chi^{M} \mid M \in M_{a \times n}]. \) Therefore we can see the equivalence \( \mathbb{K}[T_{a}^{\leq 1}(V)] \cong \mathbb{K}[\chi_{a \times n}]. \) Now if we define for \( j \in \text{seq}_{a}(n) \) the matrix \( M^{j} \) as \( M^{j}_{k,j_{k}} = 1 \) and 0 elsewhere, we can see that this equivalence is witnessed by the following map

\[
\phi_{\infty} : \mathbb{K}[T_{a}(V)] \to \mathbb{K}[\chi_{a}] \quad \text{defined as} \quad \phi_{\infty}(x_{j}) = \chi^{M^{j}}
\]

Furthermore, we can see that if we let \( \text{Subs}_{a}^{\infty}(\mathbb{N}) \) act on \( M_{a \times n} \) by acting on each column separately (or equivalently, on the sequence of rows), and define \( \sigma \chi^{M} = \chi^{\sigma M}, \) the map \( \phi_{\infty} \) is \( \text{Subs}_{a}^{\infty}(\mathbb{N}) \)-equivariant. We need the action of \( \text{Subs}_{a}^{\infty}(\mathbb{N}) \) here, and not a bigger monoid, because if \( \text{im} \sigma \neq \mathbb{N} \) then \( \sigma M \) has a row completely 0, and unless \( M \) is the zero matrix, the resulting matrix does not have constant row sums. The requirement max \( \sigma_{1} < \max \sigma_{2} < \ldots \) is necessary for the following.
By analogy with \( \text{seq}^{	ext{fn}}(n) \), for \( M \in M_{\mathbb{N} \times \mathbb{N}} \) denote \( M_\infty \) the stabilising row, i.e. \( M_{k_0} \) for \( k_0 \) such that \( \forall k \geq k_0 : M_k = M_{k_0} \). Then \( (\sigma M)_\infty = M_\infty \) for \( \sigma \in \text{Subs}^f(\mathbb{N}) \), and \( (M + N)_\infty = M_\infty + N_\infty \). Now denote

\[
s(M) = \min\{k \in \mathbb{N} | \forall l \geq k : M_l = M_\infty\}
\]

Then \( s(M) \) is the minimum index such that the matrix is constant from that index on. First we give the following lemma about \( s(M) \):

**Lemma 1.** For all \( M, N \in M_{\mathbb{N} \times \mathbb{N}} \) and \( \sigma \in \text{Subs}^f(\mathbb{N}) \) we have:

1. \( s(\sigma M) = 1 + \max \sigma s(M) - 1 \)
2. \( s(M + N) \leq \max\{s(M), s(N)\} \)

**Proof:** (1): Since \( \max \sigma s(M) - 1 \in \sigma s(M) - 1 \) we have \( (\sigma M)_{\max \sigma s(M) - 1} = M_{\sigma s(M) - 1} \neq M_\infty \), so \( s(\sigma M) > \max \sigma s(M) - 1 \). On the other hand, if \( k \geq 1 + \max \sigma s(M) - 1 \) then \( k \in \sigma M \) for some \( m \geq s(M) \). Therefore \( (\sigma M)_k = M_m = M_\infty \) because \( m \geq s(M) \). Thus also \( s(\sigma M) \leq 1 + \max \sigma s(M) - 1 \). Therefore we obtain \( s(M) = 1 + \max \sigma s(M) - 1 \).

(2): Let \( k \geq \max\{s(M), s(N)\} \). Then \( (M + N)_k = M_k + N_k = M_\infty + N_\infty = (M + N)_\infty \). Therefore \( s(M + N) \leq \max\{s(M), s(N)\} \).

We lift this notation to the monoid \( \chi_{\mathbb{N} \times \mathbb{N}} \) as \( s(\chi M) = s(M) \) and obtain the following lemma:

**Lemma 2.** For all \( x, y \in \chi_{\mathbb{N} \times \mathbb{N}} \) and \( \sigma \in \text{Subs}^f(\mathbb{N}) \) we have:

1. \( s(\sigma x) = 1 + \max \sigma x - 1 \)
2. \( s(xy) \leq \max\{s(x), s(y)\} \)

Note that it is possible that \( s(M + N) < \max\{s(M), s(N)\} \), and in fact there is no lower bound on \( s(M + N) \) in terms of \( s(M) \) and \( s(N) \) as witnessed by the following example:

**Example 1.** Let \( n = 2 \) and \( e_1 = (1, 0), e_2 = (0, 1) \in \mathbb{Z}^n_{\geq 0} \). For \( m \in \mathbb{N} \) set \( M^{(m)}_k = e_1 \) if \( k = m \) and \( e_2 \) else, and \( N^{(m)}_k = e_2 \) if \( k = m \) and \( e_1 \) else. Then \( M^{(m)} + N^{(m)} \) is the constant matrix with every row \( e_1 + e_2 \) and so \( s(M^{(m)} + N^{(m)}) = 1 \), but \( s(M^{(m)}) = s(N^{(m)}) = m + 1 \) can grow arbitrarily large.

### 3.4 Well-partial orders

We recall the notion of a well-partial order and characterisations of it, which are taken from [AP04], though a well-partial order is called a Noetherian order there. Then we present a result about Noetherianness of a monoid algebra under certain constraints. The setting of this result is close to that of [HS12] Theorem 2.12, Theorem 2.19], but these theorems do not apply directly.

**Definition 1.** A partial order \( \leq \) on a set \( S \) is called a well-partial-order if there are no infinite strictly decreasing sequences \( x_1 > x_2 > x_3 > \ldots \) and no infinite sequences \( x_1, x_2, x_3, \ldots \) of pairwise incomparable elements, i.e. such that \( x_j \not\leq x_i \) for all \( i \neq j \).

A set \( S \) with a partial order is also called an ordered set, and we call \( S \) well-partially-ordered (or w.p.o.) if \( \leq \) is a well-partial order. We call an infinite sequence \( x_1, x_2, x_3, \ldots \) good if there are indices \( i < j \) such that \( x_i \leq x_j \), and bad if there are no such indices. We note the following characterisations:

**Theorem 7.** The following are equivalent for an ordered set \( S \):

1. \( S \) is well-partially-ordered.
2. Every infinite sequence \( x_1, x_2, x_3, \ldots \) in \( S \) contains an infinite increasing subsequence.
3. Every sequence \( x_1, x_2, x_3, \ldots \) in \( S \) is good.
4. Every upward closed set \( T \) (i.e. if \( x \leq y, x \in T \) then \( y \in T \)) has a finite number of generators.
   That is, there are elements \( t_1, \ldots, t_n \in T \) such that for all \( x \in T \) there is an \( i \in [n] : t_i \leq x \).
There are more known equivalent conditions, but these will suffice for our needs. By these equivalences, if we need to show a set is well-partially-ordered we only need to show that every sequence is good, i.e. there are such indices $i < j$. However, if we know that a set it is well-partially-ordered, we can take infinite increasing sequences in any infinite sequence. This will turn out to be very useful when proving well-partially-orderedness.

The above definition and conditions still apply when the order is not a partial order, but a quasi-order, where we drop the anti-symmetry requirement. We call a quasi-order a well-quasi-order (w.q.o.) when it satisfies item 2 or equivalently item 3 or item 4 of theorem 7. A set $S$ with a well-quasi-order is called well-quasi-ordered.

We would like to use Theorem 2.12 and Theorem 2.19 from [HS12] to prove that $K[T_\leq \leq V]$ is $\text{Subs}\_f(N)$-Noetherian, but the prerequisites of neither theorem are satisfied, because it would require a well-order on $M_{N \times n}$ that is additive and such that the $\text{Subs}\_f(N)$-action preserves strict inequalities. It is not possible to satisfy all three of these constraints:

**Proof:** Let $x$ and $y$ be two distinct elements of $\mathbb{Z}_0^n$. Define $\sigma = (\{1, 2\}, \{3\}, \{4\}, \{5\}, \ldots) \in \text{Subs}_f(N)$. Then necessarily $(x, y, y, \ldots) < (x, x, y, y, \ldots)$ because otherwise we would get the infinite descending chain $(x, y, y, \ldots) > (x, x, y, y, \ldots) > (x, x, x, y, y, \ldots) > \ldots$ by the action of $\sigma$. This would contradict the fact that the order is a well-order. Similarly we also need $(y, x, x, x, \ldots) < (y, y, x, x, \ldots)$. Then by additivity of the order we get:

$$
(x + y, x + y, x + y, \ldots) =
\begin{cases}
(x, y, y, \ldots) + (y, x, x, \ldots) \\
(x, x, y, y, \ldots) + (y, x, x, \ldots) \\
(x, x, x, y, \ldots) + (y, x, x, \ldots) \\
(x + y, x + y, x + y, \ldots)
\end{cases}
$$

This is a contradiction and so no such order can exist.

However, we will still be able to prove the $\text{Subs}_f(N)$-Noetherianness of $K[T_\leq \leq V]$ by giving a slightly different setting than that of Theorem 2.19. Let $F$ be a field, $Q$ a monoid and $P$ a monoid of algebra endomorphisms of $F[Q]$. We give the following two definitions, inspired by those of $P$-orders and filtrations in [HS12, Definition 2.2, Definition 2.15]:

**Definition 2.** We call a total order $\leq$ on $Q$ a pre-$P$-order if for all $q \in Q, p \in P, f \in F[Q]$ we have $LM_{\leq}(q \cdot p \cdot f) = LM_{\leq}(q \cdot p \cdot LM_{\leq}(f))$. Here $LM_{\leq}(f)$ is the leading monomial of the polynomial $f$ under the total order $\leq$.

**Definition 3.** Suppose that $Q_n \subseteq Q, P_{n,m} \subseteq P$ for $n, m \in \mathbb{N}$. We say that $Q_n, P_{n,m}$ is a pre-filtration if the following conditions are met:

1. Each $Q_n$ is a submonoid of $Q$
2. $Q_n \subseteq Q_{n+1}$ for each $n \in \mathbb{N}$.
3. $Q = \bigcup_{n \in \mathbb{N}} Q_n, P = \bigcup_{n,m \in \mathbb{N}} P_{n,m}$
4. $P_{n,m}Q_m \subseteq F[Q_n]$ for all $n, m \in \mathbb{N}$
5. Each $P_{n,n}$ contains the identity endomorphism
6. $P_{k,n} \cdot P_{n,m} \subseteq P_{k,m}$ for $m, n, k \in \mathbb{N}$

Note that a pre-$P$-order is a $P$-order without the requirement that it is a well-order. And a pre-filtration is a filtration with the following modifications:

1. Requirements 6 and 7 are dropped.
2. Only each $P_{n,n}$ needs to include the identity endomorphism, rather than all $P_{n,m}$.
3. The transitivity requirement in item 6 is added.
Furthermore, a pre-filtration does not depend on any (pre-)\(P\)-order. Now suppose a pre-\(P\)-order on \(Q\) and a pre-filtration of \(Q, P\) is given. We define a quasi-order on \(Q' := \bigcup_{n \in \mathbb{N}} Q_n \times \{n\}\), the disjoint union of the \(Q_n\) as:

\[
(q_1, m) \leq (q_2, n) \quad \iff \quad \exists p \in P_{n,m}, q \in Q_n : q \cdot LM_\leq (p \cdot q_1) = q_2
\]

It is easy to see that this is a reflexive and transitive relation by the requirements of a pre-filtration. We give the following theorem:

**Theorem 8.** Suppose \(\leq\) is a pre-\(P\)-order on \(Q\) and \(Q_n, P_{n,m}\) a pre-filtration such that for each \(n \in \mathbb{N}\) the restriction of \(\leq\) to \(Q_n\) is a well-order, and \(|\Sigma|\) is a w.q.o.. Then every \(P\)-stable ideal \(I\) of \(F[Q]\) is generated by the \(P\)-orbits of finitely many polynomials, i.e. \(F[Q]\) is \(P\)-Noetherian.

**Proof:** Since each \(Q_n\) is a submonoid of \(Q\), the algebra \(F[Q_n]\) is a subalgebra of \(F[Q]\) and the set \(I_n := I \cap F[Q_n]\) is an ideal in \(F[Q_n]\). We see that \(I = \bigcup_{n \in \mathbb{N}} I_n\) because \(Q = \bigcup_{n \in \mathbb{N}} Q_n\) and that \(I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots\) because of the chain of monoids \(Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \cdots\).

Set \(G = \bigcup_{n \in \mathbb{N}} (LM_\leq (f), n) | f \in I_n \subseteq Q'\). Then \(G\) is upward closed, for suppose that we have \((LM_\leq (f), n) \in G\) and \((LM_\leq (f'), n) \in \Sigma(q, m)\). Let \(p \in P_{m,n}, q' \in Q_m\) such that:

\[
q = q' \cdot LM_\leq (p \cdot LM_\leq (f)) = LM_\leq (q' \cdot p \cdot LM_\leq (f)) = LM_\leq (q' \cdot p \cdot f)
\]

Then \(q = LM_\leq (g)\) for \(g = q' \cdot p \cdot f \in I_m\). So indeed \(G\) is upward closed. Therefore, since \(|\Sigma|\) is w.q.o., there is a finite number of elements \((LM_\leq (g_1), n_1), \ldots, (LM_\leq (g_k), n_k) \in G\) such that for each \(n \in \mathbb{N}\) and \(f \in I_n\) there is an \(i \in [k]\) such that \((LM_\leq (g_i), n_i)) \leq (LM_\leq (f), n)\), meaning that there is a \(p \in P_{n,n_i}, q \in Q_n\) such that \(LM_\leq (f) = q \cdot LM_\leq (p \cdot g_i)\). Since \(g_i \in I_{n_i}\), we have \(p \cdot g_i \in I_n\) and therefore \(q \cdot p \cdot g_i \in I_n\). Then there is an \(e \in F\) such that \(f' := f - cq \cdot p \cdot g_i\) has \(LM_\leq (f') < LM_\leq (f)\). However, since \(f, cq \cdot p \cdot g_i \in I_n\) we have \(f' \in I_n\). On the other hand, \(cq \cdot p \cdot g_i \in I'\) so that \(f' \not\in I'\). Therefore \(f' \in I_n \setminus I'\), but then \(LM_\leq (f') < LM_\leq (f)\) is a contradiction since we assumed \(LM_\leq (f)\) to be minimal.

Therefore \(I = I'\) and thus \(I\) is generated by the \(P\)-orbits of \(g_1, \ldots, g_k\). Therefore \(F[Q]\) is \(P\)-Noetherian. \(\blacksquare\)

Note if \(\leq\) is a pre-\(P\)-order on \(Q\), then \(Q\) is left-cancellative: If \(qq_1 = qq_2\) and \(q_1 < q_2\) then:

\[
qq_2 = LM_\leq (qq_2) = LM_\leq (q \cdot LM_\leq (q_2 - q_1)) = LM_\leq (q(q_2 - q_1)) = LM_\leq (qq_2 - qq_1) = LM_\leq (0) = 0
\]

But since \(0 \not\in Q\) this is a contradiction, and by symmetry we then get \(q_1 = q_2\). Then we can see that a pre-\(P\)-order is multiplicative. That is, \(q_1 < q_2 \Rightarrow qq_1 < qq_2\), because:

\[
LM_\leq (qq_1 + qq_2) = LM_\leq (q(q_1 + q_2)) = LM_\leq (q \cdot LM_\leq (q_1 + q_2)) = LM_\leq (qq_2) = qq_2
\]

Therefore \(qq_1 \leq qq_2\), and we cannot have \(qq_1 = qq_2\) as then \(q_1 = q_2\). So we must have \(qq_1 < qq_2\). Next for a total order \(\preceq\) on \(Q\), if we have:

\[
\forall q, q_1, q_2 \in Q, p \in P : q_1 < q_2 \Rightarrow LM_\leq (qppq_1) < LM_\leq (qppq_2)
\]

then \(\leq\) is a pre-\(P\)-order. This follows exactly the proof of Proposition 2.4 of [HST2] in the if-direction, because it does not use the properties of a well-order.

