Dry friction energy dissipation
with an appendix on stabilization via vibrations

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Preface

This master’s thesis consist of two parts which have little in common. The main part provides theoretical and experimental analysis of energy dissipation through Coulomb friction damping. The other part is a study of the stabilization of linear time-invariant (LTI) systems via parameter vibrations. So, while in the latter subject the vibrations are used to stabilize a LTI system in the former subject the vibrations are damped using dry friction.

I started my master’s thesis with the latter subject. Due to personal matters I could not treat the subject to a satisfactory level, and this part can be viewed as an introduction into the subject. This part is presented as an appendix. Subsequently, the main subject started; this project has both a theoretical and experimental part. The theoretical part demonstrates the equivalence between the model used in the literature and the model of the experimental setup. So, the experimental results can be compared to the literature simulation results. Finally, the measurements validate the simulation results. This is a pretty nice result for a quite simple setup.
Summary

The subject of the thesis is the energy dissipation through dry friction. Literature provides theoretical solutions which are based on the analysis of a 1-DOF model where a mass is subject to friction on a periodically moving base. It is shown that, in case of realistic parameters, friction models as Stribeck and viscous damping or arc-tangent in the stick-phase give energy dissipation curves similar as with Coulomb friction. The aim of this thesis is to provide experimental data to validate this theory. Therefore, experiments are performed with a mass with dry friction and the energy dissipation is measured without precise knowledge of the friction. Then, the measured energy dissipation should correspond with the theoretical solutions of energy dissipation through Coulomb friction.

The experimental setup is modeled as a mass sliding on a stationary base and excited by a periodic force. The model of the setup and the literature simulations are shown to be equivalent in relative velocity using a certain factor for the excitation force. The models are expanded by connecting the mass with a spring and a damper to the world. Equivalence in energy dissipation is shown for small values for the stiffness and damping. Since this is valid for the experimental setup the measurements are comparable with the simulation results from the literature.

The experimental setup consists of a shaker attached to a sledge via a steel needle. The sledge is slides in a parallel bearing. The friction is created by clamping a hard friction plate between two bearing balls. A force cell measures the force of the shaker to the sledge. The velocity of the sledge is measured by a laser interferometer. By increasing the force amplitude of the shaker the normalized friction force can be decreased and the energy dissipation through the dry friction can be measured for a range of normalized friction forces.

Experiments are performed and the measurements are compared to the analytical model with Coulomb friction. The normalized friction force is estimated in two ways. One is to use the transition friction of stick-slip to continuous sliding from the model as calibration and the other is to estimate the friction force using the excitation force friction at release in the stick-slip face. Using the latter method the measured energy dissipation curves are shifted to higher normalized friction forces compared with the analytical model. An explanation is that the measured friction force is the static friction which is higher than the dynamic friction which is in concern if the mass is moving and energy is dissipated.

Using the first method of estimating the normalized friction force results in a reasonably well correspondence between the measured and theoretical energy dissipation curve. The disadvantage of this method is that the measured normalized transition friction force is by definition at the theoretical transition value. For most measurements, the energy dissipation curve is similar as the theoretical curve. The maxima of energy dissipation are at corresponding height and at corresponding normalized friction force. Only the measured energy dissipation values for the stick-slip mode are significant higher than the theoretical values. This can be explained by viscous damping in the stick-phase.

Therefore, it is concluded that the free mass model with Coulomb friction is accurate enough to be used to predict the maximum energy dissipation through dry friction. This is a great advance in designing dry friction contacts to dissipate energy of undesired vibrations.

Further investigations to withdraw the dynamical friction force from the measurements are recommended. Then it is possible to compare the measured and theoretical energy dissipation curves without assuming the model to estimate the friction force. Therefore it is recommended to design a experimental setup where the mass moves really free apart from the desired dry friction and where the friction force is adjustable and observable.
Samenvatting

Het onderwerp van dit rapport is de energie dissipatie door droge wrijving. De literatuur verschaft theoretische analyses die gebaseerd zijn op een 1-dimensionaal model waarbij een massa wrijving ondervindt door de periodiek bewegende ondergrond. Simulatie resultaten in de literatuur tonen dat, voor realistische parameters, wrijvingsmodellen als Stribeck en visceuze demping of een arc-tangent functie voor de stick-fase ongeveer dezelfde energie dissipatie curves geven als die van Coulombse wrijving. Het doel van dit rapport is om deze simulatie resultaten te valideren door middel van meetgegevens. Daarvoor zijn experimenten uitgevoerd met een slede, geëxciteerd door een periodieke kracht, met een element voor droge wrijving zonder specifieke kennis van de wrijving. De excitatie kracht en snelheid van de slede worden gemeten. Verwacht wordt dat dan de daaruit gecalculeerde energie dissipatie ten opzichte van de genormaliseerde wrijvingskracht overeenkomt met die uit de theoretische analyse.

De experimentele opstelling is gomodelleerd als een massa met een wrijvingscontact met de stationaire ondergrond en geëxciteerd door een periodieke kracht. Dit model en het model dat voor de simulaties uit de literatuur is gebruikt zijn equivalent in relatieve snelheid van de massa ten opzichte van de ondergrond voor een bepaalde constante factor. Wanneer de modellen worden uitgebreid met een veer en demper aan de massa dan blijft de equivalentie gelden voor kleine veer en demping constanten. Omdat dit geldt voor de experimentele opstelling is die daardoor geschikt is om meetresultaten te geven die te vergelijken zijn met de simulatie resultaten uit de literatuur. De experimentele opstelling bestaat uit een shaker die met behulp van een stalen naald verbonden is met een slede. De slede wordt geleidt door twee parallelle kogellagers. The droge wrijving wordt verkregen door een keramische plaat tussen twee stalen kogels the klemmen. Een krachtsensor meet de excitatiekracht van de shaker op de slede en een laser interferometer meet de snelheid van de slede. Als de kracht amplitude van de shaker wordt vergroot, verkleint dit de genormaliseerde wrijvingskracht. Hierdoor kan de energiedissipatie door de wrijving worden gemeten voor een scala van genormaliseerde wrijvingskrachten.

De experimenten zijn uitgevoerd en de meetgegevens zijn vergeleken met het analytische model met Coulombse wrijving. The genormaliseerde wrijvingskracht is geschat door middel van twee methoden. De een gebruikt de wrijvingskracht van de overgang van stick-slip naar continu glijden uit het model als ijpunt. De ander sacht de wrijvingskracht door de excitatie kracht te meten op het moment dat de slede loslaat in de stick-slip fase. Bij deze laatste methode zijn de genometen energiedissipatiecurven vergeleken met die van het analytische model, verschoven naar hogere waarden van genormaliseerde wrijvingskrachten. Een verklaring hiervoor is dat de genometen wrijvingskracht de statische wrijving is in plaats van de dynamische wrijvingskracht die van belang is voor de energie dissipatie tijdens het glijden van de slede.

De eerste methode om de genormaliseerde wrijvingskracht te schatten resulteert in een redelijk goed overeenkomende curve van de gemeten genormaliseerde energiedissipatie met die van het theoretisch model. Het nadeel is alleen dat bij deze methode de transitie van stick-slip naar continu glijden per definitie op de theoretische waarde van de genormaliseerde wrijvingskracht plaats vindt. De maxima van de genormaliseerde energiedissipatiecurven komen overeen met de theoretische maxima en liggen op de theoretische waarden van de genormaliseerde wrijvingskracht. Alleen in de stick-slip fase liggen de genormaliseerde energiedissipatiecurven hoger dan de theoretische. Dit kan worden verklaard door visceuze demping in de stick fase. Hieruit wordt geconcludeerd dat dat het model met een vrijbewegende massa met Coulombse wrijv-
ing voldoende accuraat is om de maximum energiedissipatie te kunnen berekenen. Omdat dit een erg eenvoudig model is, is dit een groot voordeel bij het ontwerpen van wrijvingscontacten voor energiedissipatie.

Verdere onderzoeken om de dynamische wrijvingskracht te kunnen meten zijn aanbevolen. Dan wordt het mogelijk om de experimentele en theoretische energiedissipatiecurven met elkaar te vergelijken zonder het theoretische model nodig te hebben voor de schatting van de genormaliseerde wrijvingskracht in de meting. Daarom is het aanbevolen om een nieuwe opstelling te ontwerpen waarbij de slede, op de gewenste droge wrijving na, wrijvingsloos beweegt en waarbij die gewenste wrijving instelbaar is.
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<tr>
<td>$m$</td>
<td>mass (of the sledge)</td>
<td>$kg$</td>
</tr>
<tr>
<td>$k$</td>
<td>spring stiffness</td>
<td>$N/m$</td>
</tr>
<tr>
<td>$c$</td>
<td>damping constant</td>
<td>$N/m/s$</td>
</tr>
<tr>
<td>$\omega_n$</td>
<td>natural frequency</td>
<td>$Hz$</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>damping ratio</td>
<td>$-$</td>
</tr>
<tr>
<td>$\omega_d$</td>
<td>damped natural frequency</td>
<td>$Hz$</td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>excitation frequency</td>
<td>$Hz$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>displacement sledge</td>
<td>$m$</td>
</tr>
<tr>
<td>$x_0$</td>
<td>displacement base</td>
<td>$m$</td>
</tr>
<tr>
<td>$v_{rel}$</td>
<td>relative velocity between sledge and base</td>
<td>$m/s$</td>
</tr>
<tr>
<td>$X_0$</td>
<td>amplitude displacement base</td>
<td>$m$</td>
</tr>
<tr>
<td>$F_0$</td>
<td>excitation force</td>
<td>$N$</td>
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<tr>
<td>$F_r$</td>
<td>friction force</td>
<td>$N$</td>
</tr>
<tr>
<td>$f_r$</td>
<td>normalized friction force</td>
<td>$-$</td>
</tr>
<tr>
<td>$f_r</td>
<td>_{\text{max}}$</td>
<td>normalized friction force at maximum normalized energy dissipation</td>
</tr>
<tr>
<td>$f_r</td>
<td>_{\text{threshold}}$</td>
<td>normalized friction force at the threshold of stick-slip and continuous sliding</td>
</tr>
<tr>
<td>$E_d$</td>
<td>energy dissipation</td>
<td>$J$</td>
</tr>
<tr>
<td>$e_d$</td>
<td>normalized energy dissipation</td>
<td>$-$</td>
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<tr>
<td>$e_d</td>
<td>_{\text{max}}$</td>
<td>maximum normalized energy dissipation</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
<td>$s$</td>
</tr>
<tr>
<td>$\cdot$</td>
<td>first derivative in time</td>
<td></td>
</tr>
<tr>
<td>$\cdot\cdot$</td>
<td>second derivative in time</td>
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Chapter 1

Introduction

1.1 Frictional damping

Damping of structural vibrations is a well known application of dry friction. It is a simple and inexpensive method to increase the damping of a system. Den Hartog, [den Hartog 1931], was the first to obtain analytical solutions for a mechanical system with coulomb and viscous friction. Since then, more friction models and computational methods have been developed to describe the phenomenon of dry friction, see [Putra 2004], [Leine et al. 1998]. The computational methods presented in [Popp et al. 2003] are able to predict the frictional damping effectively so that it can be utilized to decrease vibrations in beam-like structures and turbine blades. Another area of interest concerning frictional damping is the use of friction dampers in buildings to prevent failure in case of earthquakes, see [Liao et al. 2004], [Moreschi & Singh 2003], [Ng & Xu 2006].

Furthermore, extensive investigation of friction damping specific to railway systems is done by [Lopez 1998]. On-site measurements proved that the dry friction damping is very effective in reducing squeal noise from railway wheels. In continuation to this work, the energy dissipation through coulomb friction for 1-DOF systems is investigated in [Lopez et al. 2004]. The existence and stability of a periodic solution is demonstrated and the energy dissipation per cycle is determined as a function of the system parameters. In [Lopez & Nijmeijer 2005] the energy dissipation curve of the simulated 1-DOF system with Coulomb friction is compared with energy dissipation curves of the 1-DOF system simulated with other friction models. In case of realistic parameters, only small differences are observed between the simulated energy dissipation curves of the 1-DOF system with Coulomb friction and the energy dissipation curves where the friction in the stick-phase is modeled by viscous damping or an arc-tangent function. Moreover, for friction models in the slip-phase like Stribeck effect and viscous damping, the energy dissipation curve is of similar shape as for the standard Coulomb friction model although the optimum friction force for maximum energy dissipation is shifted to higher friction values.

Although there have been measurements on train-wheels and on trains in daily environment which provided sufficient data to support the effectiveness of frictional damping, see [Lopez 1998], there is no experimental data which supports the analytic solution and simulation results of the frictional energy dissipation in a 1-DOF system.

The simulation results in [Lopez et al. 2004] show that the shape of the energy dissipation curve relative to the normalized friction force is similar for various friction models. Hence, an experiment where the energy dissipation is measured, can confirm the simulation results, even if the friction behavior of the experimental setup is not (precisely) known. If the measured energy dissipation has a similar shape as the analytical it confirms the simulation results. Then, the measurements will demonstrate the feasibility of frictional damping since the optimal energy dissipation can easily be predicted without exact knowledge of the friction behavior.
1. Introduction

1.2 Objectives

The objective of this study is to provide, using a small-scale experimental setup, empirical data on energy dissipation through dry friction and to compare the data with the analytical solution of the single degree of freedom system with coulomb friction and other friction models. The work is divided in three steps

- building an experimental setup
- performing experiments
- comparing the experimental results with the solutions of the theoretical analysis.

1.3 Outline

To achieve this objective, first the analytical solution of the 1-DOF mass-with-Coulomb-friction model of the experimental setup is provided and is proved to be equivalent with the model from [Lopez et al., 2004] in Chapter 2. Also, the effect of adding a damper or spring to the mass on the frictional energy dissipation is investigated. In Chapter 3 the experimental setup is presented. The measurement method is described and the measurement results are shown. After that in Chapter 4 the normalized measurement results are compared with the solutions of the theoretical analysis. Finally conclusions are drawn and recommendations are made.
Chapter 2

Theoretical analysis of periodically forced systems with friction

In this chapter the analytical expressions are provided for the normalized friction force and the normalized energy dissipation of two kinds of single degree of freedom (1-DOF) systems with coulomb friction. One kind is the model used in previous investigations done by Lopez (1998) and the other is the model of the experimental setup. First the two models are introduced and their correspondence is shown. After that, the analytical solutions of the models are given. From this the transition from one system to the other will be presented. This will form the basis to provide expressions to compare the experimental results with the investigations of Lopez et al. (2004). After that numerical calculations are provided which support the analytical results. Finally conclusions are derived from the analytical model especially concerning the experiments.

2.1 Introduction

The two 1-DOF systems, studied in this chapter, are depicted in Figure 2.1. The first, a mass on a periodically moving base, see Figure 2.1(a), is used in Lopez (1998), Lopez et al. (2004) and Lopez & Nijmeijer (2005). The second, a mass with a periodic excitation force, see Figure 2.1(b), is used as a model for the experimental setup. The system with periodic excitation force is equivalent in relative velocity, and thus in friction force, with the system with periodic moving base for a given scaling factor and phase shift of the periodic excitation force.

\[ m \ddot{x}_1 = F_r (v_{rel}, \dot{x}_0) \]  \hspace{1cm} (2.1)

Figure 2.1: 1-DOF systems, (a) mass on a moving base and (b) mass driven by periodic force.
2. Theoretical analysis of periodically forced systems with friction

where the friction force, \( F_r (v_{rel}, \dot{x}_0) \) is a function of the relative velocity, \( v_{rel} = \dot{x}_1 - \dot{x}_0 \) and the acceleration of the base, \( \ddot{x}_0 \). The equation of motion of the system in Figure 2.1(b) is

\[
m \ddot{x}_1 = F_0 - F_r (v_{rel}, F_0)
\]  

(2.2)

where the friction force, \( F_r (v_{rel}, F_0) \) is a function of the relative velocity, \( v_{rel} = \dot{x}_2 \) and the excitation force, \( F_0 \).

The two 1-DOF systems are extended by connecting the mass to the world by a spring (and damper) as is depicted in figure 2.2.

![Figure 2.2: 1-DOF systems with spring and damper, (a) mass on a moving base and (b) mass excited by periodic force.](image)

The equations of motion for the system depicted in Figure 2.2(a) is

\[
\ddot{x}_1 = -2\zeta \omega_n \dot{x}_1 - \omega_n^2 x_1 + \frac{1}{m} F_r (v_{rel}, \dot{x}_0)
\]  

(2.3)

and for the system in Figure 2.2(b) the equation of motion is

\[
\ddot{x}_2 = -2\zeta \omega_n \dot{x}_2 - \omega_n^2 x_2 + \frac{1}{m} (F_0 - F_r (v_{rel}, F_0))
\]  

(2.4)

where \( \omega_n = \sqrt{k/m} \) is the natural frequency and \( \zeta = c/2m\omega_n \) is defined as the damping ratio.

Equivalence in friction energy dissipation is achieved by equivalence in relative velocity, \( y(t) \). In the 1-DOF system with excitation force the relative velocity is the absolute velocity and in the 1-DOF system with moving base the relative velocity is the velocity of the mass minus the velocity of the base. In stick-phase the motion is given by

\[
\begin{align*}
    m \ddot{x}_1 &= m \ddot{x}_0 \quad \text{moving base} \\
    m \ddot{x}_2 &= 0 \quad \text{excitation force}
\end{align*}
\]

Then the friction force for the moving base system is

\[
f_r = m\ddot{x}_1 + kx_1 + c\dot{x}_1
\]  

(2.5)

in relative state, \( y(t) = x_1(t) - x_0(t) \), this is

\[
f_r = m\ddot{y} + ky + c\dot{y} + m\ddot{x}_0 + kx_0 + c\dot{x}_0
\]  

(2.6)

Substituting the friction force \( f_r = -Fr \ \text{sign} \ \dot{y} \), in (2.6) gives

\[
m\ddot{y} + ky + c\dot{y} = -Fr \ \text{sign} \ \dot{y} - m\ddot{x}_0 - kx_0 - c\dot{x}_0
\]  

(2.7)

Since \( y(t) = x_2(t) \) for the 1-DOF system with excitation force it can be concluded from (2.4) and (2.7) that the 1-DOF system with excitation force is equivalent with the 1-DOF moving base system if the excitation force is given by

\[
F_0(t) = -m\ddot{x}_0 - kx_0 - c\dot{x}_0
\]  

(2.8)
2.2 Free mass system

For $\ddot{x}_0(t) = -X_0 \omega_0^2 \sin \omega_0 t$ equation $[2.8]$ can be written as follows

$$F_0(t) = m X_0 \omega_0^2 \left[ \left( 1 - \frac{\omega_0^2}{\omega_n^2} \right) \sin \omega_0 t - 2 \zeta \frac{\omega_n}{\omega_0} \cos \omega_0 t \right] \tag{2.9}$$

This can be rewritten as an amplitude and phase shift of a sine wave as

$$F_0(t) = -m X_0 \sqrt{\left( \omega_n^2 - \omega_0^2 \right)^2 + (2 \xi \omega_n \omega_0)^2} \sin (\omega_0 t - \theta) \tag{2.10}$$

with

$$\cos \theta = \frac{\omega_n^2 - \omega_0^2}{\sqrt{\left( \omega_n^2 - \omega_0^2 \right)^2 + (2 \xi \omega_n \omega_0)^2}} \quad \text{and} \quad \sin \theta = \frac{2 \xi \omega_n \omega_0}{\sqrt{\left( \omega_n^2 - \omega_0^2 \right)^2 + (2 \xi \omega_n \omega_0)^2}} \tag{2.11}$$

Thus if the mass is free or only attached by a spring then only an amplitude scaling factor is needed to make the two systems equivalent. If a damper is involved a phase shift is also necessary. This makes a sledge that moves as a free mass or a mass attached with springs most useful as experimental setup. In the next section the differences in frictional energy dissipation between both systems is presented for the case of a free mass, a spring-mass and a spring-damper-mass.

2.2 Free mass system

In this section the analytic solution is derived of the 1-DOF system with only a friction and an excitation force working on the mass as is depicted in Figure 2.1(b). In the previous section it is shown that this system is equivalent in relative velocity to the 1-DOF mass on a periodically moving base with $x_0(t) = X_0 \sin \omega_0 t$ by choosing

$$F_0 = X_0 m \omega_0^2 \tag{2.12}$$

In this section it will be shown that the energy dissipation through friction in steady state motion for both systems is equal as well. The analytic solution of the 1-DOF system with a periodically moving base can be found in Chapter 2.2 of [Lopez 1998]. There it is proven that $2.1$ has a steady state motion with a period $t = 2 \pi / \omega_0$.

Let us analyse the response of the block during one cycle of the periodic force. It is assumed that the system has a stable periodic solution with the same period as the periodic force and initial conditions $x(t_1) = \dot{x}(t_2) = 0$ at time $t_1$ slip occur for the stick-slip mode or change of direction in case of the continuous sliding mode. At time $t_2$ stick occur for the stick-slip mode or change of direction for continuous sliding mode.

The acceleration and velocity for the mass sliding in negative direction are

$$\begin{align*}
\ddot{x}(t) &= \frac{F_0}{m} \sin \omega_0 t - F_r / m \\
\dot{x}(t) &= \frac{F_0}{m} (\cos \omega_0 t - \cos \omega_0 t_1) - F_r / m (t - t_1) \quad \text{for} \quad t_1 \leq t \leq t_2.
\end{align*} \tag{2.13}$$

In the stick slip mode $\ddot{x}(t_1) = 0$ and the periodic force on the mass is equal to the value of the friction force so that $0 = F_0 \sin \omega_0 t_1 - F_r$ from which follows that

$$F_r = \sin \omega_0 t_1 = \frac{F_r}{F_0} \tag{2.14}$$

The normalized friction parameter $f_r = F_r / F_0$ is equivalent to $f_r$ of the moving base system, depicted in Figure 2.1(a) with $F_0 = -X_0 m \omega_0^2$, see [Lopez 1998].

The mass sticks again or changes direction when the velocity becomes zero due to the force on the mass at $t = t_2$. Combining $[2.13]$ and $[2.14]$ gives

$$\omega_0 f_r (t_2 - t_1) = \cos \omega_0 t_1 - \cos \omega_0 t_2. \tag{2.15}$$
2. Theoretical analysis of periodically forced systems with friction

The behavior of the mass at \( t = t_2 \) can be separated in three modes:

\[
\begin{align*}
| \sin \omega_0 t_2 | &< f_r \quad \text{stick slip} \\
| \sin \omega_0 t_2 | & = f_r \quad \text{continuous sliding} \\
| \sin \omega_0 t_2 | & > f_r \quad \text{continuous sliding}
\end{align*}
\]

In the stick-slip mode \( t_2 \) can be calculated by a numerical tool from (2.15).

When \( | \sin \omega_0 t_2 | = f_r \) the system is on the threshold of stick-slip mode and continuous sliding mode. Using \( t_2 \) for continuous sliding with (2.15) and (2.14) derives the threshold normalized friction force.

\[
f_r |_{\text{threshold}} = \sqrt{\frac{1}{1 + \frac{\pi}{4}}} \quad (2.16)
\]

This value is constant and independent of the system parameters.

In the continuous sliding mode the sliding in one direction, from \( t_1 \) to \( t_2 \), takes half the period which gives

\[
t_2 = t_1 + \frac{\pi}{\omega_0}. \quad (2.17)
\]

When (2.17) is substituted into (2.15), the relationship between \( f_r \) and \( t_1 \) for continuous sliding can be found as

\[
f_r = \frac{2}{\pi} \cos \omega_0 t_1. \quad (2.18)
\]

Energy dissipation is calculated as 2 times the energy dissipation for half the period by

\[
E_d = 2 \times \int_{t_1}^{t_2} F_r \dot{x}_2(t) \, dt \quad (2.19)
\]

The energy dissipation in (2.19) is normalized with

\[
e_d = E_d \frac{m \omega_0^2}{F_0^2} \quad (2.20)
\]

which is the same norm as is used with the moving base system in Lopez (1998). Substituting (2.13) in (2.19) and using the normalization of (2.20) gives

\[
e_d = 2 f_r (\sin \omega_0 t_1 - \sin \omega_0 t_2) + 2 f_r \omega_0 (t_1 - t_2) \cos \omega_0 t_1 - f_r \omega_0^2 (t_2 - t_1)^2 \quad (2.21)
\]

which is equal to (2.19) in Lopez (1998). There it is shown that the maximum of (2.21) is in the continuous sliding mode. The normalized energy dissipation for the continuous sliding mode is derived by substituting (2.18) in (2.21). Described in terms of \( \omega_0 t_1 \) and of the normalized friction, \( f_r \), gives

\[
e_d = \frac{4}{\pi} \sin 2 \omega_0 t_1 = 4 f_r \sqrt{1 - \frac{\pi^2 f_r^2}{4}}. \quad (2.22)
\]

Form (2.22) the maximum normalized energy is derived as \( e_d = 4/\pi \approx 1.27 \) which occurs at a phase shift of \( \omega_0 t_1 = \pi/4 \). Substituted in (2.18) gives

\[
f_r |_{\text{max}} = \frac{\sqrt{2}}{\pi} \quad (2.24)
\]

which is the normalized friction force with the maximum energy dissipation.
2.3 Mass-spring system

An expression for the phase shift of the threshold of stick-slip and continuous sliding is derived by combining (2.16) and (2.18) which gives

\[
\cos \omega_0 t_{1 \text{threshold}} = \frac{\pi}{2} \sqrt{\frac{1}{1 + \frac{\pi^2}{4}}}. \tag{2.25}
\]

From (2.25) the phase shift at threshold is derived as \(\omega_0 t_{1 \text{threshold}} = 0.5669\). This value is lower than \(\omega_0 t_{1 \text{max}}\) which proves that the maximum energy dissipation occurs in the continuous sliding mode.

