MASTER

Regimes in shallow dipolar flows driven by time-dependent forcing

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Regimes in shallow dipolar flows driven by time-dependent forcing.

R.B.G. Balk

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Abstract

A time-dependent Lorentz force resulting from the interaction of a time-dependent electric current driven through a shallow electrolytic layer and a constant magnetic field originating from a fixed permanent square magnet, drives quasi-two-dimensional dipolar vortices. In this study we identify and characterize generic flow regimes driven by the Lorentz force. Experiments were performed using dye visualizations and numerical simulations were carried out with COMSOL Multiphysics. Three different flow regimes have been identified and characterized with the Chandrasekhar and the oscillatory Reynolds number. Maximum vorticity and horizontal divergence are used to characterize boundaries between regimes. Transient effects influencing the flow and characterizing sub-regimes have been studied numerically and analytically.
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Chapter 1

Introduction

1.1 Dipolar vortex structures

Vortices come in all sizes, ranging from $\mu m$ to thousands of km. From vortices in a bathtub to hurricanes formed in the Earth’s atmosphere. These (monopolar) vortices are characterized by closed streamlines around a common center [11]. Jupiter’s Great Red Spot existing for more than 300 years is another example of a monopolar vortex. This vortex has dimensions of 24,000-40,000 km by 12,000-14,000 km, which is 2 to 3 times the Earth’s diameter. When two monopolar vortices with oppositely signed vorticity are present simultaneously and close to each other they constitute a dipolar structure. As in monopolar vortices, there is essentially a continuous distribution of vorticity in dipolar vortexes [11]. For example, dipoles can occur at oscillating flows in narrow channels connecting a basin, e.g. a harbor, with the ocean [12].

Quasi-2-dimensional (Q2D) flows are characterized by 3-dimensional flows with one velocity component much smaller than the other two components. The Earth’s atmosphere and the Earth’s oceans are examples where Q2D flows can exist. In these relatively shallow fluid layers horizontal length scales (parallel to the Earth’s surface) can be of thousands of kilometers whereas the thickness of the fluid layer is of the order of
kilometers. Flows can never be perfectly 2-dimensional because a fluid layer will always have a finite thickness. However, in some cases it is possible to neglect one component of the 3-dimensional flow field. These particular cases are classified as Q2D [7] [3].

It is known that dipoles in shallow fluid layers exist in everyday life [12]. But why and how are they created? And which physical parameters play a role in their existence? Numerous studies on dipoles have been conducted in order to answer these questions. For example, a study by Cantwell [2], investigates the events leading towards the creation of a starting vortex from an initial state of rest. A study by Honji et al. on Q2D vortex pairs induced by a Lorentz force due to an electric current interacting with a moving magnetic field [6]. Also Afenasyev et al. studied wakes and vortex streets generated by a translating Lorentz force [1].

Different ways of generating dipoles have been described in literature i.e. by an oscillating flow through channels [12], by solid objects performing oscillatory motions in fluids [10] and by electromagnetic forcing [1] [6] [4]. In the case of electromagnetic forcing, the force driving the fluid motion is the Lorentz force. The Lorentz force results from the interaction of an electric current through the fluid and a magnetic field. The magnetic field can originate from a permanent magnet located under the fluid. The advantage of electromagnetic forcing is that no physical objects have to be placed in the fluid to force the flow. Therefore the flow will not be obstructed by such objects.

1.2 Focus of this study

In this study we focus on identifying and characterizing generic regimes in shallow (Q2D) dipolar flows that are driven by time-dependent forcing.

This study starts with theoretical work that shows how dipolar vortex structures are driven by an electromagnetic force, which is the Lorentz force. Also an approximate
analytical solution for a horizontal velocity component as function of time $t$ and height $z$ is presented.

Experimental work is carried out using a stationary magnetic field interacting with an oscillating electric (AC) current. Numerical work is conducted with COMSOL Multiphysics. To identify and characterize flow regimes experimental and numerical work is conducted for the different (externally applied) physical input parameters which are the frequency (of the applied AC current) and current density. The results are used to establish a so called 'regime-diagram’. In this diagram three different flow regimes are identified and characterized using the maximum vorticity and the horizontal divergence.

After this introduction the report will continue with general theory discussing the governing equations such as the Navier Stokes equation and the vorticity equation. The next chapter will elaborate on the experimental set up and numerical simulations with a summary of all data points used. In this chapter we also compare experimental (Lagrangian) and numerical (Eulerian) results. Chapter 4 describes the theory as it is applied to our experimental and numerical setup. Chapter 5 discusses the obtained results focussing on the identification and characterization of flow regimes. Chapter 6 focusses on the boundaries between flow regimes. The last chapter holds the conclusion.
Chapter 2

General theory

The Navier Stokes equation is the equation of motion for a shallow layer of incompressible Newtonian electrolytic fluid. Taking the curl of the Navier Stokes equation will lead to the vorticity equation expressing vorticity production. The Lorentz force is an external body force capable of inducing vorticity. Circulation \( \Gamma \) and bottom friction time decay are quantities characterizing the induced flow.

2.1 Governing equations

2.1.1 Navier-Stokes equation

Consider a shallow layer of incompressible Newtonian electrolytic fluid. The governing equations describing this fluidic layer are conservation of momentum (the Navier-Stokes equation) and conservation of mass (the continuity equation). Respectively the equations are written as:

\[
\partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} + \frac{\vec{F}}{\rho}, \quad (2.1)
\]

\[
\nabla \cdot \vec{u} = 0. \quad (2.2)
\]
The fluid velocity in 3 dimensions is represented as \( \vec{u} = (u, v, w) \), \( \rho \) is density, \( p \) is pressure, \( \nu \) is kinematic viscosity and \( \vec{F} \) are all external body forces. In (2.1) \( D_t \vec{u} = \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} \) represents the material derivative of the velocity. The material derivative of a fluid element’s velocity resembles the total acceleration of a fluid element. This is the sum of the local acceleration \( \partial_t \vec{u} \) and the convective acceleration \( \vec{u} \cdot \nabla \vec{u} \). All terms on the right hand side of (2.1) are forces acting on the fluid element; \( \nabla p \) is the pressure gradient force, \( \nu \nabla^2 \vec{u} \) the viscous forces and \( \vec{F} \) are external body forces.

### 2.1.2 Vorticity equation

The Navier Stokes equation can be rewritten into the vorticity equation by taking the curl of (2.1)

\[
D_t \vec{\omega} = \partial_t \vec{\omega} + (\vec{u} \cdot \nabla) \vec{\omega} = (\vec{\omega} \cdot \nabla) \vec{u} + \frac{\nabla p \times \nabla \rho}{\rho^2} + \nu \nabla^2 \vec{\omega} + \frac{\nabla \times \vec{F}}{\rho},
\]

where vorticity is expressed as \( \vec{\omega} = \nabla \times \vec{u} \). The first term on the right hand side \( ((\vec{\omega} \cdot \nabla) \vec{u}) \) represents stretching and tilting of vortex lines. Vorticity production due to baroclinic effects and the influence of viscous effects are respectively represented by the second and third term on the right hand side of (2.3). The last term of (2.3) signifies vorticity production due to external forces.

Two-dimensionality of shallow flows is usually attributed to vertical confinement. Vertical velocities are restrained and can be neglected when compared to horizontal velocities [7]. In quasi-2-dimensional (Q2D) flows the vertical length scale is much smaller than the horizontal length scales. It is then assumed that the vertical velocity is also substantially smaller than horizontal velocities (i.e. \( \vec{u} = (u, v, 0) \)). In such Q2D flows the vorticity is directed perpendicular to the (x,y)-plane, therefore \( \vec{\omega} = (0, 0, \omega_z) \). When density \( \rho \) is assumed to be homogeneous there is no baroclinic vorticity production. The vorticity equation (2.3) is then written as:

\[
\partial_t \omega_z + (\vec{u} \cdot \nabla) \omega_z = \nu \nabla^2 \omega_z + \frac{1}{\rho} (\nabla \times \vec{F})_z.
\]
Quasi 2-dimensional flows where velocity has two components ($\vec{u} = (u, v, 0)$), and vorticity has only one component ($\vec{\omega} = (0, 0, \omega_z)$), the stream function $\Psi$ is used to describe the fluid’s motion. This stream function owes its existence to the incompressibility of the fluid ($\nabla \cdot \vec{u} = 0$) and is defined as:

$$\vec{u} = \nabla \times \Psi \vec{e}_z.$$  \hfill (2.5)

The $x-$ and $y-$components of the velocity are now written as:

$$u = \frac{\partial \Psi}{\partial y},$$  \hfill (2.6)

$$v = -\frac{\partial \Psi}{\partial x}.$$  

Substituting (2.5) into $\vec{\omega} = \nabla \times \vec{u}$ will lead to the Poisson equation:

$$\omega_z = -\nabla^2 \Psi.$$  \hfill (2.7)

### 2.1.3 Lorentz force

The external body forces $\vec{F}$ in the Navier Stokes equation (2.1) result from the interaction of an electric current density $\vec{J}$ driven through the fluid layer, and a magnetic field $\vec{B}$ originating from a permanent magnet. The interaction results in the Lorentz force:

$$\vec{F}_L = \vec{J} \times \vec{B}.$$  \hfill (2.8)

### 2.2 Circulation

Circulation $\Gamma$ is an integral quantity characterizing the induced flow and is defined as:

$$\Gamma = \oint_C \vec{u} \cdot d\vec{r},$$ \hfill (2.9)
with \( \vec{u} = \vec{u}(x, y, z) \) the velocity, \( C \) the closed contour over which the integral is evaluated and \( d\vec{r} \) a line element of that contour (see the left image of figure 2.1). A relation between circulation and vorticity is derived applying Stokes’ theorem:

\[
\Gamma = \oint_C \vec{u} \cdot d\vec{r} \equiv \int_A (\nabla \times \vec{u}) \cdot \vec{n} dA = \int \int_A \vec{\omega} \cdot \vec{n} dA, \tag{2.10}
\]

with \( A \) the surface enclosed by contour \( C \) and \( \vec{n} \) the outward normal vector on surface \( A \). Proving that circulation around a closed contour \( C \) is equal to the flux of vorticity through a surface \( A \) spanned by contour \( C \) is expressed in (2.10) (the right image of figure 2.1).

Figure 2.1: Illustration of circulation. The left image illustrates (2.9), and the right image illustrates (2.10).

### 2.3 Bottom friction time decay

In every shallow natural flow situation and in every type of laboratory experiment on Q2D flows, 3-dimensional effects play an additional role [7] [9]. A no-slip boundary condition at the bottom implies a shear in the vertical direction. In shallow Q2D flows where vertical diffusion plays a significant role vertical shear can be modeled by a Poiseuille like profile [9]:

\[
\vec{u}(x, y, z) = \vec{u}^{*}(x, y) \sin \left( \frac{\pi z}{2h} \right), \tag{2.11}
\]
with $h$ the height of the fluid. When substituting the velocity profile (2.11) in the Navier
Stokes equation (2.1), the 3-dimensional diffusion operator can be separated in two decay
components; i.e. lateral diffusion and exponential decay due to vertical diffusion: $\nu \nabla^2_{3D} = \nu \nabla^2_H - \lambda$. The exponential decay factor $\lambda$ due to vertical diffusion is sometimes referred
to as the external friction parameter:

$$\lambda = \frac{\pi^2 \nu}{4h^2},$$

(2.12)

see Appendix B. In SI units the unit of $\lambda$ is $\frac{1}{s}$. Accordingly, the bottom friction time $T_{BF}$ is:

$$T_{BF} = \frac{1}{\lambda} = \frac{4h^2}{\pi^2 \nu}. \tag{2.13}$$

Bottom friction time $T_{BF}$ is the typical time for vertical diffusion of horizontal momentum.