We say \(P\) acts by monomial endomorphisms if it sends elements of \(Q\) to scalar multiples of elements of \(Q\). If we assume \(P\) works by monomial endomorphisms we get that \(\leq\) is a pre-\(P\)-order if \(q_1 < q_2\) implies \(qq_1 < qq_2\).
3.5 Noetherianness of $\mathbb{K}[T_{\infty}^{\leq 1}(V)]$

We will show that the coordinate ring $\mathbb{K}[T_{\infty}^{\leq 1}(V)] \simeq \mathbb{K}[\chi_{\mathbb{N} \times \mathbb{N}}]$ is Subs$^s_<(\mathbb{N})$-Noetherian. This implies that the ideal of every Subs$^s_<(\mathbb{N})$-stable sub-variety of $T_{\infty}^{\leq 1}(V)$ is generated by the Subs$^s_<(\mathbb{N})$-orbits of finitely many polynomials in $\mathbb{K}[T_{\infty}(V)]$.

Since the map $\phi_\infty : \mathbb{K}[T_{\infty}^{\leq 1}(V)] \to \mathbb{K}[\chi_{\mathbb{N} \times \mathbb{N}}]$ that induces the isomorphism is Subs$^s_<(\mathbb{N})$-equivariant, we can say that $\mathbb{K}[T_{\infty}^{\leq 1}(V)]$ is Subs$^s_<(\mathbb{N})$-Noetherian if and only if $\mathbb{K}[\chi_{\mathbb{N} \times \mathbb{N}}]$ is. We will interpret the algebra $\mathbb{K}[\chi_{\mathbb{N} \times \mathbb{N}}]$ in the setting of theorem [8] verify all properties, and in the end conclude that indeed $\mathbb{K}[T_{\infty}^{\leq 1}(V)]$ is Subs$^s_<(\mathbb{N})$-Noetherian.

In our case the monoid $Q$ is given as the monoid $\chi_{\mathbb{N} \times \mathbb{N}}$ and $P$ is given as Subs$^s_<(\mathbb{N})$. Since Subs$^s_<(\mathbb{N})$ acts on $\chi_{\mathbb{N} \times \mathbb{N}}$, which we extended to endomorphisms on $\mathbb{K}[\chi_{\mathbb{N} \times \mathbb{N}}]$, it is indeed a monoid of monomial endomorphisms. Therefore, if we have a total order $\leq$ on $Q$ it is a pre-$P$-order if $\chi^M < \chi^N$ implies $\chi^R \sigma \chi^M < \chi^R \sigma \chi^N$ for all matrices $M, N, R \in M_{\mathbb{N} \times \mathbb{N}}$ and $\sigma \in$ Subs$^s_<(\mathbb{N})$. From now on, denote $Q = \chi_{\mathbb{N} \times \mathbb{N}}$ and $P =$ Subs$^s_<(\mathbb{N})$. First we give a pre-filtration of $Q, P$:

**Theorem 9.** Set $Q_a = \{ \chi^M \in Q \mid s(\chi^M) \leq a \}$ and $P_{b,a} = \{ \sigma \in P \mid \max \sigma_a = b \}$. Then $Q_a, P_{b,a}$ is a pre-filtration of $Q, P$.

**Proof:** We will verify all properties of definition [3].

1. By lemma [1] if $s(\chi^M), s(\chi^N) \leq a$ then $s(\chi^M \chi^N) \leq \max \{ s(\chi^M), s(\chi^N) \} \leq a$. Furthermore, the zero matrix $\tilde{O}$ has $s(\tilde{O}) = 1$ and so $\tilde{O} \in Q_a$. Therefore each $Q_a$ is a submonoid of $Q$.

2. Clearly $Q_a \subseteq Q_{a+1}$ for all $a \in \mathbb{N}$ because $s(\chi^M) \leq a$ surely means $s(\chi^M) \leq a + 1$.

3. If $\chi^M \in Q$ then $\chi^M \in Q_{s(\chi^M)}$. Therefore $Q = \bigcup_{a \in \mathbb{N}} Q_a$. Furthermore, we have $\bigcup_{a,b} P_{b,a} = P$ because if $\sigma \in P$ then $\sigma \in P_{\max \sigma_a, 1}$.

4. Also by lemma [1] we have for $\sigma \in P_{b,a}, \chi^M \in Q_a$ that $s(\sigma \chi^M) = 1 + \max \sigma_{s(\chi^M) - 1} \leq \max \sigma_{s(\chi^M)} \leq \max \sigma_a = b$, meaning $\sigma \chi^M \in Q_b$, and so $P_{b,a} Q_a \subseteq Q_b$.

5. If $\sigma$ is the identity element of $P$ then $\sigma_a = \{ a \}$ and therefore $\max \sigma_a = \max \{ a \} = a$. Therefore $\sigma \in P_{a,a}$ for all $a \in \mathbb{N}$.

6. Finally, if $\sigma \in P_{c,b}, \pi \in P_{b,a}$ then:

$$\max(\sigma \pi)_a = \max \left( \bigcup_{k \in \pi_a} \sigma_k \right) = \max(\sigma_{\max \pi_a}) = \max \sigma_b = c$$

Therefore all requirements of a pre-filtration are met.

Next we give a pre-$P$-order on $Q$ that satisfies the requirements of theorem [8] with respect to the pre-filtration above. First define $T = M_{\mathbb{N} \times \mathbb{N}}$ and $T_a = \{ M \in T \mid s(M) \leq a \}$. Since $T$ is isomorphic to $Q$, it suffices to give an order on $T$ with the desired properties, because then we derive an order the $Q$ as $\chi^M \leq \chi^N \iff M \leq N$ that satisfies the requirements.

The lexicographic order $\leq_{\text{lex}}$ on $\mathbb{Z}_{\geq 0}^n$ is an additive well-order on rows of the matrices of $T$. Using this well-order we can give the following order on $T$:

**Theorem 10.** Define an order on $T$ as:

$$M < N \iff \begin{cases} M_{\infty} <_{\text{lex}} N_{\infty}, \text{or} \\ M_{\infty} = N_{\infty} \text{ and } \exists k \in \mathbb{N} : M_k <_{\text{lex}} N_k, \forall l > k : M_l = N_l \end{cases}$$

This is a strict total order. Its reflexive closure is a pre-$P$-order on $T$ that is a well-order when restricted to each $T_a$. 

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Proof: Strict total order: Note that \( M \neq M \) as that would mean \( M_\infty <_{\text{lex}} M_\infty \) or \( M_k <_{\text{lex}} M_k \) for some \( k \in \mathbb{N} \). Therefore the order is irreflexive.

To see that it is total, assume \( M \neq N \). If \( M_\infty \neq N_\infty \) then since \( \leq_{\text{lex}} \) is a total order on rows, either \( M_\infty <_{\text{lex}} N_\infty \) or \( N_\infty <_{\text{lex}} M_\infty \), meaning \( M < N \) or \( N < M \) respectively. So suppose that \( M_\infty = N_\infty \). Since \( M \neq N \) there must be a \( k \) such that \( M_k \neq N_k \). Therefore the set \( \{ k \in \mathbb{N} | M_k \neq N_k \} \) is non-empty and bounded by \( \max \{ s(M), s(N) \} \), so it has a maximum, say \( K \). Then \( M_K \neq N_K \) and \( M_l = N_l \) for \( l > K \). Then \( K \) is readily a witness to either \( M < N \) (if \( M_K <_{\text{lex}} N_K \)) or \( N < M \) (if \( N_K <_{\text{lex}} M_K \)). Therefore the order is total.

To see that the order is transitive, let \( M < N < R \). If \( M_\infty <_{\text{lex}} N_\infty \) or \( N_\infty <_{\text{lex}} R_\infty \) then by transitivity of \( \leq_{\text{lex}} \) we can see that \( M_\infty <_{\text{lex}} R_\infty \) and so \( M < R \). So assume \( M_\infty = N_\infty = R_\infty \). Let \( k_1, k_2 \) be witnesses to \( M < N \) and \( N < R \) respectively. Take \( k = \max \{ k_1, k_2 \} \). Then for \( l > k \) we have \( l > k_1, k_2 \) and so \( M_l = N_l = R_l \), meaning \( M_l = R_l \). Furthermore:

- (a) If \( k_1 < k_2 \) then \( M_k = N_k <_{\text{lex}} R_k \)
- (b) If \( k_1 > k_2 \) then \( M_k <_{\text{lex}} N_k = R_k \)
- (c) If \( k_1 = k_2 \) then \( M_k <_{\text{lex}} N_k <_{\text{lex}} R_k \)

Therefore by the transitivity of \( \leq_{\text{lex}} \), \( M_k <_{\text{lex}} R_k \) and so \( M < R \).

So consider \( \leq \), the reflexive closure of \( < \) on \( T \). This is then a total order on \( T \).

Pre-\( P \)-order: Let \( M < N \) and \( R \in T \). If \( M_\infty <_{\text{lex}} N_\infty \) then by additivity of \( <_{\text{lex}} \) we have:

\[
(\sigma M + R)_\infty = (\sigma M)_\infty + R_\infty = M_\infty + R_\infty <_{\text{lex}} N_\infty + R_\infty = (\sigma N)_\infty + R_\infty = (\sigma N + R)_\infty
\]

and so \( \sigma M + R < \sigma N + R \). So suppose \( M_\infty = N_\infty \). Then \( (\sigma M + R)_\infty = (\sigma N + R)_\infty \). Let \( k \) be a witness to \( M < N \) and set \( K = \max \sigma_k \). Then we have:

\[
(\sigma M + R)_K = (\sigma M)_K + R_K = M_K + R_K < N_K + R_K = (\sigma N)_K + R_K = (\sigma N + R)_K
\]

On the other hand if \( L > K \) then \( L \in \sigma_i \) for some \( l > k \) and so:

\[
(\sigma M + R)_L = (\sigma M)_L + R_L = M_l + R_l = N_l + R_l = (\sigma N)_L + R_L = (\sigma N + R)_L
\]

Therefore \( \leq \) is a pre-\( P \)-order.

Well-order on \( T_a \): Suppose that it is not a well-order. Then there is an infinite strictly decreasing sequence \( M^{(1)}, M^{(2)}, \ldots \) in \( T_a \). Then \( M^{(1)} \geq_{\text{lex}} M^{(2)} \geq_{\text{lex}} M^{(3)} \geq_{\text{lex}} \ldots \) and since \( \leq_{\text{lex}} \) is a well-order, this chain stabilises. So let \( i_0 \) such that \( M^{(i_0)} = M^{(i_0)} \) for all \( i \geq i_0 \). Then there exist witnesses \( k_i \) for \( i \geq i_0 \) such that \( M^{(i)}_{k_i} >_{\text{lex}} M^{(i+1)}_{k_i} \) for all \( i \geq i_0 \). These elements form an ascending chain \( k_i \leq k_{i+1} \leq \ldots \) in \( \mathbb{N} \). However, they are bounded by \( a \) because each matrix \( M^{(i)} \) stabilises at least from row \( a \) on. Therefore this ascending chain must stabilise to an element \( k \) from \( j_0 \) on. That is, for all \( j \geq j_0 \) \( M^{(j)}_{k_{j}} >_{\text{lex}} M^{(j+1)}_{k_{j}} \). Then we have an infinite strictly decreasing chain \( M^{(j_0)} >_{\text{lex}} M^{(j_0+1)} >_{\text{lex}} \ldots \). This is a contradiction with the fact that \( \leq_{\text{lex}} \) is a well-order. Therefore \( \leq \) is a well-order when restricted to \( T_a \).

So all the properties are satisfied.

Therefore we have a pre-\( P \)-order on \( Q \) that satisfies the requirements of theorem. So the only remaining thing to prove is that the following quasi-order on \( Q' := \bigcup_{a \in \mathbb{N}} Q_a \times \{ a \} \) is w.q.o.:

\[
(x^M, a) \leq_{\Sigma} (x^N, b) \iff \exists x^R \in Q_b, \sigma \in P_{b,a} : x^R \sigma x^M = x^N
\]

For notational convenience, we will prove this on the level of matrices. Then we need to prove that the following quasi-order on \( T' := \bigcup_{a \in \mathbb{N}} T_a \times \{ a \} \) is w.q.o.:

\[
(M, a) \leq_{\Sigma} (N, b) \iff \exists R \in T_b, \sigma \in P_{b,a} : R + \sigma M = N
\]

Clearly \( \leq_{\Sigma} \) is w.q.o. on \( T' \) if and only if \( |\Sigma| \) is w.q.o. on \( Q' \). If \( (M, a) \leq_{\Sigma} (N, b) \) and \( \sigma M + R = N \) for \( \sigma \in P_{b,a}, R \in T_b \), then we call \( (\sigma, R) \) a witness to \( M \leq_{\Sigma} N \). Note that since \( P_{b,a} \) is empty for \( b < a \) we have that \( (M, a) \leq_{\Sigma} (N, b) \) implies \( a \leq b \). We need the following two theorems:
Theorem 11. $\mathbb{Z}_{\geq 0}^n$ with component-wise partial order $\preceq$ is w.p.o. ("Dickson’s Lemma")

Theorem 12. The set $\mathcal{I}$ of monomial ideals in $\mathbb{K}[x_1, \ldots, x_n]$ partially ordered by reverse inclusion ($\supseteq$) is w.p.o.

The second is proven in [Mac01] and another elegant and general proof is given in [AP04]. Now we introduce notation and two further lemma’s. For $M = (M_1, M_2, M_3, \ldots) \in M_{N \times n}$, with the $M_i$ the rows of $M$, denote $\overline{M} = (M_2, M_3, \ldots) \in M_{N \times n}$, the matrix in $M_{N \times n}$ with the first row removed. Then we have $s(\overline{M}) = s(M) - 1$ unless $s(M) = 1$, in which case $s(M) = s(\overline{M}) = 1$. Therefore $s(M) \leq s(\overline{M}) + 1$. Furthermore, if $(M, a) \in T'$ with $s(M) > 1$ then $a \geq s(M) > 1$ and $a - 1 \geq s(\overline{M}) \geq 1$, so $(\overline{M}, a - 1) \in T'$.

Lemma 3. Let $(M, a), (N, b) \in T'$ with $s(M), s(N) > 1$. If $(\overline{M}, a - 1) \leq_{\Sigma} (\overline{N}, b - 1)$ and $M_1 \leq N_1$ in $\mathbb{Z}_{\geq 0}$ then $(M, a) \leq_{\Sigma} (N, b)$.

Proof: Let $(\sigma, R)$ be a witness to $(\overline{M}, a - 1) \leq_{\Sigma} (\overline{N}, b - 1)$. Then define:

$$
\pi_i = \begin{cases}
\{1\} & \text{if } i = 1 \\
\{j + 1 \mid j \in \sigma_{i-1}\} & \text{if } i > 1
\end{cases}
$$

and

$$
R'_i = \begin{cases}
N_1 - M_1 & \text{if } i = 1 \\
R_{i-1} & \text{if } i > 1
\end{cases}
$$

Indeed $R'$ has constant column sum. Note that $\overline{R'} = R$, and therefore $s(R') \leq s(R) + 1 \leq b - 1 + 1 = b$. Furthermore, $\max \pi_a = \max \sigma_{a-1} + 1 = b - 1 + 1 = b$. Therefore $R' \in T_b$ and $\pi \in P_{b,a}$ and finally, indeed we have $\pi M + R' = N$, meaning $(M, a) \leq_{\Sigma} (N, b)$.