The results of (2.22), (2.23) and (2.24) are equivalent with equation (2.21), (2.22) and (2.23) in [Lopez 1998], respectively. From this we conclude that the equivalency in frictional energy dissipation for a free mass between the moving base and the excitation force is simply achieved by scaling the amplitude of the excitation force to \(F_0 = X_0 m \omega_0^2\).

2.3 Mass-spring system

The second system to study analytically is the undamped spring-mass system excited by a periodic force as depicted in Figure 2.2(b) for \(c = 0\). The undamped spring-mass system has a natural frequency of \(\omega_n = \sqrt{k/m}\).

The normalized energy dissipation through friction in steady state motion is calculated and compared with normalized energy dissipation through friction in the spring-mass system with periodically moving base as depicted in Figure 2.2(a) with \(c = 0\). The steady state motion will have a period of \(2\pi/\omega_0\) and the downward half-cycle of motion will follow the same law as the upward half-cycle of motion. The solution of the half-cycle with negative velocity is calculated. The results presented here are limited to \(\omega_n/\omega_0 < 2\). At time \(t_1\) the mass begins to slide (stick-slip) or the velocity changes from positive to negative (continuous sliding). At time \(t = t_2\) the mass sticks again or its velocity changes again to positive. To simplify the notation \(\tilde{t}\) will be used for \(t - t_1\).

The differential equation for \(t = [t_1, t_2]\) is

\[
\ddot{x}_2 + \omega_n^2 x_2 = -\frac{F_r}{m} + \frac{F_0}{m} \sin(\omega_0 t) \tag{2.26}
\]

where \(F_0\) is the amplitude of the excitation force. The value of \(F_0\) is taken from (2.12). The homogeneous solution of (2.26) is

\[
x(t) = C_1 \cos \omega_n \tilde{t} + C_2 \sin \omega_n \tilde{t}
\]

and the particular solution is of the form

\[
x_p(t) = A \cos \omega_0 t + B \sin \omega_0 t + D (1 - \sin \omega_n \tilde{t})
\]

Then the general solution of (2.26) for \(t = t_1, t_2\) is

\[
x_2(t) = C_1 \cos \omega_n \tilde{t} + C_2 \sin \omega_n \tilde{t} + \frac{F_0}{k} \frac{\omega_n^2}{\omega_n^2 - \omega_0^2} \sin \omega_0 t - \frac{F_r}{k} (1 - \sin \omega_n \tilde{t}) \tag{2.27}
\]

where with \(\dot{x}(t_1) = 0\)

\[
C_1 = x_2(t_1) - \frac{F_0}{k} \frac{\omega_0^2}{\omega_n^2 - \omega_0^2} \sin \omega_0 t_1 + \frac{F_r}{k}
\]

\[
C_2 = -\frac{F_0}{k} \frac{\omega_n \omega_0}{\omega_n^2 - \omega_0^2} \cos \omega_0 t_1 - \frac{F_r}{k}
\]

There are three unknowns \(x_2(t_1), t_1, t_2\). They can be solved with the begin conditions

\[
\dot{x}_2(t_2) = 0 \quad \text{and} \quad x_2(t_1) = -x_2(t_2) \quad \text{(2.28)}
\]
2. Theoretical analysis of periodically forced systems with friction

and the third initial condition

\[ \dot{x}_2(t_1) = 0 \]  \hspace{1cm} (2.30)

if the system is in stick-slip mode, or else

\[ t_2 = t_1 + \pi/\omega_0 \]  \hspace{1cm} (2.31)

in case of the continuous sliding mode only.

In the stick-slip mode, by using the initial condition \((2.30)\), \(x_2(t_1)\) is derived as

\[ x_2(t_1) = -\frac{F_r}{k} + \frac{F_0}{k} \sin \omega_0 t_1. \]  \hspace{1cm} (2.32)

The remaining unknowns \(t_1\) and \(t_2\) have to be solved numerically.

In the continuous sliding mode the initial conditions from \((2.28)\), \((2.29)\) and \((2.31)\) give

\[ x_2(t_1) = \frac{F_0}{k} \omega_0^2 \left(1 + \cos \frac{\omega_0}{\omega_n} \pi \right) (\omega_0^2 - \omega_n^2) \sin \frac{\omega_0}{\omega_n} \pi \cos \omega_0 t_1. \]  \hspace{1cm} (2.33)

The normalized friction force on the threshold from stick-slip to continuous sliding mode is calculated by combining \((2.32)\), \((2.33)\) and \((2.34)\). This gives

\[ f_r \mid_{\text{threshold}} = \frac{1}{1 - \frac{\omega_n^2}{\omega_0^2}} \sqrt{\frac{1}{1 + \frac{\omega_n^2}{\omega_0^2} \sin^2 \frac{\omega_0}{\omega_n} \pi}}. \]  \hspace{1cm} (2.35)

The dissipated energy per period is known from \((2.19)\) as

\[ E_d = 2 \times \int_{t_1}^{t_2} F_r x_2(t) dt = -4x_2(t_1)F_r \]  \hspace{1cm} (2.36)

with \((2.20)\) the normalized energy dissipation is

\[ e_d = -\frac{4x_2(t_1)}{X_0} f_r. \]  \hspace{1cm} (2.37)

The normalized energy dissipation for the continuous sliding mode is derived by substituting \((2.33)\) and \((2.34)\) in \((2.37)\), which gives

\[ e_d = 2 \frac{\omega_n}{\omega_0} \left(1 + \cos \frac{\omega_0}{\omega_n} \pi \right) \sin 2\omega_0 t_1. \]  \hspace{1cm} (2.38)

When the excitation force, \(F_0\), is multiplied with the factor \(1 - \frac{\omega_n^2}{\omega_0^2}\) the equations \((2.34)\), \((2.35)\) and \((2.38)\) are equal to the corresponding equations for the spring-mass moving-base system in [Lopez 1998], which are \((2.53)\), \((2.54)\) and \((2.56)\), respectively. In this section the equivalency of the systems \((2.2(a))\) and \((2.2(b))\) is shown. The spring-mass system with a excitation force is equivalent to the spring-mass system with a moving base for an excitation force amplitude, \(F_0 = X_0 m \omega_0^2 \left(1 - \frac{\omega_n^2}{\omega_0^2}\right)\).
This means that in case of spring-stiffness in the experimental setup the measurement results are not arbitrarily comparable with the results from [Lopez et al. (2004)] where a mass on a moving-base is used.

Therefore, it is studied how the frequency ratio, \(0 < \omega_n/\omega_0 < 1\) effects the maximum normalized energy dissipation and if that maximum occurs in the continuous sliding mode. Analyses are done on the spring-mass system with a periodic force amplitude \(F_0 = X_0 m \omega_0^2\). The results are compared with the spring-mass system with moving base.

The maximum of normalized energy dissipation in continuous sliding mode, in (2.38), occurs for \(0 < \omega_n/\omega_0 < 1\) at \(\omega_0 t_{1_{\text{max}}} = 45^\circ\) and for \(1 < \omega_n/\omega_0 < 2\) at \(\omega_0 t_{1_{\text{max}}} = 135^\circ\). At frequency ratios higher than 2 the continuous sliding motion is not possible, see [Pratt & Williams (1981)]. The system is in continuous sliding mode if the normalized friction force is smaller than the threshold friction. From (2.34) and (2.35) an expression for \(\omega_0 t_{1}\), when the friction force is equal to the threshold value, can be derived.

\[
\cos \omega_0 t_{1_{\text{threshold}}} = \frac{\omega_0}{\omega_n} \frac{\sin \frac{\omega_0 \pi}{\omega_n \pi}}{1 + \cos \frac{\omega_0 \pi}{\omega_n \pi}} \left( \frac{1}{1 + \frac{\omega_0^2}{\omega_n^2} \sin^2 \frac{\omega_0 \pi}{\omega_n \pi}} \right).
\] (2.39)

If \(\omega_0 t_{1_{\text{threshold}}} < \omega_0 t_{1_{\text{max}}}\) the threshold friction force is greater than the friction force for the maximum energy dissipation and this maximum occurs in the continuous sliding motion range. In Figure 2.3 the curves of \(\omega_0 t_{1_{\text{max}}}\) and \(\omega_0 t_{1_{\text{threshold}}}\) are depicted for the frequency ratio \(\omega_n/\omega_0\) from 0 to 2.

This is equal for the moving base and periodic force system, using \(F_0 = X_0 m \omega_0^2\). Figure 2.3 shows that for \(\omega_0 t_{1_{\text{threshold}}} < 45^\circ\) and \(\omega_0 t_{1_{\text{threshold}}} > 135^\circ\) the threshold friction force is greater then the friction force for the maximum energy dissipation and thus that for those phase shifts the maximum energy dissipation occurs in the continuous sliding mode. Thus for frequency ratios \(\omega_n/\omega_0 < \sqrt{2}\) the maximum energy dissipation occurs in continuous sliding mode and the maximum energy dissipation is given by (2.38).

![Figure 2.3: Phase shifts \(\omega_0 t_{1_{\text{max}}}\) (thick line) and \(\omega_0 t_{1_{\text{threshold}}}\) (thin line).](image)
For both system the maximum energy dissipation occurs in the continuous sliding mode for the frequency ratio $\frac{\omega_n}{\omega_0}$ from 0 to 1. For this range of frequency ratios the maximum energy dissipation for both system is shown in Figure 2.4. Comparing both systems it is clear that for the periodic force system the frictional energy dissipation for the periodic force spring-mass system increases instead of decreases for increasing frequency ratios. So, that the dissipated energy goes to $\infty$ for $\frac{\omega_n}{\omega_0} = 1$. 

Figure 2.4: Maximum energy dissipation versus frequency ratio for the 1-DOF systems with spring on the moving base system(dashed line), and with the periodic force(continuous line).
2.4 Mass-spring-damper system

In this section the damped spring-mass system as depicted in Figure 2.2(b) is studied analytically to derive a solution of the normalized dissipated energy. It is shown that this solution is equivalent with the solution of the 1-DOF system with moving base depicted in Figure 2.2(a).

The results presented here are limited to \( \omega_n/\omega_0 < 1 \). The steady state motion will have a period of \( \pi/\omega_0 \) and the downward half-cycle of motion will follow the same law as the upward half-cycle.

The differential equation is for \( t = [t_1, t_2] \) is

\[
\ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = -\frac{F_r}{m} + \frac{F_0}{m} \sin(\omega_0 t) \tag{2.40}
\]

with \( \omega_n = \sqrt{k/m} \) is the natural frequency, \( \zeta = \frac{c}{2m\omega_n} \) is defined as the damping ratio and \( \omega_d = \sqrt{1 - \zeta^2} \omega_n \) is the damped natural frequency. The homogeneous solution of (2.40) is

\[ x(t) = e^{-\zeta \omega_n t} \left[ C_1 \cos \omega_n t + C_2 \sin \omega_n t \right] \]

with \( t = t - t_1 \), and the particular solution is of the form

\[ x_p(t) = A \cos \omega_0 t + B \sin \omega_0 t + D \left( 1 - e^{-\zeta \omega_n t} \sin \omega_0 t \right) \]

Then the general solution of (2.40) is

\[ x_2(t) = e^{-\zeta \omega_n t} \left[ C_1 \cos \omega_n t + C_2 \sin \omega_n t \right] + A \cos \omega_0 t + B \sin \omega_0 t + D \left( 1 - e^{-\zeta \omega_n t} \sin \omega_0 t \right) \tag{2.41}
\]

where with \( \dot{x}(t_1) = 0 \)

\[
C_1 = x_2(t_1) - A \cos \omega_0 t_1 - B \sin \omega_0 t_1 - D
\]

\[
C_2 = \frac{\omega_n}{\omega_d} x_2(t_1) + \left( -\frac{\zeta \omega_n A - \omega_0 B}{\omega_d} \right) \cos \omega_0 t_1 + \left( \frac{\omega_0 A - \zeta \omega_n B}{\omega_d} \right) \sin \omega_0 t_1 + \frac{\omega_d - \zeta \omega_n}{\omega_d} D
\]

with

\[
A = -\frac{F_r}{m} \frac{2\omega_n \omega_0 \zeta}{(\omega_d^2 - \omega_0^2)^2 + (2\omega_n \omega_0 \zeta)^2}
\]

\[
B = \frac{F_r}{m} \frac{(\omega_d^2 - \omega_0^2)}{(\omega_d^2 - \omega_0^2)^2 + (2\omega_n \omega_0 \zeta)^2}
\]

\[
D = -\frac{F_r}{m \omega_n^2}
\]

So there are three unknowns \( x_2(t_1) \), \( t_1 \) and \( t_2 \). The initial conditions are identical to those of the undamped system in the previous section.

In the continuous sliding mode \( x_2(t_1) \) and the normalized friction force, \( f_r \), are derived using the initial conditions from (2.28), (2.29) and (2.31) and as excitation amplitude \( F_0 = m \omega_n^2 X_0 \). This gives

\[
x_2(t_1) = \frac{1}{2} \omega_0 \omega_d X_0 \frac{e^{\frac{\omega_0 t_1}{\omega_0} \pi} - e^{-\frac{\omega_0 t_1}{\omega_0} \pi} - 2 \cos \frac{\omega_0 t_1}{\omega_0} \pi \omega_0^2 \left( \omega_0^2 - \omega_d^2 \right) \cos \omega_0 t_1 - 2 \omega_0 \zeta \omega_n \sin \omega_0 t_1}{\left( \omega_d^2 - \omega_0^2 \right)^2 + (2\omega_n \omega_0 \zeta)^2}
\]

\[
+ X_0 \frac{\omega_0^2 \left( \omega_0^2 - \omega_d^2 \right) \sin \omega_0 t_1 - 2 \omega_0 \zeta \omega_n \cos \omega_0 t_1}{\left( \omega_d^2 - \omega_0^2 \right)^2 + (2\omega_n \omega_0 \zeta)^2} \tag{2.42}
\]

\[
f_r = \frac{1}{2} \omega_0 \frac{e^{\frac{\omega_0 t_1}{\omega_0} \pi} + e^{-\frac{\omega_0 t_1}{\omega_0} \pi} + 2 \cos \frac{\omega_0 t_1}{\omega_0} \pi \omega_0^2 \left( \omega_0^2 - \omega_d^2 \right) \cos \omega_0 t_1 - 2 \omega_0 \zeta \omega_n \sin \omega_0 t_1}{\left( \omega_d^2 - \omega_0^2 \right)^2 + (2\omega_n \omega_0 \zeta)^2} \tag{2.43}
\]
2. Theoretical analysis of periodically forced systems with friction

Substituting (2.42) and (2.43) in (2.37) gives the normalized energy dissipation

\[ e_d = \frac{\omega_0^3 \omega_0}{\sin \frac{\omega_d}{\omega_0} \pi} \left( \frac{\omega_0^2 - \omega_n^2}{\omega_0^2 + (2 \omega_0 \xi \omega_n)^2} \right) \left( e^{\frac{\omega_0^2}{\omega_n^2} \pi} - e^{-\frac{\omega_0^2}{\omega_n^2} \pi} - \frac{2 \xi \omega_0}{\omega_n^2} \sin \frac{\omega_d}{\omega_0} \pi \cos^2(\omega_0 t_1 + \theta) + \sin (2(\omega_0 t_1 + \theta)) \right) \] (2.44)

with \( \theta \) as defined in (2.11). The expression for the normalized energy dissipation, \( e_d \) in (2.44) converts to \( e_d \) for the undamped spring-mass system in (2.37) for \( \xi \to 0 \).

The value of \( \omega_0 t_1 \) for which the energy dissipation is maximally can be obtained by taking the first derivative of (2.44) and find its zero.

\[ \tan (2(\omega_0 t_1 + \theta)) \bigg|_{\text{max}} = \frac{2 \omega_0^2}{\omega_0 \omega_0} \frac{e^{\frac{\omega_0^2}{\omega_n^2} \pi} - e^{-\frac{\omega_0^2}{\omega_n^2} \pi} - \frac{2 \xi \omega_0}{\omega_n^2} \sin \frac{\omega_d}{\omega_0} \pi}{\cos^2(\omega_0 t_1 + \theta) + \sin (2(\omega_0 t_1 + \theta))} \] (2.45)

The maximum energy dissipation calculated in the range \( 0 < \omega_n/\omega_0 < 1 \), for \( 0 < \xi < 1 \) is shown in Figure 2.5.

![Figure 2.5: Maximum energy dissipation for several damping ratios and a range of frequency ratios.](image-url)
2.5 Comparison with numerical analysis

The analytical investigation is supported by the outcome of numerical simulations. The equivalence between the moving base and the periodic excitation force system as is presented in Section 2.1 is shown by the numerical results in Figure 2.6. The maximum energy dissipation is shown for both systems for a range of frequency and damping ratios.

The used friction model is the switch model from Leine et al. (1998). It is shown there that the switch model is much more cost effective than a smooth friction model like an arc-tangents function for the stick-phase while accurateness of the solution is equal. The numerical tools to solve the equations of motion are shooting and path-following method, see Leine & van de Wouw (2001).

In Section 2.1 the equivalency of the periodic force system and the moving base system is shown. For both systems the relative displacement, relative velocity, friction force and periodic force or acceleration of the base are depicted in Figure 2.6. The equivalency in the continuous sliding mode, \( f_r = 0.3 \) and the stick-slip mode, \( f_r = 0.7 \), can be clearly seen. The numerical outcome is according the analytical solution.

Figure 2.6: State variables and friction force for 1-DOF system with periodically moving base for (a) \( f_r = 0.3 \) and (c) \( f_r = 0.7 \) and excited by periodic force for (b) \( f_r = 0.3 \) and (d) \( f_r = 0.7 \).
2. Theoretical analysis of periodically forced systems with friction

The next subject to compare the analytical solution with the numerical simulation result is the maximum energy dissipation. In the analytical calculations the maximum energy dissipation in the continuous sliding mode of the periodic force system was calculated in the range $0 < \omega_n/\omega_0 < 1$, for $0 < \zeta < 1$.

For several frequency ratios in the range $0 < \omega_n/\omega_0 < 1$ versus several damping ratios in the range $0 < \zeta < 1$ the maximum energy dissipation is numerically calculated and it is determined if the maximum occurs in stick-slip or continuous sliding mode. The results are depicted in Figure 2.7[b]. The dotted lines present the maximum energy dissipation occurring in the stick-slip mode.

For completeness the simulations are also done with $F_0(t) = -m(\ddot{x}_0 + 2\zeta\omega_n\dot{x}_0 + \omega_n^2x_0)$ so that the system is equivalent in relative velocity to the system with periodically moving base. The results of the maximum energy dissipation is set against the frequency ratio versus the damping ratio are depicted in Figure 2.7[a]. Note that simulating with the periodic force system instead of the moving base system has the advantage that it costs ten times less calculation time.

Comparing Figure 2.7[b] with 2.5 the height of the maximum energy dissipation between the numerical outcome and the analytical solution is similar. Even in the range of damping ratios versus frequency ratios at which the maximum energy dissipation occurs in the stick-slip mode, according to the numerical simulation, the analytical values, calculated for the continuous sliding mode, are equal to the numerical ones. It is expected that the range of damping ratios versus frequency ratios for which the maxima of energy dissipation occurs in stick-slip mode are equal for the periodic force system and the moving base system. Numerical simulation confirm this as clearly can be seen by comparing Figure 2.7[a] and 2.7(b).

2.6 Discussion of the modeling

The analytic solution show that the 1-DOF system with a moving base and the one with a periodic excitation force can be made equivalent by scaling the excitation amplitude if in the system the mass is free or if only a spring is attached. In case that a spring is attached to the mass, the energy dissipation of the experimental system differs with the energy dissipation of the moving base system for frequency ratio, $\omega_n/\omega_0 \rightarrow 1$. Therefore, the experiments have to be performed at a low frequency ratio, $\omega_n/\omega_0 \ll 1$.

In the case that a spring and a damper are attached to the mass, the 1-DOF system with an excitation force is equivalent to the 1-DOF system with a moving base if the phase of the excitation force is shifted. Therefore, in order to be able to compare the experimental results with the 1-DOF model investigated by Lopez [1998], during the experiments the damping should be low.

The excitation force during the experiment is shifted nor scaled: it just is at its time and with its size. The shifting and scaling takes place when the data is processed in order to compare the measurements with the analytic model. Therefore, it is desirable for the experiment to have a moving mass without spring or damper. In that case the the transformation from one model to the other is just a (constant) scaling factor.
Figure 2.7: Energy dissipation of the 1-DOF spring-mass-damper system with period force or with moving base against the frequency ratio ($\omega_n/\omega_0$) and damping ratio ($\zeta$) occurring in continuous sliding mode (continuous line) or in stick-slip mode (dashed line).
2. Theoretical analysis of periodically forced systems with friction
Chapter 3

Measurements 1-DOF experimental setup with dry friction

3.1 Experimental setup

The experimental investigation of energy dissipation of dry friction has been done at a setup where a shaker is forcing a sledge back and forth. In Figure 3.1, a top-view sketch of the setup is depicted. As friction lip a steel part is used which is stiff in the translating direction and elastic in the direction perpendicular to the translation. A bearing ball is attached to the part and used as friction tip. The bearing balls slide along a polished silicium-carbonate plate which is fixed to the underground. The normal force of the bearing balls to the plate is adjustable by turning the bolt which pushes a spring. A laser-doppler acquires the velocity and the displacement. The force cell measures the force of the shaker on the sledge and at the sledge is an acceleration sensor attached. The data-acquisition and the control of the excitation force is done with a Siglab® interface. To excite the sledge periodically by the shaker, a sinusoidal voltage signal is send from the Siglab® interface to a current amplifier. The current amplifier transforms the voltage signal to a current signal that is sent to the electromechanical shaker.

With this setup it is possible to measure the phenomena of stick-slip and continuous sliding that are associated with dry friction.

![Figure 3.1: Top-view of the experimental setup with the new friction addition.](image)
3. Measurements 1-DOF experimental setup with dry friction

3.2 Measurement description

During an experiment a sinusoidal input signal is used to let the shaker excite the sledge with a periodic force. At a defined excitation frequency $\omega_0$, the voltage amplitude, $V_0$ of the sinusoid is increased or decreased in small steps. The resulting excitation force, $F_0$ of the shaker on the sledge leads to a normalized friction force $f_r = F_r / F_0$. The range of the excitation amplitude levels used in the experiments is sufficiently large that the transition from stick-slip to continuous sliding and the maximum normalized energy dissipation can be observed. At each excitation voltage amplitude level the actual excitation force on the sledge and the acceleration, velocity and displacement of the sledge are measured for a time of about 25 excitation periods.

The frequency of the excitation is chosen such that the sledge moves as a free mass according to the frequency response function measured with the friction plate removed. The measured frequency response function is depicted in Appendix A.1. It is clear that the sledge will move as a free mass for excitation frequencies above ca. 13 Hz. This means that the sledge is comparable to the system depicted in Figure 2.1(a) that is used for the analytic modeling.

Low excitation frequencies are preferred, because at low excitation frequencies the excitation force amplitude is less sensitive to changes in the input voltage amplitude level, see Figure 3.2. This is caused by the dynamics of the electro magnetic shaker, see Figure 17 in Lang (1997) where the same type of shaker excites a cantilever beam in order to examine the behavior of the shaker.

![Graphs showing force amplitude against input voltage for increasing and decreasing input voltage at different frequencies](image)

Figure 3.2: Force amplitude against input amplitude voltage, (a) 13Hz, (b) 14Hz, (c) 15Hz and (d) 16Hz. UP for increasing and DOWN for decreasing input voltage amplitude. Data from measurement series A.
Measurements are performed in a series with excitation frequencies of 13, 14, 15, 16 and in one measurement also 17 and 18 Hertz. A series always starts at 13 Hertz increased till the highest frequency. At each excitation frequency the excitation amplitude is first increased and than decreased with small steps. At each step of the excitation amplitude the sledge is measured for about 25 excitation periods. This together is called a measurement series. After one series wear has affected the plate and bearing balls to much to be used again. The plate and the bearing balls are changed for a new series of measurements. In this report the results of four measurement series are presented, called A, B, C and D.

From the sledge without friction plate the linear relation between excitation force and the velocity is examined by measuring the coherence between both signals, see Figure A.2. From that it is concluded that the setup behaves as a linear system for the used excitation frequencies. Thus the setup is usable to compare the measurement results with the linear model.

3.3 Measurement results

The measured signals (excitation force, acceleration, velocity and displacement of the sledge) are examined for usability before the data is processed to compare the setup with the analytic model. Let us take the measurements with an excitation frequency of 13 Hertz from series A with increasing amplitude as a pilot to investigate the measured signals using its time signals as depicted in Figure 3.3.

Figure 3.3: Time plots of force and acceleration, velocity and displacement for different input amplitude voltage level (which was increasing) at a frequency of 13 Hz. Data from measurement series A
3. Measurements 1-DOF experimental setup with dry friction

In Figure 3.3(a) the system is clearly in stick-slip phase. At a higher excitation force, the system passes from stick-slip phase into continuous sliding for an input voltage between 0.109V and 0.115V. At an input voltage of 0.109V stick-slip is seen in Figure 3.3(b) where the velocity signal is sticking at zero and at the same time the acceleration is zero as well, while at an input voltage of 0.115V, depicted in Figure 3.3(c) a non-zero acceleration signal is measured at the time instant where the velocity is zero which indicates that the system is in continuous sliding.

