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Translating magnets in electromagnetically forced fluid flows

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Translating magnets in electromagnetically forced fluid flows

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Abstract

Electromagnetic forcing is a non-intrusive method of driving fluid flows using a Lorentz force. This force is induced by an electric current running through a thin layer of fluid and a magnetic field originating from a permanent magnet below the fluid. The force is localized above the magnet and drives a jet in the fluid, which forms a vortex dipole in quasi-two-dimensional flows. It has been useful in the studies of two-dimensional turbulence and understanding vortices in geophysical fluid flows. The goal of the current research is to study the effect of adding a time dependence to the forcing by translating the magnet. Four different cases have been studied, where the magnet either translates parallel or perpendicular to the dominant component of the Lorentz force and where the magnet moves either linear from one point to another or harmonically oscillates. A theoretical model is derived to describe the behavior of the linearized fluid flow. Laboratory experiments were performed using dye visualization. The theoretically predicted $\sim t^{1/2}$ dependence of the translation of a vortex dipole was confirmed. Different behaviors of the fluid flow were found for each case and three or four different regimes in the parameter space could be identified. Numerical simulations were in close agreement with the laboratory experiments. However a few questions remain open specific to each case.
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1. Introduction

1.1 Vortices in quasi-two-dimensional flows

Vortices are a common phenomenon in nature. They occur on a wide range of scales. From microscopic scales such as turbulent flow to macroscopic scales such as spiraling galaxies. Vortices may occur in quasi-two-dimensional (Q2D) flows. Q2D flows refer to three-dimensional flows where one component of the flow velocity is much smaller than the other two components. These typically occur in thin layers of fluid. A prime example is the earth’s atmosphere or oceans, for which the horizontal velocities (parallel to the earth’s plane) are much larger than the vertical velocities. Though their two-dimensional (2D) nature is also promoted by the earth’s background rotation and by density stratification.

One method of generation of vortices is by electromagnetic forcing, a method well known in the field of magnetohydrodynamics. Electromagnetic forcing refers to the method of using a Lorentz force to drive flows in fluids. This Lorentz force is formed by an electric current through a fluid combined with a magnetic field, which may originate from permanent magnets. When using a permanent magnet the largest component of Lorentz force’s strength is localized at the position of magnet. When this local (and stationary) Lorentz force is present in a Q2D fluid, a jet is produced by the force (see Figure 1). Viscous starting jets in 2D flows are known to form vortex dipole structures in the fluid, whereas in three-dimensional flows they form vortex rings (Cantwell [4]). The method of electromagnetic forcing sets itself apart from other methods due to its non-intrusive nature as it enables to drive flows and generate vorticity without inserting physical bodies into the fluid itself or by pumping extra fluid into it.

![Figure 1 A local force $F$ in a fluid produces a starting jet. These jets in 2D flows are known to form vortex dipole structures. A vortex dipole is a flow structure with both positive and negative vorticity and has zero angular momentum and non-zero linear momentum.](image)

For the last 20 years electromagnetic forcing in Q2D flows has been used in many experimental studies of 2D turbulence. One of the earliest experiments were done by Tabeling et al. [16], in which they confirmed power law exponents in decaying turbulence experimentally. This was done by placing an array of permanent magnets below a thin layer of fluid carrying an electric current density. More recently Akkermans et al. [3] performed a thorough investigation of the three-dimensionality of a vortex dipole produced by electromagnetically forcing. Research of vortices in Q2D fluid flows provides a contributing factor to the fundamental understanding of turbulence. Other recent literature includes investigation of the flows in the wakes of a moving Lorentz force (Honji et al. [9], [10], Afanasyev et al. [2]). Honji et al. observed wakes that were wavy and unstable for sufficiently high forcing. Afanasyev et al. concentrated on vortex streets in the wakes, flows analogous to Von Karman vortex streets behind a cylinder moving in a fluid.
1.2 Focus of this research

The current research in electromagnetically forced flows focuses on moving magnets. The main question in this research is: what happens when the Lorentz force in Q2D flows is not stationary? What kinds of behavior in the fluid flow are found when the force is translating in a variety of configurations and can these behaviors be explained and predicted?

Different behavior of the fluid flow may be expected when the Lorentz force is not stationary. The starting jet that is initially formed by the force may change its flow direction. Consequently, the vortex dipole may have a different shape and trajectory. Furthermore by applying an oscillatory movement to the force, the fluid flow may become chaotic near or at the region of forcing.

The focus of this research is the investigation of the resulting flow patterns when a movement is added to a permanent magnet below a thin fluid layer. Essentially, a time-dependence is introduced to the electromagnetic forcing. Different possible configurations for the movement of the magnet are shown in Figure 2. The direction of the translation can be either parallel or perpendicular to the direction of the force. The following cases are distinguished:

- **Case L⊥**: Linear perpendicular
  The magnet moves from one point to another in a linear fashion, with a constant translation speed. The current density is parallel to this direction of motion, resulting in a Lorentz force that is dominant in a direction perpendicular to the magnet’s movement.

- **Case L∥**: Linear parallel
  The magnet moves from one point to another in a linear fashion, but with the current density perpendicular to the direction of motion. The Lorentz force’s dominant component is in the parallel direction. Note that this case is the configuration used in the research of Honji et al. [9], [10] and Afanasyev et al. [2].

- **Case H⊥**: Harmonic perpendicular
  The magnet oscillates around a point with a certain frequency and magnitude. The current density is parallel to this direction of motion, resulting in a Lorentz force that is dominant in a direction perpendicular to the magnet’s movement.

- **Case H∥**: Harmonic parallel
  The magnet oscillates around a point with a certain frequency and magnitude. The current density is perpendicular to this direction of motion, resulting in a Lorentz force that is dominant in a direction parallel to the magnet’s movement.

And the case of no translation that serves as a base for comparison. This will be called case 0:

- **Case 0: No translation**
  The magnet does not move.
Figure 2 Four different setups for directing the force $F$ and the magnet velocity $V$ (either $x$-component or $y$-component). Note that the force $F$ is not actually a 2D point force as depicted in this figure. The force $F$ depicted here is the largest component (which is the $y$ component) of the actual Lorentz force.

The remainder of this report is organized in the following way:
Chapter 2 describes a theoretical model in which the flow is linearized is developed to describe the fluid flow for low forcing strength. Chapter 3 provides the details of the set up of the laboratory experiments and numerical simulations. These laboratory experiments are performed using a shallow water tank with an electric current running through the fluid. Flows are visualized using fluorescent dye. Numerical simulations are performed using COMSOL Multiphysics. These numerical simulations provide the velocity field in the fluid flow and create the possibility to change the boundary conditions in the experimental domain. Chapter 4 follows with the discussion of the results of laboratory experiments and numerical simulations. Chapter 5 encompasses the concluding remarks and discussion of the research.
2. Theory

2.1 Governing equations

Consider a thin layer of a homogeneous and incompressible Newtonian fluid. The fluid flow is defined in a Cartesian coordinate system where the vector \( \mathbf{x} = (x, y, z) \) denotes the position and \( \mathbf{u} = (u, v, w) \) the fluid flow velocity vector. The \( x \)- and \( y \)-directions will be referred to as horizontal direction and the \( z \)-direction will be referred to as the vertical direction\(^1\). The fluid layer which has a depth of \( h \) is initially at rest and extends to infinity.

The dynamics of the fluid are governed by conservation of mass and momentum. The continuity equation for an incompressible fluid and the Navier-Stokes equation respectively are (Kundu [11]):

\[
\nabla \cdot \mathbf{u} = 0 ,
\]

\[
\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \frac{\mathbf{F}}{\rho} ,
\]

where \( \rho \) is the density, \( p \) the pressure, \( \nu \) the kinematic viscosity and \( \mathbf{F} \) external body forces.

Vorticity equation

The Navier-Stokes equation may be rewritten into the vorticity equation by taking the curl of equation (1.2). Inspecting the vorticity \( \omega = \nabla \times \mathbf{u} \) of the fluid flow may help to gain more insight in the behavior of the flow:

\[
\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla) \omega = (\mathbf{\omega} \cdot \nabla) \mathbf{u} + \frac{\nabla \rho \times \nabla p}{\rho^2} + \nu \nabla^2 \omega + \frac{1}{\rho} \nabla \times \mathbf{F} .
\]

The left-hand side (LHS) represents the material derivative of vorticity. It consists of a time dependent term \( \frac{\partial \omega}{\partial t} \) and a convection term \((\mathbf{u} \cdot \nabla) \omega\). The first right hand side (RHS) term \((\mathbf{\omega} \cdot \nabla) \mathbf{u}\) expresses the change of vorticity due to stretching/squeezing and tilting of vortex tubes. The second RHS term \( \frac{\nabla \rho \times \nabla p}{\rho^2} \) represents vorticity production due to baroclinic effects. The third RHS term \( \nu \nabla^2 \omega \) is the diffusion term. The last RHS term represents the production of vorticity by the curl of the body forces. More details on the force term are described in the next paragraph.

If the vertical length scale of the fluid is much smaller than the horizontal length scales, one can consider the fluid layer to be a “thin” layer. It is then assumed that the vertical velocities are much smaller than the horizontal velocities. The fluid flow is then said to be quasi-two-dimensional (the justification of this assumption is described in chapter 3.4). The \( z \)-component of the velocity vector is neglected: \( \mathbf{u} = (u, v, 0) \). The vorticity is then directed perpendicular to the \( x,y \)-plane \( \mathbf{\omega} = (0, 0, \omega_z) \).

---

\(^1\) Though throughout the report “upper” and “lower” will refer to higher and lower \( y \)-position.
Furthermore the density $\rho$ is assumed to be homogeneous throughout the fluid and there is no baroclinic vorticity production. Equation (1.3) becomes:

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla) \omega = \nu \nabla^2 \omega + \frac{1}{\rho} (\nabla \times F) \cdot \omega. \quad (1.4)$$

When the vorticity has only a component in the $z$-direction, one can find the velocity vector by defining a stream function $\psi = \psi e_z$. The stream function and the vorticity are related by a Poisson equation:

$$u = \nabla \times \psi, \quad (1.5)$$

$$\nabla^2 \psi = -\omega_z(x,y,t). \quad (1.6)$$

**Lorentz force**

The last term in (1.4) denotes the Lorentz force from a magnetic field originating from a permanent magnet and an electrical current density flowing through the fluid:

$$F_L = j \times B. \quad (1.7)$$

The current density is assumed to be homogeneous throughout the fluid and it is defined to flow along the $x$-axis: $j = j_0 e_x$. $j_0$ is constant over time, implying continuous forcing. $B(x,y,z)$ is the magnetic field that originates from a rectangular permanent magnet. The magnetic field can be expressed analytically and is described in Appendix A. A stationary rectangular magnet is defined in the origin of the coordinate system at $t=0$ s. Its size is $2D \times 2D \times d$ (see equations (A.12) and (A.19)). For a translating magnet the $B$-field becomes time-dependent according to:

$$B(x,y,z) \rightarrow B(x-V_x t, y-V_y t, z), \quad (1.8)$$

where $V_x$ and $V_y$ are the velocity components of the moving permanent magnet. The $B$-field’s largest component is in the $z$-direction and is largest close to the magnet (see Figure 3). The area above the magnet where the Lorentz force is the strongest will be referred to as the “forcing region”. Because the $z$-component of $B$ is dominant over the other directions: $B(x-V_x t, y-V_y t, z) \approx B_z(x-V_x t, y-V_y t, z) e_z$. The resulting dominant component of Lorentz force (see equation (1.7)) will be in the $y$-direction. Then the last RHS term in equation (1.3) becomes:

$$\frac{1}{\rho} \frac{\partial}{\partial x} (\nabla \times F)_z \approx \frac{1}{\rho} \frac{\partial F_y}{\partial x} = \frac{j_0}{\rho} \frac{\partial B_z}{\partial x}. \quad (1.9)$$

This shows that vorticity production is related the $x$-directed gradient of the $z$-component of the $B$-field. Figure 3 shows $B_z$ and its derivative $\frac{\partial B_z}{\partial x}$ plotted against $x$ at $y=0$ and $z=d/5$. $B_z$ is indeed largest directly above the magnet. The $\frac{\partial B_z}{\partial x}$ curve shows two oppositely signed peaks, which indicates the production of positive and negative vorticity: a vortex dipole structure.
Figure 3 a) $B_z(x,0,d/5)$ plotted against $x/D$. The dashed blue line indicates the line $B_z=0$. $D$ is the horizontal size of a rectangular magnet and $d$ its vertical size.  

b) $d/dx B_z(x,0,d/5)$ plotted against $x/D$. The dashed blue line indicates the line $dB_z/dx =0$. The two oppositely signed peaks is an indication for the formation of a vortex dipole. The black arrows are placed to draw attention to the fact that outside the two largest peaks, the $dB_z/dx$ value does not go to zero immediately. These two smaller bumps in $dB_z/dx$ also contribute to the change in vorticity.

Figure 4 The values of $dB_z/dx$ are plotted in a color plot. The black box in the middle indicates the position of the magnet. The solid lines are positive $dB_z/dx$ contour lines and dotted lines are negative $dB_z/dx$ contour lines. According to equation (1.9) the value of $dB_z/dx$ is directly related to the change in vorticity in the fluid layer. This figure shows that a vortex dipole forms in the fluid above the magnet. The main vortex dipole core is surrounded by two satellites of oppositely signed vorticity.
Figure 4 shows a 2D color-plot where $\frac{\partial B}{\partial x}$ is plotted as function of both $x$ and $y$. The vortex dipole structure can be clearly seen. Note how in Figure 3b the $\frac{\partial B}{\partial x}$ shows two main peaks in the middle, but does not immediately go to zero outside these peaks. The $\frac{\partial B}{\partial x}$ changes sign before going asymptotically to zero. In Figure 4 this shows up as two satellites of opposite vorticity production surrounding the vortex dipole core.

The governing equations (1.1) and (1.2) combined with the expression for the $B$-field (A.12), (A.19) may be solved numerically. However, to explain certain phenomenon qualitatively an analytical solution for the flow may be more desirable. The Navier-Stokes equation, difficult to solve due to its non-linearity, may be simplified by linearization. See chapter 2.4.

### 2.2 Dimensionless groups

The number of dimensionless parameters can be identified using the Buckingham Pi theorem. The Buckingham Pi theorem states that in a physical problem where $n$ physical variables are relevant, which are expressed in $k$ number of dimensions there will be $n-k$ number of dimensionless groups. For each case the dimensional groups are identified. As it turns out, the number of dimensionless groups is also the number of parameters that are varied in the laboratory experiments.

**Case 0: No movement**

The following variables and their dimensions are relevant in this case:

- $D$: Magnet size [L]
- $\rho$: Fluid density [M/L$^3$]
- $\nu$: Fluid viscosity [L$^2$/T]
- $F$: Force (per volume) [M/L$^2$/T$^2$]

So there are four relevant physical variables and three base dimensions, leading to 4-3=1 dimensionless group:

$$Ch = \frac{FD^3}{\rho \nu^2}.$$  \hfill (1.10)

This dimensionless parameter is also known as the Chandrasekhar number. This parameter compares the Lorentz force against the viscous forces in the flow. For high Chandrasekhar number the flow is dominated by the Lorentz force and viscous effects can be neglected. For low Chandrasekhar number the flow is inertia dominated.

