Dynamics of mixing in 2D Stokes flows, numerics and experiments

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Dynamics of mixing in 2D Stokes flows, numerics and experiments

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Master’s thesis

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Summary

Mixing of very viscous fluids is typical for polymer blending, food processes etc. Although important advances in the understanding of the basic mechanisms of laminar mixing have been achieved, the work on a general framework that would allow for quantifying the dynamics of mixture quality and mixing abilities of different flows is still far from complete. The goal of this thesis is to continue the development of a general analysis, by approaching several mixing problems from a dynamics viewpoint. The flows to be considered exist of incompressible, viscous, Stokes flows. Knowing the velocity field, is enough to determine the position of a particle after an arbitrary amount of time.

Zones of chaotic mixing and regular motion are revealed by using Poincaré maps associated to the velocity field. Mappings of incompressible fluids can contain unstable periodic points or elliptic points, surrounded by periodic islands, which do not mix with the rest of the fluid. A bifurcation occurs when a small smooth change made to the parameter values of a system causes a sudden “qualitative” or topological change in the systems behavior.

The Blinking Vortex flow consists of two co-rotating point vortices that blink on and off periodically. The mapping exists of an analytical expression. It turns out that Poincaré sections do not provide all the information about the “regular” and “chaotic” regions. In order for truly understanding the mixing process, a better, analytical, approach is needed. The position of periodic points of the Blinking Vortex flow can be determined by the intersection of lines of symmetry, because the periodic points are located on these lines or in pairs on opposite sides of these lines. By varying the actuation parameter $\mu$, series of bifurcations occurs; the positions and the number of periodic points changes. Two periodic points become suddenly stable and shift towards an unstable periodic point resulting in a Pitchfork bifurcation.

The Journal Bearing flow consists of two eccentric cylinders, both actuated periodically by turns. Poincaré maps are being made by integrating the analytical expression for the velocity field. Some hyperbolic periodic points have already been located. However, no general approach has been developed yet to locate these points. Using the lines of symmetry a periodic island of period 2 has been detected. This island arises by a Saddle Node bifurcation and looses its stability again by a Period Doubling bifurcation, where a stable periodic island of period 4 arises. The position of this island is again determined by using the lines of symmetry. Another Period Doubling bifurcation leads to a periodic island of period 8. The same process results in a periodic island of period 16. By plotting a Feigenbaum tree, the process of period doubling becomes visible and shows the plausibility that the process of Period Doubling bifurcations keeps repeating until a periodic island of period infinity exists, meaning complete chaos.

The periodic islands of period 2 have been detected experimentally. Although the islands are much smaller as the numerically determined islands, they are still after 20 periods well visible. They even match quantitatively well with the numerically determined elliptic points. The periodic islands of period 4 have not been detected, because they are likely too small to notice.
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Chapter 1

Introduction

1.1 Laminar fluid mixing

Mixing of fluids is a topic of significant interest, because of both the widespread occurrence of such phenomena in nature and its importance for industrial applications. Although in many cases mixing is associated with turbulent fluid motions, mixing of very viscous fluids constitutes another important class of mixing occurrences, and is typical for polymer blending, compounding, food processes etc.

The traditional solution to many mixing problems has been to increase the energy input and to let the turbulence produce effective mixing. With the high viscosities associated with polymers, turbulent flow is not achievable and laminar flow is the only possible mechanism for mixing. Another reason why mixing processes are often laminar is when excessive stresses should be avoided, for example in bio-polymers, [6].

Although important advances in the understanding of the basic mechanisms of laminar mixing have been achieved, the work on a general framework that would allow for quantifying the dynamics of mixture quality and mixing abilities of different flows is still not completed and continues.

1.2 Dynamical systems

The flows to be considered exist of incompressible, viscous, Stokes flows. These kind of flows have theoretically no diffusion, meaning fluid-particles are expected to return to their initial locations after flow reversal. Knowing the velocity field, is enough to determine the position of a particle after an arbitrary amount of time. Therefore the possibility of making use of dynamical system tools appears.

The goal of this thesis is to continue the development of a general analysis, by approaching several mixing problems from a dynamical viewpoint. Progress has already been made by several others, for example Ottino [13], who have made use of Poincaré maps to display characteristics of some flows. Ottino has worked also on finding periodic points in these Poincaré maps, meaning a closed orbit in the flow, and deciding their stability by looking at the eigenvalues of these points.
CHAPTER 1. INTRODUCTION

This thesis will continue the work on Poincaré maps to gain more knowledge about the possibilities and impossibilities of mixing. This will be applied on two different mixing cavities; the Blinking Vortex flow, which consists of two co-rotating point vortices, separated by a fixed distance, that blink on and off periodically, and the Journal Bearing flow, which consists of two eccentric cylinders, both actuated periodically by turns. A more general approach in finding periodic points will be introduced by making use of the lines of symmetry of the flows. The stability of the periodic points will be determined by the eigenvalues of the Jacobian at these equilibria. The development of these stabilities will be defined, leading to bifurcation diagrams showing the positions of stable periodic points as a function of the parameter value. The analysis will be treated in an analytical and numerical way. At last the results will be validated experimentally.

1.3 Outline of the thesis

To develop a general analysis, at first some tools from the dynamical systems theory will be considered in chapter 2. The use of Poincaré maps in revealing the zones of chaotic mixing and regular motion will be discussed. Herein an important part is played by the stability of periodic points. In chapter 3 the first flow will be examined, the Blinking Vortex flow. Periodic points will be determined analytically in different ways using symmetry characteristics. Using the gained knowledge of the symmetry characteristics the Journal Bearing flow will be examined in chapter 4. The detected results of the Journal Bearing flow will be validated by experiments in chapter 5. Finally, chapter 6 summarizes the results and gives some recommendations for future work.
Chapter 2

Dynamical system tools

2.1 Poincaré map

The Poincaré map method, named after Henri Poincaré, helps to reveal zones of chaotic mixing and regular motion, [13]. It allows a systematic reduction in complexity of problems by means of a reduction in the number of dimensions since it converts a flow into a map.

Consider an \( n \)-dimensional system

\[
\dot{x} = f(x) \tag{2.1}
\]

Let \( S \) be an \((n-1)\)-dimensional surface of section, see figure 2.1. \( S \) is required to be transversal to the flow, i.e., all trajectories starting on \( S \), flow through it, i.e. they are not parallel to it.

The Poincaré map \( P \) is a mapping from \( S \) in itself, obtained by following trajectories from one intersection with \( S \) to the next, if this exists. If \( x_k \in S \) denotes the \( k \)th intersection, then the Poincaré map is defined by

\[
x_{k+1} = P(x_k) \tag{2.2}
\]

This equation relates two consecutive intersection points. A Poincaré map turns a continuous time dynamical system into a discrete time one. It is also called Poincaré section, referring to the surface of section \( S \).

Suppose that \( x^* \) is a fixed point of \( P \), i.e., \( P(x^*) = x^* \). Then a trajectory starting at \( x^* \) returns to \( x^* \) after one period, and is therefore a closed orbit for the original system \( \dot{x} = f(x) \).

The point \( x^* \) is called a periodic point of period 1. There also may exist periodic points of higher period. If \( P(y_k^*) = y_{k+1}^* \neq y_k^* \) and \( P(y_{k+1}^*) = y_{k+2}^* = y_k^* \), the point \( y_k^* \) is said to be a periodic point of period 2, etc.

By looking at the behavior of \( P \) near a fixed point, the stability of the closed orbit can be determined. Thus the Poincaré map converts problems about closed orbits into problems about fixed points of a mapping.

In this thesis 2-dimensional flows depending on time will be considered, hence represented by a 3-dimensional system. In the case of time-periodic systems the technique of Poincaré amounts
to take stroboscopic pictures of initial conditions at timesteps $T$, $2T$, $3T$, ..., etc. This will reduce the system to a 2-dimensional one. A single plot can now display the phase portrait, revealing the solutions belonging to all possible initial conditions.

### 2.2 Stability of periodic points

To determine the stability of a periodic point in a mapping, we first take a look at the 1-dimensional situation:

Consider the system

$$x_{n+1} = f(x_n)$$

(2.3)

with $x$ 1-dimensional and $x_p$ a periodic point of period 1. If the Jacobian of point $x_n = x_p + h_n$ will be determined, with deviation $|h_n| << 1$, then

$$x_{n+1} = x_p + h_{n+1} = f(x_p + h_n) = f(x_p) + f'(x_p) \cdot h_n + O(h_n^2)$$

$$= x_p + f'(x_p) \cdot h_n + O(h_n^2) \Rightarrow$$

$$h_{n+1} = f'(x_p) \cdot h_n + O(h_n^2)$$

(2.4)

So if $|f'(x_p)| < 1$, then the deviation $h_n$ converges to 0 and the periodic point turns out to be asymptotically stable. If $|f'(x_p)| > 1$ the deviation $h_n$ diverges and the periodic point turns out to be unstable.

We can apply the same principle for 2-dimensional systems, by looking at the eigenvalues of the Jacobian at an equilibrium, $\lambda_1$ and $\lambda_2$. This results in, [15]:

- if $|\lambda_1| < 1$ and $|\lambda_2| < 1$ the equilibrium is called a sink, meaning it is asymptotically stable; this is reflected in figure 2.2a.

- if $|\lambda_1| > 1$ and $|\lambda_2| > 1$ the equilibrium is called a source, meaning it is unstable; this is reflected in figure 2.2b.
• if $|\lambda_1| > 1$ and $|\lambda_2| < 1$, meaning contraction in one direction and stretching in the other direction, the equilibrium is called a saddle point and is also unstable; reflected in figure 2.2c.

• if $|\lambda_1| = 1$ and $|\lambda_2| = 1$ the situation is unclear, because in this case the higher order terms $O(h_0^2)$ of (2.4) cannot be neglected. The periodic point might be a center, also called elliptic periodic point, containing rotational motion in its neighborhood. This is for instance the case when (2.3) is linear with two eigenvalues $|\lambda_i| = 1$; see figure 2.2d.

An elliptic point with its rotating neighborhood forms an island of fluid, called periodic island, which does not mix with the rest of the fluid. Fluid outside a periodic island cannot penetrate the island, and vice versa.

Considering the 1-dimensional situation, we only know for sure that if a periodic point is an elliptic point, the deviation $h_{n+1}$ of (2.4) has to equal $h_n$, so the absolute value of the eigenvalues has to equal 1 and the higher order terms have to equal 0, the equilibrium will then be stable. But if the higher order terms do not equal zero, the periodic point can be either stable or unstable.

A fixed point of a map is called hyperbolic if none of the absolute values of the eigenvalues equals one. Within the Dynamical Systems theory a hyperbolic fixed point, can therefore be a saddle, a sink or a source. In mappings of incompressible fluids sinks and sources cannot exist, because the fixed point cannot absorb or produce fluid. So there are only 2 possibilities left: center or saddle, the latter is therefore simply called hyperbolic periodic point.
The Jacobian at an equilibrium can be calculated numerically and looks like as follows:

\[ J = \left( \begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array} \right) \bigg|_{(x_p,y_p)} \]  

(2.5)

The terms herein can be determined using the centered difference formula:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{f(x+\epsilon)-f(x-\epsilon)}{2\epsilon}, \\
\frac{\partial f}{\partial y} &= \frac{f(y+\epsilon)-f(y-\epsilon)}{2\epsilon}, \\
\frac{\partial g}{\partial x} &= \frac{g(x+\epsilon)-g(x-\epsilon)}{2\epsilon}, \\
\frac{\partial g}{\partial y} &= \frac{g(y+\epsilon)-g(y-\epsilon)}{2\epsilon}.
\end{align*}
\]  

(2.6)

In addition to the theory of the Poincaré map, there is one important note to be made: Consider the point \( x^* \), from figure 2.1, as the only fixed point and moreover as an elliptic one surrounded by a small elliptic island. By making a Poincaré section with a number of initial positions, except for initial positions situated within the elliptic island, the fixed point will not be discovered, although it exists. It can only be discovered by making a Poincaré section with an initial position situated within this island. This shows that a Poincaré section strongly depends on the initial conditions. In chapter 3.3 this will be shown by an example.

### 2.3 Bifurcations

A bifurcation occurs when a small smooth change made to the parameter values of a system causes a sudden “qualitative” or topological change in the systems long-term dynamical behavior. In particular, fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are called bifurcations, and the parameter values at which they occur are called bifurcation points. Bifurcations are important scientifically, they provide models of transitions and instabilities as some control parameter is varied.

Before we will look at bifurcations in discrete time, we will look at bifurcations in continuous time.

#### 2.3.1 Continuous time

In continuous time we consider the differential equation

\[ \dot{x} = F(x, \mu) \]  

(2.7)

Here the state \( x \in \mathbb{R}^n \) and the parameter \( \mu \in \mathbb{R} \). Equilibria are characterized by

\[ F(x, \mu) = 0 \]  

(2.8)

Clearly this equation may have, for fixed \( \mu \), no, one, two, ..., or infinitely many solutions. Let \( x^*(\mu) \) be a solution for fixed \( \mu \).

Assume the Jacobian of the equilibrium \( x^*(\mu) \) has no eigenvalue equal to 0. Then, by the implicit function theorem, we find a smooth branch of solutions \( x^*(\mu) \).

In case the Jacobian of the equilibrium \( x^*(\mu) \) has one or more zero eigenvalues at an equilibrium point \( x^*(\mu) \), several branches may come together. By drawing these equilibrium-solution branches in the \((x, \mu)\) space, called a bifurcation diagram, the bifurcations of a system become visible. Some well known bifurcations will be introduced now.
2.3. BIFURCATIONS

Fold bifurcation/ Saddle node bifurcation

Consider the differential equation
\[ \dot{x} = \mu + x^2 \] (2.9)

To specify the qualitative behavior of this system, the velocity field is plotted for several values of \( \mu \) in figure 2.3. When \( \dot{x} < 0 \) the velocity field is negative, indicated by arrows to the left. The arrows to the right implying a positive velocity field. The fixed points are indicated by the intersections with the x axis and the arrows show whether it is a stable or unstable fixed points. The stable points are indicated by filled dots and the unstable points are indicated by open dots. This system consist of several qualitative states:

- \( \mu < 0 \): 2 equilibria, 1 stable and 1 unstable equilibrium
- \( \mu > 0 \): no equilibrium
- \( \mu = 0 \): ‘double’equilibrium

The bifurcation diagram is represented in figure 2.4. To distinguish between stable and unstable fixed points, a solid line is used for stable points and a broken line is used for unstable ones.

Figure 2.3: Velocity fields of a Saddle Node bifurcation, \( \dot{x} = \mu + x^2 \)

Figure 2.4: Fold bifurcation/ Saddle node bifurcation
Transcritical/ Exchange of stability bifurcation

Consider the differential equation

\[ \dot{x} = \mu x - x^2 \]  

(2.10)

Figure 2.5 shows the velocity fields for several values of \( \mu \). This system contains always two equilibria; \( x = 0 \) and \( x = \mu \). The stability depends on the value of \( \mu \):

- \( \mu < 0 \) : \( x = 0 \) stable
- \( \mu > 0 \) : \( x = \mu \) stable

When \( \mu \) reaches 0, see figure 2.5 b, The arrows show contraction in one direction and stretching in the other direction. The corresponding bifurcation diagram is represented in figure 2.6.
Pitchfork bifurcation

An example of the pitchfork bifurcation is given by the differential equation

\[ \dot{x} = \mu x - x^3 \]  

(2.11)

The velocity fields of this system are shown in figure 2.7. The stable fixed point \( x = 0 \) changes in an unstable one when \( \mu \) becomes positive. At the same time two new stable fixed points show up. So there are again two qualitative states to distinguish, as can be seen also in the bifurcation diagram, see figure 2.8:

- \( \mu < 0 : 1 \) stable equilibrium
- \( \mu > 0 : 3 \) equilibria, \(-\sqrt{\mu} (\text{stable}), 0 (\text{unstable}), \sqrt{\mu} (\text{stable})\)
2.3.2 Discrete time

To analyse the bifurcations of maps, we will look at the discrete time system

\[ x_{k+1} = G(x_k, \mu) \]  

(2.12)

Periodic points of period 1 are characterized by

\[ x(\mu) = G(x(\mu), \mu) \]  

(2.13)

The functions (2.12) and (2.13) will be plotted in the same figure to visualize the periodic points. The intersections of both functions indicate the periodic points of period 1.

**Fold bifurcation/ Saddle Node bifurcation**

Consider the discrete time system

\[ x_{k+1} = x_k + \mu + x_k^2 \]  

(2.14)

The periodic points of period 1 can be determined by using equation (2.13) in (2.14)

\[ x_k = x_k + \mu + x_k^2 \]

\[ -\mu = x_k^2 \Rightarrow \]

\[ x_k = \pm \sqrt{-\mu}, \mu < 0 \]  

(2.15)

Figure 2.9a, with \( \mu = -1 \), shows these two periodic points. To determine the stability of these points, we take an arbitrary value for \( x_k \), draw a vertical line to the belonging value of \( x_{k+1} \) and take this value as a new value for \( x_k \), by drawing a horizontal line until it reaches the linear line of (2.13). If this method is repeated several times and for several initial positions of \( x_k \), represented by \( * \), it becomes clear which periodic points are repelling points and which one are attracting points.

![Figure 2.9: Saddle Node bifurcation in discrete time, \( x_{k+1} = x_k + \mu + x_k^2 \)]

We see that for \( \mu < 0 \) this systems has one stable and one unstable periodic point, for \( \mu = 0 \) the system had one saddle point and for \( \mu > 0 \) the system has no periodic points of period 1 at all. So this is Saddle Node bifurcation, which can be represented by the same bifurcation diagram as shown in figure 2.4, although the dynamics of a discrete time system and a continuous time system are of course not the same.
2.3. BIFURCATIONS

Transcritical bifurcation

An example of a Transcritical bifurcation in discrete time is the system

\[ x_{k+1} = x_k + \mu x_k + x_k^2 \]  

(2.16)

The periodic points of period 1 can be determined by using equation (2.13) in (2.18)

\[
\begin{align*}
    x_k &= x_k + \mu x_k + x_k^2 \\
    0 &= x_k(\mu + x_k) \Rightarrow \\
    x_k &= 0 \lor x_k = -\mu \\
\end{align*}
\]

(2.17)

The logistic maps, figure 2.10, of this system show 2 periodic points for \( \mu < 0 \), where the point \( x_k = 0 \) is a stable periodic point and the point \( x_k = -\mu \) is an unstable periodic point. For \( \mu = 0 \) the system contains one double periodic point \( x_k = 0 \), which is a saddle point and for \( \mu > 0 \) the system contains again 2 periodic points, which exchanged stability. This system can be represented by the same bifurcation diagram as shown in figure 2.6, the transcritical bifurcation.