A second observation is that higher input voltage amplitudes result in a larger phase shift between force and velocity. It is this phase-shift which results in the frictional energy dissipation. The phase-shift between velocity and force at maximum energy dissipation, in Figure 3.3(d), is about 0.01 second which is about one eight of the period. This value is according to the theoretical value for the free mass model as follows from (2.22).

A third observation is that for an input voltage level of 0.077V, depicted in Figure 3.3(a), the velocity signal sticks at zero before the excitation force crosses changes direction. At the higher input voltages depicted in Figure 3.3 the force signal changes direction before the velocity signal reaches the zero. This behavior is seen in the free mass model as well.

A forth observation is that the force of shaker to the sledge is not a perfect sine wave. This is due to the fact that the exerted force is altered by the velocity and friction forces on the sledge. This deviation from the sine wave is not problematic for the current study as long as the force is periodic. Moreover, the amplitude of the force is not linear with the amplitude of the input voltage control signal. This is not such a problem since the actual excitation force is measured. However, great deviations of the force amplitude in relation to the input voltage amplitude results in disturbances in the energy dissipation curve. Probably the greater deviations indicates non-stationary behavior of the dry friction or failure of the parallel bearings. Estimates of the friction force confirms these indications. Great deviations in the force-voltage curve only result in disturbances of the energy dissipation curve if there are great deviations in the estimated friction force. The friction force is estimated by the measured force to the sledge at maximum velocity for continuous sliding mode or the measured force before slip in the stick-slip mode.

At other frequencies similar behavior is seen.

3.4 Discussion of the experiments

In this section, the suitability is investigated of the experimental setup. It is shown that the sledge moves as a free mass in the applied excitation frequencies, which makes the measurement data easy to compare with the analytical model in the previous chapter. Furthermore, the phase-shift of the force and velocity signal is confirm the free mass model and the transition of stick-slip to continuous sliding can clearly be seen. Therefore, it is concluded that the experimental setup is suitable for the measurement of energy dissipation through dry friction and validation of the free-mass model.

There are a few disadvantages at the setup. The hertzian pressure at the contacts is high, resulting in wear. With less wear the measurements become more stable. Another, larger contact surface is recommended. A second problem in controlling the contact pressure is that it is hard to divide the pressure exactly between the two contacts. A change in construction is recommended.

Secondly the parallel bearings are not suitable for back and forth movements because these type of movements cause lack of lubrication. This adds an unpredictable factor to the viscous friction a bearing is causing usually. This friction interferes with the dry friction that is to be measured. Parallel leave springs can be a good replacement. With these leave springs small movements can be made without extra friction.
Chapter 4

Comparison of the measurements with the analytic solution

The measured data is processed for comparison with the analytical models. Different methods of normalization are assessed. First the normalization methods are discussed and then two of them are applied to the measurement data and the outcome is presented together with the analytic model of Figure 2.1(b). Finally the comparison is discussed.

4.1 Processing of measurement data

The measured data is processed whereby the experimental setup is modeled as in Figure 2.1(b). The friction force is calculated from

\[ F_r(t) = F_0(t) - m \, a(t) \]  

(4.1)

where \( F_0(t) \) is the measured force on the sledge, \( m \) is the mass and \( a(t) \) the measured acceleration of the sledge. The sledge is supposed to be rigid so no spring force is subtracted.

The dissipated energy is calculated by

\[ E_d = \frac{1}{n} \sum_{0}^{nT} F_r(t) \, v(t) \]  

(4.2)

where \( n \) is number of periods measured and \( v(t) \) the velocity of the sledge. The normalized dissipated energy is known from (2.20) as \( e_d = E_d / (X_0^2 m \omega_0^2) \). Since the amplitude of the excitation force is equivalent to \( m \, X_0 \, \omega_0^2 \) the normalized measured energy dissipation is

\[ e_d = \frac{m \omega_0^2 E_d}{(\text{mean amplitude } F_0(t))^2} \]  

(4.3)

To obtain the normalized friction force, \( f_r = F_r / F_0 \), the following results of the analytical model are recalled from Section 2.2

\[ f_r \mid_{\text{max}} = \frac{\sqrt{2}}{\pi} = 0.4502 \]  

(4.4)

\[ f_r \mid_{\text{threshold}} = \sqrt{\frac{1}{1 + \frac{4}{\pi}}} = 0.5370 \]  

(4.5)

which are the normalized friction force at the threshold of stick-slip to continuous sliding and the normalized friction force with the maximum normalized energy dissipation, respectively. There are three different possibilities to find the normalized friction force:
4. Comparison of the measurements with the analytic solution

i. Based on the excitation force at maximum normalized energy dissipation, $F_0(e_d\text{max})$.

The friction force can be estimated from $f_r \mid_{\text{max}} \times F_0(e_d\text{max})$ and the normalized friction force becomes

$$f_r = \frac{f_r \mid_{\text{max}} \times F_0(e_d\text{max})}{F_0}.$$  \hspace{1cm} (4.6)

This method has the disadvantage that the maximum of the normalized energy dissipation can not be obtained sufficiently accurate from the data.

ii. Based on the excitation force at the threshold of stick-slip to continuous sliding $F_0(\text{threshold})$.

The friction force can be estimated from $f_r \mid_{\text{threshold}} \times F_0(\text{threshold})$ and the normalized friction force becomes

$$f_r = \frac{f_r \mid_{\text{threshold}} \times F_0(\text{threshold})}{F_0}.$$  \hspace{1cm} (4.7)

The threshold is determined by comparing the acceleration and velocity signals. Stick is observed if the acceleration signal crosses the zero when the velocity is zero. The accuracy is limited by the increments used for the excitation amplitudes and the sample frequency. The transition form stick-slip to continuous sliding is often for all periods at the same excitation force.

iii. Based on the excitation force at the moment of release in the stick-slip phase. This gives

$$f_r = \frac{F_0(\text{release})}{F_0}.$$  \hspace{1cm} (4.8)

The great advantage of this method is that it does not assume a specific model. A disadvantage is that the static friction force is obtained which is normally higher than the dynamic friction force. Therefore the dynamic friction force is obtained by determining the excitation force at maximum velocity when the sledge is in the continuous sliding mode. However, the obtained value differ often too much for different excitation forces to obtain a reliable value for the dynamical friction force.

We conclude that the second and the third method are most interesting. The normalized energy dissipation obtained form four measurement series will be presented for each of these two methods.

4.2 Measurements compared to 1-DOF model with coulomb friction

This section compares the normalized energy dissipation calculated from the measurement data with the analytical solution of the 1-DOF model with coulomb friction, see Figure 2.1(b). In Section 4.2.1 the normalized energy dissipation is depicted with the measured friction force that is normalized using the excitation force at the threshold of stick-slip to continuous sliding (4.7). The energy dissipation depicted against the measured friction force normalized using the excitation force at the moment of release in the stick-slip phase (4.8) is presented in Section 4.2.2. Note that the energy dissipation curves are only displayed if the threshold of stick-slip to continuous sliding can be determined from the data. Seen from the side of continuous sliding, at the first measurement where the system is not in continuous sliding for all measured periods the symbol for stick-slip and continuous sliding are depicted crossed.

For reasons of readability, only the results of the measurement series at 13 Hz and 16 Hz are depicted here. The measurement results of all excitation frequencies are provided in Appendix B. The values of the maximum normalized energy dissipation, phase shift of velocity and excitation force (at maximum energy dissipation and at the threshold of stick-slip and continuous sliding), the number of measured periods and some other values tabulated in Appendix C.
4.2 Measurements compared to 1-DOF model with coulomb friction

4.2.1 Friction normalized using $F_0$ threshold stick-slip to continuous sliding

In this section the normalized friction force is estimated by normalizing the measured friction force using the excitation force at the threshold of stick-slip to continuous sliding (4.7). For each measurement series the resulting normalized energy dissipation curves with an excitation frequency of 13 and 16 Hz are depicted in the Figures 4.1 and 4.2, respectively. Clearly, the normalized energy dissipation curves agree quite well with the analytic solution, although here are some discrepancies.

For this method of normalizing the threshold from stick-slip to continuous sliding occurs by definition at $f_r = 0.537$ as prescribed by the theoretical model. It can be clearly seen that the resulting normalized friction force for maximum energy dissipation is at, or reasonably close, to the theoretical value of $f_r = 0.45$.

The maximum value of the normalized energy dissipation is for most measurements at the theoretical value and the maximum is independent of the excitation frequency which is according to the theory. Only for the measurement series A and C at 13 Hz and A and D at 16 Hz with decreasing $f_r$, the values of the maximum normalized energy dissipation are considerably higher than the analytical value. Besides, in the stick-slip mode the measured energy dissipation levels are lower than the analytic values for the increasing $f_r$ in Figure 4.1(a) and the decreasing $f_r$ in Figure 4.2(a). But for decreasing $f_r$ in Figure 4.1(b) and 4.1(c) the measured energy dissipation values are higher than proposed by the analytical model.

Figure 4.1: Normalized energy dissipation against normalized friction force calculated by (4.7) for measurement series (a) A, (b) B, (c) C and (d) D at 13 Hz.
Figure 4.2: Normalized energy dissipation against normalized friction force calculated by \(4.7\) for measurement series (a) A, (b) B, (c) C and (d) D at 16 Hz.
4.2 Measurements compared to 1-DOF model with coulomb friction

4.2.2 Friction normalized using $F_r$ measured in stick-slip phase

The friction force is normalized using the friction force in stick-slip mode \[4.8\]. This friction force is estimated by measuring the excitation force at the moment of release in the stick-slip phase. For each measurement series the resulting normalized energy dissipation curves with an excitation frequency of 13 and 16 Hz are depicted in the Figures 4.3 and 4.4, respectively. According to the proposed model the maximum energy dissipation should occur at $f_r = 0.45$ and the threshold of stick-slip to continuous sliding at $f_r = 0.537$. This is only the case with decreasing $f_r$ in Figure 4.3(a), 4.4(a) and 4.4(d). The other energy dissipation curves are shifted to higher normalized friction force values and the curves are wider than the ones in the previous section. This is caused by the fact that the excitation force at release is a good estimate for the static friction force but not for the lower dynamical friction force. The analytical model, to which the curves in the previous section are normalized, uses the dynamical friction force and therefore there is a shift in normalized friction force between the analytical model and the energy dissipation curves in this section.

It is obvious that the height of the normalized energy dissipation does not change by the normalization of friction force.

![Normalized energy dissipation against normalized friction force calculated by \[4.8\] for measurement series (a) A, (b) B, (c) C and (d) D at 13 Hz.](image)
4. Comparison of the measurements with the analytic solution

Figure 4.4: Normalized energy dissipation against normalized friction force calculated by (4.8) for measurement series (a) A, (b) B, (c) C and (d) D at 16 Hz.
4.3 Discussion

Dry friction is a non-stationary process and wear has been observed during the measurements, even though the measurements are reproducible. This makes the measurements usable to validate the analytical model. In most cases the data with increasing $f_r$ matches with the data with decreasing $f_r$. This indicates that the measurements are reliable.

First the analytical model is compared to the measurement data from which the normalized friction force is obtained by using the excitation force at the threshold of stick-slip to continuous sliding. Then there is a great correspondence between the energy dissipation curves of the measurement data and the analytical model as can be seen in the figures of Section 4.2. The normalization method sets the threshold of stick-slip to continuous sliding at the theoretical value for $f_r\mid_{\text{threshold}}$. The resulting value of the normalized friction force at maximum energy dissipation, $f_r\mid_{\text{max}}$, is in most cases near the theoretical value. Moreover, the maximum normalized energy dissipation from the measurements agrees reasonably well with the theoretical value and is independent of the excitation frequency as is expected from the theoretical model. The maximum energy dissipation is in most cases higher then the theoretical value for Coulomb friction. Viscous damping can be an explanation as Figure 4.5(a) shows.

We conclude that the analytical model gives a good estimate for the energy dissipation measured at the setup, although in the stick-slip phase the energy dissipation is often higher than predicted by the analytical model. This phenomenon is also seen by viscous damping in the stick phase or an arc-tangent function as simulation results show in Lopez & Nijmeijer (2005), see Figure 4.5(a) and 4.5(b), respectively.

![Viscous damping during stick-phase](image1)

(a) Viscous damping during stick-phase

![Arc-tangent function for stick-phase](image2)

(b) Arc-tangent function for stick-phase

Figure 4.5: Normalized energy dissipation vs normalized friction force, Lopez & Nijmeijer (2005).

Secondly the analytical model is compared to the measurement data from which the normalized friction force is obtained by using the excitation force at the moment of release in the stick-slip mode. Compared to the previous normalization method, the normalized energy dissipation curve is shifted to higher values of the normalized friction force, see the figures in Section 4.2.2. An explanation is found in the fact that in most dry contacts the static friction force is higher than the dynamical friction force. This causes the shift of the normalized energy dissipation curve to higher normalized friction values than expected according to an analytical model with standard Coulomb friction. The shift of the maximum normalized energy dissipation to higher normalized friction force is not always equal for the curves of increasing $f_r$ and decreasing $f_r$. Since the measured excitation force at release in stick-slip mode is determined separately for the decreasing $f_r$ and increasing $f_r$, they can differ and that is the cause for the shift difference. In most cases the shift is smaller than the shift in Figure 4.6(a). From this we conclude that in the experimental setup $\gamma > 0.7$, where $\gamma$ is the ratio of...
4. Comparison of the measurements with the analytic solution

dynamic and static friction force.

There is another phenomenon that indicates static friction. In e.g. Figure 4.2(a) a large fall can be seen in the energy dissipation at the threshold of continuous sliding to stick-slip. According to the analytical model in Lopez et al. (2004) this can be explained as caused by the static friction force, see Figure 4.6. Note that in the region between $f_{s1}$ and $f_{s2}$ two solutions for the energy dissipation exist. At all normalized static friction force values between $f_{s1}$ and $f_{s2}$ the solution can switch between the two solutions. More examples of this switching behavior are provided in Appendix B in Figure B.1(b) for increasing $f_r$ and Figure B.3(b) for decreasing $f_r$.

![Graph showing energy dissipation vs. normalized static friction force](a)

![Graph showing energy dissipation vs. normalized static friction force](b)

Figure 4.6: Dissipated energy versus static friction force. (a) $\gamma = 0.7$ and (b) $\gamma = 0.5$, Lopez et al. (2004). $\gamma$ is the ratio of the dynamic friction force to the static friction force.

The measurements show that the maximum energy dissipation is independent of the excitation frequency. This demonstrates that the mass moves freely, without a spring attached. If a spring would be present, the maximum energy dissipation would be dependent of the excitation frequency, see Figure 2.4. This confirms the assumption of the sledge to be rigid.

The comparison of the two applied methods to obtain a normalized friction force from the measurement data makes clear that normalized friction force can be determined best from the excitation force at the threshold of stick-slip to continuous sliding using (4.7). The analysis of the experimental data show that the coulomb friction model gives a good prediction of the energy dissipation in practical dry friction environment.
Chapter 5

Conclusions and Recommendations

In the literature, the energy dissipation of dry friction is investigated by simulating a 1-DOF model where a mass has a frictional contact with its periodically moving base. The experimental setup used in this report is modeled by a 1-DOF model where a mass, excited by a periodic force, has dry friction contact with its stationary base. Theoretic analysis in this report shows equivalence between the two models. Hence, the measurement data can be compared to the simulation results from the literature. The experimental setup consists of a shaker which excites the sledge and on the sledge a friction element is attached. In the experiments the friction force on the sledge is constant and the excitation force amplitude is increased in small steps. The excitation force on the sledge and the velocity of the sledge are measured. The experiments have produced reliable measurement data. The experimental results show that the simple experimental setup can provide suitable and reliable measurements of the energy dissipation of dry friction. Moreover, the transition from stick-slip mode to continuous sliding is precisely identified.

From the experimental results, the normalized energy dissipation and the normalized friction force are calculated for comparison with the analytical results. The normalized friction force, that is the friction force divided by the excitation force amplitude, is estimated using two methods. In the first, the excitation force at the threshold of stick-slip to continuous sliding is normalized to the normalized friction force at the threshold gained from the 1-DOF model with Coulomb friction. In the second method, the friction force is estimated from the measured excitation force at the release of the sledge in stick-slip mode. The measured normalized energy dissipation as a function of the estimated normalized friction force is compared to the analytic curves from the 1-DOF model with Coulomb friction and other friction models. From these investigation the following conclusions and recommendation can be made.

5.1 Conclusions

The best method to obtain an estimate of the normalized friction force from the measurement data is by using the excitation force on the sledge at the transition of stick-slip to continuous sliding and normalize it to the theoretical value of the normalized friction force of the 1-DOF model with Coulomb friction. Then, by definition, the transition of stick-slip to continuous sliding occurs at the theoretical transition value of the normalized friction force. Then, for most experiments the measured energy dissipation as a function of the normalized friction force corresponds reasonably well with the energy dissipation curve of the analytical model with Coulomb friction. This indicates that the transition of stick-slip to continuous sliding in the experiments occurs at a normalized friction force of about the theoretical values of the analytical model and that the measured energy dissipation curve can be described by the analytical 1-DOF model with Coulomb friction.

In Lopez & Nijmeijer (2005) it is shown by simulations that the normalized energy dissipation curves against the normalized friction force remains the same for small changes in the Coulomb fric-
5. Conclusions and Recommendations

The friction of the sledge in the experiments is also not perfectly Coulomb friction since wear of the contact surface and friction in the bearings occur. Overall, the measured normalized energy dissipation corresponds reasonably well with the outcome of the analytical model. So, the simulation results are supported by the experimental data. This proves that the simple Coulomb friction model without static friction, Stribeck effects or viscous damping is sufficiently accurate to estimate the normalized energy dissipation in an environment with dry friction. Moreover, the experimental data show that it is not necessary to measure the friction force itself to estimate the energy dissipation.

Hence, the experiments show that the simple Coulomb scheme suffices to predict the energy dissipation in a dry friction contact. This is a great advance in designing dry friction contacts to dissipate energy of undesired vibrations.

5.2 Recommendations

If the normalized friction force is estimated by calibrating the measured excitation force at the transition of stick-slip mode and continuous sliding mode to the theoretical transition value of the 1-DOF model then, by definition, the transition is at the theoretical normalized friction force. Hence, the value of normalized friction force at the transition of stick-slip to continuous sliding in the model cannot be validated by the experimental data.

Therefore another method is applied to estimate the normalized friction force. The friction force is estimated by the measured excitation force at the release of the sledge in stick-slip mode. Then the normalized friction force is obtained by dividing this friction force by the excitation force amplitude for each measurement. Applying this normalized friction force, each measured normalized energy dissipation is shifted to higher values of normalized friction force than expected from analytic solution of the 1-DOF model with Coulomb friction. The explanation for this is that the static friction force is measured while the lower dynamic friction is wanted.

However, accurate methods to obtain the dynamic friction force from the experimental data are hard to apply. Processing the measurement data to obtain the dynamic friction force gives too much distribution in the outcome to obtain a reliable value of the dynamic friction force. Therefore, we recommend to design a new experimental setup where the mass can move frictionless except for the desired dry friction. In the setup the normal force of the friction is measurable, such that:

i. a good estimate of the dynamic friction force can be achieved from the measurements,

ii. the normalized friction force can be calculated without the use of the analytic 1-DOF model with Coulomb friction model,

iii. validation of the model can be improved,

iv. the friction force can be estimated by multiplying the measured normal force with the friction coefficient for the used materials,

v. the relation between the designed friction force and the energy dissipation can be investigated whereas that relation is now based on the assumption that the transition is at the analytical predicted value.

In the experiment wear of the friction contact is observed. It is recommended to investigate to what extent wear decreases the durability of dry friction surfaces applied as energy dissipation contacts.
Appendix A

Test Measurements

A.1 Frequency response function

Figure A.1: Bode plot for the experimental setup without friction.
A. Test Measurements

A.2 Coherence force and velocity signal

Figure A.2: Coherence between force and velocity signal for the experimental setup without friction.
Appendix B

Measurements compared to 1-DOF model with coulomb friction

This appendix shows the normalized energy dissipation calculated from the measurement data compared to the analytical solution of the 1-DOF model with coulomb friction, see Figure 2.1(b). In Section 4.2.1 the normalized energy dissipation is depicted with the measured friction force that is normalized using the excitation force at the threshold of stick-slip to continuous sliding (4.7). The energy dissipation depicted against the measured friction force normalized using the excitation force at the moment of release in the stick-slip phase (4.8) is presented in Section 4.2.2. Note that the energy dissipation curves are only displayed if the threshold of stick-slip to continuous sliding can be determined from the data. Seen from the side of continuous sliding, at the first measurement where the system is not in continuous sliding for all measured periods the symbol for stick-slip and continuous sliding are depicted crossed.

The values of the maximum normalized energy dissipation, phase shift of velocity and excitation force (at maximum energy dissipation and at the threshold of stick-slip and continuous sliding), the number of measured periods and some other values tabulated in Appendix C.

B.1 Friction normalized using $F_0$ threshold stick-slip to continuous sliding

The measured friction force is normalized using the excitation force at the threshold of stick-slip to continuous sliding (4.7). Using this method of normalizing the threshold from stick-slip to continuous sliding occurs by definition at the theoretical value of the model at $f_r = 0.537$. 

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The normalized energy dissipation calculated using the measurement data of series A agrees quite well with the analytic solution as presented in figure B.1. There are some discrepancies. In measurement series A the maximum values of the normalized energy dissipation is considerable higher than the analytical values. Besides, in the stick-slip mode the measured energy dissipation levels are lower than the analytic values for the increasing \( f_r \) in Figure B.1(a) and the decreasing \( f_r \) in Figure B.1(d). Moreover, for the increasing \( f_r \) at 14 Hz in Figure B.1(b) the maximum energy dissipation is at the transition of continuous sliding to stick-slip.

![Graphs](image)

**Figure B.1:** Normalized energy dissipation against normalized friction force calculated by (4.7). (a) 13 Hz, (b) 14 Hz, (c) 15 Hz and (d) 16 Hz. 

*Measurement series A*

The curves of the measurement data in Figure B.2 fit quite well on the analytic one for the continuous sliding mode. The measured maximum energy dissipation corresponds mostly with the analytic maximum value. The excitation frequency does not influence the maximum value. In most cases for measurement series B the curves of the energy dissipation in the stick-slip mode are higher than the analytic but not at 13 Hz with increasing \( f_r \) in Figure B.2(a) and at 18 Hz in Figure B.2(f). The differences between the measured curves are probably caused by the non-stationary behavior of dry friction.
B.1 Friction normalized using $F_0$ threshold stick-slip to continuous sliding

Figure B.2: Normalized energy dissipation against normalized friction force calculated by $\frac{\Delta F}{F_0}$. (a) 13 Hz, (b) 14 Hz, (c) 15 Hz, (d) 16 Hz, (e) 17 Hz and (f) 18 Hz.

Measurement series B
B. Measurements compared to 1-DOF model with coulomb friction

The energy dissipation curves of measurement series C in Figure B.3 are similar to those of measurement series B in Figure B.2.
B.1 Friction normalized using $F_0$ threshold stick-slip to continuous sliding

Figure B.4: Normalized energy dissipation against normalized friction force calculated by (4.7), (a) 13 Hz, (b) 14 Hz, (c) 15 Hz and (d) 16 Hz.

The energy dissipation of the measurement series D in Figure B.4 fits quite well to the analytic curve. Specially in the stick-slip mode the correspondence between the analytic curve and the measurements is more than in the previous measurements. Except the measurement of decreasing $f_r$ in Figure B.4(a) which does not fit on the analytic curve because for some unknown reason the energy dissipation in continuous sliding mode decreases more for lower normalized friction forces than expected from the analytic model.
B. Measurements compared to 1-DOF model with coulomb friction

B.2 Friction normalized using $F_r$ measured in stick-slip phase

The friction force is normalized using the friction force in stick-slip mode (4.3). This friction force is estimated by measuring the excitation force at the moment of release in the stick-slip phase. According to the proposed model the maximum energy dissipation should occur at $f_r = 0.45$ and the threshold of stick-slip to continuous sliding at $f_r = 0.537$.

![Normalized energy dissipation](image)

Figure B.5: Normalized energy dissipation against normalized friction force calculated by (4.8). (a) 13 Hz, (b) 14 Hz, (c) 15 Hz and (d) 16 Hz.

Measurement series A

The tops of the curves of the energy dissipation in Figure B.5 are wider in comparison with those in Figure B.4 and the curves are shifted to higher friction force values.

In Figure B.6 the same phenomena are observable for the energy dissipation curve as in Figure B.5.
B.2 Friction normalized using $F_r$ measured in stick-slip phase

Figure B.6: Normalized energy dissipation against normalized friction force calculated by (4.8). (a) 13 Hz, (b) 14 Hz, (c) 15 Hz, (d) 16 Hz, (e) 17 Hz, and (f) 18 Hz.

Measurement series B
B. Measurements compared to 1-DOF model with coulomb friction

Figure B.7: Normalized energy dissipation against normalized friction force calculated by (4.8), (a) 13 Hz, (b) 14 Hz, (c) 15 Hz and (d) 16 Hz.

Just as in Figure B.6, the curves in Figure B.7 are shifted to higher normalized friction force.

Measurement series C
B.2 Friction normalized using $F_r$ measured in stick-slip phase

Figure B.8: Normalized energy dissipation against normalized friction force calculated by (4.8), (a) 13 Hz, (b) 14 Hz, (c) 15 Hz and (d) 16 Hz.

