MASTER

Analysis of infinite phased arrays of printed antennas

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Analysis of Infinite Phased Arrays of Printed Antennas

door

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Abstract

In this report, the infinite array model is used to analyse large arrays of tapered-slot antennas. These antennas are famous for their wide bandwidth and wide scan range in the $H$-plane. Due to the infinite array model, the analysis can be restricted to one unit cell only. The electromagnetic field in the unit cell is analysed using the method of moments including the exact Green's functions of the unit cell. Therefore, mutual coupling effects are automatically included.

The model is applied to analyse the effects of metallic walls on the $E$-plane scan behaviour of these arrays. It is shown that metallic walls, placed on the four sides of the unit cell, have a positive effect on the scan behaviour. However, when grating lobes enter the array, blind scan angles are likely to occur. This has also been observed in experimental data.

The infinite array model is also used to examine arrays, which contain two radiating elements perpendicular to one another inside the unit cell. In this way, it is expected that the polarization properties of the electric field can be controlled. The results indicate that scanning in the principal planes is very well possible. Unfortunately, for scanning in the diagonal planes strong coupling is observed between the two elements inside the unit cell.

Finally, a model to analyse single polarized arrays with a triangular grid is presented. This model has been used to design an array of so-called bunny-ear antennas. It is shown that the scan behaviour improves when the elements are placed closer to one another.
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Chapter 1

Introduction

1.1 General introduction

Nowadays, the application of antenna systems is roughly divided into two categories:

1. Telecommunication
2. Active and passive remote sensing

Ad 1: A fast growing branch of telecommunication is satellite communication. For this kind of communication, a high antenna gain and low side lobes are important. Furthermore, the polarization characteristics are of interest, since for fixed ground stations two orthogonal polarizations are used to obtain two communication channels at one frequency band and for mobile ground stations circular polarization is used to make the system independent of position and orientation.

Ad 2: An example of active remote sensing is radar, as applied in civil aviation and navigation or as in military systems. For radar in general, a high antenna gain and an accurate positioning of the main lobe are important demands, as they determine the detection range and accuracy of the target position determination, respectively. For military radar in particular, multiple target tracking, by means of multiple beams or fast scanning, and jammer suppression, by means of side-lobe cancellation or adaptive nulling, is also significant. Further, the use of dual polarization is becoming more and more an issue for radar.

A typical example of passive remote sensing is radio astronomy, where very weak radio signals from distant stars or galaxies are received to further analyse the physical laws of nature. For this kind of systems, the antenna gain determines the sensitivity of the radio telescope, which in its turn determines what kind of phenomena can be observed by the telescope. Other important issues of the antenna system are low side lobes (to prevent terrestrial signals from disturbing the analysis, for example), accurate beam positioning, large bandwidth, wide scan properties and excellent polarization characteristics, among others.

Demands like high gain, low side lobes and accurate beam positioning can be accomplished by using both reflector antennas and phased arrays. For reflector antennas, demands like fast scanning, multiple beams and side-lobe cancellation are difficult to realise, especially when these techniques have to be time variant, such as adaptive nulling or multiple beams for multiple target tracking. For phased arrays, these techniques are, at least in principle, straight forward.

Phased arrays are built in all shapes and sizes. The main difference between arrays is the type of radiating element, for example: wave guide, wire antenna, patch antenna, tapered slot antenna or even reflector antenna. In this report, phased arrays of tapered-slot antennas are analysed. These antennas consist of a conducting surface printed on a substrate that is protruding from a ground plane. The conducting surface on the substrate can have all sorts of shapes.
Some examples are linear tapered antennas, folded dipoles and bunny-ears, which are shown in figure 1.1. Tapered-slot antennas are known for their wide bandwidth and good scan characteristics in the $H$-plane (i.e. the plane perpendicular to the protruding surface). However, the $E$-plane scan behaviour is usually poor.

Figure 1.1: Tapered-slot antennas

Typical problems in the design of this kind of phase arrays are demands like large bandwidth and wide-scan angles in all planes. Also, the application of dual polarization is not straightforward due to the expected mutual coupling between two elements when they are placed perpendicular to one another.

1.2 Model description

Large phased arrays, which typically contain a thousand or more radiating elements, can be analysed by means of the infinite array model. In this model, the array is extended toward infinity in both directions, which makes it possible to formulate periodicity conditions. Due to these conditions, the analysis of the array can be restricted to a unit cell, i.e. the smallest part of the array out of which the entire array can be built by means of translation only. The infinite array model removes the boundary effects of an array. However, it is expected that in a large array the edge effects are not dominant for the behaviour of the array, since in most cases a tapering is used over the array to reduce the side lobes.

The unit cell, containing the protruding antenna, is analysed by making use of the equivalence principle. In this way, the array is divided into two regions: a region containing the groundplane and the antenna and a region consisting of free space. These two regions can be analysed separately and more easily than the entire unit cell at once. In case of metallic walls inside the unit cell, this analysis is especially convenient. Furthermore, it is assumed that the conducting surfaces of the ground plane and of the antenna are perfectly conducting and that the substrate on which the antenna is printed has no influence on the behaviour of the antenna. The last condition is valid when a very thin substrate is used or when the relative permittivity and permeability of the substrate are close to unity.

The two separate regions are analysed by means of the dyadic Green’s functions inside these regions of the unit cell and by applying the method of moments. In the method of moments, the current distributions (both electric and magnetic) are expanded into a set of basis functions with unknown mode coefficients. These expanded current distributions are substituted into the integral equations which are formulated from the boundary conditions. The integral equations are then tested by means of testing functions. In this way, a matrix equation is obtained for the unknown mode coefficients. After solving the matrix equation, the mode coefficients can be used to calculate the active input impedance, the active reflection coefficient, the element pattern or the electromagnetic field in the far field region.
Introduction

This method of modelling arrays of tapered-slot antennas has been developed by Cooley [5]. The method has also been applied by Hulshof [8], to analyse arrays of bunny-ear tapered-slot antennas.

1.3 Organisation of the report

In Chapter 2, the formulation of the unit cell model is explained together with the use of the equivalence principle. Also, the use of vector potentials and dyadic Green’s functions to describe the electromagnetic field is discussed. A model to analyse metallic walls inside the unit cell is presented in chapter 3, since metallic walls can have a positive effect on the scan behaviour of an array [6]. The analysis is applied to an array of folded dipoles. Chapter 4 contains the description of a model to analyse dual polarization by means of two radiating elements, perpendicular to one another, inside the unit cell. The effects of mutual coupling are investigated by analysing an array of dipoles and an array of bunny-ear antennas. Finally, the extension of the model, described in [8], to a triangular grid is given in chapter 5. A triangular grid allows larger element spacings compared to a rectangular grid, when grating lobes are unwanted for a certain range of scan angles.
Chapter 2

Modelling approach

2.1 Introduction

In this chapter, the basic ideas of the analysis of large phased arrays are explained, starting with a discussion about the infinite array model and followed by the model for the unit cell. Then, the electromagnetic field in terms of vector potentials is defined, after which the boundary conditions for the vector potentials inside the unit cell are discussed. The chapter is completed by two examples which use the Green's functions in order to describe the electromagnetic field as a function of the electric and magnetic current densities.

2.2 Infinite array model

The behaviour of large arrays, which typically contain a thousand or more radiating elements, can be approximated by the infinite array model. In this model, it is assumed that the array extends towards infinity in both directions and all radiating elements are the same. Furthermore, all elements are excited by sources of equal amplitude and linearly varying phase. Under these assumptions, Floquet's Theorem [7] can be applied. Due to Floquet's Theorem, the analysis can be performed on the unit cell only. The unit cell is defined as the smallest element out of which the entire array can be built, by means of translation only.

2.3 The unit cell and its equivalent structure

In principle, the unit cell anywhere in the infinite array can be used to perform the analysis. However, a unit cell near the origin of the array is used here. The unit cell consists of a perfectly conducting ground plane and a radiating element protruding from the ground plane. Other features, such as metallic walls, can be included in the unit cell. Figure 2.1 shows an example of the centre unit cell.

Figure 2.1: Configuration of the centre unit cell
Modelling approach

The ground plane is placed at \( z = -d \) and the height of the radiating element is assumed to be smaller than \( d \). Furthermore, it is assumed that the radiating element is a perfectly conducting surface. Therefore, the current density on the element will only have \( y \)- and \( z \)-directed components.

In order to analyse this kind of structure, use will be made of the uniqueness theorem and the equivalence theorem [7]. At the aperture plane \((z=0)\), a perfectly conducting surface is placed. In this way, two regions are created: the interior region, for \(-d < z < 0\), and the exterior region, for \( z > 0 \). It is obvious that this situation is not equivalent to the one presented in figure 2.1. However, by defining magnetic current densities on both sides of the aperture plane, the situation can be made equivalent to the one in figure 2.1. This configuration is shown in figure 2.2.

![Figure 2.2: Equivalent centre unit cell](image)

The arrows on the aperture plane in figure 2.2 indicate the \( x \)- and \( y \)-directed magnetic current densities on both sides of the aperture plane. The magnetic current density on the inside of the aperture plane (i.e. \( z = 0^- \)) will be indicated by \( M^{in} \), while the magnetic current density on the outside of the aperture plane (i.e. \( z = 0^+ \)) will be indicated by \( M^{ex} \). It is clear that the electric and magnetic fields in the interior region (\( E^{in} \) and \( H^{in} \), respectively) are produced by the electric current density on the radiating element and by the magnetic current density on the inside of the aperture plane, while the electric and magnetic fields in the exterior region (\( E^{ex} \) and \( H^{ex} \), respectively) are produced by the magnetic currents on the outside of the aperture plane.

The uniqueness theorem states that electric and magnetic fields in a region are entirely determined by their tangential components at the boundaries of that region. In the original situation of figure 2.1, the tangential fields at the aperture plane (i.e. \( z = 0 \)) are continuous. Therefore, the configuration of figure 2.2 is equivalent to the one of figure 2.1, if and only if Eq(2.1) is satisfied.

\[
\begin{align*}
E^{in} \big|_{z=0^-} \times e_z &= E^{ex} \big|_{z=0^+} \times e_z, \\
H^{in} \big|_{z=0^-} \times e_z &= H^{ex} \big|_{z=0^+} \times e_z.
\end{align*}
\] (2.1a) (2.1b)

Eq(2.1b) will be dealt with when the method of moments is considered, while Eq(2.1a) can be satisfied by examining the magnetic current densities on the aperture plane. This can be done...
by using the relation given in Eq(2.2)

\[ \begin{align*}
M^{in} &= -E^{in} \times \varepsilon_x \bigg|_{z=0} , \\
M^{ex} &= E^{ex} \times \varepsilon_x \bigg|_{z=0} ,
\end{align*} \] (2.2a, 2.2b)

where the minus sign in Eq(2.2a) is due to the fact that the normal vector has to point into the region in which the magnetic current density produces the electric field. Combining Eq(2.1a) and Eq(2.2), the following relation between the interior and exterior magnetic current densities can be found:

\[ M^{in} = -M^{ex} . \] (2.3)

By using an equivalence plane at the aperture plane, the original structure of the unit cell has been transformed into a geometry that can be analysed in an easier way.