Lemma 4. Let $(M, a), (N, b) \in T'$ with $s(M) > 1$. If $(M, a) \leq_{\Sigma} (N, b - 1)$ and $M_m \leq N_1$ in $\mathbb{Z}_{\geq 0}$ for some $m \in \mathbb{N}$ then $(M, a) \leq_{\Sigma} (N, b)$.

Proof: Let $(\sigma, R)$ be a witness to $(M, a) \leq_{\Sigma} (N, b - 1)$. Define:

$$
\pi_i = \begin{cases}
\{j + 1 \mid j \in \sigma_i\} & \text{if } i \neq m \\
\{j + 1 \mid j \in \sigma_i\} \cup \{1\} & \text{if } i = m
\end{cases}
$$

and

$$
R'_i = \begin{cases}
N_1 - M_m & \text{if } i = 1 \\
R_{i-1} & \text{if } i > 1
\end{cases}
$$

Indeed $R'$ has constant column sum. Note that $\overline{R'} = R$, and therefore $s(R') \leq s(R) + 1 \leq b - 1 + 1 = b$. Furthermore, $\max \pi_a = \max \sigma_{a-1} + 1 = b - 1 + 1 = b$. Therefore $R' \in T_b$ and $\pi \in P_{b,a}$ and finally, indeed we have $\pi M + R' = N$, meaning $(M, a) \leq_{\Sigma} (N, b)$.

Note that we need $s(N), s(M) > 1$ and $s(N) > 1$ respectively, otherwise we could have $a = 1$ or $b = 1$ in which case the prerequisites of the lemmas are not well-formed.

Now the proof that $\leq_{\Sigma}$ is a w.p.o. follows by an argument similar to the one in [NW63] for Kruskals Tree Theorem, by choosing a bad sequence in a suitable minimal way, and deriving a contradiction.

Proof: Suppose $\leq_{\Sigma}$ on $T'$ is not w.p.o., then there are bad sequences. For $M \in M_{N \times n}$ denote $I(M) = (x^{M_i} \mid i \in \mathbb{N}) \subset \mathbb{K}[x_1, \ldots, x_n]$, the monomial ideal generated by monomials with exponent vectors the rows of $M$. Since such monomial ideals are w.p.o., in a bad sequence $(M^{(1)}, a_1), (M^{(2)}, a_2), \ldots, (M^{(i)}, a_i), \ldots$, there is an infinite subsequence $(M^{(i_1)}, a_{i_1}), (M^{(i_2)}, a_{i_2}), \ldots$ such that $I(M^{(i_1)}) \supseteq I(M^{(i_2)}) \supseteq \cdots$, and then this subsequence is still bad. We consider the set of such bad sequences that induce such a weakly descending chain on their monomial ideals, and call such sequences “ideally bad sequences”.

Suppose that in a sequence $(M^{(1)}, a_1), (M^{(2)}, a_2), \ldots$, there are infinitely many indices $i$ such that $s(M^{(i)}) = 1$. Take the infinite subsequence of these elements. Since $\mathbb{N}$ is well-ordered (and therefore w.p.o.), and so is $\mathbb{Z}_{\geq 0}$ by component-wise ordering, we can take an two indices $i < j$ such that $M^{(i)}_1 \leq M^{(j)}_1$ and $a_i \leq a_j$. Then set the following witnesses:

$$
\sigma_k = \begin{cases}
\{k\} & \text{if } k < a_i \\
\{a_i, \ldots, a_j\} & \text{if } k = a_i \\
\{a_j - a_i + k\} & \text{if } k > a_i
\end{cases}
$$

and $\forall l \in \mathbb{N} : R_l = M^{(i)}_1 - M^{(i)}_1$.
Then we have \( \sigma \in P_{\lambda, \mu} \) and \( R \in Q_{\mu} \) with \( \sigma M^{(i)} + R = M^{(j)} \). Therefore \( M^{(i)} \leq_{\Sigma} M^{(j)} \), and so such a sequence is good. Therefore in a bad sequence there exists an index \( i_0 \) such that for \( i \geq i_0 \), 
\[ s \left( M^{(i)} \right) > 1. \]

Now choose an ideally bad sequence \( S = \left( \left( M^{(1)}, a_1 \right), \left( M^{(2)}, a_2 \right), \ldots \right) \) such that:

1. \( s \left( M^{(1)} \right) \) is minimal
2. \( s \left( M^{(i+1)} \right) \) is minimal among all ideally bad sequences that start with \( \left( M^{(1)}, a_1 \right), \ldots, \left( M^{(i)}, a_i \right) \).

By Dickson’s Lemma, there is an infinite sequence \( i_1 < i_2 < \cdots \) such that \( M^{(i_1)} \leq_{\Sigma} M^{(i_2)} \leq \cdots \), and we take this sequence such that \( i_1 \geq i_0 \). Now consider the sequence \( \left( M^{(i_1)}, a_{i_1} - 1 \right), \left( M^{(i_2)}, a_{i_2} - 1 \right), \ldots \).

This sequence is bad by lemma 3. Within this subsequence we take another infinite subsequence \( j_1 < j_2 < \cdots \) such that \( \left( M^{(j_1)}, a_{j_1} - 1 \right), \left( M^{(j_2)}, a_{j_2} - 1 \right), \ldots \) is an ideally bad sequence (i.e. \( I \left( M^{(j_1)} \right) \supseteq I \left( M^{(j_2)} \right) \supseteq \ldots \)).

We claim that the following sequence is ideally bad:
\[ S' = \left( \left( M^{(1)}, a_1 \right), \ldots, \left( M^{(j_1-1)}, a_{j_1-1} \right), \left( M^{(j_1)}, a_{j_1} - 1 \right), \left( M^{(j_2)}, a_{j_2} - 1 \right), \ldots \right) \]

To show that it is bad, note that the first part, \( \left( \left( M^{(1)}, a_1 \right), \ldots, \left( M^{(j_1-1)}, a_{j_1-1} \right) \right) \), can not be good because \( S \) is bad, and neither can the second part \( \left( \left( M^{(j_1)}, a_{j_1} - 1 \right), \left( M^{(j_2)}, a_{j_2} - 1 \right), \ldots \right) \), as shown above. Therefore if it would be a good sequence, then \( I \left( M^{(i)}, a_i \right) \leq_{\Sigma} I \left( M^{(j)}, a_j \right) \) for some \( i < j_1 \) and \( k \in \mathbb{N} \). However, by the assumption that \( S \) is ideally bad, we know \( I \left( M^{(j_1)} \right) \supseteq I \left( M^{(j_2)} \right) \), and therefore \( M^{(i)} \leq_{\Sigma} M^{(j_k)} \) for some \( m \in \mathbb{N} \), because these are monomial ideals. This means that also \( M^{(i)} \leq_{\Pi} M^{(j_k)} \) by lemma 4, which contradicts the fact that \( S \) is bad. Therefore \( S' \) is bad.

By construction we have
\[ I \left( M^{(1)} \right) \supseteq \cdots \supseteq I \left( M^{(j_1-1)} \right) \supseteq I \left( M^{(j_1)} \right) \text{ and } I \left( M^{(j_1)} \right) \supseteq I \left( M^{(j_2)} \right) \supseteq I \left( M^{(j_3)} \right) \supseteq \cdots \]

Since we remove a generator from \( I \left( M^{(j_1)} \right) \) to create \( I \left( M^{(j_1)} \right) \), we have \( I \left( M^{(j_1)} \right) \supseteq I \left( M^{(j_1)} \right) \), so we then immediately get that \( S' \) is indeed ideally bad.

Now the first \( j_1 - 1 \) elements of \( S \) and \( S' \) are the same, and \( s(M^{(j_k)}) > 1 \) for \( k \in \mathbb{N} \) because \( j_1 \geq i_0 \). Then \( s \left( M^{(j_1)} \right) = s \left( M^{(j_1)} \right) - 1 < s \left( M^{(j_1)} \right) \), which contradicts the minimality by which we chose \( S \). Therefore there are no ideally bad sequences, and therefore no bad sequences at all. Thus indeed \( \leq_{\Sigma} \) on \( T' \) is a well-ordering.

From this it immediately follows that \( \mathbb{K}[T_{\Sigma}^{\leq 1}(V)] \) is \( \text{Subs}^*_\Sigma(\mathbb{N}) \)-Noetherian. Therefore we know that the ideal of any \( \text{Subs}^*_\Sigma(\mathbb{N}) \)-stable subvariety of \( T_{\Sigma}^{\leq 1}(V) \) is generated by finitely many \( \text{Subs}^*_\Sigma(\mathbb{N}) \)-orbits of polynomials.

### 3.6 Noetherianness of \( \mathbb{K}[\prod_{i=1}^{L} T_{\Sigma}^{\leq 1}(V_i)] \)

To show that this coordinate ring, \( \mathbb{K}[\prod_{i=1}^{L} T_{\Sigma}^{\leq 1}(V_i)] \), is \( \text{Subs}^*_\Sigma(\mathbb{N}) \)-Noetherian, we reduce it to the case \( L = 1 \). First, since we take different vector spaces \( V_1, \ldots, V_L \), we have different alphabets. So let \( n_1, \ldots, n_L \) be their sizes (dimensions of the vector spaces). Then we note
\[ \mathbb{K} \left[ \prod_{i=1}^{L} T_{\Sigma}^{\leq 1}(V_i) \right] \cong \bigotimes_{i=1}^{L} \mathbb{K} \left[ T_{\Sigma}^{\leq 1}(V_i) \right] \cong \bigotimes_{i=1}^{L} \mathbb{K} \left[ M_{n \times n_i} \right] \cong \mathbb{K} \left[ \prod_{i=1}^{L} M_{n \times n_i} \right] \]

Note that we can let \( \text{Subs}^*_\Sigma(\mathbb{N}) \) act diagonally on \( \prod_{i=1}^{L} M_{n \times n_i} \), i.e. by acting on all matrices of a tuple, and this makes all of the above isomorphisms \( \text{Subs}^*_\Sigma(\mathbb{N}) \)-equivariant. The monoid \( \prod_{i=1}^{L} M_{n \times n_i} \) embeds
in $M_{\sum_{i=1}^{L} n_i}$, but is not equal to it. Denote $s_i = \sum_{k=1}^{i} n_k$ for $i = 0, \ldots, L$. The embedding is given as follows:

$$f(M^{(1)}, \ldots, M^{(L)}) = M$$

defined as

$$M_{k,i} = \begin{cases} M^{(1)}_{k,i - s_0} & \text{if } s_0 < i \leq s_1 \\ M^{(2)}_{k,i - s_1} & \text{if } s_1 < i \leq s_2 \\ \vdots & \vdots \\ M^{(L)}_{k,i - s_{L-1}} & \text{if } s_{L-1} < i \leq s_L \end{cases}$$

This map effectively glues the rows of $M^{(1)}, \ldots, M^{(L)}$ together to form a wider matrix.

**Proposition 5.** The function $f$ is an injective $\text{Subs}_<^s(\mathbb{N})$-equivariant monoid homomorphism

**Proof:** Obviously $f$ is a monoid homomorphism, because it is additive and the tuple of zero matrices maps to the zero matrix. Furthermore, it is clearly injective. To show that it is $\text{Subs}_<^s(\mathbb{N})$-equivariant, let $\sigma \in \text{Subs}_<^s(\mathbb{N})$ and denote for $l \in \mathbb{N}, k_l$ such that $l \in \sigma_{k_l}$.

Let $l \in \mathbb{N}$ and $i \in \left[\sum_{j=1}^{L} n_j\right]$ with $m$ such that $s_{m-1} < i \leq s_m$. Denote $M = f(M^{(1)}, \ldots, M^{(L)})$ and $M' = f(\sigma M^{(1)}, \ldots, \sigma M^{(L)})$. Then:

$$M'_{k,i} = (\sigma M^{(m)})_{i-s_{m-1}} = M^{(m)}_{k,i - s_{m-1}} = M_{k,i} = (\sigma M)_{l,i}$$

Therefore $f$ is indeed $\text{Subs}_<^s(\mathbb{N})$-equivariant.

Note that informally we could say, since we simply concatenate rows and $\text{Subs}_<^s(\mathbb{N})$ acts on each column separately, the map is obviously equivariant.

Now we define $A = \prod_{i=1}^{L} M_{n_i \times n_i}$ and a function $s : A \to \mathbb{N}$ as $s(M^{(1)}, \ldots, M^{(L)}) = \max_{i \in [L]} s(M^{(i)})$. Then by analogy with $T, T_a, T'$ in section 3.5 define:

$$A_a = \{ M \in A \mid s(M) \leq a \}$$

$$A' = \bigcup_{a \in \mathbb{N}} A_a \times \{a\}$$

Then we can see that $s(f(M)) = s(M)$ for $M \in A$ and therefore $f(A_a) \subseteq T_a$. Using $P$ and $P_{n,m}$ as in theorem 9, we can then see that $A_n, P_{n,m}$ is a pre-filtration of $A, P$. We induce an order $\leq_A$ on $A$ as $M \leq_A N \iff f(M) \leq f(N)$. Then since the order $\leq$ on $T$ is a pre-$P$-order and $f$ is a monoid homomorphism and $\text{Subs}_<^s(\mathbb{N})$-equivariant, we can see that $\leq_A$ is a pre-$P$-order. Furthermore, $\leq_A$ is a well-order on each $A_n$ because $\leq$ is a well-order on each $T_a$. Therefore we only need to show that the following quasi-order on $A'$ is a w.q.o.:

$$(M, a) \mid_\Sigma (N, b) \iff \exists R \in A_b, \sigma \in P_{b,a} : R + \sigma M = N$$

Note that we have:

$$R + \sigma M = N \iff f(R + \sigma M) = f(N) \iff f(R) + \sigma f(M) = f(N)$$

Then note that if a matrix $\bar{R} \in T$ satisfies $\bar{R} + \sigma f(M) = f(N)$ then for each $k \in [L]$, the row sums of the sub-matrix consisting of the columns indexed by $s_{k-1} + 1, \ldots, s_k$ are also constant. Therefore we have $\bar{R} = f(R)$ for some $R \in A$. This means that for $(M, a), (N, b) \in A'$ we have $(M, a) \mid_\Sigma (N, b)$ if and only if $(f(M), a) \leq_\Sigma (f(N), b)$. And since $\leq_\Sigma$ on $T'$ is a w.q.o., $\mid_\Sigma$ on $A'$ is a w.q.o.. Therefore we can see that the algebra $K[\prod_{i=1}^{L} M_{n_i \times n_i}]$ is $\text{Subs}_<^s(\mathbb{N})$-Noetherian. We obtain the following convenient corollary:

**Corollary 1.** For any $L \in \mathbb{N}, V_1, \ldots, V_L$ finite-dimensional vector spaces, and $X$ a $\text{Subs}_<^s(\mathbb{N})$-stable sub-variety of $\prod_{i=1}^{L} T_{\Sigma_1}^{\leq_1}(V_i)$, the ideal $I(X)$ is generated by finitely many $\text{Subs}_<^s(\mathbb{N})$-orbits of polynomials

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3.7 Varieties generated in bounded degree

We make a note on projections of \( \text{Subs}(\mathbb{N}) \)-stable subvarieties of \( \prod_{i=1}^{L} T_{a_i}^{\leq 1}(V_i) \) to \( \prod_{i=1}^{L} T_{a_i}^{\leq 1}(V_i) \). As seen before the maps \( a_{i,b} \) have left-inverses and thereby we obtain the following maps:

\[
\begin{align*}
F_{a,b} &= (f_{a_1,b_1,1}, \ldots, f_{a_L,b_L}) \\
G_{b,a} &= (g_{b_1,a_1,1}, \ldots, g_{b_L,a_L}) \\
P_a &= (p_{a_1,1}, \ldots, p_{a_L,1}) \\
R_a &= (r_{a_1,1}, \ldots, r_{a_L}) \\
Q_a &= q_{a_1,1, \ldots, q_{a_L,1}} \\
S_a &= s_{a_1} \cdots s_{a_L}
\end{align*}
\]

Note that each of these morphisms give unique morphisms on the products of these varieties and their coordinate rings for \( a \leq b \) in \( \mathbb{N}^L \):

\[
\begin{align*}
F_{a,b} &= (f_{a_1,b_1,1}, \ldots, f_{a_L,b_L}) \\
G_{b,a} &= (g_{b_1,a_1,1}, \ldots, g_{b_L,a_L}) \\
P_a &= (p_{a_1,1}, \ldots, p_{a_L,1}) \\
R_a &= (r_{a_1,1}, \ldots, r_{a_L}) \\
Q_a &= q_{a_1,1, \ldots, q_{a_L,1}} \\
S_a &= s_{a_1} \cdots s_{a_L}
\end{align*}
\]

Then we have that \( F_{a,b} \circ G_{b,a}, P_a \circ R_a \) and \( S_a \circ Q_a \) are the identities. This implies \( Q_a(f) = f \circ P_a \):

\[
\begin{align*}
Q_a(f)(t_1, \ldots, t_L) &= (q_{a_1}(f_1)(t_1)) \cdots (q_{a_L}(f_L)(t_L)) \\
&= (f_1(p_{a_1,1}(t_1))) \cdots (f_L(p_{a_L,1}(t_L))) \\
&= (f_1 \cdots f_L)(p_{a_1,1}(t_1), \ldots, p_{a_L,1}(t_L)) \\
&= f(P_a(t_1, \ldots, t_L)) \\
&= (f \circ P_a)(t_1, \ldots, t_L)
\end{align*}
\]

And similarly \( S_a(f) = f \circ R_a \). These maps are exactly the maps that occur when analysing the double system \( (\prod_{i=1}^{L} T_{a_i}(V_i), G_{b,a}, F_{a,b}) \).