Determining the friction force has been more successful for measurement series D. The curves of the energy dissipation are less shifted than for the previous measurement series. The curves of the energy dissipation fit to the analytic for the measurements with 14 and 15 Hz and for the decreasing $f_r$ at 16 Hz in Figure B.8(b), B.8(c) and B.8(d), respectively.
B. Measurements compared to 1-DOF model with coulomb friction
# Appendix C

## Measurement Data

In this appendix, data is presented which is collected from the measurements for each measurement series. Therefore the following terms are used:

<table>
<thead>
<tr>
<th>Term</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freq</td>
<td>excitation frequency in Hz</td>
</tr>
<tr>
<td>way</td>
<td>'UP' for increasing input voltage amplitude (results in decreasing $f_s$)</td>
</tr>
<tr>
<td></td>
<td>'DOWN' for decreasing input voltage amplitude (results in increasing $f_s$)</td>
</tr>
<tr>
<td>F_0thresh</td>
<td>mean excitation force at the threshold of stick-slip and continuous sliding in N</td>
</tr>
<tr>
<td>F_edmax</td>
<td>mean excitation force at maximum normalized energy dissipation in N</td>
</tr>
<tr>
<td>edmax</td>
<td>mean maximum normalized energy dissipation</td>
</tr>
<tr>
<td>phi_thresh</td>
<td>mean phase shift between force and velocity at the threshold of stick-slip and continuous sliding in rad</td>
</tr>
<tr>
<td>phi_edmax</td>
<td>mean phase shift between force and velocity at maximum normalized energy dissipation in rad</td>
</tr>
<tr>
<td>F_release</td>
<td>mean excitation force at release in stick-slip mode in N</td>
</tr>
<tr>
<td>F_min</td>
<td>mean minimum excitation force in N</td>
</tr>
<tr>
<td>F_max</td>
<td>mean maximum excitation force in N</td>
</tr>
<tr>
<td>Volmin</td>
<td>minimum input voltage amplitude in V</td>
</tr>
<tr>
<td>Volmax</td>
<td>maximum input voltage amplitude in V</td>
</tr>
<tr>
<td>periods</td>
<td>number of measured periods</td>
</tr>
</tbody>
</table>

### Measurement series = A

**Result**

<table>
<thead>
<tr>
<th>Freq: [13 13 14 14 15 15 16 ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>way: {'UP' 'DOWN' 'UP' 'DOWN' 'UP' 'DOWN' 'UP'}</td>
</tr>
<tr>
<td>F_0thresh: [0.1154 0.0993 0.1081 0.0820 0.0870 0.0649 0.0896]</td>
</tr>
<tr>
<td>F_edmax: [0.1258 0.1298 0.1295 0.0830 0.1142 0.0761 0.1097]</td>
</tr>
<tr>
<td>phi_thresh: [0.6962 0.7463 0.7144 0.6681 0.6844 0.6221 0.6895]</td>
</tr>
<tr>
<td>phi_edmax: [0.7974 0.8905 0.8297 0.8128 0.8784 0.8784 0.9168]</td>
</tr>
<tr>
<td>F_release: [0.0662 0.0724 0.0655 0.0572 0.0633 NaN 0.0515]</td>
</tr>
<tr>
<td>F_min: [0.0601 0.0602 0.0602 0.0655 0.0659 0.0649 0.0613]</td>
</tr>
<tr>
<td>F_max: [0.2962 0.2955 0.3820 0.3850 0.4876 0.4899 0.5305]</td>
</tr>
<tr>
<td>Volmin: [0.0610 0.0610 0.0610 0.0610 0.0610 0.0610 0.0610]</td>
</tr>
<tr>
<td>Volmax: [0.2660 0.2660 0.2660 0.2660 0.2660 0.2660 0.2360]</td>
</tr>
<tr>
<td>periods: [24 24 26 26 28 28 30 ]</td>
</tr>
</tbody>
</table>
C. Measurement Data

**Measurement series = B**

Result -

Freq: [13 13 14 14 15 15 16 16 17 17 18 18 ]

way: {'UP' 'DOWN' 'UP' 'DOWN' 'UP' 'DOWN' 'UP' 'DOWN' 'UP' 'DOWN' 'UP' 'DOWN' }

F_0thresh: [0.1397 0.1277 0.1254 0.1257 0.1249 0.1265 0.1273 0.1257 0.1166 0.1117 0.0929 0.0963]

F_0edmax: [0.1729 0.1706 0.1457 0.1358 0.1421 0.1590 0.1399 0.1691 0.1334 0.1260 0.1222 0.1196]

edmax: [1.3266 1.2903 1.3169 1.2829 1.3686 1.3469 1.3440 1.2544 1.3406 1.3099 1.1498 1.3424]

phi_thresh: [0.7928 0.7720 0.7688 0.7745 0.7692 0.7890 0.7767 0.7707 0.7883 0.8882 0.8909 0.8772]

phi_edmax: [0.9279 0.9235 0.8799 0.8577 0.8664 0.9442 0.8882 0.9809 0.8772 0.8637 0.8620 0.8601]

F_0release: [0.1126 0.0928 NaN 0.0834 0.0824 0.0736 0.0764 0.0836 0.0786 0.0797 0.0728 0.0737]

F_0min: [0.0618 0.0618 0.0618 0.0642 0.0637 0.0642 0.0631 0.0642 0.0631 0.0642 0.0624 0.0629]

F_0max: [0.1866 0.1882 0.2261 0.2270 0.2669 0.2665 0.2770 0.2813 0.2777 0.2776 0.2497 0.2553]

Volmax: [0.1570 0.1570 0.1570 0.1570 0.1570 0.1570 0.1570 0.1570 0.1570 0.1570 0.1570 0.1570]

Volmin: [0.0610 0.0610 0.0610 0.0610 0.0610 0.0610 0.0610 0.0610 0.0610 0.0610 0.0610 0.0610]

periods: [18 18 20 20 22 22 23 23 25 25 26 26 ]

**Measurement series = C**

Result -

Freq: [13 13 14 14 15 15 16 16 ]

way: {'UP' 'DOWN' 'UP' 'DOWN' 'UP' 'DOWN' 'UP' 'DOWN' }

F_0thresh: [0.1505 0.1362 0.1385 0.1346 0.1445 0.1294 0.1270 0.1443]

F_0edmax: [0.1787 0.1864 0.1818 0.1798 0.1956 0.1436 0.1463 0.1563]

edmax: [1.3943 1.3286 1.3085 1.2752 1.3231 1.2708 1.3180 1.2839]

phi_threshold: [0.7642 0.8038 0.7690 0.8326 0.8379 0.8171 0.7398 0.8251]

phi_edmax: [0.8418 0.8870 0.9199 0.9124 0.9556 0.9028 0.8640 0.8939]

F_0release: [0.1114 0.1057 0.0997 0.1000 0.0897 0.1020 0.0889 0.0920]

F_0min: [0.0868 0.0862 0.0856 0.0874 0.0874 0.0897 0.0887 0.0899]

F_0max: [0.1866 0.1882 0.2261 0.2270 0.2669 0.2665 0.2770 0.2813]

Volmax: [0.1570 0.1570 0.1570 0.1570 0.1570 0.1570 0.1570 0.1570]

Volmin: [0.0810 0.0810 0.0810 0.0810 0.0810 0.0810 0.0810 0.0810]

periods: [18 18 20 20 22 22 23 23 ]

**Measurement series = D**

Result -

Freq: [13 13 14 14 15 15 16 16 ]

way: {'UP' 'DOWN' 'UP' 'DOWN' 'UP' 'DOWN' 'UP' 'DOWN' }

F_0thresh: [0.1487 0.0993 0.0988 0.1076 0.1004 0.1010 0.1046 0.0837]

F_0edmax: [0.1681 0.1202 0.1720 0.1255 0.1184 0.1182 0.1170 0.1073]

edmax: [1.3234 1.2904 1.3728 1.2831 1.3453 1.3391 1.3854 1.3286]

phi_threshold: [0.8257 0.7400 0.6528 0.7551 0.6169 0.6845 0.6312 0.7197]

phi_edmax: [0.9288 0.8789 0.9350 0.8662 0.7221 0.8351 0.7059 0.8161]

F_0release: [0.0877 0.0619 0.0625 0.0550 0.0555 0.0584 0.0553 0.0585]

F_0min: [0.0868 0.0634 0.0633 0.0652 0.0639 0.0638 0.0641 0.0663]

F_0max: [0.2711 0.2692 0.3241 0.3217 0.3773 0.3602 0.4431 0.4568]

Volmax: [0.2070 0.2070 0.2070 0.2070 0.2070 0.2070 0.2070 0.2070]

Volmin: [0.0610 0.0610 0.0610 0.0610 0.0610 0.0610 0.0610 0.0610]

periods: [24 24 26 26 28 28 30 30 ]
Bibliography


Appendix D

Stabilization of Linear Systems through Parameter Vibrations
D. Stabilization of Linear Systems through Parameter Vibrations

D.1 Introduction

D.1.1 Vibrational control

Linear vibrational control is the method to change the system dynamics significantly by adding periodic or stochastic vibrations to the system parameters. The possibility of parametric periodic vibrations to stabilize a linear system is known for almost a century [Fradkov & Evans 2001]. Linear vibrational control can be used to stabilize linear systems or to change the system dynamics in such a way that the system can be stabilized using common control techniques. A linear system is modeled by a finite number of coupled first-order differential equations which parameters determines the stability of the equilibrium point.

A simple example of vibrational control is stabilization of the unstable equilibrium of a pendulum by vertical oscillation at sufficiently high frequency and low amplitude [Shapiro & Zinn 1997]. Vibrational control can also be used in chemical reactors [Cinar et al. 1987], where periodic input flow rates stabilize an unstable steady-state of the reactor-system. This results in a higher production rate or lower energy expenditure compare to the stable reactor state with fixed flow rates. Other practical examples are underactuated mechanical systems [Hong et al. 1998] and computing trajectories for robotic locomotion systems [Bullo 2002].

Compared to linear feedback control, the advantage of vibrational control is that measurements of deviations and disturbances are not required [Meerkov 1980] and that underactuated systems due to either design or failure, still can be controlled. In Meerkov 1980 the author mentioned that in stead of vibrations white noise could have similar stabilizing effect. In Townley et al. 2003 the stabilizing effect of white noise is proven. Still, the stabilization of systems by vibration of the parameters with periodic or random vibrations is quite an unknown area within mechanical engineering.

D.1.2 Objectives and outline

The aim of this work is to explore when linear vibrational control can be used to stabilize linear time-invariant systems. Further, we will investigate whether stabilization by periodic excitation gives comparable results as with stabilization by linear white noise excitation. To achieve this aim,

- various methods for determining stability properties of systems with linear time-periodic or stochastic vibrating parameters are studied,
- the usefulness of these methods is illustrated with an example,
- the example system (and varieties of it) is simulated.

The outline of this part of the report will be as follows: in the next chapter, a literature survey is presented. The theory of stabilization by stochastic vibration is provided. Various methods of determining the stability properties of linear systems with time-varying parameters are presented. Also, several strategies of linear vibrational control are provided. In Chapter D.3, an example is used to apply the obtained methods of determining stability properties in order to examine their usefulness. In Chapter D.4, simulation results are presented of band-limited white noise with a high and a low frequency band. The low frequency band seems more adequate in stabilizing the system. Next, the same system is used for time-periodic parameter vibrations. Various vibration amplitudes and frequencies are simulated to examine their influence on the stabilizing effect of the vibration. Then these simulations are also performed for some other system matrices. The results are compared to the theory. Finally, in Chapter D.5 the conclusions and recommendations are presented.
D.2 Literature survey of linear vibrational systems

The possibilities of linear vibrational control are achieved by a literature survey. First, the stabilization of a linear time-invariant system by noise is provided. Next, various theories to determine stability properties of linear time-periodic (LTP) systems described in literature will be treated. Furthermore, stabilization or control strategies for linear time-invariant systems by means of linear time-periodic vibrations. Finally, the usefulness of the provided theories is discussed.

D.2.1 Stabilization via white noise vibration

Noise assisted stabilization of linear time-invariant (LTI) systems is shown by Townley et al. (2003). Consider a general linear stochastic differential equation (LSDE) of the form

\[ dx = Ax dt + u B \circ dW(t) \]  

(D.1)

where \( \circ \) denotes that the equation is interpreted in Stratonovich sense, \( u \) denotes the control, \( A, B \) are matrices of suitable dimensions and \( W \) is an independent Wiener process.

There are several notions of exponential growth rates for a LSDE. The almost surely exponentially growth rate is determined by the leading Lyapunov exponent. The leading Lyapunov exponent of (D.1) is defined as

\[ \lambda = \lim_{t \to \infty} \frac{1}{t} \log \left( \sup_{\|x_0\|=1} \|x(t, x_0)\| \right) \]  

(D.2)

where \( \| \cdot \| \) is any norm on \( \mathbb{R}^d \). The LSDE is said to be almost surely exponentially stable if \( \lambda < 0 \).

Another important notion is the exponential growth rate in the \( p \)th mean of a LSDE is given by

\[ g(p) = \lim_{t \to \infty} \frac{1}{t} \log E \left( \sup_{\|x_0\|=1} \|x(t, x_0)\|^p \right) \]  

(D.3)

where \( E \) denotes expectation. In particular, the growth rate in the \( p \)th mean is greater than or equal to the almost sure growth rate for \( p > 0 \).

For (D.1) with \( B \) is skew-symmetric the following limiting behavior of the leading Lyapunov exponent is obtained

\[ \lambda \longrightarrow \frac{1}{n} \operatorname{tr} A = \frac{1}{n} \sum_{i=1}^{n} a_{ii} \quad \text{as} \quad u \longrightarrow \infty. \]  

(D.4)

where \( a_{ii} \) denotes the elements of the \( n \times n \) matrix \( A \). This means that a deterministic system \( \dot{x} = Ax \) with a negative trace, it suffices to agitate the system by noise of sufficiently high intensity to achieve stability with a probability of one. Consider the two dimensional version of (D.1)

\[ dx = \begin{pmatrix} a - k & b \\ c & d \end{pmatrix} x dt + \sigma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x \circ dW(t). \]  

(D.5)

where \( a, b, c, d \) are fixed parameters. The drift matrix \( \begin{pmatrix} a - k & b \\ c & d \end{pmatrix} \) in (D.5) arises from the 2-dimensional system \( \dot{x} = Ax + Bu, \ y = Cx \) in the case \( u = -ky \) and \( CB > 0 \).

In Townley et al. (2003) it is shown that the leading Lyapunov exponent of (D.5) is given by

\[ \lambda_{k, \sigma} = d - \frac{\sigma^2}{2} + \left( bc - \frac{\sigma^4}{4} \right) k^{-1} + O(k^{-2}) \]  

(D.6)

for large gain \( k \) and every \( a, b, c, d \in \mathbb{R} \). Therefore, (D.5) is almost surely exponentially stabilizable for \( k \) and \( \sigma \) high enough. For high-gain it is shown that \( \lim_{k \to \infty} \lambda_{k, \sigma} = \frac{1}{2} (2d - \sigma^2) \), so that (D.5) is high-gain almost surely exponentially stabilizable if and only if \( d < \frac{1}{2} \sigma^2 \).
D. Stabilization of Linear Systems through Parameter Vibrations

The exponential growth rate of the second mean of (D.5), which is \( g(2)/2 \) with \( g(2) = g_{k, \sigma}(2) \) given by (D.3), satisfies

\[
g_{k, \sigma}(2) = (2d - \sigma^2) + \left( \frac{2bc - \sigma^4}{2} \right) k^{-1} + O(k^{-2}) = 2\lambda_{k, \sigma} + \sigma^4 k^{-1} + O(k^{-2}). \tag{D.7}
\]

for large \( k \) and for every \( a, b, c, d \in \mathbb{R} \). This shows that the difference between almost sure and second mean exponents disappears asymptotically with order \( 1/k \).

For (D.5) with a low gain \( k \), the limit of the leading Lyapunov exponent for \( \sigma \) to infinity is

\[
\lim_{\sigma \to \infty} \lambda = \frac{1}{2} (a + d). \tag{D.8}
\]

So, almost sure stability is obtained if the mixing of the negative trace by noise is strong enough.

Consider again the Stratonovich LSDE (D.5) but now with non-skew-symmetric noise such that

\[
dx = \begin{pmatrix} a-k & b \\ c & d \end{pmatrix} x dt + \begin{pmatrix} \gamma & -\sigma \\ \sigma & \gamma \end{pmatrix} x \circ dW(t). \tag{D.9}
\]

Then the following holds:

- The almost sure exponential growth rate of (D.9), the Lyapunov exponent \( \lambda_{k, \sigma} \), is independent of \( \gamma \). In particular,

\[
\lambda_{k, \sigma} = d - \frac{\sigma^2}{2} + O(k^{-1}), \quad \text{and therefore} \quad \lim_{k \to \infty} \lambda_{k, \sigma} = d - \frac{\sigma^2}{2}. \tag{D.10}
\]

for every \( \gamma \in \mathbb{R} \).

- The exponential growth rate of the second mean, \( g_{k, \sigma, \gamma}(2)/(2) \), is given by

\[
g_{k, \sigma, \gamma}(2) = 2d + 2\gamma^2 - \sigma^2 + O(k^{-1}) \tag{D.11}
\]

for large \( k \). In particular,

\[
\lim_{k \to \infty} g_{k, \sigma, \gamma}(2) = 2d + 2\gamma^2 - \sigma^2. \tag{D.12}
\]

This shows that (D.9) for \( \gamma < \sigma/\sqrt{2} \) and large \( k \) is almost sure exponential stable with \( \lambda_{k, \sigma} \) as in (D.10). However, the LSDE (D.9) for \( \gamma > \sigma/\sqrt{2} \) and large \( k \) is unstable in the second mean exponential with \( g_{k, \sigma, \gamma}(2) \) as in (D.11). Combining (D.10) and (D.12) gives the following asymptotic relation between the exponential growth rates \( g(2)/2 = \lambda + \gamma^2 \).

It is shown that noise is capable to stabilize the unstable zero dynamics of a LTI system by noise assisted high-gain feedback if the trace of the system matrix is smaller than zero.

D.2.2 Stability of linear vibrational systems

In this section a literature survey of stability theories of linear time-varying (LTV) systems is provided with the focus on linear time-periodic (LTP) systems. The most common definition of stability of an equilibrium is the stability in the sense of Lyapunov, see Khalil (1992). An equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise it is unstable. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity.

Consider the linear time-varying system

\[
\dot{x}(t) = A(t)x(t) \tag{D.13}
\]

with an equilibrium point at \( x = 0 \). Let \( A(t) \) be piecewise continuous for all \( t \geq 0 \). Then the equilibrium point \( x = 0 \) of (D.13) is
D.2 Literature survey of linear vibrational systems

- (globally) uniformly stable if and only if for each $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$, independent of $t_0$, such that
  \[ \|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0 \geq 0 \]
- (globally) uniformly asymptotically stable if and only if it is uniformly stable and there is $c > 0$, independent of $t_0$, such that
  \[ x(t) \to 0 \quad \text{as} \quad t \to 0 \]
uniformly in $t_0$, for all $\|x(t_0)\| < c$.

Due to the linear dependence of $x(t)$ on $x(t_0)$, if the origin is uniformly asymptotically stable it is globally so.

For the linear time-varying system (D.13) the stability of the origin as an equilibrium point can be completely defined in terms of the state transition matrix, $\Phi(t, t_0)$. The state transition matrix describes the solution of (D.13) as a linear function of $x(t_0)$, such that the solution of (D.13) can be written as
\[ x(t) = \Phi(t, t_0)x(t_0). \] (D.14)

The following theory characterizes uniform asymptotic stability in terms of $\Phi(t, t_0)$.

The equilibrium point $x = 0$ of (D.13) is (globally) uniformly asymptotically stable if and only if the state transition matrix satisfies the inequality
\[ \|\Phi(t, t_0)\| \leq ke^{\gamma(t-t_0)}, \quad \forall t \geq t_0 \geq 0 \] (D.15)
for some positive constants $k$ and $\gamma$.

The stability of an equilibrium point of linear time-varying (LTV) systems can not be determined by the methods used for linear time-invariant (LTI) systems, e.g. the eigenvalues of the system matrix, $A(t)$, see example 4.5 in [Khalil 1992] and [Unbehauen et al. 2009].

Various methods to determine the stability of an equilibrium point of a linear time-varying system are treated in terms of the stability definitions described above.

D.2.2.1 Lyapunov’s direct method

The classical Lyapunov theory for autonomous systems to determine the stability is very well known, as mentioned in [Aeyels 1993], [Khalil 1992]. An equilibrium of an autonomous system is stable if there exist a positive-definite Lyapunov function with a negative-definite derivative. A common Lyapunov function is the energy of the system. If the energy dissipates the system will stop moving and hence the equilibrium is stable. The extended theory for non-autonomous systems, see [Khalil 1992], [Slotine & Li 1991], will be presented below.

To characterize the stability of LTV systems a special class of scalars is defined: a continuous $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Consider the system (D.13) with $x = 0$ is the equilibrium point. Let $D = \{x \in \mathbb{R}^n \mid \|x\| < r\}$ and let $V : 0, \infty) \times D \to \mathbb{R}$ be a continuously differentiable function such that
\[ \alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|), \] (D.16)
\[ \dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \dot{x} \leq -\alpha_3(\|x\|) \] (D.17)
\[ \forall t \geq 0, \forall x \in D, \text{where } \alpha_1(\cdot), \alpha_2(\cdot) \text{ and } \alpha_3(\cdot) \text{ are class } \mathcal{K} \text{ functions defined on } 0, r. \]

Then $x = 0$ is uniformly asymptotically stable [Khalil 1992].

If the assumptions are satisfied globally (for all $x \in \mathbb{R}^n$) and $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ belong to class $\mathcal{K}_\infty$, then $x = 0$ is globally uniformly asymptotically stable.

The great advantage of the direct Lyapunov method is that the stability is defined without solving (D.13). Moreover, the theorem of Lyapunov’s direct method exists in converse. That is, if the equilibrium point of (D.13) is stable, then there exist a function $V(t, x)$ satisfying the conditions (D.16) and (D.17).
A suitable candidate for the Lyapunov function $V(t, x)$ for system (D.13) with a stable equilibrium point $x = 0$ can be found as follows. Suppose there is a piecewise continuously differentiable, symmetric, bounded, positive definite matrix $P(t)$ which satisfies the inequality

$$0 < c_1 I \leq P(t) \leq c_2 I, \quad \forall t \geq 0$$

for some positive constants $c_1$ and $c_2$, as well as the matrix differential equation

$$-\dot{P}(t) = \dot{P}(t)A(t) + A^T(t)P(t) + Q(t)$$

where $Q(t)$ is a bounded, continuous, symmetric matrix which satisfies the inequality

$$c_4 I \geq Q(t) \geq c_3 I > 0, \quad \forall t \leq 0,$$

with $c_3$ and $c_4$ are some positive constants. Consider a Lyapunov function candidate $V(t, x) = x^T P(t) x$. The function $V(t, x)$ is positive definite and decrescent since

$$c_1 \|x\|^2_2 \leq V(t, x) \leq c_2 \|x\|^2_2.$$

Moreover, it is radially unbounded since the function $c_1 \|x\|^2_2$ belongs to class $\mathcal{K}_\infty$. The derivative of $V(t, x)$ is given by

$$\dot{V}(t, x) = x^T \dot{P}(t) x + x^T \dot{P}(t) + \dot{x}^T P(t) x$$

$$= x^T \left[ P(t) + \dot{P}(t) A(t) + A(t)^T P(t) \right] x$$

$$= -x^T Q(t) x \leq -c_3 \|x\|^2_2.$$

Hence, $\dot{V}(t, x)$ is negative definite. Thus the assumptions for of the Lyapunov’s direct theory, (D.16) and (D.17), are satisfied globally with $\alpha_i = c_i r^2$ for $i = 1, 2, 3$. Therefore the equilibrium of (D.13) is globally exponentially stable.

Thus by solving (D.18) for a continuously differentiable, symmetric, bounded, positive definite matrix $P(t)$ and a bounded continuous, symmetric, positive definite matrix $Q(t)$ a Lyapunov function can be calculated for the stable equilibrium $x = 0$ of (D.13). If the equilibrium point of the linear system (D.13) is uniformly asymptotically stable, then there is a solution for (D.18) that possesses the desired properties.

The invariance theory of La Salle’s describes an alternative approach to conclude asymptotic stability of the equilibrium point based on the trajectories of the system. The equilibrium point of (D.13) is also asymptotically stable if the trajectories of the system to the equilibrium point results in $V(t, x) = 0$ when the integral of $V(t, x)$ satisfies a certain inequality.

Suppose $Q(t)$ in (D.18) can be written as $C(t) C(t)$ where $C(t)$ is continuous in $t$. The derivative of the Lyapunov candidate along the trajectories of the system is

$$\dot{V}(t, x) = -x^T C(t)^T C(t) x \leq 0.$$

The solution of the linear system, (D.13) is given by $\Phi(t, t, x) = \Phi(t, t) x$ where $\Phi(t, t)$ is the transition matrix, so that

$$\int_t^{t+\delta} \dot{V}(t, \Phi(t, t, x)) d\tau = -x^T \int_t^{t+\delta} \Phi(t, \tau) C(t)^T C(t) \Phi(t, \tau) d\tau$$

$$= -x^T W(t, t+\delta) x$$

Suppose there is a positive constant $k < c_2$ such that

$$W(t, t+\delta) \geq k I, \quad \forall t \geq 0$$

then

$$\int_t^{t+\delta} \dot{V}(t, \Phi(t, t, x)) d\tau \leq -k \|x\|^2_2 \leq -\frac{k}{c_2} V(t, x).$$
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In the case that
\[ \alpha_i(r) = c_i r^2, \quad i = 1, 2, \quad \lambda = \frac{k}{c_2} < 1 \]
the origin of (D.15) is globally exponentially stable.