2.4 Electromagnetic fields in terms of vector potentials

The analysis of the unit cell will concentrate on finding the electric and magnetic current densities on the radiating element and on the aperture plane, respectively. In order to find the current densities, use will be made of the Green’s function for the electric and magnetic vector potentials. In order to use the vector potentials, it is necessary to derive the relations between the electric and magnetic fields on the one hand and the electric and magnetic vector potentials on the other hand.

For a vacuum right handed coordinate system and an assumed time dependence \( e^{j\omega t} \), Maxwell’s laws, including the magnetic current density, can be written as

\[ \begin{align*}
\nabla \times E (r) &= -j\omega \mu_0 H - M (r) , \\
\nabla \times H (r) &= j\omega \varepsilon_0 E + J (r) ,
\end{align*} \] (2.4a, 2.4b)

where \( J (r) \) is the electric current density, \( \varepsilon_0 \) is the permittivity in vacuum and \( \mu_0 \) is the permeability in vacuum. Due to the linearity of Maxwell’s laws, the electric and magnetic fields can be written in terms of fields generated by an electric current density (indicated by a sub-index \( e \)) and fields generated by a magnetic current density (indicated by a sub-index \( m \)), as in Eq(2.5) (superposition).

\[ \begin{align*}
E (r) &= E_e (r) + E_m (r) , \\
H (r) &= H_e (r) + H_m (r) .
\end{align*} \] (2.5a, 2.5b)

Maxwell’s laws for the electric and magnetic fields generated by an electric current density, can now be written as:

\[ \begin{align*}
\nabla \times E_e (r) &= -j\omega \mu_0 H_e , \\
\nabla \times H_e (r) &= j\omega \varepsilon_0 E_e + J (r) ,
\end{align*} \] (2.6a, 2.6b)

while for the fields generated by a magnetic current density, Maxwell’s laws can be written as:

\[ \begin{align*}
\nabla \times E_m (r) &= -j\omega \mu_0 H_m - M (r) , \\
\nabla \times H_m (r) &= j\omega \varepsilon_0 E_m .
\end{align*} \] (2.7a, 2.7b)
Modelling approach

Observing Eq(2.6a), it can be seen that the magnetic field generated by an electric current density is divergence free. Therefore, $H_e$ can be written as the curl of some other vector, according to Eq(2.8).

$$H_e(r) = \nabla \times A(r), \tag{2.8}$$

where $A$ is coined the magnetic vector potential. From Eq(2.6a) it can be seen that, $E_e$ can be written as:

$$E_e(r) = -j\omega\mu_0 A(r) - \nabla \Phi(r), \tag{2.9}$$

where $\Phi$ is the scalar electric potential. Since only the curl of $A$ has been defined, the divergence of $A$ can still be defined. In this case, the best choice for the divergence is the so-called Lorentz-gauge, which is defined as

$$\nabla \cdot A(r) = -j\omega\varepsilon_0 \Phi(r). \tag{2.10}$$

The electric field generated by an electric current density can now be written as:

$$E_e(r) = -j\omega\mu_0 A(r) + \frac{1}{j\omega\varepsilon_0} \nabla (\nabla \cdot A(r)). \tag{2.11}$$

When Eq(2.6b), Eq(2.8) and Eq(2.11) are combined, the relation between the electric current density and the magnetic vector potential is obtained:

$$\nabla^2 A(r) + k_0^2 A(r) = -J(r), \tag{2.12}$$

where $k_0^2 = \omega\sqrt{\varepsilon_0\mu_0}$. Eq(2.12) is known as the inhomogeneous Helmholtz equation.

When a similar analysis is done for the electric and magnetic fields generated by a magnetic current density, by means of an electric vector potential $E$, the following set of equations can be found:

$$E_e(r) = -\nabla \times E(r), \tag{2.13a}$$

$$H_m(r) = -j\omega\varepsilon_0 E(r) + \frac{1}{j\omega\mu_0} \nabla (\nabla \cdot E(r)), \tag{2.13b}$$

$$\nabla^2 E(r) + k_0^2 E(r) = -M(r). \tag{2.13c}$$

If the coordinate system is Cartesian, the fields can be written as:

$$H_e(x, y, z) = \begin{bmatrix} \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \\ \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \\ \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} \end{bmatrix}, \tag{2.14a}$$

$$E_e(x, y, z) = -\frac{j\omega\mu_0}{k_0^2} \begin{bmatrix} (k_0^2 + \partial_x^2)A_x + \partial_x \partial_y A_y + \partial_x \partial_z A_z \\ (k_0^2 + \partial_y^2)A_y + \partial_y \partial_z A_z + \partial_y \partial_x A_x \\ (k_0^2 + \partial_z^2)A_z + \partial_z \partial_x A_x + \partial_z \partial_y A_y \end{bmatrix}, \tag{2.14b}$$
Modelling approach

\[ E_m(x, y, z) = - \begin{bmatrix} \partial_y F_z - \partial_z F_y \\ \partial_z F_x - \partial_x F_z \\ \partial_x F_y - \partial_y F_x \end{bmatrix}, \]

(2.14c)

\[ H_m(x, y, z) = \frac{j \omega \varepsilon_0}{k_0^2} \begin{bmatrix} (k_0^2 + \alpha_x^2) F_x + \partial_x F_y + \partial_y F_z \\ (k_0^2 + \alpha_y^2) F_y + \partial_y F_x + \partial_x F_z \\ (k_0^2 + \alpha_z^2) F_z + \partial_z F_x + \partial_x F_y \end{bmatrix}. \]

(2.14d)

2.5 Boundary conditions of the unit cell

A typical feature of phased arrays is the fact that the direction of the main beam can be controlled in an electronic way. This can be done by applying a linearly varying phase to the radiating elements with respect to their position. In order to describe the direction of the main beam, the coordinate system of figure 2.3 is used.

![figure 2.3: Coordinate system](image)

For a phased array with a rectangular grid, the vector potentials (and thus the electric and magnetic fields) must obey the following relations:

\[ \Delta(x + a, y, z) = \Delta(x, y, z) e^{-j\Psi_x}, \]

(2.15a)

\[ \Delta(x, y + b, z) = \Delta(x, y, z) e^{-j\Psi_y}, \]

(2.15b)

\[ E(x + a, y, z) = E(x, y, z) e^{-j\Psi_x}, \]

(2.15c)

\[ E(x, y + b, z) = E(x, y, z) e^{-j\Psi_y}, \]

(2.15d)

where \( \Delta \) is the magnetic vector potential, \( E \) is the electric vector potential, \( a \) is the dimension of the unit cell in the \( x \)-direction, \( b \) is the dimension of the unit cell in the \( y \)-direction and \( \Psi_x \) and
Modelling approach

\( \Psi_y \) are given by

\[
\Psi_x &= k_0 a \sin(\theta) \cos(\varphi), \\
\Psi_y &= k_0 b \sin(\theta) \sin(\varphi).
\]  

These relations are due to the fact that all radiating elements of the array are equal and all sources have equal amplitude and a linearly varying phase with respect to their position. The relations are the so-called periodicity conditions and can be used as boundary conditions for the vector potentials inside the unit cell.

2.6 The electromagnetic field in terms of Green’s functions

When an electric dipole inside the unit cell is considered, the magnetic vector potential can be determined by means of the boundary conditions of the unit cell and the inhomogeneous Helmholtz equation. The electric and magnetic field, caused by the electric dipole, can be found by applying Eq(2.14a) and Eq(2.14b) and are the so-called dyadic Green’s functions of the electric dipole. For example, suppose \( G_x^m \) is the magnetic vector potential for an \( x \)-directed dipole inside the unit cell, i.e.

\[
\nabla^2 G_x^m + k_0^2 G_x^m = -\delta (x-x', y-y', z-z') \hat{e}_x,
\]

where the unprimed coordinates are the coordinates of observation and the primed coordinates are the source coordinates. \( G_x^m \) satisfies the boundary conditions of the unit cell. Furthermore, let \( G_y^m \) and \( G_z^m \) be defined as the magnetic vector potential for a \( y \)- and \( z \)-directed electric dipole inside the unit cell, respectively. Then, the dyadic Green’s functions in cartesian components (indicated by a double line over the function) can be written as

\[
\mathbb{H}_e = \begin{bmatrix}
(\partial_y G^m_{yx} - \partial_z G^m_{yz}) & (\partial_y G^m_{zy} - \partial_z G^m_{yz}) & (\partial_y G^m_{yz} - \partial_z G^m_{yz}) \\
(\partial_z G^m_{yx} - \partial_x G^m_{zx}) & (\partial_z G^m_{zy} - \partial_x G^m_{zx}) & (\partial_z G^m_{zx} - \partial_x G^m_{zx}) \\
(\partial_x G^m_{yx} - \partial_y G^m_{xy}) & (\partial_x G^m_{zy} - \partial_y G^m_{zy}) & (\partial_x G^m_{zy} - \partial_y G^m_{zy})
\end{bmatrix},
\]  

\[
\mathbb{E}_e = -\frac{j\omega}{k_0^2} \begin{bmatrix}
E_{xx} & E_{xy} & E_{xz} \\
E_{yx} & E_{yy} & E_{yz} \\
E_{zx} & E_{zy} & E_{zz}
\end{bmatrix},
\]

where

\[
\begin{align*}
E_{xx} &= (k_0^2 + \partial_x^2) G^m_{xx} + \partial_x \partial_y G^m_{yx} + \partial_x \partial_z G^m_{xz}, \\
E_{yx} &= \partial_y \partial_x G^m_{yx} + (k_0^2 + \partial_y^2) G^m_{yx} + \partial_y \partial_z G^m_{yz}, \\
E_{zx} &= \partial_z \partial_x G^m_{zx} + \partial_z \partial_y G^m_{zy} + (k_0^2 + \partial_z^2) G^m_{xz}.
\end{align*}
\]
\begin{align}
\begin{bmatrix}
E_{xy} \\
E_{yy} \\
E_{zz}
\end{bmatrix}
&= 
\begin{bmatrix}
(k_0^2 + \partial_x^2)G_{xy}^m + \partial_x \partial_y G_{xy}^m + \partial_x \partial_y G_{xy}^m \\
(k_0^2 + \partial_y^2)G_{xy}^m + \partial_x \partial_y G_{xy}^m + \partial_x \partial_y G_{xy}^m \\
(k_0^2 + \partial_z^2)G_{xy}^m + \partial_x \partial_y G_{xy}^m + \partial_x \partial_y G_{xy}^m
\end{bmatrix}, \\
(2.19b) \\
\begin{bmatrix}
E_{xz} \\
E_{yz} \\
E_{zz}
\end{bmatrix}
&= 
\begin{bmatrix}
(k_0^2 + \partial_x^2)G_{xz}^m + \partial_x \partial_y G_{xz}^m + \partial_x \partial_y G_{xz}^m \\
(k_0^2 + \partial_y^2)G_{xz}^m + \partial_x \partial_y G_{xz}^m + \partial_x \partial_y G_{xz}^m \\
(k_0^2 + \partial_z^2)G_{xz}^m + \partial_x \partial_y G_{xz}^m + \partial_x \partial_y G_{xz}^m
\end{bmatrix}, \\
(2.19c)
\end{align}

and

The dyadic Green's functions are functions of both primed and unprimed coordinates. The first sub-index of the matrix elements refers to the component of the magnetic vector potential and the second sub-index refers to the direction of the electric dipole, (e.g. $G_{xz}$ refers to the x-component of $G_z^m$).