The above results are summarised in the following theorem:

**Theorem 13.** Let \( (X_a, G_{b,a}, F_{a,b}) \) be a subsystem of \( (\prod_{i=1}^{L} T_{a_i}^{\leq 1}(V_i), G_{b,a}, F_{a,b}) \) such that the Zariski-closure \( \overline{X} \) of the inverse limit \( X \) is \( \text{Subs}^{a,b}(\mathbb{N}) \)-stable. Then there is a constant \( C \) such that for each \( b \in \mathbb{N}^L \) the ideal \( I(X_b) \) is generated by polynomials of degree at most \( C \).

**Proof:** Since each \( q_{a_i}^{L} \) and \( s_{a_i}^{L} \) send variables to variables, they preserve degree, and therefore so do \( Q_b \) and \( S_b \) for each \( b \). Now as shown in proposition \( \square \) if \( G \) is a generating set for the ideal \( I(X) \) then \( S_b(G) \) is a generating set for \( I(X_b) \) for each \( b \). The closure \( \overline{X} \) is a \( \text{Subs}^{a,b}(\mathbb{N}) \)-stable sub-variety of \( \prod_{i=1}^{L} T_{a_i}^{\leq 1}(V_i) \), so its ideal is generated by the \( \text{Subs}^{a,b}(\mathbb{N}) \)-orbits of finitely many polynomials by corollary \( \square \). Let \( f_1, \ldots, f_k \) be these polynomials. Since \( \text{Subs}(\mathbb{N}) \) acts on each \( \mathbb{K}[T_{\infty}(V_i)] \) by sending variables to variables, the polynomials in the orbit of a single polynomial have the same degree. So let \( C = \max_{j \in [k]} \deg(f_j) \). Then the \( \text{Subs}^{a,b}(\mathbb{N}) \)-orbits of \( f_1, \ldots, f_k \) contain only polynomials of degree at most \( C \). So \( I(\overline{X}) = I(X) \) has a generating set \( G \) of polynomials of degree at most \( C \). Now let \( b \in \mathbb{N}^L \). Since \( S_b \) preserves degree, the polynomials in \( S_b(G) \) also have degree at most \( C \). Since \( S_b(G) \) is a generating set of \( I(X_b) \), it is generated by polynomials of degree at most \( C \), and the bound \( C \) is independent of \( b \). \( \square \)
Chapter 4

The toric fibre product

The toric fibre product is a construction of an ideal from two ideals in polynomial $\mathbb{K}$-algebras. This construction is elaborated on from various angles in [Sul07, KR14, EKS14]. We describe a different construction on the level of affine varieties and show that this construction is equivalent to the toric fibre product when applied to radical ideals. For the remainder of this chapter, $\mathbb{K}$ is an algebraically closed field.

4.1 The toric fibre product

First we recall two equivalent definitions of the toric fibre product as in [EKS14]. Let $r > 0$, $s, t \in \mathbb{Z}_{\geq 1}$ and define three polynomial algebras

$$\mathbb{K}[x] := \mathbb{K}[x_i^j | j \in [r], i \in [s_j]]$$

$$\mathbb{K}[y] := \mathbb{K}[y_k^j | j \in [r], k \in [t_j]]$$

$$\mathbb{K}[z] := \mathbb{K}[z_{i,k} | j \in [r], i \in [s_j], k \in [t_j]]$$

We identify $\mathbb{K}[x] \otimes \mathbb{K}[y] = \mathbb{K}[x, y]$, the algebra generated by all variables $x_i^j, y_k^j$. Define

$$\phi: \mathbb{K}[z] \to \mathbb{K}[x, y] \quad \text{as} \quad z_{i,k}^j \mapsto x_i^j y_k^j$$

Then for ideals $I \subseteq \mathbb{K}[x]$ and $J \subseteq \mathbb{K}[y]$ we define $I \times_A J := \phi^{-1}(I^e + J^e)$, where $I^e, J^e$ are the extensions of $I$ and $J$ to ideals in $\mathbb{K}[x, y]$ respectively. Note that an equivalent definition of $I \times_A J$ is as the kernel of the map $\overline{\phi}: \mathbb{K}[z] \to \mathbb{K}[x]/I \otimes \mathbb{K}[y]/J$ defined as

$$z_{i,k}^j \mapsto \overline{x_i^j} \otimes \overline{y_k^j}$$

where $\overline{x_i^j}$ and $\overline{y_k^j}$ are the equivalence classes of $x_i^j$ and $y_k^j$ respectively. This is exactly the map $\phi$ composed with the projection $\pi: \mathbb{K}[x] \otimes \mathbb{K}[y] \to \mathbb{K}[x]/I \otimes \mathbb{K}[y]/J$, and therefore $\phi^{-1}(I^e + J^e) = I \times_A J = \ker \overline{\phi} = \phi^{-1}(\ker(\pi))$.

Typically the sum of two radical ideals need not be radical. However, in this case, because $I$ and $J$ come from “different parts of $\mathbb{K}[x, y]$”, we can show that indeed $I^e + J^e$ is radical if $I$ and $J$ are:

**Proof:** Since $I$ and $J$ are radical, the algebras $\mathbb{K}[x]/I$ and $\mathbb{K}[y]/J$ are reduced, i.e. have no nilpotent elements other than 0. Then by [Bou03, Chapter 5], because $\mathbb{K}$ is algebraically closed (and thus perfect), the algebra $\mathbb{K}[x]/I \otimes_{\mathbb{K}} \mathbb{K}[y]/J$ is reduced. In [Sul07, Proposition 8] it is already noted that $\mathbb{K}[x]/I \otimes_{\mathbb{K}} \mathbb{K}[y]/J \simeq \mathbb{K}[x, y]/(I^e + J^e)$. Therefore this algebra is reduced, and thus $I^e + J^e \subseteq \mathbb{K}[x, y]$ is radical.

Now suppose $I_1, \ldots, I_s$ are ideals in such algebras and $a \in \mathbb{N}^s$. Then we denote for the iterated toric fibre product that iterates ideal $I_k a_k$ times:

$$\left( \bigotimes_{A} I_1 \right) \times_A \cdots \times_A \left( \bigotimes_{A} I_s \right)$$
We note that in most references we give on the toric fibre product, inherently a grading of $\mathbb{K}[x]$ and $\mathbb{K}[y]$ w.r.t. the same affine semigroup is chosen such that $\deg(x_i^j) = a_j$ for all $j \in [r], i \in [s_j], k \in [t_j]$ and $I$ and $J$ are homogeneous with respect to this grading. The notation is then that $\mathcal{A} = \{a_1, \ldots, a_r\}$, the set of all (multi-)degrees of these variables, from which the notation $I \times_\mathcal{A} J$ stems. However, apart from the requirement that $x_i^j$ and $y_k^j$ have the same degree, the grading itself is not used in the definition of the toric fibre product. We do in fact need this requirement, because we create a variable $z$ for each pair of variables in $\mathbb{K}[x]$ and $\mathbb{K}[y]$ that have the same degree. This is why we have exactly one variable for each combination of $i, j, k$. The notation hides the fact that $x_i^j$ and $y_k^j$ have the same degree by giving them the same superscript, and creates a variable $z$ for each combination of variables with the same superscript. But in fact there is a $z$-variable for every combination of $x$- and $y$-variables that have the same degree. So although the grading itself is not important, the way it is assigned is. However, for consistency, we will still adopt the notation $I \times_\mathcal{A} J$.

We will not bother with gradings and homogeneity of ideals until later, when looking at applications.

### 4.2 Product of varieties

Let $r > 0$ an integer and $V_{1,1}, \ldots, V_{1,r}, V_{2,1}, \ldots, V_{2,r}$ finite-dimensional vector spaces over $\mathbb{K}$. Let $X \subseteq V_1 := V_{1,1} \times \cdots \times V_{1,r}$ and $Y \subseteq V_2 := V_{2,1} \times \cdots \times V_{2,r}$ be two affine varieties. We define a map:

$$\psi : V_1 \times V_2 \rightarrow \prod_{j=1}^r (V_{1,j} \otimes V_{2,j})$$

$$\psi((v_{1,1}, \ldots, v_{1,r}), (v_{2,1}, \ldots, v_{2,r})) = (v_{1,1} \otimes v_{2,1}, \ldots, v_{1,r} \otimes v_{2,r})$$

Then we define $X \ast Y := \overline{\psi(X \times Y)}$, the Zariski-closure of the image of $X \times Y$ under $\psi$. Note that $X \ast Y$ is naturally isomorphic to $Y \ast X$ because $V \otimes W \simeq W \otimes V$ in a natural way, and also $(X \ast Y) \ast Z \simeq X \ast (Y \ast Z)$ because of the natural isomorphism $(V \otimes W) \otimes U \simeq V \otimes (W \otimes U)$.

We will now show that the pullback of the map $\psi$ constructed here is exactly the $\mathbb{K}$-algebra homomorphism $\phi$ used to define the toric fibre product. However, to do so we need to choose coordinates. Let $s_j = \dim(V_{1,j}), t_j = \dim(V_{2,j})$ and identify $V_{1,j} = \mathbb{K}^{s_j}$ and $V_{2,j} = \mathbb{K}^{t_j}$ for $j \in [r]$. Then we can also directly identify $V_{1,j} \otimes V_{2,j}$ with the space of $s_j \times t_j$-matrices. For $j \in [r], i \in [s_j], k \in [t_j]$, define the morphisms:

$$a_{j,k}^i : \prod_{j=1}^r (V_{1,j} \otimes V_{2,j}) \rightarrow \mathbb{K} \quad \text{as} \quad a_{i,k}^j(v_{1,1}, \ldots, v_{1,r}) = (v_j)_i^k$$

These are exactly the coordinate projections of the vector space $\prod_{j=1}^r (V_{1,j} \otimes V_{2,j})$, and therefore we can identify the two algebras:

$$\mathbb{K}[a] := \mathbb{K}[a_{j,k}^i \mid j \in [r], i \in [s_j], k \in [t_j]] \simeq \mathbb{K}\left[\prod_{j=1}^r (V_{1,j} \otimes V_{2,j})\right]$$

Similarly, we define $b_j^i(v_{1,1}, \ldots, v_{1,r}) = (v_j)_i^k$ and $c_k^j(v_{1,1}, \ldots, v_{1,r}) = (v_j)_k^i$ as coordinate projections of $V_1$ and $V_2$, and therefore identify

$$\mathbb{K}[b] := \mathbb{K}[b_j^i \mid j \in [r], i \in [s_j]] \simeq \mathbb{K}[V_1]$$

$$\mathbb{K}[c] := \mathbb{K}[c_k^j \mid j \in [r], k \in [t_j]] \simeq \mathbb{K}[V_2]$$

$$\mathbb{K}[b, c] := \mathbb{K}[b_j^i, c_k^j \mid j \in [r], i \in [s_j], k \in [t_j]] \simeq \mathbb{K}[V_1 \times V_2]$$

Here the the elements $b_j^i, c_k^j$ of $\mathbb{K}[b, c]$ are their respective functions in $\mathbb{K}[b]$ and $\mathbb{K}[c]$ after the projection of the input from $V_1 \times V_2$ to $V_1$ or $V_2$. We then have the following obvious isomorphisms:

$$\mathbb{K}[z] \simeq \mathbb{K}[a], \quad \mathbb{K}[x] \simeq \mathbb{K}[b], \quad \mathbb{K}[y] \simeq \mathbb{K}[c] \quad \text{and} \quad \mathbb{K}[x, y] \simeq \mathbb{K}[b, c]$$

Now we show what the pullback of $\psi$ is:
Proposition 6. The pullback $\psi^* : \mathbb{K}[a] \to \mathbb{K}[b,c]$ of $\psi$ is given as $\psi^*(a_{i,k}^j) = b_i^j c_k^j$.

Proof: Let $v_l = (v_{l,1}, \ldots, v_{l,r}) \in V_l$ for $l = 1, 2$.

$$
\begin{align*}
\psi^*(a_{j,k}^j)(v_1, v_2) &= \psi^*(a_{j,k}^j)((v_{1,1}, \ldots, v_{1,r}), (v_{2,1}, \ldots, v_{2,r})) \\
&= a_{j,k}^j(\psi((v_{1,1}, \ldots, v_{1,r}), (v_{2,1}, \ldots, v_{2,r}))) \\
&= a_{j,k}^j(v_{1,j} \otimes v_{2,j}) \\
&= (v_{1,j} \otimes v_{2,j})_{i,k} \\
&= (v_{1,j})_i \cdot (v_{2,j})_k \\
&= b_i^j(v_{1,1}, \ldots, v_{1,r}) \cdot c_k^j(v_{2,1}, \ldots, v_{2,r}) \\
&= b_i^j((v_{1,1}, \ldots, v_{1,r}), (v_{2,1}, \ldots, v_{2,r})) \cdot c_k^j((v_{1,1}, \ldots, v_{1,r}), (v_{2,1}, \ldots, v_{2,r})) \\
&= (b_i^j c_k^j)((v_{1,1}, \ldots, v_{1,r}), (v_{2,1}, \ldots, v_{2,r})) \\
&= (b_i^j c_k^j)(v_{1}, v_{2})
\end{align*}
$$

Therefore we can readily see that the $\mathbb{K}$-algebra homomorphisms $\phi$ and $\psi^*$ are the same. Note that for a polynomial map $\eta : V \to W$ between finite-dimensional vector spaces (morphism of affine varieties) and a subset $X \subset V$ we have $I(\eta(X)) = (\eta)^{-1}(I(X))$:

$$
\begin{align*}
f \in I(\eta(X)) \iff & \forall x \in X : f(\eta(x)) = 0 \\
\iff & \forall x \in X : \eta^*(f)(x) = 0 \\
\iff & \eta^*(f) \in I(X) \\
\iff & f \in (\eta^*)^{-1}(I(X))
\end{align*}
$$

Therefore we get:

$$
\begin{align*}
I(X \ast Y) &= I(\psi(X \times Y)) \\
&= \phi^{-1}(I(X \times Y)) \\
&= \phi^{-1}(I((X \times V_2) \cap (V_1 \times Y))) \\
&= \phi^{-1} \left( \sqrt{I(X \times V_2) + I(V_1 \times Y)} \right) \\
&= \phi^{-1} \left( \sqrt{I(X)^e + I(Y)^e} \right) \\
&= \phi^{-1}(I(X)^e + I(Y)^e) \\
&= I(X) \times_A I(Y)
\end{align*}
$$

Thus the ideal of $X \ast Y$ is the toric fibre product of the ideals $I(X)$ and $I(Y)$, because $I(X)^e + I(Y)^e$ is radical. However, for general ideals $I$, $J$, $V(I)$ and $V(J)$ are varieties, and with the above we can see that $I(V(I) \ast V(J)) = \sqrt{I} \times_A \sqrt{J}$. Therefore our product is not exactly the same as the toric fibre product, only when applied to radical ideals.