Note that \( W(t, t + \delta) \) is the observability Grammian of the pair \((A(t), C(t))\), with \(C(t)\) the output matrix. If the pair \((A(t), C(t))\) is uniformly observable, then the positive definiteness requirement of \(Q(t)\) can be replaced by the weaker requirement \(Q(t) = C^T(t)C(t)\) while the conclusion of stability holds.

D.2.2.2 Floquet transition matrix

The fundamental technique to determine stability properties of LTP systems is the calculation of the monodromy matrix, also known as the Floquet Transition Matrix. This matrix is, in short, the fundamental matrix of the homogeneous time-periodic system after one period of time with begin position \(X_n(0) = I_n\). To be complete:

The linear time-periodic system
\[ \dot{x}(t) = A(t)x(t) \] (D.19)

where \(A(t)\) is a \(n \times n\) \(T\)-periodic matrix, so \(A(t) = A(t + T)\), with the initial conditions \(x(0) = x_0\), has a standard solution
\[ x(t) = x_0 + \int_0^t A(s)x(s)ds. \] (D.20)

A matrix of solutions is created. Let \(x_j(t)\) (\(j = 1, ..., n\)) be \(n\) solutions of (D.19) (\(x_j(t)\) is a column vector) with initial conditions \(x_j(t_0) = e_j\), \((j = 1, ..., n)\) where \(e_j\) is the \(j\)th column of \(I_n\). Then the \(n \times n\) matrix
\[ X(t) = [x_1(t), ..., x_n(t)] \] (D.21)
satisfying the matrix differential equation
\[ \dot{X}(t) = A(t)X(t), \quad X(t_0) = I_n \] (D.22)
is the fundamental matrix or the state transition matrix of (D.19) over the interval \((t_0, t)\).

In short \(X(t, t_0 = 0)\) will be denoted as \(X(t)\).

Since \(A(t)\) in (D.19) is homogeneous \(T\)-periodic, the state transition matrix of (D.19) will satisfy the identity
\[ X(t + T) = X(t)X(T), \quad \forall t. \] (D.23)

By definition \(X(T) \triangleq X\) is called the Monodromy Matrix or the Floquet Transition Matrix and its eigenvalues are the (Floquet) Multipliers.

Thus, the monodromy matrix is defined as the state transition matrix \(X(t, 0)\) defined by the initial condition \(X(0, 0) = I_n\) at \(t = T\) (the “end” of a period). The multipliers are the roots of the equation
\[ \det[X - \rho I_n] = 0 \] (D.24)

which is the characteristic equation of system (D.19), see [Yakubovich & Starzhinskii 1975]. If all eigenvalues of the monodromy matrix are within the unity circle, the equilibrium point of the system (D.19) is globally uniformly asymptotically stable. If one eigenvalue of the monodromy matrix is outside the unity circle, the equilibrium point of (D.19) is unstable.

In the simulations of linear vibrational systems in Chapter D.4 the stability is determined by numerically calculating the monodromy matrix.

It van be proven that a monodromy matrix always consist of an exponential drift times a periodic function. Let \(\Lambda\) denote the constant \(n \times n\) matrix
\[ \Lambda = \frac{1}{T} \ln X \] (D.25)
and $F(t)$ denotes the matrix function

$$F(t) = X(t)e^{-t\Lambda}.$$  \hspace{1cm} (D.26)

The matrix $A(t)$ in (D.19) is real and $T$-periodic. Hence, $X$ is also a real matrix and all eigenvalues of $X$ are positive real. Therefore $\Lambda$ and $F(t)$ will be real. Using the identity (D.23), it follows that

$$F(t + T) = X(t + T)e^{-T\Lambda} = X(t)X(T)X(T)^{-1}e^{-t\Lambda} = F(t).$$

Thus the state transition matrix of (D.19) in (D.21) can be written as

$$X(t) = F(t)e^{t\Lambda}.$$  \hspace{1cm} (D.27)

From this the Floquet-Lyapunov theorem, see Yakubovich & Starzhinskii [1975], is stated as follows: The state transition matrix of a system of differential equations (D.19) with $T$-periodic coefficients may be expressed in the form (D.27), where $F(t)$ is a $T$-periodic $n \times n$ matrix-function, non-singular for all $t$, continuous, with an integrable piecewise continuous derivative, where $F(0) = I_n$, and $\Lambda$ is a constant $n \times n$ matrix.

The Floquet-Lyapunov theorem has another version which is the so-called Lyapunov reducibility theorem. The problem of integration of (D.19) can be rewritten by differentiating (D.26) and substituting that in (D.22) such that

$$\frac{dF(t)}{dt} = A(t)F(t) - F(t)\Lambda.$$  \hspace{1cm} (D.28)

The integration of (D.26) is equivalent to solving the following problem: find a constant $n \times n$ matrix $\Lambda$ such that (D.28) has a $T$-periodic solution $F(t)$ for which $\det F(t) \neq 0$.

Given system (D.19) with $T$-periodic elements and the derivative of $F(t)$ as above, the substitution

$$x(t) = F(t)z(t)$$

reduces the original time-varying system (D.19) in to the time invariant system

$$\dot{z}(t) = \Lambda z(t).$$  \hspace{1cm} (D.29)

In some literature, e.g. Joseph & Sinha [1993], this is called the Floquet-Lyapunov transformation. The stability properties of (D.29) are equal to that of the original time-periodic system (D.19). Thus the stability behavior of (D.19) can be determined by examining (D.29) with use of the common techniques for LTI systems.


If the periodic system (D.22) is uniquely solvable, the Liouville-Jacobi formula says that

$$\det(X(t, 0)) = \det(X(0, 0)) \exp(\text{tr} \int_0^t A(\tau)d\tau).$$  \hspace{1cm} (D.30)

Using the monodromy matrix $\overline{X}$ in (D.30) gives

$$\det(\overline{X}) = \exp(\text{tr} \int_0^T A(\tau)d\tau).$$  \hspace{1cm} (D.31)

The determinant of a matrix is equal to the product of the eigenvalues, see (D.24). This gives

$$\rho_1 \rho_2 \cdots \rho_n = \exp(\int_0^T \text{tr} A(\tau)d\tau).$$  \hspace{1cm} (D.32)

Thus only if $\int_0^T \text{tr} A(\tau)d\tau < 0$ all Floquet multipliers can be smaller than 1. Consider the LTI system $\dot{x} = Ax$ with the unstable equilibrium point $x = 0$. Let $B(t)$ be matrix whose elements are $T$-periodic zero mean, then the equilibrium point $x = 0$ of $\dot{x} = (A + B(t))x$ can only be stabilized by
Theorem can also be found in [Sarychev, 2001] but is not valid for non-linear systems [Shapiro & Zhou & Hagiwara, 2005].

An approximation method for the monodromy matrix is found in [Unbehauen et al., 2000]. The state transition matrix $\Phi(t, 0)$ of (D.15) is approximately obtained from the Peano-Baker series

$$\Phi(t, 0) = I + \int_0^t A(\tau_1)d\tau_1 + \int_0^t \int_0^{\tau_1} A(\tau_2)d\tau_2d\tau_1 + \int_0^t A(\tau_1)\int_0^{\tau_1} A(\tau_2)\int_0^{\tau_2} A(\tau_3)d\tau_3d\tau_2d\tau_1 + \ldots.$$  

(D.33)

Evaluation of $\Phi(t, 0)$ from (D.33) is a difficult task, even for simple cases! It is obvious that $\Phi(T, 0)$ is the approximated monodromy matrix.

Another method to obtain a monodromy matrix is by expansion of the periodic system matrix in terms of Chebyshev polynomials over the principal period. Such an expansion reduces the original problem to a set of linear algebraic equations from which the solution in the interval of one period can be obtained. This is described in [Joseph & Sinha, 1993], [Sinha & Wu, 1991] and provides an efficient computational scheme with accurate results. The idea is similar to that of the harmonic balance approach, which provides a frequency response function for linear time-periodic systems. The harmonic balance approach is used in [Zhou & Hagiwara, 2005] to provide a 2-regularized Nyquist criterion to determine the stability properties of a linear time-periodic system.

### D.2.2.3 2-regularized Nyquist criterion

The Nyquist criterion is a well-known method to analyse the stability properties of a LTI system. In this section a generalized Nyquist criterion for LTP systems is provided. Therefore the state of the LTP system is expanded in terms of the Fourier series and so the harmonic state operator is derived, [Wereley & Hall, 1990]. Then a Nyquist-type criterion can be defined based on the 2-regularized determinant of the harmonic state operator and stability properties of a LTP system can be determined, see [Zhou & Hagiwara, 2005].

Consider a linear time periodic system expressed in the state-space form

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x,$$  

(D.34)

where $A(t), B(t)$ and $C(t)$ are $h$-periodically time-varying matrices. Recall from Section D.2.2.2 that the transition matrix of the LTP system (D.34) is has a Floquet factorization $\Phi(t, t_0) = P(t, t_0)e^{Q(t-t_0)}$, where $P(t, t_0)$ is absolutely continuous in $t$, nonsingular and $h$-periodic in $t$ and $t_0$, and $Q$ is a constant matrix. The stability of (D.34) is determined by the eigenvalues of $Q$. We assume $t_0 = 0$ without loss of generality.

Now we review the Toeplitz transformation of periodic functions. The $h$-periodic dynamics matrix, $A(t)$ is expressed in terms of its harmonics, $[A_m|m \in \mathbb{Z}]$ by expanding $A(t)$ to its Fourier series

$A(t) = \sum_{m=-\infty}^{+\infty} A_m e^{im\omega_0 t}$ with $\omega_0 = 2\pi/h$. The Toeplitz transformation on $A(t)$, denoted by $T\{A(t)\}$, maps $A(t)$ onto a doubly infinite-dimensional block Toeplitz operator of the form

$$T\{A(t)\} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots \\ \ddots & A_0 & A_{-1} & \cdots \\ \cdots & A_{1} & A_0 & A_{-1} & \cdots \\ \cdots & A_{2} & A_{1} & A_0 & \cdots \\ \ddots & \ddots & \ddots & \ddots \end{bmatrix} = A.$$  

(D.35)
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We further define $\mathcal{B} = \mathcal{T}\{B(t)\}$, $\mathcal{C} = \mathcal{T}\{C(t)\}$, $\mathcal{P} = \mathcal{T}\{P(t, 0)\}$, $\hat{\mathcal{B}} = \mathcal{T}\{P^{-1}(t, 0)B(t)\}$, and $\mathcal{Q} = \mathcal{T}\{C(t)P(t, 0)\}$ and $\mathcal{Q} = \text{diag}[\ldots, \mathcal{Q}, \mathcal{Q}, \ldots]$. Finally, define the modulation frequency matrix,$$
abla = \text{diag} [\ldots, \varphi_{-2}(s)I, \varphi_{-1}(s)I, \varphi_{0}(s)I, \varphi_{1}(s)I, \varphi_{2}(s)I, \ldots],$$where $\varphi_m(s) = s + j m \omega_h$, $m \in \mathbb{Z}$, $s \in \mathbb{C}$. It follows that $\nabla(s) = \nabla(j0) + s I$, where $I = \mathcal{T}\{I\}$.

Now we introduce the harmonic transfer operator of (D.34) given by
$$\mathcal{g}(s) = \mathcal{C}(\nabla(s) - \mathcal{A})^{-1}\mathcal{B}$$
(D.36)
in which $\mathcal{A} - \nabla(s)$ is called the harmonic state operator of (D.34). The relation between the Floquet state operator, $\mathcal{Q} - \nabla(s)$, and the harmonic state operator is such that
$$\mathcal{P}(\nabla(j0) - \mathcal{Q})\mathcal{P}^{-1} = \nabla(j0) - \mathcal{A}.$$ (D.37)

Furthermore, system (D.34) is asymptotically stable if and only if the set $\Lambda$ of all eigenvalues of $\mathcal{Q} - \nabla(j0)$ lies in the open left-half plane [Zhou & Hagiwara, 2002].

Next, we consider the domain $\Omega \subset \mathbb{C}$ and assume that $s \in \Omega$. Furthermore we assume that $\Omega$ is closed and has a simple boundary denoted by $\delta \Omega$ which contains no points in $\Lambda$. $\nabla(s)$ is invertible on the domain $\Omega$ and $\nabla(s) - \mathcal{Q}$ is invertible for each $s \in \delta \Omega$.

Subsequently relation (D.37) tells us that
$$\mathcal{P}(\nabla(j0) - \mathcal{Q})^{-1}\mathcal{P}^{-1} = (\nabla(j0) - \mathcal{A})^{-1}$$ (D.38)
for all $s \in \Omega \setminus \Lambda$. (D.38) says that the harmonic transfer operator $\mathcal{g}(s)$ is well defined for all $s \in \Omega \setminus \Lambda$. Furthermore, $\|\mathcal{g}(s)\|_2$ has a uniform upper bound over $s \in \delta \Omega$.

Next, the 2-regularized determinant of $(I + \mathcal{A})$ is defined as
$$\det_2(I + \mathcal{A}) = \prod \left[ (I + \lambda_i(\mathcal{A})) \exp(-\lambda_i(\mathcal{A})) \right],$$ (D.39)
where $\lambda_i(\mathcal{A})$ denote the $i$th eigenvalue of the linear operator $\mathcal{A}$ and consider the following property
$$\det_2(I + \mathcal{A}) \det_2(I + B) = \det_2[(I + \mathcal{A})(I + B)] \exp\{\text{tr}(AB)\}.$$ (D.40)

In the case of LTP systems a Nyquist-type criterion is obtained by using the 2-regularized determinant of the modified Harmonic state operator $I - \nabla^{-1}(s)\mathcal{A}$ or the modified Floquet operator $I - \nabla^{-1}(s)\mathcal{Q}$. The relation of the 2-regularized determinant of the modified harmonic state operator and 2-regularized determinant of the modified Floquet operator is known as
$$\det_2[I - \nabla^{-1}(s)\mathcal{A}] = g_A(s) \det_2[I - \nabla^{-1}(s)\mathcal{Q}],$$ (D.41)
where the function $g_A(s)$ does not vanish for each $s \in \Omega$ and is analytic over $\Omega$.

Let $\lambda_k(Q)$ denote the $k$th eigenvalue of the $n \times n$ matrix $Q$. Then, the function
$$f_Q(s) = \det_2[I - \nabla^{-1}(s)\mathcal{Q}]$$
$$= \prod_{k=1}^{n} \prod_{m=-\infty}^{\infty} \left( 1 - \frac{\lambda_k(Q)}{s + jm\omega_h} \right) \exp\left\{ \frac{\lambda_k(Q)}{s + jm\omega_h} \right\}$$ (D.42)
is analytic on $\Omega$, which has a zero at each point $\lambda_k(Q) - jm\omega_h$, $k = 1, 2, \ldots, n$, and $m \in \mathbb{Z}$ and has no other zeros on the complex plane [Zhou & Hagiwara, 2002].

From this we conclude that the set of all zeros of $f_Q(s)$ is equal to the set of $\Lambda$ of all eigenvalues of $\mathcal{Q} - \nabla(s)$. Thus the asymptotic stability of (D.34) can be reflected by the function $f_Q(s)$. This is the basis to create a Nyquist criterion based on the 2-regularized determinant. The Nyquist contour $N_r$.
will be formed by $\delta \Omega$. $N_{c}$ needs to directly pass trough the origin or to include the origin in the interior of the region by bypassing the origin in case of an eigenvalue of $Q - \mathcal{E}(j)0$ at the origin. But then the inverse of $\mathcal{E}(s)$ does not exist on $\Omega$. To surmount this problem a shift factor $\rho$ is introduced. Then on the domain $\Omega$, $\mathcal{E}(s + \rho) = \mathcal{E}(s) + \rho I$ is invertible for $\rho > 0$.

Now that $\rho > 0$ is introduced, $\det_{2}\left[I - \mathcal{E}^{-1}(s + \rho)(\mathcal{A} + \rho I)\right]$ and $\det_{2}\left[I - \mathcal{E}^{-1}(s + \rho)(\mathcal{Q} + \rho I)\right]$ are considered instead of similar relations in (D.41) and (D.42). Therefore the terms $\mathcal{E}(s)$, $g_{A}(s)$, $\mathcal{A}$, $Q$, $\lambda_{k}(Q)$, $f_{Q}(s)$ and $jm_{0}h$ are replaced by $\mathcal{E}(s + \rho)$, $g_{A + \rho I}(s + \rho)$, $\mathcal{A} + \rho I$, $Q + \rho I$, $\lambda_{k}(Q + \rho I)$, $f_{Q + \rho I}(s + \rho)$ and $\rho + jm_{0}h$, respectively. Then the statements made for $g_{A}(s)$ and $f_{Q}(s)$ hold for $g_{A + \rho I}(s + \rho)$ and $f_{Q + \rho I}(s + \rho)$, respectively.

The 2-regularized Nyquist criterion is developed to determine the stability of a closed-loop LTP system. The open-loop Nyquist system [D.34] is formed to a closed-loop by introducing a feedback $u = -K(t)v$, where $k(t)$ is $h$-periodic and $v$ is the new reference input. Then the harmonic transfer function for the closed-loop system is defined by

$$
\mathcal{G}_{c}(s) = \mathcal{E}(s) - \mathcal{A}_{c})^{-1}\mathcal{B}
$$

(D.43)

where $\mathcal{A}_{c} = T\{\mathcal{A}(t)\}$ where $\mathcal{A}(t) = A(t) - B(t)K(t)C(t)$. The statements made above for the open-loop system do apply to the closed-loop system as well.

As known from the LTI case, a relationship between the open- and closed-loop pole polynomials must be obtained to achieve a Nyquist criterion. Therefore, a 2-regularized relation between the open- and closed-loop modified state operators is provided. To derive such relationship, we define $K = T\{K\}$ and we compute the $\det_{2}$ of the return difference operator $I + \mathcal{K}\mathcal{G}$ for each $s \in \Omega \setminus \Delta$, which is defined as

$$
\det_{2}\left[I + \mathcal{K}\mathcal{G}(s)\right] = \det_{2}\left[\left(I - \mathcal{E}^{-1}(s + \rho)(\mathcal{A} + \rho I)\right)^{-1}\left(I - \mathcal{E}^{-1}(s + \rho)(\mathcal{A}_{c} + \rho I)\right)\right]
$$

$$
= \exp(\Delta(s + \rho)) \frac{\det_{2}\left[I - \mathcal{E}^{-1}(s + \rho)(\mathcal{A}_{c} + \rho I)\right]}{\det_{2}\left[I - \mathcal{E}^{-1}(s + \rho)(\mathcal{A} + \rho I)\right]} 
$$

(D.44)

where the scalar function $\Delta(s + \rho)$ is given by

$$
\Delta(s + \rho) = -\text{tr}\left(\mathcal{E}^{-1}(s + \rho)(\mathcal{A} - \rho I)(\mathcal{E}(s) - \mathcal{A})^{-1}\mathcal{B}\mathcal{K}\mathcal{E}\right)
$$

(D.45)

Next an appropriate Nyquist contour is described. Note that all eigenvalues of $Q - \mathcal{E}(s)$ are located in a vertical strip region parallel to the imaginary axis and the eigenvalue distribution in horizontal strip

$$
C_{F} = \{s \in \mathbb{C} : -\omega_{h}/2 < m(s) < \omega_{h}/2\}
$$

(D.46)

which is called the fundamental strip, unfolds itself vertically to both $-j\infty$ and $j\infty$ with the period $j\omega_{h}$ [Zhou & Hagiwara 2002]. In other words, if we understand the eigenvalue distribution pattern in $C_{F}$ then the whole eigenvalue structure is clarified. Then a possible Nyquist contour would be the boundary of the right-half fundamental strip, $\{s : \text{Re}(s) \geq 0, s \in C_{F}\}$. However, the Nyquist contour must avoid $s \in \Lambda$ since $\mathcal{G}(s)$ is not well defined for these points in $\Lambda$. The Nyquist contour $N_{c}$ is plotted in Figure [D.1] where the crosses (\times) denote possible eigenvalues of $Q - \mathcal{E}(j)0$. The possible eigenvalues of $Q - \mathcal{E}(s)$ on $N_{c}$ are bypassed via a semicircle with radius $r$ in the way that is shown in Figure [D.7]. Assume that $N_{cd}$ is chosen far enough from the imaginary axis that there are no eigenvalues of $Q - \mathcal{E}(j)0$ are on it and so that $N_{cd}$ encloses all unstable closed-loop eigenvalues on the fundamental strip in the interior of $N_{c}$. Then the union of $N_{c}$ satisfies the properties for the domain $\Omega$. In [Zhou & Hagiwara 2002] it is demonstrated that the plot segments of $\det_{2}\left[I + \mathcal{K}\mathcal{G}(s)\right] \exp(-\Delta(s + \rho))$ on $N_{cd}$, $N_{cd}$ and $N_{cd}$ neither goes through the origin nor contributes to encirclements around the origin. So, to investigate the encirclements of $\det_{2}\left[I + \mathcal{K}\mathcal{G}(s)\right] \exp(-\Delta(s + \rho))$ around the origin on $N_{c}$, it is enough to see the plot of $\det_{2}\left[I + \mathcal{K}\mathcal{G}(s)\right] \exp(-\Delta(s + \rho))$ corresponding to $N_{ab}$, which is given by $N_{ab} = -\omega_{h}/2 < m(\lambda) < \omega_{h}/2$ from $a = -j\omega_{h}/2$ to $b = j\omega_{h}/2$ and bypassing eigenvalues of
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\[ Q - \mathcal{E}(j0) \] from the left-hand side via a semi-circle with radius \( r \) that is small enough.

Then the following 2-regularized Nyquist criterion can be stated: Let \( n_{us} \) be the number of unstable eigenvalues of the open loop state operator \( A - \mathcal{E}(j0) \) or its Floquet equivalent \( Q - \mathcal{E}(j0) \) with any \( \rho > r \geq 0 \) for any positive \( \rho \) and \( r \). Then \( G_c \) is asymptotically stable if and only if the Nyquist locus \( \det_2 \left[ I + KG(s) \right] e^{-\Delta_1(s+\rho)} : \mathcal{N}_{ab} \to C \), vanishes nowhere on \( \mathcal{N}_{ab} \) and encircles the origin \( n_{us} \) times in counterclockwise sense.

It is shown in Zhou & Hagiwara (2002) that this criterion is also true for

\[
\det_2 \left[ I + K\hat{g}(s) \right] \exp\{\tilde{\Delta}(s+\rho)\} = \frac{\det_2 \left[ I - \mathcal{E}^{-1}(s+\rho)(\tilde{A}_c + \rho I) \right]}{\det_2 \left[ I - \mathcal{E}^{-1}(s+\rho)(Q + \rho I) \right]}
\]

where \( \tilde{A}_c = Q - \hat{B}K\hat{C} \) and

\[
\tilde{\Delta}(s+\rho) = -\text{tr} \left( \mathcal{E}^{-1}(s+\rho)(Q - \rho I)(\mathcal{E}(s) - Q)^{-1} \hat{B}K\hat{C} \right)
\]

with the harmonic transfer operator \( \hat{g}(s) \) been rewritten as

\[
\hat{g}(s) = \hat{C} (\mathcal{E}(s) - Q)^{-1} \hat{B}.
\]

Note that any periodically time-varying state matrix \( A(t) \) can be written in the form \( A_{constant} + \tilde{A}(t) \). Hence, the stability of any LTP system, no matter if the LTP system itself is open- or close-loop, can easily tested by recasting the stability problem as a closed-loop stability problem with \( (A_{constant}, I, I) = (Q, I, I) \) is the open-loop system matrices and \( -\tilde{A}(t) \) being treated as the feedback gain.

Now we have provided a Nyquist-type criterion for LTP system to determine its stability properties. To implement this theory the infinite-dimensional matrices are truncated in two steps. First \( \hat{g}(s) \) to
\[ g_{[N]}(s) = \hat{\mathcal{C}}_{[N]} (E(s) - \mathcal{Q})^{-1} \hat{\mathcal{B}}_{[N]} \quad (D.49) \]

where \( \hat{\mathcal{B}}_{[N]} = T \{ \hat{B} \} \). Here \( \hat{B} = \sum_{m=-N}^{N} \hat{B}_m e^{in \omega_0 t} \) with \( \hat{B}_m \) being the Fourier coefficient sequence of \( \hat{B} \). Similar \( \hat{\mathcal{B}}_{[N]} \) is constructed. Then

\[ g_{[N,M]}(s) = \hat{\mathcal{C}}_{[N,M]} (E_M(s) - \mathcal{Q}_M)^{-1} \hat{\mathcal{B}}_{[N,M]} \quad (D.50) \]

where the infinite-dimensional matrix \( \hat{\mathcal{B}}_{[N,M]} = \text{diag} \left[ \ldots, \hat{B}_{N,M}, \hat{B}_{N,N}, \ldots \right] \)

\[ \hat{\mathcal{B}}_{N,M} = \begin{bmatrix} \hat{B}_0 & \cdots & \hat{B}_{-N} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \hat{B}_N & \cdots & \hat{B}_{-N} & 0 \\ 0 & \hat{B}_N & \cdots & \hat{B}_{-N} \end{bmatrix} \]

\((2M+1)\) blocks

where we assume \( M \geq N + 1 \). The infinite-dimensional matrix \( \hat{\mathcal{C}}_{[N,M]} \) is build similarly but in terms of the Fourier coefficients of \( \hat{\mathcal{C}}(t) \). The infinite-dimensional block-diagonal operators \( E(s) \) and \( \mathcal{Q} \) are partitioned into diagonal blocks by

\[ \mathcal{Q}_M = \text{diag} \left[ \ldots, Q_M, Q_M, Q_M, \ldots \right] (= \mathcal{Q}) \]

\[ E_M(s) = \text{diag} \left[ \ldots, E_{M-1}(s), E_{M0}(s), E_{M1}(s), \ldots \right] (= E(s)) \]

with \( \mathcal{Q}_M = \text{diag} \left[ Q, Q, \ldots, Q \right] \) being \((2M+1) \times (2M+1)\) and

\[ E_{Mm}(s) = \text{diag} \left[ \psi_m(2M+1-M)(s)I, \ldots, \psi_m(2M+1)(s)I, \ldots, \psi_m(2M+1+M)(s)I \right], \]

where \( m \in \mathbb{Z} \).