Now, suppose there is an electric current density $J(\mathbf{r})$ (in cartesian components) inside the unit cell. This current density can also be written as an infinite summation of electric dipoles:

$$J(\mathbf{r}) = \int_{V_{cell}} J(\mathbf{r}') \delta((\mathbf{r}-\mathbf{r}')) dV,$$

(2.20)

where $\mathbf{r} = (x,y,z)^T$ and $\mathbf{r}' = (x',y',z')^T$ and $V_{cell}$ is the volume of the unit cell. Due to the linearity of Maxwell's laws, the electric and magnetic fields inside the unit cell, caused by the electric current density, can also be written as an infinite summation:

$$H(\mathbf{r}) = \int_{V_{cell}} \bar{H}_e \cdot J(\mathbf{r}') dV,$$

(2.21a)

$$E(\mathbf{r}) = \int_{V_{cell}} \bar{E}_e \cdot J(\mathbf{r}') dV.$$

(2.21b)

A similar derivation can be done for a magnetic dipole. If $G_x^e$, $G_y^e$ and $G_z^e$ are the electric vector potentials for an $x$-, $y$- and $z$-directed magnetic dipole inside the unit cell, respectively, than the dyadic Green's functions in cartesian coordinates are given by

$$\bar{E}_m = -
\begin{bmatrix}
(\partial_{y xx} G_x^e - \partial_{y xy} G_x^e) & (\partial_{y yx} G_x^e - \partial_{y yy} G_x^e) & (\partial_{y z x} G_x^e - \partial_{y z y} G_x^e) \\
(\partial_{y xx} G_y^e - \partial_{y xy} G_y^e) & (\partial_{y yx} G_y^e - \partial_{y yy} G_y^e) & (\partial_{y z x} G_y^e - \partial_{y z y} G_y^e) \\
(\partial_{y z x} G_z^e - \partial_{y z y} G_z^e) & (\partial_{y z y} G_z^e - \partial_{y z z} G_z^e)
\end{bmatrix},
$$

(2.22a)
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\[ \mathbb{H}_m = \frac{j\omega \varepsilon_0}{k_0^2} \begin{bmatrix} H_{xx} & H_{xy} & H_{xz} \\ H_{yx} & H_{yy} & H_{yz} \\ H_{zx} & H_{zy} & H_{zz} \end{bmatrix}, \]  
\tag{2.22b}

where

\[ \begin{bmatrix} H_{xx} \\ H_{yx} \\ H_{zx} \end{bmatrix} = \begin{bmatrix} (k_0^2 + \partial_x^2)G^e_{xx} + \partial_x \partial_y G^e_{yx} + \partial_x \partial_z G^e_{xz} \\ \partial_x \partial_y G^e_{yx} + (k_0^2 + \partial_y^2)G^e_{yy} + \partial_y \partial_z G^e_{yz} \\ \partial_x \partial_z G^e_{xz} + \partial_y \partial_z G^e_{yz} + (k_0^2 + \partial_z^2)G^e_{zz} \end{bmatrix}, \]  
\tag{2.23a}

\[ \begin{bmatrix} H_{xy} \\ H_{yy} \\ H_{yz} \end{bmatrix} = \begin{bmatrix} \partial_x \partial_y G^e_{xx} + \partial_x \partial_y G^e_{yy} + \partial_x \partial_y G^e_{yz} \\ \partial_x \partial_y G^e_{yx} + \partial_y \partial_y G^e_{yy} + \partial_y \partial_y G^e_{yz} \\ \partial_x \partial_y G^e_{xy} + \partial_y \partial_y G^e_{yx} + (k_0^2 + \partial_z^2)G^e_{zz} \end{bmatrix}, \]  
\tag{2.23b}

and

\[ \begin{bmatrix} H_{xz} \\ H_{yz} \\ H_{zz} \end{bmatrix} = \begin{bmatrix} \partial_x \partial_z G^e_{xx} + \partial_x \partial_z G^e_{xz} + \partial_x \partial_z G^e_{zz} \\ \partial_x \partial_z G^e_{xz} + \partial_y \partial_z G^e_{yz} + \partial_y \partial_z G^e_{zz} \\ \partial_x \partial_z G^e_{xz} + \partial_y \partial_z G^e_{yz} + (k_0^2 + \partial_z^2)G^e_{zz} \end{bmatrix}. \]  
\tag{2.23c}

The first sub-index of \(G^e\) refers to the component of the vector potential and the second sub-index refers to the direction of the dipole.

Now, suppose there is magnetic current density \(\mathbf{M}(r)\) (in cartesian components) inside the unit cell. Since the current density can be written as an infinite summation of dipoles, according to Eq(2.24), the electric and magnetic fields inside the unit cell can be written as in Eq(2.25).

\[ \mathbf{M}(\tau) = \int_{V_{cell}} \mathbf{M}(r') \delta(\tau - r') dV, \]  
\tag{2.24}

\[ \mathbf{H}(\tau) = \int_{V_{cell}} \mathbb{H}_m \cdot \mathbf{M}(r') dV, \]  
\tag{2.25a}

\[ \mathbf{E}(\tau) = \int_{V_{cell}} \mathbb{E}_m \cdot \mathbf{M}(r') dV, \]  
\tag{2.25b}

where \(\mathbf{r} = (x,y,z)^T\) and \(\mathbf{r}' = (x',y',z')^T\).

Although in general the dyadic Green's functions are \(3 \times 3\) matrices, the dyadic Green's func-
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tions in the next chapters will be $2 \times 2$ matrices. This is due to the fact that all sources are 2-dimensional and the boundary conditions are formulated on horizontal or vertical planes. Therefore, only 4 components of the dyadic Green's function are of interest in the analyses.
Chapter 3

Metallic walls

3.1 Introduction

It has long been known that metallic walls between elements of an array influence the radiation properties of these elements [6]. Metallic walls are usually applied to reduce the mutual coupling between the elements. In this chapter three configurations of metallic walls will be considered: metallic walls parallel to the x-axis, parallel to y-axis and parallel to both x- and y-axis.

3.2 Configurations

In order to analyse the effects of metallic walls on the radiating properties of an array, three different configurations have to be analysed: metallic walls parallel to the x-axis, parallel to the y-axis and parallel to both x- and y-axis. The three possible configurations are presented in figure 3.1a, b, and c, respectively.

Figure 3.1a: Metallic walls parallel to the x-axis

Figure 3.1b: Metallic walls parallel to the y-axis
Metallic walls

Figure 3.1c: Metallic walls parallel to x- and y-axis

The equivalence plane is placed on top of the metallic walls at \( z = 0 \) and the height of the radiating elements is assumed to be smaller than \( d \). Furthermore, the metallic walls and the ground plane are assumed to be perfectly conducting.

3.3 Green's functions

3.3.1 Metallic walls parallel to the x-axis

First, an array of electric dipoles in the interior region \((-d < z < 0)\) is considered. The metallic walls are placed on the \( x = ma \) planes, where \( m \) is an integer and \( a \) is the distance between two elements in the \( x \)-direction. Since the unit cell is a homogeneous region, it is sufficient to consider a magnetic vector potential with only a component in the direction of the dipole.

Recalling the position of the ground plane, the equivalence plane and the metallic walls, the homogeneous boundary conditions of Eq(3.1) are obtained.

\[
\begin{align*}
E_{in}^x \times \hat{n}_{z=-d} &= 0, \quad \text{(3.1a)} \\
E_{in}^x \times \hat{n}_{z=0} &= 0, \quad \text{(3.1b)} \\
E_{in}^x \times \hat{n}_{y=0} &= 0, \quad \text{(3.1c)} \\
E_{in}^x \times \hat{n}_{y=b} &= 0. \quad \text{(3.1d)}
\end{align*}
\]

The formulation of the problem is completed by obeying the periodicity condition in the \( x \)-direction and the source jump condition.

Observing the homogeneous boundary conditions, the magnetic vector potential for a \( y \)-directed electric dipole can be described by Eq(3.2).

\[
A_y = \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} f_{np}(x) \cos \left( \frac{n \pi y}{b} \right) \sin \left( \frac{p \pi z}{d} \right).
\]
This vector potential satisfies all homogeneous boundary conditions. In order to obey the periodicity condition and the source jump condition, the interval \(-a/2 < x < a/2\) is considered first. The vector potential has to satisfy the following inhomogeneous Helmholtz equation:

\[
\nabla^2 A_y + k_0^2 A_y = -\delta (z-z') \delta (y-y') \delta (x). 
\]  
(3.3)

Substituting Eq(3.2) in Eq(3.3) yields

\[
\sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \left( \frac{d^2}{dx^2} f_{np}(x) + \left( k_0^2 - \left( \frac{n\pi}{b} \right)^2 - \left( \frac{p\pi}{d} \right)^2 \right) f_{np}(x) \right) \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{p\pi z}{d} \right)
\]
\[
= -\delta (z-z') \delta (y-y') \delta (x). 
\]  
(3.4)

This equation can be reduced by using the orthogonality relations of Eq(3.5).

\[
\int_{0}^{b} \cos \left( \frac{n\pi y}{b} \right) \cos \left( \frac{n'\pi y}{b} \right) dy = \frac{b\delta_{nn'}}{2} 
\]
\[
\int_{0}^{d} \sin \left( \frac{p\pi z}{d} \right) \sin \left( \frac{p'\pi z}{d} \right) dz = \frac{d\delta_{pp'}}{2} 
\]

In Eq(3.5) \(\delta_{nn'}\) and \(\delta_{pp'}\) are Kronecker delts and \(e_n = 2\) for \(n = 0\) and \(e_n = 1\) for \(n = 1, 2, \ldots\). Applying these relations, Eq(3.4) becomes the following ordinary differential equation:

\[
\frac{d^2}{dx^2} f_{np}(x) + \beta_{np}^2 f_{np}(x) = -F_{np} \delta (x), \quad (3.6a)
\]

\[
\beta_{np}^2 = k_0^2 - \left( \frac{n\pi}{b} \right)^2 - \left( \frac{p\pi}{d} \right)^2 \quad \text{(Im} (\beta_{np}) \leq 0), \quad (3.6b)
\]

\[
F_{np} = \frac{4}{bde_n} \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{p\pi z}{d} \right). \quad (3.6c)
\]

The solution is the sum of a homogeneous and a particular solution. The particular solution is given by

\[
f_{np}^{par}(x) = \frac{F_{np}}{2\beta_{np}} e^{-\beta_{np}x}, \quad (3.7)
\]

which satisfies the source jump condition at \(x = 0\). The homogeneous solution is of the form

\[
f_{np}^{hom}(x) = K^+ e^{-\beta_{np}x} + K^- e^{\beta_{np}x}. \quad (3.8)
\]

The constants \(K^+\) and \(K^-\) can be determined by applying the remaining boundary condition: periodicity in the \(x\)-direction as described by Eq(3.9).

\[
f_{np}(a/2) = e^{-\beta_{np}a/2} f_{np}(a/2), \quad (3.9a)
\]

\[
\frac{d}{dx} f_{np}(a/2) = e^{-\beta_{np}a/2} \frac{d}{dx} f_{np}(a/2). \quad (3.9b)
\]
Applying these boundary conditions yields:

\[
K_{np}^+ = \frac{F_{np}}{2j\beta_{np}(e^{j(\beta_{np}a - \Psi_a)} - 1)},
\]
(3.10a)

\[
K_{np}^- = \frac{F_{np}}{2j\beta_{np}(e^{j(\beta_{np}a + \Psi_a)} - 1)}.
\]
(3.10b)