Note the correspondence between the grading of the toric fibre product and the vector spaces introduced here: We have $X$ and $Y$ varieties inside vector spaces that are the product of $r$ vector spaces, the number of different degrees of variables for the toric fibre product. The vector spaces in $V_1$ have dimension $s_j$ for $j \in \mathbb{Z}$ and therefore we get $s_j$ $x$-variables of the same degree, and similarly in the vector space $V_2$, with dimensions $t_j$. The vector space $V_{1,j} \otimes V_{2,j}$ has dimension $s_j t_j$, and therefore we get $s_j t_j$ $z$-variables of the same degree. Therefore we see that the choice of splitting the vector spaces $V_1$ and $V_2$ corresponds to deciding which variables have the same degree.
4.2.1 The iterated product

We shall give notation for the iterated product construction.

Let \( r, s > 0 \) positive integers and \( V_{i,j} = \mathbb{K}^{k_{i,j}} \) be finite-dimensional vector spaces for \( i \in [s], j \in [r] \).
We define \( V_i = \prod_{j=1}^s V_{i,j} \). Let \( X_i \subseteq V_i \) be a variety for each \( i \in [s] \), and let \( a \in \mathbb{N}^s \).
We make the following definitions:

\[
V^a := \prod_{i=1}^s \prod_{k=1}^{a_i} V_i = \prod_{i=1}^s \prod_{k=1}^{a_i} ^r V_{i,j}
\]

\[
X^a := \prod_{i=1}^s \prod_{k=1}^{a_i} X_i = \prod_{i=1}^s \prod_{k=1}^{a_i} X_i^{a_i}
\]

\[
W_a(V) := \prod_{j=1}^r \prod_{i=1}^s \prod_{k=1}^{a_i} V_{i,j}
\]

Now we can define the iterated version of the map \( \psi \):

\[
\psi_a : V^a \to W_a(V) \quad \text{defined as} \quad (v_{i,k,j})_{i \in [s], \ k \in [a_i], \ j \in [r]} \mapsto \left( \bigotimes_{i=1}^s \bigotimes_{k=1}^{a_i} v_{i,k,j} \right)_{j \in [r]}
\]

If we define \( X^{sa} = \overline{\psi_a(X^a)} \), we can see that these definitions give a concise notation to the iterated product. Note that \( V^{sa} := \overline{\psi_a(V^a)} = \psi_a(V^a) \) is exactly the set

\[
\{(t_1, \ldots, t_r) \in W_a(V) | \forall j \in [r] : rk(t_i) \leq 1\}
\]

of tuples of rank-at-most-1 tensors in \( W_a(V) \).

4.2.2 Reduction to \( s = 1 \)

When computing the inverse limit of \( X^{sa} \) under certain maps we can choose a cofinal subset of \( \mathbb{N}^s \) and use this to obtain the inverse limit. By associativity and commutativity of the product \( X \ast Y \) we see that \( X^{sb} = (X_1 \ast \cdots \ast X_s)^{sa} \) for \( b = (a, \ldots, a) \). Therefore we only need to consider the case of \( s = 1 \) by choosing the diagonal cofinal subset \( \{(a, \ldots, a) | a \in \mathbb{N} \} \) of \( \mathbb{N}^s \).

When \( s = 1 \) the situation becomes simpler. Let \( X \subseteq V := V_1 \times \cdots \times V_r \) an affine variety and \( a \in \mathbb{N} \). The definitions for the iterated products are then:

\[
V^a = \prod_{k=1}^a V = \prod_{k=1}^a \prod_{j=1}^r V_j
\]

\[
X^a = \prod_{k=1}^a X
\]

\[
W_a(V) = \prod_{j=1}^r \prod_{k=1}^a V_j
\]

Using the notation \( T_a(V) \) and \( T_{<1}^a(V) \) from section 3.2 and section 3.3 we can see that \( W_a(V) = \prod_{j=1}^r T_{<1}^a(V_j) \). Note that we use the same \( a \in \mathbb{N} \) for all factors of the product. The map \( \psi_a \) is then given by

\[
\psi_a : V^a \to W_a(V) \quad \text{defined as} \quad (v_{k,j})_{k \in [a], \ j \in [r]} \mapsto \left( \bigotimes_{k=1}^a v_{k,j} \right)_{j \in [r]}
\]

The definition of \( X^{sa} \) is still \( \overline{\psi_a(X^a)} \). We can see that \( V^{sa} = \prod_{j=1}^r T_{<1}^a(V_j) \) and obviously \( X^{sa} \subseteq V^{sa} \). Therefore for each \( a \in \mathbb{N} \) we have \( X^{sa} \subseteq \prod_{j=1}^r T_{<1}^a(V_j) \). In the next chapter we will study the ideals of these varieties.
Chapter 5

Stabilisation of the product

We will now show that the iterated product stabilises when applied to finite number of varieties from a certain class. We will first describe the class of varieties, after which we will show the stabilisation. For the remainder of this chapter, \( K \) is an algebraically closed field.

5.1 Hadamard-closed varieties

We call an affine variety \( X \subseteq V := \mathbb{K}^d \) a Hadamard-closed variety if \( X \) is stable under coordinate-wise multiplication, and \( X \) contains \((1, \ldots, 1)\), the unit of coordinate-wise multiplication. That is, \( X \) is a Zariski-closed submonoid of the (multiplicative) monoid \( V \).

Proposition 7. Any affine variety cut out by binomials is a Hadamard-closed variety.

Proof: Suppose \( X \subseteq \mathbb{K}^n \) and \( X = V(x^{a_1} - x^{b_1}, \ldots, x^{a_m} - x^{b_m}) \). Let \( y, z \in X \) and \( i \in [m] \). Then

\[
\prod_{j=1}^{n} (y_j z_j)^{a_{i,j}} = \left( \prod_{j=1}^{n} y_j^{a_{i,j}} \right) \left( \prod_{j=1}^{n} z_j^{a_{i,j}} \right) = \left( \prod_{j=1}^{n} y_j^{b_{j,i}} \right) \left( \prod_{j=1}^{n} z_j^{b_{j,i}} \right) = \prod_{j=1}^{n} (y_j z_j)^{b_{j,i}}
\]

Therefore \( y \cdot z \in X \). Furthermore, since \((1, \ldots, 1)\) is a zero of each such binomial, \( X \) is a Hadamard-closed variety.

In particular, toric varieties are Hadamard-closed varieties.

Note that \( \mathbb{K}^{d_1} \otimes \mathbb{K}^{d_2} \) admits coordinate-wise multiplication as \((v_1 \otimes v_2)(w_1 \otimes w_2) = ((v_1 w_1) \otimes (v_2 w_2))\). Therefore, if \( X, Y \) are Hadamard-closed varieties inside \( V_1 := \prod_{j=1}^{n-1} V_{1,j} \) and \( V_2 := \prod_{j=1}^{n-1} V_{2,j} \) respectively, then \( X \ast Y \) is a Hadamard-closed variety in \( \prod_{j=1}^{n-1} V_{1,j} \otimes V_{2,j} \), because the map \( \psi \) defining \( X \ast Y \) is a monoid homomorphism with respect to coordinate-wise multiplication. To see this, let \( v^1, v^2 \in V_1, w^1, w^2 \in V_2 \):

\[
\psi(v^1 \cdot v^2, w^1 \cdot w^2) = \psi((v_1^1, \ldots, v_1^n), (v_2^1, \ldots, v_2^n), (w_1^1, \ldots, w_1^n), (w_2^1, \ldots, w_2^n))
= \psi((v_1^1 \cdot v_2^1, \ldots, v_1^n \cdot v_2^n), (w_1^1 \cdot w_2^1, \ldots, w_1^n \cdot w_2^n))
= \psi((v_1^1 \cdot v_2^1) \otimes (w_1^1 \cdot w_2^1), \ldots, (v_1^n \cdot v_2^n) \otimes (w_1^n \cdot w_2^n))
= \psi((v_1^1 \otimes w_1^1, \ldots, v_1^n \otimes w_1^n), (v_2^1 \otimes w_2^1, \ldots, v_2^n \otimes w_2^n))
= \psi((v_1^1, \ldots, v_1^n), (w_1^1, \ldots, w_1^n)) \cdot \psi((v_2^1, \ldots, v_2^n), (w_2^1, \ldots, w_2^n))
= \psi(v^1, w^1) \cdot \psi(v^2, w^2)
\]

Thus if \( X, Y \) are Hadamard-closed varieties, so is \( X \ast Y \). Therefore if we want to take iterated products of Hadamard-closed varieties \( X_1, \ldots, X_s \), we can again restrict to the case \( s = 1 \) because \( X_1 \ast \cdots \ast X_s \) is also Hadamard-closed.
5.2 Iterated products of Hadamard-closed varieties

Let \( r > 0 \) be an integer and \( V_j = \mathbb{K}^{d_j} \) a finite-dimensional vector space for each \( j \in [r] \), and \( X \) be a Hadamard-closed variety in \( V := \prod_{j=1}^r V_j \). We consider iterated toric fibre products of these varieties, i.e. \( X^{*a} \) for \( a \in \mathbb{N} \). Since the spaces \( W_{a}(V) \) are equal to the spaces \( \prod_{j=1}^r T_{a}(V_j) \), we can use the maps \( G_{b,a} \) and \( F_{a,b} \) from section 3.7 and obtain maps \( P_{a}, R_{a}, Q_{a} \) and \( S_{a} \). These maps use elements \( a, b \in \mathbb{N}^r \) as subscripts, but since we use the same \( a \in \mathbb{N} \) for all \( j \in [r] \) these can be taken to be elements of the diagonal of \( \mathbb{N}^r \), so we can simply use natural numbers as subscripts.

Since \( X \) is a Hadamard-closed variety, we have \( F_{a,b}(X^{sb}) \subseteq X^{*a} \) and \( G_{b,a}(X^{*a}) \subseteq X^{sb} \). Therefore we have a double system \((W_{a}(V), G_{b,a}, F_{a,b})\) with two subsystems \((V^{*a}, G_{a,b}, F_{a,b})\) and \((X^{*a}, G_{b,a}, F_{a,b})\).

Let \( \overline{V} \) be the inverse and direct limit respectively of \((V^{*a}, G_{a,b}, F_{a,b})\), and \( \overline{X} \) the inverse and direct limit respectively of \((X^{*a}, G_{b,a}, F_{a,b})\).

Since the varieties \( V^{*a} \) are simply \( \prod_{j=1}^r T_{a}^{\leq 1}(V_j) \) we have that \( \overline{V} = \prod_{j=1}^r T_{\infty}^{\leq 1}(V_j) \), and therefore \( \overline{X} \) is a subset of \( \prod_{j=1}^r T_{\infty}^{\leq 1}(V_j) \). To invoke theorem 13 we need to show that its closure is \( \text{Subs}_\mathbb{N}(\mathbb{N}) \)-stable. As noted in section 3.2 by theorem 6, we only need to show that a dense subset of \( \overline{X} \) is \( \text{Subs}_\mathbb{N}(\mathbb{N}) \)-stable, because then its closure is.

5.2.1 Parametrisation of \( \overline{X} \)

Let \( e_j \) be the unit of coordinate-wise multiplication in \( V_j \). We define a map \( H_{b,a} : V^{a} \rightarrow V^{b} \) as:

\[
(v_{1,j}, \ldots, v_{a,j}) \in [r] \mapsto (v_{1,j}, \ldots, v_{a,j}, e_j, \ldots, e_j) \in [r]
\]

We can easily verify that then \( \psi_b \circ H_{b,a} = G_{b,a} \circ \psi_a \). Furthermore, we can see that \( H_{c,b} \circ H_{b,a} = H_{c,a} \) for \( a \leq b \leq c \). Therefore \((V^{a}, H_{b,a})\) forms a direct system. Using theorem 2, the direct limit can be seen to be:

\[
V^{\infty} := \left\{ (v_{l,j})_{l \in \mathbb{N}, j \in [r]} \in \prod_{l \in \mathbb{N}} V \mid \exists a \in \mathbb{N} : \forall k \geq a, j \in [r] : v_{k,j} = e_j \right\}
\]

with morphisms \( Z_{a} : V^{a} \rightarrow V^{\infty} \) defined as appending the elements \( e_j \) ad infinitum. Now we can give the analogue \( \psi_{\infty} : V^{\infty} \rightarrow \overline{V} \) of the maps \( \psi_{a} \):

\[
(v_{l,j})_{l \in \mathbb{N}, j \in [r]} \mapsto \left( \prod_{l \in \mathbb{N}} (v_{l,j})_{l_i} \right)_{i \in \text{seq}^{\infty(\{d_j\})} j \in [r]}
\]

Each of these products is well-defined because almost all of their factors are 1. First we give the following lemmas about \( \psi_{\infty} \):

**Lemma 5.** \( \forall a \in \mathbb{N} : \psi_{\infty} \circ Z_{a} = R_{a} \circ \psi_{a} \)

**Proof:** Let \( v \in V^{a} \). Then for each \( j \in [r] \) and \( i \in \text{seq}^{\infty(\{d_j\})} \), the infinite product \((\psi_{\infty}(Z_{a}(v)))_{j})_{i}\) reduces to the following finite product:

\[
((\psi_{\infty}(Z_{a}(v)))_{j})_{i} = \prod_{l \in \mathbb{N}} (v_{l,j})_{l_i} = \prod_{l=1}^{a} (v_{l,j})_{l_i} = \prod_{l=1}^{a} (v_{l,j})_{\delta_{a}(i)}_{l_i}
\]

Furthermore, we have

\[
((R_{a}(\psi_{a}(v)))_{j})_{i} = ((\psi_{a}(v)))_{j})_{\delta_{a}(i)}_{l_i}
\]

Therefore we can see that \( \psi_{\infty}(Z_{a}(v)) = R_{a}(\psi_{a}(v)) \).}

**Lemma 6.** \( \text{im}(\psi_{\infty}) = \overline{V} \)

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Proof: Since $\overrightarrow{V}$ is the direct limit, it is the union: $\overrightarrow{V} = \bigcup_{a \in \mathbb{N}} R_a(V^a)$, and similarly we have $V^\infty = \bigcup_{a \in \mathbb{N}} Z_a(V^a)$. Therefore we have:

$$
\begin{align*}
    x \in \overrightarrow{V} &\Leftrightarrow \exists a \in \mathbb{N}, v \in V^a : x = R_a(v) \\
    &\Leftrightarrow \exists a \in \mathbb{N}, v \in V^a : x = R_a(\psi_a(v)) \\
    &\Leftrightarrow \exists a \in \mathbb{N}, v \in V^a : x = \psi_\infty(Z_a(v)) \\
    &\Leftrightarrow x \in \text{im}(\psi_\infty)
\end{align*}
$$

And so $\text{im}(\psi_\infty) = \overrightarrow{V}$.