Next, a convergence lemma associated with the staircase truncation on the harmonic transfer operator in the \( \det_2 \) and the trace sense is presented. The 2-regularized determinant of the difference operator \( [D.44] \), rewritten in truncated terms, is given by

\[ \det_2 \left[ I + K_{[N,M]} g_{[N,M]}(s) \right] \exp \left\{ -\hat{\Delta}_{[N,M]}(s + \rho) \right\} = \prod_{m} \prod_{k} \left( 1 + \lambda_k (G_{m,[N,M]}(s)) \right) \exp \left\{ -\lambda_k (G_{m,[N,M]}(s)) \right\} \times \exp \left\{ -\sum_{m} \text{tr} \left( \hat{\Delta}_{m,[N,M]}(s + \rho) \right) \right\}, \quad m, k \in \mathbb{Z} \quad (D.51) \]

where

\[ \hat{\Delta}_{m,[N,M]}(s + \rho) = E_{mM}^{-1}(s + \rho)(Q - \rho I_M)(E_{Mm}(s) - \mathcal{Q}_M)^{-1} \hat{\mathcal{B}}_{[N,M]} \quad (D.52) \]

where the finite-dimensional matrix \( G_{m,[N,M]}(s) \) is given by

\[ G_{m,[N,M]}(s) = \hat{\mathcal{C}}_{[N,M]} (E_{Mm}(s) - \mathcal{Q}_M)^{-1} \hat{\mathcal{B}}_{[N,M]} \quad (m \in \mathbb{Z}) \quad (D.53) \]

and \( \mathcal{B} \mathcal{K} \mathcal{C}^2_{[N,M]} \) is constructed similar as \( \hat{\mathcal{B}}_{[N,M]} \) but in terms of the Fourier coefficients of \( \hat{\mathcal{B}}(t)K(t)\hat{\mathcal{C}}(t) \).

In Zhou & Hagiwara [2002] it is proved that for any \( \mu > 0 \) and fixed \( N > 0 \), there exists an integer \( M_2(N, \mu) \) such that \( \forall M \geq M_2(N, \mu) \) and \( \forall \omega \in \Omega \)

\[ |\det_2 \left[ I + K_{[N,M]} g_{[N,M]}(s) \right] \exp \left\{ \hat{\Delta}_{[N,M]}(s + \rho) \right\} - \det_2 \left[ I_M + K_{[N,M]} G_{[N,M]}(s) \right] \exp \left\{ -\text{tr} \left( \hat{\Delta}_{[N,M]}(s + \rho) \right) \right\} | < \mu \quad (D.54) \]
Let \( \alpha \) be characteristic exponents of \( \text{(D.13)} \) with the minimum and maximum real parts, respectively. For any \( \epsilon > 0 \), there is a matrix function \( G(t) \) as above such that

\[
\frac{1}{2T} \int_0^T q_1 dt \leq \text{real}(\alpha_1) < \frac{1}{2T} \int_0^T q_1 dt + \epsilon, \quad \frac{1}{2T} \int_0^T q_2 dt - \epsilon < \text{real}(\alpha_2) \leq \frac{1}{2T} \int_0^T q_2 dt
\]

Then the following statement can be made:

Let \( \alpha_1 \) and \( \alpha_2 \) be characteristic exponents of \( \text{(D.13)} \) with the minimum and maximum real parts, respectively. For any \( \epsilon > 0 \), there is a matrix function \( G(t) \) as above such that

\[
\frac{1}{2T} \int_0^T q_1 dt \leq \text{real}(\alpha_1) < \frac{1}{2T} \int_0^T q_1 dt + \epsilon, \quad \frac{1}{2T} \int_0^T q_2 dt - \epsilon < \text{real}(\alpha_2) \leq \frac{1}{2T} \int_0^T q_2 dt
\]

Therefore, with \( \text{(D.56)} \) the characteristic exponents can be estimated arbitrarily sharp. The solution \( x(t) = 0 \) of \( \text{(D.13)} \) is stable if and only if the characteristic exponents are negative real \( \text{(Yakubovich & Starzhinskii 1975)}. \)

For a second order LTP system with real coefficients the real parts of the characteristic exponents of \( \text{(D.13)} \) are defined by

\[
\text{real}(\alpha_{1,2}) = \frac{1}{2T} \left[ \int_0^T \text{tr} A dt + \int_0^T \sqrt{(\delta + \text{tr} A)^2 - e^{-2\delta} \Lambda dt} \right]
\]

where \( e^{2\delta} = \text{det} G \) and \( \Delta = \text{det} Q \). The difficulty of this method is to find a proper \( G(t) \).

D.2.2.5 Computing the transition matrix via algebraic equations

In \( \text{Wu (1974)} \) a method is described to calculate the fundamental matrix via a set of algebraic equations if the eigenvalues of \( A(t) \) in \( \text{(D.13)} \) are time-invariant. The system matrix of \( \text{(D.13)} \) does not need to be periodic for this method. The stability of the equilibrium point of \( \text{(D.13)} \) can be determined without solving the differential equations.
Consider the system (D.13) with \( A(t) \) is continuously differentiable. Let there exist a differentiable one-to-one mapping of \( t \) into \( \tau \) and vice versa, such that,

\[
\tau = f^{-1}(t) \triangleq g(t).
\]  

and

\[
t = f(\tau).
\]

Let the system (D.19) be transformed with the mapping (D.58) into

\[
\dot{z}(\tau) = \bar{A}(\tau)z(\tau)
\]

where \( z(\tau) = x(t), \dot{z}(\tau) = \frac{dx(t)}{d\tau} \) and \( \bar{A}(\tau) = \frac{d}{d\tau} A(f(\tau)) \), such that the eigenvalues of \( \bar{A}(\tau) \) are independent of \( \tau \). If so, then there exists a constant matrix \( A_1 \) such that

\[
A_1 \bar{A}(\tau) - \bar{A}(\tau) A_1 = \dot{\bar{A}}(\tau) \quad \forall \tau,
\]

and a constant matrix \( A_2 \) defined as

\[
A_2 = \bar{A}(0) - A_1.
\]

In that case the state transition matrix \( \Phi(t, t_0) \) of (D.19) is defined as

\[
\Phi(t, t_0) = \Phi(\tau, \tau_0) \bigg|_{\tau=g(t)}^{\tau=g(t_0)}
\]

with

\[
\Phi(\tau, \tau_0) = e^{A_1 \tau} e^{A_2 (\tau - \tau_0)} e^{A_1 \tau_0}.
\]

The proof of this part can be found in Wu (1974b). The equilibrium point of the linear time-varying system (D.19) is

a stable if

\[
Re \lambda^j_1 + Re \lambda^k_2 \leq 0 \quad \forall j, k = 1, 2, \ldots, n
\]

b asymptotically stable if

\[
Re \lambda^j_1 + Re \lambda^k_2 < 0 \quad \forall j, k = 1, 2, \ldots, n
\]

c unstable if

\[
Re \lambda^j_1 + Re \lambda^k_2 > 0 \quad \text{for some} \ j, k = 1, 2, \ldots, n
\]

where \( \lambda^j_1 \) and \( \lambda^k_2 \) are the eigenvalues of \( A_1 \) and \( A_2 \), respectively.

Note that if all eigenvalues of \( A(t) \) are independent of \( t \) then an one to one mapping for \( t \) on \( \tau \) can be used.

### D.2.2.6 Vibration in terms of a small parameter

The vibration in the coefficients of the system (D.13) can be described in terms of a small parameter. There are three classes of such systems.

1 slowly varying systems:

\[
\dot{x} = A(t)x, \ \text{with} \ \|A(t)\| \sim \epsilon
\]
2 low gain, fast periodic (perturbed) systems:
\[ \dot{x} = (A + a B(t/\epsilon)) x \quad \text{or} \quad \dot{x}(\epsilon \tau) = \epsilon (A + a B(\tau)) x(\epsilon \tau) \quad \text{with} \quad \tau = t/\epsilon \]

3 high gain, fast periodic systems:
\[ \dot{x} = (A + a/\epsilon B(t/\epsilon)) x \quad \text{or} \quad \dot{x}(\epsilon \tau) = \epsilon (A + a B(\tau)) x(\epsilon \tau) \quad \text{with} \quad \tau = t/\epsilon \]

where \( 0 < \epsilon \ll 1 \) is a small parameter.

These three classes of systems have different stability properties. Stabilization and control via vibrations in terms of a small parameter is only possible with high gain, fast periodic terms (case 3).

**Slowly varying systems**

For the linear \( T \)-periodic system (D.19) with \( \|\dot{A}(t)\| \sim \epsilon \ll 1 \) stability properties are determined by the eigenvalues of \( A(t) \) for all \( 0 \geq n > T \). This is called the “frozen time” approach. Skoog & Lau (1972) show that the “frozen time” approach only works if the system variations are slow and if no eigenvalues of \( A(t) \) cross the imaginary axes. Desoer (1969) and Shamma & Athans (1987) provide the “frozen time” approach for the linear time-varying system, (D.13) and also show what is “slow enough”.

Assume that \( A(t) \) in (D.13) is bounded, continuously differentiable, and for some \( k_A \geq 0 \), \( \|\dot{A}(t)\| \leq k_A, \forall t \geq 0 \). Assume that at each instant in time \( A(t) \) is a stable matrix, and that there exist positive constants \( m \) and \( \lambda \) such that

\[
\| e^{A(p)\tau} \| \leq me^{-\lambda \tau}, \quad \forall \tau \geq 0, \forall p \in \mathbb{R}_+ \tag{D.65}
\]

then, given any \( \eta \in (0, \lambda) \) for which

\[
k_A \leq \frac{(\lambda - \eta)^2}{4m \ln(m)} \tag{D.66}
\]

implies

\[
\| x(t) \| \leq me^{-\eta t} \| x_0 \| \quad \forall t \geq 0, \forall x_0 \in \mathbb{R}^n.
\]

The proof is provided in Shamma & Athans (1987). The strategy to determine if the system, (D.13) is “slow enough”, is first to determine a minimum value for \( k_A \) by the maximum of \( |\dot{A}(t)| \). Next a maximum for \( |e^{A(p)\tau}| \) is calculated to obtain the minimum values for \( \lambda \) and \( m \). Then the system (D.13) is “slow enough” if for the obtained values of \( k_A, \lambda \) and \( m \) there exists an \( \eta \in (0, \lambda) \) which satisfies the inequality (D.66). Note that in the special case where \( m = 1 \), the time-variations may be arbitrarily fast as (D.66) shows.

If the eigenvalues of \( A(t) \) are not bounded away from the imaginary axis for all \( t \geq 0 \) or if \( \|\dot{A}(t)\| > k_A \) then the solution of system (D.13) is not necessarily unstable.

**Perturbed systems**

For perturbed or low-gain fast-periodic systems the stability of the equilibrium point is determined by the eigenvalues of time-invariant part of the system matrix.

Consider the perturbed system

\[
\dot{x} = (A + a B(t/\epsilon)) x \tag{D.67}
\]

with \( A \) is a constant matrix, \( B(t/\epsilon) \) a matrix whose coefficients are zero mean periodic with period \( T \epsilon \) and \( a \) is a scalar. It has been shown that for \( \epsilon \) sufficiently small, the stability of the equilibrium point \( x(t) = 0 \) of (D.67) is determined by the eigenvalues of matrix \( A \), see Bellman et al. (1985). The same kind of results are described by the perturbation theory in Khalil (1992) which use the converse Lyapunov theorem to show that periodic perturbations added to an autonomous system do not influence the stability properties.
Next, the stability properties of the fast periodic vibrations itself is provided. Let $A$ in (D.67) be zero, such that
\[
\dot{x}(t) = B\left(\frac{t}{\epsilon}\right)x(t)
\]
which is mapped to timescale $\tau = \frac{t}{\epsilon}$, such that
\[
\frac{dx}{d\tau} = \epsilon B(\tau)x(\tau).
\]
Let (D.69) be reducible in the sense of Lyapunov, see section D.2.2.2. Assume that the time-invariant system
\[
\dot{z} = \epsilon Fz
\]
has no imaginary eigenvalues. Then there exist $\epsilon_0$ such that for any $0 < \epsilon \leq \epsilon_0$:
\[
\epsilon F
\]
with $\epsilon F$ has at least one eigenvalue in $\mathbb{Re}\lambda > 0$ then the equilibrium point of (D.69) will be unstable. If all eigenvalues of $\epsilon F$ are in $\mathbb{Re}\lambda < 0$ then the equilibrium point of (D.70) is asymptotically stable and then the equilibrium point of (D.69) is asymptotically stable too, see Bellman et al. (1985) and Skoog & Lau (1972).

If the equilibrium point of (D.70) is asymptotically stable then there exist positive constants $\lambda$ and $m$ such that
\[
\|z(t)\| \leq \|z_0\|e^{-\lambda t}.
\]
Then the parameter $\epsilon_0$ can be estimated by (Bellman et al. 1985):
\[
\epsilon_0 = \left(\frac{e^{-\gamma}}{10(1 + \beta)\gamma m^1+4\gamma/\lambda2^{\gamma/\lambda}}\right)^2
\]
where $\gamma$ and $\beta$ are defined by
\[
\gamma \triangleq \sup_t \|B(t)\|, \quad \beta \triangleq \max_{i,j} \max_t b_{ij}(t)
\]
where $b_{ij}(t)$ are the elements of $B(t)$. The estimate of the parameter $\epsilon_0$ is extremely conservative. It is often orders of magnitude lower than values obtained from numerical experiments.

**Fast periodic systems**

Consider a system with $\epsilon T$-periodic elements in the form
\[
\dot{x} = \left( A + \frac{\alpha}{\epsilon} B\left(\frac{t}{\epsilon}\right)\right)x
\]
where $\alpha$ is a small positive constant. Assume that the state transition matrix of
\[
\frac{dx}{d\tau} = \alpha B(\tau)x, \quad \tau = \frac{t}{\epsilon}
\]
exist and is bounded for all $\tau \in (-\infty, \infty)$. Then there exists an $\epsilon_0$ such that for any $0 < \epsilon \leq \epsilon_0$ the stability of (D.72) can be analyzed by the eigenvalues of a constant matrix
\[
R = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi^{-1}(\tau, 0) A \Phi(\tau, 0)d\tau
\]
where $\Phi(\tau, 0)$ is the state transition matrix of (D.73). The system (D.72) is asymptotic stable if and only if $R$ is a Hurwitz matrix. If at least one eigenvalue has a positive real part (D.72) is unstable. The
parameter $\epsilon_0$ can be estimated by (D.71), see Bellman et al. (1985). To prove this (D.72) is rewritten as

$$\frac{dx}{d\tau} = (\epsilon A + \alpha B(\tau))x.$$  

(D.75)

Under the assumption that $\Phi(\tau, 0)$, the transition matrix of (D.73), is bounded, the relationship

$$x(\tau) = \Phi(\tau, 0)y(\tau)$$

(D.76)

can be viewed as a Lyapunov substitution. Using (D.75), from (D.76) an equation in standard form is obtained:

$$\frac{dy}{d\tau} = \epsilon \Phi^{-1}(\tau, 0)A\Phi(\tau, 0)y$$

(D.77)

the average of which is $dz/d\tau = \epsilon Rz$ where $R$ is defined in (D.74). Since (D.75) is periodic and $\Phi(\tau, 0)$ is bounded, (D.77) is a reducible system. Now the statements of stability follow directly from those made on (D.70).

In Meerkov (1980, 1973) another averaging technique is described which can be applied if $B(t)$ in (D.72) has a quasi-triangular structure such that

$$B(t) = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
k_{21} \sin \omega_{21} t & 0 & \cdots & 0 & 0 \\
k_{n1} \sin \omega_{n1} t & k_{n2} \sin \omega_{n2} t & \cdots & k_{n,n-1} \sin \omega_{n,n-1} t & 0
\end{bmatrix}$$

(D.78)

where $k_{ij} \geq k_0 = \alpha/\epsilon$ and $\omega_{ij} \gg \omega_0$ for all $i, j$. Consider the time-periodic system

$$\dot{x} = B(t)x$$

(D.79)

where $B(t)$ is as defined in (D.78). First determine the solution of the first two equations of (D.79)

$$x_1(t) = x_1(t_0), \quad x_2(t) = x_2(t_0) + (F_{21}(t) - F_{21}(t_0))x_1(t_0)$$

where $F_{21}(t)$ is a periodic function of $t$ with zero mean. Take up the third equation of (D.79) and determine its solution with

$$x_1(t) = x_1(t_0), \quad x_2(t) = x_2(t_0) + F_{21}(t)x_1(t_0).$$

Apply an analogous procedure to each equation of (D.79) with $B(t)$ as in (D.78). That is, if by the $i$th step of the procedure the functions

$$x_k(t) = x_k^0 + \sum_{j=1}^{k-1} (F_{kj}(t) - F_{kj}(t_0))x_j^0, \quad k = 1, \ldots, i - 1$$

are found, then a solution of the $i$th equation of (D.79) is sought by substituting

$$x_k(t) = x_k^0 + \sum_{j=1}^{k-1} F_{kj}(t)x_j^0, \quad k = 1, \ldots, i - 1$$

where $F_{ij}(t)$ are almost periodic functions of $t$ with zero mean. This results in a matrix $F(t) = \|F_{ij}\|_{t,j=1}$. Let $C$ be a matrix which elements $c_{ij}$ are determined by $F_{ij}$ such that

$$c_{ij} = \lim_{T \to \infty} \frac{1}{T} \int_0^T F_{ij}^2(t)dt.$$  

The averaged form of $B(t)$ is $\bar{B}$ whose elements are defined by

$$\bar{b}_{ij} = -a_{ji}c_{ij}.$$  

(D.80)
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In matrix notation \[ (D.80) \] is written as

\[
\bar{B} = - \left( A^T \otimes C \right)
\]

where \( \otimes \) is the Kronecker product which denotes element-by-element multiplication.

Consider the system with constant coefficients

\[
\dot{\mathbf{y}} = (A + \bar{B}) \mathbf{y}.
\]  \hspace{1cm} (D.81)

The solutions \( x(t) \) and \( y(t) \) of system \[ (D.72) \] and \[ (D.81) \] are, for identical initial conditions \( x(0) = y(0) \), related by the expressions

\[
\bar{x}(t) = (I + F(t)) y(t), \quad t \in [0, \infty) \]

\[
\|x(t) - \bar{x}(t)\| \leq \frac{1}{\kappa_0}
\]  \hspace{1cm} (D.82)

where \( F(t) \) is a matrix computed through the above described procedure and \( \bar{x}(t) \) an approximation of the solution of \[ (D.72) \] constructed of the solution of the averaged system \[ (D.81) \] with the vibrations added to it. If \[ (D.81) \] is asymptotically stable, relation \[ (D.83) \] holds for all \( t \in [0, \infty) \); in the opposite case it holds for \( t \in [0, \kappa_0] \) only. The proof of this can be found in Sarychev (2001) and Meerkov (1986). And stability properties of \[ (D.72) \] can be determined with the methods for LTI-systems on \[ (D.81) \].

Note that the sum of the averaged matrix \( \bar{B} \) and the system matrix \( A \) is equal to the matrix \( R \). The use of calculating \( R \) is that with it the stability of \[ (D.72) \] can be determined for any \( B(t) \) matrix for which a state transition matrix can be calculated and not only for systems with a quasi-triangular \( B(t) \).

The next method to determine stability is concerning the Monodromy matrix. In Sarychev (2001) an approximation method for the monodromy matrix, \( M_\epsilon \), with respect to \( \epsilon \) is given. For sufficiently small \( \epsilon \), \( M_\epsilon \) can be represented as convergent power series in \( \epsilon \). Therefore the matrix \( \Lambda_\epsilon \) is token.

The Floquet theorem describes the fundamental solution \( X_\epsilon(t) \) as

\[
X_\epsilon(t) = e^{\lambda_\epsilon} \Phi_\epsilon(t)
\]  \hspace{1cm} (D.84)

where \( \Phi_\epsilon(t) \) is a periodic matrix function. This is equivalent with \[ (D.27) \].

It is obvious that \( X_\epsilon(T) = X_\epsilon(\tau = 1) = M_\epsilon \).

The matrix \( \Lambda_\epsilon \) admits a power series representation

\[
\Lambda_\epsilon = \sum_{i=1}^{\infty} \epsilon^i \Lambda^{(i)}
\]  \hspace{1cm} (D.85)

where the first terms of \[ (D.85) \] are

\[
\Lambda^{(1)} = \int_0^1 A(\tau) d\tau
\]

\[
\Lambda^{(2)} = \frac{1}{2} \int_0^1 \left[ \int_0^{\tau} A(\tau_1) d\tau_1 , A(\tau) \right] d\tau
\]

\[
\Lambda^{(3)} = -\frac{1}{2} \left[ \Lambda^{(1)} , \Lambda^{(2)} \right] + \frac{1}{3} \int_0^1 \text{ad}^2 \left( \int_0^{\tau} A(\tau_1) d\tau_1 , A(\tau) \right) d\tau
\]

where \( \cdot , \cdot \) are Lie-brackets so that \( a,b = ab - ba \), and \( \text{ad} a b = a, \text{ad}^i a b = a, \text{ad}^{i+1} a b \) for \( i \geq 2 \).

Note that the expression of \( \Lambda^{(3)} \) is a contracted version of its original using integration by parts and the Jacobi identity.

The matrix \( \Lambda_\epsilon \) provides the complete information on the averaged behavior of the solution of the time-varying system. If \( \Lambda_\epsilon \) is Hurwitz than the system is asymptotic stable. If all eigenvalues of \( \Lambda_\epsilon \) are in the left-half-plane but at least one is zero the system is stable else the system is unstable. The approximation order of the algorithm \( \Lambda_\epsilon \) is high enough if higher order terms do not change the stability properties of \( \Lambda_\epsilon \).
D. Stabilization of Linear Systems through Parameter Vibrations

D.2.3 Linear vibrational control

The methods to determine the stability properties of vibrational systems are explored in the previous section. Next, control strategies are provided to stabilize an equilibrium of a linear system by vibrations of the coefficients. Vibrational control is of course only useful when the stability of the vibrational system can be determined. In literature two classes of vibrational control can be found. Vibrational control can be done by assigning the eigenvalues of the monodromy matrix or, in case the vibrations are approximated in terms of a small parameter, by assigning the eigenvalues of the averaged system. Vibrational control is sometimes only used to stabilize the system. To assign the desired poles the vibrational stabilized system is controlled by a linear feedback controller.

D.2.3.1 Assigning poles of the monodromy matrix

Consider the system
\[ \dot{x} = A(t)x(t) + B(t)u(t) \]  \hspace{1cm} (D.86)

where \( A(t) \) and \( B(t) \) are T-periodic continuous matrices. First, the controllability or stabilizability of (D.86) is provided. The system is controllable if it is controllable at each time. Define the controllability or reachability Grammian on \( t_1, t_2 \)
\[ G(t_1, t_2) = \int_{t_1}^{t_2} X(t_2, \tau)B(\tau)B^T(\tau)X^T(t_2, \tau) d\tau \]  \hspace{1cm} (D.87)

where \( X(t_2, \tau) \) is the transition matrix of (D.86). The system (D.86) is controllable on \( t_1, t_2 \) if \( G(t_1, t_2) \) is nonsingular. The system (D.86) is controllable if and only if the pair \((G(T, 0), X(T, 0))\) is controllable.

There are other methods of approach to define stabilizability, see e.g. Weiss. (1968). It is proven that they are equivalent to each other, see Bittanti & Bolzern (1985).

Now a control strategy is provided which uses the reachability Grammian, presented in Kabamba (1986).

Assignment of the eigenvalues of the monodromy matrix using continuous periodic feedback (CPF) control of the form
\[ u(t) = F(t)x(t), \quad F(t + T) = F(t) \]  \hspace{1cm} (D.88)

leads to a nontrivial boundary value problem. This can be avoided by using the concept of sampled state periodic hold (SSPH) control of the form
\[ u(t) = F_s(t)x(iT), \quad t \in [iT, (i + 1)T], \quad F_s(t + T) = F_s, \quad i \geq 0. \]  \hspace{1cm} (D.89)

Moreover SSPH control is slightly more general than CPF control. (D.88) and (D.89) are equivalent if \( F(t) \) is defined as
\[ F(t) = F_s(t)\left[ X(t, 0) + \int_0^T X(t, \tau)B(\tau)F_s(\tau)d\tau \right]^{-1}, \quad t \in [0, T] \]  \hspace{1cm} (D.90)

at all times when the inverse in (D.90) is defined. SSPH control can be implemented as CPF control using (D.90) except in the neighborhood of singularities of (D.90) when (D.89) should be used. An important remark for SSPH control is that the closed loop system remains linear and periodic. The state transition obeys the law
\[ x(iT + \sigma) = \left[ X(\sigma, 0) + \int_0^\sigma X(\sigma, \tau)B(\tau)F_s(\tau)d\tau \right]x(iT), \quad \sigma \in [0, T]. \]  \hspace{1cm} (D.91)

The closed-loop monodromy matrix becomes
\[ \bar{\Psi} = X(T, 0) + \int_0^T X(T, \tau)B(\tau)F_s(\tau)d\tau. \]  \hspace{1cm} (D.92)
D.2 Literature survey of linear vibrational systems

Suppose (D.86) is controllable. Use SSPH control with $F_\epsilon$ defined by

$$F_\epsilon(t) = B^T(t)X^T(T, t)F_\epsilon$$

with $F_\epsilon$ is constant such that $\Psi = X(T, 0) + G(T, 0)F_\epsilon$ has the desired eigenvalues. The whole monodromy matrix of (D.86) is assignable by SSPH if and only if (D.86) is controllable in one period. In this case, a feedback gain is

$$F_\epsilon = B^T(t)X^T(T, t)G^{-1}(T, 0)\Psi$$

where $\Psi$ denotes the desired monodromy matrix.