![Figure 2.10: Logistic maps of a Transcritical bifurcation, \( x_{k+1} = x_k + \mu x_k + x_k^2 \)](image)

Pitchfork bifurcation

A pitchfork bifurcation is represented by the system

\[ x_{k+1} = x_k + \mu x_k + x_k^3 \] 

(2.18)

Which contains the following periodic points of period 1

\[
\begin{align*}
    x_k &= x_k + \mu x_k + x_k^3 \\
    0 &= x_k(\mu + x_k^2) \Rightarrow \\
    x_k &= 0 \lor x_k = \pm \sqrt{-\mu}, \mu < 0 \\
\end{align*}
\]

(2.19)

This system contains 3 periodic points of period 3 for \( \mu < 0 \), visualised by the logistic map, figure 2.11. The points \( x_k = \pm \sqrt{-\mu} \) are unstable periodic points and the point \( x_k = 0 \) is a stable one. When \( \mu \) reaches 0 the unstable points disappear and the stable one becomes unstable. This system can be represented by a pitchfork bifurcation diagram as shown in figure 2.8.
Figure 2.11: Logistic maps of a Pitchfork bifurcation, $x_{k+1} = x_k + \mu \cdot x_k + x_k^3$

**Period doubling bifurcation**

The bifurcations of discrete time systems considered until now, have one characteristic in common; the eigenvalue of the linearized system at the bifurcation point equals 1 in all cases

$$\frac{\partial G}{\partial x_k}(x_k = 0, \mu = 0) = 1$$  \hspace{1cm} (2.20)

There exists another interesting situation; a bifurcation point containing an eigenvalue of -1. An example of such a bifurcation is represented by the system

$$G(x_k, \mu) = x_{k+1} = -x_k - \mu x_k + x_k^3$$  \hspace{1cm} (2.21)

This system contains the following periodic points of period 1

$$x_k = -x_k - \mu x_k + x_k^3$$
$$0 = -2x_k - \mu x_k + x_k^3$$
$$0 = x_k(-2 - \mu + x_k^2) \Rightarrow$$
$$x_k = 0 \quad \text{or} \quad x_k = \pm \sqrt{2 + \mu}, \quad \mu \leq -2$$  \hspace{1cm} (2.22)

The eigenvalue of the linearized system at the periodic points can be determined by differentiating equation (2.21)

$$\frac{\partial G}{\partial x_k}(x_k, \mu) = -1 - \mu + 3x_k^2$$  \hspace{1cm} (2.23)

Using the periodic points of (2.22), the bifurcation points with eigenvalues of +1 or -1 can be determined. For the periodic point of $x_k = 0$ this results in

$$\frac{\partial G}{\partial x_k}(x_k = 0, \mu) = -1 - \mu$$
$$-1 - \mu = -1 \quad \Rightarrow \quad \mu = 0$$
$$-1 - \mu = 1 \quad \Rightarrow \quad \mu = -2$$  \hspace{1cm} (2.24)
2.3. BIFURCATIONS

And for the periodic point of \( x_k = \pm \sqrt{(2 + \mu)} \) this results in

\[
\frac{\partial G}{\partial x_k}(x_k = \pm \sqrt{(2 + \mu)}, \mu) = -1 - \mu + 3(2 + \mu) = 5 + 2\mu
\]

\[
5 + 2\mu = 1 \rightarrow \mu = -2
\]

\[
5 + 2\mu = -1 \rightarrow \mu = -3, \text{ not possible}
\]

We notice 2 bifurcation points;

- \( x_k = 0, \mu = -2 \) with an eigenvalue of 1
- \( x_k = 0, \mu = 0 \) with an eigenvalue of -1.

Figure 2.12, shows an unstable periodic point for \( \mu < -2 \) which changes in a stable periodic point when \( \mu \) reaches -2. At the same time a couple of unstable periodic points show up, another example of a pitchfork bifurcation.

When \( \mu \) reaches 0, the stable periodic point \( x_k = 0 \) becomes unstable and a periodic point of period 2 arises. This can be seen by the 2 rectangles in the figure; a point moves to another point after 1 period, but returns back to its original position after 2 periods. This bifurcation is an example of a Period Doubling bifurcation.

The period doubling becomes clearer by computing \( x_{k+2} \) as a function of \( x_k \), yielding the iterated map

\[
x_{k+2} = G(G(x_k, \mu), \mu)
\]

\[
= -(G(x_k, \mu)) - \mu(G(x_k, \mu)) + (G(x_k, \mu))^3
\]

\[
= -(x_k - \mu x_k + x_k^3) - \mu(-x_k - \mu x_k + x_k^3) + (-x_k - \mu x_k + x_k^3)^3
\]

\[
x_{k+2} = (1 + 2\mu + \mu^2)x_k + (-2 - 5\mu - 3\mu^2)x_k^3 + (3 + 7\mu + 2\mu^2)x_k^5 + (-3 - 3\mu)x_k^7 + x_k^9
\]

(2.26)

with an eigenvalue of the linearized system at the bifurcation point equal to 1

\[
\frac{\partial G(G(x_k, \mu), \mu)}{\partial x_k}(x_k = 0, \mu = 0) = 1
\]

(2.27)

Figure 2.12: Logistic maps of a Period Doubling bifurcation, \( x_{k+1} = -x_k - \mu \cdot x_k + x_k^3 \)
This map is shown in figure 2.13. For $\mu < 0$, the point $x = 0$ is a stable periodic point. This point being a stable periodic point of period 1 was already known, so of course this is also a stable periodic point of period 2. When $\mu$ becomes positive, the stable periodic point $x = 0$ changes in an unstable one and two other stable periodic points arise. When we look at the map of (2.26) a Pitchfork bifurcation occurs at $\mu = 0$. So this means that for the map (2.21) a stable periodic point of period 1 changes in an unstable one and meanwhile a pair of stable periodic points of period 2 arise, meaning a Period Doubling bifurcation.

Figure 2.13: Period Doubling bifurcation, $x_{k+1} = -x_k - \mu \cdot x_k + x_k^3$
Chapter 3

The Blinking Vortex flow

3.1 Introduction

The Blinking Vortex flow (BV), introduced by Aref [1], consists of two co-rotating point vortices A and B, separated by a fixed distance 2a, that blink on and off periodically, each with a constant period T (Figure 3.1). At any given time only one of the vortices is on, so that the motion is made up of consecutive twist maps about different centers. During the protocol of the BV, the right vortex is switched on first with a counter-clockwise rotation. After a period T the right vortex is turned off and the left vortex is turned on for period T with also a counter-clockwise rotation. Hence, the total period time equals 2T. This protocol is defined in (3.1).

\[
\begin{aligned}
0 + 2nT < t < T + 2nT & : \text{vortex A actuated counter-clockwise} \\
T + 2nT < t < 2T + 2nT & : \text{vortex B actuated counter-clockwise}
\end{aligned}
\]

(3.1)

with \( n = 1, 2, 3, \ldots \) and period time T.

Figure 3.1: Schematic diagram of the blinking vortex system


3.2 Numerical model

The velocity field due to a single point vortex at the origin is given by:

\[
\begin{aligned}
  v_r &= 0 \\
  v_\theta &= \Gamma / 2\pi r^2
\end{aligned}
\] (3.2)

where \( \Gamma \) is the strength of the vortex.

The mapping consists of two parts, each of the form

\[
\mathbf{G}(r, \theta) = (r, \theta + \Delta \theta)
\] (3.3)

Radius \( r \) is measured with respect to the center of the vortex and the angle \( \Delta \theta \) is given by

\[
\Delta \theta = \Gamma T / 2\pi r^2
\] (3.4)

Placing two vortices at distances \((-a, 0)\) and \((a, 0)\) in a Cartesian coordinate system (Figure 3.1), the complete mapping, in dimensionless form, is given by:

\[
\begin{aligned}
  x_{k+1} &= \xi_i + (x_k - \xi_i) \cos \Delta \theta_k - y_k \sin \Delta \theta_k \\
  y_{k+1} &= (x_k - \xi_i) \sin \Delta \theta_k + y_k \cos \Delta \theta_k
\end{aligned}
\] (3.5)

\( \xi_i \) denotes the position of the vortex \( i (i=A,B) \), \( \Delta \theta = \frac{\mu T}{r} \), with \( r = ((x - \xi_i)^2 + y^2)^{1/2} \), and \( \mu = \frac{\Gamma T}{2\pi a^2} \), Ottino [13]. Since \( \mu \) is proportional to the strength of the vortex \( \Gamma \) and the time \( T \), \( \mu \) can be interpreted as a dimensionless parameter for the actuation.

The distances are made dimensionless with respect to \( a \) so that the vortices A and B are placed at \( \xi_i = \pm 1 \), so the vortex at \( \xi_i = A \) = 1 is switched on first. Using this mapping a script is written to make Poincaré sections of the Blinking Vortex flow (Appendix A). The equations are written in a different way in this script to avoid numerical inaccuracies.

\[
\begin{aligned}
  x_{k+1} &= \xi_i - \xi_i \cos \Delta \theta + x_k \cos \Delta \theta - y_k \sin \Delta \theta \\
  y_{k+1} &= x_k \sin \Delta \theta - \xi_i \sin \Delta \theta + y_k \cos \Delta \theta
\end{aligned}
\] (3.6)

The Poincaré sections can be made with a variable parameter \( \mu \), variable initial positions, variable number of periods and a variable protocol. For this analysis only the protocol defined in section 3.1 is used.

3.3 Poincaré sections

Figure 3.2 shows several Poincaré sections of the Blinking Vortex flow, \( t=2nT \) with \( n=1,2,3,... \), for different values of the actuation parameter \( \mu \). The flow consists of only 'regular' regions for very small values of the actuation parameter \( \mu \). As \( \mu \) increases, regions of chaos appear first near the vortices, then in the center region, until for \( \mu=0.5 \) they seem to occupy the entire region, Doherty and Ottino, 1988 [2].

As mentioned already in section 2.2, it has to be noticed that a Poincaré section strongly depends on the initial positions. Although the Poincaré section of \( \mu = 0.3 \) in figure 3.2 shows
3.3. POINCARÉ SECTIONS

a) $\mu = 0.01$

b) $\mu = 0.1$

c) $\mu = 0.3$

d) $\mu = 0.05$

Figure 3.2: Poincaré plots of the Blinking vortex; $\mu$ is varied from 0.01 till 0.5

a) $\mu = 0.03$

b) $\mu = 0.03$, zoom

Figure 3.3: Poincaré plots of the Blinking vortex; $\mu$ equals 0.3
only “regular” and “chaotic” regions, another Poincaré section of $\mu = 0.3$ (figure 3.3) shows some interesting patterns. It turns out that there are several islands of high period present. To show these periodic islands in a Poincaré section, a lot of patience is needed and some good luck, because the correct initial positions have to be chosen. Poincaré sections only provide some of the information about the limits of possible mixing, but in order for truly understanding, a better, analytical, approach is needed. Mixing is limited by the appearance of periodic islands, especially of low period. So it is of great importance to find these periodic islands if they exist.

### 3.4 Conjugated lines

Figure 3.4 shows two candidates for period-1 periodic points. The simplest one is a point on the y-axis, A’. First vortex A moves A’ to B’ and then vortex B moves B’ back to A’. So a point starting on A’ moves back to its initial position at A’ after one period, therefore point A’ is called a periodic point of period 1, also called a period-1 point. But there are many other period-1 points. For example, the point A” moves in more than one rotation due to vortex A to position B” and moves then in less than one rotation back to position A” due to vortex B. Again a point moves back to its initial position after one period.

To find these periodic points in an analytical way, we need to find an analytical expression for the mapping. In generally a mapping can be written in the following way:

$$ x_{n+1} = M(x_n) \quad (3.7) $$

This denotes, a particle which is located at $x_n$ at the $n$th cycle can be found at $M(x_n)$ at the $(n+1)$th cycle. In this notation, the vector function $M$ is operating on $x_n$. Mappings can be composed, i.e., if $x_1 = M(x_0)$ and $x_2 = N(x_1)$, then the composition of $M$ with $N$ results in

$$ x_2 = N \circ M(x_1) \quad (3.8) $$

where the right-hand operator is acting first. The identity map is denoted $1$ and it has the property that $1(x) = x$ for any $x$. Because the mapping $M^{-1}$, which is the inverse mapping of $M$, takes a particle back to its initial position after being mapped by $M$, we can say $M \circ M^{-1} = 1$.

The mapping of the Blinking Vortex is composed of two operations, i.e., operation $A_T$, the counter-clockwise mapping of vortex A for a duration of time $T$ and operation $B_T$, the counter-clockwise mapping of vortex B for a duration of time $T$. The complete mapping of the Blinking Vortex becomes now:

$$ x_{k+2} = B_T \circ A_T(x_k) \quad (3.9) $$

To find the periodic points in an analytical way, we take a look at the mapping of a period-1 periodic point of the BV flow. If we call this periodic point $(x*, y*)$, then it is necessary that

$$ A_T(x*, y*) = (x*, -y*) \quad (3.10) $$
$$ B_T(x*, -y*) = (x*, y*) \quad (3.11) $$

Rearranging (3.10) results in the following equation;

$$ A_T^{-1}(x*, 0) = (x_A, 0) $$

Applying $A_T^{-1}$ to both sides $\Rightarrow$

$$ A_T^{-1} \circ A_T(x*, y*) = A_T^{-1}(x_A, 0), \text{ with } A_T^{-1} \circ A_T = 1 $$

$$ A_T^{-1}(x_A, 0) = (x*, y*) $$

$$ (3.12) $$
3.5. EIGENVALUES AND STABILITY

where \( \mathbf{A}_{T/2}^{-1}(x, y) \) represents the mapping of point \((x, y)\) with vortex A actuated clockwise for an amount of time \(T/2\).

Rearranging (3.11) results in

\[
\mathbf{B}_{T/2}(x_B^*, 0) = (x^*, y^*)
\] (3.13)

So to find the period-1 points at first the x-axis has to be mapped by \( \mathbf{A}_{T/2}^{-1} \), clockwise by vortex A for an amount of time \(T/2\), see (3.12). By this mapping the point \( A'^* \) will be displaced to the point \( A' \) and the point \( A''^* \) will be displaced to the point \( A'' \). Then the x-axis also has to be mapped by \( \mathbf{B}_{T/2} \), counter-clockwise by vortex B for an amount of time \(T/2\), see (3.13). By this mapping the point \( B'^* \) will be displaced to the point \( A' \) and the point \( B''^* \) will be displaced to the point \( A'' \). The intersections of these two lines, called the conjugated lines, result in the period-1 points.

For several values of \( \mu \) the conjugated lines are determined in figure 3.5. This figure shows some bifurcations, because the number of period-1 points depends on the value of the parameter \( \mu \). For low values of \( \mu \) there are a lot of intersections, meaning period-1 points, above the x-axis. As \( \mu \) increases the periodic points disappear except for 3 points; one point situated on top of the y-axis and 2 points mirrored in the y-axis, see the figure for \( \mu = 3 \). As \( \mu \) increases even more, other intersections show up. We have now noticed bifurcations of the numbers of periodic points, next it is interesting to find out if there are also bifurcations of the stability of these periodic points.

3.5 Eigenvalues and stability

To determine the stability of the periodic points the eigenvalues of the Jacobian at the periodic points have to be calculated. Therefore the exact positions of the points have to be known. The positions have been graphically found and these inaccurate descriptions have been used to compute the numerical estimations by calculating the zeros by the Newton-Raphson method.
Figure 3.5: Conjugated lines of the Blinking Vortex flow for different values of $\mu$
of the following system of equations:

\[
\begin{align*}
    x_{k=2}(x_{k=0}, y_{k=0}) - x_{k=0} &= 0 \\
y_{k=2}(x_{k=0}, y_{k=0}) - y_{k=0} &= 0
\end{align*}
\] (3.14)

The Matlab script of this calculation is shown in appendix C.

Now the numerical estimations of the positions of the period-1 points can be calculated accurately, the stability can be determined by calculating the eigenvalues of the Jacobian at the equilibria. Tables 3.1 and 3.2 show that all of these periodic points have one \(|\text{eigenvalue}| > 1\), so all these points are unstable periodic points of period 1, also called hyperbolic points of period 1. Table 3.3 shows something interesting; 2 points have imaginary eigenvalues with an absolute value of exactly one. So these points might be elliptical periodic points of period 1.

This assumption is confirmed by producing a Poincaré section, see figure 3.6. The initial points near the left and right periodic points create periodic islands, so this proves that they are indeed elliptical periodic points. The initial point near the periodic point on top of the y-axis creates a ‘chaotic’ region, so this one is indeed an unstable periodic point.

<table>
<thead>
<tr>
<th>Coordinates</th>
<th>(\lambda_1)</th>
<th>(\lambda_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-1.4499; 0.3031)</td>
<td>-0.16</td>
<td>-6.26</td>
</tr>
<tr>
<td>((-1.1711; 0.3399)</td>
<td>-0.08</td>
<td>-12.26</td>
</tr>
<tr>
<td>((-0.8641; 0.3910)</td>
<td>0.06</td>
<td>16.89</td>
</tr>
<tr>
<td>(0.0000; 0.6091)</td>
<td>0.17</td>
<td>6.02</td>
</tr>
</tbody>
</table>

Table 3.1: Eigenvalues of the periodic points of period 1 for \(\mu=1.5\)

<table>
<thead>
<tr>
<th>Coordinates</th>
<th>(\lambda_1)</th>
<th>(\lambda_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-1.5623; 0.5671)</td>
<td>-0.1678</td>
<td>-5.9588</td>
</tr>
<tr>
<td>(0.0000; 0.9721)</td>
<td>0.1293</td>
<td>7.7325</td>
</tr>
</tbody>
</table>

Table 3.2: Eigenvalues of the periodic points of period 1 for \(\mu=3\)
Table 3.3: Eigenvalues of the periodic points of period 1 for $\mu=9$

<table>
<thead>
<tr>
<th>Coordinates</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\pm 1.4349; 1.4961)$</td>
<td>$-1.0000$</td>
<td>$\pm 0.0041$</td>
</tr>
<tr>
<td>$(\pm 0.1718; 0.1196)$</td>
<td>$0.0252$</td>
<td>$39.6128$</td>
</tr>
<tr>
<td>$(\pm 0.2030; -0.3655)$</td>
<td>$0.0121$</td>
<td>$82.9117$</td>
</tr>
<tr>
<td>$(0.0000; 1.7979)$</td>
<td>$0.2319$</td>
<td>$4.3123$</td>
</tr>
<tr>
<td>$(0.0000; 0.4986)$</td>
<td>$0.0063$</td>
<td>$159.7162$</td>
</tr>
<tr>
<td>$(0.0000; -0.9410)$</td>
<td>$0.0116$</td>
<td>$86.4849$</td>
</tr>
</tbody>
</table>

These elliptic periodic points have been investigated for several values of $\mu$. Figure 3.7 shows the results graphically, see appendix E for the exact values. For $\mu < 9$ the points are unstable. When $\mu$ approaches 9 the points become stable with $\lambda_{1,2} = -1$. As $\mu$ increases the eigenvalues split up and rotate around the complex unit circle. The negative real part becomes smaller and the imaginary part becomes larger, until the real part reaches 0 and the imaginary part reaches $\pm 1$, then the real part becomes positive and becomes larger and the imaginary part becomes smaller until both eigenvalues reach a value of 1 when $\mu$ approaches 15.