Lateral diffusion and vertical diffusion are characterized by two different Reynolds
numbers, i.e.

$$Re = \frac{L^2 \omega}{\nu},$$

$$Re_\lambda = \frac{\omega}{\lambda} = \omega z T_{BF}, \tag{2.14}$$

with $Re$ and $Re_\lambda$ respectively the ordinary Reynolds number and a Reynolds number
associated with damping due to bottom friction. The typical value of vorticity is presented
by $\omega z$. Typical horizontal length scales in the flow are represented by $L$. Since $h << L$ for
shallow flows, $Re_\lambda << Re$ implying that these flows are dominated by bottom friction.
Chapter 3

Experimental procedures and numerical simulations

This chapter will elaborate on the experimental setup and the numerical tool COMSOL Multiphysics, a summary of the conducted experiments and simulations is given. Also Matlab, quantifying experimental results and an analytical solution of the equation of motion (i.e. $v(z,t)$) is presented.

3.1 Experimental setup

Laboratory experiments have been carried out using a $520 \times 520 \times 25 \text{mm}^3$ tank filled with an electrolytic fluid and a $16 \times 16 \times 5 \text{mm}^3$ permanent square magnet placed at a fixed position underneath the tank. The bottom of the tank separating the electrolytic fluid from the magnet is approximately $1 \text{mm}$ thick (see figures 3.1 and 3.2). The magnetic field strength has been measured using a Gauss/Tesla meter connected to an analog voltage meter. The magnetic field strength is $0.225T$ right above the magnet and $0.210T$ at the bottom of the tank. The electrolytic fluid filling the tank is a salt solution capable of conducting an electric current. The fluid consists of NaCl 12% brix and will fill the
tank to a height of \( h = 4\text{mm} \). At both sides of the tank electrodes are placed covering the entire length of the tank. The electrodes are connected to a power supply driving a time-dependent current (density) through the fluid. The applied current is controlled with the software program 'Kepco'. The current density is directed in the \( x \)-direction, is assumed to be uniform and of the form:

\[
\vec{J}(t) = J_0 \sin(\omega t)\vec{e}_x.
\]  

(3.1)

The amplitude of the applied current density is \( J_0 \) and \( \omega = 2\pi f \) is the angular frequency. The frequency \( f \) is an input variable that is changed for different experiments. See figure 3.1 for a top view of the experimental setup. Figure 3.2 shows a side view of the experimental setup. Both figures are not to scale.

In cartesian coordinates the origin of the experimental setup \( ((x, y, z) = (0, 0, 0)) \) is located at the center of the magnet on the bottom of the tank (see figures 3.1 and 3.2). A high resolution camera of type 'THORLABS DCC1645C', with a lens of 1.4/6mm is mounted at a height of 770mm above the fluid. The camera will record images with a framerate of 10Hz. The maximum frequency of the applied current density used during experiments is about \( f = 1.5\text{Hz} \). A framerate of 10Hz will record about 7 images per characteristic time period \( T_f \). For visualization purposes fluorescent dye is added to the fluid. A light source installed above the tank increases the contrast between the luminescent dye and the black background, which is the bottom of the tank.

During experiments the amplitude \( I_0 \) and frequency \( f \) of the externally applied current \( I(t) = I_0 \sin(\omega t) \) are changed. An overview of the conducted experiments is given in table 3.1.
Figure 3.1: Top view of the experimental setup showing the tank with electrolytic fluid, the square permanent magnet, the electrodes and the power supply. Note: this figure is not to scale.

<table>
<thead>
<tr>
<th>current $I_0$ in [A]</th>
<th>frequency $f$ in [mHz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>10</td>
</tr>
<tr>
<td>0.05</td>
<td>5 - 10 - 15 - 20 - 25 - 50 - 100</td>
</tr>
<tr>
<td>0.10</td>
<td>5 - 10 - 20 - 30 - 50 - 75 - 200</td>
</tr>
<tr>
<td>0.25</td>
<td>10 - 15 - 25 - 30 - 35 - 40 - 50 - 75 - 100 - 300</td>
</tr>
<tr>
<td>0.50</td>
<td>10 - 25 - 50 - 75 - 100 - 125 - 150 - 175 - 200 - 250 - 300 - 400</td>
</tr>
<tr>
<td>1.0</td>
<td>25 - 50 - 75 - 100 - 125 - 135 - 150 - 200 - 300 - 400 - 600 - 800</td>
</tr>
<tr>
<td>2.0</td>
<td>25 - 50 - 75 - 100 - 150 - 300 - 350 - 400 - 450 - 500 - 550 - 600 - 700 - 1000</td>
</tr>
<tr>
<td>4.0</td>
<td>100 - 200 - 300 - 400 - 500 - 600 - 700 - 800 - 900 - 1000 - 1200</td>
</tr>
</tbody>
</table>

Table 3.1: Overview of all different experiments conducted using the experimental setup.
Figure 3.2: Side view of the experimental setup additionally showing the light source, the high resolution camera and indicating the height of the fluid. Note: this figure is not to scale.

3.2 Numerical simulations: COMSOL Multiphysics

COMSOL multiphysics is a finite element simulation software package used in this study for performing numerical simulations on dipole formations. This tool will compute solutions of the Navier Stokes equation (2.1).

The 'main' computational domain (see figure 3.3) measures 0.235m in the $x -$ and $y -$ direction, and 0.004m in the $z -$ direction. The origin of the computational domain ($(x, y, z) = (0, 0, 0)$) is at the center of the focus domain at the bottom of the simulated tank (see figure 3.3). Horizontal length scales of the main computational domain outrange the dipole propagation distances so that interactions with the edges of the domain are avoided. A 'focus' computational domain that measures 0.064m, 0.085m and 0.004m in respectively the $x, y$ and $z -$ direction is located in the center of the main domain. In the focus domain the grid size is smaller (maximum grid size 0.003m) than in the main domain.
(maximum grid size 0.0235 m), see figure 3.3. Due to a smaller grid size results gathered from the focus domain (where dipoles propagate) are more accurate. Computation times can be limited because of the larger grid size in the main domain.

Figure 3.3: Visualization of the differences in grid size for the main and the focus domain. Solid red lines indicate the edges between the main and the focus domain.

At the bottom of the computational domain a no-slip boundary condition is applied, and at the top a stress free boundary condition is used. Vertical edges of the main domain are also set to stress free. In order to reduce the influence of the edges between the focus and the main domain on the evolution of the dipole, these edges are set at continuity.

The magnetic field originating from the permanent square magnet is available in analytical form (see Appendix A). Together with the electric current density \( \vec{J} \) this results in a Lorentz force given by:

\[
\vec{F}_L(x, y, z, t) = -J_0 \sin(\omega t) B_z(x, y, z) \vec{e}_y + J_0 \sin(\omega t) B_y(x, y, z) \vec{e}_z, \quad (3.2)
\]
with $B_z$ and $B_y$ respectively given by (A.18) and (A.12).

Numerical simulations were performed for $t = T_{\text{sim}} = 5 T_f = \frac{5}{f}$, i.e. for 5 characteristic time periods of the applied alternating current. A summary of all the simulations conducted with COMSOL Multiphysics is given in the following table (i.e. table 3.2).

<table>
<thead>
<tr>
<th>current $I_0$ in $[A]$</th>
<th>frequency $f$ in [mHz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.85 ·10^{-2}</td>
<td>10 - 50 - 80 - 100 - 200</td>
</tr>
<tr>
<td>4.39 ·10^{-2}</td>
<td>10</td>
</tr>
<tr>
<td>5.34 ·10^{-2}</td>
<td>10</td>
</tr>
<tr>
<td>5.50 ·10^{-2}</td>
<td>25 - 50 - 80 - 100 - 200 - 500</td>
</tr>
<tr>
<td>8.25 ·10^{-2}</td>
<td>50 - 80 - 100</td>
</tr>
<tr>
<td>0.25</td>
<td>50 - 80 - 150 - 200 - 500 - 1500</td>
</tr>
<tr>
<td>1.10</td>
<td>50 - 100 - 250 - 400 - 500 - 625 - 800 - 1000 - 1250 - 1500</td>
</tr>
<tr>
<td>2.00</td>
<td>25 - 50 - 150 - 300 - 400 - 600 - 700 - 900 - 1000</td>
</tr>
<tr>
<td>2.75</td>
<td>50 - 100 - 250 - 400 - 500 - 625 - 800 - 1000 - 1250 - 1500</td>
</tr>
<tr>
<td>3.85</td>
<td>50 - 100 - 250 - 400 - 500 - 625 - 800 - 1000 - 1250 - 1500</td>
</tr>
</tbody>
</table>

Table 3.2: Summary of all numerical simulations conducted with COMSOL Multiphysics.

### 3.3 Comparing experimental results with numerical results

Even though experimental and numerical results are obtained for the same fluidic flow, they should be interpreted differently.

Dipoles that are created experimentally by the Lorentz force are visible due to advection of the added dye. Studying the fluid’s motion by following an individual fluid parcel as it moves through space and time is the Lagrangian approach. In this approach the position of the fluid parcel $A$ is given by:

$$\vec{r}_A(t) = (x_A(t), y_A(t), z_A(t)).$$  \hspace{1cm} (3.3)

In numerical results, the fluid motion is studied by focussing on a specific location through which fluid flows as time passes. Numerical results give an Eulerian specification.
of the flow field, where the fluid velocity at position \((x, y, z)\) and at time \(t\) is given by:

\[
\vec{u}(x, y, z, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)).
\] (3.4)

The Eulerian and Lagrangian approach are connected through the following relations:

\[
\frac{dx_A}{dt} = u(x_A, y_A, z_A, t);
\]
\[
\frac{dy_A}{dt} = v(x_A, y_A, z_A, t);
\]
\[
\frac{dz_A}{dt} = w(x_A, y_A, z_A, t).
\] (3.5)

### 3.4 Quantification of experimental results using Matlab

In order to quantify experimental results we use Matlab to convert recorded dye visualizations into high contrast black and white images (see figure 3.4) that can then be used to determine the dipole’s frontal velocity and it’s width. As an example, figure 3.5 shows the distance \(|\vec{r}|\) a certain dipole has traveled in about 55 seconds, accordingly the frontal velocity of the dipole can be plotted: \(|\vec{u}| = |\frac{d\vec{r}}{dt}|\). Another example is visualized in figure 3.6 that shows the width of the dipole.

The border where Matlab distinguishes between 'dye' and 'no-dye' is chosen manually and is therefore subjectively set.
Figure 3.4: The right image is recorded during experiments, and the left shows the same image in black and white. Matlab is used to increase contrast in the image of ($I_0 = 0.05A$, and $f = 10mHz$).