Now, a control procedure can be defined for (D.86). First, the controllability of (D.86) is checked by calculating $G(T, 0)$. If (D.86) is controllable and the monodromy matrix of (D.86) can be determined, then the input signal is determined by (D.95) with $F(t)$ in (D.94) where $\Psi$ denotes the desired monodromy matrix.

Note that the closed-loop monodromy matrix $\Psi$ in (D.92) can be singular because SSPH control allows assignment of the whole monodromy matrix to any value. The number of periods required to control (D.86) is at the most the controllability index of the pair $(X(T, 0), G(T, 0))$. Thus in some cases the SSPH control can be used to dead-beat the system in a finite time (Kabamba 1986).

D.2.3.2 Assigning poles of averaged vibrational systems

The problem of controlling linear systems with fast zero-averaged parametric oscillation is formulated as follows: Given the system $\dot{x} = Ax$, does there exist a control matrix $B(t)$ with periodic zero mean elements such that the equilibrium $x = 0$ of the system

$$\dot{x} = (A + B(t))x$$

is asymptotically stable?

Requirements for vibrational stabilizability are provided in Meerkov (1980). One is that a row $c$ exists such that the pair $(A, c)$ is observable and second that the $tr(A) < 0$. Then and only then, a vibrational control matrix $B(t)$ as mentioned above exists. It is also shown that if $B(t)$ oscillates fast enough, the stability properties of (D.93) can be described by the averaged system $\dot{x} = (A + \bar{B})x$. This implies that $trA = tr(A + \bar{B})$. Using averaging techniques, vibrational control cannot relocate poles arbitrarily.

In Kabamba et al. (1998) the possibilities of pole assignment by vibrational control are shown for the canonical controller form of $A$ and $B(t)$.

Thus

$$A = \begin{bmatrix}
0 & 1 & \ldots & 0 \\
0 & 0 & \ldots & 1 \\
-a_n & -a_{n-1} & \ldots & -a_1 \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
-b_n(\omega t) & -b_{n-1}(\omega t) & \ldots & -b_1(\omega t)
\end{bmatrix}$$

$$B = \begin{bmatrix}
0 & 1 & \ldots & 0 \\
0 & 0 & \ldots & 1 \\
-a_n & -a_{n-1} & \ldots & -a_1 \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
-b_n(\omega t) & -b_{n-1}(\omega t) & \ldots & -b_1(\omega t)
\end{bmatrix}$$

where

$$b_i(\omega t) = \sum_{i=1}^{\infty} k_i^s \sin(s\omega t + \phi_i), \quad i = 1, \ldots, n.$$
Then there exist positive constants $k_0$ and $\omega_0$ such that if $k_i^1 > k_0, i = 1, \ldots, n,$ and $\omega > \omega_0,$ matrix $\bar{B}$ is defined by formula

$$B = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
-b_n & -b_{n-1} & \cdots & -b_1 \\
\end{bmatrix}$$

which elements are defined by

$$b_i = \sum_{s=1}^{\infty} \frac{k_i^2 k_i^2}{2\omega^2 s^2} \cos(\phi_i^2 - \phi_i^s), \quad i = 2, \ldots, n.$$ 

From this it follows that the characteristic polynomials of $A$ and $(A + \bar{B})$ have, respectively, the form

$$p_0(s) = s^n + a_{10} s^{n-1} + a_{20} s^{n-2} + \cdots + a_n$$

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$$

with

$$a_1 = a_{10} \quad \text{ (D.96)}$$

$$a_2 \geq a_{20} \quad \text{ (D.97)}$$

and $a_j, 3 \leq j \leq n$ can be assigned arbitrarily. Next, necessary and sufficient conditions are provided for pole placement in the desired region of the complex plane within the constraints (D.96) and (D.97). The desired region of the pole within the complex plane is defined as $D(\sigma, \omega)$ where $\sigma$ is the real value and $\omega$ the imaginary value.

The real pole assignment of the $n$th order system (D.95) in the region $D(\sigma, 0)$ is possible using vibrational control if and only if

$$-a_{10} \leq n\sigma$$

$$a_{20} \leq \frac{n-1}{2n} a_{10}^2$$

The complex pole assignment the $n$th order system (D.95) in the region $D(\sigma, \omega)$ is possible using vibrational control if and only if

1. for $n$ is even

   $$-a_{10} \leq n\sigma$$

   and if $|\sigma| \geq \omega$

   $$a_{20} \leq \frac{a_{10}}{2n\sigma^2} \left( (n-1)\sigma^2 + \omega^2 \right)$$

   or if $|\sigma| \leq \omega$

   $$a_{20} \leq \left( \omega^2 - \sigma^2 \right) \left( \frac{n+1}{2} \right) \left( \frac{n-2}{2} + \frac{a_{10}}{\sigma} \right) + \left( \omega^2 + \sigma^2 \right) \left( \frac{a_{10}}{2\sigma} \right)^2$$

2. for $n$ is odd with one pole placed at $p < \sigma$

   $$-a_{10} \leq (n-1)\sigma + p$$

   and if $|\sigma| \geq \omega$

   $$a_{20} \leq -p(a_{10} + p) + \frac{(a_{10} + p)^2}{2(n-1)\sigma^2} \left( (n-2)\sigma^2 + \omega^2 \right)$$

   or if $|\sigma| \leq \omega$

   $$a_{20} \leq p(a_{10} - p) + \left( \omega^2 - \sigma^2 \right) \left( \frac{n-1}{2} \right) \left( \frac{n-3}{2} + \frac{a_{10} - p}{\sigma} \right) + \left( \omega^2 + \sigma^2 \right) \left( \frac{a_{10} + p}{2\sigma} \right)^2$$
D.2 Literature survey of linear vibrational systems

In Lee et al. (1987) a more advanced vibrational feedback controller is used. The key value of that article is to show that with this controller the open-loop zeros can be relocated and, under certain conditions, assigned arbitrarily. This is a mayor advance of vibrational feedback control to time-invariant feedback control. This way of vibrational control can help to increase the bandwidth of the system and hence higher gains can be used (Kabamba et al. 1998).

D.2.4 Discussion literature survey

The literature survey revealed that the unstable systems can be stabilized by noise-assisted high-gain feedback control. For that, the trace of the system matrix must be negative.

Furthermore, the literature survey provides various methods to determine the stability characteristics of systems with time-varying coefficients.

The Lyapunov direct method extended for time-varying systems gives a clear definition of stability for an equilibrium of a periodic time-varying system but this method can be hard to apply since a good candidate for the Lyapunov function is not always close at hand.

The Floquet theory has the advantage that it splits the solution of a LTP system in a drift and a vibration term so that the stability properties can be assessed easily. The disadvantage is that the transition matrix has to be found, so this technique requires calculation of the system’s solution. If the system’s solution cannot be calculated, an approximation of the Floquet transition matrix with the Péano-Baker series can give some usable results.

The 2-regularized Nyquist criterion defines a Nyquist-like locus for LTP systems. A LTP system is determined as stable if the locus encircles the origin \( n \) times in counterclockwise sense where \( n \) is the number of unstable open-loop eigenvalues. The method is a useful numerical tool to determine the stability properties and stability robustness of the LTP system.

A method is provided for a LTV system with constant eigenvalues to transform the system into a set of algebraic equations. Then, via algebraic equations the stability properties of the LTV system can be determined easily.

Stability properties of LTP system can be determined by expressing the periodic elements in terms of a small parameter. The stability is determined by means of the approximated or averaged solution. Using these approximation the stabilization can only be achieved at high frequencies and high amplitudes.

Strategies of linear vibrational control are provided which shows that the utilization of linear vibrational (feed-forward) control can be interesting since it has the advantage over linear feed-back control that it can assign the whole monodromy matrix.

This overview shows that there are several analytic methods available to determine the stability properties of LTP systems. However, most of the methods are not usable for all LTP systems. The usability of the methods is unclear. It is unlike LTI systems for which the stability properties can always be determined by the eigenvalues of the system matrix.
D.3 Analytical analysis of an example of vibrational control

In the previous section literature methods on the stability properties of LTI systems with random or time-periodic vibrations are presented. In this chapter the comparability of stabilization by random and time-periodic vibrations is analyzed analytically. Therefore, the LSDE \([D.5]\), used to demonstrate stabilization via noise, is taken and the noise is replaced by time-periodic vibrations such that

\[
\dot{X} = \begin{bmatrix} 0 & 1 + a \sin(\omega \cdot t) \\ 101 - a \sin(\omega \cdot t) & -101 \end{bmatrix} X.
\]  

(D.98)

where \(a\) and \(\omega\) are some positive constants which denote the amplitude and frequency of the vibration, respectively. Analytical methods to determine stability of linear vibrational systems are utilized to determine the stabilization of the time-periodic vibrations. Finally the outcome of these analytical analysis is discussed.

D.3.1 Lyapunov's direct method

The stability properties of \([D.98]\) are determined by using Lyapunov’s direct method to find an appropriate Lyapunov candidate function. A proper \(P(t)\) is sought for the Lyapunov candidate \(x^T P(t) x\) using \([D.18]\). Consider the symmetric matrices

\[
P(t) = \begin{bmatrix} P_1(t) & P_2(t) \\ P_2(t) & P_3(t) \end{bmatrix}, \quad Q(t) = \begin{bmatrix} Q_1(t) & Q_2(t) \\ Q_2(t) & Q_3(t) \end{bmatrix}
\]

(D.99)

where \(P_1(t), P_2(t), P_3(t), Q_1(t), Q_2(t)\) and \(Q_3(t)\) are unknown functions such that the matrices \(P(t)\) and \(Q(t)\) are positive definite. Substitute these matrices and \(A(t)\) from \([D.98]\) in \([D.18]\) gives

\[
\dot{P}(t) = \begin{bmatrix} -2P_2(t)(101 - a \sin(\omega t)) - Q_1(t) & -P_2(t)(101 - a \sin(\omega t)) - P_1(t)(1 + a \sin(\omega t)) + 101P_3(t) - Q_2(t) \\ -P_3(t)(101 - a \sin(\omega t)) - P_2(t)(1 + a \sin(\omega t)) + 101P_3(t) - Q_2(t) & -2P_3(t)(1 + a \sin(\omega t)) + 202P_3(t) - Q_3(t) \end{bmatrix}.
\]

(D.100)

The symbolic toolbox of Matlab is used to solve the coupled differential equation of \([D.100]\) but a solution is not found. Even simplifications of \([D.100]\) with \(Q_2(t) = 0, Q_{1,4}(t) = 1, P_3(t) = 0\) and \(P_1(t) = 1\) or combinations do not lead to a solution of \([D.100]\). The functions \(P_{1,2,4}(t)\) remain unknown. Thus the Lyapunov direct method is not able to determine the stability properties of \([D.98]\).

D.3.2 Floquet transition matrix

The Floquet-Lyapunov theorem in Section \(D.2.2.2\) shows that the stability of \([D.98]\) can be determined by calculating the monodromy matrix \(X(T)\). This is the transition matrix over one period. But the transition matrix is not constructible analytically because \([D.98]\) is not analytically solvable. The Liouville-Jacobi formula in \(D.32\) proves that if the trace of the system matrix is negative then a system can be made stable by adding vibration with zero mean. This means that it is possible that \([D.98]\) becomes stable when the vibrations has a proper combination of amplitude and frequency.

According to literature this should be at high amplitudes and frequencies. Since the monodromy matrix is not constructible analytically it is impossible to determine analytically for which amplitude and frequency of the vibration the solution \(x(t) = 0\) of \([D.98]\) is stable.

In the next chapter numerical simulation are used to determine the monodromy matrix for a range of amplitudes and frequencies of the vibration.

D.3.3 Estimate the characteristic exponents

Since \(X(T)\) is not available analytically, an estimate is searched for the characteristic exponents, \(\alpha_n\) of system \([D.98]\) with the literature method provided in Section \(D.2.2.4\). A proper \(G(t)\) is sought such that the stability properties are comparable to the simulation results in the next chapter, see Section \(D.4.2.1\).
Consider the matrix $G = bI$. With this $G$ the real parts of the characteristic exponents are calculated using \[ D.57 \]. The resulting real characteristic exponents are independent of the vibration. Therefore the used matrix $G$ is unsuitable to demonstrate the stabilization of \[ D.98 \] via the parameter vibrations. Other matrices $G$ which are considered to be unsuitable to demonstrate stabilization of \[ D.98 \] are

$$ G = \begin{bmatrix} b & d \\ d & c \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} b(1 + \sin(\omega t)) & 0 \\ 0 & c \end{bmatrix} $$

for some real constants $c$, $b$ and $d$. Also with these matrices the resulting real characteristic exponents are independent of the vibration.

Furthermore, the following matrices $G(t)$ are investigated to be suitable:

i. $$ G(t) = \begin{bmatrix} c & b \sin(\omega t) \\ b \sin(\omega t) & c \end{bmatrix} $$

ii. $$ G(t) = \begin{bmatrix} b(2 + \sin(\omega t)) & 0 \\ 0 & c(2 + \sin(\omega t)) \end{bmatrix} $$

iii. $$ G(t) = \begin{bmatrix} b(1 + \sin(\omega t)) & c \\ c & d \end{bmatrix} $$

iv. $$ G(t) = \begin{bmatrix} c \sin(\omega t) & b \sin(\omega t) \\ b \sin(\omega t) & c \sin(\omega t) \end{bmatrix} $$

v. $$ G(t) = \begin{bmatrix} c \sin(\omega t) & b \sin(\omega t) \\ b \sin(\omega t) & d \sin(\omega t) \end{bmatrix} $$

for some real constants $c$, $b$ and $d$. With these matrices, $G$, the real characteristic exponents, real characteristic exponents, are calculated by solving the integral of \[ D.57 \]. This is done by using the symbolic toolbox of Matlab which, for the matrices 1-5 above, result in a warning that the explicit integral cannot be found. Therefore, a suitable matrix $G$ to determine the real parts of the characteristic exponents of \[ D.98 \] is not found.

D.3.4 Péano-Baker series

Using the Péano-Baker series to construct the transition matrix of the system gives a transition matrix which is very hard to interpret. To test the stability characteristics numerical calculations are achieved using the approximated transition matrix for different frequencies and amplitudes of the vibrations. Plots are constructed of stability/unstability of the solution of \[ D.98 \] for various vibrations in a range frequencies and amplitudes where the stability was determined by the eigenvalues of monodromy matrix which was constructed by the Péano-Baker series, see \[ D.33 \]. Comparing these plots of stability for the third to the sixth order of the Péano-baker series, the stability properties of monodromy matrix does not converge to the outcome of numerical simulations, see Section \[ D.4.2.1 \] when the order of the serie is increased. Moreover, the eigenvalues of the approximated monodromy matrix are increasing as the order of the serie increases. Therefore the method is unsuitable.
D. Stabilization of Linear Systems through Parameter Vibrations

D.3.5 Computing the transition matrix via algebraic equations

According to Section [D.2.2.5] system [D.98] can only be transformed into a set of algebraic equation if \( A(t) \) exist and if the eigenvalues of \( A(t) \) are constant. The former condition is fulfilled but the latter not. Moreover, there does not exist a mapping of \( t \) into \( \tau, t = f(\tau) \), such that the eigenvalues of \( A(\tau) = f(\tau)/dt\) are constant for any \( a \) or \( f \) unequal to zero. Therefore this method is not suitable to determine the stability properties of [D.98].

D.3.6 Vibration in terms of a small parameter

D.3.6.1 Slowly varying

The system [D.98] is slowly varying if \( \| \dot{A}(t) \| \sim \epsilon \ll 1 \). Thus for \( \| \dot{A}(t) \| = |a \cdot \omega| \ll 1 \) the stability of system [D.98] can be determined by the eigenvalues of \( A(t) |_{t=n} \) for all \( 0 \geq n > T \). Since the unstable pole of \( A(t) |_{t=0} \) is 1 a vibration with a very small amplitude can not stabilize the system.

D.3.6.2 Perturbed systems

Following Section [D.2.2.6] system [D.98] could be defined as a perturbed system if the vibration frequency is a factor \( 1/\epsilon \) higher than the vibration amplitude where \( \epsilon \ll 1 \). Perturbations do not change the stability properties of the system since the stability is determined by the eigenvalues of \( A \). Thus to stabilize the system with the time-periodic parameter vibrations the vibration frequency should be of the same order or smaller than the vibration amplitude.

D.3.6.3 Fast periodic systems

Averaging via the method expelled in section [D.2.2.6] is not possible. The matrix \( B(t) \) is not quasi-triangular and the transition matrix of [D.73] can not be constructed analytically with

\[
B = a \sin(\omega \cdot t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_1 \]

Next, the power series of the exponential power, \( \Lambda_\epsilon \), of the monodromy matrix in \( \epsilon \) are calculated to determine the stability properties of [D.98]. A third order power series representation of \( \Lambda_\epsilon \) is calculated using the terms as described by [D.85]. This gives

\[
\Lambda_\epsilon = \epsilon \Lambda^{(1)} + \epsilon^2 \Lambda^{(2)} + \epsilon^3 \Lambda^{(3)}
\]

\[
= 2\pi \epsilon \omega \left[ \begin{array}{cc} 0 & 1 \\ 101 & -101 \end{array} \right] + \frac{\pi a \epsilon^2}{\omega^2} \left[ \begin{array}{cc} 102 & -101 \\ -101 & -102 \end{array} \right] + \frac{\epsilon^3 a \pi}{\omega^3} \left[ \begin{array}{cc} -1010\pi - 68/3e \left( -31815 + 8e^2 + 600\pi \right) \omega & 10405\pi + 202/9e \left( -31815 + 8e^2 + 600\pi \right) \omega \\ -30805\pi + 202/9e \left( -31815 + 8e^2 + 600\pi \right) \omega & 10100\pi + 68/3e \left( -31815 + 8e^2 + 600\pi \right) \epsilon \end{array} \right].
\] (D.101)

Substituting various values for \( a, f \) and \( \epsilon \) in [D.101] gives \( \Lambda_\epsilon < 0 \) for \( \epsilon = 0.001, f < 1 \) and some \( a \). Thus a third order power serie of the exponential of the monodromy matrix shows stabilization of [D.98] by the vibrations of the parameters. However, the third order serie gives different stability properties than the second order power serie hence the approximation of the third order is not high enough to describe the stability properties of [D.98] accurately. Higher order approximations are recommended.

D.3.7 Discussion

The literature theory in Section [D.84] states that linear parameter vibrations can stabilize a linear time-invariant system if the trace of the system matrix of the LTI is negative. Therefore it is expected that system [D.98] can stabilized by the vibrations, \( n(t) \). However, the methods to define stability of linear vibrational systems, summarized in Section [D.2.2] are not. Moreover, there does not exist a mapping of \( t \) into \( \tau, t = f(\tau) \), such that the eigenvalues of \( A(\tau) = f(\tau)/dt \) are constant for any \( a \) or \( f \) unequal to zero. Therefore this method is not suitable to determine the stability properties of [D.98].

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are not capable to determine the stability properties of (D.98) analytically. The analytical analysis in this chapter demonstrates the difficulty of determining the stability properties of linear vibrational systems. Approximation of the vibration in terms of a small parameter, $\epsilon$, and using the power series representation of the monodromy matrix shows that stabilization in (D.98) can be obtained by the parameter vibrations but higher order approximations are required to obtain reliable results. Numerical simulations are recommended to determine the stability properties of (D.102).

This section shows how difficult it is to determining the stability properties of (D.102). It is therefore recommended to apply LVC by using the control strategies described in Section D.2.3.
D.4 Numerical analysis of stabilization via linear vibrational control

Numerical simulations are performed to demonstrate the literature theory of the stability properties of linear vibrational system provided in Section D.2.2. In the previous chapter the analytical analysis of an example, system (D.98), of linear vibrational control did not clarify the theories and utility of LVC. The example system (D.98) is rewritten in the form

\[ \dot{x} = Ax + n(t)Bx \]  

such that

\[ A = \begin{bmatrix} 0 & 1 \\ 101 & -101 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \]

and where \( n(t) \) can be a random vibration or a time-periodic vibration. In the performed simulations various \( A \) matrices are used.

First, simulations are performed of (D.102) with \( A \) as in (D.102b) and for \( n(t) \) frequency bandwidth-limited white noise. The stability is determined by examining the simulation outcome for a sufficiently long period. These simulations give the remarkable result that a lower frequency band is more successful in stabilization than a high frequency band for the same frequency bandwidth and noise intensity. The same phenomenon is provided by the simulation results of (D.102) with for \( n(t) \) a time-periodic vibration using various amplitudes and frequencies. For the time-periodic vibrations the stability of the solutions is determined by calculating the monodromy matrix from the simulations. Next, the literature theory of the Liouville-Jacobi formula in (D.30) is validated. It is demonstrated by simulations that stabilization of a linear time-varying system via parameter vibration is only possible if the trace of the system matrix is negative. This is achieved by simulation of (D.102) with different other system matrices, \( A \), with a trace of \(-1\), 0 and 101, respectively. Furthermore, system (D.102) is simulated with two different stable \( A \) matrices to demonstrate that perturbations can not change the stability properties of a system.

Then the 2-regularized Nyquist locus is calculated of (D.102) with \( n(t) \) a sinusoid. The numerically calculated Nyquist loci show stability properties of (D.102) corresponding with the previous simulations.

Finally, the simulation results are discussed.

D.4.1 Frequency bandwidth-limited white noise vibration

Simulations are performed to show the stabilizing properties of white noise as stated in [Townley et al. (2003)] which is presented in Section D.2.1. Since white noise can not be realized a frequency band-limited white noise is used as excitation signal in the simulations which are done in Simulink. The realizations of the band-limited white noise are created by the method developed by Shinozuka and Yang, see [van de Wouw (1999)]. Since these realizations do not contain infinite high frequencies like white noise, (D.102) with for \( n(t) \) these realizations is not a stochastic differential equation but deterministic.

The stabilization properties of frequency band-limited white noise is simulated for two frequency bands, one with a band of \([0 - 100] \) Hz and a second with \([100 - 200] \) Hz with equivalent intensity. The used sample frequency is 3200 Hz.

For each frequency-band a realization of frequency band-limited white noise is used for two simulations with different initial conditions, that is with \( x_0 = 0 \) 1 and \( x_0 = 1 \) 0. This is done to make the simulations comparable with the simulation with time-periodic vibrations where the stability is determined by the monodromy matrix for which \( x(0) = I \). The state variables in Figures D.2(a) and D.2(b) converge exponentially to the equilibrium \( x(t) = 0 \). So, the equilibrium point of (D.102) is exponentially stable with the used frequency bandwidth-limited white noise. The state variables in Figures D.2(c) and D.2(d) vibrate randomly about an arbitrary value. Other realizations of the bandwidth-limited white noise results in the simulations in other values around which the state variables vibrate. In some sense the system is stable since the state variables are bounded. However, the equilibrium point of system (D.102) is not exponentially stable with the used frequency bandwidth of...
D.4 Numerical analysis of stabilization via linear vibrational control

Figure D.2: The simulated state variables of the system (D.102) excited by a band-limited white noise signal for two different frequency bands and begin positions, $x_0$.

Thus the conclusion can be made that for the noise intensity the low frequency band of $0 – 100 \text{ Hz}$ converges better than the high frequency band of $100 – 200 \text{ Hz}$ for both begin conditions. Some more simulations are done with other realizations of the bandwidth-limited white noise. The results are comparable with the results shown here. Other simulations show that the system stabilizes better if excited by bandwidth-limited white noise for a bandwidth of $0 – 100 \text{ Hz}$ instead of a bandwidth of $0 – 200 \text{ Hz}$. Since the simulated system is not a stochastic system it is not clear if in the stochastic system the low frequencies stabilize more than the high frequencies. The excitation with frequency bandwidth limited white noise is equivalent to a sum of sinusoids with frequencies in the range of the band. It is interesting to investigate if simulations with a time-periodic parameter vibrations show the same phenomenon that low frequencies stabilize better than high frequencies.

D.4.2 Linear time-periodic vibrational control

The stabilization of (D.102) via a vibration of frequency bandwidth-limited white noise is shown by simulations in the previous section. It is shown there that the stability properties of (D.102) are dependent of the height of the frequency band of the white noise. Since frequency bandwidth-limited white noise can be seen as a sum of sinusoids, the frequency dependency of the vibrational stabilization of (D.102) is investigated via simulation of (D.102) with for $n(t)$ a time-periodic vibration.
system is simulated for various frequencies and amplitudes of \( n(t) \).

The stability of the solutions of the simulations of \( D.102 \) is determined by constructing the monodromy matrix, see Section \( D.2.2.2 \) and calculate its eigenvalues. This monodromy matrix is equal to the transition matrix after one period. Hence, the monodromy matrix is constructed by solving the system numerically over one period with \( x_0 = I \). Then, the solution of \( D.102 \) is stable if and only if both eigenvalues of the monodromy matrix are within the unity circle, else the solution is unstable.