Meanwhile, the two points have been shifted towards the period-1 point situated on the $y$ axis. When $\mu$ approaches 15 they have reached the position of this periodic point, see figure 3.8a. The two stable periodic points convert in the unstable periodic point, which now becomes stable. So this turns out to be a Pitchfork bifurcation, see figure 2.8, as explained in section 2.3.

Figure 3.6: Poincaré plots of the stable periodic points of period 1
3.5. **EIGENVALUES AND STABILITY**

![Diagram showing eigenvalues as a function of $\mu$, view from above.](image1)

Figure 3.7: Eigenvalues of the right and left upper periodic points of period 1

![Diagram showing conjugated lines and a Poincaré plot.](image2)

Figure 3.8: Detection of the stable periodic point of period 1 for $\mu=15$

The eigenvalues of the Jacobian at the periodic point situated on top of the y-axis have real values before $\mu$ has reached a value of 15, one is larger than 1 and the other is smaller than 1. When the Pitchfork bifurcation occurs, the eigenvalues have both a value of 1. The periodic point becomes an elliptic point, as can be seen in figure 3.8b. When $\mu$ increases, the eigenvalues rotate again around the complex unit circle, see figure 3.9 and appendix E for the exact values. The positive real part becomes smaller and the imaginary part becomes larger, until the real part reaches 0 and the imaginary part reaches $\pm 1$, then the real part becomes negative and becomes larger and the imaginary part becomes smaller until both eigenvalues reach a value of -1. Then the eigenvalues continue rotating around the complex unit circle, the negative real part becomes smaller and the imaginary part becomes larger, until the real part reaches 0 and the imaginary part reaches $\pm 1$, then the real part becomes positive and becomes larger and the imaginary part becomes smaller until both eigenvalues reach a value of 1 again,
and the point becomes unstable again. This happens when $\mu$ has reached a very large value of $1e+10$.

Although some good insight has already been obtained by determining the position and stability of the period-1 points, it would also be useful to determine the location of periodic points of higher order, provided they exist. This is however likely, because we saw already periodic points of higher order in a Poincaré plot, see figure 3.3. Symmetries will be utilized in the next section to achieve this.

![Eigenvalues as function of $\mu$.](image)

Figure 3.9: Eigenvalues of the upper periodic point of period 1 on the y-axis

### 3.6 Symmetry lines of the Blinking Vortex flow

To determine the position of the period-1 points, the conjugated lines have been constructed in section 3.4. In fact, this is nothing more than making use of the lines of symmetry. Periodic points are located on lines of symmetry or in pairs on opposite sides of the line. To find periodic islands it is therefore of great advantage to determine these symmetry-lines.

A set of points that remains invariant under mapping $M$, forms a line of symmetry, also called fixed line, of this mapping. A map $M$ possesses a symmetry $S$ if there exists a mapping $S$ such that $(M \circ S)^2 = 1$, with $S^2 = 1$. By making use of the lines of symmetry, not only period-1 points can be determined, but also periodic points of higher period. A theorem formulated by de Vogelaere [16] states that a periodic point of order $(k-j)$ is found at the intersection of the fixed lines of symmetries $M^k \circ S$ and $M^j \circ S$.

The lines of symmetry will be constructed making use of geometrical insight. The complete mapping consists of two “sub-mappings” Usually, the symmetries of the “sub-mappings” are more easily determined, and this information can be used to deduce the symmetries of the overall map, Franjione et al. [4]
A fluid particle starting at location C is moved by actuator A for a duration of time t. It moves along the streamline until it reaches location D, see figure 3.10. This movement is symbolic written as $A_t$.

The same motion can be accomplished by again starting at location C, reflecting across the x axis location E, moving the particle by actuator A in the reversed direction to location F for an amount of time t and finally reflecting back across the x axis to location D. This movement is symbolically written as $S_x \circ A_{-1} \circ S_x$, where $S_x$ represents the mapping of point $(x, y)$ to the point $(x, -y)$ and $A_{-1}$ represents the mapping by actuator A in the reversed direction for a duration of time t. Equating these movements results in

$$A_t = S_x \circ A_{-1} \circ S_x$$

This is simply a statement of the definition of symmetry, showing that the flow $A_t$ is symmetric about reflections across the x axis. A similar relation can be deduced for the left actuator flow:

$$B_t = S_x \circ B_{-1} \circ S_x$$

An additional symmetry relation can be deduced between the two flows. The movement $A_t$ can also be accomplished by starting at position C and first reflecting across the y axis to location G, moving the particle by actuator B in the reversed direction during time t and finally reflecting back across the y axis to location D. Symbolically, this is written as

$$A_t = S_y \circ B_{-1} \circ S_y$$

where $S_y$ is the transformation that takes a point $(x, y)$ to $(-x, y)$. Here we say that the two flows are symmetric with respect to each other about reflections across the y axis. The relations (3.16) and (3.17) can be combined to yield a third useful relation:

$$A_t = R \circ B_t \circ R$$
where \( R = S_y S_x = S_x S_y \) is a transformation that takes point \((x, y)\) to \((-x, -y)\).

The symmetries of the flows generated by the individual vortex motions can be used to deduce symmetries of more complicated mappings which are composed of various combinations of \(A\) and \(B\) flows. The mapping, denoted as \(F_{2T}\), is written as

\[
F_{2T} = B_T \circ A_T
\]  

(3.19)

Deducing the symmetries of this map is simply a matter of algebra. Using relation (3.17) and substituting in relation (3.19), we obtain

\[
F_{2T} = (S_y \circ A_T^{-1} \circ S_y) \circ (S_y \circ B_T^{-1} \circ S_y)
\]

\[
= S_y \circ (A_T^{-1} \circ B_T^{-1}) \circ S_y
\]

\[
= S_y \circ (F_{2T}^{-1}) \circ S_y
\]  

(3.20)

where \(F_{2T}^{-1}\) represents the reversed mapping of the blinking vortex flow, i.e., \(A_T^{-1} \circ B_T^{-1}\). Hence can be concluded that the Blinking Vortex mapping is symmetric about reflections across the \(y\)-axis.

However, \(S_x\) is not the only fundamental symmetry of the flow. Substituting (3.15) and (3.16) in relation (3.19) results in

\[
F_{2T} = (S_x \circ B_T^{-1} \circ S_x) \circ (S_x \circ A_T^{-1} \circ S_x)
\]

\[
= S_x \circ (B_T^{-1} \circ A_T^{-1}) \circ S_x
\]

\[
= (S_x \circ A_T) \circ A_T^{-1} \circ B_T^{-1} \circ (A_T^{-1} \circ S_x)
\]

\[
= S^* \circ (F_{2T}^{-1}) \circ S^*
\]  

(3.21)

Where \(S^*\) equals \(S_x \circ A_T = S_x \circ A_T \circ S_x \circ S_x = A_T^{-1} \circ S_x\), with 3.15. So \(S^*\) is another line of symmetry of the Blinking Vortex flow, it remains invariant under mapping with \(F_{2T}\).

The fixed line of \(A_T^{-1} \circ S_x\) is the line which remains invariant upon transformation. That is, we want to find the set \(C\) such that

\[
C = \{ x | A_T^{-1} \circ S_x(x) = x \}
\]  

(3.22)

The flow \(A_T^{-1}\) can be written as the composition of two flows, each run for half the duration of time. The defining equation for the fixed line can then be written as

\[
A_T^{-1/2} \circ A_T^{-1/2} \circ S_x(C) = C
\]  

(3.23)

Applying \(A_T^{-1/2}\) to both sides of the equation results in

\[
A_T^{-1/2} \circ S_x(C) = A_T^{-1/2}(C)
\]  

(3.24)

Because the vortex flow is symmetric about reflections across the \(x\) axis, \(A_T^{-1} \circ S_x\) is equivalent to \(S_x \circ A_T\), see (3.15), so that

\[
S_x \circ A_T(C) = A_T(C)
\]  

(3.25)

Consider the set \(D\) given by the transformation of the set \(C\) by \(A_T^{-1/2}\), i.e.,

\[
D = A_T^{-1/2}(C)
\]  

(3.26)
Substituting (3.26) in (3.25) results in

\[ S_x(D) = (D) \] (3.27)

Meaning the set \( D \) remains invariant upon transformation by \( S_x \), so the set \( D \) has to be equal to the \( x \) axis. The set \( C \), i.e., the fixed line of the symmetry \( A_T^{-1} \circ S_x \), is obtained by mapping the \( x \) axis with \( A_T^{-1} \), see (3.26), i.e., vortex A in reversed direction, for an amount of only half the usual actuating time. Note that the fixed line of this symmetry is not a straight line, but a curve.

The symmetry \( F_{2T} \circ S^* \) is given by

\[ F_{2T} \circ S^* = B_T \circ A_T \circ A_T^{-1} \circ S_x \]

= \( B_T \circ S_x \) (3.28)

and the fixed line is formed by mapping the \( x \) axis with \( B_T \), i.e., vortex B for only an amount of half the usual actuating time. By the theorem of Vogelaere the positions of periodic points of order 1 can be determined by the intersections of these two fixed lines. Note that these lines are the same as the conjugated lines, constructed in section 3.4, so the results are already known. However, this is still a very useful method, because now also periodic points of higher order can be determined, as will be used in the next chapter about the Journal Bearing flow.

### 3.7 Conclusions

The position of periodic points of several order of the Blinking Vortex flow can be determined by the intersection of lines of symmetry. The Blinking Vortex flow has at least three of these fixed lines:

- The \( y \) axis
- \( A_T^{-1} \) (\( x \) axis), clockwise mapping of the \( x \) axis by vortex A over a time \( T/2 \) (3.29)
- \( B_T \) (\( x \) axis), counter-clockwise mapping of the \( x \) axis by vortex B over a time \( T/2 \)

By varying the parameter \( \mu \), a number of bifurcations occurs; the positions and the number of periodic points differs when the variable parameter \( \mu \) increases. For low values of \( \mu \) there are a lot of periodic points above the \( x \)-axis. As \( \mu \) increases the periodic points disappear except for 3 points; one point situated on top of the \( y \)-axis and 2 points mirrored in the \( y \)-axis. These three points can become elliptic periodic points, surrounded by elliptic islands. As \( \mu \) increases even more, other intersections show up.

The two points mirrored in the \( y \)-axis become stable when \( \mu \) reaches a value of 9. When \( \mu \) increases, the eigenvalues rotate around the complex unit circle for half a circle both, starting at -1 and ending at +1. Meanwhile the pair of two points shifts towards the position of the periodic point situated on the \( y \)-axis, resulting in a pitchfork bifurcation as \( \mu \) approaches 15.

The eigenvalues of the periodic point situated on top of the \( y \)-axis start to rotate around the complex unit circle after \( \mu \) becomes 15. They start at +1 and rotate both for a complete circle until they reach +1 again, then the point becomes unstable again. This occurs however only when \( \mu \) has reached a very large value of \( 1e+10 \).
Chapter 4

The Journal Bearing flow

4.1 Introduction

The Journal Bearing flow (JB) consists of a time-periodic Stoke’s flow (Re << 1) of a viscous incompressible Newtonian fluid in the gap between two eccentric cylinders. The flow between eccentric cylinders is already considered in classical works on lubrication theory since 1887, [8] and [9], because it describes the idealized flow of a lubricating fluid between a rotating journal and its cylindrical support. In these early works approximate solutions for the velocity field were used. Later a closed analytical expression for the velocity field was developed for a given geometry and velocity of the inner- and outer cylinder, [17]

\[ U_x = -2A* \left( \frac{S+y}{x^2+(S+y)^2} + \frac{S-y}{x^2+(S-y)^2} \right) - B* \left( \frac{(S+2y)*(x^2+(S+y)^2) - 2y*(S+y)^2}{(x^2+(S+y)^2)^2} \right) - C* \left( \frac{(S-2*y)*(x^2+(S-y)^2) + 2y*(S-y)^2}{(S-y)^2} \right) - D - 2E*y \]

\[ U_y = -A* \left( \frac{8Sxy}{(x^2+(S+y)^2)(x^2+(S-y)^2)} - B* \frac{2xy(S+y)}{(x^2+(S+y)^2)^2} - C* \right) \]

with

\[ A = -0.5*(D_1D_2-S^2)*Curlb \]
\[ B = (D_1+S)*(D_2+S)*Curlb \]
\[ C = (D_1-S)*(D_2-S)*Curlb \]
\[ D = (D_1L_2-D_2L_1)*R_1V_1 + R_2V_2 \]
\[ E = 0.5*(L_1-L_2)(R_1V_1 + R_2V_2) \]
\begin{align*}
F &= \frac{Ecc \cdot (R_1 V_1 + R_2 V_2)}{Den} \\
D_1 &= \frac{R_2^2 - R_1^2}{2 \cdot Ecc} - 0.5 \cdot Ecc \\
D_2 &= D_1 + Ecc \\
S &= \sqrt{(R_2 - R_1 - Ecc) \cdot (R_2 - R_1 + Ecc) \cdot (R_2 + R_1 + Ecc) \cdot (R_2 + R_1 - Ecc)} \\
L_1 &= \log \frac{D_1 + S}{D_1 - S} \\
L_2 &= \log \frac{D_2 + S}{D_2 - S} \\
Den &= (R_2^2 + R_1^2) \cdot (L_1 - L_2) - 4 S \cdot Ecc \\
Curlb &= 2 \cdot (D_2^2 - D_1^2) \cdot \frac{R_1 V_1 + R_2 V_2}{(R_2^2 + R_1^2) \cdot Den} + R_1^2 R_2^2 \cdot \frac{(V_1/R_1 - V_2/R_2)}{(S \cdot (R_2^2 + R_1^2) \cdot (D_2 - D_1))} \tag{4.3}
\end{align*}

where \(V_1\) and \(V_2\) represent the velocity of the inner resp. outer cylinder, \(R_1\) and \(R_2\) represent the radius of the inner resp. the outer velocity and \(Ecc\) represents the eccentricity. Poincaré maps are being made by integrating this velocity field, see appendix G.

The geometry consists of the radius of the outer cylinder \(r_{\text{out}}\), the radius of the inner cylinder \(r_{\text{in}}\) and the distance between the centers of the two cylinders \(d\), see figure 4.1a. This geometry is characterized by two dimensionless parameters: the ratio of the radii of the inner and outer cylinder, \(r_{\text{in}}/r_{\text{out}}\), and the dimensionless eccentricity \(e = d/r_{\text{out}}\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{flow_domain.png}
\caption{Schematic representations of geometry and flow domain of the Journal Bearing}
\end{figure}

The streamline patterns created by the rotation of the inner and outer cylinder are shown in figures 4.1b and 4.1c. Note that each streamline pattern exist of two different parts: the vortex and the main flow, separated bij the separatrix. The direction of the vortex is opposite to the direction of the cylinder and the main flow.
4.2 Lines of Symmetry of the Journal Bearing Flow

In case of a Stokes flow the result of the fluid motion is completely defined by the angle of rotation of the cylinders. Thus, the flow protocol can be described by two dimensionless parameters: rotation angle $\theta$ of the outer cylinder and the ratio $\Omega$ of the rotation angles of both cylinders. Following Muzzio et al. [11], we fix the geometric parameters as $r_{in}/r_{out} = 1/3$, $e = 0.3$ and the ratio of the rotation angles is kept $\Omega = 3.0$.

The periodic flow is induced by a discontinuous two-step protocol: during the first half-period the outer cylinder is rotated counter-clockwise with a variable angle $\theta$, while the inner remains stationary; during the second half-period the outer cylinder is fixed and the inner one is rotating counter-clockwise with an angle $\theta \cdot \Omega$. This protocol is defined in (4.4).

$$\begin{align*}
\begin{cases}
n = 1, 3, 5, \ldots & \text{: Counter-clockwise rotation of } \theta \\
n = 2, 4, 6, \ldots & \text{: Counter-clockwise rotation of } \theta \cdot \Omega
\end{cases}
\end{align*}$$

with $r_{in}/r_{out} = 1/3$, $e = 0.3$, $\Omega = 3.0$

Some hyperbolic first-order periodic points have already been located and classified by Galaktionov et. al., [5], using the algorithm that exploits the symmetry properties of the particle trajectories during each half-period, [10]. Using Poincaré maps, also some periodic islands of higher period have already been located by Galaktionov et. al., [5]. However no general approach has been developed yet to locate the periodic points of several order. Using the lines of symmetry it is attempted to develop such a general approach.

4.2 Lines of Symmetry of the Journal Bearing Flow

Just like in chapter 3.6, the position of the periodic points will be determined by making use of the lines of symmetry, because the periodic points are located on lines of symmetry or in pairs on opposite sides of the line.