Figure 3.5: Plot of a dipole’s traveled distance as a function of time ($I_0 = 2A$, and $f = 100mHz$).
Figure 3.6: Plot of a dipole’s width as a function of time ($I_0 = 4A$, and $f = 200mHz$).

3.5 Approximate analytical solution of the $y$–velocity for a non-transient flow

In order to find an approximate analytical solution of the $y$–velocity for a non-transient flow we assume a Stokes flows. Therefore we leave the pressure gradient ($-\frac{1}{\rho}\nabla p$) and non-linear terms ($\vec{u} \cdot \nabla \vec{u}$) out of the Navier Stokes equation (2.1). This results in:

$$\frac{\partial \vec{u}}{\partial t} = \nu \nabla^2 \vec{u} + \frac{\vec{F}_L}{\rho}.$$  \hfill (3.6)

In a Stokes flow (equation (3.6)) the advective inertial forces ($\vec{u} \cdot \nabla \vec{u}$) are small compared to viscous forces [4] [8], resulting in the linear equation (3.6). The alternating current that is injected in the $x$–direction (3.1), is responsible for oscillations in the Lorentz force. Equation (3.6) is solved in the forcing region (i.e. $(x, y) = (0, 0)$) where the velocity in the $x$–direction is zero ($u = 0$).

Around the center of the permanent square magnet (at $z = 0$) the magnetic field is strongest in the $z$–direction. Here, the $x$– and $y$–components of the magnetic field
can be neglected. Therefore, \( \vec{B}(x, y, z) \approx B_z(z) \hat{e}_z \). The oscillating Lorentz force is now directed strictly in the \( y \)-direction (\( \vec{F}_L = F_y \hat{e}_y \)). The equation of motion (3.6) can then be written as:

\[
\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} - \frac{J_0}{\rho} \sin(\omega t) B_z(z).
\]  

(3.7)

The \( z \)-component of the magnetic field is approximated by:

\[
B_z(z) = B_0 \exp(-\gamma z),
\]  

(3.8)

where \( B_0 = 0.225T \) and \( \gamma = 138 m^{-1} \) is the decay rate of the magnetic field in the \( z \)-direction. The \( z \)-component of a permanent square magnet is derived in Appendix A (A.18). However, due to its simplicity the approximated magnetic field (3.8) is used in the equation of motion (3.7).

Figure 3.7 shows both the \( z \)-component of the magnetic field above the center of the magnet as a function of \( z \) for the analytical solution (A.18), indicated by the blue line, and for the approximated field (3.8), indicated by the red line. The height \( z \) in figure 3.7 is limited to \( z = h = 4 mm \), which is the maximum height of the fluid. Clearly the proposed field is a good approximation of the real magnetic field within the range \( 0 < z < h \).

The boundary conditions originate from a no-slip condition at the bottom and the absence of shear stress at the free surface. Respectively the boundary conditions are:

\[
v(z = 0) = 0,
\]

\[
\left. \frac{dv}{dz} \right|_{z=h} = 0.
\]  

(3.9)

Implementing the boundary conditions (3.9) into the equation of motion (3.7) an analytical steady state solution for the velocity in the \( y \)-direction as function of time \( t \) and
Figure 3.7: $z-$Component of the magnetic field for the analytical solution (A.18) (the blue line) and the approximated field (3.8) (the red line) as a function of height $z$ above the magnet at $(x, y) = (0, 0)$.

Height $z$ is found (see Appendix C):

$$v(z, t) = \Re \left( i \frac{\delta_c B_0 \exp(i \omega t)}{\rho \nu (\gamma^2 - 2i \delta_c^2)} \right)$$

$$\left[ \frac{(\sqrt{2}/\delta_c) \cosh(\sqrt{2}(z-h)/\delta_c) - \gamma \exp(-\gamma h) \sinh(\sqrt{2}z/\delta_c)}{(\sqrt{2}/\delta_c) \cosh(\sqrt{2}h/\delta_c)} - \exp(-\gamma z) \right],$$

where $\delta_c = \sqrt{2\nu/\omega}$ is the so called Stokes’ layer thickness and $\Re$ the real part of equation (3.10).
Chapter 4

Applied theory

This chapter discusses the theory applied to the experimental and numerical experiments carried out for this study. This chapter will clarify why and how the Lorentz force is responsible for creating dipoles. Also time scales related specifically to this research are presented, e.g. the characteristic time period $T_f = \frac{1}{f}$. Dimensionless numbers characterizing flow regimes and arguments for a 2-dimensional approach are given at the end of this chapter.

4.1 Applied Lorentz force

The externally applied current density $\vec{J}(t)$ inside the fluid layer is assumed to be uniform in space and given by $\vec{J}(t) = J(t)\vec{e}_x$, with $J(t)$ the time-dependent function (3.1) . For a rectangular permanent magnet the magnetic field $\vec{B}$ can be expressed analytically (see Appendix A) and is dependent on the position in 3D: $\vec{B} = \vec{B}(x, y, z)$.

Vorticity production due to external body forces corresponds to the last term on the right hand side of equation (2.3). Substituting the Lorentz force (equation (2.8)) into this
term will result in:

$$\nabla \times \vec{F}_L = \nabla \times (\vec{J} \times \vec{B})$$

$$= \vec{J}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{J}) + (\vec{B} \cdot \nabla)\vec{J} - (\vec{J} \cdot \nabla)\vec{B}. \quad (4.1)$$

Since $\nabla \cdot \vec{B} = 0$ and $\vec{J}$ is homogeneous, vorticity production in the $z$–direction $\omega_z$ due to external body forces is written as:

$$(\nabla \times \vec{F}_L)_z = -J_x \frac{\partial B_z}{\partial x}. \quad (4.2)$$

Equation (4.2) shows that a gradient in the $x$–direction of the $z$–component of the magnetic field contributes to the production of vorticity. Figure 4.1 shows the $z$–component of the magnetic field of a permanent square magnet at $(y, z) = (0, 0)$ (left image of figure 4.1), and the gradient of the magnetic field $B_z$ in relation to $x$ in the right image of figure 4.1. The right image of figure 4.1 illustrates the voricity production $\omega_z$ (see equations (2.4) and (4.2)) in the form of peaks as $\frac{dB_z}{dx}$. The two peaks represent respectively positive and negative vorticity. Two patches of equal but oppositely signed vorticity is referred to as a dipole.

![Figure 4.1](image)

**Figure 4.1:** Left: $z$–component of the magnetic field at $y = 0, z = 0$ versus $x/2R$, $2R$ is the horizontal length scale of the magnet (see Appendix A). Right: gradient of the magnetic field $B_z$ to $x$ at $y = 0, z = 0$ versus $x/2R$. 

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4.2 Absolute circulation

Introduced for this study is ’absolute circulation’ \( \Gamma_{abs} \). In section 2.2 circulation is defined by (2.9) and (2.10). However, instead of monopolar vortexes, dipolar structures are present in this study. Two patches of equal but oppositely signed vorticity form dipolar structures. When equation (2.10) is applied to dipoles the circulation will always turn out to be zero. To overcome this problem \( \Gamma_{abs} \) is defined as:

\[
\Gamma_{abs} = \int \int_A \sqrt{\omega_z^2} dA,
\]

with \( A \) the integral area. Calculating the absolute circulation \( \Gamma_{abs} \) over the entire dipole will give the same result as calculating twice the circulation \( \Gamma \) over only half of the dipole (i.e. the individual monopoles).

4.3 Time scales

Time scales pertinent to this study are the advection time \( T_{\text{adv}} \), the characteristic time period \( T_f \) and the forcing time period \( T_F \). These time periods are determined by the experimental and numerical setup.

4.3.1 Advection time

Advection is a transport mechanism of fluidic packages due to the fluid’s bulk motion in a particular direction. In this research advection time \( T_{\text{adv}} \) is the time it takes for a dipole to travel the distance of it’s own length (see figure 4.2). The ratio of the propagation velocity \( V_{\text{dipole}} \) to the width/length of the dipole \( L_{\text{dipole}} \) determines \( T_{\text{adv}} \). Since the dimension of \( T_{\text{adv}} \) is \( s \), \( T_{\text{adv}} \) can be written as \( 1/\omega_z \), with \( \omega_z \) the vorticity of the vortexes constructing the dipole.

\[
T_{\text{adv}} = \frac{L_{\text{dipole}}}{V_{\text{dipole}}} \sim \frac{1}{\omega_z}.
\]
Figure 4.2 shows a dipole at \( t = t_0 \) on the left. This dipole has a typical length/width \( L_{\text{dipole}} \) and a propagation velocity \( V_{\text{dipole}} \). Time \( t = t_1 \) (the right image of figure 4.2) shows the dipole moved over a distance equal to it’s own length \( L_{\text{dipole}} \). The typical time difference \( t_1 - t_0 \) equals the advection time \( T_{\text{adv}} \).

4.3.2 Characteristic time period

The characteristic time period \( T_f \) relates to the angular frequency \( \omega = 2\pi f \) of the applied current density i.e.

\[
T_f = \frac{1}{f} = \frac{2\pi}{\omega}. \tag{4.5}
\]

Because the applied current density is a sine-form \((3.1)\) the direction of the Lorentz force will change sign once within each characteristic time period. A time scale relating to half of the characteristic time period is introduced. This time scale is referred to as
the forcing time period:

\[ T_F = \frac{1}{2} T_f. \]  

During a forcing time period the sign of the Lorentz force will not change. Every induced dipole (see section 4.1) will correspond with a forcing time period. See figure 4.3 for a visualization of the characteristic time period and the forcing time period for \( \omega = 1/s \).

![Figure 4.3: Visualization of the characteristic time period \( T_f \) and the forcing time period \( T_F \) for \( \omega = 1/s \).](image)

Induced flow can be characterized by the dimensionless ratio of the characteristic time period \( T_f \) to the advection time \( T_{adv} \):

\[ \delta_f = \frac{T_f}{T_{adv}}. \]  

### 4.4 Dimensionless numbers characterizing the flow

A systematic study (using the Buckingham theorem) in which the relevant parameters of the problem are varied lead to dimensionless parameters describing the fluidic motion.
One of the relevant parameters is the frequency $f = \frac{\omega}{2\pi}$ which is externally applied. Other parameters are the fluid’s density $\rho$ and kinematic viscosity $\nu$ with dimensions $[kg/m^3]$ and $[m^2/s]$ respectively. The height of the electrolytic fluid $h$, the width of the magnet $2R$ and the height of the magnet $d$ are all expressed in meters $[m]$.

The Lorentz force (2.8) results from the interaction of current density $\vec{J}$ and magnetic field strength $\vec{B}$. Current density $\vec{J}$ in $[A/m^2]$ is expressed as the externally applied current $\vec{I}$ in $[A]$ per unit surface; $\vec{J} = \frac{\vec{I}}{L_y h}$, with $L_y$ the length of the tank. Magnetic field strength $\vec{B}$ is expressed in Tesla $[T] = \left[ \frac{kg}{m^2A} \right]$. Table 4.1 contains all parameters with corresponding symbols and dimensions. The parameters indicated with superscript $\ast$ are kept constant.