In a similar fashion we can define $X^\infty$ and see that it is the direct limit of $(X^a, H_{b,a})$. The following lemma shows why $\psi_\infty$ is interesting:

Lemma 7. $\psi_\infty(X^\infty)$ is dense in $\overrightarrow{X}$ and then in $\overleftarrow{X}$.

Proof: We only need to show that $\psi_\infty(X^\infty)$ and $\overrightarrow{X}$ have the same closure, because then $\psi_\infty(X^\infty)$ has the same closure as $\overrightarrow{X}$, and so $\psi_\infty(X^\infty)$ is dense in $\overrightarrow{X}$. Note that $\overrightarrow{X} = \bigcup_{a \in \mathbb{N}} R_a(X^a)$ and $\psi_\infty(X^\infty) = \psi_\infty(\bigcup_{a \in \mathbb{N}} Z_a(X^a)) = \bigcup_{a \in \mathbb{N}} R_a(\psi_a(X^a))$ by lemma 3. Therefore we have $\psi_\infty(X^\infty) \subseteq \overrightarrow{X}$. Since the maps $R_a$ are continuous, we have $R_a(Y) \subseteq R_a(\overrightarrow{Y})$ for all $Y \subseteq W_a(V)$. Therefore

$$
\psi_\infty(X^\infty) = \bigcup_{a \in \mathbb{N}} R_a(\psi_a(X^a)) \supseteq \bigcup_{a \in \mathbb{N}} R_a(\psi_a(X^a)) \supseteq \bigcup_{a \in \mathbb{N}} R_a(\psi_a(X^a)) = \bigcup_{a \in \mathbb{N}} R_a(X^a) = \overrightarrow{X}
$$

Therefore $\psi_\infty(X^\infty) \subseteq \overrightarrow{X} \subseteq \overrightarrow{\psi_\infty(X^\infty)}$, meaning they have the same closure and the same vanishing ideal. Therefore $\psi_\infty(X^\infty)$ is dense in $\overrightarrow{X}$. ■

We can define a right-$\text{Subs}^e_N$-action on $V^\infty$ as

$$(v_{l,j})_{l \in \mathbb{N}, j \in [r]} \cdot \sigma = \left( \prod_{k \in \sigma_l} v_{k,j} \right)_{l \in \mathbb{N}, j \in [r]}$$

Indeed this is an action, because the identity $(\{1\}, \{2\}, \{3\}, \ldots)$ maps $v \in V^\infty$ to itself, and:

$$
\begin{align*}
    ((v_{l,j})_{l \in \mathbb{N}, j \in [r]} \cdot \sigma) \cdot \pi &\equiv \left( \prod_{k \in \sigma_l} v_{k,j} \right)_{l \in \mathbb{N}, j \in [r]} \\
    &\equiv \left( \prod_{m \in \sigma_l, k \in \sigma_m} v_{k,j} \right)_{l \in \mathbb{N}, j \in [r]} \\
    &\equiv \left( \prod_{k \in \bigcup_{m \in \sigma_l} \sigma_m} v_{k,j} \right)_{l \in \mathbb{N}, j \in [r]} \\
    &\equiv \left( \prod_{k \in (\sigma \pi)_l} v_{k,j} \right)_{l \in \mathbb{N}, j \in [r]} \\
    &\equiv (v_{l,j})_{l \in \mathbb{N}, j \in [r]} \cdot (\sigma \pi)
\end{align*}
$$

Since $X$ is Hadamard-closed, $X^\infty$ is stable under this action. Now we can give the final result that we need:

Theorem 14. The map $\psi_\infty$ is $\text{Subs}^e_N$-equivariant.
Proof: Let \( \sigma \in \text{Subs}_s(\mathbb{N}) \), \( v = (v_{ij})_{i \in \mathbb{N}, j \in [r]} \in V^\infty, j \in [r] \) and \( i \in \text{seq}^\infty(d_j) \). Then:

\[
((\psi_\infty(v) \cdot \sigma)_j)_i = \prod_{l \in \mathbb{N}} (v_{lj})_i \sigma_l = \prod_{l \in \mathbb{N}} (v_{lj})_i \sigma_l = \prod_{l \in \mathbb{N}} \left( \prod_{k \in \sigma_l} v_{kj} \right)_i = \left( (\psi_\infty(v \cdot \sigma))_j \right)_i
\]

The third equality follows because \( \text{im} \sigma = \mathbb{N} \).

Then since \( X^\infty \) is \( \text{Subs}_s(\mathbb{N}) \)-stable, so is \( \psi_\infty(X^\infty) \), and therefore also the closure of \( \overline{X} \). Therefore we can invoke theorem 13 and see that there is a constant \( C \in \mathbb{N} \) such that all the ideals \( I(X^a) \) are generated by polynomials of degree at most \( C \).

The full result is given as follows:

**Theorem 15.** Let \( r, s > 0 \) be integers and \( X_i \subseteq \prod_{j=1}^r V_{i,j} \) be a Hadamard-closed variety for \( i \in [s] \). Then there is a constant \( C \in \mathbb{N} \) such that for each \( a \in \mathbb{N}^s \), the ideal \( I(X^a) \) is generated by polynomials of degree at most \( C \).
Chapter 6

Applications

In this chapter we shall see how the stabilisation of the ideal $I(X^a)$ of the iterated product proves conjecture 56 in [RS16], and how this results in stabilisation of certain families of graphical models.

6.1 Iterated toric fibre products of matrices

An $m \times n$ matrix $B$ gives rise to a toric ideal $I_B = (x^a - x^b | a, b \in \mathbb{Z}_{\geq 0}^n : Ba = Bb) \subseteq \mathbb{K}[x_1, \ldots, x_n]$, which is a prime ideal and thus radical. As already noted in [RS16], we have $I_{B_1 \times_A B_2} = I_{B_1} \times_A I_{B_2}$. That is, the toric fibre product of toric ideals is the toric ideal associated to the toric fibre product of the matrices. When applying this to our product construction, we see that the product $T_1 \ast T_2$ of two toric varieties yields a toric variety. Then the ideal of this product is exactly the toric fibre product of the ideals of the varieties. This is still true for iterated toric fibre products of finitely many matrices.

As shown in proposition [7], toric varieties are Hadamard-closed varieties. The results on stabilisation of the product on Hadamard-closed varieties therefore directly translate to stabilisation of the iterated toric fibre product of toric ideals. The result is that for toric ideals $I_1, \ldots, I_s$, there is a constant $C \in \mathbb{N}$ such that for every $a \in \mathbb{N}^s$, the iterated toric fibre product $(\times_A^a I_1) \times_A \cdots \times_A (\times_A^a I_s)$ is generated by binomials of degree at most $C$. By the Fundamental theorem of Markov Bases, this degree is an upper bound for the Markov degree of the associated matrix, and we get the result that for matrices $B_1, \ldots, B_s$ there is a constant $C \in \mathbb{N}$ such that for all $a \in \mathbb{N}^s$, the matrix $(\times_A^a B_1) \times_A \cdots \times_A (\times_A^a B_s)$ has Markov degree at most $C$. This is exactly the statement in conjecture 56 of [RS16].

6.2 Stabilisation in graphical models

Graphical models are a special type of hierarchical model, defined using the clique complex of a graph. We show that the ideal of the parametrisation of the glueing of two graphs over a common vertex-set is the same as the toric fibre product over some grading of the ideals of the parametrisations of the separate graphs. Then we can show that stabilisation of the Markov degree occurs. The motivating example of this stabilisation is in [RS14], where for the family of binary graphical models over $\{K_3, N\}$ the uniform bound on the Markov degree is shown to be at most 12.

6.2.1 Graph glueing

To formalise glueing two graphs, we let Graph be the category of finite simple undirected loopless graphs with graph homomorphisms. A graph homomorphism is a map $f : V(H) \to V(G)$, satisfying $uv \in E(H) \Rightarrow f(u)f(v) \in E(G)$, and we denote it $f : H \to G$. We call such a homomorphism injective if it is on the level of vertices.

The formalisation of glueing graphs is given as the pushout of two graph homomorphisms. The pushout is the colimit of a functor from the (small) category $\bullet \leftarrow \bullet \rightarrow \bullet$ to Graph. Concretely, we are given two graph homomorphisms, $f_i : H \to G_i$. The pushout is a graph $G$ with two homomorphisms $g_i : G_i \to G$ such that $g_1 \circ f_1 = g_2 \circ f_2$ and $(G, g_1, g_2)$ is universal; For any other such tuple $(X, h_1, h_2)$ there exists a unique homomorphism $u : G \to X$ such that $u \circ g_i = h_i$.  

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We note that the pushout does not exist for all pairs of homomorphisms, unless we allow loops in our graphs, as witnessed by the following example:

**Example 2.** Let \( H = (\{1,2\},\emptyset) \), the graph with only two vertices, and define \( G_1 = (\{1\},\emptyset) \) and \( G_2 = (\{1,2\},\{\{1,2\}\}) \), the graph with 1 vertex and the graph with 2 vertices and 1 edge. Let \( f_1 : H \to G_1 \) be given as \( f_1(1) = f_1(2) = 1 \), and \( f_2 : H \to G_2 \) as \( f_2(1) = 1, f_2(2) = 2 \). Then these are obviously graph homomorphisms because \( H \) contains no edges. However, there is not even a cocone to this diagram:

Suppose \((G,g_1,g_2)\) is a cocone. Then \( g_1 \circ f_1 = g_2 \circ f_2 \), meaning \( g_2(1) = g_2(f_2(1)) = g_1(f_1(1)) = g_1(f_1(2)) = g_2(f_2(2)) = g_2(2) \). Therefore both vertices of \( G_2 \) must be mapped to the same vertex. However, since \( G_2 \) has an edge between its two vertices and \( g_2 \) is a graph homomorphism, this means that \( G \) must have a loop from \( g_2(1) \) to itself. Since objects in \( \text{Graph} \) are loopless, this is impossible. So no cocone exists, and thus in particular no colimit. \( \square \)

However, the pushout does exist when both homomorphisms are injective, as shown in the following construction. Let \( G_i = (V_i,E_i) \). Define \( V' = \{(v,i) \mid i = 1,2, v \in V_i\} \), the disjoint union of \( V_1 \) and \( V_2 \), and define an equivalence relation on \( V' \) as:

\[
(v,i) \sim (w,j) \quad \Leftrightarrow \quad (i = j \text{ and } v = w) \text{ or } (i \neq j \text{ and } \exists u \in V(H) : f_i(u) = v, f_j(u) = w)
\]

We use injectivity of both \( f_i \) to show that this relation is transitive. Then we set \( V = V' / \sim \), which is actually the pushout of the sets \( V_1, V_2 \) with \( f_1, f_2 \) considered as functions \( V(H) \to V_1 \) and \( V(H) \to V_2 \). This can be seen by the an argument similar to the one in section 2.3.1 where the direct limit of sets is constructed. Denote \([v,i] \) as the equivalence class of \((v,i) \). Now we define \( g_i : V_i \to V \) as \( v \mapsto [v,i] \). Then we can see that \( g_i(f_1(v)) = [f_1(v),1] = [f_2(v),2] = g_2(f_2(v)) \). Now define \( E = \{(g_i(u)g_i(v)) \mid i = 1,2, uv \in E_i\} \) and \( G = (V,E) \). Then clearly \( g_1 : G_1 \to G \) and \( g_2 : G_2 \to G \) are graph homomorphisms, and we can see that each \( g_i \) is injective, and therefore \( E \) contains no loops. Then indeed \( G \) is a graph and \( g_1 \circ f_1 = g_2 \circ f_2 \), meaning \((G,g_1,g_2)\) is a cocone.

To see that it is universal, suppose \((X,h_1,h_2)\) is also a cocone. Define \( u : G \to X \) as \( u([v,i]) = h_i(v) \). Then \( u \) is well-defined and \( u(g_i(v)) = h_i(v) \). Furthermore \( u \) is a graph homomorphism because if \( g_i(w)g_i(v) \in E \) then \( uv \in E_i \), so \( h_i(u)h_i(v) = u(g_i(w))u(g_i(v)) \) is an edge in \( X \). We can easily see that \( u \) is the unique such morphism because we need \( g_i \circ u = h_i \circ g_i \circ w \) and \( g_i \) is injective.

We can see that this construction does not use the edges of \( H \), and so by replacing \( H \) by a subgraph on the same vertices, we obtain the same pushout. This means that the pushout actually only depends on the maps \( f_i : H \to G_i \) on the level of vertices, i.e. the pushout is independent of the edges of \( H \).

We denote the glueing of \( G_1 \) and \( G_2 \) over \( H \) as \( G_1 +_H G_2 \).

**Example 3.** Let \( G_1 = K_{k,n} \) and \( G_2 = K_{k,m} \) be two complete bipartite graphs with vertex sets \( \{a_1,\ldots,a_k,1,\ldots,n\} \) and \( \{b_1,\ldots,b_k,1,\ldots,m\} \) respectively. Let \( H \) be the graph with only \( k \) vertices, and map these \( k \) vertices onto \( \{a_1,\ldots,a_k\} \) and \( \{b_1,\ldots,b_k\} \) respectively. Then the pushout of this diagram is \( K_{k,n+m} \).

The disjoint union \( V' \) is \( \{(1,a_1),\ldots,(1,a_k),(2,b_1),\ldots,(2,b_k),(1,1),\ldots,(1,n),(2,1),\ldots,(2,m)\} \) and the equivalence relation simply identifies each \( (1,a_i) \) with \( (2,b_i) \), resulting in:

\[
V = \{c_1,\ldots,c_k,(1,1),\ldots,(1,n),(2,1),\ldots,(2,m)\}
\]

The edges \( E \) are then given as \( \{(c_i,1,p),\{c_i,2,q\} \mid i \in [k], p \in [n], q \in [m]\} \) and the graph \( (V,E) \) can indeed be seen to be the glueing of \( G_1 \) and \( G_2 \) over the vertices \( \{a_1,\ldots,a_k\} \) and \( \{b_1,\ldots,b_k\} \). \( \square \)

For graphs \( G_1,\ldots,G_s \) with injective graph homomorphisms \( f_i : H \to G_i \), and \( a \in \mathbb{N}^s \), we denote the iterated glueing with graph \( G_k \) iterated \( a_k \) times as:

\[
\left( \sum_H G_1 \right) +_H \cdots +_H \left( \sum_H G_s \right)
\]
6.2.2 The Hammersley-Clifford theorem

For the following results from [DSS09, Chapter 3], consider the general setting of \( m \) random variables \( X_1, \ldots, X_m \) with supports \( \mathcal{X}_1, \ldots, \mathcal{X}_m \) and joint probability distribution \( f \) on \( \mathcal{X} := \prod_{i=1}^{m} \mathcal{X}_i \). We denote \( \mathcal{X}_C = \prod_{i \in C} \mathcal{X}_i \) and \( x_C \) the restriction of \( x \in \mathcal{X} \) to \( \mathcal{X}_C \) for \( C \subseteq [m] \). Given a graph \( G \) on vertices \([m]\), denote \( \text{mcl}(G) \) the set of its maximal cliques. We say \( f \) factorises according to \( G \) if for each \( C \subseteq \text{mcl}(G) \) there exist a continuous positive function \( \psi_C : \mathcal{X}_C \to \mathbb{R} \) such that for each \( x \in \mathcal{X} \) we have:

\[
  f(x) = \frac{1}{Z} \prod_{C \in \text{mcl}(G)} \psi_C(x_C)
\]

where \( Z \) is a normalising constant. Now we want the joint probability density function of \((X_i)_{i \in [m]}\) to obey the structure of the graph, in some sense. This is formalised by the Markov properties from [DSS09, Section 3.2], which are conditional independence statements in terms of the random variables and the structure of the graph. Then we have the following theorem, known as the Hammersley-Clifford theorem:

**Theorem 16.** A continuous positive probability density function \( f \) on \( \mathcal{X} \) satisfies the pairwise Markov properties if and only if \( f \) factorises according to \( G \).