Note that only a single length scale is taken into account (the magnet size) and the vertical length scale $h$ is ignored for now. If the vertical length scale is taken, an extra dimensionless group $\delta = \frac{h}{D}$ would be found. This group is important for the justification of the quasi-two-dimensional nature of the flow. This is described in chapter 3.4. The only parameter that is varied in the experiments is the Lorentz force.
**Case L⊥ & L∥: Linear movement**

The following variables and their dimensions are relevant in this case:

- **$D$: Magnet size** [L]
- **$\rho$: Fluid density** [M/L³]
- **$\nu$: Fluid viscosity** [L²/T]
- **$F$: Force (per volume)** [M/L²/T²]
- **$V$: Magnet translation velocity** [L/T]

In this case there are five different physical variables. The number of dimensional groups to find is therefore 5-3=2. The most meaningful dimensionless groups are found to be:

\[
\Pi_1 = Q = \frac{DF}{V^2 \rho} \\
\Pi_2 = Re_v = \frac{DV}{\nu}
\]  

(1.11)

In which a Reynolds number $Re_v$ is recognized. Note that this Reynolds number contains $V$, the magnet translation velocity, rather than the fluid velocity which is more common in fluid dynamics. Furthermore another dimensionless parameter is found which will be called $Q$. This number represents the ratio between the Lorentz force and the inertia force. Besides the magnitude of the Lorentz force, the magnet velocity is varied in experiments.

**Case H⊥ & H∥: Harmonic movement**

The following variables and their dimensions are relevant in this case:

- **$D$: Magnet size** [L]
- **$\rho$: Fluid density** [M/L³]
- **$\nu$: Fluid viscosity** [L²/T]
- **$F$: Force (per volume)** [M/L²/T²]
- **$A$: Magnet amplitude (for oscillatory movements)** [L]
- **$f$: Magnet frequency (for oscillatory movements)** [1/T]

When the magnet performs a harmonic translation, the magnet’s velocity is not constant. Instead the variables amplitude $A$ and frequency $f$ become relevant. In these cases there will be 6 physical variables with 3 different dimensions and thus 6-3=3 dimensionless parameters are found:

\[
\Pi_1 = S = \frac{A}{D} \\
\Pi_2 = Q_f = \frac{F}{Df^2 \rho} \\
\Pi_3 = Re_f = \frac{D^2 f}{\nu}
\]  

(1.12)

$S$ is a parameter that compares the amplitude of the oscillatory movement with the typical size of the magnet. Two other parameters are found that resemble the two
dimensionless groups found in the previous case. These parameters are denoted as $Q_f$ and $Re_f$. In experiments $F, A$ and $f$ are varied.

### 2.3 Typical timescales

Consider equation (1.4) for the $z$-component of the vorticity equation (1.3). One may write this equation in a dimensionless form by defining the following dimensionless variables:

$$\omega'_z = \frac{D \omega_z}{V}, \quad t' = \frac{t}{T}, \quad u' = \frac{u}{V}, \quad \nabla' = D\nabla, \quad F' = \frac{F}{F}, \quad (1.13)$$

where $T$ is a typical timescale. Substituting these variables into equation (1.4):

$$\frac{V}{T D} \frac{\partial \omega'_z}{\partial t'} + \frac{V^2}{D^2} (u \cdot \nabla') \omega'_z = \frac{V V}{D^3} \frac{\partial^2 \omega'_z}{\partial r^2} + \frac{F}{\rho D} (\nabla \times F')_z. \quad (1.14)$$

The apostrophes are omitted as of now and the equation is divided by $\frac{V}{D^2}$, the characteristic magnitude of the convection term:

$$\frac{D}{V T} \frac{\partial \omega_z}{\partial t} + (u \cdot \nabla) \omega_z = \frac{1}{Re_v} \frac{\nabla^2 \omega_z}{\rho V^2} + \frac{1}{Q} (\nabla \times F)_z, \quad (1.15)$$

where the dimensionless groups $Re_v$ and $Q$ are identified on the right-hand side. On the left-hand side the first term, the time derivative term, contains a group $\frac{D}{V T}$. A number that is similar to the Strouhal number, which parameterizes oscillations in a flow. Recall that $V$ is not the fluid velocity, but the magnet translation velocity.

**Advection by force timescale**

One may find a timescale by comparing the characteristic sizes of the terms in (1.15). By comparing the time derivative term with the forcing term, one finds the typical timescale for the advection of fluid particles by means of the forcing:

$$\tau_f = T = \frac{\rho V}{F}. \quad (1.17)$$

This timescale may be interpreted as the time it takes for the fluid’s vorticity $\omega_z$ to reach a certain magnitude above the magnet.

**Diffusion timescales**

Bottom friction may play a large role in shallow fluid layers. The fluid velocity at the bottom is assumed to be zero (no-slip boundary). As a result the fluid flow is expected to have a shear in the $z$-direction and a three dimensional structure. However the flow
may still be considered quasi-two-dimensional under certain circumstances. Refer to Satijn et al. [15] and Durán Mutate [12] for more details and to chapter 3.4.

In order to quantify the timescales for vertical and horizontal diffusion, the same approach as Satijn et al. and Durán Mutate is used. The three dimensional diffusion operator (the Laplacian) \( \nabla^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \) is separated in terms of horizontal diffusion and vertical diffusion:

\[
\nu \nabla^2 \omega_z = \nu \nabla_h^2 \omega_z + \nu \frac{\partial^2 \omega_z}{\partial z^2}.
\] (1.18)

The vertical diffusion is related to the bottom friction and can be parameterized as a linear term: \( \nu \frac{\partial \omega_z}{\partial z} \sim -\lambda \omega_z \) with \( \lambda = \nu \left( \frac{\pi}{2h} \right)^2 \) the bottom drag coefficient. \( h \) is the typical vertical length scale: the fluid layer depth.

Starting from (1.14) this equation becomes:

\[
\frac{V}{TD} \frac{\partial \omega_z}{\partial t} + \frac{V^2}{D^2} (u \cdot \nabla) \omega_z = \nu \frac{V}{D^2} \nabla_h^2 \omega_z - \frac{\lambda V}{D} \omega_z + \frac{F}{\rho D} \left( \nabla \times F \right)_z,
\] (1.19)

and dividing by the characteristic size of the convection term:

\[
\frac{D}{VT} \frac{\partial \omega_z}{\partial t} + (u \cdot \nabla) \omega_z = \frac{1}{Re_H} \nabla_h^2 \omega_z - \frac{1}{Re_B} \omega_z + \frac{1}{Q} \left( \nabla \times F \right)_z,
\] (1.20)

where the Reynolds number is now separated into the horizontal Reynolds number

\( Re_H = \frac{DV}{\nu} \)

and vertical Reynolds number

\( Re_B = \frac{V}{\nu D} \left( \frac{2h}{\pi} \right)^2 \).

Two timescales can be derived from (1.20) by comparing the diffusion terms with the time derivative term. A timescale for horizontal diffusion and a timescale for vertical diffusion: denoted as bottom friction timescale. The horizontal diffusion timescale is:

\[
\tau_d = \frac{D^2}{\nu}.
\] (1.21)

The bottom friction timescale is:

\[
\tau_B = \frac{4h^2}{\pi^2 \nu}.
\] (1.22)

These timescales can be interpreted as the time before a typical change is found in the vorticity \( \omega_z \) due to the diffusion in horizontal and vertical directions respectively.

The typical sizes of the timescales \( \tau_f \), \( \tau_B \) and \( \tau_d \) can be evaluated from the typical values of the relevant parameters used in laboratory experiments. It was found that \( \tau_f \) was up to four orders of magnitude smaller than \( \tau_B \) and that \( \tau_B \) is one order of
magnitude smaller than \( \tau_d \). This indicates that the flow in the fluid is mostly dominated by the Lorentz force. In absence of the Lorentz force the bottom friction will dominate the flow. The bottom friction has a damping effect on the vorticity as can be seen in equation (1.20), where the sign before the bottom friction term is negative. Therefore outside the forcing region, where the Lorentz force is not present, the flow is expected to damp out.

### 2.4 A linearized model of the fluid flow

Solving the vorticity equation (1.4) would give an analytical description of the Q2D fluid flow. Unfortunately this equation is non-linear, making it difficult to solve analytically. This chapter describes the solution through simplification of equation (1.4) by:

- A linearization by removing the non-linear convection term.
- Simplifying the \( z \)-component of the magnetic field to a Dirac Delta function.

The result of this model is the solution for the vorticity distribution and velocity field in linear approximation. The linearization of the vorticity equation implies that velocities in the fluid flow are small, such that the non-linear convection term can be neglected. Expected is that results from laboratory experiments and numerical simulations for very low forcing (low \( Q \)) will be comparable to the solutions of the linearized model. When \( Q \) is low, velocities that are induced by the Lorentz force should stay relatively small.

As mentioned in chapter 1.2, the Lorentz force generates starting jets in the fluid flow. These jets form dipolar vortex structures in (quasi) two-dimensional fluid flows. Starting jets in viscous fluids in linear approximation have been extensively described by Cantwell [4] for different configurations of forces in two and three dimensions. In the same context, Afanasyev et al. [1], [2] have also done numerical simulations and laboratory experiments with electromagnetic forcing in two-dimensional flows. This model for the vorticity distribution and velocity field is based on the 2D model by Afansyev et al. [2], but now includes a translational component in the force term to account for the movement of the magnet.

The non-linear term is neglected in equation (1.4) and equation (1.9) is used for the force. The \( z \)-component of the vorticity equation reduces to:

\[
\frac{\partial \omega_z}{\partial t} - \nabla^2 \omega_z = \frac{J_0}{\rho} \frac{\partial}{\partial x} B_z(x, y, t) .
\]  

(1.23)

The \( z \)-component of \( B \)-field can be found in Appendix A. However, equation (1.23) is difficult to solve. Instead of using the model for the rectangular permanent magnet, the magnetic field is described as a Dirac delta function, as this will greatly simplify the calculus later on:

\[
B_z(x-V_zt, y-V_yt) = B_0 \delta(x-V_zt)\delta(y-V_yt) .
\]  

(1.24)

Note that \( B_0 \) has the dimension of Tm\(^2\). Equation (1.23) takes on the form of a classical problem, the diffusion equation with a (localized) source term. The diffusion equation is left in its dimension-full form and may be solved using the Green’s function. The Green’s function for this problem is:
\[ G(x, y, t | x', y', \tau) = \frac{1}{4\pi \nu(t-\tau)} \exp \left( -\frac{(x-x')^2 + (y-y')^2}{4\nu(t-\tau)} \right). \tag{1.25} \]

So that the solution for the linearized vorticity equation becomes:

\[ \omega_z = \frac{j_0 B_0}{\rho} \int_0^\infty \int_0^\infty G(x, y, t, x', y', \tau) \frac{\partial}{\partial x} \left( \delta(x-V_x \tau) \delta(y-V_y \tau) \right) dx' dy' d\tau. \tag{1.26} \]

This integral is solved (see Appendix C.1 for the derivation) and an expression for the vorticity distribution is found:

\[ \omega_z(x, y, t) = \frac{j_0 B_0}{\rho} \int_0^\infty \frac{2(x-V_x \tau)}{4\pi \nu^2 (t-\tau)^2} \exp \left( -\frac{(x-V_x \tau)^2 + (y-V_y \tau)^2}{4\nu(t-\tau)} \right) d\tau. \tag{1.27} \]

The velocity field can be found by calculating the stream function \( \psi \), see equations (1.5) and (1.6). The solution to this Poisson equation is given by Cantwell [4]:

\[ \psi(x, y, t) = -\frac{2j_0 B_0}{\pi \rho} \int_0^\infty \frac{1-\exp \left( -\frac{(x-V_x \tau)^2 + (y-V_y \tau)^2}{4\nu(t-\tau)} \right)}{(x-V_x \tau)^2 + (y-V_y \tau)^2} d\tau. \tag{1.28} \]

The velocity field is found from differentiating (1.28) to \( x \) and \( y \):

\[ u = \frac{\partial \psi}{\partial y}, \tag{1.29} \]

\[ v = -\frac{\partial \psi}{\partial x}. \]

Different cases for \( V_x \) and \( V_y \) are considered. First the case of no movement of the magnet. Then followed by the four cases described in chapter 1.2.
Case 0: No magnet translation
As comparison with a translating magnet, the case of a stationary magnet is considered first. Here $V_x = V_y = 0$. The vorticity and stream function become:

$$\omega(x, y, t) = \frac{j_0 B_0}{\pi \rho} \cdot \frac{2x}{x^2 + y^2} \exp\left(-\frac{x^2 + y^2}{4\nu t}\right),$$

$$\psi(x, y, t) = -\frac{2j_0 B_0}{\pi \rho} \int_{0}^{t} x \frac{1-\exp\left(-\frac{x^2 + y^2}{4\nu(t-\tau)}\right)}{x^2 + y^2} d\tau.$$  \hspace{1cm} (1.31)

$\omega$ depends on both position and time. An $\omega(x,0,1)$ profile with $V_x = V_y = 0$ is plotted against $x$ in Figure 5. Notice the large difference with Figure 3b, due to the choice of the (unrealistic) magnetic field $B$. The vorticity is singular in the origin and has a vertical asymptote at $x=0$. Still, the $B$-field as a Dirac delta function produces a vortex dipole structure (two oppositely signed peaks in vorticity) and the effect of translation of the magnet may be investigated using this model for the magnetic field.

Following from the stream function (1.31), the velocity field is found with equation (1.29). Figure 6 shows the velocity field and instantaneous streamlines (figures from here on will only show the streamlines). The streamlines clearly show the dipole structure in the fluid.
Figure 5 A cross section of the vorticity $\omega_z(x,y,t)$ at $y=0$ and $t=1s$ is plotted here. The magnet does not move. The $B$-field is modeled as a Dirac delta function, which results in the vorticity being singular in the position of the magnet.

Figure 6 Velocity field acquired from equations (1.29) and (1.31) for $j_0 B_0'/\rho=10^{-5}$ m$^3$/s$^2$ at $t=1s$. The arrows represent the velocity. Also included are instantaneous streamlines showing the vortex dipole structure. Dotted red lines are the streamlines for negative vorticity, the blue ones for positive vorticity. The magnet does not move and is positioned in the origin. The main component of the Lorentz force is directed in the positive y-direction.
Case \(L_\perp\): Linear perpendicular movement \(V_x = \text{Constant}, V_y = 0\)

Figure 7 shows the development of the instantaneous streamlines in time for the case of a translating magnet with a constant speed, perpendicular to the forcing direction (\(y\)-direction). Figure 8 shows the vorticity distribution and Figure 9 shows a cross section of the vorticity distribution at \(y=0\). The figures are obtained by solving equations (1.27) to (1.29). The magnet starts at \(x=0, y=0\) (also applies to the other cases) and moves at \(V_x=-5 \text{ mm/s}\). Note that the figures with vorticity distribution often show the color scale capped off at random values. This is due to the singular nature of the magnet model.

The figures show that initially a vortex dipole is formed at \(x=0, y=0\), the initial position of the magnet. As the magnet translates in the negative \(x\)-direction the two centers of the closed streamlines move away from each other, which can be seen in Figure 7. This seems to indicate that the two oppositely signed vortex patches become separated. Figure 8 and Figure 9 show exactly how the vorticity is distributed in the domain. One may have expected the vortex dipole to be simply translated with the movement of the magnet. However this is not completely the case. Figure 8 and Figure 9 do show that the vortex dipole translates with the magnet, however there is also a persistent patch of vorticity that remains at the origin after the magnet has moved away from the initial position.