At first, the mapping $M$ of the Journal Bearing flow needs to be specified, so the mapping can be written in an analytical way like the analytical expression of the Blinking Vortex flow in (3.9). The mapping of the Journal Bearing flow is composed of two operations, i.e., operation $O_\theta$, the counter-clockwise mapping of the outer cylinder over an angle $\theta$ and operation $I_\theta$, the counter-clockwise mapping of the inner cylinder over an angle $\theta \cdot \Omega$. The complete mapping of the Journal Bearing flow becomes now:

$$x_{k+1} = I_\theta \circ O_\theta(x_k)$$

(4.5)

Where the mapping is defined $F$, so

$$F = I_\theta \circ O_\theta$$

(4.6)

Note that every mapping what will be discussed in this chapter only acts on points in the cavity of the Journal Bearing. Also all points and set of points discussed in this chapter are defined to be situated in this cavity. Operation $I_\theta$ is symmetric about reflections across the y axis, as can be seen by the streamline pattern of this operation in figure 4.1b. The operation $O_\theta$ is also symmetric about reflections across the y axis, as can be seen by the streamline pattern of this
operation in figure 4.1c. This results in the following equations:

\[
O_\theta = S_y \circ O_{\theta}^{-1} \circ S_y \quad (4.7)
\]
\[
I_\theta = S_y \circ I_{\theta}^{-1} \circ S_y \quad (4.8)
\]

where \( S_y \) represents the mapping of point \((x, y)\) to \((-x, y)\), just like in chapter 3.6. Again we can apply some algebra on these relations. Using (4.7) and (4.8), we obtain:

\[
O_\theta \circ I_\theta = S_y \circ (O_{\theta}^{-1} \circ S_y) \circ (I_{\theta}^{-1} \circ S_y) = S_y \circ (I_\theta \circ O_\theta)^{-1} \circ S_y \Rightarrow
\]
\[
O_\theta \circ I_\theta = S_y \circ (F)^{-1} \circ S_y \quad (4.9)
\]

Forming a composition of \( I_\theta \) with both sides of 4.9 with \( I_{\theta}^{-1} \) results in

\[
I_\theta \circ O_\theta \circ I_\theta \circ I_{\theta}^{-1} = I_\theta \circ S_y \circ (F)^{-1} \circ S_y \circ I_{\theta}^{-1} \Rightarrow
\]
\[
F = S^* \circ F^{-1} \circ S^* \quad (4.10)
\]

where

\[
S^* = I_\theta \circ S_y
\]
\[
S^* = S_y \circ I_\theta \circ S_y \Rightarrow
\]
\[
S^* = S_y \circ I_{\theta}^{-1} \quad (4.11)
\]

So \( S^* \) is a line of symmetry of the Journal Bearing flow. The fixed line of \( I_\theta \circ S_y \) is the line which remains invariant upon transformation \( F \). Now we are looking for the set \( A \), such that

\[
A = \{ x | I_\theta \circ S_y(x) = x \} \quad (4.12)
\]

The flow \( I_\theta \) will be written as the composition of two flows, each run for half the duration of time.

\[
I_{\theta/2} \circ I_{\theta/2} \circ S_y(A) = A \quad (4.13)
\]

Applying \( I_{\theta/2}^{-1} \) to both sides of the equation results in

\[
I_{\theta/2} \circ S_y(A) = I_{\theta/2}^{-1}(A) \quad (4.14)
\]

Applying \( S_y \circ I_{\theta/2} \circ S_y = 1 \) to the left side of the equation leads to

\[
S_y \circ S_y \circ I_{\theta/2} \circ S_y(A) = I_{\theta/2}^{-1}(A) \quad (4.15)
\]

Because the vortex flow is symmetric about reflections across the \( y \) axis, \( S_y \circ I_{\theta/2} \circ S_y \) is equivalent to \( I_{\theta/2}^{-1} \), see (4.8), so that

\[
S_y \circ I_{\theta/2}^{-1}(A) = I_{\theta/2}^{-1}(A) \quad (4.16)
\]
4.2. LINES OF SYMMETRY OF THE JOURNAL BEARING FLOW

Consider the set $B$, given by the transformation of the set $A$ by $I_{\theta/2}^{-1}$, i.e.

$$B = I_{\theta/2}^{-1}(A) \quad (4.17)$$

Substituting (4.17) in (4.16) results in

$$S_y(B) = B \quad (4.18)$$

This implies the set $B$ remains invariant upon transformation by $S_y$, so the set $B$ has to be equal to $[y \text{ axis}]$, the interval of the $y$ axis which is situated in the cavity of the Journal Bearing. So (4.17) becomes

$$[y \text{ axis}] = I_{\theta/2}^{-1}(A) \quad (4.19)$$

Applying $I_{\theta/2}$ to both sides of the equation reveals the set $A$

$$I_{\theta/2}[y \text{ axis}] = A \quad (4.20)$$

The fixed line of the symmetry $I_{\theta} \circ S_y$, the set $A$, is obtained by mapping the $[y \text{ axis}]$ with $I_{\theta/2}$, see (4.20), i.e., the actuation of the inner cylinder counter-clockwise, for half the usual actuating time.

Another line of symmetry can be found by the composition of $O_{\theta}^{-1}$ with both sides of (4.9) with $O_{\theta}$. This results in

$$O_{\theta}^{-1} \circ O_{\theta} \circ I_{\theta} \circ O_{\theta} = O_{\theta}^{-1} \circ S_y \circ (F)^{-1} \circ S_y \circ O_{\theta}$$

$$I_{\theta} \circ O_{\theta} = O_{\theta}^{-1} \circ S_y \circ (F)^{-1} \circ S_y \circ O_{\theta}$$

$$F = S^{**} \circ F^{-1} \circ S^{**} \quad (4.21)$$

where

$$S^{**} = O_{\theta}^{-1} \circ S_y$$

$$= S_y \circ O_{\theta} \quad (4.22)$$

So $S^{**}$ is another line of symmetry of the Journal Bearing flow. The fixed line of $O_{\theta}^{-1} \circ S_y$ is the line which remains invariant upon transformation $F$. Now we are looking for the set $C$, such that

$$C = \{x | O_{\theta}^{-1} \circ S_y(x) = x\} \quad (4.23)$$

The flow $O_{\theta}^{-1}$ will be written as the composition of two flows, each run for half the duration of time.

$$O_{\theta/2}^{-1} \circ O_{\theta/2} \circ S_y(C) = C \quad (4.24)$$

Applying $O_{\theta/2}$ to both sides of the equation results in

$$O_{\theta/2}^{-1} \circ S_y(C) = O_{\theta/2}(C) \quad (4.25)$$
Applying $S_y \circ S_y = 1$ to the left side of the equation leads to
\[ S_y \circ S_y \circ O_{\theta/2}^{-1} \circ S_y(C) = O_{\theta/2}(C) \] (4.26)
Because the vortex flow is symmetric about reflections across the $y$ axis, $S_y \circ O_{\theta/2}^{-1} \circ S_y$ is equivalent to $O_{\theta/2}$, see (4.7), so that
\[ S_y \circ O_{\theta/2}(C) = O_{\theta/2}(C) \] (4.27)
Consider the set $(D)$, given by the transformation of the set $C$ by $O_{\theta/2}$, i.e.,
\[ D = O_{\theta/2}(C) \] (4.28)
Substituting (4.28) in (4.27) results in
\[ S_y(D) = D \] (4.29)
This implies that the set $D$ remains invariant upon transformation by $S_y$, so the set $D$ has to be equal to the $[y \text{ axis}]$. So (4.28) becomes
\[ [y \text{ axis}] = O_{\theta/2}(C) \] (4.30)
Applying $O_{\theta/2}^{-1}$ to both sides of the equation reveals the set $C$
\[ O_{\theta/2}^{-1}[y \text{ axis}] = C \] (4.31)
The fixed line of the symmetry $O_{\theta}^{-1} \circ S_y$, the set $C$, is obtained by mapping the $[y \text{ axis}]$ with $O_{\theta/2}^{-1}$, see (4.31), i.e., the actuation of the outer cylinder clockwise, for half the usual actuating time. So the intersections of the lines of symmetry, (4.31) and (4.17), indicate the position of the periodic points of period 1.
\[ \left\{ \begin{array}{l}
I_{\theta/2} [y \text{ axis}] \\
O_{\theta/2}^{-1} [y \text{ axis}]
\end{array} \right. \] (4.32)
Figure 4.2 shows the lines of symmetry, (4.32), for several values of $\theta$. The intersections of these lines indicate the positions of periodic points of period 1. However, all these intersections turn out to be hyperbolic periodic points. The Poincaré plot of $\theta = \pi$, figure 4.3a, shows the existence of an elliptic periodic point of period 2, as already discovered by Galaktionov et al, [5], by making a Poincaré plot with a small number of markers. The position of these elliptic islands is indicated in figure 4.3b by dots. It turns out that they are situated of the line of symmetry $I_{\theta/2}[y \text{ axis}]$. The assumption is that they are situated on opposite sides of the other line of symmetry, the line $O_{\theta/2}^{-1}[y \text{ axis}]$. This will be investigated next.

Figure 4.2: Lines of symmetry for several values of $\theta$
4.3 Periodic islands of period 2

To detect the positions of the periodic points of period 2, the line of symmetry \( I_{\theta/2} \) has to be reflected in the other line of symmetry \( O_{\theta/2} \). So we want to define the mapping \( P_O \) for this reflection. This mapping \( P_O \) reflects \( p_1 \), an arbitrary point in the cavity, in the line \( O_{\theta/2} \) to the point \( p_2 \)

\[
p_2 = P_O(p_1), \quad p_1, p_2 \in R^2
\]  

(4.33)

Applying \( O_{\theta/2} \) to both points and the line of symmetry \( O_{\theta/2}^{-1} \), shows the points \( O_{\theta/2}(p_1) = p_1^* \) and \( O_{\theta/2}(p_2) = p_2^* \) are situated on opposite sides of the line of symmetry \( O_{\theta/2} \circ O_{\theta/2}^{-1} \) \( y \) axis \( = [y \text{ axis}] \), so

\[
O_{\theta/2}(p_1) = p_1^* \\
O_{\theta/2}(p_2) = p_2^* \\
S_y(p_1^*) = p_2^*
\]

(4.34)

Equation (4.33) can now be rewritten

\[
O_{\theta/2}^{-1}(p_2^*) = P_O \circ O_{\theta/2}^{-1}(p_1^*)
\]

(4.35)

Applying \( O_{\theta/2} \) to both sides leads to

\[
p_2^* = O_{\theta/2} \circ P_O \circ O_{\theta/2}^{-1}(p_1^*)
\]

(4.36)

Using (4.34) results in

\[
S_y(p_1^*) = O_{\theta/2} \circ P_O \circ O_{\theta/2}^{-1}(p_1^*) \Rightarrow \\
S_y = O_{\theta/2} \circ P_O \circ O_{\theta/2}^{-1}
\]

(4.37)

Composing \( O_{\theta/2}^{-1} \) with both sides of 4.37 with \( O_{\theta/2} \) results in

\[
O_{\theta/2}^{-1} \circ S_y \circ O_{\theta/2} = P_O
\]

(4.38)
The positions of the periodic points of period 2 are indicated by the intersections of the line of symmetry \( I_{\theta/2} \) [\( y \) axis] and this line mirrored in the line of symmetry \( O_{-\theta/2} \) [\( y \) axis], in other words, the line \( P_O \circ I_{\theta/2} \) [\( y \) axis]. So the lines of symmetry to determine the position of the periodic islands of period 2 becomes

\[
\begin{align*}
{P_O \circ I_{\theta/2} \ [y \text{ axis}] \quad & I_{\theta/2} \ [y \text{ axis}]} \\
\end{align*}
\]

Figure 4.4 shows the lines of symmetry of (4.39). The two elliptic points of period 2, determined by the Poincaré plot of figure 4.3 are indicated by the dots and match indeed the intersection of the lines.

### 4.4 Eigenvalues and stability

The eigenvalues of the linearization of the Poincaré map at the periodic points are determined in the same way as in chapter 3.5. The eigenvalues have not only been determined of the periodic points for \( \theta = \pi \), but also for other values of \( \theta \). The position of the periodic islands have been indicated by the intersections of the lines of symmetry, like figure 4.4. For values of \( \mu \) lower than 0.97812\( \pi \) the lines do not intersect and the periodic islands do not exist. Figure 4.5a shows the smallest value of \( \theta \), \( \theta = 0.97812\pi \), where the lines still intersect. The Poincaré plot in figure 4.5b and the enlarged islands in figure 4.5c and d, show how these islands arise.

The eigenvalues of these intersections have been calculated and are visualized in figure 4.6, see appendix H for the exact results. When \( \theta = 0.97812\pi \), where the lines intersect for the first time, the eigenvalues equal 1. When \( \mu \) increases, the eigenvalues rotate around the complex unit circle. The positive real part becomes smaller and the imaginary part becomes larger, until the real part reaches 0 and the imaginary part reaches \( \pm1 \), then the real part becomes negative and becomes larger and the imaginary part becomes smaller until both eigenvalues reach a value of -1.
This figure shows eigenvalues entering and leaving the complex unit circle at 1 and -1. It looks like no bifurcations can occur for eigenvalues with an imaginary part $\neq 0$. This can be physically explained by realizing the volume needs to be conserved. If a bifurcation would occur for an eigenvalue on the complex unit circle with an imaginary part $\neq 0$, the conjugated pair of eigenvalues would both go inwards the complex unit circle, creating a sink, or both go outwards the complex unit circle, creating a source. In chapter 2.2 it has already been shown sinks and sources cannot occur in incompressible fluids.

The appearance of the periodic 2 islands occurs with an eigenvalue equals to 1 and in accordance with the Dynamical Systems theory of section 2.3 a Saddle Node, Transcritical or Pitchfork bifurcation is expected. By looking closely to the lines of symmetry, a couple of double intersections appear, as can be seen in figure 4.7. Computing the eigenvalues of the linearization of the Poincaré map at these intersections, see table 4.1, shows that a couple of stable and unstable periodic points arise "out of the blue", meaning this is a Saddle Node bifurcation.

In accordance with the Dynamical Systems theory of section 2.3.2, a Period Doubling bifurcation occurs at an eigenvalue of -1. For a value of $\theta = 1.12\pi$, the eigenvalues just leave
4.5 Periodic islands of higher period

To determine the positions of the periodic islands of period 4, the lines of symmetry will again be explored. Figure 4.9 illustrates the positions of the periodic islands in relation to the lines of symmetry. The upper periodic islands are assumed to be situated on the line of symmetry $I_{\theta/2} \parallel [y\text{ axis}]$ and on opposite sides of the other line of symmetry, the line $P_O \circ I_{\theta/2} \parallel [y\text{ axis}]$. To detect the positions of the periodic points of period 4, the line of symmetry $I_{\theta/2}(y\text{ axis})$ has to be reflected in the other line of symmetry $P_O \circ I_{\theta/2} \parallel [y\text{ axis}]$. So we want to define the mapping $P$ for this reflection. This mapping $P$ reflects $r_1$, an arbitrary point in the cavity, in the line

Figure 4.7: Couple of stable and unstable periodic points of period 2, $\theta = 0.98\pi$
4.5. PERIODIC ISLANDS OF HIGHER PERIOD

Table 4.1: eigenvalues of the linearization of the Poincaré map at the periodic points of period 2 for $\theta = 0.98\pi$

<table>
<thead>
<tr>
<th>Coordinates</th>
<th>eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.5424; -2.8298)</td>
<td>0.8147, 1.2275</td>
</tr>
<tr>
<td>(0.0269; 2.8767)</td>
<td>0.8147, 1.2275</td>
</tr>
<tr>
<td>(0.0450; 2.7622)</td>
<td>0.9384 ± 0.3456i</td>
</tr>
<tr>
<td>(0.9028; -2.6246)</td>
<td>0.9384 ± 0.3456i</td>
</tr>
</tbody>
</table>

$\mathbf{P}_O \circ \mathbf{I}_{\theta/2} \; [y \text{ axis}]$ to the point $r_2$

$$r_2 = \mathbf{P}(r_1),$$

$$r_1, r_2 \in R^2$$

(4.40)

Applying $\mathbf{I}_{\theta/2}^{-1} \circ \mathbf{P}_O$ to both points and to the line of symmetry $\mathbf{P}_O \circ \mathbf{I}_{\theta/2} \; [y \text{ axis}]$ shows the points $\mathbf{I}_{\theta/2}^{-1} \circ \mathbf{P}_O(r_1) = r_1^*$ and $\mathbf{I}_{\theta/2}^{-1} \circ \mathbf{P}_O(r_2) = r_2^*$ are situated on opposite sides of the line of symmetry $\mathbf{I}_{\theta/2}^{-1} \circ \mathbf{P}_O \circ \mathbf{I}_{\theta/2} \; [y \text{ axis}] = [y \text{ axis}]$, so

$$\mathbf{I}_{\theta/2}^{-1} \circ \mathbf{P}_O(r_1) = r_1^*$$

(4.41)

$$\mathbf{I}_{\theta/2}^{-1} \circ \mathbf{P}_O(r_2) = r_2^*$$

(4.42)

$$\mathbf{S}_y(r_1^*) = r_2^*$$

(4.43)

Equation (4.40) can now be rewritten by using (4.41) and (4.42).

$$\mathbf{P}_O \circ \mathbf{I}_{\theta/2}(r_2^*) = \mathbf{P} \circ \mathbf{P}_O \circ \mathbf{I}_{\theta/2}(r_1^*)$$

(4.44)

Using (4.43) this results in

$$\mathbf{P}_O \circ \mathbf{I}_{\theta/2} \circ \mathbf{S}_y(r_1^*) = \mathbf{P} \circ \mathbf{P}_O \circ \mathbf{I}_{\theta/2}(r_1^*) \Rightarrow$$

$$\mathbf{P}_O \circ \mathbf{I}_{\theta/2} \circ \mathbf{S}_y = \mathbf{P} \circ \mathbf{P}_O \circ \mathbf{I}_{\theta/2}$$

(4.45)

a) Initial points: $[0.0319; 2.2259] - [0.0384; 2.2570]$

b) Zoom of upper islands

Figure 4.8: Poincaré plot with $\theta = 1.12\pi$
Letting both sides operate on $I_{\theta/2}^{-1}P_O$ results in

$$P_O \circ I_{\theta/2} \circ S_y \circ I_{\theta/2}^{-1} \circ P_O = P$$

(4.46)

By calling $I_{\theta/2} \circ S_y \circ I_{\theta/2}^{-1} = P_I$, the lines of symmetry to determine the periodic points of period 4 becomes

$$\begin{cases} P_O \circ P_I \circ P_O \circ I_{\theta/2} & [y \text{ axis}] \\ I_{\theta/2} & [y \text{ axis}] \end{cases}$$

(4.47)

Figure 4.10 shows the lines of symmetry of (4.47). It turns out that the two periodic points of period 4 situated above the $x$ axis, are indeed indicated by the intersection of these lines. The location of the two periodic point situated beneath the $x$ axis cannot be determined by this figure, but making a Poincaré plot with the upper periodic points as initial positions will of course also reveal the location of the lower periodic points.

The eigenvalues of these periodic points are represented in figure 4.11, the exact values are shown in appendix H. Again we see the eigenvalues rotating around the complex unit circle, starting at 1 and leaving the complex unit circle at -1, and again a Period Doubling bifurcation occurs, as is shown in figure 4.12.

We already have seen a Period Doubling bifurcation from a periodic island of period 2 to a periodic island of period 4, and now we see also a Period Doubling bifurcation from a periodic island of period 4 to a periodic island of period 8.