The oscillation Reynolds number is found applying the parameters from table 4.1 in the Buckingham theorem;

$$Re_\omega = \frac{\omega h^2}{\nu}.$$  \hspace{1cm} (4.8)

The oscillation Reynolds number $Re_\omega$ quantifies the ratio of the viscous time $h^2/\nu$ to the characteristic oscillation period $T_f = \frac{1}{f} = \frac{2\pi}{\omega}$. Bottom friction time $T_{BF} = \frac{4h^2}{\pi^2 \nu}$ is viscous time $h^2/\nu$ multiplied with $\frac{4}{\pi^2}$. The Chandrasekhar number $Ch$ is a dimensionless number commonly used in the field of electromagnetic forcing [7]. The Chandrasekhar number quantifies the ratio of the Lorentz force to the viscous forces, i.e.

$$Ch = \frac{IBh}{\rho \nu^2}.$$  \hspace{1cm} (4.9)

A high Chandrasekhar number indicates that the Lorentz force dominates the flow, and that viscous forces can be neglected. For low Chandrasekhar numbers the flow is inertia dominated.
<table>
<thead>
<tr>
<th>parameter</th>
<th>symbol</th>
<th>dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>angular frequency</td>
<td>$\omega$</td>
<td>$[\frac{1}{s}]$</td>
</tr>
<tr>
<td>density*</td>
<td>$\rho$</td>
<td>$[\frac{kg}{m^3}]$</td>
</tr>
<tr>
<td>kinematic viscosity*</td>
<td>$\nu$</td>
<td>$[\frac{m^2}{s}]$</td>
</tr>
<tr>
<td>fluid height*</td>
<td>$h$</td>
<td>$[m]$</td>
</tr>
<tr>
<td>magnet width*</td>
<td>$2R$</td>
<td>$[m]$</td>
</tr>
<tr>
<td>magnet height*</td>
<td>$d$</td>
<td>$[m]$</td>
</tr>
<tr>
<td>current</td>
<td>$I$</td>
<td>$[A]$</td>
</tr>
<tr>
<td>magnetic field strength*</td>
<td>$B$</td>
<td>$[\frac{kg}{m^2 A}]$</td>
</tr>
</tbody>
</table>

Table 4.1: All relevant parameters for this study are shown on the left with the corresponding dimensions on the right. Parameters indicated with superscript * are kept constant.
Chapter 5

Experimental and numerical results - characterizing flow regimes

Experimental and numerical results show different flow regimes that are illustrated in a regime diagram. With the used range in Chandrasekhar and oscillatory Reynolds numbers (tables 3.1 and 3.2) three main regimes are distinguished and sub divided into a total of seven sub-regimes.

5.1 Regime diagram

Laboratory experiments and numerical simulations show that for different values of the Chandrasekhar number (4.9) and the oscillatory Reynolds number (4.8), three different flow regimes can be distinguished (see figure 5.1).

The regime diagram also shows all experimental and numerical data points presented in the tables 3.1 and 3.2. In the three identified regimes the flow shows characteristic behavior. The regimes are labeled as regime I, II and III. Regime I is divided into two
Figure 5.1: Regime diagram showing three main regimes divided into seven sub-regimes. Experimental and numerical data points are presented in the regime diagram.

Regime I can be described as a dipolar 'cannon' forcing away dipoles. Other characteristics of regime I are:

- an immediate steady state;
- an increasing maximum vorticity for an increasing oscillatory Reynolds number;
- a relatively large horizontal divergence.

Regime II is illustrated as a potential dipole 'cannon'. But due to high frequencies
(high $Re_\omega$) dipoles do not detach from the forcing region and influence consecutive dipoles. Characteristic for regime II is:

- a steady state after multiple characteristic time periods;
- a decreasing maximum vorticity for an increasing oscillatory Reynolds number;
- a relatively large horizontal divergence.

When multiple forcing time periods have passed, an accumulation of dipolar structures is illustrated in figure 5.2 for regime III. Dipolar structures are created, but due to very low frontal velocities the dipolar structures remain in the forcing region. More generic characteristics of regime III are:

- small maximum vorticities;
- a relatively small horizontal divergence.

A detailed description of the generic different behaviors is given in the sections 5.2, 5.3 and 5.4.
Figure 5.2: Illustration of regimes I, II and III for the course of 2 characteristic time periods.
5.2 Regime I

In regime I startup phenomena have no significant influence on the fluid’s flow. An immediate steady state will set in once the Lorentz force is activated. Regime I can be seen as a dipole ‘cannon’, where dipoles detach from the forcing region (i.e. \((x, y) = (0, 0)\)) and consecutive dipoles are not influenced by former dipoles. A relatively large horizontal divergence is a measure for the ability of dipoles to detach from the forcing region. The maximum vorticity of dipoles will increase for increasing oscillatory Reynolds numbers.

The steady state in regime I can be distorted by an unstable jet. For very low frequencies dipoles will detach from the forcing region, but the Lorentz force is still forcing fluid particles parallel to the propagating dipole in the form of a jet. Small disturbances in the jet can grow in time and result in an unstable state. Whether or not the flow becomes unstable sub-divides regime I. When the flow remains stable it is characterized as regime I.A, if the flow becomes unstable it characterizes as regime I.B.

5.2.1 Regime I.A

Figure 5.3 shows for regime I.A a series of pictures depicting the experimental formation of a dipole under the influence of the Lorentz force. This experimentally created dipole is created at \(Ch = 1.41 \cdot 10^6\) and \(Re_\omega = 30\). When the dipole’s vorticity has reached a maximum, dependent on a combination of the Chandrasekhar number and the oscillatory Reynolds number, it propagates away from the center. When the Lorentz force changes sign the dipole has traveled far enough to not be influenced by the oppositely signed Lorentz force. A consecutive dipole formed during the consecutive forcing time period \(T_F\) propagates in the opposite direction. The consecutive dipole travels an equal distance from the center and is of the same width as the former dipole. Since this event already applies to the first characteristic time period \(T_f\), the flow will almost instantaneously reach a steady state. This indicates that even though experiments start with \(\vec{u}(t = 0) = 0 m/s\), startup phenomena don’t influence the fluid’s motion substantially.

Figure 5.4 shows numerical results in the form of vorticity \(\omega_z\) for regime I.A. This
Figure 5.3: Experimental results characterizing regime LA for $Ch = 1.41 \cdot 10^6$ and $Re_\omega = 30$. The picture series show a dipole that is created in the forcing region (left), and detaches from the forcing region (right) before the Lorentz force changes sign.

A numerical result is obtained with $Ch = 1.41 \cdot 10^6$ and $Re_\omega = 30$. The images in figure 5.4 are at the same time $t$ as the pictures in figure 5.3. The image series in figure 5.4 shows that a dipole in the form of two patches of oppositely signed vorticity is created (left-hand side image). The series of images show the focus computational domain. The right-hand side image shows that the area of vorticity has increased compared to the image on the left. The middle and right-hand side images show that the centers of the vortexes have propagated away from the magnet. Propagation of the center of the vortexes indicates that the dipole is moving away from the forcing region where it was created.
Figure 5.4: Numerical results characterizing regime I.A for $Ch = 1.41 \cdot 10^6$ and $Re_\omega = 30$ at $z = 4mm$. The left image shows the existence of two patches of equal but oppositely signed vorticity. The right-hand side image shows that vorticity has spread and that the centre of the dipole has propagated away from the forcing region.

5.2.2 Regime I.B

Regime I.B is very similar to regime I.A in the sense that for both regimes a dipole is created by the Lorentz force that detaches from the forcing region before the Lorentz force changes sign. Figure 5.5 shows a series of pictures characterizing regime I.B at a Chandrasekhar number of $Ch = 1.41 \cdot 10^6$ and an oscillation Reynolds number of $Re_\omega = 2.5$. The first two pictures show the experimental formation of the dipole and the detachment from the forcing region. The picture on the right shows the unstable state of the dipole at $t = T_F$. Regimes I.A and I.B both classify as a regime I fluid flow, because in both regimes the created dipoles detach from the forcing region. But comparing the right-hand side pictures of figure 5.5 and 5.3 (both taken just before the Lorentz force changes sign) it is evident that the dipole in regime I.B becomes unstable under the influence of a strong unstable jet forming the tail of the dipole.

When comparing numerical results of regime I.A and regime I.B it is evident that regime I.B is unstable; compare the right-hand side images of figure 5.6 and 5.4. In the image on the right in figure 5.6 there is no symmetrical distribution of vorticity, indicating
Figure 5.5: Experimental results characteristic for regime I.B for $Ch = 1.41 \cdot 10^6$ and $Re_\omega = 2.5$. The first two pictures show the formation and detachment of the dipole from the forcing region, resembling to regime I.A. The right-hand side picture shows a dipolar structure disturbed by instabilities.

the unstable character of the flow.
Figure 5.6: Numerical results characterizing regime I.B for $Ch = 1.41 \cdot 10^6$ and $Re_\omega = 2.5$ at $z = 4\,mm$. The left-hand side and middle image show the creation and detachment of the dipole similar to regime I.A. The image on the right shows the disturbances in the vorticity distribution in the flow of regime I.B.

5.3 Regime II

Because the forcing time periods are too short (higher frequencies), dipoles formed in regime II can not detach from the forcing region. Therefore, dipoles will influence consecutive dipoles formed when the Lorentz force changes sign. In regime II we see maximum vorticity $\omega_z$ decreasing for increasing oscillatory Reynolds numbers $Re_\omega$.

Startup phenomena are influencing the fluid’s flow and a steady state is reached after multiple characteristic time periods $T_f$. Regimes II.A and II.B are characterized by the number of characteristic time periods it takes to reach a steady state. When the flow reaches a steady state within 5 characteristic time periods it is classified as regime II.A, otherwise as regime II.B.

5.3.1 Regime II.A

Dipoles in regime II.A gain a smaller maximum vorticity than dipoles created in regime I.A. Figure 5.7 shows the experimentally obtained evolution of a dipole in regime II.A for $Ch = 1.41 \cdot 10^6$ and $Re_\omega = 40$. The first two pictures show the formation of a dipole,
with the picture in the middle showing the dipole at maximum width. The right-hand side picture in figure 5.7 shows the dipole during the consecutive forcing time period influenced by the oppositely signed Lorentz force. Figures 5.8 and 5.9 show numerical results of regime II.A for \( Ch = 1.41 \cdot 10^6 \) and \( Re_\omega = 40 \). The images in figure 5.8 are at the same time \( t \) as the pictures in figure 5.7.

Comparing the second picture in figure 5.7 (at \( t = T_F \)) with the third picture in figure 5.3 (at \( t = T_F \)) the dipole in regime II.A is significantly smaller in size than the dipole of regime I.A.

Dipoles in regime II.A are not capable of propagating away from the forcing region. Therefore they are affected by the Lorentz force when it changes sign (see the right-hand side picture in figure 5.7). The dipole is affected in a sense that fluid particles used to build the consecutive dipole are being extracted from the former dipole. This phenomenon presents itself as the narrowing of the back of the dipole in figure 5.7. The dipole in regime I.A has a more circular shape indicating that it is not influenced by the oppositely signed Lorentz force. Numerical results of this regime (figure 5.8 created at \( Ch = 1.41 \cdot 10^6 \) and \( Re_\omega = 50 \)) show the disability of dipoles to detach from the forcing region. Figure 5.8 does not show the extraction of fluid particles from one dipole to form the other, this phenomenon is only visible in experimental results.