It was found that this “persistent patch of vorticity” is a direct effect of the movement of the magnet. A mathematical representation of this observation is derived in Appendix C.2. In this derivation a rectangular magnet is used instead of a Dirac Delta function, but the result is applicable to this situation nonetheless. It is shown that when neglecting both non-linear terms and viscous terms in the NS equation, the resulting flow consists of two swirling structures. One that moves with the translating magnet, the other that stays at the initial position of the magnet. The swirling structure that remains at the origin can be appointed to the persistent patch of vorticity in Figure 8 and Figure 9.
Figure 7 Streamlines of the fluid flow for a moving magnet at $V_x = -5$ mm/s and $j_0B_0'/\rho = 10^{-5}$ m$^3$/s$^2$ at two points in time: $t=1$s (left) and $t=4$s (right).

Figure 8 Vorticity distribution $\omega_z(x,y,t)$ for a moving magnet at $V_x = -5$ mm/s and $j_0B_0'/\rho = 10^{-5}$ m$^3$/s$^2$ at two points in time: $t=1$s (left) and $t=4$s (right).

Figure 9 A cross section of the vorticity distribution $\omega_z(x,y,t)$ at $y=0$ at $t=1$s (left) and $t=4$s (right). At the position of the magnet the vorticity becomes arbitrarily large due to the singular nature of the magnet. The black arrows are placed to draw attention to the persistent patch of negative vorticity at the origin.
**Case L₁:** Linear parallel movement $V_x = 0, V_y = \text{Constant}$

The development of instantaneous streamlines of the fluid flow for a magnet translating in the positive $y$-direction is shown in Figure 10. Figure 11 shows the vorticity distribution. The force and velocity of the magnet are parallel here.

As the magnet moves in the $y$-direction, the vortex dipole structure becomes elongated in the $y$-direction. Figure 12 shows a cross section of the vorticity distribution at $y=0$. The source of the production of vorticity moves with the magnet, the vorticity that was initially produced by the Lorentz force diffuses over time.
Figure 10 Streamlines of the fluid flow for a translating magnet at $V_x=5\text{mm}/\text{s}$ and $j_0B_0'/\rho=10^{-5} \text{m}^4/\text{s}^2$ at $t=1\text{s}$ (left) and $t=4\text{s}$ (right).

Figure 11 Vorticity distribution for a translating magnet at $V_x=5\text{mm}/\text{s}$ and $j_0B_0'/\rho=10^{-5} \text{m}^4/\text{s}^2$ at $t=1\text{s}$ (left) and $t=4\text{s}$ (right).

Figure 12 A cross section of the vorticity distribution $\omega_z(x,y,t)$ at $y=0$ plotted at several times.
Case $H_{\perp}$: Harmonic perpendicular movement $V_x \sim \sin(t)$, $V_y = 0$

For the oscillatory translation of the magnet, the velocity $V$ in the $x$-direction is taken as $V_x = A \sin(2\pi ft) \ast 2\pi f \, t$. The resulting position of the magnet will go as $x_{magnet} = -A \cos(2\pi ft)$. See Figure 13 shows the instantaneous streamlines of the flow and Figure 14 shows the vorticity distribution.

Figure 13 shows a vortex dipole with the centers of the circular streamlines separated by approximately the size of $A$, the amplitude of oscillatory movement of the magnet. Figure 14 shows that there are actually two pairs of oppositely signed vorticity patches. The inner two vortices appear to be smaller in magnitude and do not are not visible in the streamline plots. An explanation for this behavior can be derived from the "persistent vorticity patch" in case $L_{\perp}$. As the magnet moves back and forth, at each end there is a persistent patch of vorticity that is strengthened each time the magnet passes by. As a result, the outer two vortices become more pronounced than the two inner vortices. Figure 15 shows that indeed the outer vorticity peak is larger than the inner ones (though in this figure the left vortex pair is singular due the magnet that is positioned there at that time).
Figure 13 Streamlines of the fluid flow for a moving magnet after 5 oscillations with $A=10\text{mm}$, $f=0.5 \text{ Hz}$ and $j_0B_0/\rho = 10^{-5} \text{ m}^3/\text{s}^2$ at $t=10s$. The magnet oscillates in the $x$-direction and a vortex dipole is seen with the vortex patches separated by a distance that is proportional to $A$.

Figure 14 Snapshot of the vorticity distribution $A=10\text{mm}$, $f=0.5 \text{ Hz}$ and $j_0B_0/\rho = 10^{-5} \text{ m}^3/\text{s}^2$ at $t=10s$. The magnet is currently at $x=-0.01\text{m}$. Two vortex pairs are seen. The two outer vortex patches form the large vortex dipole structure seen in Figure 13.

Figure 15 A cross section of the vorticity distribution $\omega_z(x,y,t)$ at $y=0$ at $t=10s$. 
**Case H\(_\parallel\): Harmonic parallel movement** \( V_x = 0, \ V_y \sim \sin(t) \)

In case H\(_\parallel\) the magnet oscillates in the y-direction as \( y_{\text{magnet}} = -A \cos(2\pi ft) \). Figure 16 shows the streamlines for the flow after several oscillations. Figure 17 shows the vorticity distribution. As the magnet moves up and down in the y-direction, the vortex dipole structure becomes elongated in this direction. Figure 17 shows that at \( y=-10\text{mm} \) and \( y=10\text{mm} \) the vorticity is larger (in absolute sense). This is because on average the magnet stays at those positions slightly longer, as the translation speed goes as \( \sim \sin(t) \).

![Flow streamlines at t=10.00 s](image1)

**Figure 16** Streamlines of the fluid flow for a moving magnet with \( A=10\text{mm}, \ f=0.5 \text{ Hz} \) and \( j_0B_0/\rho=10^{-5} \text{ m}^3/\text{s}^2 \) at \( t=10\text{s} \).

![Vorticity distribution at t=10.00s](image2)

**Figure 17** Vorticity distribution for a moving magnet with \( A=10\text{mm}, \ f=0.5 \text{ Hz} \) and \( j_0B_0/\rho=10^{-5} \text{ m}^3/\text{s}^2 \). Left: \( t=10\text{s} \). Right: \( t=11\text{s} \)
Summary and discussion

In the present section a model was derived for the vorticity and velocity of the fluid flow. This model solves the linearized NS equation and uses a Dirac delta function for the magnetic field. Different behaviors of the fluid flow were observed for each case of the magnet translation.

One component in this model compared to the model of Afanasyev et al. was not included. A vortex dipole structure is known to have a net linear momentum and the structure has a translational motion. The vortex dipole in this model does not translate, due to the removed non-linear convective term. Afansyev et al. do include an extra translation term in their model to account for the translation of the vortex dipole. In the given derivation, this translation term must be included in the Green’s function (1.25). The translation of the vortex dipole is calculated from the potential flow outside the dipole structure. Details are given in the next chapter.
2.5 Vortex dipole translation

Vortex dipoles in (quasi) two-dimensional flows are known to have a net linear momentum and they translate as a whole. Finding the translation velocity and trajectory is of interest. How does the vortex dipole translate while the Lorentz force is translating as well?

Typically, fluid flows may be decomposed in two kinds of flows: rotational and irrotational. When a flow is irrotational, it can be expressed by a potential. Afanasyev et al. [1] state that the vorticity is concentrated inside the dipole. Just in front of the dipole the flow is approximately irrotational. The vortex dipole’s translation speed is the velocity of the potential flow just outside the vortex dipole front, meaning the vortex dipole follows the potential flow. So in order to find the trajectory and translation speed of the vortex dipole, the potential flow velocity needs to be calculated.

The Navier-Stokes equation (1.2) is linearized to simplify the equation, as in the previous chapter. This linearization implies low velocities in the flow. The divergence of every term is then taken:

\[
\nabla \cdot \left( \frac{\partial \mathbf{u}}{\partial t} \right) = -\frac{1}{\rho} \nabla^2 p + \nabla \cdot (\nabla \times \mathbf{u}) + \frac{1}{\rho} \nabla \cdot (\mathbf{j} \times \mathbf{B}).
\]

(1.32)

So that by applying the continuity equation (1.1) a Poisson equation for the pressure is found:

\[
\nabla^2 p = \nabla \cdot (\mathbf{j} \times \mathbf{B}),
\]

(1.33)

where \( \mathbf{j} = j_0 \mathbf{e}_x \). Again, the \( \mathbf{B} \)-field is chosen as a Dirac delta function to stay consistent with the previous chapter. However, it will soon be clear that this choice as a model for the \( \mathbf{B} \)-field is not adequate.

**B-field modeled as a Dirac Delta function**

In two dimensions, the Dirac Delta function can be written as:

\[
\delta(x-Vt) = -\nabla^2 \frac{\ln|x-Vt|}{2\pi},
\]

(1.34)

Recall that \( \mathbf{j} = j_0 \mathbf{e}_x \) and \( \mathbf{B} = B_0 \mathbf{e}_z \), so that:

\[
(\mathbf{j} \times \mathbf{B}) = -j_0 B_0 \delta(x-Vt) \mathbf{e}_z = -j_0 B_0 \nabla^2 \ln|x-Vt| \mathbf{e}_z.
\]

(1.35)

This is substituted in (1.33):

\[
\nabla^2 \left( p + \frac{j_0}{2\pi} \nabla \cdot \left( \ln|x-Vt| \mathbf{e}_y \right) \right) = 0
\]

(1.36)

and pressure becomes:
\[ p = \frac{j_0 B_0 \delta(y-V_x t)}{2\pi \left((x-V_x t)^2 + (y-V_y t)^2\right)}. \]  

(1.37)

The potential flow \( u \) is found by looking at the far field of the linearized NS equation, outside the vortex dipole:

\[ \frac{\partial u}{\partial t} + \frac{1}{\rho} \nabla p = 0. \]  

(1.38)

Substituting equation (1.37) for the pressure here and integrating with respect to time, the potential flow velocity is found to be:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{j_0 B_0 \delta}{2\pi \rho} \nabla \left( \frac{y-V_y t}{(x-V_x t)^2 + (y-V_y t)^2} \right), \\
u &= \int \frac{j_0 B_0 \delta}{2\pi \rho} \nabla \left( \frac{y-V_y t}{(x-V_x t)^2 + (y-V_y t)^2} \right) dt.
\end{align*}
\]  

(1.39)

The position of the vortex dipole is then found by solving

\[ u = \frac{dx_v}{dt}, \]  

(1.40)

where \( x_v \) is the position vector of the vortex dipole. For a stationary magnet: \( V_x = V_y = 0 \), with the initial condition \( x_v(0) = y_v(0) = 0 \), the system of two differential equations are easily solved analytically:

\[
\begin{align*}
y_v(t) &= \left( \frac{3 j_0 B_0 \delta}{4\pi \rho} \right)^{1/3} t^{2/3}, \\
x_v(t) &= 0.
\end{align*}
\]  

(1.41)

The vortex dipole is found to translate as \( \sim t^{2/3} \) in the \( y \)-direction and is stationary in the \( x \)-direction. This solution is also found by Afanasyev et al. [1]. However for the case of a moving magnet \( V_x \neq 0 \) and with initial conditions \( x_v(0) = y_v(0) = 0 \), finding a solution suddenly becomes problematic. A nonlinear system of differential equations needs to be solved (see equation (1.39)). A numerical solver should be able to produce a solution for the trajectory and velocity of the vortex dipole. However, the right-hand side of the equation system is singular for the initial conditions. This is a consequence of the choice for the \( B \)-field, namely a Dirac delta function. Using a different, more realistic model for the \( B \)-field that does not contain a singularity is more appropriate.
**B-field originating from a rectangular magnet**

Consider the model for the \( B \)-field of a rectangular magnet, as described in Appendix A. Equation (1.33) may be reformulated using a vector potential. The vector potential is defined as follows:

\[
B = \nabla \times A, \\
\nabla \cdot A = 0, \tag{1.42}
\]

where the Coulomb gauge has been assumed. Rewriting equation (1.33) using this vector potential and the Coulomb gauge leads to:

\[
\nabla^2 p = \nabla \cdot (j \times B) = \nabla \cdot (j \times (\nabla \times A)) \\
= (\nabla \times A) \cdot (\nabla \times j) - j \cdot (\nabla \times (\nabla \times A)) \tag{1.43}
\]

A particular solution is:

\[
p = j \cdot A. \tag{1.44}
\]

The expression for the vector potential \( A(x, y, z) \) of a rectangular magnet can be found in Appendix B, where the translational component must be added to account for the movement of the magnet: \( A(x, y, z) \rightarrow A(x-V_y t, y-V_x t, d) \).

Substituting this into (1.38):

\[
\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho} \nabla p = -\frac{j_0}{\rho} \nabla A, (x-V_y t, y-V_x t, d). \tag{1.45}
\]

The vector potential \( A \) may be simplified by taking the lowest order multipole term (a dipole) for \( A \), which applies for sufficiently large distances from the magnet:

\[
A = \frac{\mu_0}{4\pi} \frac{m \times r}{r^3}, \tag{1.46}
\]

where \( m \) is the dipole moment and is chosen to be in the \( z \)-direction: \( m = m_0 e_z \). The pressure caused by a stationary magnet \( (A = A(x, y, z)) \) becomes:

\[
p = j \cdot A = j_0 e_x \frac{\mu_0 m_0}{4\pi r^2} e_z = \frac{\mu_0 m_0 j_0}{4\pi r^2} \sin \varphi \tag{1.47}
\]

and in case for a moving magnet \( (A \rightarrow A(x-V_y t, y-V_x t, d)) \), the translational components are added:
\[ p = \frac{\mu_0 m_i j_0}{4\pi} \frac{y-V_x t}{\left( (y-V_y t)^2 + (x-V_x t)^2 \right)^{3/2}}. \quad (1.48) \]

Solving (1.38) for a stationary magnet \( V_x = V_y = 0 \) and with initial conditions \( x_i(0) = y_i(0) = 0 \):

\[
y_i(t) = \left( \frac{4 j_0 \mu_0 m_i}{4\pi \rho} \right)^{1/4} t^{1/2},
\]
\[
x_i(t) = 0.
\]

The vortex dipole is now found to translate as \( \sim t^{1/2} \) rather than \( \sim t^{2/3} \), which was found by Afansyev et al. It is expected that the latter result \( \sim t^{1/2} \) is a more realistic one.

However, solving (1.38) with \( V_x \neq 0 \) and with initial conditions \( x_i(0) = y_i(0) = 0 \) becomes problematic again. Another singularity prevents a solution by numerical methods. The singularity problem arises from the singularity in the far field approximation, as \( A \sim \frac{1}{r^2} \). Therefore, the vector potential of the rectangular magnet in its full form must be taken into account.