By these observations the assumption is made that the process of Period Doubling bifurcations keeps continuing until a periodic island of period infinity exists, this would mean complete chaos. To prove the plausibility of this assumption, one more Period Doubling bifurcation will be demonstrated.

Figure 4.13 shows that one pair of the 4 periodic islands of period 8, is situated on top of the line $I_{\theta/2} [y \text{ axis}]$ and on opposite sides of the line $P_O P_I P_O I_{\theta/2} [y \text{ axis}]$. Another pair of the 4 periodic islands of period 8 is situated on top of the line $P_O P_I P_O I_{\theta/2} [y \text{ axis}]$ and on opposite
sides of the line $I_{\theta/2} [y \text{ axis}]$. The arbitrary choice is made to construct the lines to indicate the position of the periodic islands last mentioned.

In the same way as already has been explained to indicate the positions of the periodic points of period 4 by the lines of symmetry, the lines to indicate the position of the periodic points of period 8 have been determined. The periodic points of period 8 are situated on the intersections of the lines of symmetry

\[
\begin{align*}
\{ P_O \circ P_I \circ P_O \circ I_{\theta/2} [y \text{ axis}] \\
(P_I \circ P_O)^2 \circ I_{\theta/2} [y \text{ axis}]
\end{align*}
\]

(4.48)

The lines of symmetry of (4.48) are shown in figure 4.14. This figure does indeed indicate the position of the periodic points of period 8. Once more we see the eigenvalues rotate around the complex unit circle, starting at 1 and leaving the complex unit circle at -1 with a Period Doubling Bifurcation. The assumption that the process of Period Doubling bifurcations repeats itself until a periodic island of period infinity exists, is shown to be plausible.

To illustrate the beauty of the technique of the lines of symmetry to detect the periodic islands, another Poincaré plot is represented in figure 4.16a. Although this Poincaré plot exists of 40 initial positions and 5000 steps, the small periodic islands of period 16 are still not visible. These islands are shown in figure 4.16b, where a Poincaré map is made with initial positions obtained by the intersections of the lines of symmetry. A zoom of the 4 upper periodic points of period 16 is represented in figure 4.16d. The same area of figure 4.16d is used to zoom on figure 4.16a, resulting in figure 4.16c. This figure should contain the 4 upper periodic points of period 16, nevertheless these islands are still not visible. Although Poincaré maps gain a lot of insight in the mixing behavior, it is not sufficed to detect small periodic islands, which are easily to detect by using the lines of symmetry.

Figure 4.10: Lines of symmetry to determine position of periodic islands of period 4
4.6 Feigenbaum

To gain more insight in the process of repeating Period Doubling bifurcations, figure 4.17 shows a so called Feigenbaum tree, [3]. In a Feigenbaum tree the location of the periodic islands are shown dependently of the parameter $\theta$. Because we are dealing with a 2d flow, the x value of the location is shown separately of the y value of the location. Furthermore there is a separate figure for the upper islands and for the lower islands. Now the period doublings become visible. One island splits up into two other islands, than these two islands each split up into two new ones. This process is repeated several times, so it is likely that this will be repeated until periodic islands of period infinity will have risen.

Figure 4.11: Eigenvalues of the period 4 points

Figure 4.12: Poincare plot of $\theta = 1.148$ with the period 8 points
4.7 Conclusions

Like the Blinking Vortex flow, the position of periodic points of several order of the Journal Bearing flow can also be determined by the intersection of lines of symmetry. The Journal Bearing flow has at least two fixed lines

\[
\left\{ \begin{array}{l}
I_{\theta/2} \quad [y \text{ axis}] \\
O_{\theta/2}^{-1} \quad [y \text{ axis}]
\end{array} \right.
\]

(4.49)

However the intersections of these lines do not indicate the positions of periodic islands. Knowing that periodic points are located on lines of symmetry or in pairs on opposite sides of the line, one fixed line is mirrored in the other fixed line. The intersections result in the detection of a periodic island of period 2. This island appears by a Saddle Node bifurcation, where one stable and one unstable periodic point of period 2 appears “out of the blue”.

Figure 4.13: Position of the periodic points of period 8 on the lines of symmetry to detect periodic points of period 4

Figure 4.14: Position of the periodic points of period 8 on the lines of symmetry to detect periodic points of period 8
Figure 4.15: Eigenvalues of the period 8 points

As the variable parameter $\theta$ increases, the eigenvalues rotate around the complex unit circle for half a circle both, starting at $+1$ and leaving the complex unit circle at $-1$ with a Period Doubling bifurcation. At this point the stable periodic point of period 2 becomes unstable and a stable periodic point of period 4 appears.

The location of these periodic islands can be determined by again mirroring one line of symmetry in the other. The intersection of these lines indicate the position of the periodic points of period 4. Again the eigenvalues rotate around the complex unit circle from $+1$ to $-1$, where another Period Doubling bifurcation leads to a periodic island of period 8. The same process results in a periodic island of period 16.

By plotting a Feigenbaum tree, the process of period doubling becomes visible and shows the assumption that the process of Period Doubling bifurcations keeps repeating until a periodic island of period infinity exists, meaning complete chaos, to be very plausible.
4.7. CONCLUSIONS

a) 40 Initial positions, 5000 iterations
b) Intersections of the lines of symmetry as initial positions

c) Zoom of figure a, same dimensions as figure d
d) Zoom of the 4 upper periodic islands

Figure 4.16: Poincaré plots with $\theta = 1.523\pi$ to detect the periodic islands of period 16
Figure 4.17: Feigenbaum tree, illustrating the positions of periodic islands as function of $\theta$
Chapter 5

Experiments with the Journal Bearing flow

The previous chapter revealed a useful analysis to detect periodic islands of several order using the lines of symmetry. This analysis needs to be validated experimentally. The goal of these experiments is to check whether the periodic islands of period 2 really exist. A quantitative similarity would be best, but a qualitative similarity is already sufficient.

The periodic islands of period 2 arise at \( \theta = 0.97812\pi \) and they become unstable for a value of \( \theta = 1.115\pi \). A save value for \( \theta \) where the periodic points are likely to be stable is a value in the middle of these two values, so the experiments are done with \( \theta = 1.05\pi \).

It would be best if the split up of the periodic island of period 2 to a periodic island of period 4 could be observed. But because the periodic islands of period 4 are quite small, this is not expected.

5.1 Experimental setup

To approach a Stokes flow, the Reynolds number has to be very small, \( Re \ll 1 \). The Reynolds number can be calculated using the following formula

\[
Re = \frac{\rho \cdot V \cdot L}{\eta}
\]  

(5.1)

with

- \( Re \) = Reynolds number [-]
- \( \rho \) = fluid density [kg/m\(^3\)]
- \( V \) = mean fluid velocity [m/s]
- \( L \) = characteristic length [m] (equal to diameter if cross-section is circular)
- \( \eta \) = Dynamical fluid viscosity [Pa \cdot s]
Figure 5.1: Experimental setup, with A: Outer cylinder, B: Inner cylinder, C: mechanism to adjust the eccentricity, D: support for the Inner cylinder with actuator and eccentricity unit, E: mirror, F: actuator of the Inner cylinder, G: gear of the Inner cylinder, H: encoder of the Inner cylinder, I: actuator of the Outer cylinder, J: gear of the Outer cylinder, K: encoder of the Outer cylinder, L: Transmission belt
The outer cylinder has a diameter of 75mm and the inner cylinder has a diameter of 25mm. These dimensions serve the condition $r_{in}/r_{out}=25\text{mm}/75\text{mm}=1/3$. The maximal characteristic length equals the diameter of the outer cylinder, $75\cdot10^{-3}\text{m}$. To obtain a small Reynolds number, the dynamical fluid viscosity needs to be very large. Therefore Polydimethylsiloxabe Trimethylsiloxy has been used with

$$\eta = 10.88[\text{Pa} \cdot \text{s}] @ 23^\circ \text{C} \quad (5.2)$$

The fluid density of Polydimethylsiloxabe Trimethylsiloxy equals $974 \text{kg/m}^3$. As a contrast fluid some Sudanrot 7B or graphite powder will be solved in the Polydimethylsiloxabe Trimethylsiloxy. Because the amount of powder is negligible compared to the amount of Polydimethylsiloxabe Trimethylsiloxy, the values for the fluid density and the dynamical fluid viscosity remain the same.

The last influence on the Reynolds number is the mean fluid velocity. This velocity can be regulated by the controller. To obtain a very small Reynolds number a very low velocity is desired. Though, it cannot be too low, because the period time will become too large then and the gravity will make the contrast fluid drop down, wherefore 3D effects will influence the streaming.

Knowing this, the best fluid velocity is determined experimentally, resulting in a period time of 40 seconds, with a displacement of $1.05\pi$ radians of the outer cylinder. The main fluid velocity becomes now

$$V_{max} = X_{max}/T \Rightarrow$$
$$V_{max} = 1.05\pi \cdot r_{out}/T \Rightarrow$$
$$V_{max} = 1.05\pi \cdot 37.5 \cdot 10^{-3}\text{m}/20\text{s} = 0.006185\text{m/s} \quad (5.3)$$

The Reynolds number results in

$$Re_{max} = \frac{\rho \cdot V_{max} \cdot L_{max}}{\eta}$$
$$Re_{max} = \frac{974 \cdot 0.006185 \cdot 75 \cdot 10^{-3}}{10.88}$$
$$Re_{max} = 0.0415 \quad (5.4)$$

This value for the Reynolds number seems small enough to achieve successful experiments.

The experimental setup is shown in figure 5.1. Several components are mentioned by arrows in this figure. The cylinders are actuated by Maxon DC motors, the outer cylinder is connected to the motor with a transmission belt to obtain free sight from beneath the cylinder. By placing a mirror under the cylinder, it is easy to make pictures of it. Incremental encoders with 500 pulses are used to feedback the position of the motors. The real time control is implemented by a Simulink program. For the data acquisition a TUeDAC has been used. The signals have been strengthened by two amplifiers, as is shown in figure 5.2.

5.2 Results

It is assumed that the considered Stokes flows have no diffusion, meaning fluid-particles are expected to return to their initial locations after flow reversal. To verify this assumption a drop
of contrast-fluid has been added in the cavity, the protocol has been executed for two periods and after that the reverse protocol has been executed for two periods. This process is shown in figure 5.4. The last photo approximates the first photo, but does not match completely the first one. Hereby we conclude the assumptions are rather satisfied.

As already mentioned in the introduction of this chapter, the experiments to detect the periodic islands of period 2 will be done with $\theta = 1.05\pi$. First the locations of the islands are determined numerically. This is shown in figure 5.3 and has resulted in the numerical positions

$$A_{num} = (0.058; 2.41)$$
$$B_{num} = (1.510; -1.91)$$

Figure 5.3: Elliptic periodic points of period 2 for $\theta = 1.05\pi$
5.3. **CONCLUSIONS OF THE EXPERIMENTS**

The difference of dimensions of the diameter $D$ of the numerical model and the experimental setup needs to be taken into account. The positions of the numerical model have to be scaled to the positions of the experimental setup.

$$D_{\text{Outer Cylinder num}} \approx D_{\text{Outer Cylinder exp}}$$

This results in the experimental positions:

$$A_{\text{exp}} = (0.73; 30.10)$$
$$B_{\text{exp}} = (18.90; -23.90)$$

A fluid drop containing Sudanrot 7B dissolved in Polydimethylsiloxabe Trimethylsiloxy is injected at point A. The protocol is repeated several times, as is shown in figure 5.5. After 2, 4 and 10 periods, a part of the drop returns to its initial position. The part of the fluid drop what was injected outside of the island undergoes stretching and folding. After 20 periods not only island A is visible, but island B becomes also visible by the lack of contrast fluid; a small white island appears within the red fluid at position B. One period later island B is marked by the concentrated red fluid and island A by the lack of red fluid. Although the islands are much smaller as the numerically determined islands, they are well visible. They even match quantitatively very well with the numerically determined elliptic points. These measurements took about 20 minutes. Because of the gravity the contrast fluid has dropped 3 cm, meaning 3D effects have occurred. So this flow is not perfectly represented by a 2D flow.

The periodic islands of period 4 have not been detected yet. These islands are likely too small to notice in such a small setup. By making a setup with larger radii, the islands will become larger, so easier to detect. Also it would than be easier to inject the fluid on the desired position. This is now rather difficult, because the fluid distorts the sight. By making the setup also deeper, it is expected that 3D effects will be less disturbing. At last it is desired to investigate which viscous fluid can be better used to approach a Stokes flow without diffusion.

5.3 **Conclusions of the experiments**

The fluid Polydimethylsiloxabe Trimethylsiloxy approaches a Stokes flow without diffusion quite well, although the fluid-particles do not return exactly to their initial locations after flow reversal. To obtain better results, the use of another viscous fluid to approach a Stokes flow without diffusion should be investigated.

The periodic islands of period 2 have been detected experimentally. One island is indicated by a concentration of red fluid and the other island is indicated by the lack of red fluid. Although the islands are much smaller as the numerically determined islands, they are after 20 periods still well visible. They even match quantitatively very well with the numerically determined elliptic points.

The periodic islands of period 4 have not been detected, because they are likely too small to notice in such a small setup. It is recommended to make a setup with larger radii and to make the setup also deeper.
(a) Initial conditions $X$

(b) $I\theta O\theta(X)$

(c) $(I\theta O\theta)^2(X)$

(d) $I^{-1}\theta O^{-1}(I\theta O\theta)^2(X)$

(e) $(I^{-1}\theta O^{-1})^2(I\theta O\theta)^2(X)$

Figure 5.4: Flow reversal
Figure 5.5: Experimental results, $\theta = 1.05\pi$, protocol = $I_\theta O_\theta$
Chapter 6

Conclusions and recommendations

6.1 Conclusions

This thesis has shown that dynamical systems tools provide a lot of opportunities in understanding the mechanisms of laminar mixing. Linking the Poincaré sections to the lines of symmetries shows the existence and largeness of periodic islands. By looking at the eigenvalues of the linearization of the Poincaré map at the periodic points, not only the stability of this point for the current parameters can be determined, but also assumptions can be made, about what will happen with the stability of this point by changing these parameters. Although this has not been proven yet, a start to validate this analysis experimentally has already been made.

The position of periodic points of the Blinking Vortex flow can be determined by the intersection of lines of symmetry. By varying the actuation parameter $\mu$, series of bifurcations occurs; the positions and the number of periodic points changes. Two periodic points become suddenly stable and shift towards an unstable periodic point resulting in a Pitchfork bifurcation. Like the Blinking Vortex flow, the position of periodic points of the Journal Bearing flow can also be determined by the intersections of the lines of symmetry. However, the intersections of these lines do not indicate the positions of periodic islands. Knowing that periodic points are located on lines of symmetry or in pairs on opposite sides of the line, one fixed line is mirrored in the other fixed line. The intersections result in the detection of a periodic island of period 2. This island arises by a Saddle Node bifurcation, where one stable and one unstable periodic point of period 2 appears ”out of the blue”. It looses its stability again by a Period Doubling bifurcation, where a stable periodic island of period 4 arises. The position of this island is again determined by using the lines of symmetry. Another Period Doubling bifurcation leads to a periodic island of period 8. The same process results in a periodic island of period 16. By plotting a Feigenbaum tree, the process of period doubling becomes visible and shows the plausibility that the process of Period Doubling bifurcations keeps repeating until a periodic island of period infinity exists, meaning complete chaos.

The periodic islands of period 2 have been detected experimentally. Although the islands are much smaller as the numerically determined islands, they are still after 20 periods well visible. They even match quantitatively well with the numerically determined elliptic points. The periodic islands of period 4 have not been detected, because they are likely too small to notice.
6.2 Recommendations

- The developed approach seems very promising. To validate its correctness further research is recommended.

- It would be interesting to indicate the position of periodic islands of period 3 using the lines of symmetry. It is assumed that this will result in another serie of period doubling bifurcations, resulting in a Feigenbaum tree leading to chaos.

- It is recommended to demonstrate the period doubling bifurcation of the Journal Bearing flow experimentally. This can be achieved by scaling up the experimental set-up.

- The use of another fluid should be investigated to obtain a flow which approaches a Stokes flow without diffusion better.

- It is assumed that the process of repeatedly Period Doubling bifurcations occurs in most cavities. This should be investigated.