The obtained solution is valid for \(|x| < a/2\). In order to find a solution for \(0 < x < a\), the solution is written as a function of 2 components: one for the interval \([-a/2, 0]\) and one for the interval \([0, a/2]\). Now, the component for the interval \([-a/2, 0]\) is shifted to the interval \([a/2, a]\), including a phase shift \(e^{jn\Psi_\alpha}\). The magnetic vector potential for a \(y\)-directed electric dipole in the interior region is now given by Eq(3.11).

\[
A_y = \sum_{n = 0}^{\infty} \sum_{p = 1}^{\infty} \left( Q_{np}^+ e^{-j\beta_{np}x} + Q_{np}^- e^{j\beta_{np}x} \right) \cos\left( \frac{n\pi y}{b} \right) \cos\left( \frac{n\pi y}{b} \right) \frac{1}{\epsilon_n} \sin\left( \frac{p\pi z}{d} \right) \sin\left( \frac{p\pi z}{d} \right),
\]
(3.11)

where

\[
Q_{np}^+ = \frac{2}{j\beta_{np}b \left( 1 - e^{j(\Psi_a - \beta_{np}a)} \right)},
\]
(3.12a)

\[
Q_{np}^- = \frac{2}{j\beta_{np}b \left( 1 - e^{j(\Psi_a + \beta_{np}a)} \right)}.
\]
(3.12b)

For a \(z\)-directed electric dipole, an analogous derivation can be done, resulting in Eq(3.13).

\[
A_z = \sum_{n = 1}^{\infty} \sum_{p = 0}^{\infty} \left( Q_{np}^+ e^{-j\beta_{np}x} + Q_{np}^- e^{j\beta_{np}x} \right) \sin\left( \frac{n\pi y}{b} \right) \sin\left( \frac{n\pi y}{b} \right) \cos\left( \frac{p\pi z}{d} \right) \cos\left( \frac{p\pi z}{d} \right) \frac{1}{\epsilon_p}.
\]
(3.13)

For an \(x\)- or \(y\)-directed magnetic dipole on the inside of the aperture plane, the homogeneous boundary conditions 3.1a, c and d remain valid, as is the periodicity condition in the \(x\)-direction. Observing the homogeneous boundary conditions and the periodicity in the \(x\)-direction, the electric vector potentials are sought in the form

\[
F_{xn}^+ = \sum_{m = -\infty}^{\infty} \sum_{n = 0}^{\infty} A_{mn} e^{-jk_{mn}x} \cos\left( \frac{n\pi y}{b} \right) \sin\left( \frac{p\pi z}{d} \right),
\]
(3.14a)

for an \(x\)-directed dipole,

\[
F_{yn}^+ = \sum_{m = -\infty}^{\infty} \sum_{n = 1}^{\infty} B_{mn} e^{-jk_{mn}x} \sin\left( \frac{n\pi y}{b} \right) \sin\left( \frac{p\pi z}{d} \right),
\]
(3.14b)

where

\[
k_{mn}^x = \frac{2\pi m}{a},
\]
(3.15a)

\[
(k_{mn}^{wx})^2 = k_0^2 - (k_{mn}^{x})^2 = \left( \frac{\pi}{b} \right)^2 (\text{Im}(k_{mn}^{wx}) \leq 0).
\]
(3.15b)

The source jump condition is replaced by

\[
-E_{mn}^{\text{in}} e_i |_{z=0^-} = \frac{\partial}{\partial z} F_{mn}^{\text{in}} e_i |_{z=0^-} = \delta(x-x') \delta(y-y') e_i, \quad i = x, y,
\]
(3.16)

where the minus sign appears because the normal vector at the equivalence plane was defined
as pointing into the interior region. Using the orthogonality relations of Eq.(3.5) and the orthogonality relation for the Floquet modes
\[ \int_0^a e^{-jk_{ex}x} e^{jk_{mx}x} dx = a \delta_{mm'} , \] (3.17)
the constants \( A_{mn} \) and \( B_{mn} \) are determined as
\[ A_{mn} = \frac{-2}{ab k_{mn}^w \sin (k_{mn}^w d)} e^{jk_{mx} \cos \left( \frac{n \pi y}{b} \right)} \frac{1}{e_n} , \] (3.18a)
\[ B_{mn} = \frac{-2}{ab k_{mn}^w \sin (k_{mn}^w d)} e^{jk_{mx} \sin \left( \frac{n \pi y}{b} \right)} . \] (3.18b)

The electric vector potentials are thus given by
\[ F_{ex}^{in} = \frac{-2}{ab} \sum_{m = -\infty}^{\infty} \sum_{n = 0}^{\infty} e^{-jk_{mn}^w (x-x')} \cos \left( \frac{n \pi y}{b} \right) \cos \left( \frac{n \pi y'}{b} \right) \frac{1}{e_n} \cos \left( k_{mn}^w (z + d) \right) , \] (3.19a)
\[ F_{ey}^{in} = \frac{-2}{ab} \sum_{m = -\infty}^{\infty} \sum_{n = 0}^{\infty} e^{-jk_{mn}^w (x-x')} \sin \left( \frac{n \pi y}{b} \right) \sin \left( \frac{n \pi y'}{b} \right) \frac{1}{e_n} \cos \left( k_{mn}^w (z + d) \right) \frac{k_{mn}^w \sin (k_{mn}^w d)}{k_{mn}^w \sin (k_{mn}^w d)} . \] (3.19b)

In the exterior region \((d > 0)\) nothing has changed, compared to [8, p. 16], so the electric vector potentials for an \( x- \) or \( y- \)directed magnetic dipole is given by
\[ E_{ex}^{ex} = \frac{1}{ab} \sum_{m = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} \frac{1}{jk_{mn}} e^{-jk_{mn}^w (x-x')} e^{-jk_{mn}^w (y-y')} e^{-jk_{mn}^z} \quad i = x, y , \] (3.20)
where
\[ k_{mx} = \frac{2 \pi m + \Psi_x}{a} , \] (3.21a)
\[ k_{ny} = \frac{2 \pi n + \Psi_y}{b} , \] (3.21b)
\[ k_{mn}^2 = k_0^2 - (k_{mx})^2 - (k_{ny})^2 \quad (\text{Im} \ (k_{mn}) \leq 0) . \] (3.21c)

### 3.3.2 Metallic walls parallel to the \( y- \)axis

The main difference between this configuration and the previous one, is that the radiating element is placed inside the unit cell instead of at the sides. For this configuration, the homogeneous boundary conditions are formulated as
\[ E_{ex}^{in} \times \hat{n} = \begin{cases} 0, & \text{at } z = -d, \\ 0, & \text{at } z = 0, \\ 0, & \text{at } x = 0, \\ 0, & \text{at } x = a. \end{cases} \] (3.22a-d)
The set of boundary conditions for the magnetic vector potential of a \( y- \) or \( z- \)directed electric dipole is completed by the periodicity condition in the \( y- \)direction and the source jump condition at \( x = x^b \).
Metallic walls

Observing the periodicity condition for the y-direction and observing the homogeneous boundary conditions at \( z = -d \) and \( z = 0 \), the vector potential for a y-directed electric dipole for \( 0 < x < a \) is sought in the form

\[
A_y = \sum_{n = -\infty}^{\infty} \sum_{p = 1}^{\infty} f_{np}(x) e^{-j\beta_y^p y} \sin \left( \frac{npz}{d} \right),
\]

where

\[
k_y^n = \frac{2\pi n + \Psi_y}{b}.
\]

This vector potential has to satisfy the inhomogeneous Helmholtz equation

\[
\nabla^2 A_y + k_y^2 A_y = -\delta (z - z') \delta (y - y') \delta (x - x^b).
\]

Using the orthogonality relations of Eq(3.5) and the orthogonality relation analogous to Eq(3.17) for the y-directed Floquet modes, the following ordinary differential equation is obtained

\[
\frac{d^2}{dx^2} f_{np}(x) + \beta_y^2 f_{np}(x) = -F_{np} \delta (x - x^b),
\]

\[
\beta_y^n = k_0^2 - (k_y)^2 - \left( \frac{p\pi}{d} \right)^2 \quad \text{(Im} (\beta_y^n) \leq 0),
\]

\[
F_{np} = \frac{2}{bd} e^{j\beta_y^p y} \sin \left( \frac{p\pi y}{d} \right).
\]

The total solution can be written as

\[
f_{np}(x) = K_{np}^+ e^{-j\beta_y^p x} + K_{np}^- e^{j\beta_y^p x} + K_{np}^{+*} e^{-j\beta_y^p |x - x^b|}.
\]

The remaining constants can be determined by applying the homogeneous boundary conditions at \( x = 0 \) and \( x = a \) and by satisfying the source jump condition at \( x = x^b \).

The magnetic vector potential for a y-directed electric dipole can now be written as

\[
A_y = \sum_{n = -\infty}^{\infty} \sum_{p = 1}^{\infty} \left( Q_{np}^+ e^{-j\beta_y^p x} + Q_{np}^- e^{j\beta_y^p x} + Q_{np}^{+*} e^{-j\beta_y^p |x - x^b|} \right) e^{-j\beta_y^p (y - y')} \sin \left( \frac{p\pi y}{d} \right) \sin \left( \frac{p\pi z}{d} \right),
\]

where

\[
Q_{np}^+ = \frac{1}{j\beta_{np} bd} \frac{\sin \left( \beta_{np} (x - a) \right)}{\sin \left( \beta_{np} a \right)},
\]

\[
Q_{np}^- = \frac{2}{b \beta_{np} (1 - e^{j\beta_{np} x})},
\]

\[
Q_{np}^{+*} = \frac{1}{j\beta_{np} bd}.
\]
Metallic walls

For a z-directed electric dipole, an analogous derivation can be done, resulting in

\[
A_z = \sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} \left( Q_{np} e^{-jp \beta p x} + Q_{np} e^{j \beta p x} + Q_{np}^{\ast} e^{-j \beta p x} - y^4 \right) \]

\[
\cdot e^{-jk_p^G (y-y')} \cos \left( \frac{p \pi z}{d} \right) \cos \left( \frac{p \pi z}{d} \right) \frac{1}{\varepsilon_p}.
\]  (3.30)

For an x- or y-directed magnetic dipole on the inside of the aperture plane, the situation is completely analogous to that of the metallic walls parallel to the x-axis, except for the fact that \( x \) and \( y \) need to be interchanged, as well as \( a \) and \( b \). Hence, the electric vector potentials for an x- or y-directed magnetic dipole are given by

\[
F_{in}^{x} = \frac{-2}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{m \pi x}{a} \right) e^{-jk_p^G (y-y')} \cos \left( \frac{k_{mn}^y (z + d)}{k_{mn}^y} \right),
\]  (3.31a)

\[
F_{in}^{y} = \frac{-2}{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos \left( \frac{m \pi x}{a} \right) \cos \left( \frac{m \pi x}{a} \right) \frac{1}{\varepsilon_m} e^{-jk_p^G (y-y')} \cos \left( \frac{k_{mn}^x (z + d)}{k_{mn}^x} \right),
\]  (3.31b)

where

\[
k_p^x = \frac{2 \pi n + \Psi_y}{b},
\]  (3.32a)

\[
(k_{mn}^y)^2 = k_0^2 - \left( \frac{m \pi x}{a} \right)^2 - (k_p^G)^2 \quad \text{Im} \left( k_{mn}^y \right) \leq 0.
\]  (3.32b)

In the exterior region, again nothing has changed, so the electric vector potential for an x- or y-directed current is given by Eq(3.20).