Consider the case of \( V_y = 0 \), \( V_x \neq 0 \). Going back to equation (1.45), a variable transformation simplifies the problem:

\[
\tilde{x} = x - V_x t,
\]
\[
\tilde{y} = y,
\]
\[
\tilde{z} = z,
\]
\[
\tilde{t} = t.
\]

So that equation (1.45) may be rewritten as:

\[
\frac{\partial u}{\partial \tilde{t}} - V_x \frac{\partial u}{\partial \tilde{x}} = -\frac{j_0}{\rho} \nabla A_x(\tilde{x}, \tilde{y}, \tilde{t}). \quad (1.51)
\]

Solving this equation gives the potential flow \( u \). In the \( x \)-direction, the particular solution is:

\[
\frac{\partial u}{\partial \tilde{t}} - V_x \frac{\partial u}{\partial \tilde{x}} = -\frac{j_0}{\rho} \frac{\partial}{\partial \tilde{x}} A_x(\tilde{x}, \tilde{y}, \tilde{t})
\]
\[
\rightarrow u_p = \frac{j_0 A_x}{\rho V_x}
\]

and the homogeneous solution is:

\[
u_H = F_x(\tilde{x} + V_x t), \quad (1.53)
\]
where $F(\tilde{x}+V_x t)$ is any kind of function. This function is found by inspecting the initial condition:

\[ u(t=0) = 0 \]
\[ \frac{j_0 A_x}{\rho V_x} + F(\tilde{x}) = 0 \]
\[ F(\tilde{x}) = -\frac{j_0 A_x}{\rho V_x}. \]  

(1.54)

In the y-direction, the particular solution is:

\[ \frac{\partial v}{\partial t} - V_x \frac{\partial v}{\partial \tilde{x}} = -\frac{j_0}{\rho} \frac{\partial}{\partial \tilde{y}} A_x(\tilde{x}, \tilde{y}, \tilde{z}) \]
\[ \rightarrow v_p = \frac{j_0}{\rho \gamma_x} \frac{\partial}{\partial \tilde{y}} \int A_x d\tilde{x}. \]  

(1.55)

And the homogeneous solution:

(1.56)

The total solution is then:

\[ u(x, y, z, t) = \frac{j_0}{\rho V_x} \left( A_x(x-V_x t, y, z) - A_x(x, y, z) \right), \]
\[ v(x, y, z, t) = \frac{j_0}{\rho V_x} \frac{\partial}{\partial \tilde{y}} \left( \int_0^\infty A_x(\tilde{x}, \tilde{y}, \tilde{z}) - A_x(\tilde{x}-V_x t, \tilde{y}, \tilde{z}) d\tilde{x} \right). \]  

(1.57)

Equation (1.57) represents the potential flow velocity in the far field of the flow. This velocity field is plotted in Figure 18 for a stationary magnet, where $V_x$ was taken arbitrarily small. The velocity field of the potential flow resembles that of a source-sink dipole, where the upper side of the magnet appears as the source and the lower side of the magnet appears as the sink.

To calculate the trajectory of the dipole, equation (1.40) is solved where $u$ is given in equation (1.57). Even for $V_x = 0$, $V_y \neq 0$ and initial conditions $x_v(0) = 0$, $y_v(0) = D$ this system of nonlinear differential equations is now numerically solvable. The initial condition $y_v(0) = D$ is changed compared to the previous $y_v(0) = 0$, as the frontal position of the vortex dipole is now located at the upper side of the rectangular magnet at $y=D$.

Figure 19 shows the velocity field of the potential flow when the magnet is translating. Also plotted is the trajectory of a vortex dipole. The vortex dipole does not stay at $x=0$ anymore, but moves in the $x$-direction opposite of the magnet’s translation direction.
Figure 18 Velocity field of the potential flow. The arrows represent the velocity and the black box denotes the position of the magnet. The magnet is stationary with its center at the origin and $D$ was taken to be 8 mm.

Ultimately, combining this model for the translation of the vortex dipole and the model for the vorticity distribution would be desirable. This was possible for Afanasyev et al. [1] where the force is stationary. However when the magnet is translating, solving (1.39) becomes problematic and a Dirac delta function for the $B$-field can no longer be used. Using a more realistic model for the $B$-field aided in solving (1.39), but this $B$-field was not used in the first place as it increases complexity to the equation (1.23). Finding an analytical solution to equation (1.23) with the $B$-field from appendix A would solve this dilemma.

Figure 19 The blue arrows represent the potential flow for a moving magnet (black square). The blue circle represents the frontal position of the vortex dipole at $t=10s$. Red line represents the trajectory that it has taken. The magnet speed $V=-5$ mm/s (going to the left) and $j_0=23.8$ A/m$^2$. 
Discussion of the models

Summarized, the last two chapters encompass the following on theoretical models: A model for the fluid flow was developed using a linearization and using a Dirac delta function as $B$-field. The solution to this vorticity equation predicts a vorticity distribution that represents a dipolar vortex structure. When the magnet is moving, this vorticity distribution consequently changes. In certain cases, patches of one sign of vorticity move away with the magnet from its initial position.

A consequence of linearizing the vorticity equation was that the vortex dipoles do not translate. To account for this, a separate model for the translation for the vortex dipole was developed. The fluid flow is expected to be irrotational outside the vortex dipole. It is expected that the fluid particles inside vortex dipole, which is in the rotational flow region, will follow the potential flow in the irrotational flow region. Thus by calculating the potential flow, one finds the movement of the vortex dipole. However, the choice for a Dirac delta function for the $B$-field became problematic. Using a more realistic model for the $B$-field lead to a solution and another consequence: the $\sim t^{2/3}$ dependence for the dipole position as described by Afanasyev et al. was found to be $\sim t^{1/2}$ instead. In any case, the model describes the following for the movement of vortex dipole when the magnet translates: the vortex dipole will not strictly have a momentum in the parallel direction of the Lorentz force, but also gains momentum in the perpendicular direction. The trajectory bends in the direction opposite to the translational direction of the magnet. This is purely a consequence of the movement of the magnet, as vortex dipole’s relative position from the magnet in the potential flow changes.
3. Experiments and numerical simulations

3.1 Experimental Set Up

For the laboratory experiment a shallow 52 x 52 cm² water tank and a translation stage is used. The bottom plate of the tank is 1 mm thick. The water tank is filled with salt solution (NaCl 12% brix) to a height of \( h = 7 \) mm. Electrodes are placed on the sides of the water tank. These electrodes are connected to a current source for which the current through the fluid may be controlled. A rectangular magnet is placed below the water tank and is moved by the translation stage driven by a stepping motor. This magnet is sized 16 x 16 x 5 mm³. The water tank (which includes the electrodes) is detachable from the main framework and the tank can be rotated such that the resulting current density is effectively turned 90°. A top view of the set up is shown in Figure 20.

![Figure 20 Top view of the experimental set up. The water tank includes two electrodes to send an electric current through the fluid. A translation stage is placed next to the tank to move the magnet, which is placed directly below the water tank.](image)

Figure 21 shows a side view of the set up. A high resolution camera is mounted above the tank and records images at a framerate of 10 Hz. This framerate is sufficiently high as the time period between frames is 0.1s, about an order of magnitude smaller than the observed eddy turnover times \( \tau_e \sim O(1) \) s. A light source is also mounted above the tank, a slide projector is used for this. This is needed for the luminescent dye, which is used to visualize the flow. The green colored dye and black colored bottom of the tank should then have enough contrast for a proper visualization.

During measurements of the dye visualization, the current density is kept constant over time. This means there is a continuous forcing in the fluid flow.

A limitation of the stepping motor is that it is not able to start and stop accurately at positions for movements with high velocities. This has consequences for
cases $H_\perp$ and $H_\parallel$, the harmonic movements, as the average magnet position starts drifting to one direction. For example, the average magnet position for harmonic oscillations at a frequency of 1 Hz with an amplitude of 5 mm eventually starts drifting to the negative $x$ direction.

Another problem for the set up is that for a certain orientation of the water tank, a light reflection can be seen from the light source. However, the qualitative descriptions of the observations should not be affected by this.

![Figure 21 Side view of the set up. A camera is mounted above the tank to take images of the fluid flow, visualized by luminescent dye.](image)

**3.2 Numerical simulations**

Numerical simulations are performed in COMSOL Multiphysics. The computational domain is a rectangular volume, with a constant height of 7mm. The horizontal size of the domain is varied when needed and is usually taken large enough so the starting jets formed from the Lorentz force do not reach the edge of the domain. Edge interactions are not the topic of this research and are undesirable. COMSOL uses a finite element solver to compute solutions of the Navier-Stokes equation (1.2) on the domain. The domain is discretized by a mesh of typically 10,000 to 50,000 or more elements.

The top of the domain has a free stress boundary condition. The bottom of the domain has a no-slip boundary condition. Though it would be intuitive to set the horizontal edges of the domain with a no-slip boundary, a free stress boundary condition was taken instead in order to minimize the effects of the edges.

The $B$-field is calculated in accordance with Appendix A. In order for the Lorentz force to be of the correct order of magnitude, the current density is calibrated as described in the next chapter.

**3.3 Calibrating the numerical simulations**

The exact strength of the magnetic field originating from the permanent magnet is unknown. As a consequence, the exact Lorentz force is unknown. At most, the maximum strength of the magnetic field may be measured using a Gauss/Tesla meter. Thus a value of 0.225T directly above the magnet was found (with an unknown, but relatively high margin of error). However, this method is deemed not accurate enough. Another method is finding some sort of measure for the Lorentz force in order to calibrate the theoretical and numerical simulations with the laboratory experiments. The theory described in the previous chapter provides a way to do this. Fluid particles
inside the vortex dipole follow the potential flow (1.38) and frontal position of the vortex dipole goes as:

\[ L(t) = \left( \frac{4\mu_0 j_0 m_0}{4\pi \rho} \right)^{1/4} t^{1/2} = at^{1/2}, \]  

(3.1)

where \( L \) is the distance travelled by the vortex dipole in the y-direction and 
\[ a = \left( \frac{4\mu_0 j_0 m_0}{4\pi \rho} \right)^{1/4} \]

can be used as a measure for the Lorentz force. The distance travelled by a vortex dipole is measured as a function of time and electric current density. In the numerical simulations the same distance is calculated in order to set the equivalent Lorentz force used in the laboratory experiments.

The vortex dipole is generated by a stationary magnet under the fluid with a constant current density. The distance defined to be difference between the y-position of the magnet (not visible in Figure 22) and the top of the dye distribution. The measurement of the distance is then repeated for different current densities. Images captured by the digital camera are converted to black and white image with an empirically chosen threshold.

Figure 22 A vortex dipole structure is created in the fluid with a current density of \( j = 476.2 \, \text{A/m}^2 \) and fluid depth \( h = 7 \, \text{mm} \). The green dye is converted to white with a threshold value of 0.3.

The distance \( L \) as function of time is then fitted against \( y = at^{1/2} + b \). The starting position \( b \) is also taken as a fit parameter, as the exact start point of the vortex dipole front is unknown. This fit parameter is bounded within 8 mm (half the size of the magnet), which was taken as the margin of error for the starting position. The initial dye placement affects the fit, as dye is placed on the fluid around the position above the magnet. The initial dye placement does not represent the actual vortex dipole position. Therefore the initial data points at small times were discarded (see Figure 23) in the fitting procedure.
The fit procedure allows $a$ to be measured. As $a^4 = \frac{4\mu_0 j_0 m_0}{4\pi \rho}$ has a linear relation with the current density $j_0$, $a$ was measured for several values $j_0$. Figure 24 confirms this linear relationship.
As a check for the $\sim t^{1/2}$ dependence, the vortex dipole displacements are plotted in a log scale. (see Figure 25). The two dotted lines denote the $\sim t^{1/2}$ and $\sim t^{2/3}$ dependence. It appears that most curves follow the $\sim t^{1/2}$ dependence. It was measured that at higher current densities, the curves start to become more like $\sim t^{2/3}$. The jets produced from the Lorentz force for these current densities were observed to be unstable and wavy. At higher current densities the non-linear terms play a larger role, for which the theory does not hold. Typically in the laboratory experiments, current densities larger than 500 A/m$^2$ were not used.

The same procedure is followed for numerical simulations. Dipolar vortex structures are observed in the numerical solutions by COMSOL (see Figure 26). The distance travelled by the vortex dipole is now defined as the difference between $y$-position of the front of the dipole and the magnet’s $y$-position (which is well defined in numerical simulations). This “front of the dipole” was taken as the top position where the level of the $z$-component of the vorticity reached the value of $|\omega_z| \leq 0.1$ s$^{-1}$ (if the value does not exist, the threshold value of 0.1 s$^{-1}$ is consistently divided by two until an existing value for $|\omega_z|$ is found). The distance was measured as function of time (see Figure 27). $a$ was then determined for different current densities $j$ (see Figure 28). The Lorentz force in numerical simulations is kept proportional to the laboratory experiments by adjusting the current density in the numerical simulations such that $a$ becomes equal to the value found in laboratory experiments.
Figure 26 Numerical solution for the \( z \)-component of the vorticity distribution by COMSOL at \( t=10 \)s and \( j=250 \) A/m\(^2\). The black box represents the position of the magnet.

Figure 27 Vortex dipole position in a numerical simulation with \( j=250 \) A/m\(^2\), fitted to \( \sim t^{1/2} \).
3.4 Two-dimensionality of the flow

It was mentioned in chapter 2 that the two-dimensional nature of the flow was merely an assumption. However it is possible to quantify the two-dimensionality of the flow. This quantification is described by Durán Mutate [12], where three regimes were identified for the two-dimensionality of the fluid flow. Mutate uses $\delta^2 Re$ as a measure, where $\delta = \frac{h}{D}$ and $Re = \frac{UD}{\nu}$. Note that $U$ is the typical velocity of the fluid flow. The regimes are defined as (see [12] for an extensive description of all regimes):

- Q2D flow regime $\delta^2 Re \leq 6$
- Transitional flow regime $6 < \delta^2 Re \leq 15$
- Three-dimensional flow regime $\delta^2 Re \geq 15$

The typical velocity $U$ was measured from the numerical simulations. Measuring from simulations with a stationary magnet at $j=500 \text{ A/m}^2$ (the maximum is $j=1357 \text{ A/m}^2$ in all numerical simulations), it appeared that $\delta^2 Re = 15.3$, just inside the three-dimensional flow regime. However with moving magnets, the typical fluid velocity $U$ decreases as the forcing $Q$ or $Q_f$ decreases as well. Measuring the typical velocity $U$ in a few numerical simulations with moving magnets, it appears that most cases are in the transitional flow regime $6 \leq \delta^2 Re \leq 15$. A few exceptions with high $Q$ (in the order $\sim 10^4$), it was seen that $\delta^2 Re \geq 15$. Therefore the assumption of a Q2D flow is reasonable, but three-dimensional effects may not be neglected at high $Q$. 

![Figure 28 a^4 fitted against the current density with a linear curve for numerical simulations.](image-url)
4. Experimental and Numerical Results

4.1 Case 0: No movement

The case of no magnet translation will serve as base for comparison with the moving magnet cases. This case has been studied before by Emonts [6], though slightly different experimental parameters were used. A disk shaped magnet was used rather than a rectangular magnet and the fluid depth was 5mm. Nonetheless, the resulting flows should be comparable. 2 regimes were identified in [6], for continuous forcing:

- “Regime C: A stable double jet structure”
- “Regime D: An unstable double jet structure”

Regimes “A” and “B” that were identified in [6] apply for cases of a non-continuous forcing and are deemed irrelevant, as a continuous forcing was used in all experiments. The regimes C and D were identified in the laboratory experiments.

Figure 29 Left: Initial dye distribution at $t=0$ s. Right: Dye distribution at $t=10$ s. A stable vortex dipole is produced by the Lorentz force (directed upwards in this figure) at $Ch=2.6\times10^{11}$. This flow is identified as “regime C”.

Figure 29 shows a typical Regime C flow. A stable vortex dipole is observed. But at higher $Ch$ numbers, the jet produced from the Lorentz force becomes unstable. This is shown in Figure 30, a typical Regime D flow. This instability is ascribed to the high velocity shear in the fluid, caused by the strongly localized force. The growing perturbations in the jet are identified as the Kelvin-Helmholtz instability.

Note that in Figure 30 the dye is mostly advected to the right (actually up, but the figure was rotated 90°), but some dye is also advected in the opposite direction. The reason for this becomes clear in Figure 31. The numerical simulation shows the vorticity distribution at $t=1$ for $Ch=2.6\times10^{11}$. The flow structure created by the Lorentz force consists of a vortex dipole core, two oppositely signed patches of vorticity, but also two satellites of oppositely signed vorticity surrounding the vortex dipole core. While the main vortex dipole core is advected to the positive $y$-direction, the satellites form a weaker vortex pair that translates to the negative $y$-direction.
Figure 30 Upper: Dye distribution at \( t=0 \). Lower: An unstable jet is observed at \( Ch=1.5*10^{12} \). This flow can be identified as “regime D”. The Lorentz force is directed to the right (the image was rotated 90° clockwise). The jet is seen to move to the right, but notice how some dye is also transported to the left side.