More numerical results of two consecutive dipoles are presented in figure 5.9. The first image shows a dipole formed during the first forcing time period \( (0 < t < T_F) \) and the second image shows the consecutive dipole during the second forcing time period \( (T_F < t < 2T_F) \). The first dipole has a higher vorticity (color intensity) and is larger (larger diameter) than the consecutive dipole. Since the dipoles differ in size, startup phenomena (transient effects) are still influencing the flow. When the Lorentz force is activated the fluid is initially at rest. Therefore the velocity in the \( y \)-direction creating the first dipole is larger compared to the velocity in the opposite direction during the second forcing time period. When the Lorentz force changes sign, the fluid particles still
Figure 5.7: Experimental results characterizing regime II.A for $Ch = 1.41 \cdot 10^6$ and $Re_\omega = 40$. The picture in the middle shows a dipole just when the Lorentz force is about to change sign. This picture shows a smaller dipole when compared to regime I. Dipoles in regime II can not detach from the forcing region. See the right picture showing the dipole during the consecutive forcing time period.

Figure 5.8: Numerical results characteristic for regime II.A for $Ch = 1.41 \cdot 10^6$ and $Re_\omega = 40$ at $z = 4mm$. The images show that the center of the dipole remains in the forcing region (left and middle image). The right image shows the consecutive dipole.
carry momentum that has to be reversed. This results in a smaller $y -$velocity during the consecutive forcing time period. Within 5 characteristic time periods $T_f$ the fluid flow in regime II.A will reach a steady state where startup phenomena are no longer influencing the fluid’s motion.

Figure 5.9: Numerically obtained snapshots of the vorticity $\omega_z$ (for $Ch = 1.41 \cdot 10^6$ and $Re_\omega = 40$ at $z = 4mm$) at $t = T_f$ and $t = 2T_f$. The images illustrate the influence of startup phenomenon (transient effects) in the flow. The left-hand side image shows two oppositely signed vortices forming the dipole. The right-hand side image shows the spread of vorticity of the first dipole during the consecutive forcing time period. The right-hand side image also shows that the consecutive dipole is smaller in size.

5.3.2 Regime II.B

In regime II.B it takes more than 5 characteristic time periods before startup phenomena no longer influence the fluid’s motion. When the Lorentz force is switched on, a dipolar structure becomes visible. But after a short time the Lorentz force changes sign and the dipolar structure disappears. Figure 5.10 shows experimental results of a dipolar structure created with a Chandrasekhar number of $Ch = 1.41 \cdot 10^6$ and an oscillating Reynolds number of $Re_\omega = 70$. Figure 5.11 shows numerically obtained vorticity $\omega_z$ for $Ch = 1.41 \cdot 10^6$ and $Re_\omega = 70$. It shows vorticity at times $t = T_f$ (left), $t = 9T_f$ (middle) and $t = 10T_f$ (right).
Figure 5.10 shows the experimentally obtained evolution of dipolar structures for the course of one characteristic time period. The area lined-in in red shows where the flow changes for 2 consecutive dipolar structures. The series of pictures in figure 5.10 show that a small dipolar structure is created (first three pictures). During the consecutive forcing time period (last two pictures) a similar structure is created in opposite direction. These dipolar structures are relatively small and fluid particles are not capable of making at least one complete rotation. Regime II.B is a mere oscillatory state than a dipolar state. The pictures show an overall 'blob' created of layers of dye from every oscillation. The size of every dipolar structure created at every new forcing time period is smaller than in all other flow regimes.

![Figure 5.10: Experimental results characterizing regime II.B for $Ch = 1.41 \cdot 10^6$ and $Re_\omega = 70$. In the by red lines enclosed area the evolution of a dipolar structure is shown during 1 characteristic time period ($5T_F < t < 7T_F$).](image)

The influence of startup phenomena is visible in figure 5.11. The size (diameter) and strength (color intensity) of the ninth and tenth numerical dipolar structures are much smaller than the first. Comparing the ninth and tenth dipolar structure, a difference in strength and size is still visible but already much smaller. This indicates that the fluid’s motion is reaching a steady state. In a steady state, consecutive dipolar states will be of equal size and have similar vorticities.

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5.4 Regime III

Dipoles created in regime III do not detach from the forcing region and will accumulate. Small horizontal divergence in regime III indicate the disability of dipoles to detach from the forcing region. After multiple characteristic time periods the fluid’s flow is in a mere oscillatory state and only vaguely reminds of the dipole ‘cannon’. Experimental and numerical results presented in this section are presented for different Chandrasekhar and oscillatory Reynolds numbers. However, the differences are very small.

5.4.1 Regime III.A

Characteristic for regime III is the relative low Chandrasekhar number. Experimental results for regime III.A are presented for $Ch = 1.76 \cdot 10^5$ and $Re_\omega = 2.5$ in figure 5.12. This figure shows that dipoles in regime III.A are incapable of detaching from the forcing region. Different dye filaments curling up around each other indicate that the dye makes more than one rotation in the individual cores of the dipole before the Lorentz force changes sign. The right-hand side picture of figure 5.12 shows the consecutive dipole.
This dipole appears to have the same width and comparable maximum vorticity as the former dipole (see figure 5.13). For this reason startup phenomena don’t have a substantial influence in the fluid’s motion.

The incapability of dipoles to detach from the forcing region is what differentiates regime III.A from I.A. The capability of dipoles to propagate away from the forcing region in regime I.A is obtained by using horizontal divergence (see section 6.2).

Figure 5.12: Experimental results characterizing regime III.A for $Ch = 1.76 \cdot 10^5$ and $Re_\omega = 2.5$. The left image shows the first dipole that is created and the right image the consecutive dipole. The repetition of dye filaments curling around each other indicate multiple rotations. The right-hand side image shows that the first dipole does not propagates away from the forcing region, and it’s fluid particles are used to form the consecutive dipole.

![Image](image.png)

Figure 5.12: Experimental results characterizing regime III.A for $Ch = 1.76 \cdot 10^5$ and $Re_\omega = 2.5$. The left image shows the first dipole that is created and the right image the consecutive dipole. The repetition of dye filaments curling around each other indicate multiple rotations. The right-hand side image shows that the first dipole does not propagates away from the forcing region, and it’s fluid particles are used to form the consecutive dipole.

![Image](image.png)

$t = 1.0T_F$

$t = 2.0T_F$

Numerical results characterizing regime III.A are shown in figure 5.13 for a Chandrasekhar number of $Ch = 7.76 \cdot 10^4$ and oscillation Reynolds number of $Re_\omega = 5$. Figure 5.13 shows two pictures of the first dipole (picture on the left and in the middle). The pictures show a relatively small vorticity of only $\omega_z = 8.11/s$. The second picture shows
that the center of the vortexes does not move. Proving that the dipole does not propagate away from center.

Figure 5.13: Numerical results characterizing regime III.A for $Ch = 7.76 \cdot 10^4$ and $Re_\omega = 5$ at $z = 4\text{mm}$. The left-hand side image shows the oppositely signed vortices that form the dipole. The centers of the vortices remain in the forcing region (middle image). The right-hand side image shows the consecutive dipole.

5.4.2 Regime III.B

Regime III.B is a transition between regime III.A and regime III.C. In regime III.B startup phenomena are present, but within 5 characteristics time periods a steady state is reached. Experimental results with $Ch = 1.76 \cdot 10^5$ and $Re_\omega = 10$ are characterized as regime III.B. Figures of these experimental results are shown in figure 5.14, with the picture on the left showing the first dipole created and in the picture on the right showing the consecutive dipole influencing the first. Numerical results, obtained for $Ch = 2.72 \cdot 10^4$ and $Re_\omega = 15$, are shown in figure 5.15.

Compared to the dipole in regime III.A, the dipole in regime III.B is considerably smaller (compare figures 5.12 and 5.14). The right-hand side picture in figure 5.14 shows the consecutive dipole extracting fluid particles from the former dipole to grow in size. This indicates that dipoles in regime III.B can not detach from the forcing region. When
a symmetrical steady state is reached, dipolar structures will be constructed (grow in size) during one forcing time period and then decrease in size during the consecutive forcing time period.

Figure 5.14: Experimental result characterizing regime III.B for $Ch = 1.76 \cdot 10^5$ and $Re_{\omega} = 10$. Similar to figure 5.12 the dipole does not propagate away from the forcing region. Respectively the first and the consecutive dipoles are shown in this picture series.

Comparing the first ($t = T_F$) and second ($t = 2T_F$) image of figure 5.15 a difference in dipolar size is apparent. This indicates that startup phenomena are influencing the motion. Figure 5.15 shows numerical dipoles during the first and second forcing time periods.
Figure 5.15: Numerical results characterizing regime III.B for $Ch = 7.76 \cdot 10^4$ and $Re_\omega = 15$ at $z = 4mm$. The consecutive dipole shown in the right image is weaker than the first dipole which indicates the influence of transient effects.

### 5.4.3 Regime III.C

In regime III.C a steady state is reached after more than 5 characteristic time periods. The steady state is characterized as a mere oscillatory state with only small resemblances to dipolar structures. Experimental results (see figure 5.16) show dye filaments being pushed forward periodically. In figure 5.16 the experimentally obtained evolution of a dipolar structure (during the course of one characteristic time period) is shown within the red lined-in area. In numerical results (figure 5.17) the difference in maximum vorticity between regime II.B (figure 5.11) and III.C is visible (color intensity). Regime III.C is identical to regime II.B, only with smaller dipolar structures and smaller maximum vorticities. Both regimes are in an oscillatory state when startup phenomena no longer influence the fluid’s motion. Regime III.C is established for a lower horizontal divergence.
Figure 5.16: Experimental result characterizing regime III.C for $Ch = 1.76 \cdot 10^5$ and $Re_\omega = 25$. The pictures show the fluid’s flow during one characteristic time period.

Figure 5.17: Numerical results characterizing regime III.C for $Ch = 7.76 \cdot 10^4$ and $Re_\omega = 50$ at $z = 4mm$. The color intensity indicates very weak vortices. The images also show the presence of a mere oscillatory state.
Chapter 6

Experimental and numerical results - Differentiating between flow regimes

Characteristics of all different regimes have been discussed in the former chapter. In the present chapter the border boundaries (differentiation) between different regimes will be quantified. Differentiating between regime I and II is done according to maximum vorticity $\omega_z$. Horizontal divergence is used to differentiate regime III from regimes I and II. The influence of startup phenomena on the fluid’s motion is used to differentiate regime II.B from II.A and III.C from III.B. Bottom friction time decay is a measure to differentiate regime I.B from I.A.
6.1 Differentiating regime I from regime II - maximum vorticity

The capability of a dipole to whether or not detach from the forcing region differentiates regime I from regime II. A physical characteristic that was observed in numerical simulations is the significant decrease in maximum vorticity $\omega_z$ when going from regime I to II. Figure 6.1 shows the maximum vorticity $\omega_z$ computed by COMSOL Multiphysics for varying Chandrasekhar and oscillatory Reynolds numbers. The numerical results are shown for Chandrasekhar numbers of $Ch = 2.72 \cdot 10^6$, $Ch = 1.94 \cdot 10^6$ and $Ch = 7.76 \cdot 10^5$, where the oscillatory Reynolds number was varied from $Re_\omega = 20$ to $Re_\omega = 150$.