- Further research will have to concentrate on the possibilities of controlling to achieve a better and faster mixing. Using the techniques discussed in this thesis the zones of bad mixing can be detected more easily, wherefore they can be eliminated by a controller. The difficulty with this is to use a controller whereafter the flow can still be analyzed to detect new periodic islands.
Bibliography


Appendix A

Script numerical model BV

```matlab
%%% poincare_BV_gray.m
%%% To produce Poincare maps of the Blinking Vortex Flow
%%% Variable protocol, Left or Right, direction +1 or -1
%%% and variable parameter mu and number of periods
close all;
clear all;
clc;
global x0 y0 mu
xbegin=['0.0';'1.0';'0.0';'0.8'];
ybegin=['0.0';'0.4';'0.4';'0.0'];
protocol=['Right';'+1 ';'Left ';'+1 '];% +1=counter-clockwise, -1=clockwise
mu=0.1;
steps=1000;
sizebegin=size(ybegin,1);
sizeprot=size(protocol,1);
prot=cellstr(protocol);
figure;
hold on;
xw(2,steps,sizebegin)=0;
for j=1:sizebegin
    x0=str2num(xbegin(j,:)); % decide initial x position
    y0=str2num(ybegin(j,:)); % decide initial y position
    k=0;
    for i=1:steps
        func_BV(char(prot(((k+1)*2)-1,1)),char(prot(((k+1)*2),1)),1);
        func_BV(char(prot(((k+1)*2)+1,1)),char(prot(((k+1)*2)+2,1)),1);
        xw(:,i,j)=[x0;y0];
        k=mod(k+2,(sizeprot/2));
    end
end
for i=1:sizebegin
    a=0.9/(sizebegin-1);
j=i-1;
plot(xw(1,:,i),xw(2,:,i),'.','color',[a*j a*j a*j],'MarkerSize',5);
```
end %plot different mappings in greyscale

title(['\mu = ', num2str(mu),', number of periods = ', num2str(steps)]);
ylabel('y')
xlabel('x')
whitebg([1 1 1])
set(gcf,'color',[1 1 1])
axis equal
a=axis;
xmin=((a(2)-a(1))/15+a(1));
ymin1=((a(4)-a(3))/20+a(3));
ymin2=((a(4)-a(3))/10+a(3));
text(xmin,ymin2,'x initial= 0.0; 1.0; 0.0; 0.8')
text(xmin,ymin1,'y initial= 0.0; 0.4; 0.4; 0.0')

%%%%%% func_BV.m
%%%%%% Function to produce Poincare maps of the Blinking Vortex Flow
function func_BV(action,direction,timepart)
% action='right' or 'left', vortexindication
% direction='+1' or '-1', vortexdirection counter-clockwise or clockwise
% timepart, value between 0 and 1, 1=complete blinkingtime

global x0 y0 mu
switch action,
case 'Left',
ksi=-1;
t=timepart;
d=str2num(direction);
rkwadraat=(x0-ksi)^2+y0^2;
deltaphi=d*mu*t/rkwadraat;
x1=ksi-ksi*cos(deltaphi)+x0*cos(deltaphi)-y0*sin(deltaphi);
y1=x0*sin(deltaphi)-ksi*sin(deltaphi)+y0*cos(deltaphi);
x0=x1;
y0=y1;
case 'Right',
ksi=+1;
t=timepart;
d=str2num(direction);
rkwadraat=(x0-ksi)^2+y0^2;
deltaphi=d*mu*t/rkwadraat;
x1=ksi-ksi*cos(deltaphi)+x0*cos(deltaphi)-y0*sin(deltaphi);
y1=x0*sin(deltaphi)-ksi*sin(deltaphi)+y0*cos(deltaphi);
x0=x1;
y0=y1;
end
Appendix B

Script conjugated lines of period 1 BV

```matlab
%%%%%% Conjugated_period_1_BV.m
%%%%%% To produce conjugated lines of period 1
%%%%%% With variables \( \mu, \text{xmin}, \text{xmax} \) and \text{stepsize}

close all;
clear all;
clc;
format long
global x0 y0 mu
mu=3; xmin=-2.5; xmax=2.5;
xstepsize=0.001;
steps=(xmax-xmin)/xstepsize;

xA=[]; xB=[];
figure;
for x0=xmin:xstepsize:xmax
    xoud=x0;
y0=0;
    func_BV(’Right’,’-1’,1/2);
    xA=[xA,[x0;y0]];
y0=0;
    func_BV(’Left’,’+1’,1/2);
    xB=[xB,[x0;y0]];
end
plot(xA(1,:),xA(2,:),’color’,[0 0 0]);
hold on;
plot(xB(1,:),xB(2,:),’--’,’color’,[0 0 0]);
title(’Conjugated lines with \( \mu = \)’, num2str(mu));
axis equal
ylabel(’y’)
xlabel(’x’) 
whitebg([1 1 1])
set(gcf,’color’,[1 1 1])
legend(’A^{-1}_{T/2} x-axis’,’B_{T/2} x-axis’);
```
Appendix C

Script position period 1 points BV

%%% Newton_Raphson.m
%%% To determine the exact position of a periodic point of period 1
clear all
close all
clc;
format long e;
Syms x y
maxit = 50;
tol = 1e-8;
k = 0;
xold = 1.45105124834276;
yold = 0.30322795045212;
conv = 0;
mu=1.5;
DA = mu/((x-1)^2 + y^2);
xk1 = 1 + (x - 1)*cos(DA) - y*sin(DA);
yk1 = (x - 1)*sin(DA) + y*cos(DA);
DB = mu/((xk1 + 1)^2 + yk1^2);
xk2=-1+(1+(x-1)*cos(DA)-y*sin(DA)+1)*cos(DB)-((x-1)*sin(DA)+y*cos(DA))*sin(DB);
yk2=(1+(x-1)*cos(DA)-y*sin(DA)+1)*sin(DB)+((x-1)*sin(DA)+y*cos(DA))*cos(DB);
J = jacobian([xk2-x; yk2-y], [x y]);
F=[xk2-x; yk2-y];
while ~conv,
    k = k+1;
    F2=subs(F,{x,y},{xold,yold});
    J2=subs(J,{x,y},{xold,yold});
    s=J2^-1-F2;
    xnew=xold+s(1,1); ynew=yold+s(2,1);
    res=norm(F2,2);
    conv = ( res<tol ) || ( k==maxit );
    xold = xnew; yold = ynew;
end
xnew
ynew
Appendix D

Script Jacobian of period 1 points BV

```matlab
%%%%%% JacobianBVperiod1.m
%%%%%% To calculate numerically the eigenvalues of periodic points of period 1
clear all;
close all;
clc;
global mu
x0=1.449904641012846e+000;
y0= 3.030523070102435e-001;
epsilon=1e-8; mu=1.5;
A=[x0;y0];
Axp=[x0+epsilon;y0];
Ayp=[x0;y0+epsilon];
Axn=[x0-epsilon;y0];
Ayn=[x0;y0-epsilon];
Bxp=MappingBV(MappingBV(Axp,1),-1);
Bxn=MappingBV(MappingBV(Axn,1),-1);
Byp=MappingBV(MappingBV(Ayp,1),-1);
Byn=MappingBV(MappingBV(Ayn,1),-1);
J11=(Bxp(1)-Bxn(1))/(2*epsilon);
J12=(Byp(1)-Byn(1))/(2*epsilon);
J21=(Bxp(2)-Bxn(2))/(2*epsilon);
J22=(Byp(2)-Byn(2))/(2*epsilon);
A=[J11,J12; J21,J22];
[u,v]=eig(A);
lambdas1=v(1,1)
lambdas2=v(2,2)

%%%%%% function MappingBV
function f=MappingBV(f0,ksi)
global mu
xk=f0(1); yk=f0(2);
phi=mu/((xk-ksi)^2 + yk^2);
f= [ksi + (xk-ksi)*cos(phi)-yk*sin(phi);
     (xk-ksi)*sin(phi)+yk*cos(phi)];
```
Appendix E

Script plot eigenvalues BV

```matlab
%%%%%% Eigenvalues_plot_BV.m
clear all;
close all;
clc;

mu=[8.6;8.9;9.1;9.5;10;11;11.5;12;13;14;14.5;14.7;14.9];

lambda1_imag=[0;0;0.0042;0.2642;0.5680;0.7619;0.9531;0.9911;0.9994;0.9313;
0.7274;0.5360;0.4213;0.2459];

lambda1_real=[-0.5897;-0.7665;-0.99999;-0.9645;-0.8230;-0.6477;-0.3026;
-0.1330;0.03477;0.3644;0.6862;0.8442;0.9070;0.9693];

lambda2_real=[-1.6957;-1.3047;-0.99999;-0.9645;-0.8230;-0.6477;-0.3026;
-0.1330;0.03477;0.3644;0.6862;0.8442;0.9070;0.9693];

lambda2_imag=-1*lambda1_imag;

min=floor(mu(1,1)); % lowest value of mu round towards minus infinity
max=ceil(mu(size(mu,1),1)); % highest value of mu round towards plus infinity

%%%%%% 3-Dimensional plot
plot3(lambda1_real,lambda1_imag,mu,'k*',lambda2_real,lambda2_imag,mu,'ok')
hold on
plot3(lambda1_real,lambda1_imag,mu,'k',lambda2_real,lambda2_imag,mu,'k')
hold on
[x,y,z]=cylinder(1);
surf(x,y,(max-min)*z+min)
colormap([0 0 0]);
shading flat
alpha(.1)
legend('</\lambda1', '</\lambda2')
xlabel('Real axis')
ylabel('Imaginary axis')
```

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APPENDIX E. SCRIPT PLOT EIGENVALUES BV

zlabel('{\mu}')
title('Eigenvalues as function of {\mu}')
whitebg([1 1 1])
set(gcf,'color',[1 1 1])
axis equal
axis([-1.5,1.5,-1.5,1.5,min,max])

%%%%%% 2-Dimensional plot
figure
plot(lambda1_real,lambda1_imag,'k*',lambda2_real,lambda2_imag,'ok')
axis equal
title('Eigenvalues as function of {\mu}, view from above')
xlabel('Real axis')
ylabel('Imaginary axis')
legend('{\lambda}_1','{\lambda}_2')
hold on
a=0:0.1:2*pi;
plot(sin(a),cos(a),'k');
plot([-2.2,1.2],[0,0],'k');
plot([0,0],[-1.2,1.2],'k');
whitebg([1 1 1])
set(gcf,'color',[1 1 1])
## Appendix F

### Eigenvalues BV for several values of $\mu$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>x value</th>
<th>y value</th>
<th>eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.00</td>
<td>1.5030</td>
<td>1.3499</td>
<td>-0.4390, -2.2778</td>
</tr>
<tr>
<td>8.60</td>
<td>1.4644</td>
<td>1.4380</td>
<td>-0.5897, -1.6957</td>
</tr>
<tr>
<td>8.90</td>
<td>1.4426</td>
<td>1.4816</td>
<td>-0.7665, -1.3047</td>
</tr>
<tr>
<td>9.00</td>
<td>1.4349</td>
<td>1.4961</td>
<td>-1.0000 ± 0.0042i</td>
</tr>
<tr>
<td>9.10</td>
<td>1.4270</td>
<td>1.5106</td>
<td>-0.9645 ± 0.2642i</td>
</tr>
<tr>
<td>9.50</td>
<td>1.3936</td>
<td>1.5684</td>
<td>-0.8230 ± 0.5680i</td>
</tr>
<tr>
<td>10.00</td>
<td>1.3470</td>
<td>1.6400</td>
<td>-0.6477 ± 0.7619i</td>
</tr>
<tr>
<td>11.00</td>
<td>1.2362</td>
<td>1.7817</td>
<td>-0.3026 ± 0.9531i</td>
</tr>
<tr>
<td>11.50</td>
<td>1.1704</td>
<td>1.8519</td>
<td>-0.1330 ± 0.9911i</td>
</tr>
<tr>
<td>12.00</td>
<td>1.0963</td>
<td>1.9216</td>
<td>0.0348 ± 0.9994i</td>
</tr>
<tr>
<td>13.00</td>
<td>0.9151</td>
<td>2.0597</td>
<td>0.3644 ± 0.9313i</td>
</tr>
<tr>
<td>14.00</td>
<td>0.6604</td>
<td>2.1962</td>
<td>0.6862 ± 0.7274i</td>
</tr>
<tr>
<td>14.50</td>
<td>0.4713</td>
<td>2.2639</td>
<td>0.8442 ± 0.5360i</td>
</tr>
<tr>
<td>14.70</td>
<td>0.3662</td>
<td>2.2909</td>
<td>0.9070 ± 0.4213i</td>
</tr>
<tr>
<td>14.90</td>
<td>0.2113</td>
<td>2.3178</td>
<td>0.9693 ± 0.2459i</td>
</tr>
</tbody>
</table>

Table F.1: Eigenvalues of the pair of possible stable periodic points, situated on opposite sides of the y axis, of period 1 for several values of $\mu$
<table>
<thead>
<tr>
<th>$\mu$</th>
<th>eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.0</td>
<td>0.57, 1.76</td>
</tr>
<tr>
<td>14.5</td>
<td>0.67, 1.49</td>
</tr>
<tr>
<td>14.999</td>
<td>1.0000 $\pm$ 0.008 i</td>
</tr>
<tr>
<td>15</td>
<td>0.9998 $\pm$ 0.0197 i</td>
</tr>
<tr>
<td>17.2</td>
<td>0.6916 $\pm$ 0.7223i</td>
</tr>
<tr>
<td>19</td>
<td>0.4819 $\pm$ 0.8762i</td>
</tr>
<tr>
<td>20</td>
<td>0.379 $\pm$ 0.925 i</td>
</tr>
<tr>
<td>25</td>
<td>-0.02 $\pm$ 0.9997i</td>
</tr>
<tr>
<td>35</td>
<td>-0.4911 $\pm$ 0.8711i</td>
</tr>
<tr>
<td>45</td>
<td>-0.7315 $\pm$ 0.6818i</td>
</tr>
<tr>
<td>55</td>
<td>-0.8631 $\pm$ 0.5050i</td>
</tr>
<tr>
<td>75</td>
<td>-0.9756 $\pm$ 0.2195i</td>
</tr>
<tr>
<td>100</td>
<td>-0.9994 $\pm$ 0.0352i</td>
</tr>
<tr>
<td>120</td>
<td>-0.9829 $\pm$ 0.1842i</td>
</tr>
<tr>
<td>150</td>
<td>-0.9268 $\pm$ 0.3499i</td>
</tr>
<tr>
<td>200</td>
<td>-0.8459 $\pm$ 0.5333i</td>
</tr>
<tr>
<td>250</td>
<td>-0.7584 $\pm$ 0.6518i</td>
</tr>
<tr>
<td>300</td>
<td>-0.6797 $\pm$ 0.7225i</td>
</tr>
<tr>
<td>400</td>
<td>-0.55 $\pm$ 0.84 i</td>
</tr>
<tr>
<td>500</td>
<td>-0.44 $\pm$ 0.896 i</td>
</tr>
<tr>
<td>750</td>
<td>-0.25 $\pm$ 0.97 i</td>
</tr>
<tr>
<td>1000</td>
<td>-0.127 $\pm$ 0.99 i</td>
</tr>
<tr>
<td>1250</td>
<td>-0.033 $\pm$ 0.9994i</td>
</tr>
<tr>
<td>1500</td>
<td>0.03966 $\pm$ 0.992 i</td>
</tr>
<tr>
<td>2000</td>
<td>0.147 $\pm$ 0.989 i</td>
</tr>
<tr>
<td>3000</td>
<td>0.284 $\pm$ 0.9587i</td>
</tr>
<tr>
<td>5000</td>
<td>0.43 $\pm$ 0.90 i</td>
</tr>
<tr>
<td>1e+4</td>
<td>0.58 $\pm$ 0.81 i</td>
</tr>
<tr>
<td>5e+4</td>
<td>0.807 $\pm$ 0.59 i</td>
</tr>
<tr>
<td>1e+5</td>
<td>0.86 $\pm$ 0.51 i</td>
</tr>
<tr>
<td>5e+5</td>
<td>0.938 $\pm$ 0.35 i</td>
</tr>
<tr>
<td>1e+6</td>
<td>0.956 $\pm$ 0.294 i</td>
</tr>
<tr>
<td>1e+7</td>
<td>0.988 $\pm$ 0.1557i</td>
</tr>
<tr>
<td>1e+9</td>
<td>0.984 $\pm$ 0.176 i</td>
</tr>
<tr>
<td>1e+11</td>
<td>0.9653, 1.0360</td>
</tr>
<tr>
<td>1e+12</td>
<td>0.9522, 1.0502</td>
</tr>
<tr>
<td>1e+15</td>
<td>0.5486, 1.8227</td>
</tr>
</tbody>
</table>

Table F.2: Eigenvalues of the possible stable periodic point of period 1, situated on top of the y axis, for several values of $\mu$.
Appendix G

Script velocity field JB

PROGRAM JBMAP
IMPLICIT NONE

REAL*8 EPS, H0, HMIN
COMMON /PRECIS/ EPS, H0, HMIN

REAL*8 R1,R2,ECC,D1,D2,S,V1,V2,AIN,BIN,CIN,DIN,EIN,FIN,
,AEX,BEX,CEX,DEX,EEX,FEX

COMMON /COEF/ R1,R2,ECC,D1,D2,S,AIN,BIN,CIN,DIN,EIN,FIN,
,AEX,BEX,CEX,DEX,EEX,FEX,V1,V2

INTEGER NUM
COMMON /NPTS/ NUM

INTEGER NOK, NBAD,N2
EXTERNAL BSSTEP,JBVELXY,RKQS,MOVEPAR
DOUBLE PRECISION XYZWORK(6006), TS, TF, EPSD, H0D, HMIND

REAL*8 THETA,TAU,OMEGA,XN(10000),YN(10000),TCURR,TFIN,PI,Y1,Y2,
,YSTART,YEND,Y0,RESOL,X1,X2,X0,F(2,2),DIVERG,JACOB,TRACE,SHIFT

INTEGER*4 I,NDIV,N,SWITCH,KITER,NITER

PI=4.D+00*DATAN(1.D+00)

NITER=5000

TAU=1.25D+00*PI

OMEGA=3.D+00

THETA=1.1523D+00

OPEN(22,FILE='pmap.asc')
CALL MOVEPAR

NDIV=20
YSTART=R1-ECC
YEND=R2
DO I=1,NDIV-1
  XN(I)=0.D+00
  YN(I)=YSTART+((YEND-YSTART)*I)/NDIV
ENDDO
YSTART=-R2
YEND=-R1-ECC
DO I=1,NDIV-1
  XN(I+NDIV-1)=0.D+00
  YN(I+NDIV-1)=YSTART+((YEND-YSTART)*I)/NDIV
ENDDO

Fill array XYZWORK with x and y coordinates of points

NUM=2*NDIV-2
DO I=1,NUM
  XYZWORK((I-1)*2+1)=XN(I)
  XYZWORK((I-1)*2+2)=YN(I)
END DO

WRITE(22,*) NUM, NITER
WRITE(22,*) THETA, OMEGA

Write initial points to file pmap.asc

DO I=1,NUM
  WRITE(22,*) XYZWORK((I-1)*2+1),XYZWORK((I-1)*2+2)
END DO

H0D = H0
HMIND = HMIN
EPSD = EPS

------ tracking the set of points
DO KITER=1,NITER
  WRITE(*,*) ' Iteration = ',KITER
  N2=2*NUM
  TCURR=0.D+00
  TFIN=THETA
  V1=0.D+00
  V2=R2*PI
CALL ODEINT(XYZWORK,N2,TCURR,TFIN,EPSD,H0D,HMIND,
    NOK,NBAD,JBVELXY,RKQS)

TCURR=0.D+00
TFIN=THETA
V1=R1*PI*OMEGA
V2=0.D+00
CALL ODEINT(XYZWORK,N2,TCURR,TFIN,EPSD,H0D,HMIND,
    NOK,NBAD,JBVELXY,RKQS)

DO I=1,NUM
    XN(I)=XYZWORK((I-1)*2+1)
    YN(I)=XYZWORK((I-1)*2+2)
    WRITE(22,*) XN(I),YN(I)
END DO

ENDDO

CLOSE(22)
STOP
END

SUBROUTINE MOVEPAR
IMPLICIT NONE
CHARACTER*12 FINAME
CHARACTER*4 MASK
REAL*8 EPS, H0, HMIN, R1, R2, E
COMMON /PRECIS/ EPS, H0, HMIN

R2=3.D+00
R1=3.D+00/3.D+00
E=0.9D+00
EPS=1.D-08
H0=0.001D+00
HMIN=0.D+00

CALL JBSETCOEF(R1,R2,E)
RETURN
END

SUBROUTINE MOVEPAR
IMPLICIT NONE
CHARACTER*12 FINAME
CHARACTER*4 MASK
REAL*8 EPS, H0, HMIN, R1, R2, E
COMMON /PRECIS/ EPS, H0, HMIN