Figure 31 Vorticity distribution obtained from a numerical simulation at \( t=2 \) s and \( Ch=3.82*10^{11} \). The Lorentz force is directed in the positive y-direction. The black lines are isolines for which the solid contours represent positive vorticity and dotted lines represent negative vorticity. The satellites form a vortex pair that translates in the negative y-direction, whereas the vortex dipole core translates in the positive y-direction.
4.2 Case $L_\perp$: Linear perpendicular movement

Case $L_\perp$ is the set of experiments with a constant velocity for the translation of the magnet and perpendicular to the direction of the force ($V_z \neq 0$). Two parameters have been varied: current density $j$ and magnet velocity $V$. Varying these two parameters effectively varies the dimensionless numbers $Re_V$ and $Q$.

Before each experiment, luminescent dye is placed on the surface of the fluid. While the magnet translates from one side of the water tank to the other the current source is turned on. As the moving Lorentz force adds momentum to the fluid the dye visualizes the resulting flow patterns. Several regimes have been identified from the different flow patterns. These regimes show typical behavior of the flow for certain combinations of $Re_V$ and $Q$. All of these regimes in this case are identified by the starting jet that is induced by the Lorentz force:

- **Regime $L_\perp$ A**: Small/no jet
- **Regime $L_\perp$ B**: Stable jet
- **Regime $L_\perp$ C**: Meandering jet
- **Regime $L_\perp$ D**: Unstable jet

Regimes A to D are ordered by increasing $Q$. A regime diagram is plotted in Figure 32. The boundaries of the regimes may be unclear as the defined regimes do not have a clear transition from one to another. An explanation of each regime follows below.

![Regime diagram for case $L_\perp$: linear perpendicular magnet movement](image)

**Figure 32** Regime diagram for case $L_\perp$: linear perpendicular magnet movement
Regime L⊥A: Small/no jet
The first regime “small or no jet” refers to the behavior where the flow is not forced long or strongly enough to form a noticeable jet flow that transports fluid particles out of the forcing region. The fluid is observed to rotate shortly when the magnet is passed by and does not clearly show a vortex dipole in the fluid flow. This behavior seems to fit the description by the theory described in chapter 2.4. This is backed up by Figure 37, which shows the result of the numerical simulation. A small patch of negative vorticity persists at the initial position of the magnet.

Regime L⊥B: Stable jet
The second regime “stable jet” refers to the flow that does create a clear jet upwards. Initially a vortex dipole is seen and is later stretched and tilted in the direction away from the magnet’s translation direction. This typical type of flow is shown in Figure 34.

This regime is also confirmed in the numerical simulations. Figure 38 shows a typical vorticity distribution for a regime L⊥B. The same behavior is observed: a stable jet is formed above the position of the magnet and tilts in the direction opposite of the magnet’s translation direction.

Regime L⊥C: Meandering jet
The third regime “meandering jet” is where the jet starts to become unstable and show wavy patterns. The dye has the tendency to curl up at several points in the jet. These wavy patterns occur due to the higher Q value. Again, as in “Regime D” for the non-moving magnet, this is triggered by the shear velocity and is recognized as Kelvin-Helmholtz instabilities. This typical type of flow is shown in Figure 35. Figure 39 shows the typical vorticity from the numerical simulations.

Regime L⊥D: Unstable jet
The last regime shows the jet is becoming clearly unstable and shows regions of dye curling up, indicating vortex shedding. This is due to the high values of Q and the effect of non-linearity. This typical type of flow is shown in Figure 36. The vortex shedding is confirmed with the numerical simulations, showing isolated patches of negative vorticity in Figure 40.

Also seen in Figure 36 and Figure 40 is the peculiar vortex that seems to follow the magnet position below and is spatially larger than any other vortex patch. This vortex patch is one of the satellites that appear next to the vortex dipole core. The other satellite is not visible in the dye patterns. As the vorticity of the satellites is much lower than the core, their shape may be explained by the linearized model in chapter 2.4. Figure 40 shows how the negative vorticity satellite is dragged behind and stretched, much like how vortex patches are dragged and stretched in regime L⊥A.

Notes about the figures: All following figures showing vorticity distributions are generated by COMSOL Multiphysics and the color ranges are automated. Text may become very small and hard to read. Some images of laboratory experiments and numerical simulations are flipped or rotated for convenience’s sake or to create a better comparison with other images. In this report the x- and y-directions are always defined as depicted in Figure 2 unless specified otherwise.
Figure 33 Regime 1-A: Dye pattern for $Q = 2.59 \times 10^3$ and $Re = 160$. The magnet moves to the left while the force is directed up. The dye is shortly seen swirling clockwise at the initial position of the magnet. At the current position of the magnet, the dye is seen swirling counter-clockwise shortly.

Figure 34 Regime 1-B: Dye pattern for $Q = 1.30 \times 10^4$ and $Re = 32$ after the start of the magnet movement. The magnet moves to the left while the force is directed up. A vortex dipole is seen in the dye pattern. Upper image: dye pattern at $t=1s$. Lower image: dye pattern at $t=4s$.

Figure 35 Regime 1-C: Dye pattern for $Q = 1.04 \times 10^4$ $Re = 160$. The magnet moves to the left. The jet starts becoming unstable and shows a wavy pattern.
Figure 36 Regime 1-D: Dye pattern for $Q = 6.22 \times 10^4$, $Re_1 = 80$. The magnet moves to the left. A highly unstable jet is observed and vortex shedding occurs. Notice there is a large vortex below where the black arrow is pointing at. This vortex is a “stretched” satellite. A white band of light can be seen at the bottom left corner (left from the arrow), which is a reflection of the light source and should be ignored.
Figure 37 Regime 1-A: Numerically obtained vorticity distribution for $Q=3\times10^2$ and $Re_V=80$. The magnet starts at $x=-0.15m$ and moves to the left while the Lorentz force is directed upwards.

Figure 38 Regime 1-B: Numerically obtained vorticity distribution for $Re_V=80$ and $Q=3.06\times10^3$. The magnet moves to the left while the Lorentz force is directed upwards.
Figure 39 Regime 1-C: Numerically obtained vorticity distribution for $Re_V=80$ and $Q=7.64 \times 10^3$. The jet starts meandering. The magnet moves to the left while the Lorentz force is directed upwards.

Figure 40 Regime 1-D: Numerically obtained vorticity distribution for $Re_V=80$ and $Q=1.53 \times 10^4$. Vortex shedding occurs in the jet. The magnet moves to the left while the Lorentz force is directed upwards.
Persistent patch of vorticity
In chapter 2.4 there was a mention of a “persistent patch of vorticity” which remains at the initial position of the magnet. Figure 33 seems to suggest that this phenomenon is also present in the laboratory experiments. It is confirmed that it is present in the numerical simulations. Figure 41 shows the cross section of $\omega_z(x, y, t)$ at $y = y_{magnet} = 0.08\text{m}$. A patch of negative vorticity remains at the initial position of the magnet.

Figure 41 Numerically obtained vorticity for $Q=3\times10^2$ and $Re_V=80$ (also see Figure 37) along the $y$-position of the magnet at different moments of time. As described by the theory, a persistent patch of vorticity remains at the initial position of the magnet (indicated with the black arrow).

Jet height
Besides the regimes, an interesting observation is the jet height. The theoretical model for the vortex dipole’s movement predicts a trajectory for the vortex dipole that bends to the opposite direction of the magnet velocity. This has also been observed in the laboratory experiments. In order to compare theory and experimental results, the “jet height” is quantified. This is roughly defined as the maximum distance between the magnet’s $y$-position and the dipole’s trajectory, divided by the magnet size $D$ (see Figure 42). For increasing $Q$, the measurement of jet height becomes more difficult in laboratory experiments as the distance of the dipole’s trajectory is less clear. Nonetheless attempts have been made to quantify the jet height and it is compared with the theoretic jet height. This jet height in laboratory experiment is plotted against the dimensionless parameter $Q$ in Figure 43.
Figure 42 Jet height in laboratory experiments. As $Q$ is increased, the vortex dipole’s trajectory (blue line) becomes harder to define. The magnet’s $y$-position (red line) is known. The jet height (yellow line) is defined as the maximum distance between the magnet’s $y$-position and the dipole’s trajectory.

Figure 43 Measured jet height as function of $Q$.

In the theoretical model for the movement of the vortex dipoles, the jet height is defined as the vortex dipole’s $y$-position divided by $D$. However in the model for the vortex dipole movement, the flow is 2D and bottom friction was not taken into consideration. As the dipole moves with less friction a time constant is chosen to measure the vortex dipole’s $y$-position at $\tau = 5D/V$ (see Figure 44). The factor 5 is an empirically chosen constant, as this resulted in a realistic timescale (ranging from 4 to 80 seconds) for the vortex dipole’s movement to sufficiently slow down. The resulting jet height plotted against $Q$ in Figure 45. The dimensionless parameter $Q$ was varied by varying both $V$ and $j$.

Both experimentally determined values and theoretical values are plotted in Figure 46. Though there is a difference of approximately a factor of 2, both curves seem to have a $\sim Q^{0.3}$ dependence.
Figure 44 The arrows depict the velocities in the potential flow, the blue dot the vortex dipole’s position and the red curve the trajectory it has taken. The theoretical jet height is defined as the vortex dipole’s $y$-position at $t=5*D/V$.

Figure 45 Jet height from theoretical calculations. Both the magnet velocity $V$ and current density $j$ were varied to obtain this plot.

Figure 46 Theoretical and experimental values plotted in one figure. The dashed line indicates a $-Q^{0.3}$ dependence.
4.3 Case $L_\parallel$: Linear parallel movement

Measurements for the case of linear movement with parallel alignment to the current density were limited. This case has been extensively studied by Afanasyev et al. and Honji et al. However there is a clear difference between these two studies. Honji et al. describe different regions with wavy and meandering wakes (see Figure 47). Whereas Afanasyev et al. observes vortex streets and ejecting vortex dipoles in the wakes behind the moving magnet (see Figure 48).

During the laboratory experiments, the same patterns as those of Honji et al. and Afanasyev et al. were found for low $Q$: “straight” and “regular wavy” wakes (see Figure 49a and b). However when increasing $Q$, the third regime “irregular meandering” wake was not found. Vortex streets as those observed by Afanasyev et al. were not found either. Instead, vortex dipoles are observed to be ejected from the forcing region (see Figure 49c). This behavior is also observed by Afanasyev et al. for high $Q$ values (Figure 48d). Most likely, the reason for not finding the “irregular meandering” regime by Honji et al. is due to the width of the water tank. In comparison Honji et al. used a water tank that is much narrower: 20 cm and their flow visualizations often show the dye being very close with the boundaries of the water tank.

Only Afanasyev et al. managed to observe vortex streets in the wake of the magnet. The reason for not finding the vortex streets in the laboratory experiments also lies in the differences between the experimental setups. Both the experimental setup in this research and that of Honji et al. consists of a single layer of fluid. Afansyev et al. use two layers of fluid in order to reduce the bottom friction. When the bottom friction is reduced vortex shedding seems to occur at a much earlier time. This is confirmed by the numerical simulations. Figure 50 shows a numerical simulation of a flow with $Q=2.07*10^4$ and with a no-slip condition on the bottom. The vortex dipole that is formed above the magnet is seen to start swaying from side to side and is most likely the starting behavior for the “ejecting dipole” regime as seen in Figure 49c. Vortex shedding does not occur. However when changing the boundary condition at the bottom from no-slip to a slip bottom, a different flow is observed. Figure 51 shows the swaying of the vortex dipole and vortex shedding at a much earlier time step. This is an indication that the bottom friction plays a large role in the vortex shedding process. No clear vortex streets as by Afanasyev et al. were found in the numerical simulations, but a striking difference between the simulations with different boundary conditions is the much higher rate of vortex shedding. This may lead to vortex streets under the right conditions.
Figure 47 Flow patterns for increasing $Q$ from top to bottom, found by Honji et al. (image taken from [10]). Honji et al. used colored dye to visualize the flow. Honji et al. named these patterns: a) “straight wake” b) “regular wavy wake” and c) “irregular meandering”.

Figure 48 Flow patterns for increasing $Q$ from top to bottom, found by Afanasyev et al. (image taken from [2]). Afanasyev et al. used the ph-indicator thymol blue to visualize the flow. For low $Q$ a) and b) are the same regimes observed by Honji et al.: a straight and wavy wake. c) Shows a vortex street in the wake. d) Shows ejecting vortex dipoles.
Figure 49 Flow patterns visualized by dye where $Re=160$ and varying $Q$. The magnet moves from right to left while the Lorentz force is directed to the left.

a) Flow pattern for $Q=5.18 \times 10^2$. For low values of $Q$, the flow shows a straight wake.

b) Flow pattern for $Q=2.60 \times 10^3$. A wavy wake appears for higher $Q$.

c) Flow pattern for $Q=5.19 \times 10^3$. At even higher $Q$, several vortex dipoles are formed and are seen to be ejected from the forcing region.
Figure 50 Numerically obtained vorticity distribution for $Q=2.07 \times 10^4$. The bottom boundary condition is a no-slip condition. Both the magnet translation and the direction of the Lorentz force are in the positive $y$-direction (the image was turned $90^\circ$ counterclockwise). The vortex dipole structure is seen to start swaying from side to side.

Figure 51 Numerically obtained vorticity distribution for $Q=2.07 \times 10^4$. The boundary condition of the bottom was changed to a slip condition. The magnet translation and the direction of the Lorentz force are in the positive $y$-direction. Vortex shedding occurs in the flow. In the wake of the flow, complex vorticity distribution structures are seen. Left image: vorticity distribution at $t=10$. Right image: vorticity distribution at $t=28$. 

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4.4 Case H⊥: Harmonic perpendicular movement

The laboratory experiments for the harmonic movements have some limitations in the possible combinations of frequency and amplitude, as explained in chapter 3.1. The translation stage starts drifting to one side for too high frequencies. Therefore certain regimes are not accessible experimentally. In order to construct a regime diagram including higher frequencies, the results of the laboratory experiments are supplemented with results of the numerical simulations.

An important difference between harmonic cases (H⊥ and H∥) and linear cases (L⊥ and L∥) is that the harmonic cases have three parameters to be varied: amplitude A, frequency f and current density j. These parameters vary the dimensionless parameter S, Qf and Reff. This in comparison to the two parameters in the linear cases j and V, which vary Q and Rev. Finding regimes in a three-dimensional parameter space would be a time consuming undertaking. Instead two slices are taken in the parameter space by holding the dimensionless number S constant at two levels: S = 0.625 and S = 1.875 (A = 10 mm and A = 30 mm respectively). Only the other two parameters f and j are then varied, effectively changing Qf and Reff.

This case is split into two subcases for which S<1 will be referred to as case H⊥ 1 and S>1 as case H⊥ 2.