### 3.3.3 Metallic walls parallel to both x- and y-axis

In this case, the radiating element is placed inside the unit cell at \( x = x^b \). The homogeneous boundary conditions for the magnetic vector potential of a y- or z-directed electric dipole are formulated as

\[
E_{e}^{in} \times \hat{n}|_{x=0} = 0,
\]  (3.33a)

\[
E_{e}^{in} \times \hat{n}|_{x=a} = 0,
\]  (3.33b)

\[
E_{e}^{in} \times \hat{n}|_{y=0} = 0,
\]  (3.33c)

\[
E_{e}^{in} \times \hat{n}|_{y=b} = 0,
\]  (3.33d)

\[
E_{e}^{in} \times \hat{n}|_{z=0} = 0,
\]  (3.33e)

The formulation of the problem is completed by the source jump condition at \( x = x^b \). Observing the homogeneous boundary conditions of Eq(3.33c, d, e, f), the magnetic vector potential for a y-directed electric dipole for \( 0 < x < a \) is sought in the form
Metallic walls

\[ A_y = \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} f_{np}(x) \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{p\pi z}{d} \right), \]  

(3.34)

which satisfies the homogeneous boundary conditions of Eq(3.33c, d, e, f). This vector potential has to satisfy the following inhomogeneous Helmholtz equation

\[ \nabla^2 A_y + k_0^2 A_y = -\delta(z-z') \delta(y-y') \delta(x-x^b). \]  

(3.35)

Applying Eq(3.35) to Eq(3.34) and using the orthogonality relations of Eq(3.5), the ordinary differential equation of Eq(3.36) is obtained.

\[ \frac{d^2}{dx^2} f_{np}(x) + \beta_{np}^2 f_{np}(x) = -F_{np} \delta(x-x^b), \]  

(3.36a)

\[ \beta_{np}^2 = k_0^2 - \left( \frac{n\pi}{b} \right)^2 - \left( \frac{p\pi}{d} \right)^2 \quad (\text{Im} (\beta_{np}) \leq 0), \]  

(3.36b)

\[ F_{np} = \frac{4}{b d e_n} \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{p\pi z}{d} \right). \]  

(3.36c)

The solution of this differential equation can be written as

\[ f_{np}(x) = K^{+}_{np} e^{-j\beta_{np} x} + K^{-}_{np} e^{j\beta_{np} x} + K^{+\prime}_{np} e^{-j\beta_{np} |x-x^b|}, \]  

(3.37)

where

\[ K^{\pm}_{np} = \frac{F_{np}}{2j\beta_{np}}. \]  

(3.38)

The remaining constants can be determined by applying the homogeneous boundary conditions at \( x = 0 \) and \( x = a \). The magnetic vector potential for a \( y \)-directed electric dipole can now be written as

\[ A_y = \sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} \left( Q^+_{np} e^{-j\beta_{np} x} + Q^-_{np} e^{j\beta_{np} x} + Q^{+\prime}_{np} e^{-j\beta_{np} |x-x^b|} \right) \cdot \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{p\pi z}{d} \right), \]  

(3.39)

where

\[ Q^{+}_{np} = \frac{2}{j\beta_{np} b d} \sin \left( \beta_{np} (x^b - a) \right), \]  

(3.40a)

\[ Q^{-}_{np} = \frac{4}{b d \beta_{np} (1 - e^{2j\beta_{np} a})}, \]  

(3.40b)

\[ Q^{+\prime}_{np} = \frac{2}{j\beta_{np} b d}. \]  

(3.40c)

For a \( z \)-directed electric dipole, an analogous derivation can be done. The obtained vector potential is

\[ A_z = \sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} \left( Q^+_{np} e^{-j\beta_{np} x} + Q^-_{np} e^{j\beta_{np} x} + Q^{+\prime}_{np} e^{-j\beta_{np} |x-x^b|} \right) \cdot \sin \left( \frac{n\pi y}{b} \right) \sin \left( \frac{p\pi z}{d} \right), \]  

(3.41)
For the electric vector potential of an x- or y-directed magnetic dipole, on the inside of the aperture plane, the homogeneous boundary conditions of Eq(3.33a, b, c, d, e) remain valid. The electric vector potential is sought in the form

\[ F_{m}^{in} = \sum_{m = -\infty}^{\infty} \sum_{n = 0}^{\infty} A_{mn} \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) \cos \left( k_{mn}^{wy} (z + d) \right), \]  

(3.42a)

\[ F_{m}^{in} = \sum_{m = -\infty}^{\infty} \sum_{n = 0}^{\infty} B_{mn} \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \cos \left( k_{mn}^{wy} (z + d) \right), \]  

(3.42b)

where

\[ (k_{mn}^{wy})^2 = k_0^2 - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2 \quad \text{Im}(k_{mn}^{wy}) \leq 0. \]  

(3.43)

The inhomogeneous boundary condition at the aperture plane, which has to be satisfied by the electric vector potentials is

\[ -E_{m}^{in} \times \epsilon_{z} \bigg|_{z = 0} = \partial F_{m}^{in} \cdot \epsilon_{j} \bigg|_{z = 0} = \delta (x - x') \delta (y - y') \epsilon_{i} \quad i = x, y . \]  

(3.44)

Substituting Eq(3.42) into Eq(3.44) and using the orthogonality relations of Eq(3.5) and the analogous relations for the x-coordinate, the constants \( A_{mn} \) and \( B_{mn} \) are determined as

\[ A_{mn} = \frac{-4}{ab k_{mn}^{wy} \sin (k_{mn}^{wy} d)} \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) \frac{1}{\epsilon_{n}}, \]  

(3.45a)

\[ B_{mn} = \frac{-4}{ab k_{mn}^{wy} \sin (k_{mn}^{wy} d)} \cos \left( \frac{m\pi x}{a} \right) \frac{1}{\epsilon_{m}} \sin \left( \frac{n\pi y}{b} \right). \]  

(3.45b)

Hence, the electric vector potentials are given by

\[ F_{m}^{in} = \frac{-4}{ab} \sum_{m = -\infty}^{\infty} \sum_{n = 0}^{\infty} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{m\pi x'}{a} \right) \cos \left( \frac{n\pi y}{b} \right) \cos \left( \frac{n\pi y'}{b} \right) \frac{1}{\epsilon_{n}} \cos \left( k_{mn}^{wy} (z + d) \right), \]  

(3.46a)

\[ F_{m}^{in} = \frac{-4}{ab} \sum_{m = -\infty}^{\infty} \sum_{n = 1}^{\infty} \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{m\pi x'}{a} \right) \frac{1}{\epsilon_{m}} \sin \left( \frac{n\pi y}{b} \right) \sin \left( \frac{n\pi y'}{b} \right) \cos \left( k_{mn}^{wy} (z + d) \right). \]  

(3.46b)

The electric vector potential in the exterior region for an x- or y-directed dipole is described by Eq(3.20), since nothing has changed in the exterior region.

### 3.4 Method of moments formulation

#### 3.4.1 Integral equations

In order to find the electric currents on the radiating element and the magnetic currents on the aperture, two boundary conditions remain to be satisfied: continuity of the tangential magnetic field over the equivalence plane and a disappearing tangential electric field on the radiating element. These conditions are described by Eq(3.47).

\[ E_{m}^{ex} \times \epsilon_{z} \bigg|_{x = x'} = 0, \]  

(3.47a)

\[ H_{m}^{ex} \times \epsilon_{z} \bigg|_{z = 0} = H_{m}^{ex} \times \epsilon_{z} \bigg|_{z = 0}. \]  

(3.47b)
where

\[ x^s = \begin{cases} 
0 & \text{for metallic walls parallel to the x-axis} \\
1 & \text{else}
\end{cases} \]  

Eq(3.47) can be written in tangential components (indicated by a sub-index \( t \)) as

\[
(E_{t,f}^{\text{in}} + E_{t,a}^{\text{in}}) |_{x = x^s} = -E_{g}^{\text{in}},
\]

\[
(H_{t,f}^{\text{in}} + H_{t,a}^{\text{in}} - H_{t,a}^{\text{ex}}) |_{z = 0} = 0,
\]

where the indices \( f \) and \( a \) refer to the source plane: the fin (radiating element) and the aperture, respectively. The indices \( \text{in} \) and \( \text{ex} \) refer to the interior and exterior region, respectively. The field \( E_{g}^{\text{ex}} \) is the excited field due to an impressed source for the transmitting case. For the receiving case, an incident magnetic field in the exterior region can be included in Eq(3.49b). However, only the transmitting case will be addressed here.

In Eq(3.50), the tangential fields have been written in terms of the dyadic Green’s functions and the electric and magnetic current distributions on the fin and the aperture, respectively.

\[
E_{t,f}^{\text{in}} = \int_{S_f} E_{f}^{\text{in}} \cdot J(y', z') \, dS_f',
\]

\[
E_{t,a}^{\text{in}} = \int_{S_a} E_{a}^{\text{in}} \cdot M_{\text{in}}^{\text{in}}(x', y') \, dS_a',
\]

\[
H_{t,f}^{\text{in}} = \int_{S_f} H_{f}^{\text{in}} \cdot J(y', z') \, dS_f',
\]

\[
H_{t,a}^{\text{in}} = \int_{S_a} H_{a}^{\text{in}} \cdot M_{\text{in}}^{\text{in}}(x', y') \, dS_a',
\]

\[
H_{t,a}^{\text{ex}} = \int_{S_a} H_{a}^{\text{ex}} \cdot M_{\text{ex}}^{\text{ex}}(x', y') \, dS_a',
\]

where \( S_f \) and \( S_a \) are the surfaces of the fin and the aperture, respectively, and the primed coordinates refer to the source coordinates. The dyadic Green’s functions are defined as

\[
\bar{\mathbb{E}}^{\text{in}} = \begin{bmatrix} E_{f,xy}^{\text{in}} & E_{f,yz}^{\text{in}} \\ E_{f,yx}^{\text{in}} & E_{f,zz}^{\text{in}} \end{bmatrix},
\]

\[
\bar{\mathbb{E}}^{\text{in}} = \begin{bmatrix} E_{a,xy}^{\text{in}} & E_{a,yz}^{\text{in}} \\ E_{a,yx}^{\text{in}} & E_{a,zz}^{\text{in}} \end{bmatrix},
\]

\[
\bar{\mathbb{H}}^{\text{in}} = \begin{bmatrix} H_{f,xy}^{\text{in}} & H_{f,yz}^{\text{in}} \\ H_{f,yx}^{\text{in}} & H_{f,zz}^{\text{in}} \end{bmatrix},
\]

\[
\bar{\mathbb{H}}^{\text{in}} = \begin{bmatrix} H_{a,xy}^{\text{in}} & H_{a,yz}^{\text{in}} \\ H_{a,yx}^{\text{in}} & H_{a,zz}^{\text{in}} \end{bmatrix},
\]

\[
\bar{\mathbb{H}}^{\text{ex}} = \begin{bmatrix} H_{a,ex,xy} & H_{a,ex,yz} \\ H_{a,ex,yx} & H_{a,ex,zy} \end{bmatrix},
\]

\[
\bar{\mathbb{H}}^{\text{ex}} = \begin{bmatrix} H_{a,ex,xy} & H_{a,ex,yz} \\ H_{a,ex,yx} & H_{a,ex,zy} \end{bmatrix},
\]
with the dyadic components derived as in appendix A1. Recalling the fact that $\mathbf{M}^{ex} = -\mathbf{M}^{in}$, to ensure continuity of the electric field over the aperture plane, and substituting Eq(3.50) into Eq(3.49) yields

\[
\left( \begin{array}{c}
\int_{S_f}^{\mathbf{E}^{ff, in}} \cdot j (y', z') \, dS_f + \int_{S_a}^{\mathbf{M}^{in}} (x', y') \, dS_a' \\
\int_{S_f}^{\mathbf{H}^{af, in}} \cdot j (y', z') \, dS_f + \int_{S_a}^{\mathbf{M}^{in}} (x', y') \, dS_a'
\end{array} \right) \bigg|_{x = x'} = -\mathbf{E}^{ef} \bigg|_{x = x'},
\]