The trend for every Chandrasekhar number is that for increasing oscillatory Reynolds numbers the maximum vorticity increases followed by a rapid decrease. The peak value (for every Chandrasekhar number) of maximum vorticity is where the flow enters regime II. When maximum vorticity starts decreasing, dipoles can no longer detach from the forcing region. Increase and subsequent decreasing of vorticity (for increasing oscillatory Reynolds numbers) is most pronounced for $Ch = 2.72 \cdot 10^6$. For $Ch = 1.94 \cdot 10^6$ and $Ch = 7.76 \cdot 10^5$ it is less obvious that the vorticity first increases to a maximum and then decreases for increasing oscillatory Reynolds numbers.
6.2 Regime III differentiated from regime I and II -
horizontal divergence

Differentiating regime III from regime I and II is done using horizontal divergence. Since
the fluid is incompressible the total divergence is zero (i.e. $\nabla \cdot \vec{u} = 0$). Therefore the
horizontal divergence equals the negative gradient of vertical velocity, that is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}. \quad (6.1)$$

If $\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y} \neq 0$ at $z = h$ then $\frac{\partial w}{\partial z} \neq 0$ at $z = h$.

Figure 6.2 shows the numerically obtained horizontal divergence for three different
oscillatory Reynolds numbers (i.e. $Re_\omega = 10, 50$ and 150) and four Chandrasekhar numbers (i.e. $Ch = 2.72 \cdot 10^6$, $Ch = 1.94 \cdot 10^6$, $Ch = 7.76 \cdot 10^5$ and $Ch = 7.76 \cdot 10^4$). Note that the y-axis is in logarithmic scale.

Figure 6.2 shows that the horizontal divergence for $Ch = 7.76 \cdot 10^5$ at $Re_\omega = 10$ is at least a factor ten times higher than the horizontal divergence for $Ch = 7.76 \cdot 10^4$. For increasing oscillatory Reynolds numbers, the horizontal divergence is decreasing for all Chandrasekhar numbers. For $Ch = 7.76 \cdot 10^4$ horizontal divergence decreases fastest. At $Re_\omega = 150$ the horizontal divergence for $Ch = 7.76 \cdot 10^5$ is already a hundred times larger than for $Ch = 7.76 \cdot 10^4$. And compared to $Ch = 2.72 \cdot 10^6$ the horizontal difference is a factor thousand times smaller. The large difference in horizontal divergence is a physical characteristic that is applied to differentiate between hydrodynamical regimes.

The obtained results show high horizontal divergence for the regimes I and II and low horizontal divergence for regime III.

6.3 Instability - differentiating regime I.A from I.B

The instability of the flow in regime I.B (see figure 5.5) is what differentiates regime I.A from I.B. The steady state in regime I (maintained in regime I.A) can be distorted by an unstable jet. For very low frequencies dipoles will detach from the forcing region while the Lorentz force remains forcing fluid particles in the form of a jet parallel to the propagating dipole. Small disturbances in the jet can grow in time and result in an unstable flow. The instability of the flow can be explained by the long forcing time $T_F$. For $Ch = 1.41 \cdot 10^6$ and $Re_\omega = 2.5$ dipoles detach from the forcing region after about $0.25T_F$, but the lorentz force doesn’t change sign for another $0.75T_F$. The bottom friction time (2.13) (with $h = 4mm$ and $\nu = 10^{-6}m^2s$) is $T_{BF} = 6.5s$. This bottom friction time relates to an oscillatory Reynolds number of $Re_\omega = \frac{\nu^3}{T^2} = 15.5$ (see appendix D), differentiating regime I.B from I.A and vice versa.
Figure 6.2: Horizontal divergence versus the oscillatory Reynolds number for Chandrasekhar numbers of \( Ch = 2.72 \cdot 10^6 \), \( Ch = 1.94 \cdot 10^6 \), \( Ch = 7.76 \cdot 10^5 \) and \( Ch = 7.76 \cdot 10^4 \). Note that the horizontal divergence on the y-axis has a logarithmic scale.

6.4 Startup phenomenon - differentiating regime II.A from II.B and III.B from III.C

Startup phenomenon are observed both experimentally and numerically in the form of non-symmetry between consecutive dipoles. The first dipole that is created by the Lorentz force will be largest, and the vortices will be stronger than the ones forming the consecutive dipole. In section 5.3.1 we described the influence of startup phenomenon in numerical results (see figure 5.9). In figure 5.9 it is evident that the consecutive vortices are considerably smaller than the first vortices created. The influence of startup phenomenon is present at the start of the experiment when the flow is in a non-stationary state. After a couple characteristic time periods (depending on the Chandrasekhar number and the oscillatory Reynolds number) the flow will reach a steady state (a non-transient state).
In regime I the influence of startup phenomenon vanishes instantaneously when the Lorentz force changes sign for the first time. However in regime II and regime III.B and III.C startup phenomenon influence the flow for multiple forcing time periods. The differences in vortex strength are caused by a decrease in the maximum y-velocity for consecutive dipoles. Velocity profiles of \( v(z, t) \) in the center of the focus domain (i.e. \( (x, y) = (0, 0) \)) illustrate the y-velocity at different heights \( z \) for time \( t \). In regimes II.A and III.B a non-transient state is reached within 5 characteristic time periods. Velocity profiles of \( v(z, t) \) show when a quasi stationary state is reached and so differentiate between regimes.

Besides y-velocity profiles, absolute circulation (see section 4.2) can be used to identify steady states.

### 6.4.1 Numerical solution of \( v(z, t) \) at \( (x, y) = (0, 0) \)

Figure 6.3 shows the numerical y−velocity profile \( (v(z, t)) \) at \( (x, y) = (0, 0) \) for different time steps of \( \Delta t = T_f/40 \). Distortions in the velocity profiles are caused by a limited resolution in the vertical grid distribution. The \( v(z, t) \) profile for the first (red profile) and the second (blue profile) forcing time periods are shown in figure 6.3. Figure 6.3 shows the transition of the y−velocity in time during the first characteristic time period \( (0 < t < T_f) \). In this figure we see that the maximum y−velocity is about \( 0.034 m/s \) at \( z = 1.6 \cdot 10^{-3} m \) during the first forcing time period.

Studying figure 6.3 shows that even while the Lorentz force has changed sign, the y−velocity is still positive for most of the consecutive forcing time period. The maximum y−velocity reached during the consecutive forcing time period is \( v = 0.0065m/s \) of opposite sign. This is a lot smaller than the \( v = 0.034m/s \) established during the first forcing time period. During the first forcing time period the fluid particles gain a large momentum that has to be compensated during the second forcing time period. Therefore the fluid particles will gain a smaller maximum (oppositely signed) y−velocity during the consecutive forcing time period.
Figure 6.3: Numerical solution of the $v(z,t)$ profile, showing the $y$–velocity during the first (red profile) and the second (blue profile) forcing time periods.

During the first and fifth forcing time period the Lorentz force has the same direction. Figure 6.4 shows the $v(z,t)$ profile for both forcing time periods. The red profile indicates the first forcing time period ($0 < t < T_F$) and the blue profile indicates the fifth forcing time period ($4T_F < t < 5T_F$). The velocity profile changes when time evolves from the first to the fifth forcing time period. This difference in the velocity profiles identifies the startup phenomena. The smaller $y$–velocity during the fifth forcing time period characterizes the smaller vorticities seen in experimental and numerical results (see section 5.3.1). The blue profile shows that startup phenomena still influence the flow during the fifth forcing time period. The absolute value of the $y$–velocity at $t = 4T_F$ is smaller than the absolute value of the $y$–velocity at $t = 5T_F$. 

55
Figure 6.4: This $v(z,t)$ plot compares the $y-$velocity profile of the first forcing time period (red profile) with the fifth forcing time period (blue profile). Lorentz force has the same direction during both periods, but the difference in the $v(z,t)$ profiles is evident.

Figure 6.5: Comparison of the seventh forcing time period (green profile) and ninth forcing time period (blue profile) $y-$velocity profiles. This plot shows that after a couple of forcing time periods a symmetrical steady state will set in.
Figure 6.5 shows the overlap of the $v(z, t)$ profile for the seventh and ninth forcing time periods. The green profile indicates the seventh period ($6T_F < t < 7T_F$) and the blue profile the ninth ($8T_F < t < 9T_F$). The overlap is not yet but almost exact, indicating a near non-transient state. Compared to figure 6.4 it is evident that the profiles have almost reached a symmetrical state. Once the flow is symmetrical in time it is called a steady state. Also looking at the absolute values of the $y-$velocities at the first and last time step $\Delta t$ show that they are almost identical, indicating a steady state.

6.4.2 Analytical solution of $v(z, t)$ at $(x, y) = (0, 0)$

Figures 6.3 to 6.5 show numerically obtained $y-$velocity profiles (i.e. $v(z, t)$) using COMSOL Multiphysics. The analytical solution of a Q2D flow in a steady state (3.10), section 3.5, is also a $y-$velocity profile as a function of fluid height $z$ and time $t$. In comparison, numerical results include the effect of startup phenomena, while the analytical solution of $v(z, t)$ is based on a steady state flow with no influence of startup phenomena.

Figure 6.6 shows the $v(z, t)$ profile for numerical (left image) and analytical (3.10) (right image) solutions for $Ch = 7.76 \cdot 10^4$ and $Re_\omega = 20$. The image is plotted for times $t = T_f/6$ and $t = 2T_f/3$. The figures show a lot of resemblances, e.g. the maximum $y-$velocity and the height at which the $y-$velocity is maximum are similar. This comparison shows that the assumption made by Cuevas [4] for the magnetic field (3.8) gives similar results in the form of $v(z, t)$ at $(x, y) = (0, 0)$ as when the full magnetic field (expressed by (A.18)) is used. Concluding that for heights up to 4mm an exponential decay of magnetic field strength can be used to analytically solve the Navier Stokes equation.

Since the resemblances are apparent for varying Chandrasekhar and oscillatory Reynolds numbers, the analytical solution can be used to study differences in the $y-$velocity profile in steady states (non-transient states). This solution can be used to differentiate between flow regimes by comparing them with numerical results. Because the computational time to solve the analytical solution is less than a second and running an entire numerical simulation (reaching a non-transient state) can take days, using the analytical solution is
Figure 6.6: Plot of the numerical and analytical solution of the $v(z,t)$ profile at $(x,y) = (0,0)$ in the left-hand side and right-hand side image respectively. The plots were created for times $t = T_f/6$ and $t = 2T_f/3$.

preferred over running numerical simulations.

6.4.3 Absolute circulation: $\Gamma_{abs}$

Another possibility to characterize a steady state is to study the absolute circulation (4.3) in a predefined area. In this part of the study the integration area is chosen to be the focus domain of the total computational domain (see figure 3.3).