4.4.1 Case H⊥ 1: S<1

4 regimes are identified for case H⊥ 1. The regime diagram is shown in Figure 52. The explanation for each regime is given hereafter:

- **Regime H⊥ 1-A**: Vortex dipole
- **Regime H⊥ 1-B**: Strong mixing
- **Regime H⊥ 1-C**: Mixing
- **Regime H⊥ 1-D**: Vortex street

![Phase diagram](image)
Regime H⊥1-A Vortex dipole
The first regime is the regime for low forcing $Q_f$. The dye pattern shows a single vortex dipole structure. This pattern was expected from the vorticity distribution model described in the theory, which predicts two oppositely signed vorticity patches are separated by a distance proportional to $A$ (see Figure 13). Though this separation can not clearly be seen in Figure 53, as $A$ is relatively small. Figure 57 shows a slight deformation in the vortex dipole. The separation will become more clear in case 2 where $S=1.875$.

Regime H⊥1-B Strong mixing
The second regime is called “strong mixing”, as the dye pattern becomes well mixed. This regime only appears for low $Re_f$ numbers (at low magnet oscillation frequencies), and large $Q_f$.

Figure 54 shows a complicated pattern, as patches of dye are constantly being stretched by the fluid motion while swaying left and right, effectively folding it. This kind of motion resembles the Baker’s transformation. The fluid filaments soon return to the forcing region above the magnet to be stretched again. Eventually, the dye pattern becomes well mixed.

Regime H⊥1-C Mixing
The third regime is a transition between the previous regime and the next regime: the “mixing” regime. The regime resembles the next regime “vortex street” as the dye patterns show an alternating pattern of swirling dye. The resemblance with the previous regime is that patches of dye get stretched by the fluid motion while swaying side to side. The patches of dye eventually move back to the forcing region, as is shown in Figure 55. Figure 59 shows the vorticity distribution which has a very similar structure to the dye distribution in Figure 55.

Regime H⊥1-D Vortex street
A typical dye pattern for the last regime is shown in Figure 56. This regime is called the “vortex street” regime. The choice of calling this “vortex street” is purely based on the dye pattern observed in the laboratory experiments. Also note that even though this pattern is described as a vortex street, the formation of this vortex street is completely different than, for example, the Von Karman vortex street. Whereas a Von Karman vortex street is formed behind an obstacle in a steady 2D flow, this vortex street appears solely due to the oscillatory motion of the Lorentz force. As the Lorentz force moves quickly side to side, the jet makes short swaying motions while at the same time being propelled upwards in a jet. Doing this for the right values for frequency and current density, patches of fluid particles will form to swirl in alternating directions.

Figure 60 shows the vorticity distribution. From Figure 56 one would perhaps expect the vorticity to be well isolated into patches of negative and positive vorticity (a street of vorticity). However this is not the case, but small peaks are seen in the vorticity distribution, which are sufficient to form the dye pattern seen in Figure 56.
Figure 53 Regime $H_{\perp}$ 1-A: Dye pattern for $Re_f=128$, $Q_f=8.33 \times 10^5$. The force is directed up. A vortex dipole is seen.

Figure 54 Regime $H_{\perp}$ 1-B: Upper image: Dye pattern for $Re_f=25.6$, $Q_f=4.02 \times 10^5$. The Lorentz force is pointed in the $y$-direction while the magnet moves in the $x$-direction. Lower image: the same set up, but after several more oscillations. Eventually, the dye pattern becomes well mixed. (The light band below is the reflection of the light source and should be ignored.)
Figure 55 Regime $H_{\perp}^1$-C: Left: Dye pattern for $Re_f=64, Q_f=6.49 \times 10^4$ for a few oscillations (less than 10). Right: pattern for more oscillations (more than 10).

Figure 56 Regime $H_{\perp}^1$-D: Left: Dye pattern for $Re_f=128, Q_f=1.62 \times 10^4$. Right picture: Pattern for the same set up, but after many more oscillations. The jet above the forcing region may start meandering.
Figure 57 Regime $H_\perp$ 1-A: Numerically obtained vorticity distribution for $Re_f=128$, $Q_f=8.33\times10^2$. A vortex dipole is observed here.

Figure 58 Regime $H_\perp$ 1-B: Numerically obtained vorticity distribution for $Re_f=25.6$, $Q_f=4.02\times10^5$. The jet from the Lorentz force sways left and right heavily.
Figure 59 Regime $H_{\perp}1$-C: Numerically obtained vorticity distribution for $Re_f=64, Q_f=6.49\times10^4$.

Figure 60 Regime $H_{\perp}1$-D: Numerically obtained vorticity distribution for $Re_f=128, Q_f=1.62\times10^4$. 
Time averaged flow
The flow patterns in case H⊥ are often complex looking and seem to promote a high rate of mixing. The areas above the forcing region are subject to a strong time dependence as the position of the force is varying over time. In the case of linear movement (case L⊥) it was seen that when a perpendicular movement is added to the magnet perpendicular to the Lorentz force, the resulting jets not only have momentum in the y-direction, but gain momentum in the x-direction. So the perpendicular movement of the magnet results in the swaying of the jet in the harmonic case H⊥ as the momentum flux in the x-direction constantly changes over time.

Despite the harmonic movement, on average most of the fluid at the forcing region is still transported in the positive y-direction (ignoring the jet in the negative y-direction due to the vorticity satellites surrounding the forcing region). Higher frequencies promote more transport into the y-direction. The far field flow may be approximated by a source-sink dipole. The jet above the forcing region can be interpreted as a source whereas directly below the forcing region fluid is being drawn to as a sink. Figure 61 shows two typical dye patterns for regime H⊥ 1-B and regime H⊥ 1-C.

Figure 61 Left: Typical regime H⊥ 1-B pattern with magnet frequency $f=0.1$ Hz after 3 periods.
Right: Typical regime H⊥ 1-C pattern with magnet frequency $f=0.5$ Hz after 15 periods.

The source-sink dipole approximation can be confirmed by checking the streamlines of the flow. These streamlines are obtained from the velocities in the x- and y-directions from numerical simulations. The velocities are averaged over time to inspect the time averaged streamlines. Dye particles, as shown in Figure 61, follow the field lines in the flow known as streaklines. As a direct comparison the time averaged streamlines are overlaid on Figure 61 and is shown Figure 62. This figure shows that near the forcing region there a net transport of fluid mostly in the positive y-direction. Outside this region, the flow behaves like a sink-flow and the fluid particles return to the lower side of the forcing region. The streamlines coincide quite well with the streaklines. This indicates a time independent, steady flow in the far field of the flow. The streamlines inside the forcing regions do not coincide well, indicating a very time dependent, unsteady flow.
Figure 62 Time averaged streamlines obtained from numerical simulations are overlaid on the results from the laboratory experiments. Edge effects can also be seen in the streamlines, as a consequence of the domain of the numerical simulations.
Dye surface growth

In chapter 2 it was derived that for a stationary magnet the distance travelled by the vortex dipole goes as $\sim t^{1/2}$ and this relation was confirmed in laboratory experiments. In case $H_\perp$, however, it seems that the dye positions grow in both $x$- and $y$-directions. It was established earlier that a moving magnet also adds momentum to the fluid in the $x$-direction. From the observations of the laboratory experiments there is a suspicion that the width and height of the dye surface grows as $\sim t^b$ as well, with $b$ an unknown growth factor. Therefore the width and height of the dye surface above the forcing region was measured as function of time, see Figure 63. The same method is used as described in chapter 3.3: the images are converted to black and white images. The top and sides of the white dye areas are then tracked and measured.

The width and height were measured for laboratory experiments with different frequencies, but at constant current density and amplitude. Some results can be seen in Figure 64 and Figure 65. The width $\Delta x$ and height $\Delta y$ is seen to have an algebraic growth after several oscillations. These are then fitted as $\Delta x \sim at^{b_{x\text{fit}}}$ and $\Delta y \sim at^{b_{y\text{fit}}}$ where $a$ is the value obtained from the calibrations (see chapter 3.3). The results of the fits are listed in Table 1 and Table 2.

Figure 63 A regime $H_\perp \cdot B$ pattern. Blue arrows indicate the height and width of the dye surface above the forcing region.

![Figure 63](image)

Figure 64 Two examples of the dye height as function of time for different frequencies. Blue lines represent the measurements, red lines the fit. The height is fitted after several oscillations.
Figure 65 Two examples of the dye width as function of time for different frequencies.

<table>
<thead>
<tr>
<th>$f$ (Hz)</th>
<th>$b_{\text{height}}$</th>
<th>$b_{\text{width}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1 / 3.78</td>
<td>1 / 5.31</td>
</tr>
<tr>
<td>0.2</td>
<td>1 / 3.74</td>
<td>1 / 3.96</td>
</tr>
<tr>
<td>0.3</td>
<td>1 / 3.82</td>
<td>1 / 3.29</td>
</tr>
<tr>
<td>0.4</td>
<td>1 / 2.58</td>
<td>1 / 2.25</td>
</tr>
<tr>
<td>0.5</td>
<td>1 / 4.32</td>
<td>1 / 2.59</td>
</tr>
<tr>
<td>0.6</td>
<td>1 / 3.44</td>
<td>1 / 2.42</td>
</tr>
<tr>
<td>0.7</td>
<td>1 / 3.76</td>
<td>1 / 2.07</td>
</tr>
</tbody>
</table>

Table 1 Growth factor of the dye height as function of frequency.

Case H⊥: Growth factor of dye width

<table>
<thead>
<tr>
<th>$f$ (Hz)</th>
<th>$b_{\text{width}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1 / 5.31</td>
</tr>
<tr>
<td>0.2</td>
<td>1 / 3.96</td>
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<tr>
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<tr>
<td>0.4</td>
<td>1 / 2.25</td>
</tr>
<tr>
<td>0.5</td>
<td>1 / 2.59</td>
</tr>
<tr>
<td>0.6</td>
<td>1 / 2.42</td>
</tr>
<tr>
<td>0.7</td>
<td>1 / 2.07</td>
</tr>
</tbody>
</table>

Table 2 Growth factor of the dye width as function of frequency.

The growth factor of the height of the dye surface seems approximately constant whereas the growth of the width increases with the frequency. The sum of these factors $b_{\text{height}} + b_{\text{width}}$ gives an indication of the growth of the dye surface $\Delta x \Delta y \sim t^{b_{\text{total}}}$. Figure 66 shows the sum of the growth factors as function of the frequency.

Note that inspecting the growth of the dye surface is fundamentally different than inspecting the movement of the vortex dipole formed by a stationary magnet. The dye surface’s width and height cannot be described by integrating the potential flow velocity over time, as this would only follow the movement of the frontal position of a single vortex dipole. The increasing dye surface should be interpreted as continuous mass flux coming from outside the mixing region. This continuously mixes dyed and undyed fluid and grows. Possibly, for high frequencies the moving magnet could be interpreted as a broader magnet and thus has an increased mass flux.
towards the mixing region. Further research would need to be done to confirm this conjecture.

Finally, it is noted that the dye patterns presented here are very similar to those observed by Wells et al. [18]. In the experiments by Wells et al. a periodic flow in a narrow channel was generated by water elevations to simulate tidal changes. Dye visualization of the flow was applied at the end of the channel. Though in [18] the flow was periodic by changing the magnitude of the momentum flux, which is fundamentally different to the periodicity applied here (by moving the source of momentum flux). The similarity in the dye visualizations is simply due to the existence of the time dependence in the fluid flow.

![Surface growth as function of frequency](image)

Figure 66 Growth factor of the dye surface as function of frequency. There seems to be a linear dependence.
4.4.2 Case $H_{⊥2}: S>1$

The second sub-case discusses the flow patterns where the amplitude is $A=30\text{mm}$, larger than the magnet size. The following regimes are identified. With the exception of the “vortex dipole” regime, the regimes in this case are completely different from case $H_{⊥1}$. The description of every regime follows below. The regimes A to C are ordered by increasing $Q_i$ and the regime diagram is shown in Figure 67.

- **Regime $H_{⊥2}$-A**: Vortex dipole
- **Regime $H_{⊥2}$-B**: Small double vortex dipole
- **Regime $H_{⊥2}$-C**: Double vortex dipole

![Regime diagram](image_url)

*Figure 67 Regime diagram for case $H_{⊥2}$.*
Regime H⊥2-A Vortex dipole

The regime “vortex dipole” for case 2 is essentially the same as the “vortex dipole” in case 1. The forcing parameter $Q_f$ is low in this regime and a vortex dipole emerges with the vorticity patches separated by a distance $A$ as described by the theory. In case 1 the amplitude $A$ was simply too small to notice a clear separation in vorticity patches. This separation can now clearly be seen in Figure 68.

This separation may be explained by the linearized model and Appendix C.2. Again, one patch of vorticity moves with the magnet as the other one stays in place. The magnet moves back and forth, leaving a patch of vorticity at each end. This results in each vortex patch staying in place, separated by a distance proportional to $A$. Each vortex patch is temporarily strengthened when the magnet passes by.

Figure 71 shows the vorticity distribution from a numerical simulation. This vorticity distribution has a high resemblance to Figure 14. The vorticity patches on the left and right side are the two vortices observed in Figure 68. In-between the vorticity distribution is more complex and several small patches of oppositely signed vorticity can be seen. The complex vorticity distribution between the two main vortices can not be distinguished in laboratory experiments using dye visualization.

Regime H⊥2-B: Small double vortex dipole

Because the amplitude $A$ is larger than the magnet size $D$, the fluid has the tendency to form two vortex dipoles. The two pairs of vorticity are not equally sized spatially. This regime is the transition between the single “separated” dipole and the double vortex dipole regimes. As $Q_f$ increases (the frequency lowers or the Lorentz force increases) the fluid flow has more time to produce a vortex dipole at each end of the oscillatory movement. In the next regime, regime H⊥2-C, the double vortex dipoles are formed at each end with vorticity patches that are approximately equally sized spatially. This regime however, describes the transition where the secondary patches of vorticity are smaller than the main ones, but are still noticeable in the fluid visualizations.

Regime H⊥2-C Double vortex dipole

The last regime “double vortex dipole” describes the flow patterns where two vortex dipoles are produced from the Lorentz force (see Figure 70). With sufficient forcing $Q_f$, a vortex dipole is able to emerge from each end of the oscillatory movement. As the magnet moves under the fluid, new vortex dipoles are constantly being formed.

Each vortex dipole is seen to translate away from the magnet at an angle. This may be explained by the same reasons given for regime L⊥A. As the magnet moves in one direction, the vortex dipole translates in the opposite direction due to the potential flow outside the vortex dipole. This potential flow arises from the gradient of the pressure inside the fluid, which has a spatial dependence due to the presence of Lorentz force.
Figure 68 Regime $H_\perp 2$-A: Dye pattern for $Re_f=128$, $Q_f=8.10*10^2$. Two oppositely signed vorticity patches are separated by a distance.

Figure 69 Regime $H_\perp 2$-B: Dye pattern for $Re_f=64$ and $Q_f=3.2414*10^4$. Two vortex start to emerge. One patch of each vortex dipole is smaller than the other.

Figure 70 Regime $H_\perp 2$-C: Dye pattern for $Re_f=25.6$ and $Q_f=2.02*10^5$. Two vortex dipoles are formed by the harmonic movement of the magnet.
Figure 71 Regime H⊥2-A: Numerically obtained vorticity distribution for $Re_f=128$, $Q_f=8.10 \times 10^2$.

Figure 72 Regime H⊥2-B: Numerically obtained vorticity distribution for $Re_f=64$ and $Q_f=3.2414 \times 10^4$.

Figure 73 Regime H⊥2-C: Numerically obtained vorticity distribution for $Re_f=25.6$ and $Q_f=2.02 \times 10^5$. 
4.5 Case $H_{\parallel}$: Harmonic parallel movement

The last case is where the magnet’s movement is parallel to the direction of the Lorentz force. Again, this case is separated into two sub-cases: case $H_{\parallel}1$ with $S<1$ and case $H_{\parallel}2$ with $S>1$. This case however will show that the regimes for the two sub-cases will not be drastically different from each other as was they were in case $H_{\perp}$.