(3.52a)

\[
\left( \begin{array}{c}
\int_{S_f}^{\mathbf{H}^{af, in}} \cdot j (y', z') \, dS_f + \int_{S_a}^{\mathbf{M}^{in}} (x', y') \, dS_a' \\
\int_{S_f}^{\mathbf{H}^{af, in} + \mathbf{H}^{aa, ex}} \cdot j (y', z') \, dS_f + \int_{S_a}^{\mathbf{M}^{in}} (x', y') \, dS_a'
\end{array} \right) \bigg|_{z = 0} = 0,
\]

(3.52b)

where the first index of the dyadic Green's function refers to the location of the tangential components (at the fin or at the aperture) and the second index refers to the location of the current source causing the field (electric currents on the fin or magnetic currents on the aperture).

### 3.4.2 Matrix equation

The next step in the formulation of the method of moments is to expand the electric and magnetic current densities into a set of basis functions. The current densities can now be written as

\[
j (y', z') = \sum_{j=1}^{N_y} j_{yq} J_{yq} (y', z') \, \varepsilon_y + \sum_{q=1}^{N_z} j_{zq} J_{zq} (y', z') \, \varepsilon_z,
\]

(3.53a)

\[
\mathbf{M}^{in} = \sum_{s=1}^{N_{ap}} m_{xs} M_{xs} (x', y') \, \varepsilon_x + m_{ys} M_{ys} (x', y') \, \varepsilon_y,
\]

(3.53b)

where $N_y$ and $N_z$ are the number of electric expansion modes in the $y$- and $z$-direction, respectively, $N_{ap}$ is the number of magnetic expansion modes, $j_{yq}$ and $j_{zq}$ are the unknown mode coefficients for the electric currents on the fin in the $y$- and $z$-direction, respectively, and $m_{xs}$ and $m_{ys}$ are the unknown mode coefficients of the magnetic currents on the aperture in the $x$- and $y$-direction, respectively. Note that the number of magnetic expansion modes in the $x$- and $y$-direction is the same. The expansions of the current densities are substituted into Eq(3.52), which yields

\[
\left( \begin{array}{c}
\sum_{q=1}^{N_y} \int_{S_f}^{\mathbf{E}^{ff, in}} \cdot j_{yq} J_{yq} \, dS_f \\
\sum_{q=1}^{N_z} \int_{S_a}^{\mathbf{E}^{ff, in}} \cdot j_{zq} J_{zq} \, dS_a'
\end{array} \right) = -\mathbf{E}^{ef} \bigg|_{x = x'},
\]

(3.54a)

\[
\left( \begin{array}{c}
\sum_{q=1}^{N_y} \int_{S_f}^{\mathbf{H}^{af, in}} \cdot j_{yq} J_{yq} \, dS_f + \sum_{q=1}^{N_z} \int_{S_a}^{\mathbf{H}^{af, in}} \cdot j_{zq} J_{zq} \, dS_a' \\
\sum_{q=1}^{N_y} \int_{S_f}^{\mathbf{H}^{af, in} + \mathbf{H}^{aa, ex}} \cdot j_{yq} J_{yq} \, dS_f + \sum_{q=1}^{N_z} \int_{S_a}^{\mathbf{H}^{af, in} + \mathbf{H}^{aa, ex}} \cdot j_{zq} J_{zq} \, dS_a'
\end{array} \right) \bigg|_{z = 0} = 0.
\]

(3.54b)

In order to be able to determine the coefficients $j_{yq}, j_{zq}, m_{xs}$ and $m_{ys}$, a total number of $N_y + N_z + 2N_{ap}$ equations is necessary. These equations are obtained by testing (weighing) Eq(3.54) with testing functions. This means that the inner product of the equation and the testing func-
tions is taken. The inner product is defined as
\[ \langle f, g \rangle = \int f \cdot g dS. \] (3.55)

The used testing functions have the same form as the expansion functions, which means that Galerkin's method is used to formulate the method of moments. The testing functions are discriminated from the basis functions by an index \( w \) (weighing). Since the left-hand sides of Eq(3.54) are functions of the unprimed coordinates, the testing functions are defined in unprimed coordinates. The testing functions for the fin are given by
\[ J_w^f(y, z) = J_{y1}^f(y, z) \xi_y \cdots J_{yN}^f(y, z) \xi_y \} z \xi_z, \] (3.56)
and the testing functions for the aperture are given by
\[ M_w^a(x, y) = M_{x1}^a(x, y) \xi_x \cdots M_{xN_{ap}}^a(x, y) \xi_x \} M_{y1}^a(x, y) \xi_y \cdots M_{yN_{ap}}^a(x, y) \xi_y. \] (3.57)

Using Eq(3.56) to test Eq(3.54a) and Eq(3.57) to test Eq(3.54b), the following set of equations is obtained:

\[
\sum_{q=1}^{N_y} j_{yq} \int \frac{E_{zy}}{S_y} J_{yq}^w J_{yq}^w dS_f + \sum_{q=1}^{N_y} j_{zq} \int \frac{E_{zy}}{S_z} J_{zq}^w J_{zq}^w dS_f \\
+ \sum_{s=1}^{N_z} \left[ m_{x} \int \frac{E_{xa}}{S_x} J_{x}^w J_{x}^w dS_f + m_{y} \int \frac{E_{ya}}{S_y} J_{y}^w J_{y}^w dS_f \right] \right|_{x=x'} = -J_{yq}^w E_y \cdot \xi_y dS_f \right|_{x=x'} \] (3.58a)

\[
\sum_{q=1}^{N_x} j_{yq} \int \frac{E_{zy}}{S_y} J_{yq}^w J_{yq}^w dS_f + \sum_{q=1}^{N_x} j_{zq} \int \frac{E_{zy}}{S_z} J_{zq}^w J_{zq}^w dS_f \\
+ \sum_{s=1}^{N_y} \left[ m_{x} \int \frac{E_{xa}}{S_x} J_{x}^w J_{x}^w dS_f + m_{y} \int \frac{E_{ya}}{S_y} J_{y}^w J_{y}^w dS_f \right] \right|_{x=x'} = -J_{zq}^w E_x \cdot \xi_z dS_f \right|_{x=x'} \] (3.58b)

\[
\sum_{q=1}^{N_x} j_{yq} \int \frac{H_{ax}}{S_x} J_{yq}^w M_{x}^a dS_a + \sum_{q=1}^{N_x} j_{zq} \int \frac{H_{ax}}{S_z} J_{zq}^w M_{x}^a dS_a \\
+ \sum_{s=1}^{N_y} m_{x} \int \frac{H_{ax}}{S_x} J_{x}^w M_{x}^a dS_a + \sum_{s=1}^{N_y} m_{y} \int \frac{H_{ax}}{S_y} J_{y}^w M_{x}^a dS_a \right|_{z=0} = 0, \] (3.58c)

for \( t = 1, 2, \ldots, N_{ap} \).
This set of equations can be written in the form of a matrix, as was done in Eq(3.59).

\[
\left[\begin{array}{cccc}
Z_{ff y_y, rq} & Z_{ff y_z, rq} & T_{ff y_x, rs} & T_{ff y_y, rs} \\
Z_{ff y_z, rq} & Z_{ff y_z, rq} & T_{ff y_z, rs} & T_{ff y_z, rs} \\
T_{af y_y, rq} & T_{af y_z, rq} & Y_{aa y_x, rs} & Y_{aa y_y, rs} \\
T_{af y_z, rq} & T_{af y_z, rq} & Y_{aa y_z, rs} & Y_{aa y_y, rz} \\
\end{array}\right]
\left[\begin{array}{c}
j_{y_q} \\
j_{z_q} \\
m_{x_s} \\
m_{y_s} \\
\end{array}\right]
= \left[\begin{array}{c}
V_{r_t}^{yy} \\
V_{r_t}^{zz} \\
V_{x_t}^{xy} \\
V_{y_t}^{yz} \\
\end{array}\right],
\tag{3.59}
\]

where \( Z_{ff} \) is the self-impedance submatrix of the fin modes, \( T_{af} \) is the aperture-fin submatrix, \( T_{af} \) is the fin-aperture submatrix and \( Y^{aa} \) is the self-admittance submatrix of the aperture modes. The vector on the right-hand side of Eq(3.59) is the source vector and the remaining vector describes the mode coefficients of the current densities, as defined in Eq(3.53). The expressions for the matrix elements are given in appendix A2.

### 3.4.3 Basis and testing functions

The matrix elements of Eq(3.59) can be calculated when a specific choice for the basis and testing functions has been made. Since the shape of the fin can be changed, a great flexibility of the basis and testing functions is needed. Therefore, local basis and testing functions are used on the fin (i.e. the functions for the electric currents). Observing the dyadic Green’s functions, which contain a large number of trigonometrical functions, piecewise sinusoidal (PWS) functions are chosen. The \( q \)-th \( y \)-directed basis function is defined as

\[
J_{y_q}(y', z') = \frac{\sin k_q (W_y - |y' - y_k|)}{W_z \sin (k_q W_y)} e_{y'",
\]

where

\[
q \in \{ 1, \ldots, N_y \},
k_q \in \{ 2, \ldots, S_y \},
l_q \in \{ 2, \ldots, S_z + 1 \},
\tag{3.61}
\]
and the \( q' \)-th \( z \)-directed basis function is defined as
\[
J_{zq'}(y', z') = \frac{\sin k_y (W_z - |z' - z_{lq'}|)}{W_y \sin (k_y W_z)} e_z,
\]
where
\[
y_{kq'} - W_y \leq y' \leq y_{kq'},
\]
\[
z_{lq'} - W_z \leq z' \leq z_{lq'} + W_z,
\]
and the numbers \( S_y \) and \( S_z \) are the number of subdomains on the grid in the \( y \)- and \( z \)-direction, respectively.