If the state spaces \( \mathcal{X}_i \) are finite then the continuity condition does not impose any restrictions on \( f \). In this case, \( f \) factorises according to \( G \) if and only if there is a constant \( \theta^C_x \) for each \( C \in \text{mcl}(G) \), \( x \in \mathcal{X}_C \) such that

\[
  \forall x \in \mathcal{X} : \quad f(x) = \prod_{C \in \text{mcl}(G)} \theta^C_x
\]

We are interested in such probability distributions on discrete random variables. Note that there can be probability density functions that obey the Markov properties but do not factorise according to \( G \). These density functions must then be zero on some \( x \in \mathcal{X} \). However, we will only concern ourselves with positive density functions, and therefore density functions that factorise according to \( G \). More on the difference between the sets of probability density functions that satisfy the Markov properties of \( G \), and those that also factorise according to \( G \) can be found in [KRS14].

6.2.3 Ideals of graphical models

Let \( X_1, \ldots, X_m \) be random variables with finite support. We will analyse the set of all positive probability density functions \( f \) on \( X_1, \ldots, X_m \) that factorise according to a given graph \( G \). First we introduce some notation. Fix a graph \( G \). The random variables take on a finite number of states, but we abstract from which exact states these are, and simply label them. To this end, let \( d \in \mathbb{Z}^{V(G)} \) be the vector denoting for each vertex \( v \in G \) the number of states of the random variable \( X_v \). We can assume each \( d_v \) is at least 2, for otherwise the random variable \( X_v \) would be constant, and we could remove \( X_v \) and the vertex \( v \) from the model.

We denote \( \text{cl}(G) \) and \( \text{mcl}(G) \) as the set of cliques and maximal cliques of \( G \) respectively. For \( S \subseteq V(G) \), we denote \( D_S = \prod_{v \in S} [d_v] \), and by abuse of notation set \( D_G := D_V(G) \). Then for \( T \subseteq S \subseteq V(G) \) and \( i \in D_S \), we denote \( i|T \in D_T \) the restriction of \( i \) to \( T \), i.e. the element \( j \in D_T \) with \( j_v = i_v \) for \( v \in T \). Furthermore, if \( S, T \) are two disjoint subsets of \( V(G) \), and \( i \in D_S, j \in D_T \), then \( i \oplus j \in D_{S \cup T} \) is the concatenation of \( i \) and \( j \), i.e. the element \( k \in D_{S \cup T} \) such that \( k_v = i_v \) for \( v \in S \) and \( k_v = j_v \) for \( v \in T \).

From now on, whenever we have a graph \( G \), we implicitly also assume such a vector \( d \) is given, and denote it \( d^G \) whenever we need to explicitly denote to which graph it belongs. The pair \((G, d)\) is the combinatorial data underlying a graphical model. The probability density functions are given as tuples \((p_i)_{i \in D_G}\).

Note a factorisation of a probability density function can be interpreted as a monomial map \( \mathbb{C}\prod_{C \in \text{mcl}(G)} D_C \to \mathbb{C}^{D_G} \). Here we forget that the conditions that all probabilities must be positive real numbers and must sum to one. This viewpoint gives rise to the interpretation as the set of such density functions as a subset of \( \mathbb{C}^{D_G} \), and we will mainly consider its Zariski-closure \( \overline{X}_G \). This affine
variety $X_G$ is what is called the Graphical Model. In [KRST14] it is noted that indeed this variety $X_G$ is typically not the set of all probability density functions that obey the Markov properties. This is the case if and only if the graph $G$ is chordal, i.e. every cycle of length at least 4 has two vertices that are not adjacent in the cycle, but are connected in $G$ (a chord).

Let $\mathbb{C}[p_G] := \mathbb{C}[p_i \mid i \in D_G]$ and $\mathbb{C}[\theta_G] = \mathbb{C}[\theta_G^C \mid C \in \text{mcl}(G), j \in D_C]$. When dealing with different graphs, we use $p, q, r, ...$ for variables of the first algebra, and $\theta, \lambda, \mu, ...$ for variables of the second algebra. The parametrisation $(p_i)$ in terms of $(\theta_C^j)$ gives rise to a monomial $\mathbb{C}$-algebra homomorphism $\phi_G : \mathbb{C}[p_G] \to \mathbb{C}[\theta_G]$. This map is given on the generators of $\mathbb{C}[p_G]$ as

$$p_i \mapsto \prod_{C \in \text{mcl}(G)} \theta_C^j$$

Then we define the ideal $I_G := \ker(\phi_G)$. This ideal is the vanishing ideal of all possible positive probability density functions on $G$ satisfying the Markov properties of $G$. The variety corresponding to this ideal is exactly $X_G$. The next proposition gives a characterisation of this ideal. For a given $n$, we let for $u \in \mathbb{Z}^n$ denote $u^+, u^- \in \mathbb{Z}_{\geq 0}^n$ such that $u^+, u^-$ have disjoint support and $u = u^+ - u^-$.

**Proposition 8.** $I_G = \left(\left. p_{i}^{u^+} - p_{i}^{u^-} \right| \forall C \in \text{mcl}(G), j \in D_C : \sum_{i \in D_C : i|C = j} u_i = 0 \right) \subset \mathbb{C}[p_G]$, the ideal generated by such binomials.

**Proof:** Note that $\phi_G$ is a monomial map and therefore $I_G$ is a toric ideal. Let $m_i \in \prod_{C \in \text{mcl}(G)} \mathbb{Z}^{D_C}$ such that $p_i \mapsto \theta^{m_i}$. Then

$$(m_i)_{(C, j)} = \begin{cases} 1 & \text{if } i|C = j \\ 0 & \text{else} \end{cases} \quad \text{thus} \quad m_i = \sum_{C \in \text{mcl}(G)} e_{(C, i|C)}$$

Here we denote $\{e_{(C, j)} \mid C \in \text{mcl}(G), j \in D_C\}$ the standard basis of $\prod_{C \in \text{mcl}(G)} \mathbb{Z}^{D_C}$ as a $\mathbb{Z}$-module. Then by [CLS11] Proposition 1.1.9, for $u \in \mathbb{Z}^{D_C}$ we have:

$$p_{u^+} - p_{u^-} \in I_G \iff \sum_{i \in D_G} u_i m_i = 0$$

Therefore we have:

$$p_{u^+} - p_{u^-} \in I_G \iff \sum_{i \in D_G} u_i m_i = 0$$

$$\iff \sum_{i \in D_G} u_i \left( \sum_{C \in \text{mcl}(G)} e_{(C, i|C)} \right) = 0$$

$$\iff \sum_{i \in D_G} \sum_{C \in \text{mcl}(G)} u_i e_{(C, i|C)} = 0$$

$$\iff \sum_{C \in \text{mcl}(G)} \sum_{i \in D_G} u_i e_{(C, i|C)} = 0$$

$$\iff \sum_{C \in \text{mcl}(G)} \sum_{j \in D_C} \sum_{i \in D_G : i|C = j} u_i e_{(C, j)} = 0$$

$$\iff \forall C \in \text{mcl}(G), j \in D_C : \sum_{i \in D_G : i|C = j} u_i = 0$$

Since $\phi_G$ is a monomial map, the ideal $I_G$ is toric and thus generated by binomials. Since we have characterised which binomials are in $I_G$, we know that indeed it has the desired form. \[\square\]
Next we show that we may use any intermediate set $mcl(G) \subseteq M \subseteq \text{cl}(G)$ for the parametrisation, as well as in the characterisation of the ideal $I_G$. For $mcl(G) \subseteq M \subseteq \text{cl}(G)$, denote
\[
\phi^M_G : \mathbb{C}[p_G] \to \mathbb{C} \left[ \lambda^C_j \mid C \in M, j \in D_C \right] \text{ defined as } p_i \mapsto \prod_{C \in M} \lambda^C_{i|C}
\]
Then the proof above generalises directly to the analogue of the proposition with $mcl(G)$ replaced by $M, \phi_G$ replaced by $\phi^M_G$ and $\theta$ replaced by $\lambda$. Furthermore, by replacing $mcl(G)$ in the constraints of $I_G$ by a larger set $M$, we can only make the ideal smaller, i.e. $I_G = \ker(\phi_G) \supseteq \ker(\phi^M_G) \supseteq \ker(\phi^{\text{cl}(G)}_G)$ for all $M$ with $mcl(G) \subseteq M \subseteq \text{cl}(G)$. However, these inclusions collapse to equalities, as shown by the following proposition:

**Proposition 9.** $I_G = \ker(\phi^{\text{cl}(G)}_G)$

**Proof:** The inclusion $I_G \supseteq \ker(\phi^{\text{cl}(G)}_G)$ is shown above: If the constraints hold for all $C \in \text{cl}(G)$, then they certainly hold for all $C \in mcl(G)$.

For the inclusion $I_G \subseteq \ker(\phi^{\text{cl}(G)}_G)$, we show that for all $C \in \text{cl}(G), j \in D_C$ and $u \in \mathbb{Z}^{D_C}$ with $p^{u^+} - p^{u^-} \in I_G$, we have $\sum_{i \in D_C, i|C = j} u_i = 0$. To this extent, let $C \in \text{cl}(G), j \in D_C$ and let $C'$ be a maximal clique containing $C$. We do the proof by induction on $|C' \setminus C|$. If this cardinality is 0, then $C = C'$ is a maximal clique, so the constraint is true by proposition 8. So let the size be greater than 0, and let $v \in C' \setminus C$. Then $C \cup \{v\} \subseteq C'$ is also a clique, with $|C' \setminus (C \cup \{v\})| < |C' \setminus C|$, meaning:

$$\forall k \in [d_v] : \sum_{i \in D_G, i|C = j, i_v = k} u_i = 0$$

Therefore

$$0 = \sum_{k \in [d_v]} \sum_{i \in D_G, i|C = j, i_v = k} u_i = \sum_{i \in D_G, i|C = j} u_i$$

This proposition shows that we can use whichever characterisation of $I_G$ we prefer in a situation, as all such parametrisations give the same ideal. Furthermore, if we let $M$ be a multiset that contains $mcl(G)$, it also holds:

**Proposition 10.** Let $M = \{(C, n_C) \mid C \in \text{cl}(G), n_C \in \mathbb{Z}_{\geq 0}\}$ with $n_C \geq 1$ for $C \in mcl(G)$. Then $M$ is a multiset of elements of $\text{cl}(G)$ that contains $mcl(G)$. Then the following map $\phi^M_G$ has $\ker(\phi^M_G) = I_G$:

$$\phi^M_G : \mathbb{C}[p_G] \to \mathbb{C} \left[ \lambda^C_j \mid C \in \text{cl}(G), j \in D_C \right] \text{ defined as } p_i \mapsto \prod_{(C, n_C) \in M} \left( \lambda^C_{i|C} \right)^{n_C}$$

**Proof:** If we let $n_i \in \prod_{C \in \text{cl}(G)} \mathbb{Z}^{D_C}$ such that $p_i \mapsto \lambda^{n_i}$, then we can see that:

$$n_i_{(C, j)} = \begin{cases} n_C & \text{if } i|C = j \\ 0 & \text{else} \end{cases} \text{ thus } n_i = \sum_{C \in \text{cl}(G)} n_C e_{(C, i|C)}$$

So in a similar way as in the proof of proposition 8, we can see that $p^{u^+} - p^{u^-} \in \ker(\phi^M_G)$ if and only if for each $C \in \text{cl}(G), j \in D_C$ we have $n_C \sum_{i \in D_G, i|C = j} u_i = 0$.

Suppose $p^{u^+} - p^{u^-} \in I_G$ then for each $C \in \text{cl}(G), j \in D_C$ we have $\sum_{i \in D_G, i|C = j} u_i = 0$. Therefore for each such $C, j$ we have $n_C \sum_{i \in D_G, i|C = j} u_i = 0$, meaning $p^{u^+} - p^{u^-} \in \ker(\phi^M_G)$.

On the other hand, if $p^{u^+} - p^{u^-} \in \ker(\phi^M_G)$, then if for each $C \in \text{cl}(G), j \in D_C$ we have $n_C \sum_{i \in D_G, i|C = j} u_i = 0$. Since $n_C \geq 1$ for $C \in mcl(G)$, we have $\sum_{i \in D_G, i|C = j} u_i = 0$ for $C \in mcl(G), j \in D_C$. Therefore $p^{u^+} - p^{u^-} \in I_G$.

So parametrisations that use multisets $M$ that contain $mcl(G)$ give the same ideal $I_G$. 

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6.2.4 Toric fibre products of graphical models

We show how the toric fibre product applies to the toric ideals \( I_G \). First we give a grading and show some of its desired properties.

**Definition 4.** An injective graph homomorphism \( f : H \hookrightarrow G \) is called a graphical homomorphism if \( d^H_v = d^G_{f(v)} \) for all \( v \in V(H) \). If such a homomorphism exists, we call \( G \) \( H \)-graded. \( \blacksquare \)

Since \( f \) is injective, we can interpret this as \( H \) being a subgraph of \( G \) and \( d^G_v = d^H_v \) for all \( v \in V(H) \). Now let \( G \) be \( H \)-graded. Define a map:

\[
gr_H : D_G \rightarrow \prod_{v \in V(H)} \mathbb{Z}^{d^v_v} \quad \text{as} \quad i \mapsto \sum_{v \in V(H)} e_{(v,i_v)}
\]

with \( \{e_{(v,j)} \mid v \in V(H), i \in [d_v]\} \) the standard basis of \( \prod_{v \in V(H)} \mathbb{Z}^{d^v_v} \) as a \( \mathbb{Z} \)-module. This map will define our grading of \( \mathbb{C}[p_G] \) with respect to which \( I_G \) will be homogeneous, so as to use it in the toric fibre product. This explains the terminology of \( G \) being \( H \)-graded. For \( i \in D_G \) denote \( i|_H \) as the restriction of \( i \) to \( V(H) \).

**Proposition 11.** The grading \( \deg(p_i) = gr_H(i) \) makes \( I_G \) homogeneous, and for the algebras \( \mathbb{C}[p_{G_1}] \) and \( \mathbb{C}[q_{G_2}] \) of two \( H \)-graded graphs \( G_1,G_2 \) we then have \( \deg(p_i) = \deg(q_j) \iff i|_H = j|_H \).

**Proof:** The latter fact is clear. To see that \( I_G \) is homogeneous, we note that it is toric, thus generated by binomials of the form \( p^u - p^v \). If we let \( \pi : \mathbb{Z}^{D_G} \rightarrow \prod_{v \in V(H)} \mathbb{Z}^{d^v_v} \) be the linear map that sends \( u \) to \( \deg(p^u) \), then for \( I_G \) to be homogeneous it is sufficient to show that \( \pi(u^+) = \pi(u^-) \) for all \( u \) with \( p^u - p^v \in I_G \), i.e. that \( \pi(u) = 0 \) for all such \( u \).