4.5.1 Case $H_{\parallel}1$: $S<1$

Three regimes are identified for case $H_{\parallel}1$ and are ordered by increasing $Q_f$:

- **Regime $H_{\parallel}1$-A**: Vortex Dipole
- **Regime $H_{\parallel}1$-B**: Multiple Vortex Dipoles
- **Regime $H_{\parallel}1$-C**: Unstable Vortex Dipoles

Each regime is described below. The regime diagram is shown in Figure 74.

![Regime diagram for case $H_{\parallel}1$](image-url)
Regime H // 1-A: Vortex dipole
In case of low $Q_f$ a single vortex dipole is formed in the fluid flow. This structure seems to exhibit the same pattern as the theoretically predicted: an “elongated vortex dipole” (see Figure 16). Though its “elongation” was not verified quantitatively. As the magnet moves parallel to the Lorentz force, the source of vorticity spreads along the $y$-direction. Figure 75 shows the dye pattern for this regime where $Q_f$ is low. Figure 78 shows that the vorticity distribution has extra peaks, but does not present a particularly high resemblance to Figure 17.

Regime H // 1-B: Multiple Vortex dipoles
In this regime new dipolar vortex structures are constantly being formed. During each oscillation, when the magnet moves in the same direction as the Lorentz force, a vortex dipole is created and temporarily strengthened as the magnet moves along with it. Afterwards the magnet moves back in the opposite direction, while the previously formed vortex dipole continues to translate in the $y$-direction. The dye pattern for this regime is shown in Figure 76. Each time a new vortex dipole is formed, it is seen to catch up with the previously formed vortex dipole. This process is very similar to the “leap frogging” observed by Yamada & Matsui (1978)\(^3\), though the new vortex dipole was never seen to completely catch up with the previous one. The leap frogging is therefore not complete and the dye from previous vortex dipoles is consistently “pushed” to the sides.

Figure 79 shows the vorticity distribution from a numerical simulation. Each vortex pair is not well isolated from the other vortex pairs, but still several peaks in vorticity are observed indicating the formation of separate vortex dipoles. The peaks never seem to completely catch up with each other, which would result in leap frogging. Instead the vorticity is often seen to weaken and diffuse for every vorticity peak that comes by.

Regime H // 1-C: Unstable vortex dipoles
This regime shows vortex dipoles constantly being formed as in regime H // 1-B. However the difference here is that the vortex dipoles are unstable due to the high values of forcing $Q_f$. Kelvin-Helmholtz instabilities lead to irregular flow in the jet and any following vortex dipole is affected by this flow. The leap frogging effect is not observed at all, as the vortex dipoles move irregularly due to the instabilities. After several oscillations, the dye patterns show complicated structures as the vortex dipoles have the tendencies to sway left and right, which is shown in Figure 77.

\(^3\) Refer to [8] for more details on leap frogging. Essentially leap frogging is the interaction between vortex rings, where one pair of rings is caught up by another pair. In this interaction the front vortex ring pair is slowed down by the back pair and vice versa the back vortex ring pair is sped up by the front pair. Eventually the back vortex ring pair slips through the front pair and the whole process repeats itself.
Figure 75 Regime H||1-A: Dye pattern for $Re_f=128$ and $Q_f=8.10^2$. A vortex dipole is formed.

Figure 76 Regime H||1-B: Dye pattern for $Re_f=64$ and $Q_f=1.62*10^4$. The oscillatory movement causes a succession of vortex dipole formations.

Figure 77 Regime H||1-C: Left: Dye pattern for $Re_f=25.6$, $Q_f=2.03*10^5$. The Lorentz force is pointed upwards while the magnet now moves up and down. Vortex dipoles are seen swaying left and right due to instabilities in the flow. Right picture: the same set up, but after several more oscillations.
Figure 78 Regime H || 1-A: Numerically obtained vorticity distribution for $Re_f=128$ and $Q_f=8.10 \times 10^2$.

Figure 79 Regime H || 1-B: Numerically obtained vorticity distribution for $Re_f=64$ and $Q_f=1.62 \times 10^4$.

Figure 80 Regime H || 1-C: Numerically obtained vorticity distribution for $Re_f=25.6$ and $Q_f=2.03 \times 10^5$. 
More details on regime $H_{\parallel} 1$-B

The fluid flow in this regime shows new vortex dipole structures being created each period of the oscillatory movement of the magnet. One may expect that the first vortex dipole that is formed translates as $\sim t^{1/2}$, as in case of a stationary magnet. The Lorentz force and the vortex dipole remain aligned on the same $x$-position and further away from the forcing region the flow may become steady as in case $H_{\perp} 1$. However Figure 81 shows the vortex dipole does not translate as $\sim t^{1/2}$. The vortex dipoles seem to slow down in time. This may be an indication that the leap frogging motion plays a large role: the newer vortex dipoles continuously slow down the previously formed vortex dipoles and break the $\sim t^{1/2}$ dependence.

![Vortex dipole position in Case IV](image)

Figure 81 Vortex dipole position in log scale for case $H_{\parallel} 1 \, Re_f=64$ and $Q_f=1.62\times10^4$. There is no $t^{1/2}$ dependence.

It was mentioned before that the vortex dipoles never completely catch up with each other, making a full leap frogging motion. Possibly, the reason for vortex dipoles not to make a full leap frogging motion may lie in the bottom friction effect. As the bottom friction has the tendency to slow down the vortex dipole translation velocity, leap frogging may be observed if the effect of bottom friction is reduced. Therefore numerical simulations were performed where the bottom boundary condition was changed to a slip condition. Figure 82 shows the result of this simulation. Still, no leap frogging was observed. The vortex dipoles don’t seem to catch up with each other. No further attempts were done to simulate a leap frogging effect.
Figure 82 Numerically obtained vorticity distribution for $Re=64$ and $Q=1.62 \times 10^4$ with the bottom boundary condition changed to a slip condition. Vortex dipoles are formed but the pairs do not perform a leap frogging motion.
4.5.2 Case $H_{\parallel} 2$: $S>1$

In case $H_{\parallel} 2$, the magnet’s oscillatory movement’s amplitude is larger. However, the regimes identified are the same as that of case $H_{\parallel} 1$. The regime diagram is shown in Figure 83. In comparison with regime $H_{\perp}$, the regimes in case $H_{\parallel}$ do not show such different regimes. The reason for this lies in the way the vortex dipoles are formed, as vortex dipoles are formed by the gradient of the $B$-field in the $x$-direction. Then, a harmonic movement in the $x$-direction is more susceptible to creating different flow structures with different values for $S$ than moving the magnet in the $y$-direction with different values for $S$.

- **Regime $H_{\parallel} 1$-A**: Vortex Dipole
- **Regime $H_{\parallel} 1$-B**: Multiple Vortex Dipoles
- **Regime $H_{\parallel} 1$-C**: Unstable Vortex Dipoles

![Figure 83 Regime diagram for case $H_{\parallel} 2$](image-url)
Regime $H_{2-A}$ Vortex dipole
Regime $H_{2-A}$ resembles a vortex dipole again. Figure 84 shows a result from the laboratory experiments and Figure 87 shows the result of a numerical simulation. The vortex dipole appears to be slightly elongated (again, no quantitative verification was done).

Regime $H_{2-B}$ Multiple vortex dipoles
This regime is very similar to regime $H_{1-B}$. Figure 85 shows a typical dye pattern and Figure 88 shows the vorticity distribution obtained from numerical simulations. Again, vortex dipoles are formed as the magnet moves under the fluid. The only difference with regime $H_{1-B}$ is that the magnet now has more time to strengthen the forming vortex dipole. The incomplete leap frogging motion is observed again and is not much different from case $H_{1}$.

Regime $H_{2-C}$ Unstable vortex dipoles
The last regime is again similar to the previous case, regime $H_{1-C}$. As $Q_f$ is increased, the jet become unstable eventually leading to complicated looking dye patterns. Figure 86 shows results from the laboratory experiments. No numerical simulations were done.
Figure 84 Regime H || 2-A: Dye pattern for regime H || 2-A. \( Re = 64 \), \( Q_i = 3.24 \times 10^3 \).

Figure 85 Regime H || 2-B: Typical dye pattern for regime H || 2-B. \( Re = 64 \), \( Q_i = 3.24 \times 10^4 \).
Figure 86 Regime H 1-2-C: Dye pattern for $Re = 128$, $Qf = 1.62 \times 10^4$. Left: Dye pattern after a few oscillations. Dipole vortices are formed. Right: Dye pattern when the jet become unstable after several more oscillations.
Figure 87 Regime H\textsubscript{2}-A: Numerically obtained vorticity distribution for $Re_f=64$, $Q_f=3.24\times10^3$.

Figure 88 Regime H\textsubscript{2}-B: Numerically obtained vorticity distribution for $Re_f=64$, $Q_f=3.24\times10^4$. The domain is slightly too small as the vortex dipole is seen to reach the edge, though simulations on a larger domain were not done.
5. Conclusions and discussion

Electromagnetic forcing was applied to a thin layer of fluid using a rectangular magnet that was moving below the fluid layer. The magnet was moved according to four different protocols, denoted as case L⊥, L∥, H⊥ and H∥. Using dye visualization and numerical simulations a wide variety of flow structures were observed for every case. Several regimes in the parameter space have been identified in each case.

Characteristic for all cases is the presence of both stable and unstable flow regimes for low and high forcing (Q or Qf) respectively. The unstable flows structures are ascribed to the well known Kelvin-Helmholtz instabilities: growing perturbations in flows with a velocity shear. At low forcing the flow is quite well described by the theoretical model that neglects the non-linear advection term in the Navier-Stokes equation. Most flow structures described by the models were observed in the laboratory experiments and numerical simulations.

Case L⊥

In case L⊥ the theoretical model showed that the vortex dipole changes trajectory when the magnet moves. The vortex dipoles’ trajectory bend to a direction opposite to that of the magnet translation direction, which was also observed in laboratory experiments and numerical simulations. The rate at which the trajectory bends was quantified with the “jet height”. However, the theoretical jet height was off by approximately a factor of 2 compared to the laboratory experiments. It is most likely that non-linear effects and bottom friction play a large role in the trajectory of the vortex dipole, two factors that were not included in the theoretical model.

The theoretical vorticity distribution for a linear moving magnet as shown in Figure 8 is an important result, as this phenomenon appears in several cases. It was shown in Appendix C.2 that the movement of the magnet leads to a separation of vorticity patches. These vorticity patterns are observed experimentally and in numerical simulations for low Q. It was also observed in the shape of the satellites surrounding the vortex dipole core at higher Q, as seen in Figure 36 and Figure 40. Regimes H⊥ 1-A and H⊥ 2-A show vortex dipoles whose vorticity patches are separated with a distance proportional to the amplitude.

Case L∥

In case L∥, no vortex streets like those observed by Afanasyev et al. were found in the laboratory experiments. At high Q vortex shedding occurs in the flow, but the vortices are ejected from the forcing region instead of rearranging themselves into a vortex street. Numerical simulations seem to suggest that this is the consequence of the bottom friction. When the bottom boundary condition was changed to a slip-condition in the numerical simulations, vortex shedding seemed to occur at a much earlier time. Numerical computations at higher values of Q were not done, though these could show the formation vortex streets in the vorticity distribution.

Case H⊥

At high forcing the flow seems to exert a chaotic behavior for case H⊥ 1 (subcase 1 denotes S<1 and subcase 2 denotes S>1). The dye pattern on the fluid surface becomes well mixed after several oscillations of the magnet. It may be possible to
confirm the chaotic nature by identifying Lyapunov exponents and mixing rates (refer to [13] and [17]), however this was not done in the current research.

Outside the mixing region the flow was found to be time independent. The coinciding streaklines and time averaged streamlines suggest a steady flow, whereas inside the mixing region the flow is chaotic and unsteady.

The width and height of the (mixing) dye surface were measured as function of time. The dye surface was found to grow faster for higher frequencies. However, more research is needed to explain why higher frequencies promote the growth of the dye surface.

Furthermore it was observed that the dimensionless group $S$ plays an important role in case $H \perp$, as the regimes for $S < 1$ are completely different from $S > 1$. For $S > 1$ two vortex dipoles may be formed in after one oscillation of the magnet’s movement.

**Case $H \parallel$**

In case $H \parallel$ a particular regime has a resemblance to leap frogging observed by Yamada & Matsui, but a complete leap frogging motion was never observed. The reasons may lie not only in the bottom friction, but also other conditions in the experiments. The vortex dipoles themselves may not be strong or isolated enough. More research and experiments would need to be done if one would try to achieve leap frogging in the flow using translating magnets in electromagnetically forced fluid flows.
6. Bibliography


Appendix A: Magnetic field of a rectangular magnet

Consider a rectangular permanent magnet with its upper surface positioned on the x,y-plane and the center of this surface positioned at the origin. The magnet is $2D$ wide and $d$ thick. We search the analytical expression for the $B$-field around this magnet.

![Diagram of a rectangular magnet with magnetic field lines]

The Maxwell equations describe the magnetic field:

$$
\nabla \cdot B = 0 \\
\n\nabla \cdot H = J_{\text{free}} = 0 \rightarrow H = -\nabla \Psi \\
B = \mu_0 H + \mu_0 M
$$

(A.1)

Where $B$ is the magnetic fields and $M$ is the magnetic dipole moment per volume, or simply called magnetization. $H$ is the magnetic field associated with the free electric current $J_{\text{free}}$. Since there is no free current running through the permanent magnet, $H$ has a potential denoted as $\Psi$.