Original from patrick

R2=3.D+00
R1=3.D+00/3.D+00
E=0.9D+00
EPS=1.D-08
H0=0.001D+00
HMIN=0.D+00

CALL JBSETCOEF(R1,R2,E)
RETURN
END

JBVEL - routines to compute the stream function and velocity
APPENDIX G. SCRIPT VELOCITY FIELD JB

components in Journal bearing flow (following G.H.Wannier, Quarterly of Appl. Math., vol VIII, No.1, April 1950)

SUBROUTINE JBPSI(A,B,C,D,E,F,S,X,Y,PSI,UX,UY)
IMPLICIT NONE
REAL*8 A,B,C,D,E,F,S,X,Y,PSI,UX,UY,SPY,SMY,ZP,ZM,ZR,L

SPY=S+Y
SMY=S-Y
ZP=X*X+SPY*SPY
ZM=X*X+SMY*SMY
L=DLOG(ZP/ZM)
ZR=2.D+00*(SPY/ZP+SMY/ZM)

--- stream function ----
PSI=A*L + B*Y*SPY/ZP + C*Y*SMY/ZM + D*Y + E*(X*X+Y*Y+S*S) + F*Y*L

--- Ux ----
UX=-A*ZR - B*((S+2.D+00*Y)*ZP-2.D+00*SPY*SPY*Y)/(ZP*ZP) -
   - C*((S-2.D+00*Y)*ZM+2.D+00*SMY*SMY*Y)/(ZM*ZM) - D -
   - E*2.D+00*Y - F*(L+Y*ZR)

--- Uy ----
UY=-A*8.D+00*S*X*Y/(ZP*ZM) - B*2.D+00*X*Y*SPY/(ZP*ZP) -
   - C*2.D+00*X*Y*SMY/(ZM*ZM) + E*2.D+00*X -
   - F*8.D+00*S*X*Y*Y/(ZP*ZM)

RETURN
END

SUBROUTINE JBCOEF(R1,R2,ECC,D1,D2,S,V1,V2,A,B,C,D,E,F)
IMPLICIT NONE
REAL*8 R1,R2,ECC,D1,D2,S,V1,V2,A,B,C,D,E,F,L1,L2,DEN,CURLB

D1=(R2*R2-R1*R1)/(2.D+00*ECC)-0.5D+00*ECC
D2=D1+ECC
S=DSQRT((R2-R1-ECC)*(R2-R1+ECC)*(R2+R1+ECC)*(R2+R1-ECC))/
   / (2.D+00*ECC)

L1=DLOG((D1+S)/(D1-S))
L2=DLOG((D2+S)/(D2-S))
DEN=(R2*R2+R1*R1)*(L1-L2)-4.D+00*S*ECC
CURLB=2.D+00*(D2*D2-D1*D1)*((R1*V1+R2*V2)/(R2*R2+R1*R1)*DEN)+
+ R1*R1*R2*R2*(V1/R1-V2/R2)/(S*(R1*R1+R2*R2)*(D2-D1))

A=-0.5D+00*(D1*D2-S*S)*CURLB
B=(D1+S)*(D2+S)*CURLB
C=(D1-S)*(D2-S)*CURLB
D=(D1*L2-D2*L1)*(R1*V1+R2*V2)/DEN -
- 2.D+00*S*((R2*R2-R1*R1)/(R2*R2+R1*R1))*(R1*V1+R2*V2)/DEN -
- R1*R1*R2*R2*(V1/R1-V2/R2)/(S*(R1*R1+R2*R2)*ECC)
E=0.5D+00*(L1-L2)*(R1*V1+R2*V2)/DEN
F=ECC*(R1*V1+R2*V2)/DEN

RETURN
END

c===================================================================
c JBSETCOEF - sets the geometrical parameters, computes coefficients
separately for rotating inner cylinder (V1=1, V2=0) -
coefficients "*IN", and for rotating outer cylinder (V1=0, V2=1)
- coefficient "*ex". Positive velocities mean counterclockwise
rotation of the cylinders. Coefficients are then stored in
common block "COEF".
===================================================================

SUBROUTINE JBSETCOEF(R10,R20,ECC0)
IMPLICIT NONE
REAL*8 R1,R2,ECC,D1,D2,S,V1,V2,AIN,BIN,CIN,DIN,EIN,FIN,
,AEX,BEX,CEX,DEX,EEX,FEX
COMMON /COEF/ R1,R2,ECC,D1,D2,S,A1N,B1N,C1N,D1N,E1N,F1N,
,AEX,BEX,CEX,DEX,EEX,FEX,V1,V2
REAL*8 PI,R10,R20,ECC0

PI=4.D+00*DATAN(1.D+00)
R1=R10
R2=R20
ECC=ECC0

V1=1.D+00
V2=0.D+00
CALL JBCOEF(R1,R2,ECC,D1,D2,S,V1,V2,AIN,BIN,CIN,DIN,EIN,FIN)

V1=0.D+00
V2=1.D+00
CALL JBCOEF(R1,R2,ECC,D1,D2,S,V1,V2,AEX,BEX,CEX,DEX,EEX,FEX)

------for inner cylinder rotating
V2=0.D+00
V1=R1*PI

--- for inner cylinder rotating

V1=0.D+00

V2=R2*PI

RETURN

END

SUBROUTINE JBVELXY(T,XY,UV)

IMPLICIT NONE

REAL*8 R1,R2,ECC,D1,D2,S,A1,B1,C1,D1,E1,F1,A2,B2,C2,D2,E2,F2
COMMON /COEF/ R1,R2,ECC,D1,D2,S,A1,B1,C1,D1,E1,F1,A2,B2,C2,D2,E2,F2

REAL*8 T,XY(10000),UV(10000),X,Y,UX,UY,PSI

INTEGER N,I

COMMON /NPTS/ N

REAL*8 JUMP,RR,TINY,UXIN,UYIN,UXEX,UYEX,R

PARAMETER (JUMP=1.D+00, TINY=1.D-12)

DO I=1,N

X=XY(2*I-1)

Y=XY(2*I)

IF ((X*X+Y*Y).GT.(R2*R2)) THEN

R=DSQRT(X*X+Y*Y)

UX=-V2*Y/R2-JUMP*X/R

UY=V2*X/R2-JUMP*Y/R

ELSEIF (((Y+ECC)*(Y+ECC)+X*X).LT.(R1*R1)) THEN

RR=DSQRT((Y+ECC)*(Y+ECC)+X*X)

UX=-V1*(Y+ECC)/R1+JUMP*X/RR

UY=V1*X/R1+JUMP*(Y+ECC)/RR

ELSE

UX=0.D+00

UY=0.D+00

Y=Y+D2

IF (DABS(V1).GE.TINY) THEN
CALL JBPSI(AIN, BIN, CIN, DIN, EIN, FIN, S, X, Y, PSI, UXIN, UYIN)
UX=V1*UXIN
UY=V1*UYIN
ENDIF
IF (DABS(V2).GE.TINY) THEN
   CALL JBPSI(AEX, BEX, CEX, DEX, EEX, FEX, S, X, Y, PSI, UXEX, UYEX)
   UX=UX+V2*UXEX
   UY=UY+V2*UYEX
ENDIF
ENDIF
UV(2*I-1)=UX
UV(2*I)=UY
ENDDO
RETURN
END
SUBROUTINE ODEINT(YSTART,NVAR,X1,X2,EPS,H1,HMIN,NOK,NBAD,DERVS,
                   RKQS)
    INTEGER NBAD,NOK,NVAR,KMAXX,MAXSTP,NMAX
    DOUBLE PRECISION EPS,H1,HMIN,X1,X2,YSTART(NVAR),TINY
    EXTERNAL DERVS,RKQS
    PARAMETER (MAXSTP=100000000,NMAX=6006,KMAXX=200,TINY=1.D-30)
    C Runge-Kutta driver with adaptive stepsize control. Integrate the
    C starting values YSTART(1:NVAR) from X1 to X2 with accuracy EPS,
    C storing intermediate results in the common block /PATH/. H1 should be
    C set as a guessed first stepsize, HMIN as the minimum allowed stepsize
    C (can be zero). On output NOK and NBAD are the number of good and bad
    C (but retried and fixed) steps taken, and YSTART is replaced by values
    C at the end of the integration interval. DERVS is the user-supplied
    C subroutine for calculating the right hand side derivative, while RKQS
    C is the name of the stepper routine to be used. /PATH/ contains its
    C own information about how often an intermediate value is to be stored.
    INTEGER I,KMAX,KOUNT,NSTP
    DOUBLE PRECISION DXSAV,H,HDID,HNEXT,X,XSAV,DYDX(NMAX),XP(KMAXX),Y
    COMMON /PATH/ KMAX,KOUNT,DXSAV,XP,YP
    c
    c
    X=X1
    H=DSIGN(H1,X2-X1)
    NOK=0
    NBAD=0
    KOUNT=0
    DO 11 I=1,NVAR
           Y(I)=YSTART(I)
    11 CONTINUE
    C Assure storage of first step:
    IF (KMAX.GT.0) XSAV=X-2.D0+DXSAV
    C Take at most MAXSTP steps
    DO 16 NSTP=1,MAXSTP
           CALL DERVS(X,Y,DYDX)
    C If stepsize causes Y to fall outside domain, decrease stepsize
    DO 12 I=1,NVAR
           Scaling used to monitor accuracy. This general purpose choice
    C can be modified if needed.
           YSCAL(I)=ABS(Y(I))+ABS(H*DYDX(I))+TINY
           YSCAL(I)=1.D+00
    12 CONTINUE
    IF(KMAX.GT.0)THEN
           IF(ABS(X-XSAV).GT.ABS(DXSAV)) THEN
    C Store intermediate results
           IF(KOUNT.LT.KMAX-1)THEN
KOUNT = KOUNT + 1
XP(KOUNT) = X
DO 13 I = 1, NVAR
   YP(I, KOUNT) = Y(I)
13 CONTINUE
XSAV = X
ENDIF
ENDIF
ENDIF
C If stepsize can overshoot, decrease
IF((X + H - X2) * (X + H - X1) .GT. 0.D0) H = X2 - X
CALL RKQS(Y, DYDX, NVAR, X, H, EPS, YSCAL, HDID, HNEXT, DERVS)
IF(HDID .EQ. H) THEN
   NOK = NOK + 1
ELSE
   NBAD = NBAD + 1
ENDIF
IF((X - X2) * (X2 - X1) .GE. 0.D0) THEN
C Are we done?
DO 14 I = 1, NVAR
   YSTART(I) = Y(I)
14 CONTINUE
IF(KMAX .NE. 0) THEN
C Save final step
KOUNT = KOUNT + 1
XP(KOUNT) = X
DO 15 I = 1, NVAR
   YP(I, KOUNT) = Y(I)
15 CONTINUE
ENDIF
RETURN
ENDIF
IF(ABS(HNEXT) .LT. HMIN) PAUSE
* 'Stepsize smaller than minimum in ODEINT'
H = HNEXT
16 CONTINUE
PAUSE 'Too many steps in ODEINT'
RETURN
END
C (C) COPR. 1986-92 NUMERICAL RECIPES SOFTWARE 45%.
C
C
SUBROUTINE PZEXTR(IEST, XEST, YEST, YZ, DY, NV)
INTEGER IEST, NV, IMAX, NMAX
DOUBLE PRECISION XEST, DY(NV), YEST(NV), YZ(NV)
PARAMETER (IMAX = 13, NMAX = 6006)
C Use polynomial extrapolation to evaluate NV functions at x=0 by
C fitting a polynomial to a sequence of estimates with progressively
C smaller values x=XEST, and corresponding function vectors YEST(1:NV).
C This call is number IEST in a sequence of calls. Extrapolated
C function values are output as YZ(1:NV), and their estimated error is
C output as DY(1:NV).
C Parameters: Maximum expected value of IEST is IMAX; of NV is NMAX
INTEGER J,K1
DOUBLE PRECISION DELTA,F1,F2,Q,D(NMAX),QCOL(NMAX,IMAX),X(IMAX)
SAVE QCOL,X
C Save current independent variable
   X(IEST)=XEST
   DO 11 J=1,NV
       DY(J)=YEST(J)
       YZ(J)=YEST(J)
11 CONTINUE
C Store first estimate in first column
   IF(IEST.EQ.1) THEN
       DO 12 J=1,NV
           QCOL(J,1)=YEST(J)
       12 CONTINUE
   ELSE
       DO 13 J=1,NV
           D(J)=YEST(J)
       13 CONTINUE
       DO 15 K1=1,IEST-1
           DELTA=1.D0/(X(IEST-K1)-XEST)
           F1=XEST*DELTA
           F2=X(IEST-K1)*DELTA
           C Propagate tableau 1 diagonal more
           DO 14 J=1,NV
               Q=QCOL(J,K1)
               QCOL(J,K1)=DY(J)
               DELTA=D(J)-Q
               DY(J)=F1*DELTA
               D(J)=F2*DELTA
               YZ(J)=YZ(J)+DY(J)
14 CONTINUE
15 CONTINUE
   DO 16 J=1,NV
       QCOL(J,IEST)=DY(J)
16 CONTINUE
END
C (C) COPR. 1986-92 NUMERICAL RECIPES SOFTWARE 45%.
SUBROUTINE BSSTEP(Y,DYDX,NV,X,HTRY,EPS,YSCAL,HDID,HNEXT,DERVS)
INTEGER NV,NMAX,KMAXX,IMAX
DOUBLE PRECISION EPS,HDID,HNEXT,HTRY,X,DYDX(NV),Y(NV),YSCAL(NV)
*,SAFE1,SAFE2,
*REDMAX,REDMIN,TINY,SCALMX
PARAMETER (NMAX=6006,KMAXX=8,IMAX=KMAXX+1,SAFE1=.25D0,SAFE2=.7D0,
*REDMAX=1.D-5,REDMIN=.7D0,TINY=1.D-30,SCALMX=.1D0)
C USES DERVS,MMID,PZEXTR
C Bulirsh-Stoer step with monitoring of local truncation error to ensure
C accuracy and adjust stepsize. Input are the independent variable
C vector Y(1:NV) and its derivative DYDX(1:NV) at the starting value of
C the independent variable X. Also input are the stepsize to be
C attempted HTRY, the required accuracy EPS, and the vector YSCAL(1:NV)
C against which the error is scaled. On output, Y and X are replaced by
C their new values, HDID is the stepsize that was actually accomplished,
C and HNEXT is the estimated next stepsize. DERVS is the user supplied
C subroutine that computes the right-handside derivatives. Be sure to
C set HTRY on successive steps to the value of HNEXT returned from the
C previous step, as is the case if the routine is called by ODEINT.
C Parameters: NMAX is the maximum value of NV; KMAXX is the maximum row
C number used in the extrapolation; IMAX is the next rownumber; SAFE1
C and SAFE2 are safety factors; REDMAX is the maximum factor used when a
C stepsize is reduced, REDMIN the minimum; TINY prevents division by
C zero; 1/SCALMX is the maximum factor by which a stepsize can be
C increased.

INTEGER I,IQ,K,KK,KM,KMAX,KOPT,NSEQ(IMAX)
DOUBLE PRECISION EPS1,EPSOLD,ERRMAX,FACT,H,RED,SCALE,WORK,WRKMIN
*,XEST,XNEW,
*A(IMAX),ALF(KMAXX,KMAXX),ERR(KMAXX),YERR(NMAX),YSAV(NMAX),
*YSEQ(NMAX)
LOGICAL FIRST,REDUCT
SAVE A,ALF,EPSOLD,FIRST,KMAX,KOPT,NSEQ,XNEW
EXTERNAL DERVS
DATA FIRST/.TRUE./,EPSOLD/-1.D0/
DATA NSEQ /2,4,6,8,10,12,14,16,18/
C New accuracy; reinitialise
IF(EPS.NE.EPSOLD)THEN
C 'Impossible' values
HNEXT=-1.D29
XNEW=-1.D29
EPS1=SAFE1*EPS
C Compute work coefficients
A(1)=NSEQ(1)+1
DO 11 K=1,KMAXX
   A(K+1)=A(K)+NSEQ(K+1)
11 CONTINUE
C Compute alpha(k,q)
APPENDIX G. SCRIPT VELOCITY FIELD JB

DO 13 IQ=2,KMAXX
    DO 12 K=1,IQ-1
        ALF(K,IQ)=EPS1**((A(K+1)-A(IQ+1))/((A(IQ+1)-A(1)+1.D0)*(2*K+1)))
    12 CONTINUE
13 CONTINUE
EPSOLD=EPS
C Determine optimum row number for convergence
DO 14 KOPT=2,KMAXX-1
    IF(A(KOPT+1).GT.A(KOPT)*ALF(KOPT-1,KOPT))GOTO 1
14 CONTINUE
1 KMAX=KOPT
ENDIF
H=HTRY
C Save the starting values
DO 15 I=1,NV
    YSAV(I)=Y(I)
15 CONTINUE
C A new stepsize or a new integration; re-establish the order window
IF(H.NE.HNEXT.OR.X.NE.XNEW)THEN
    FIRST=.TRUE.
    KOPT=KMAX
ENDIF
REDUCT=.FALSE.
C Evaluate the sequence of modified midpoint integrations
2 DO 17 K=1,KMAX
    XNEW=X+H
    IF(XNEW.EQ.X)PAUSE 'Step size underflow in BSSTEP'
    CALL MMID(YSAV,DYDX,NV,X,H,NSEQ(K),YSEQ,DERVS)
C Squared, since error series is even
    XEST=(H/NSEQ(K))**2
C Perform extrapolation
    CALL PZEXTR(K,XEST,YSEQ,Y,YERR,NV)
C Compute normalized error estimate eps(k)
    IF(K.NE.1)THEN
        ERRMAX=TINY
        DO 16 I=1,NV
            ERRMAX=MAX(ERRMAX,ABS(YERR(I)/YSCAL(I)))
        16 CONTINUE
    ENDIF
C Scale relative to tolerance
    ERRMAX=ERRMAX/EPS
    KM=K-1
    ERR(KM)=(ERRMAX/SAFE1)**(1.D0/(2*KM+1))
ENDIF
C In order window
IF(K.NE.1.AND.(K.GE.KOPT-1.OR.FIRST))THEN
C Converged
IF(ERRMAX.LT.1.D0)GOTO 4

C Check for possible stepsize reduction
IF(K.EQ.KMAX.OR.K.EQ.KOPT+1)THEN
   RED=SAFE2/ERR(KM)
   GOTO 3
ELSE IF(K.EQ.KOPT)THEN
   IF(ALF(KOPT-1,KOPT).LT.ERR(KM))THEN
      RED=1.D0/ERR(KM)
      GOTO 3
   ENDIF
ELSE IF(KOPT.EQ.KMAX)THEN
   IF(ALF(KM,KMAX-1).LT.ERR(KM))THEN
      RED=ALF(KM,KMAX-1)*SAFE2/ERR(KM)
      GOTO 3
   ENDIF
ELSE IF(ALF(KM,KOPT).LT.ERR(KM))THEN
   RED=ALF(KM,KOPT-1)/ERR(KM)
   GOTO 3
ENDIF
ENDIF
17 CONTINUE

C Reduce stepsize by at least REDMIN and at most REDMAX
3 RED=MIN(RED,REDMIN)
   RED=MAX(RED,REDMAX)
   H=H*RED
   REDUCT=.TRUE.