We can see that:

\[
\pi(u) = \sum_{i \in D_G} u_i \pi(e_i) = \sum_{i \in D_G} u_i \deg(p_i) = \sum_{i \in D_G} u_i \left( \sum_{v \in V(H)} e_{(v,i_v)} \right) = \sum_{i \in D_G} \sum_{v \in V(H)} u_i e_{(v,i_v)} = \sum_{v \in V(H)} \sum_{i \in D_G} u_i e_{(v,i_v)} = \sum_{v \in V(H)} \sum_{j \in [d_v]} \sum_{i \in D_G : i_v = j} u_i e_{(v,j)} = \sum_{v \in V(H)} \sum_{j \in [d_v]} e_{(v,j)} \left( \sum_{i \in D_G : i_v = j} u_i \right)
\]

So for this to be 0, we only need:

\[
\forall v \in V(H), j \in [d_v] : \sum_{i \in D_G : i_v = j} u_i = 0
\]

This is guaranteed by proposition \( \ref{prop:6.2.3} \) as \( \{v\} \) is a clique of \( G \). \( \blacksquare \)

This grading allows us to take the toric fibre product of two \( H \)-graded models \( G_1,G_2 \). The next proposition shows why this grading is a useful one.

**Proposition 12.** Let \( G_1, G_2 \) be \( H \)-graded. Then \( I_{G_1 \times H G_2} = I_{G_1 \times A} I_{G_2} \) with \( A = \{\deg(p_i),\deg(q_j)|i \in D_{G_1}, j \in D_{G_2}\} \).
Proof: We will use the following characterisation of the toric fibre product of kernels of homomorphisms from \[\text{EKS14}].

Suppose \(\mathbb{C}[x] = \mathbb{C}[x_i], \mathbb{C}[y] = \mathbb{C}[y_i]\) are \(A\)-graded as \(\deg(x_i) = \deg(y_i) = a_i\), and we are given two \(\mathbb{C}\)-algebra homomorphisms \(\phi: \mathbb{C}[x] \to \mathbb{C}[s], \psi: \mathbb{C}[y] \to \mathbb{C}[t]\). Then \(\ker(\phi) \times_A \ker(\psi) = \ker(\phi \times_A \psi)\), where
\[
\phi \times_A \psi: \mathbb{C}[z] = \mathbb{C}[z_{i,k}] \to \mathbb{C}[s,t] \quad \text{defined by} \quad z_{i,k}^j \mapsto \phi(x_i^j)\psi(y_k^j)
\]

\[\square\]

Now let \(G = G_1 + H G_2\). We will create a map \(\nu: \mathbb{C}[r_G] \to \mathbb{C}[\mu_G]\) using the map above defining the toric fibre product of \(I_{G_1}\) and \(I_{G_2}\). The kernel of this map will be the toric fibre product \(I_{G_1} \times_A I_{G_2}\), and it will also be equal to \(\ker(\phi_G) = I_G\). This shows that then \(I_G = I_{G_1} \times_A I_{G_2}\).

Let \(V_i = V(G_i) \setminus V(H)\), and consider the following isomorphisms:
\[
\mathbb{C}[p_{G_i}] = \mathbb{C}[p_i | i \in D_{G_1}] \simeq \mathbb{C}[p_{G_1}^j | i \in D_{H}, j \in D_{V_i}] \quad \text{where} \quad p_i \mapsto p_{i|V_i}^j
\]

And similarly for \(\mathbb{C}[q_{G_2}]\). With gradings on these algebras induced by the isomorphisms, we can then see \(\deg(p_{G_1}^j) = \deg(q_{G_2}^k)\) if and only if \(i_1 = i_2\). Therefore \(I_{G_1} \times_A I_{G_2}\) is the kernel of the following map:
\[
\phi_{G_1} \times_A \phi_{G_2}: \mathbb{C}[r_{G_1}^j | i \in D_{H}, j \in D_{V_1}, k \in D_{V_2}] \to \mathbb{C}[\theta_{G_1}, \lambda_{G_2}]
\]
\[
\text{defined by} \quad r_{G_1}^j \mapsto \phi_{G_1}(p_{G_1}^j) \phi_{G_2}(q_{G_2}^k)
\]

We can easily see that \(\mathbb{C}[r_{G_1}^j] \simeq \mathbb{C}[r_G]\). Define a map \(\chi: \mathbb{C}[\theta_{G_1}, \lambda_{G_2}] \to \mathbb{C}[\mu_G]\) as \(\lambda_{G_1}^C \mapsto \mu_{G_1}^C\) for \(C \in \text{mcl}(G_2), j \in D_{C}, \text{and} \theta_{G_1}^C \mapsto \mu_{G_2}^C\) for \(C \in \text{mcl}(G_1), j \in D_{C}\), and define \(\nu = \chi \circ (\phi_{G_1} \times_A \phi_{G_2})\). Then we can see that the map \(\nu\) is defined as:
\[
r_i \mapsto \left(\prod_{C \in \text{mcl}(G_1)} \left(\prod_{i \in D_{C}} (\theta_{G_1}^C)^{u_+} (\lambda_{G_1}^C)^{u_-}\right)\right)
\]
\[
\left(\prod_{C \in \text{mcl}(G_2)} \left(\prod_{j \in D_{C}} (\theta_{G_2}^C)^{u_+} (\lambda_{G_2}^C)^{u_-}\right)\right)
\]

This map is almost the same as \(\phi_G\), except that variables \(\mu_{G_1}^C\) occur twice in each \(\nu(r_i)\). However, by proposition \[\text{10}\] we can also take multisets for the parametrisation and obtain the same ideal, and so \(I_G = \ker(\nu)\). Therefore we have
\[
I_G = \ker(\nu) = \nu^{-1}(\{0\}) = (\phi_{G_1} \times_A \phi_{G_2})^{-1}(\ker(\chi)) \supseteq (\phi_{G_1} \times_A \phi_{G_2})^{-1}(\{0\}) = I_{G_1} \times_A I_{G_2}
\]

Furthermore, if \(r_{u_+} - r_{u_-} \in I_G\) then for all \(C \in \text{cl}(G)\) and \(j \in D_{C}\) we have \(\sum_{i \in D_{C} : i | C = j} u_i = 0\), or, equivalently, that:
\[
\sum_{i \in D_{C} : i | C = j} u_i^+ = \sum_{i \in D_{C} : i | C = j} u_i^-
\]

Then we can see that:
\[
(\phi_{G_1} \times_A \phi_{G_2})(r_{u_+}) = \left(\prod_{i \in D_{G_1} \text{cl}(G_1)} \left(\prod_{j \in D_{G_1}} (\theta_{G_1}^C)^{u_+} (\lambda_{G_1}^C)^{u_-}\right)\right)
\]
\[
= \left(\prod_{C \in \text{mcl}(G_1)} \left(\prod_{i \in D_{C}} (\theta_{G_1}^C)^{u_+} (\lambda_{G_1}^C)^{u_-}\right)\right)
\]
\[
= \left(\prod_{C \in \text{mcl}(G_1)} \left(\prod_{i \in D_{C}} (\theta_{G_1}^C)^{u_+} (\lambda_{G_1}^C)^{u_-}\right)\right)
\]
\[
= \left(\prod_{C \in \text{mcl}(G_1)} \left(\prod_{j \in D_{C}} (\theta_{G_1}^C)^{u_+} (\lambda_{G_1}^C)^{u_-}\right)\right)
\]
\[
= \left(\prod_{C \in \text{mcl}(G_1)} \left(\prod_{j \in D_{C}} (\theta_{G_1}^C)^{u_+} (\lambda_{G_1}^C)^{u_-}\right)\right)
\]
\[
= \left(\prod_{C \in \text{mcl}(G_1)} \left(\prod_{j \in D_{C}} (\theta_{G_1}^C)^{u_+} (\lambda_{G_1}^C)^{u_-}\right)\right)
\]
\[
= \left(\prod_{C \in \text{mcl}(G_1)} \left(\prod_{j \in D_{C}} (\theta_{G_1}^C)^{u_+} (\lambda_{G_1}^C)^{u_-}\right)\right)
\]
\[
= \left(\prod_{C \in \text{mcl}(G_1)} \left(\prod_{j \in D_{C}} (\theta_{G_1}^C)^{u_+} (\lambda_{G_1}^C)^{u_-}\right)\right)
\]
Then we can work back the same way replacing every occurrence of $u^+$ by $u^-$ and obtain that 
$(\phi_{G_1} \times_A \phi_{G_2})(r^{u^+} - r^{u^-}) = 0$. Therefore $I_G \subseteq \ker(\phi_{G_1} \times_A \phi_{G_2}) \subseteq I_G$, meaning they are the same. Therefore, the toric fibre product of $I_{G_1}$ and $I_{G_2}$ under the given grading is equal to the ideal $I_{G_1 + H G_2}$. That is, $I_{G_1 + H G_2} = I_{G_1} \times_{gr_H} I_{G_2}$.

In the proof, we need that $\deg(p_i) = \deg(q_j) \iff i|_H = j|_H$ when constructing the algebra $C[r_{j,k}^i]$, but the actual degrees are not of interest as noted in section 4.1. However, to prove that there is a grading such that $\deg(p_i) = \deg(q_j) \iff i|_H = j|_H$, it is convenient to create an actual grading.

The result of these two propositions is that glueing of two graphs gives a vanishing ideal that is the toric fibre product of the vanishing ideals of the two graphs. Quite obviously, a glueing $G = G_1 + H G_2$ is also $H$-graded, and we can therefore use $G$ again in such a glueing. This allows us to iterate the glueing process.

6.2.5 Iterated glueing of graphical models

Since graphical models give rise to toric ideals, and glueing of graphical models corresponds to taking the toric fibre product of the corresponding toric ideals, we can see that for any finite number of graphical models $G_1, \ldots, G_s$ with a common subgraph $H$ and state vectors $d_1, \ldots, d_s$ that agree on $H$, there is a constant $C \in \mathbb{N}$ such that for any $a \in \mathbb{N}^s$, the graphical model with graph $G = (\sum_{i=1}^s G_i + H \cdots + H (\sum_{i=1}^s G_i))$ and state vector $d$ coming from $d_1, \ldots, d_s$ according to the glueing, the ideal of $(G, d)$ is generated by binomials of degree at most $C$.

The following example shows the occurrence of this stabilisation in a specific case:

**Example 4.** This example is shown to stabilise in [KRST]. Let $G = K_{2,N-2}$, the complete bipartite graph on 2 and $N-2$ vertices (with independent sets $A = \{v_1, v_2\}, B = \{v_3, \ldots, v_N\}$) and state vector $d$ such that $d_{v_1} = d_{v_2} = 2$. The bound on the degree of the polynomials is explicitly shown to be 4, which is independent of $N$ and even independent of $d_{v_3}, \ldots, d_{v_N}$. For $i, j \in [2], K \in D_B$ we denote $p_{i,j,K}$ the probability that $X_{v_1} = i, X_{v_2} = j$ and $X_{v_k} = K_{vk}$ for $k = 3, \ldots, N$. Then the generating polynomials are the following quadratics and quartics:

$$p_{1,1,K} \cdot p_{2,2,K} - p_{1,2,K} \cdot p_{2,1,K}$$

$$\forall K \in D_B$$

$$p_{i,j,K+L} \cdot p_{i,j,K'+L} - p_{i,j,K+L'} \cdot p_{i,j,K',L}$$

for all bipartitions $(I, J)$ of $B$ and $i, j \in [2], K, K' \in D_1, L, L' \in D_J$

$$p_{1,1,k_1+L_1,1} \cdot p_{1,2,k_2+L_1,2} \cdot p_{2,1,k_1+L_2,1} \cdot p_{2,2,k_2+L_2,2} - p_{1,1,k_1+L_{1,1}} \cdot p_{2,1,k_1+L_{1,2}} \cdot p_{2,1,k_1+L_{2,1}} \cdot p_{2,2,k_2+L_{2,2}}$$

$$\forall a \in \{3, \ldots, N\}, k_1, k_2 \in [d_a], L_{1,1}, L_{1,2}, L_{2,1}, L_{2,2} \in D_B \setminus \{a\}$$

The quadratic polynomials can be interpreted as follows. Define $X^K$ as the $2 \times 2$-table $(p_{i,j,K})_{i,j}$ for $K \in D_B$ (the slice of the table along $K$), and $Y^{i,j}$ as the $d_{v_3} \times \cdots \times d_{v_N}$-table $(p_{i,j,K})_{K \in D_B}$ (the slice along $(i, j)$). Then the quadratics are exactly the $2 \times 2$-determinants of all tables $X^K$ for $K \in D_B$ and all $2 \times 2$-subdeterminants of all possible flattenings of all tables $Y^{i,j}$.

The quartics do not have such a direct interpretation as subdeterminants of flattenings of slices. However, a clarification can be made as follows: We pick a vector $v_a$ in $B$ and two possible configurations $k_1, k_2$ for $v_a$. Then we augment the 4 configurations $(1, 1), (1, 2), (2, 1), (2, 2)$ of the vertices $(v_1, v_2)$ to arbitrary configurations of all vectors except $v_a$ (the configuration $L_{i,j}$ is the augmentation of configuration $(i, j)$). The quartics encode certain dependencies between combinations of these configurations.

In fact, the quadratics are binomial generators for the vanishing ideal of the set of all probability density functions that obey the Markov properties of $K_{2,N-2}$. With the additional quartics we obtain the vanishing ideal $I_{K_{2,N-2}}$ of all such functions that are positive.
Chapter 7

Conclusions

In this chapter we summarise the most important results of this thesis and show where there is room for further research and improvements.

7.1 Summary

In this thesis we have analysed the iterated toric fibre product applied to a finite number of ideals. In doing so we have considered the inverse limit of rank-at-most-1 tensors, and seen that its coordinate ring is Noetherian up to the action of the monoid $\text{Subs}_<(\mathbb{N})$. To prove this we have found new criteria (theorem 5) for monoid algebras to be Noetherian up to the action of a monoid of algebra endomorphisms.

The main result is theorem 15, which shows that for a finite number of Hadamard-closed varieties, the iterated toric fibre products of their ideals are generated by polynomials of uniformly bounded degree. This implies theorem 1 which is a restatement of [RS16, Conjecture 56]. We have specifically seen how this implies such a uniform bound for the binomials generating the ideals of glueings of graphical models. Since generating binomials of these ideals correspond one to one to Markov bases of the graphical models, and their maximal degree corresponds to the Markov degree of the basis, we have found a uniform bound on the Markov degree of such families of glueings.

7.2 Improvements

This thesis leaves room for improvements and further research. First off, the uniform bound we have found is existential, so we know it exists, but not what it might be in terms of the ideals or varieties, or how to construct Markov bases that obey the bound.

Related, but slightly different is the following question. We have shown that there exist a finite number of polynomials, the orbits of which generate the ideal of the projective limit of the iterated toric fibre product. However, it would be beneficial to know how the ideals and their generating sets relate to each other. Results in [RS16] show how to lift such bases from the generators of the pieces in certain cases, such as when the set $\mathcal{A}$ of degrees is linearly independent, or (in the case of matrices, as in section 6.1), when the associated semigroups are normal or have a finite number of holes.

Another improvement, which might in turn lead to the above, is to notice that the Noetherianness of the algebra $\mathbb{K}[T^\leq_1(V)]$ does not actually depend on the action of the entire monoid $\text{Subs}_<(\mathbb{N})$ but rather only uses certain surjections $[b] \to [a]$, which embed as elements $\sigma \in P_{b,a}$. Those surjections could lead to insight on how the ideals interact.

Finally something that does not directly relate to the main results, but has led to it, is the result on Noetherianness of monoid algebras. We have shown new criteria for a monoid algebra to be Noetherian up to the action of a monoid of algebra endomorphisms. It might be interesting to see how these new criteria fit in with other results from [HS12], what other criteria are sufficient, and what is necessary. Specifically, it would be interesting to see how $P$-invariant chains behave under the new criteria, and how these might in turn be interpreted as chains of ideals of iterated toric fibre products.
Bibliography