Assume a homogenous magnetization inside the magnet in the $z$-direction:

$$
M = M_0 e_z \text{ for } |x| < D, \quad |y| < D, \quad -d < z < 0
$$

(A.2)

Combining Maxwell equations from (A.1), the following Poisson equation holds for the potential and magnetization:

$$
\nabla^2 \Psi = \nabla \cdot M
$$

(A.3)

Before continuing, the following dimensionless quantities are defined:
\[ \ddot{x}, \ddot{y}, \ddot{z} = x / D, y / D, z / D \]
\[ \ddot{\Psi} = \Psi / (DM_0) \]
\[ \ddot{M} = M / M_0 \]
\[ \ddot{B} = B / (\mu_o M_0) = B / B_0 \]  \hspace{1cm} (A.4)

The tildes are left out and the dimensionless quantities are assumed from this point onward:

\[ \nabla^2 \dddot{\Psi} = \nabla \cdot M \]
\[ M = e_z \text{ for } |x|<1, |y|<1, \ -\delta < z < 0 \]  \hspace{1cm} (A.5)

where \( \delta = d / R \). For \( \nabla \cdot M \) holds:

\[ \nabla \cdot M = \begin{cases} 
\delta(z+\delta) - \delta(z) & \text{if } |x|,|y|<1 \\
0 & \text{everywhere else} 
\end{cases} \]  \hspace{1cm} (A.6)

Note that \( \delta(z) \) is the dirac delta function (not to be confused with \( \delta \)). The solution of the Poisson equation is:

\[ \dddot{\Psi}(x, y, z) = -\frac{1}{4\pi} \iiint \frac{\nabla \cdot M dx' dy' dz'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \]
\[ = -\frac{1}{4\pi} \int_{-1}^{+1} dx' \int_{-1}^{+1} dy' \left[ \left\{ (x-x')^2 + (y-y')^2 + (z+\delta)^2 \right\}^{-1/2} - \left\{ (x-x')^2 + (y-y')^2 + z^2 \right\}^{-1/2} \right] \]  \hspace{1cm} (A.7)

\[ = F(x, y, z) - F(x, y, z + \delta) \]

where \[ F(x, y, z) = \frac{1}{4\pi} \int_{-1}^{+1} dx \int_{-1}^{+1} dv \frac{1}{\sqrt{u^2 + v^2 + z^2}} \]

This integral can be solved. For example by integrating over \( v \) first:

\[ F(x, y, z) = \frac{1}{4\pi} \int_{-1}^{+1} dx \ln \left( v + \sqrt{u^2 + v^2 + z^2} \right) \bigg|_{v=-1}^{v=+1} \]
\[ \frac{1}{4\pi} \int_{-1}^{+1} dx \left\{ \ln \left( y+1+\sqrt{u^2+(y+1)^2+z^2} \right) - \ln \left( y-1+\sqrt{u^2+(y-1)^2+z^2} \right) \right\} \]  \hspace{1cm} (A.8)

Or integrate over \( u \) first:

\[ F(x, y, z) = \frac{1}{4\pi} \int_{-1}^{+1} dv \left\{ \ln \left( x+1+\sqrt{u^2+(x+1)^2+z^2} \right) - \ln \left( x-1+\sqrt{u^2+(x-1)^2+z^2} \right) \right\} \]  \hspace{1cm} (A.9)

Equations (A.8) and (A.9) are sufficient to determine \( B_x \) and \( B_y \):
\[ B_i = -\frac{\partial \Psi}{\partial x} \]
\[ B_j = -\frac{\partial \Psi}{\partial y} \]  
(A.10)

Define \( P \) for the purpose of a simplified notation:

\[ P(x, y, z) = \frac{1}{4\pi} \ln(y + \sqrt{x^2 + y^2 + z^2}) \]  
(A.11)

Then the \( B \)-field can be written as:

\[ B_x = \begin{align*} 
&= P(x+1, y+1, z+\delta) - P(x+1, y-1, z+\delta) + P(x-1, y-1, z+\delta) - P(x-1, y+1, z+\delta) \\
&= -P(x+1, y+1, z) + P(x+1, y-1, z) - P(x-1, y-1, z) + P(x-1, y+1, z) 
\end{align*} \]  
(A.12)

In order to find the component \( B_z \), the derivative \( \frac{\partial F}{\partial z} \) is needed:

\[ \frac{\partial F}{\partial z} = \frac{1}{4\pi} \int_{-1}^{1} du \left( \frac{1}{y+1+\sqrt{u^2+(y+1)^2+z^2}} \frac{z}{\sqrt{u^2+(y+1)^2+z^2}} \\
- \frac{1}{y-1+\sqrt{u^2+(y-1)^2+z^2}} \frac{z}{\sqrt{u^2+(y-1)^2+z^2}} \right) \]  
(A.13)

And the following integral needs to be solved:

\[ \int \frac{du}{\eta \sqrt{u^2 + \eta^2 + z^2 + u^2 + \eta^2 + z^2}} \]  
(A.14)

Suppose that:

\[ u = \sqrt{\eta^2 + z^2} \sinh(t) \]
\[ du = \sqrt{\eta^2 + z^2} \cosh(t) dt = \sqrt{u^2 + \eta^2} dt \]  
(A.15)

Then the integral becomes:
Define the function \( Q(x, y, z) \) for a simplified notation:

\[
Q(x, y, z) = \frac{1}{2\pi} \arctan \left( \frac{\sqrt{y^2 + z^2} - y \sqrt{x^2 + y^2 + z^2} - \sqrt{y^2 + z^2}}{z} \right) - \frac{1}{2\pi} \arctan \left( \frac{\sqrt{y^2 + z^2} - z \sqrt{x^2 + y^2 + z^2} - \sqrt{y^2 + z^2}}{x} \right)
\]  (A.17)

Then:

\[
\frac{\partial F}{\partial z} = Q(x+1, y+1, z) - Q(x-1, y+1, z) - Q(x+1, y-1, z) + Q(x-1, y-1, z) \]  (A.18)

So \( B_z \) can be expressed as:

\[
B_z = Q(x+1, y+1, z + \delta) - Q(x-1, y+1, z + \delta) - Q(x+1, y-1, z + \delta) + Q(x-1, y-1, z + \delta)
\]  (A.19)

Equations (A.12) and (A.19) form a complete analytical description of a rectangular permanent magnet.
Appendix B: Vector potential of a rectangular magnet

Consider the same magnet from Appendix A. The vector potential $\mathbf{A}$ is found from the surface magnetization current:

$$
A(x, y, z) = \frac{\mu_0}{4\pi} \iint \frac{M \times \mathbf{n}}{(x-x')^2 + (y-y')^2 + (z-z')^2} \, dA'
$$  \hspace{1cm} (B.1)

This magnetization current runs in the $x$-$y$ plane on the four sides of the magnet parallel to the $z$-axis. Note that the term $M \times \mathbf{n}$ does not have any $z$-components and is directed purely in the horizontal directions. Integrate over the four sides of the magnet:

$$
A(x, y, z) = \frac{\mu_0}{4\pi} \int_{-D}^{D} \int_{-D}^{D} \frac{M_z e_y}{(u^2 + v^2 + w^2)} \, dv \, dw + \frac{\mu_0}{4\pi} \int_{-D}^{D} \int_{-D}^{D} \frac{-M_x e_y}{(u^2 + v^2 + w^2)} \, du \, dw + \frac{\mu_0}{4\pi} \int_{-D}^{D} \int_{-D}^{D} \frac{-M_y e_x}{(u^2 + v^2 + w^2)} \, du \, dw + \frac{\mu_0}{4\pi} \int_{-D}^{D} \int_{-D}^{D} \frac{M_x e_x}{(u^2 + v^2 + w^2)} \, du \, dw
$$  \hspace{1cm} (B.2)

Where $u = x - x', v = y - y', w = z - z'$. In a shorter notation:

$$
F(x, y, z) = \int_{-D}^{D} \int_{-D}^{D} \frac{1}{(x^2 + y^2 + w^2)} \, dv \, dw \hspace{1cm} (B.3)
$$

$$
G(x, y, z) = \int_{-D}^{D} \int_{-D}^{D} \frac{1}{(u^2 + v^2 + w^2)} \, du \, dw
$$

$$
A = \frac{\mu_0 M_0}{4\pi} \left( (G(x, y + D, z) - G(x, y - D, z)) e_x + (F(x - D, y, z) - F(x + D, y, z)) e_y \right) \hspace{1cm} (B.4)
$$

Though the integrals are not written out in full, they can be solved analytically. Maple 14 by Maplesoft was used to find the full expression of the vector potential.
Appendix C: Other derivations

C.1 Derivation of $\omega_z$

The integral in (1.26) can be solved by integrating by parts. This is done for each case:

**Case 0: No magnet translation $V_x = V_y = 0$**

$$
\omega_z = \frac{j_0 B_0}{\rho} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{1}{4\pi \nu (t-\tau)} \exp \left( -\frac{(x-x')^2 + (y-y')^2}{4\nu (t-\tau)} \right) \frac{\partial}{\partial x'} (\delta(x') \delta(y')) dx' dy' d\tau
$$

$$
= \frac{j_0 B_0}{\rho} \int_0^\infty d\tau \int_{-\infty}^\infty \left[ G(x', y', \tau) \delta(x') \delta(y') \right]_{-\infty}^\infty - \int_{-\infty}^\infty \delta(x') \delta(y') \frac{\partial}{\partial x'} G(x', y', \tau) dx'
$$

$$
= \frac{j_0 B_0}{\rho} \int_0^\infty d\tau \left[ 0 - \int_{-\infty}^\infty \delta(x') \delta(y') \frac{-2(x-x')}{4\pi \nu (t-\tau)^2} \exp \left( -\frac{(x-x')^2 + (y-y')^2}{4\nu (t-\tau)} \right) dx' \right] dy' \quad \text{(B.1)}
$$

$$
= \frac{j_0 B_0}{\pi \rho} \int_0^\infty \frac{2x}{\pi^2 (t-\tau)^2} \exp \left( -\frac{x^2 + y^2}{4\nu t} \right) d\tau
$$

$$
= \frac{j_0 B_0}{\pi \rho} \int_0^\infty \frac{2x}{\pi^2 (t-\tau)^2} \exp \left( -\frac{x^2 + y^2}{4\nu t} \right) d\tau
$$

Note that the last integral for time may be solved. However, this integral becomes more difficult for the other cases.

**Case L $\perp$ & H $\perp$: Perpendicular movement $V_x \neq 0, V_y = 0$**

Equation (1.26) is solved again, but for $V_x \neq 0$:

$$
\omega_z = \frac{j_0 B_0}{\rho} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{1}{4\pi \nu (t-\tau)} \exp \left( -\frac{(x-x')^2 + (y-y')^2}{4\nu (t-\tau)} \right) \frac{\partial}{\partial x'} (\delta(x'-V_x \tau) \delta(y')) dx' dy' d\tau
$$

$$
= \frac{j_0 B_0}{\rho} \int_0^\infty d\tau \int_{-\infty}^\infty \left[ G(x', y', \tau) \delta(x'-V_x \tau) \delta(y') \right]_{-\infty}^\infty - \int_{-\infty}^\infty \delta(x'-V_x \tau) \delta(y') \frac{\partial}{\partial x'} G(x', y', \tau) dx'
$$

$$
= \frac{j_0 B_0}{\rho} \int_0^\infty d\tau \left[ 0 - \int_{-\infty}^\infty \delta(x'-V_x \tau) \delta(y') \frac{-2(x-x')}{4\pi \nu (t-\tau)^2} \exp \left( -\frac{(x-x')^2 + (y-y')^2}{4\nu (t-\tau)} \right) dx' \right] dy' \quad \text{(B.2)}
$$

$$
= \frac{j_0 B_0}{\rho} \int_0^\infty \frac{2(x-V_x \tau)}{4\pi \nu^2 (t-\tau)^2} \exp \left( -\frac{(x-V_x \tau)^2 + y^2}{4\nu (t-\tau)} \right) d\tau
$$

**Case L $\parallel$ & H $\parallel$: Parallel movement $V_x = 0, V_y \neq 0$**

Solving the equation for this case is completely analogous to the previous case:

$$
\omega_z = \frac{j_0 B_0}{\rho} \int_0^\infty \frac{2x}{4\pi \nu^2 (t-\tau)^2} \exp \left( -\frac{x^2 + (y-V_y \tau)^2}{4\nu (t-\tau)} \right) d\tau \quad \text{(B.3)}
$$
Combining the equations (B.1) to (B.3) leads to the generalized solution (1.27).

### C.2 Moving patches of vorticity in case $L_\perp$

Figure 7 shows a typical flow structure in case $L_\perp$, where it seems that one patch of vorticity moves with the magnet while the other patch of oppositely signed vorticity remains at the initial position of the magnet. It can be shown that this effect is purely the consequence of the movement of the Lorentz force. Consider expression (1.43) for the pressure. This term may be a substitution for the pressure term in the Navier-Stokes:

$$
\frac{\partial u}{\partial t} + (\nu \cdot \nabla) v = -\frac{1}{\rho} \nabla (j \cdot A) + \frac{1}{\rho} (j \times B) + \nu \nabla^2 v \quad (B.4)
$$

The first term on the right-hand side may be decomposed into two terms:

\[ \nabla (j \cdot A) = j \times (\nabla \times A) + A \times (\nabla \times j) + (j \cdot \nabla) A + (A \cdot \nabla) j \]

\[ = j \times (\nabla \times A) + 0 + (j \cdot \nabla) A + 0 \quad (B.5) \]

Which results in:

$$
\frac{\partial u}{\partial t} + (\nu \cdot \nabla) v = -\frac{1}{\rho} (j \cdot \nabla) A + \nu \nabla^2 v \quad (B.6)
$$

To simplify this equation, the nonlinear convection term is neglected. In addition, the diffusion term is also neglected (recall from chapter 2.2 that the advection by force timescale is the smallest and dominates over the other timescales). What remains is the change of the flow velocity only due to the Lorentz force and the movement of the magnet:

$$
\frac{\partial u}{\partial t} = -\frac{j_0}{\rho} \frac{\partial}{\partial x} A(x-V_t, y, z) \quad (B.7)
$$

The variable transformation (1.50) is applied and the equation becomes:

$$
\frac{\partial u}{\partial \tilde{t}} - \nu \frac{\partial u}{\partial \tilde{x}} = -\frac{j_0}{\rho} \frac{\partial}{\partial \tilde{x}} A(\tilde{x}, \tilde{y}, \tilde{z}) \quad (B.8)
$$

The particular solution and homogeneous solution are:

$$
\mathbf{u}_p = \frac{j_0}{\rho} \frac{A}{V} \quad (B.9)
$$

$$
\mathbf{u}_h = F(\tilde{x} + Vt) \quad (B.10)
$$

So that the full solution becomes:
\[ u = \frac{j_0}{\rho} \frac{A}{V} + F(\tilde{x} + Vt) \]  

(B.11)

Where \( F \) is any kind of function that translates with a velocity \( V \). This function is derived from the initial condition:

\[ u(t = 0) = 0 \]
\[ \frac{j_0}{\rho} \frac{A}{V} + F(\tilde{x}) = 0 \]  

(B.12)

\[ F(\tilde{x}) = -\frac{j_0}{\rho} \frac{A}{V} \]

So the solution for the velocity field that is only affected by the Lorentz force is as follows:

\[ u(x, y, z, t) = \frac{j_0}{\rho} \frac{A(\tilde{x}, \tilde{y}, \tilde{z})}{V} - \frac{j_0}{\rho} \frac{A(\tilde{x} + Vt, \tilde{y}, \tilde{z})}{V} \]
\[ = \frac{j_0}{\rho} \frac{A(x - Vt, y, z) - A(x, y, z)}{V} \]  

(B.13)

This solution for the velocity is a linear combination of two vector potential terms: one stationary and one moving. Figure 89 shows the vector potential of the magnet at \( z=0 \). This is indicating that the flow consists of swirling structures. Figure 90 shows the velocity field for a moving magnet.

![Vector potential A(x,y,0)](image_url)

Figure 89  An arrow plot of the vector potential of a rectangular magnet. The black box shows the position of the magnet.
Figure 90 A velocity plot where the arrows indicate the flow velocity. The translation speed of the magnet is $V = D/\tau$ where $\tau = 1\text{s}$. The black box indicates the position of the magnet at the current timestep. At $t=0.01\text{s}$ the flow shows a vortex dipole. As the magnet translates one swirling structure moves with the magnet while the stays at the origin.

It was found that indeed the flow consists of two “moving patches of vorticity”. One that moves with the magnet and one that stays stationary. This result was obtained by neglecting viscous effects and non-linear effects of the fluid.
Appendix D Technology Assessment

Vortices in Q2D flows have been the subject in many studies as they are commonly found in nature. Large vortex structures are often found in the atmosphere. For example the polar vortex which has received considerable attention due to ozone depletion. Other examples are the clockwise and anti-clockwise circulations around high and low pressure areas, a phenomenon that can determine local weather up to a week. Vortices are also found in oceans. An example is the Gulf Stream at the east coast of North America which frequently sheds vortices that can persist for several months. The interactions between vortex dipoles form the basis for 2D turbulence. The understanding of this phenomenon is fundamental in studies of meteorology and oceanography. This report describes the study of the dynamics of vortex dipoles generated by electromagnetic forcing in quasi-two-dimensional flows, where the focus is mainly on the behaviour of the fluid flow when the forcing has a time dependence.