Next, simulations are achieved to investigate that linear vibrational control can only stabilize a solution if the trace of the system matrix is negative. Furthermore, it is investigated if perturbation (high-frequent vibration with small amplitude) of the system parameters do effect stability properties. Therefore, simulations are performed of \( D.102 \) with a system matrix with both eigenvalues on the open left half-plane.

D.4.2.1 Frequency dependency of the stabilization via vibrational control

As mentioned above, the frequency dependency of the stabilizing effect of linear vibrational control is investigated. Therefore, simulations are performed of system \( D.102 \) with a time-periodic vibration, \( n(t) = a \sin(\omega \cdot t) \). These simulations are repeated for various vibration amplitudes and frequencies to obtain at which amplitude and frequency the vibration stabilizes the system. A plot of the boundary of stabilizing vibration amplitudes and frequencies for system \( D.102 \) with time-periodic vibrations, collected with the monodromy matrix and by a look at the simulated state variables.

![Figure D.3: Boundary of stabilizing vibration amplitudes and frequencies for system \( D.102 \) with time-periodic vibrations, collected with the monodromy matrix and by a look at the simulated state variables.](image-url)

of the vibration amplitude and frequency which stabilizes the solution of \( D.102 \) is shown in Figure D.3. On and above the boundary the time-periodic vibration stabilizes the system. The nodes are found by taking a vibration frequency and amplitude which stabilizes the system and than lower the amplitude with steps of 0.01 until the system is not stabilized anymore. Solutions with lower vibration amplitudes for that vibration frequency are unstable. The boundary of stability extracted from the monodromy matrix calculation is equal to the boundary extracted from running simulations and look if the solution is stable or not. This provides sufficient evidence that the monodromy matrix method is correctly implemented. Performing the simulation for different tolerances of the ODE45 solver in Matlab shows that at default tolerances the simulation creates accurate results. Note that at 30 rad/sec
the monodromy matrix calculation gives another vibration amplitude at which the system’s solutions is stabilized than a look at the simulations outcome. This is incomprehensible and the monodromy matrix is declared false since it is not on the curve of the boundary of stabilizing vibrations.

Figure D.4: Maximum modulus of the eigenvalues of the monodromy matrix of (D.102) for a range of amplitudes and frequencies of the time-periodic vibration.

The dominant modulus of the monodromy eigenvalues of (D.102), which determines stability of the solution, is shown in Figure D.4. A dominant modulus of the eigenvalues below one indicates that the solution of (D.102) is stabilized by the parameter vibration else the solution of (D.102) is unstable. From Figure D.3 it can be concluded that the solution of (D.102) is stable at low vibration frequencies and a vibration amplitude of about 10 or more. The higher the vibration frequency the higher the vibration amplitude must be to achieve stabilization. Figure D.4 shows that for vibration amplitudes above ca. 10 the dominant modulus decreases for a decreasing vibration frequency. The vibration amplitude of about 10 appears to be a threshold for stabilization via time-periodic parameter vibrations. For vibration amplitudes below about 10 the dominant modulus increases for a decreasing vibration frequency. At very low frequencies of below 0.01 rad/sec the simulation outcome becomes unreliable. For vibration frequencies below 0.01 rad/sec and vibration amplitudes above 10 system (D.102) is a slowly varying system so that the vibrations do not stabilize the system.

Since the zero dynamics of (D.102) are unstable, it is incomprehensible that for vibration amplitudes above ca. 10 the converging effect of the vibrations is stronger for smaller vibration frequencies.

D.4.2.2 Relation trace system matrix and stabilization via vibrational control

In the previous section the stabilization of (D.102) via LVC is demonstrated. This is according to the Liouville-Jacobi formula (D.32) because the trace of (D.102) is negative. The relation of the trace of the system matrix and stabilizibility via LVC is further investigated. Therefore, simulations are performed of (D.102) with three different A-matrices with a trace of −1, 1 and 0, respectively. For these A-matrices
the stabilization of (D.102) by LVC is determined for a range of vibration frequencies and amplitudes. Consider (D.102) with an $A$-matrix with a trace of $-1$, such that

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 10 & -1 \end{bmatrix} x + n(t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$  (D.103)

with $n(t) = a \sin(\omega \cdot t)$. The zero dynamics of (D.102) without vibration are unstable. Simulations are performed for a range of vibration frequencies. The resulting dominant modulus of the monodromy eigenvalues is shown in Figure D.5. In the low flat surface in the front of Figure D.5 the dominant moduli of the monodromy eigenvalues of (D.103) are below one. Thus, this surface represents the parameter vibrations which stabilize the solution of (D.103). Also some points in the first deep valley at the right of the plot in Figure D.5 have dominant moduli of the monodromy eigenvalues below zero. Because the valley is quit narrow the grid of vibrations doesn't demonstrate everywhere the true depth the valley. That is why for only some points of the valley stabilization is of the solution of (D.103) is demonstrated.

Compared to the simulation results of the original system (D.102) with the more negative trace in Figure D.3, the area of stabilizing vibrations is more restricted for system (D.103). Although stabilization is possible for vibration amplitudes below ca. 10 which does appear as a threshold of stabilizing vibration amplitudes for system (D.102). The low flat surface in the front of Figure D.3 is not as flat as it appears to be. To show that, the dominant moduli of the stable monodromy eigenvalues are plotted apart against the vibration frequency in Figure D.6. The plot shows that for the stabilized solutions lower vibration frequencies gives better convergence. Moreover, it can be clearly seen that for the stable monodromy eigenvalues the vibration amplitude does not influence the modulus of the eigenvalues of the monodromy matrix. An explanation of this phenomenon can be sought in the fact that the stable eigenvalues of the monodromy matrix are complex conjugated. The two points in Figure D.6 which are outside the curve are on the edge where vibration stabilizes the solution. The eigenvalues of these two point are real. Note that the points of stabilizing vibration in the valley at the right of Figure D.5

Figure D.5: Maximum modulus of the eigenvalues of the monodromy matrix of system (D.103) with a time-periodic vibration for a range of amplitudes and frequencies.
are on the curve of dominant stable moduli in Figure D.6.

Figure D.6: Modulus of the stable eigenvalues of the monodromy matrix of (D.103) for a range of amplitudes and frequencies of the time-periodic vibration plotted against the vibration frequency.

From Liouville-Jacobi formula in (D.32) it is concluded that zero mean vibrations can only stabilize the system if the trace of the system matrix is negative. This is validated in the previous simulations where the trace of the system matrix was negative and stabilization through vibrations was shown to be possible. Next, simulations are performed to demonstrate that a linear system with positive trace cannot be stabilized by LVC. The simulations of (D.102) are repeated with a system matrix with a positive trace such that

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x + a \sin(\omega \cdot t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x. \quad (D.104)$$

The 3D-plot of the dominant modulus of the monodromy eigenvalues of (D.104) against the vibration frequency and amplitude appears to be similar to Figure D.5. However, none of the simulated vibrations stabilizes the solution of (D.104). This is according to the theory mentioned above.

Simulations have shown that vibrations can not stabilize (D.102) with A has a positive trace. Next, simulations are performed of (D.102) with a zero trace for the A matrix, such that

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 101 & 0 \end{bmatrix} x + a \sin(\omega \cdot t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x. \quad (D.105)$$

Again, the 3D graphic of the dominant modulus of the monodromy eigenvalues of (D.105) appears to be similar to Figure D.5. However, for (D.105) the flat lower part in Figure D.5 is numerical exactly 1. This means that the solution of (D.105) is bounded but does not converge to the equilibrium point. This is according the theory derived from the Liouville-Jacobi formula (D.32). The stabilizing vibrations appear to mix the trace over the eigenvalues as the white noise does for the low-gain case in (D.8).
According to the theory derived from the Liouville-Jacobi formula \([D.32]\), the trace of the system matrix determines the stabilizing possibilities of LVC. Only if the trace of \([D.102]\) is below zero the linear time-periodic parameter vibrations can stabilize solutions of \([D.102]\) exponentially stable.

**D.4.2.3 Perturbation of a stable system**

Literature theory in Section \([D.2.2.6]\) proves that small perturbations do not have an effect on the stability of the system. Simulations are performed of two stable linear systems with time-periodic parameter vibrations to investigate this.

Consider the linear vibrational system \([D.102]\) with an \(A\)-matrix such that the system has stable zero-dynamics and with a time-periodic vibration given by

\[
\dot{x} = \begin{bmatrix}
0 & -1 \\
101 & -101
\end{bmatrix} x + a \sin(\omega \cdot t) \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} x. \tag{D.106}
\]

Simulations are performed of \([D.106]\) for a wide range of vibration amplitudes and frequencies to see if the solution can be destabilized by parameter vibrations. The dominant modulus of the resulting monodromy eigenvalues is shown in Figure \([D.7]\).

![Graph showing the dominant modulus of the monodromy eigenvalues for various vibration amplitudes and frequencies.](image)

**Figure D.7:** Maximum modulus of the eigenvalues of the monodromy matrix of system \([D.106]\) with a time-periodic vibration for a range of vibration amplitudes and frequencies.

The dominant modulus of the monodromy eigenvalues in Figure \([D.7]\) is below one for all applied vibration frequencies and amplitudes. Hence, the simulations demonstrate that for wide range of vibration amplitudes and frequencies the stability properties of system \([D.106]\) can be determined by neglecting the vibration. However, for relative small vibration frequencies the eigenvalues of the monodromy matrix are zero. This means that the solution goes to the equilibrium point \(x = 0\) in finite time, also known as dead-beat.
Next, consider the linear part of system (D.102) is described by an $A$-matrix with stable poles at $-0.5 \pm 10i$, such that

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 101 & -1 \end{bmatrix} x + a \sin(\omega \cdot t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x.$$  \hspace{1cm} (D.107)

The trace of the system matrix of (D.107) is less negative than the trace of the system matrix of (D.106). This results in less convergent eigenvalues for (D.107). This gives a total different impact of the perturbations to the solution of the system. In Figure D.8 is shown the dominant modulus of the monodromy eigenvalues for a range of vibration amplitudes and frequencies. The flat surface in the front represents vibrations for which the solution of (D.107) remain stable. Also, the bottom of the first two valleys are below one so also the vibrations with these amplitudes and frequencies do not destabilize the solution of (D.107). Figure D.8 shows that vibrations with a relative high frequency compared to the amplitude do not influence the stability properties of (D.107). This confirms the stability theory for perturbed systems in Section D.2.2.6.

At last the destabilization of (D.107) for low frequency vibration is investigated by taking a closer look at the monodromy eigenvalues. The modulus of the eigenvalues of the monodromy matrix of (D.107) with a vibration amplitude of 1 over a range of vibration frequencies is depicted in Figure D.9. Figure D.9 shows an asymptote to 1 broken by ellipses of higher and lower absolute values of the monodromy eigenvalues. The monodromy eigenvalues on the asymptote curve are complex conjugated while the eigenvalues at the ellipses are real. Thus, unstable monodromy eigenvalues are real and stable monodromy eigenvalues are complex conjugated. This phenomenon is seen earlier by the systems in Section D.4.2.2.
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Figure D.8: Dominant modulus of the eigenvalues of the monodromy matrix of (D.107) for a range of vibration amplitudes and frequencies.

Figure D.9: Modulus of the eigenvalues of the monodromy matrix of (D.107) for a range of vibration frequencies and a vibration amplitude of 1.
D.4.3 2-regularized Nyquist criterion

The stability properties of (D.102) are determined using the 2-regularized Nyquist criterion as described in Section D.2.2.3. Then, the constant part is treated as $Q$ and evidently $P(t, t_0) = I$. The time-periodic part is treated as feedback control matrix. Since the Fourier series expansion of time-periodic part has only nonzero terms up to the first harmonic wave, the Nyquist locus is calculated with $N = 1$. In the computation, the staircase truncation parameter $M = 20$, the shift factor $\rho = 0.1$ and the bypassing radius $r = 0.05$ are taken. System (D.102) has one open-loop pool in the right-half plane. If a Nyquist contour is taken as described by Figure D.3, the corresponding region has one unstable open-loop eigenvalue of the operator $Q - E(j\omega)$. Then the vibrational control stabilizes (D.102) if the Nyquist locus encircles the origin ones in counterclockwise sense.

In Figure D.10 some Nyquist loci are plotted for various vibration amplitudes and frequencies. The arrows denote the direction of the loci. In Figure D.10(a) Nyquist loci are plotted for a vibration amplitude $a = 11$ and frequencies $\omega$ is 6, 40 and 60. Stabilization is shown for $\omega$ is 6 and 40. In Figure D.10(b) Nyquist loci are plotted for a vibration frequency $\omega = 11$ and amplitude $a$ is 10, 10.1 and 11. At a vibration amplitude $a$ is 10.1 and 11 stabilization is shown. The results shown in Figure D.10 correspond with the results of the simulations, depicted in Figure D.3. In Figure D.10(a) at $a = 11$ and increasing the vibration frequency a clear development of the Nyquist loci is observed. The loci at $\omega$ is 6 or 40 are around zero in counterclockwise sense so the system is stable with these vibrations. For an vibration amplitude of 11, the locus at $\omega = 40$ is closer to the origin then the locus at $\omega = 6$ which corresponds with the fact that the vibration of $\omega = 40$ is closer to the boundary of stabilizing vibrations then with $\omega = 6$, see Figure D.3. Thus, the stabilization of (D.102) is more robust at $\omega = 6$ then at $\omega = 40$ for an amplitude of $a = 11$. This is also valid for the Nyquist loci in Figure D.10(b) where the stability of (D.102) is more robust at $a = 11$ that at $a = 10.1$. So, the advantage of the 2-regularized Nyquist criterion is that a few Nyquist loci show the robustness of the stabilization of the vibrational control without the necessity of simulating the system for a whole range of vibration frequencies and amplitudes.

D.4.4 Discussion

Simulations with frequency bandwidth-limited white noise give the surprising result that, for equal intensity and frequency bandwidth, the noise with a high-frequency band stabilizes worse than the noise with a low-frequency band. This is against the intuitive since the literature theory provides that...
slowly varying parameter changes do not effect the stability properties of a system and the averaging method provides stabilization of linear systems for high frequency and high amplitude vibration only. The simulations with periodic vibration confirm that vibrations with lower frequencies results in more converging solutions. On the other hand, higher vibration amplitudes can increase the range of stabilizing vibration frequencies. Thus a high performance with linear vibrational control can be gained by searching the lowest vibration frequency for which there is a vibration amplitude which stabilizes the system. However, this will not always be a robust solution.

The simulations of linear vibrational control demonstrate that stabilization by linear parameter vibrations is only possible if the trace of the system matrix is negative. This validates the theory obtained from the Liouville-Jacobi formula in Section D.2.2.2 that the exponent of the trace of the system matrix is equal to the product of the monodromy eigenvalues.

Perturbed systems are linear periodic systems where the high vibration frequency is relative high compared to the vibration amplitude. The stability properties of these systems is determined by neglecting the vibrations, see Section D.2.2.6. Simulation results confirm this theory. Finally, the 2-regularized Nyquist criteria shows stabilization for corresponding vibrations as the simulation do. Moreover, the robustness of the stabilization is shown.
D.5 Conclusions and Recommendations

In this appendix, system stabilization by white noise and time-periodic vibration is studied. Literature on stabilization of LTI systems by white noise vibration is discussed, using a second order LTI example system with white noise added to the system parameters. Various literature methods to determine stability of linear time-periodic (LTP) systems and some control strategies concerning linear vibrational control are presented as well. The example of stabilization of a second order system by white noise vibrations is used to examine the usability of the literature methods to determine the stability properties of LTP systems. Therefore, the white noise is replaced by time-periodic vibrations. None of the methods is able to determine the stability properties of the example system analytically. To compare stabilization by white noise with stabilization by time-periodic vibration the example system is simulated. The simulations demonstrate stabilization by frequency bandwidth-limited white noise and by time-periodic vibrations. The results are compared with the theory.

The conclusions and recommendations that can be drawn based on this work are presented here.

D.5.1 Conclusions

In literature the following methods to determine the stability properties of LTP are found: Lyapunov's Direct method, the Floquet transition matrix of monodromy matrix, the Peano-Baker approximation series of the monodromy matrix, the 2-regularized Nyquist criterion, a trick to compute the transition matrix via algebraic equations and the averaging method which approximates the vibrations in terms of a small parameter. None of the methods is able determine the stability properties of the example system. This shows clearly that determining the stability properties is much more complicated for LTP systems than for LTI systems. It is not straightforward which method to use for a certain LTP system and it is even possible that no method is capable to determine the stability properties analytically. In that case, numerical methods are available.

However, linear vibrational (feed-forward) control has great advantages over the linear feedback control. With the described control strategies it is possible to control a linear system without measuring or actuating each state. Moreover, SSPH control is able to assign the whole monodromy matrix. New developments are described in Montagnier et al. (2004)

One clear condition for stabilizability of a LTI system by linear zero-mean parameter vibrations exists: the trace of the system matrix must be negative. This is achieved by applying the Liouville-Jacobi formula with the Floquet transition matrix. The averaging method comes to the same conclusion.

The values of vibration amplitude and frequency that stabilizes the example system are determined by numerical simulation. The simulations with frequency bandwidth-limited white noise and with time-periodic vibrations show both that low vibration frequencies have stabilizes the solution better in the sense that the solution is more converging to the equilibrium point. The simulations demonstrate that time-periodic and random vibrations have comparable results on stabilization of LTI systems. The simulations demonstrate that the trace of the matrix must be negative for stabilization by zero-mean parameter vibrations, which is concluded from the literature methods as discussed above.

Numerical computation of the 2-regularized Nyquist loci for various vibration shows stabilization at equivalent vibration as the numerical simulations. A Nyquist locus has the advantage over simulation that is gives information about the robustness of the stability. Calculation of the monodromy matrix by numerical simulation has the advantage that the convergence is known.

From the discussed matter above it is concluded that the simulations corresponds to the theory of stabilization of LTI systems via linear parameter vibrations. However, there is no theory which describes the more converging solutions for lower vibration frequencies as the simulations have shown.
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D.5.2 Recommendations

The 2-regularized Nyquist criterion is a promising technique to determine the stability properties of LTP systems. The usability of this stability criterion can be improved by developing rules which define how to determine the robustness of the stability.

Further investigation of the comparability of random and time-periodic parameter vibrations is recommended. Therefore it is recommended to simulate the example system with frequency bandwidth-limited white noise using various other frequency bands and several intensities. Simulation of the other systems with various frequency bandwidth-limited white noise signals are recommended as well.

Furthermore, an experimental setup is recommended to examine the theory and the simulation results, e.g. an underactuated mechanical system with a free joint as is used as theoretical application in [Hong et al. 1998].
D.6 Bibliography


D. Stabilization of Linear Systems through Parameter Vibrations


Summary of Appendix D

In Appendix D, stabilization of linear time-invariant (LTI) systems by adding white noise or time-periodic vibrations to the parameters is studied.

Literature on stabilization of LTI systems via white noise parameter vibrations is reviewed, using a second order LTI example system with white noise added to the system parameters. This example is used in the investigation of this appendix. Furthermore, from the literature various methods are provided to determine the stability of LTP systems. These are: Lyapunov’s direct method, the Floquet transition matrix or monodromy matrix, the Péano-Baker approximation series of the monodromy matrix, the 2-regularized Nyquist criterion, a trick to compute the transition matrix via algebraic equations and the averaging method which approximates the vibrations in terms of a small parameter. The analytic analyses did not succeed in determining the stability properties of the example of stabilization via white noise where the white noise is replaced by time-periodic vibrations. Still, some conclusions can be made. The Liouville-Jacobi formula applied to the Floquet monodromy matrix proves that linear parameter vibration can stabilize a linear system if the trace is negative. Since the trace of the example system is negative, stabilization via the vibrations is expected. The stability properties of the parameter vibrations are investigated by numerical simulations. It is investigated if stabilization properties of the white noise parameter vibration are comparable with those of the time-periodic parameter vibration. Simulations with frequency bandwidth-limited white noise give the surprising result that, for equal intensity and frequency bandwidth, the noise with a lower frequencies stabilizes better than the noise with higher frequencies. This is against the intuitive since the literature theory provides that slowly varying parameter changes do not effect the stability properties of a system and the averaging method provides stabilization of linear systems for high frequency and high amplitude vibration only.

The simulations with periodic vibration confirm that vibrations with lower frequencies results in more converging solutions. On the other hand, higher vibration amplitudes can increase the range of stabilizing vibration frequencies. Thus a high performance can be gained with linear vibrational control by searching the lowest vibration frequency for which there is a vibration amplitude which stabilizes the system. However, this will not always be a robust solution.

The simulations of linear vibrational control demonstrate that stabilization by linear parameter vibrations is only possible if the trace of the system matrix is negative. This validates the theory obtained from the Liouville-Jacobi formula which says that the exponent of the trace of the system matrix is equal to the product of the monodromy eigenvalues.

Perturbed systems are linear periodic systems where the high vibration frequency is relative high compared to the vibration amplitude. The stability properties of these systems is determined by neglecting the vibrations. Simulation results confirm this theory.

The numerical calculation of the 2-regularized Nyquist locus demonstrates stabilization for the equivalent vibration amplitudes and frequencies as derived from the simulations. The Nyquist locus develops when the vibration frequency or amplitude is changed. In this way stabilization can be predicted.

We conclude that there are two additionally useful numerical methods to determine the stability properties of a linear time-periodic continuous system. The numerical calculation of the monodromy matrix provides convergence or divergence of the system and the second demonstrates the stability properties and its robustness by the 2-regularized Nyquist locus.
Samenvatting van Appendix D

In Appendix D wordt studie gedaan naar de stabilisatie van Lineair Tijd Invariante (LTI) systemen door witte ruis of tijd-periodieke vibraties aan de systeemparameters toe te voegen. De behandelde literatuur over stabilisatie van LTI systemen door witte ruis maakt gebruik van een twee dimensionaal voorbeeld systeem. Dit systeem wordt ook gebruikt in het onderzoek van Appendix D. Verder zijn in de literatuur verschillende methoden beschreven die de stabiliteitseigenschappen kunnen bepalen van LTI system met tijd-periodieke parameter vibraties. Dit zijn: Lyapunov’s direct methode, de Floquet transitie matrix of monodromy matrix, de Péano-Baker benaderingsseries van de monodromy matrix, de 2e-gereguleerde Nyquist criterium, een truc om de transitie matrix te berekenen door middel van algebraïsche vergelijkingen en de averaging methode die de oplossing van het systeem benadert door de vibraties in termen van een kleine parameter te beschrijven. De analytische analyses van het voorbeeld systeem waarbij de witte ruis is vervangen door tijd-periodieke vibraties slagen er niet in om de stabiliteitseigenschappen ervan te bepalen. Toch zijn er enige conclusies mogelijk. De Liouville-Jacobi formule toegepast op de Floquet monodromy matrix bewijst dat lineaire parameter vibraties een lineair systeem kunnen stabiliseren als het spoor van de transitie matrix negatief is. Omdat het spoor van de matrix van het voorbeeld systeem negatief is wordt stabilisatie door de vibraties verwacht. De stabiliteitseigenschappen zijn onderzocht door middel van numerieke simulaties. Het is onderzocht of de stabiliserende werking van de witte ruis vergelijkbaar is met die van de tijd-periodieke parameter vibraties. Simulaties met frequentie bandbreedte gelimiteerde witte ruis geven het verrassende resultaat dat, voor gelijke ruisintensiteit en frequentie bandbreedte, de ruis een lagere frequenties stabiliseert beter dan de ruis met hogere frequenties. Dit gaat tegen de intuïtie in op grond van wat in de literatuur bekend is. Daar staat immers dat langzame parameters vibraties geen invloed hebben op de stabiliteit en de averaging methode geeft alleen stabiliserende effecten voor hoge frequencies en hoge amplitudes. De simulaties met tijd-periodieke vibraties bevestigen dat vibraties met lagere frequenties resulteren in meer convergerende oplossingen. Wel kan er een minimum amplitude zijn waarbij stabilisatie optreedt en ook vergroot een hogere vibratie amplitude het scala aan stabiliserende frequenties. Dus, door middel van lineaire parameter vibraties kan een hoge convergencie worden bereikt door de lagste vibratie frequentie te zoeken waarbij er nog een vibratie amplitude is die het systeem stabiliseert. Een nadeel is dat deze oplossing niet altijd even robuust zal zijn. De simulaties van de lineaire parameter vibraties demontreren dat stabilisatie door lineaire parameter vibraties alleen mogelijk is als het spoor van de systeemmatrix negatief is. Dit valideert de theorie verkregen uit de Liouville-Jacobi formule. Deze formule houdt in dat de exponent van het spoor van de systeemmatrix gelijk is aan het product van de eigenwaarden van de monodromy matrix. Perturbed systemen zijn lineair periodieke systemen waar de vibratie frequentie relatief hoog is ten opzichte van de vibratie amplitude. De stabiliteitseigenschappen worden bepaald door de invloed van de vibraties te verwaarlozen. Dit wordt bevestigd door de uitgevoerde simulaties. De numerieke berekening van de 2e-gereguleerde Nyquist locus laat stabilisatie zien voor overeenkomstige systeem en parameter vibraties als de hierboven beschreven simulaties. Duidelijk was een ontwikkeling van de Nyquist locus te zien als de vibratie frequentie of amplitude veranderd werd. Op deze manier kan stabilisatie worden voorspeld.

Wij concluderen dat er twee additionele bruikbare numerieke methoden zijn om de stabiliteitseigenschappen van een lineair tijd-periodiek continu systeem te bepalen. Met de numerieke bereken-
Samenvatting van Appendix D

ing van de monodromy matrix wordt de convergentie verkregen en met de 2e gereguleerde Nyquist locus wordt de robustheid van de stabilisatie verkregen.