The grid for \( y_{lq} \) and \( z_{lq} \) is shown in figure 3.2.

\[
y_1 \quad y_2 \quad y_3 \quad y_{S_y-1} \quad y_{S_y} \quad y_{S_y+1}
\]
\[
z_1 \quad z_2 \quad z_3 \quad z_{S_z-1} \quad z_{S_z} \quad z_{S_z+1}
\]

**Figure 3.2: Grid definition on the fin**

The grid is defined in such a way that there is overlap between two successive basis functions in one direction. By choosing the proper grid coordinates, the shape of the fin can be approximated by a set of subdomains, on which the basis and testing functions are defined. For the testing functions, Eq(3.60) to Eq(3.63) remain valid, except for the change from \( q \) and \( q' \) into \( r \) and \( r' \) and the change from primed coordinates into unprimed coordinates. Furthermore, the definition of the grid on the fin is the same.

For the aperture, there is no need to use local basis and testing functions, since the shape of the aperture does not change. For the basis and testing functions, Floquet modes are used. By using these entire domain functions, the calculation of certain submatrices (e.g. \( Y^{\alpha \alpha} \)) will be less time consuming, due to orthogonality between the Green’s functions and the basis or test-
Metallic walls

ing functions. The Floquet modes for the basis functions are defined as

\[
\mathcal{M}_{xs} = e^{-j(k_{msx}x + k_{nsy}y)}e^{y},
\]

\[
\mathcal{M}_{ys} = e^{-j(k_{msx}x + k_{nsy}y)}e^{-y},
\]

where

\[
0 \leq x' \leq a,
\]

\[
0 \leq y' \leq b,
\]

\[
k_{sx}^m = \frac{2\pi m_s + \Psi_s}{a},
\]

\[
k_{sy}^n = \frac{2\pi n_y + \Psi_y}{b},
\]

and

\[
s \in \{1, \ldots, N_{ap}\},
\]

\[
m_s \in \{-m_{ap}, \ldots, m_{ap}\},
\]

\[
n_s \in \{-n_{ap}, \ldots, n_{ap}\},
\]

\[
(2m_{ap} + 1) \cdot (2n_{ap} + 1) = N_{ap}.
\]

For the testing functions, the conjugates of Eq(3.64) and Eq(3.65) are taken. Furthermore, \(s\) is replaced by \(t\) and the primed coordinates are replaced by unprimed coordinates.

The matrix elements of Eq(3.59), when these basis and testing functions are used, are given in appendix A4. In this appendix, use has been made of the integral definitions of appendix A3.

### 3.5 Computational and numerical details

When the integrals of appendix A3 and the matrix elements of appendix A4 are calculated, several numerical problem have to be dealt with. This involves numerical stability of the matrix elements, singular points and symmetry in certain sub-matrices, among others.

#### 3.5.1 Numerical stability of the matrix elements

Expressions containing \(k_{mn}\) or \(k_{mn}\) (possibly with an upper index \(wx wy\) or \(wx\)) combined with an exponential function, including sine and cosine, can cause numerical instabilities since both \(k_{mn}\) and \(k_{mn}\) can be real or imaginary. However, all problems of this kind can be dealt with.

The first problem of this kind occurs in the expressions for the self-impedance matrix. The essential parts are given by Eq(3.68), for walls parallel to the \(x\)-axis, for walls parallel to the \(y\)-axis and parallel to both \(x\)- and \(y\)-axis, respectively.

\[
Q_{np}^+ + Q_{np}^- = \frac{2}{b d \sin (\beta_{np} a)} \frac{\sin (\beta_{np} a)}{\cos (\beta_{np} a) - \cos (\Psi_s)},
\]  

(3.68a)
The extra factor two in Eq(3.68c) compared to Eq(3.68b) is due to the extra metallic walls parallel to the x-axis. This factor can also be found when Eq(3.29) is compared to Eq(3.40). The numerical instabilities occur when \( \beta_{np} \) is imaginary. For this case, both numerator and denominator become very large and can even become larger than the largest number that can be handled by the numerical program used. This causes inaccuracies in the matrix elements. To prevent this problem, asymptotic expressions of Eq(3.68) can be used for large \( \beta_{np} \). The asymptotic expressions are obtained by recognizing the dominant terms in both numerator and denominator. In order to find these dominant terms it is recalled here that the imaginary part of \( \beta_{np} \) is negative or zero. In this way, the following expressions are obtained:

\[
Q_{np}^+ + Q_{np}^- = -\frac{2}{\text{Im}(\beta_{np})b'd'}
\]

for walls parallel to the x-axis

\[
Q_{np}^+ e^{j\beta_{np} x^b} + Q_{np}^- e^{j\beta_{np} x^b} + Q_{np}^{+\ell} - \frac{1}{\text{Im}(\beta_{np})b'd'}
\]

for walls parallel to the y-axis

\[
Q_{np}^+ e^{j\beta_{np} x^b} + Q_{np}^- e^{j\beta_{np} x^b} + Q_{np}^{+\ell} - \frac{2}{\text{Im}(\beta_{np})b'd'}
\]

for walls parallel to both x- and y-axis.

A similar problem occurs in the expressions for the self-admittance sub-matrix. The essential part of the problem is:

\[
\frac{\cos (k_{mn}d)}{k_{mn} \sin (k_{mn}d)}.
\]

Numerical inaccuracies occur when the imaginary part of \( k_{mn} \) becomes large. This problem can be solved by using the asymptotic expression of Eq(3.71), when \(-\text{Im}(k_{mn}d) >> 1\).

\[
\frac{\cos (k_{mn}d)}{k_{mn} \sin (k_{mn}d)} - \frac{1}{\text{Im}(k_{mn})}.
\]

Other stability problems occur in the sub-matrices \( T_{fa}^{f} \) and \( T_{af}^{f} \). For \( T_{fa}^{f} \) the instability is caused by the factor

\[
\frac{1}{\sin (k_{mn}d)},
\]

which can become very small when \(-\text{Im}(k_{mn}d) >> 1\), and the factor

\[
\begin{cases}
I_3(z, m, n) & \text{for } T_{fa}^{f}, \\
I_4(z, m, n) & \text{for } T_{fa}^{f} \text{ and } T_{af}^{f},
\end{cases}
\]
Metallic walls

which can become very large when \(-\text{Im}(k_{mn}d) >> 1\).

By combining both factors and dividing both numerator and denominator by the dominant terms of the numerator and denominator, the combination of expression (3.72) and (3.73) can be written as in Eq(3.74).

\[
\begin{align*}
I_{3,y}(z_{m}, m, n) &= \frac{e^{kz_{i} - e^{k(z_{i} - W_{j})} - e^{-k(z_{i} - W_{j} + 2d)} + e^{-k(z_{i} + 2d)}}}{k (1 - e^{-2kd})}, \\
I_{4,y}(z_{m}, m, n) &= \frac{j k}{W_{j} \sin(k_{m}d) (k^{2} + k^{2}) (1 - e^{-2kd})} \left( e^{k(z_{i} + W_{j})} + e^{k(z_{i} - W_{j})} \\
&+ e^{-k(z_{i} + 2d - W_{j})} + e^{-k(z_{i} + 2d + W_{j})} - 2 \cos(k_{e}W_{j}) (e^{kz_{i}} + e^{-k(z_{i} + 2d)}) \right),
\end{align*}
\]

\[ (3.74a) \]

where

\[ k = -\text{Im}(k_{mn}) \quad (k \geq 0). \]

The right-hand side of this equation is a useful expression when \(-\text{Im}(k_{mn}d) >> 1\), while the left-hand side is stable otherwise.

The numerical problems of the T^a sub-matrix are located within the integrals \(I_{3,y}^{W}(m, n, p)\) and \(I_{4,y}^{W}(m, n, p)\). The closed form expressions for these integrals, given in appendix A3, become numerically unstable when \(-\text{Im}(\beta_{np}d) >> 1\). These problems are circumvented by rewriting the closed form expressions for the integrals as in Eq(3.76).

\[
I_{3,y}^{W}(m, n, p) = -\frac{2}{bd (1 - e^{-2\beta a})} \left( e^{\beta(x + a)} - e^{-\beta(x + a)} \right) \left( e^{j\theta} (e^{x} - e^{-x}) \right) \left( 1 - e^{-2\beta a} \right), \]

\[ (3.76a) \]

\[ \left\{ \begin{array}{l}
1 \text{ for walls parallel to the y-axis,} \\
2 \text{ for walls parallel to both x- and y-axis,}
\end{array} \right. \]

\[
I_{4,x}^{W}(m, n, p) = -jk \cdot I_{3,x}^{W}(m, n, p), \]

\[ (3.76b) \]

where

\[ \beta = -\text{Im}(\beta_{np}) \quad (\beta \geq 0). \]

3.5.2 Singular points

Most of the closed form expressions of the integrals defined in appendix A3 have one or more singularities, most of which are removable. Appendix A5 contains a list of removable singularities, that are likely to occur in the computations. Although most singularities are removable, there are still non-removable singularities, which need to be avoided.

In the case of walls parallel to the x-axis, a non-removable singularity occurs when \(\beta_{np} = \pm k_{m}^{m}\) or when \(\cos(\beta_{np}x) = \cos(\Psi_{x})\), which are equivalent. The singularity occurs in the self-impedance sub-matrix in the form of \(Q_{np}^{+} + Q_{np}^{-}\) and in the integrals \(I_{3,x}^{W}(m, n, p)\) and \(I_{4,x}^{W}(m, n, p)\). An other non-removable singularity occurs when \(\sin(k_{mn}x) = 0\). This singu-
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Singularity is located in the $T^{fa}$ and $Y^{aa}$ sub-matrices. These singularities can be avoided by a small change in frequency or scan angle.

In the case of walls parallel to the $y$-axis or parallel to both $x$- and $y$-axis, a non-removable singularity occurs in the self-impedance sub-matrix when $\sin(\beta_{np}d) = 0$. This singularity is located in the expression $Q_{np}^+ e^{-j\beta_{np}x^+} + Q_{np}^- e^{j\beta_{np}x^-} + Q_{np}^*$ and in the integrals $I_{zy,x}^{wy}(m_p,n,p)$ and $I_{zy,x}^{zy}(m_p,n,p)$. An other non-removable singularity exist in the $T^{fa}$ and $Y^{aa}$ sub-matrices. This singularity appears when $\sin(k_{mn} d) = 0$ or $\sin(k_{mn} y d) = 0$, for walls parallel to the $y$-axis and walls parallel to both $x$- and $y$-axis, respectively. These singularities can be avoided by a small change in frequency and in the case of walls parallel to the $y$-axis, a change in scan angle can also be used to prevent these singularities.

The last non-removable singularity is located in $Y^{aa}$ for all configurations. It occurs when $k_{mn} = 0$. This singularity is avoided by a small change in frequency or scan angle.