Figures 6.7, 6.8 and 6.9 show the absolute circulation $\Gamma_{abs}$ versus 5 characteristic time periods. The figures are created for $Ch = 2.72 \cdot 10^6$ and respective oscillatory Reynolds numbers $Re_\omega = 25$, $Re_\omega = 80$ and $Re_\omega = 150$.

In figure 6.7 the absolute circulation is oscillating around an averaged value of about $\Gamma_{abs} = 0.019 m^2/s$. This shows that within the first time period the flow already reaches a steady state, classifying it as being in regime I.

Figure 6.8 shows that after the second characteristic time period the absolute circulation starts oscillating around an averaged value of about $\Gamma_{abs} = 0.017 m^2/s$. The absolute circulation at $Re_\omega = 80$ is slightly lower than the absolute circulation for $Re_\omega = 25$. This is expected after studying the maximum vorticities in regime I and regime II. Also the
fluctuations in the absolute circulation are substantially smaller. This is caused by the ability of dipoles in regime I to detach from the forcing region and propagate outside the focus domain (integral area). Once the dipole has exit the focus domain, the dipole center (where vorticity is maximum) will not be part of the integral area and a steep decrease in absolute circulation is present. In regime II dipoles don’t detach from the forcing region and will be part of the integral area at all times.

The last figure of this series (figure 6.9) shows (similar as in figure 6.8) a trend where the absolute circulation is increasing to (presumably) a steady state. For $Re_\omega = 80$ the steady state is reached after the second characteristic time period. However at $Re_\omega = 150$ the steady state is not reached within five characteristic time periods. This classifies the fluid’s flow as being in regime II.B.

![Figure 6.7](image)

Figure 6.7: Absolute circulation at $Re_\omega = 25$ for 5 characteristic time periods. Large fluctuations in the absolute circulation suggest that the dipole propagates outside the integral area; characteristic for regime I.
Figure 6.8: Absolute circulation at $Re_\omega = 80$ for 5 characteristic time periods. A steady state is reached just after the second characteristic time period. Small fluctuations suggest the dipole doesn’t detach from the forcing region. The steady state reached at $t/T_f = 2$ in combination with small fluctuations characterizes regime II.A.
Figure 6.9: Absolute circulation at $Re_\omega = 150$ for 5 characteristic time periods. A steady state is not reached within these 5 characteristic time periods. The figure shows that the flow is still influenced by startup phenomena. This figure characterizes regime II.B.
6.5 Timescale

The dimensionless timescale \( \delta_t = \frac{T_f}{T_{adv}} = 2\pi \frac{\omega_z}{\omega} = \frac{2\pi L_{dipole}}{v_{dipole}} \) is the ratio of the characteristic time period to the advection time. It contains both frequency \( f \) (explicitly) and Lorentz force \( F_L \) (implicitly in \( \omega_z \)). Figure 6.10 shows the dimensionless time for varying Chandrasekhar and oscillatory Reynolds numbers. Experimental results quantified with MATLAB (see section 3.4 and 4.3.1) are used for this figure.

Note that both the \( x \)- and \( y \)-axis are in logarithmic scale. The relation between \( \delta_t \) and \( Re_\omega \) is given by:

\[
\log(\delta_t) = \log(A - B \cdot Re_\omega),
\]

with \( A \) and \( B \) constants defined by the experimental results. From figure 6.10 we deduce that \( A = 5800 \) and \( B = 297 \). Conclusive for this plot: the relation between \( \delta_t \) and \( Re_\omega \) is independent of the Chandraskher number, vorticity \( \omega_z \) can be measured (externally changed) with frequency squared \( (f^2) \).
Figure 6.10: Ratio of the characteristic time period to the advection time (\( \delta_t = \frac{T_f}{T_{adv}} \)) versus the oscillatory Reynolds number \( Re_\omega \). A logarithmic behavior between the dimensionless time and the oscillatory Reynolds number is evident.
Chapter 7

Conclusion

Experimental and numerical simulations show that a flow in a shallow fluid layer that is driven by a periodic-in-time Lorentz force can be classified in different regimes depending on the values of the Chandrasekhar and the oscillatory Reynolds number. Three different regimes with their own characteristics are identified and subdivided into a total of seven sub-regimes.

Characteristic for regime I is the ability of formed dipoles to detach from the forcing region. Dipolar structures in regimes II and III remain in the forcing region. The inability of dipoles in regime II to detach from the forcing region is caused by the high frequency of the periodic Lorentz force, and for dipoles in regime III it results from a weak Lorentz force.

Horizontal divergence acts as border boundary between regimes I and II (strong Lorentz force) and regime III (weak Lorentz force). For relatively high Chandrasekhar numbers we see an increase and subsequent decrease of maximum vorticity $\omega_z$ for increasing oscillatory Reynolds numbers. The peak value of maximum vorticity is where the flow enters regime II.

Transient effects influence the flow, however steady states will set in. The time after which a steady state sets in is characteristic for different (sub-)regimes. With a $y-$
velocity profile dependent on time $t$ and height $z$ in the forcing region we were able to identify steady states.

A remarkable result showing that there is a relation between vorticity and frequency independent of the Lorentz force is obtained. A mere thorough study to this result is suggested before new statements are projected.
Bibliography


Appendix A

Analytical solution for the magnetic field of a square permanent magnet

Consider a square permanent magnet with horizontal length scales $2R$ and thickness $d$ (see figure A.1). The magnetic field is denoted by $\vec{B}$ and the magnetization (magnetic dipole moment per unit volume) is $\vec{M}$. The relevant Maxwell equations are:

\[
\nabla \cdot \vec{B} = 0, \quad \text{(A.1)}
\]

\[
\nabla \times \vec{H} = \vec{J}_f = 0 \rightarrow \vec{H} = -\nabla \Psi_M, \quad \text{(A.2)}
\]

\[
\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}, \quad \text{(A.3)}
\]

with $\vec{H}$ the auxiliary field [5], $\vec{J}_f$ is the free current and $\Psi_M$ is the magnetic potential.

When the magnet has a homogeneous magnetization in the z-direction (for $|x| < R, |y| < R$ and $-d < z < 0$) the magnetization is written as $\vec{M} = M_0 \hat{e}_z$. By taking the divergence
of (A.3) and substituting (A.2) a Poisson equation for $\Psi_M$ is obtained:

$$\nabla^2 \Psi_M = \nabla \cdot \vec{M}. \quad (A.4)$$

Next, we introduce dimensionless quantities (indicated by a tilde) according to table A.1. However, in what follows the tilde is left out. Magnetization $\vec{M}$ is now written as

<table>
<thead>
<tr>
<th>dimensionless quantities</th>
<th>$\tilde{x} = \frac{x}{R}$</th>
<th>$\tilde{y} = \frac{y}{R}$</th>
<th>$\tilde{z} = \frac{z}{R}$</th>
<th>$\tilde{\Psi}_M = \frac{\Psi}{RM_0}$</th>
<th>$\tilde{M} = \frac{M}{M_0}$</th>
<th>$\tilde{\vec{B}} = \frac{\vec{B}}{\mu_0 M_0}$</th>
<th>$\Delta z = \frac{d}{R}$</th>
</tr>
</thead>
</table>

Table A.1: This table shows how quantities are made dimensionless.
\( \vec{M} = \vec{e}_z \) for \( |x| < 1, |y| < 1 \) and \(-\Delta < z < 0\). Then, for \( \nabla \cdot \vec{M} \) we obtain:

\[
\nabla \cdot \vec{M}_z = \begin{cases} 
\delta(z + \Delta) - \delta(z) & \text{for } |x|, |y| < 1, \\
0 & \text{otherwise},
\end{cases}
\] (A.5)

where \( \delta(z) \) represents the dirac delta function. The solution of the Poisson equation is:

\[
\Psi_M(x, y, z) = -\frac{1}{4\pi} \int \int \frac{\nabla \cdot \vec{M} \, dx' \, dy'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\
= \frac{-1}{4\pi} \int_{-1}^{1} dx' \int_{-1}^{1} dy' [[(x-x')^2 + (y-y')^2 + (z+\Delta z)^2]^{-\frac{1}{2}}} \\
-[[(x-x')^2 + (y-y')^2 + z^2]^{-\frac{1}{2}}].
\] (A.6)

Using Green’s function in 3-dimensions the solution for the magnetic potential is written as \( \Psi_M = G(x, y, z) - G(x, y, z + \Delta z) \) with

\[
G(x, y, z) = -\frac{1}{4\pi} \int_{x-1}^{x+1} du \int_{y-1}^{y+1} dv \frac{1}{\sqrt{u^2 + v^2 + z^2}}.
\] (A.7)

When integrating (A.7) over \( v \), the solution becomes:

\[
G(x, y, z) = \frac{1}{4\pi} \int_{x-1}^{x+1} du \ln(v + \sqrt{u^2 + v^2 + z^2})|_{y-1}^{y+1} \\
= \frac{1}{4\pi} \int_{x-1}^{x+1} du \ln\left(\frac{y+1+\sqrt{u^2+(y+1)^2+z^2}}{y-1+\sqrt{u^2+(y-1)^2+z^2}}\right).
\] (A.8)

Instead of integrating over \( v \), it would have been possible to integrate over \( u \) instead. The outcome would be the same with \( x \) and \( y \) interchanged. With (A.8) it is now possible to determine the magnetic field in the \( x - \) and \( y - \) direction, i.e.

\[
B_x = -\frac{\partial \Psi_M}{\partial x}, \\
B_y = -\frac{\partial \Psi_M}{\partial y}.
\] (A.9)

A function \( P(x, y, z) \) is introduced:

\[
P(x, y, z) = \frac{1}{4\pi} \ln(y + \sqrt{x^2 + y^2 + z^2}).
\] (A.10)
The solutions for $B_x$ and $B_y$ are respectively written as:

$$B_x = P(x + 1, y + 1, z + \Delta_z) - P(x + 1, y - 1, z + \Delta_z)$$
$$+P(x - 1, y - 1, z + \Delta_z) - P(x - 1, y + 1, z + \Delta_z)$$
$$-P(x + 1, y + 1, z) + P(x + 1, y - 1, z)$$
$$-P(x - 1, y - 1, z) + P(x - 1, y + 1, z), \quad (A.11)$$

$$B_y = P(y + 1, x + 1, z + \Delta_z) - P(y + 1, x - 1, z + \Delta_z)$$
$$+P(y - 1, x - 1, z + \Delta_z) - P(y - 1, x + 1, z + \Delta_z)$$
$$-P(y + 1, x + 1, z) + P(y + 1, x - 1, z)$$
$$-P(y - 1, x - 1, z) + P(y - 1, x + 1, z). \quad (A.12)$$

To find the magnetic field in the $z$-direction the derivative of $G(x, y, z)$ with respect to $z$ is determined: $B_z = \partial_z G(x, y, z)$,

$$\frac{\partial G}{\partial z} = \frac{1}{4\pi} \int_{x-1}^{x+1} du \left( \frac{1}{y+1+\sqrt{u^2+(y+1)^2+z^2}} \right) \frac{z}{\sqrt{u^2+(y+1)^2+z^2}}$$
$$- \frac{1}{y-1+\sqrt{u^2+(y-1)^2+z^2}} \frac{z}{\sqrt{u^2+(y-1)^2+z^2}}. \quad (A.13)$$