C Try again
GOTO 2

C Successful step taken
4 X=XNEW
   HDID=H
   FIRST=.FALSE.
   WRKMIN=1.D35

C Compute optimal row for convergence and corresponding stepsize
DO 18 KK=1,KM
   FACT=MAX(ERR(KK),SCALMX)
   WORK=FACT*A(KK+1)
   IF(WORK.LT.WRKMIN)THEN
      SCALE=FACT
      WRKMIN=WORK
      KOPT=KK+1
   ENDIF
18 CONTINUE

C Check for possible order increase, but not if stepsize was just
C reduced
HNEXT=H/SCALE
IF(KOPT.GE.K.AND.KOPT.NE.KMAX.AND..NOT.REDUCT)THEN
FACT = MAX(SCALE/ALF(KOPT-1, KOPT), SCALMX)
IF(A(KOPT+1)*FACT.LE.WRKMIN) THEN
   HNEXT = H/FACT
   KOPT = KOPT + 1
ENDIF
ENDIF
RETURN
END

C (C) COPR. 1986-92 NUMERICAL RECIPES SOFTWARE 45%.
C
SUBROUTINE MMID(Y, DYDX, NVAR, XS, HTOT, NSTEP, YOUT, DERVS)
INTEGER NSTEP, NVAR, NMAX
DOUBLE PRECISION HTOT, XS, DYDX(NVAR), Y(NVAR), YOUT(NVAR)
EXTERNAL DERVS
PARAMETER (NMAX=6006)

C Modified Midpoint Step. Dependent variable vector Y(1:NVAR) and its
derivative vector DYDX(1:NVAR) are input at XS. Also input is HTOT,
the total step to be made, and NSTEP, the number of substeps to be
used. The output is returned as YOUT(1:NVAR), which need not be a
distinct array from Y; if it is distinct, however, then Y and DYDX are
returned undamaged.
INTEGER I, N
DOUBLE PRECISION H, H2, SWAP, X, YM(NMAX), YN(NMAX)

C Stepsize this trip
H = HTOT/NSTEP
DO 11 I = 1, NVAR
   YM(I) = Y(I)
11 CONTINUE
   X = XS + H
C Will use YOUT for temporary storage of derivatives
CALL DERVS(X, YN, YOUT)
H2 = 2.0*D0*H
C General step
DO 13 N = 2, NSTEP
   DO 12 I = 1, NVAR
      SWAP = YM(I) + H2*YOUT(I)
      YM(I) = YN(I)
      YN(I) = SWAP
12 CONTINUE
   X = X + H
   CALL DERVS(X, YN, YOUT)
13 CONTINUE
C Last step
DO 14 I = 1, NVAR
YOUT(I)=0.5D0*(YM(I)+YN(I)+H*YOUT(I))

14 CONTINUE
RETURN
END

C (C) COPR. 1986-92 NUMERICAL RECIPES SOFTWARE 45%.
C
C--------------------------------------------------------------------
SUBROUTINE rkqs(y,dydx,n,x,htry,eps,yscal,hdid,hnext,dervs)
INTEGER n,NMAX
DOUBLE PRECISION eps,hdid,hnext,htry,x,dydx(n),y(n),yscal(n)
EXTERNAL dervs
PARAMETER (NMAX=6006)
CU USES dervs,rkck
INTEGER i
DOUBLE PRECISION errmax,h,htemp,xnew,yerr(NMAX),ytemp(NMAX),SAFETY *
*,PGROW,
*PSHRNK,ERRCON
PARAMETER (SAFETY=0.9d0,PGROW=-.2d0,PSHRNK=-.25d0,ERRCON=1.89d-4)
h=htry
1 call rkck(y,dydx,n,x,h,ytemp,yerr,dervs)
errmax=0.d0
do 11 i=1,n
   errmax=max(errmax,abs(yerr(i)/yscal(i)))
11 continue
errmax=errmax/eps
if(errmax.gt.1.d0)then
   htemp=SAFETY*h*(errmax**PSHRNK)
   h=sign(max(abs(htemp),0.1d0*abs(h)),h)
   xnew=x+h
   c if(xnew.eq.x)pause 'stepsize underflow in rkqs'
goto 1
else
   if(errmax.gt.ERRCON)then
      hnext=SAFETY*h*(errmax**PGROW)
   else
      hnext=5.d0*h
   endif
   hdid=h
   x=x+h
do 12 i=1,n
      y(i)=ytemp(i)
12 continue
return
END

C (C) Copr. 1986-92 Numerical Recipes Software 45%.
c
SUBROUTINE rkck(y,dydx,n,x,h,yout,yerr,ders)
INTEGER n,NMAX
DOUBLE PRECISION h,x,dydx(n),y(n),yerr(n),yout(n)
EXTERNAL dervs
PARAMETER (NMAX=6006)
IN USES dervs
INTEGER i
DOUBLE PRECISION ak2(NMAX),ak3(NMAX),ak4(NMAX),ak5(NMAX),ak6(NMAX)
*,
* ytemp(NMAX),A2,A3,A4,A5,A6,B21,B31,B32,B41,B42,B43,B51,B52,B53,
* B54,B61,B62,B63,B64,B65,C1,C3,C4,C6,DC1,DC3,DC4,DC5,DC6
PARAMETER (A2=.2d0,A3=.3d0,A4=.6d0,A5=1.d0,A6=.875d0,B21=.2d0,B31
**=3.d0/40.d0,
* B32=9.d0/40.d0,B41=.3d0,B42=-.9d0,B43=1.2d0,B51=-11.d0/54.d0,B52
**=2.5d0,
* B53=-70.d0/27.d0,B54=35.d0/27.d0,B61=1631.d0/55296.d0,B62=175.d0
*/512.d0,
* B63=575.d0/13824.d0,B64=44275.d0/110592.d0,B65=253.d0/4096.d0,CI
**=37.d0/378.d0,
* C3=250.d0/621.d0,C4=125.d0/594.d0,C6=512.d0/1771.d0,DC1=C1-2825.d0
*/27648.d0,
* DC3=C3-18575.d0/48384.d0,DC4=C4-13525.d0/55296.d0,DC5=-277.d0
*/14336.d0,
* DC6=C6-.25d0)
do 11 i=1,n
  ytemp(i)=y(i)+B21*h*dydx(i)
11 continue
call dervs(x+A2*h,ytemp,ak2)
do 12 i=1,n
  ytemp(i)=y(i)+h*(B31*dydx(i)+B32*ak2(i))
12 continue
call dervs(x+A3*h,ytemp,ak3)
do 13 i=1,n
  ytemp(i)=y(i)+h*(B41*dydx(i)+B42*ak2(i)+B43*ak3(i))
13 continue
call dervs(x+A4*h,ytemp,ak4)
do 14 i=1,n
  ytemp(i)=y(i)+h*(B51*dydx(i)+B52*ak2(i)+B53*ak3(i)+B54*ak4(i))
14 continue
call dervs(x+A5*h,ytemp,ak5)
do 15 i=1,n
  ytemp(i)=y(i)+h*(B61*dydx(i)+B62*ak2(i)+B63*ak3(i)+B64*ak4(i)+
* B65*ak5(i))
15 continue
call dervs(x+A6*h,ytemp,ak6)
do 16 i=1,n
    yout(i)=y(i)+h*(C1*dydx(i)+C3*ak3(i)+C4*ak4(i)+C6*ak6(i))
16  continue

do 17 i=1,n
    yerr(i)=h*(DC1*dydx(i)+DC3*ak3(i)+DC4*ak4(i)+DC5*ak5(i)+DC6*
*ak6(i))
17  continue

return

END

C (C) Copr. 1986-92 Numerical Recipes Software 45%.
## Appendix H

### Eigenvalues JB for several values of $\theta$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>x value</th>
<th>y value</th>
<th>eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.97812</td>
<td>0.0368</td>
<td>2.8191</td>
<td>0.9986 ± 0.0537i</td>
</tr>
<tr>
<td>0.97812</td>
<td>0.7387</td>
<td>-2.7297</td>
<td>0.9985 ± 0.0540i</td>
</tr>
<tr>
<td>0.98</td>
<td>0.0450</td>
<td>2.7622</td>
<td>0.9384 ± 0.3456i</td>
</tr>
<tr>
<td>0.98</td>
<td>0.9028</td>
<td>-2.6246</td>
<td>0.9384 ± 0.3456i</td>
</tr>
<tr>
<td>0.99</td>
<td>0.0553</td>
<td>2.6667</td>
<td>0.7519 ± 0.6593i</td>
</tr>
<tr>
<td>0.99</td>
<td>1.1266</td>
<td>-2.4398</td>
<td>0.7519 ± 0.6592i</td>
</tr>
<tr>
<td>1.00</td>
<td>0.0594</td>
<td>2.6068</td>
<td>0.5858 ± 0.8105i</td>
</tr>
<tr>
<td>1.00</td>
<td>1.2407</td>
<td>-2.3202</td>
<td>0.5859 ± 0.8104i</td>
</tr>
<tr>
<td>1.03</td>
<td>0.0615</td>
<td>2.4782</td>
<td>0.1262 ± 0.9920i</td>
</tr>
<tr>
<td>1.03</td>
<td>1.4335</td>
<td>-2.0561</td>
<td>0.1265 ± 0.9920i</td>
</tr>
<tr>
<td>1.05</td>
<td>0.0584</td>
<td>2.4113</td>
<td>-0.1560 ± 0.9878i</td>
</tr>
<tr>
<td>1.05</td>
<td>1.5107</td>
<td>-1.9147</td>
<td>-0.1556 ± 0.9878i</td>
</tr>
<tr>
<td>1.07</td>
<td>0.0530</td>
<td>2.3528</td>
<td>-0.4204 ± 0.9073i</td>
</tr>
<tr>
<td>1.07</td>
<td>1.5668</td>
<td>-1.7883</td>
<td>-0.4202 ± 0.9074i</td>
</tr>
<tr>
<td>1.10</td>
<td>0.0415</td>
<td>2.2758</td>
<td>-0.7850 ± 0.6195i</td>
</tr>
<tr>
<td>1.10</td>
<td>1.6256</td>
<td>-1.6177</td>
<td>-0.7842 ± 0.6206i</td>
</tr>
<tr>
<td>1.11</td>
<td>0.0369</td>
<td>2.2524</td>
<td>-0.8983 ± 0.4395i</td>
</tr>
<tr>
<td>1.11</td>
<td>1.6402</td>
<td>-1.5648</td>
<td>-0.8974 ± 0.4412i</td>
</tr>
<tr>
<td>1.115</td>
<td>0.0345</td>
<td>2.2410</td>
<td>-0.9534 ± 0.3018i</td>
</tr>
<tr>
<td>1.115</td>
<td>1.6467</td>
<td>-1.5389</td>
<td>-0.9524 ± 0.3049i</td>
</tr>
<tr>
<td>1.12</td>
<td>0.0319</td>
<td>2.2298</td>
<td>-0.8851, -1.1298</td>
</tr>
<tr>
<td>1.12</td>
<td>1.6527</td>
<td>-1.5134</td>
<td>-0.8926, -1.1204</td>
</tr>
<tr>
<td>1.13</td>
<td>0.0267</td>
<td>2.2082</td>
<td>-0.6248, -1.6006</td>
</tr>
<tr>
<td>1.13</td>
<td>1.6634</td>
<td>-1.4636</td>
<td>-0.6250, -1.5999</td>
</tr>
</tbody>
</table>

Table H.1: Eigenvalues of the pair of possible stable periodic points, situated on the intersection of the lines of symmetry $I_{\theta/2} y$ axis and $POL_{\theta/2} y$ axis for several values of $\theta$.  

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Table H.2: Eigenvalues of the periodic points of period 4, situated on the intersection of the lines of symmetry $I_{\theta/2}$ $y$ axis and $P_O P_I P_O I_{\theta/2}$ $y$ axis for several values of $\theta$. 

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>x value</th>
<th>y value</th>
<th>eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.12</td>
<td>0.0392</td>
<td>2.2612</td>
<td>0.9407 ± 0.3392i</td>
</tr>
<tr>
<td>1.12</td>
<td>0.0242</td>
<td>2.1997</td>
<td>0.9407 ± 0.3392i</td>
</tr>
<tr>
<td>1.125</td>
<td>0.0492</td>
<td>2.3116</td>
<td>0.5273 ± 0.8497i</td>
</tr>
<tr>
<td>1.125</td>
<td>0.0051</td>
<td>2.1359</td>
<td>0.5273 ± 0.8497i</td>
</tr>
<tr>
<td>1.13</td>
<td>0.0536</td>
<td>2.3369</td>
<td>0.1412 ± 0.98999i</td>
</tr>
<tr>
<td>1.3</td>
<td>-0.0085</td>
<td>2.0970</td>
<td>0.1412 ± 0.98999i</td>
</tr>
<tr>
<td>1.135</td>
<td>0.0565</td>
<td>2.3549</td>
<td>-0.2178 ± 0.9760i</td>
</tr>
<tr>
<td>1.135</td>
<td>-0.0208</td>
<td>2.0653</td>
<td>-0.2178 ± 0.9760i</td>
</tr>
<tr>
<td>1.135</td>
<td>1.8045</td>
<td>-0.9029</td>
<td>-0.2178 ± 0.9760i</td>
</tr>
<tr>
<td>1.135</td>
<td>1.2556</td>
<td>-2.0513</td>
<td>-0.2178 ± 0.9760i</td>
</tr>
<tr>
<td>1.14</td>
<td>0.0587</td>
<td>2.3689</td>
<td>-0.5488 ± 0.8360i</td>
</tr>
<tr>
<td>1.14</td>
<td>-0.0325</td>
<td>2.0374</td>
<td>-0.5488 ± 0.8360i</td>
</tr>
<tr>
<td>1.14</td>
<td>1.8060</td>
<td>-0.8098</td>
<td>-0.5488 ± 0.8360i</td>
</tr>
<tr>
<td>1.14</td>
<td>1.1780</td>
<td>-2.1178</td>
<td>-0.5488 ± 0.8360i</td>
</tr>
<tr>
<td>1.145</td>
<td>0.0603</td>
<td>2.3803</td>
<td>-0.8507 ± 0.5256i</td>
</tr>
<tr>
<td>1.145</td>
<td>-0.0440</td>
<td>2.0121</td>
<td>-0.8507 ± 0.5256i</td>
</tr>
<tr>
<td>1.145</td>
<td>1.8028</td>
<td>-0.7273</td>
<td>-0.8507 ± 0.5256i</td>
</tr>
<tr>
<td>1.145</td>
<td>1.1059</td>
<td>-2.1731</td>
<td>-0.8507 ± 0.5256i</td>
</tr>
<tr>
<td>1.147</td>
<td>0.0609</td>
<td>2.3843</td>
<td>-0.9634 ± 0.26773i</td>
</tr>
<tr>
<td>1.147</td>
<td>-0.0485</td>
<td>2.0025</td>
<td>-0.9634 ± 0.26773i</td>
</tr>
<tr>
<td>1.147</td>
<td>1.8005</td>
<td>-0.6966</td>
<td>-0.9634 ± 0.26773i</td>
</tr>
<tr>
<td>1.147</td>
<td>1.0784</td>
<td>-2.1927</td>
<td>-0.9634 ± 0.26773i</td>
</tr>
<tr>
<td>1.148</td>
<td>0.0612</td>
<td>2.3862</td>
<td>-1.2093, -0.8269</td>
</tr>
<tr>
<td>1.148</td>
<td>-0.0508</td>
<td>1.9978</td>
<td>-1.2081, -0.8277</td>
</tr>
<tr>
<td>1.15</td>
<td>0.0617</td>
<td>2.3897</td>
<td>-1.6369, -0.61092</td>
</tr>
<tr>
<td>1.15</td>
<td>-0.0553</td>
<td>1.9885</td>
<td>-1.6366, -0.61102</td>
</tr>
</tbody>
</table>
Appendix I

Manual to run the experiments

At first the file parameters.m needs to be run to initialize the parameters. Than the Simulink model, see figure I.1, can be opened, ”exp1.mdl” for the normal protocol and ”exp2.mdl” for the reversed protocol. The model has to be build first:
Ctrl B
Than it needs to be connected to the hardware

>> wt_startup('exp1 -w')

After this is finished choose the option ”Connect to target” in the ”Simulation menu”, at last choose the option ”Start simulation” in the same menu.

Once the simulation is ended, or earlier if you wish, choose the option ”stop simulation”. The simulation can be started again by connecting it to the hardware, choosing ”Connect to target” and ”Start simulation”, as long as nothing is changed in the Simulink model. If you applied some changes in the model, it has to be build again first.

%%%%%%% parameters.m
%%%%%%% This file initializes the parameters for the simulink
%%%%%%% files exp1.mdl and exp2.mdl

clear all;
close all;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
theta=pi*1.05
n=10; % number of periods

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
delayfactor=20;
% If the delayfactor remains the same, the Reynolds number
% remains the same independently of the value of theta

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
t=delayfactor*theta/pi
The following parameters are used in the Simulink models:

**Pulse generator 2**
- Amplitude: $3\theta/t$
- Period (secs): $2t+4$
- Pulse Width (% of period): $t*100/(2t+4)$
- Phase delay (secs): $t+2$

**Pulse generator 1**
- Amplitude: $\theta/t$
- Period (secs): $2t+4$
- Pulse Width (% of period): $t*100/(2t+4)$
- Phase delay (secs): $0$

**Saturation**
- Upper limit: $\theta n*3$
- Lower limit: $-\theta n*3$

**Saturation1**
- Upper limit: $\theta n$
- Lower limit: $-\theta n$

**Configuration parameters**
- Start time: $0.0$
- Stop time: $100000$
- Type: Fixed-step
- Solver: discrete (no continuous states)
- Periodic sample time constraint: Unconstrained
- Fixed-step size (fundamental sample time): $0.001$
- Tasking mode for periodic sample times: auto
Figure 1.1: Simulink model