The integral that needs to be calculated is:

$$\int \frac{du}{\eta \sqrt{u^2 + \eta^2 + z^2} + u^2 + \eta^2 + z^2}. \quad (A.14)$$

Introducing:

$$u = \sqrt{\eta^2 + z^2} \sinh(t), \quad (A.15)$$
$$du = \sqrt{\eta^2 + z^2} \cosh(t) dt.$$

The integral then becomes:

$$\int \frac{dt}{\eta + \sqrt{\eta^2 + z^2} \cosh(t)} = \frac{2}{z} \arctan\left( \frac{\sqrt{\eta^2 + z^2}}{\eta \sqrt{u^2 + \eta^2 + z^2} - \eta \sqrt{u^2 + \eta^2 + z^2} - \sqrt{u^2 + \eta^2}} \right). \quad (A.16)$$
A function $Q(x, y, z)$ is introduced:

$$Q(x, y, z) = \frac{1}{2\pi} \arctan \left( \frac{\sqrt{y^2 + z^2} - y \sqrt{x^2 + y^2 + z^2} - \sqrt{y^2 + z^2}}{x} \right).$$  \hspace{1cm} (A.17)

The analytical solution of the magnetic field in the $z$-direction is now:

$$B_z = Q(x + 1, y + 1, z + \Delta z) - Q(x - 1, y + 1, z + \Delta z) - Q(x + 1, y - 1, z + \Delta z) + Q(x - 1, y - 1, z + \Delta z) - Q(x + 1, y + 1, z) + Q(x - 1, y + 1, z) + Q(x + 1, y - 1, z) - Q(x - 1, y - 1, z).$$ \hspace{1cm} (A.18)
Appendix B

Bottom friction parameter

In Q2D flows where bottom friction plays a dominant role, the velocity profile is described as: [9]

\[ \vec{u}(x, y, z) = \vec{u}^* (x, y) \sin \frac{\pi z}{2h}, \]  

(B.1)

with \( h \) the height of the fluid layer. When this velocity profile is substituted in the Navier Stokes equation (2.1) the diffusion term on the right hand side can be written as:

\[ \nu \nabla^2 3D \vec{u}(x, y, z) = \nu \nabla^2_H \vec{u}(x, y, z) + \nu \frac{\partial^2}{\partial z^2} \vec{u}(x, y, z) \]

(B.2)

\[ = \nu \nabla^2_H \vec{u}(x, y, z) + \nu \vec{u}^* (x, y) \frac{\partial^2}{\partial z^2} \sin \frac{\pi z}{2h} \]

with \( \nu \nabla^2_H \) the horizontal diffusion parameter. The 3-dimensional diffusion operator is now written in terms of a 2-dimensional diffusion operator (horizontal) and a vertical diffusion operator \( \lambda \):

\[ \nu \nabla^2_{3D} = \nu \nabla^2_H - \lambda, \]  

(B.3)

with \( \lambda \) the external friction parameter written as:

\[ \lambda = \frac{\pi^2 \nu}{4h^2}. \]  

(B.4)
Appendix C

Approximate analytical solution of the $y-$velocity for a non-transient Q2D flow

To find an analytical solution for the $y-$velocity as a function of time and height (i.e. $v(z, t)$), the Navier-Stokes equation in the form of (3.7) is used with magnetic field strength (3.8). The solution also has to satisfy a no-slip condition at the bottom and a stress-free condition at the free surface. These boundary conditions are written as:

$$v(z = 0) = 0,$$
$$\frac{dv}{dz}|_{z=h} = 0.$$\tag{C.1}

The equation of motion that is solved is:

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} - \frac{J_0 B_0}{\rho} \sin (\omega t) \exp (-\gamma z),$$\tag{C.2}
with $\gamma$ the decay rate of the magnetic field strength in the $z-$direction. The second term on the right hand side contains a '$\sin(\omega t)$' term, therefore it is assumed that the steady-state solution is in the form of:

$$v(z, t) = \Re[\tilde{v}(z) \exp(i\omega t)]. \quad (C.3)$$

Since

$$\frac{\partial v}{\partial t} = \Re[i\omega \tilde{v} \exp(i\omega t)],$$
$$\nu \frac{\partial^2 v}{\partial z^2} = \Re[\nu \frac{\partial^2}{\partial z^2} \tilde{v} \exp(i\omega t)],$$
$$\sin(\omega t) = \Re[-i \exp(i\omega t)], \quad (C.4)$$

the equation of motion (C.2) is written as:

$$i\omega \tilde{v} - \nu \frac{\partial^2 \tilde{v}}{\partial z^2} = i \frac{J_0 B_0}{\rho} \exp(-\gamma z). \quad (C.5)$$

The $\exp(i\omega t)$ term has been factored out from the equation of motion. Equation (C.5) is an inhomogeneous ordinary differential equation, and its solution is given by $\tilde{v}(z) = \tilde{v}_h(z) + \tilde{v}_p(z)$, where $\tilde{v}_h(z)$ is the homogenous solution and $\tilde{v}_p(z)$ the particular solution.

In order to find the homogeneous solution the equation of motion (C.5) is written as:

$$i\omega \tilde{v}_h - \nu \frac{\partial^2 \tilde{v}_h}{\partial z^2} = 0. \quad (C.6)$$

Using standard calculus, the homogeneous solution is written as:

$$\tilde{v}_h(z) = \alpha \exp\left(\sqrt{\frac{i\omega}{\nu}} z\right) + \beta \exp\left(-\sqrt{\frac{i\omega}{\nu}} z\right). \quad (C.7)$$

In order to find the particular solution of (C.5), a solution in the following form is proposed:

$$\tilde{v}_p(z) = V_0 \exp(-\gamma z). \quad (C.8)$$
In equation (C.8) \( V_0 \) is a constant that can be found by substituting the proposed particular solution (C.8) into the equation of motion (C.5). The particular solution is now written as:

\[
\tilde{v}_p(z) = -i \frac{J_0 B_0}{\rho \nu (\gamma^2 - i \omega/\nu)} \exp(-\gamma z).
\] (C.9)

The total solution for \( \tilde{v}(z) = \tilde{v}_h(z) + \tilde{v}_p(z) \) is now written as:

\[
\tilde{v}(z) = \alpha \exp\left(\sqrt{\frac{i \omega}{\nu}} z\right) + \beta \exp\left(-\sqrt{\frac{i \omega}{\nu}} z\right) - i \frac{J_0 B_0}{\rho \nu (\gamma^2 - i \omega/\nu)} \exp(-\gamma z).
\] (C.10)

The solution contains two unknowns (namely \( \alpha \) and \( \beta \)) that can both be found by applying the boundary conditions (C.1). In order to apply the boundary conditions the derivative of (C.10) to \( z \) is required:

\[
\frac{d\tilde{v}}{dz} = \alpha \sqrt{\frac{i \omega}{\nu}} \exp\left(\sqrt{\frac{i \omega}{\nu}} z\right) - \beta \sqrt{\frac{i \omega}{\nu}} \exp\left(-\sqrt{\frac{i \omega}{\nu}} z\right) - i \frac{J_0 B_0 \gamma}{\rho \nu (\gamma^2 - i \omega/\nu)} \exp(-\gamma z).
\] (C.11)

The two following equations will be found when applying the boundary conditions:

\[
\alpha + \beta = i \frac{J_0 B_0}{\rho \nu (\gamma^2 - i \omega/\nu)} = \chi
\]

\[
\alpha \exp\left(\sqrt{\frac{i \omega}{\nu}} h\right) + \beta \exp\left(-\sqrt{\frac{i \omega}{\nu}} h\right) = -i \frac{J_0 B_0 \gamma}{\rho \nu (\gamma^2 - i \omega/\nu)} \exp(-\gamma h) = \chi \gamma \exp(-\gamma h)
\] (C.12)

Equation (C.12) contains two unknowns and two equations. Therefore (C.12) is solvable for \( \alpha \) and \( \beta \), i.e.

\[
\alpha = \frac{\sqrt{(i \omega/\nu) \exp\left(-\sqrt{(i \omega/\nu)} h - \gamma \exp(-\gamma h)\right)}}{2 \sqrt{(i \omega/\nu) \cosh\left(\sqrt{(i \omega/\nu)} h\right)}},
\]

\[
\beta = \frac{\sqrt{(i \omega/\nu) \exp\left(\sqrt{(i \omega/\nu) h} + \gamma \exp(-\gamma h)\right)}}{2 \sqrt{(i \omega/\nu) \cosh\left(\sqrt{(i \omega/\nu)} h\right)}} \chi.
\] (C.13)

Substituting (C.13) into (C.10) and using some goniometric such as: \( \cosh(x) = \frac{1}{2}(e^x + e^{-x}) \) and \( \sinh(x) = \frac{1}{2}(e^x - e^{-x}) \), the following solution for \( \tilde{v}(z) \) is found:

\[
\tilde{v}(z) = \frac{i \frac{J_0 B_0}{\rho \nu (\gamma^2 - i \omega/\nu)} \sqrt{\frac{i \omega}{\nu} \cosh\left(\sqrt{\frac{i \omega}{\nu}} (z-h)\right) - \gamma \exp(-\gamma h \cosh\left(\sqrt{\frac{i \omega}{\nu}} h\right))}}{\sqrt{\frac{i \omega}{\nu} \cosh\left(\sqrt{\frac{i \omega}{\nu}} h\right)}} - \exp(-\gamma z)].
\] (C.14)
Now by substituting (C.14) into (C.3) and using Stokes’s layer thickness $\delta_c = \sqrt{2\nu/\omega}$, the analytical solution for the velocity in the y-direction at $(x, y) = (0, 0)$ is written as:

$$v(z, t) = \Re\left\{i J_0 B_0 \exp\left(\frac{\omega t}{\rho}\right) \rho \mu (\gamma^2 - 2i\delta_c) \right.$$ 

$$\left[ (\sqrt{2i}/\delta_c) \cosh (\sqrt{2i}(z-h)/\delta_c) - \gamma \exp (-\gamma h) \sinh (\sqrt{2i}z/\delta_c) \right.$$ 

$$\left. (\sqrt{2i}/\delta_c) \cosh (\sqrt{2i}h/\delta_c) - \exp (-\gamma z) \right\}.$$

(C.15)
Appendix D

Relation between bottom friction time $T_{BF}$ and the oscillatory Reynolds number $Re_\omega$

Bottom friction time $T_{BF}$ is defined as:

$$T_{BF} = \frac{1}{\lambda} = \frac{4h^2}{\pi^2 \nu}. \quad (D.1)$$

An assumed bottom friction frequency $f_{BF}$ relates to bottom friction time as:

$$f_{BF} = \frac{1}{T_{BF}} = \frac{\pi^2 \nu}{4h^2}, \quad (D.2)$$

with an angular frequency of

$$\omega_{BF} = 2\pi f_{BF}. \quad (D.3)$$
This angular frequency can be substituted into the oscillatory Reynolds number (4.8), resulting in:

\[ Re_\omega = \frac{\omega_B h^2}{\nu} = \frac{\pi^3}{2}. \]  

(D.4)