### 3.5.3 Reduction of computation time

The computation of the matrix elements which have double summations, can be quite time consuming. A reduction in the computation time of the self-impedance matrix is possible by recognizing the symmetry properties of this sub-matrix, described by Eq(3.78) in case of walls parallel to the $y$-axis and by Eq(3.79) in case of walls parallel to the $x$-axis or parallel to both $x$- and $y$-axis.

$$Z_{yy, rq}^{ff} = - (Z_{yy, qr}^{ff})^*, \quad (3.78a)$$

$$Z_{yz, rq}^{ff} = - (Z_{yz, qr}^{ff})^*, \quad (3.78b)$$

$$Z_{zz, rq}^{ff} = - (Z_{zz, qr}^{ff})^*, \quad (3.78c)$$

where $^*$ means the complex conjugate.

$$Z_{yy, rq}^{ff} = Z_{yy, qr}^{ff}, \quad (3.79a)$$

$$Z_{yz, rq}^{ff} = Z_{yz, qr}^{ff}, \quad (3.79b)$$

$$Z_{zz, rq}^{ff} = Z_{zz, qr}^{ff}. \quad (3.79c)$$

In case of walls parallel to both $x$- and $y$-axis, the self-admittance sub-matrix is completely filled and all elements are computed by means of double summations. Therefore, use can be made of the symmetry properties given in Eq(3.80), which will also reduce the computation time.

$$Y_{aa_{xs, ts}}^{xx} = - (Y_{aa_{xx, st}}^{xx})^* (s \neq t), \quad (3.80a)$$

$$Y_{aa_{ys, ts}}^{xy} = - (Y_{aa_{yx, st}}^{xy})^* (s \neq t), \quad (3.80b)$$

$$Y_{aa_{yy, ts}}^{yy} = - (Y_{aa_{yy, st}}^{yy})^* (s \neq t), \quad (3.80c)$$
where use has been made of the following relations:

\begin{align}
I_{1a,x}^x(m_p, m) &= (I_{b,a}^x(m_p, m))^*, \\
I_{2a,x}^x(m_p, m) &= (I_{b,a}^x(m_p, m))^*, \\
I_{2a,y}^x(h_n, n) &= (I_{3b,y}^x(n, n))^*, \\
I_{4a,y}^x(n, n) &= (I_{4b,y}^x(n, n))^*.
\end{align}

The computation time is further reduced by calculating all relevant integrals in advance, except when the matrices become very large, which is the case when the integrals have three arguments (e.g. $I_{3,b,y}^x(m_p, n, p)$).

### 3.5.4 Source modelling details

In Appendix A3, the integrals $I_{1,y}^x(y_k, n)$ and $I_{1,y}^x(y_k, n)$ are also defined for $y_{c-1}$, $y_c$ and $y_{c+1}$. This was done to model the feed strip between to the fins of a bunny ear antenna. The feed strip width is usually not equal to the width of the subdomains, which are used to model the fins. By allowing a different width of sub-domains, certain integrals must be redefined and recalculated for the grid coordinates $y_{c-1}$, $y_c$, and $y_{c+1}$, since other basis and testing functions are used. These functions are adapted to the width of the feed strip. The basis and testing functions are given in Eq(3.82) and the situation is drawn in Figure 3.3.

![Figure 3.3: Definition of the feed strip modelling](image)

\begin{align}
J_y(y, y_{c-1}) &= \begin{cases} 
  \frac{\sin k_e (W_y + y - y_{c-1})}{W_z \sin (k_e W_y)} & (y_{c-1} - W_y \leq y \leq y_{c-1}) \\
  \frac{\sin k_e (W_{yf} - y + y_{c-1})}{W_z \sin (k_e W_{yf})} & (y_{c-1} \leq y \leq y_{c-1} + W_{yf}) 
\end{cases}, \\
J_y(y, y_{c+1}) &= \begin{cases} 
  \frac{\sin k_e (W_{yf} - y - y_{c+1})}{W_z \sin (k_e W_{yf})} & (y_{c+1} - W_{yf} \leq y \leq y_{c+1}) \\
  \frac{\sin k_e (W_y - y + y_{c+1})}{W_z \sin (k_e W_y)} & (y_{c+1} \leq y \leq y_{c+1} + W_y)
\end{cases}.
\end{align}

The integrals $I_{2,y}^x(y_k, n)$ and $I_{2,y}^x(y_k, n)$ have to be changed when $y_k = y_c$ or when $y_k = y_{c+1}$. However, this is easily done by replacing $W_y$ by $W_{yf}$.
In order to model the feed lines coming from behind the ground plane, so-called half modes are used to model the \( z \)-directed current density at \( z = -d \). For this case, the basis and testing function is defined as
\[
J_z(z) = \frac{\sin k_e (W_z - z + d)}{W_y \sin (k_e W_z)} \quad (-d \leq z \leq -d + W_z). \tag{3.83}
\]
When the relevant integrals (i.e. \( I_{f,z} \) and \( I_{d,z} \)) are determined, it is found that the closed form expression of these integrals in appendix A3 should be divided by 2 when \( z = -d \).

For the source itself, use has been made of the delta-gap voltage generator. In case of a \( y \)-directed source, the electric field generated by this source (i.e. \( E_y \)) can be written as
\[
E_y = -V_g \delta (y - y_g) \varepsilon_y \quad (z_g - W_z \leq z \leq z_g \wedge x = x^g), \tag{3.84}
\]
where \( V_g \) is the generator voltage, \( y_g \) and \( z_g \) are the grid coordinates of the generator and \( x^g \) as defined in Eq(3.48). It is noted that the delta-gap generator is always located at the boundary between two sub-domains.

The source vector, defined in Eq(3.59) can now be written as
\[
V_{yz} = \begin{cases} V_g & \text{when } y_k = y_g, \\ 0 & \text{else}, \end{cases} \tag{3.85a}
\]
\[
V_{zx} = 0, \tag{3.85b}
\]
\[
V_{tx} = 0, \tag{3.85c}
\]
\[
V_{ty} = 0. \tag{3.85d}
\]

### 3.6 Infinite array performance parameters

When an array is analysed, certain quantities are needed to express the performance of the array. Furthermore, the design specifications of a phased array are formulated in such terms. The quantities that will be used here are: the active input impedance, the active reflection coefficient and the active element pattern. The term ‘active’ refers to the fact that all elements in the array are excited.

The use of the delta-gap generator makes the expression for the active input impedance \( Z_{in} \) very simple. It is stated as generator voltage divided by the current crossing the delta-gap. The current \( I_g \) crossing the delta-gap can be found by integrating the basis function multiplied with its mode coefficient, at the source grid point, over its height \( W_z \):
\[
I_g = \int_{z_s - W_z}^{z_s} j_y y_g (y_g, z) dz = j_y z_s, \tag{3.86}
\]
where \( y_g \) and \( z_g \) are the grid coordinates of the delta-gap generator. Hence, it is seen that the current crossing the delta-gap equals the mode coefficient of that basis function. The active
input impedance is now formulated as
\[ Z_{in} = \frac{V_{g}}{I_{y}} = R_{in} + jX_{in}, \] (3.87)
where the minus sign appears due to the fact that the current flows through the source:

\[ \begin{array}{c}
V_{g} \\
\downarrow \\
I_{g}
\end{array} \]

Figure 3.4: Definition of the generator current

\[ R_{in} \text{ and } X_{in} \text{ in Eq(3.87) are called the resistance and reactance, respectively.} \]

The active reflection coefficient \( R \) expresses the ratio of the amplitude of an outgoing wave \( V^+ \) and the amplitude of the incoming wave \( V^- \):
\[ R = \frac{V^+}{V^-}. \] (3.88)

When it is assumed that the radiating elements are connected to a feeding network with characteristic impedance \( Z_0 \), the active reflection coefficient can be expressed in terms of the active input impedance, as was done in Eq(3.89).
\[ R = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}. \] (3.89)

Finally, the (normalised) active element pattern \( G_e \) is given by [11]
\[ G_e = (1 - |R|^2) \cos (\theta). \] (3.90)

3.7 Results

In [4], [6] and [10] metallic walls parallel to the \( x \)-axis have been analysed for an array of dipoles. In [6] a network approach is used, while in [4] and [10] a full wave analysis is performed. Analyses of metallic walls parallel to the \( y \)-axis and parallel to both \( x \)- and \( y \)-axis were not found in the literature. The analysis in this report is done for a dipole array, as was done in [4], and for an array of folded dipoles, for which measurements also have been performed.

For the dipole array in [4], the dipole is given in figure 3.5. Furthermore, the parameters of the unit cell are \( a = b = 0.5 \text{ m} \) and \( d = 0.30 \text{ m} \) (i.e. the height of the walls). The array operates at a frequency of 300 MHz and a \( \theta \)-scan is performed in the \( \phi = 90 \text{ degrees} \) plane (the \( E \)-plane). On the equivalence plane, 13 Floquet modes are used for both the \( x \)- and \( y \)-directed magnetic current densities. The dipole is divided into 8 subdomains on which 7 \( y \)-directed basis functions are used. The basis functions in the center of the dipole have the same dimension as the other basis functions. The results of the analysis are shown in figure 3.6 and were found to be in good agreement with the results of [4].
Metallic walls

Figure 3.5: Configuration of the dipole inside the unit cell

For the array of folded dipoles, the dipole dimensions, in terms of $\lambda_0$, and the arrangement of the subdomains is given in figure 3.7. The white horizontal line in figure 3.7 indicates an infinitely thin separation of the subdomains in the z-direction, which implies that no z-directed currents can cross this line. This is accomplished by removing those z-directed basis functions, which cross the white line. The unit cell dimensions are $a = 0.499\lambda_0$, $b = 0.715\lambda_0$, $d = 0.390\lambda_0$ and, in case of metallic walls parallel to the y-axis or parallel to x- and y-axis, $x^b = 0.564\lambda_0$.

The centre frequency is $f_0 = c_0/\lambda_0$, where $c_0$ is the velocity of light. Scanning is performed in the $\varphi = 90$ degrees plane ($E$-plane scan). The number of Floquet modes on the aperture plane is 13 in both x- and y-direction. For all summations an upper summation index of 200 is used to ensure good convergence and when needed, a lower summation index of -200 is used. Previous work by Hulshof [8] has shown that such an array has a poor scan behaviour in the $E$-plane for the frequency $f_0$, when no metallic walls are included in the unit cell. The numerical results in [8] are repeated here and are shown in figure 3.8. The poor scan behaviour is seen from the fact that the real part of the active input impedance decreases rapidly for $\theta > 40$ degrees and from

Figure 3.6: Active input impedance of a dipole array
the fact that the imaginary part is large in amplitude. When the folded dipole was placed along the x-axis, it was observed that the H-plane scan (\( \varphi = 90 \) degrees) has a good scan behaviour.

![Folded dipole configuration](image)

**Figure 3.7: Folded dipole configuration**

In an attempt to improve the scan behaviour in the E-plane at the frequency \( f_0 \), the effects of metallic walls were examined. The results are presented in figure 3.9a, b and c for metallic walls parallel to the x-axis, parallel to the y-axis and parallel to both x- and y-axis, respectively. From figure 3.9a it is seen that the effect of metallic walls parallel to the x-axis is that the real and imaginary part of the active input impedance exhibit a more flat behaviour, compared to figure 3.8. However, the amplitude of the imaginary part is still too large for a good scan behaviour. Figure 3.9b shows that the imaginary part of the active input impedance has a

---

**Figure 3.8: Scan performance of the folded dipole array without metallic walls**
smaller amplitude, but the real part decreases rapidly for $\theta > 20$ degrees. Therefore, the configuration of metallic walls parallel to the $y$-axis will have very poor scan characteristics. Finally, the combination of metallic walls parallel to the $x$-axis and $y$-axis shows a very good scan behaviour: the real part of the impedance fluctuates slowly around 250 Ohm over a large scan angle (up to 60 degrees) and the imaginary part has a much lower amplitude compared to the configuration without metallic walls.

Figure 3.9a: Metallic walls parallel to the $x$-axis

Figure 3.9b: Metallic walls parallel to the $y$-axis
Since the configuration of metallic walls parallel to the x- and y-axis was the most promising, the scan behaviour of this configuration has also been analysed for the frequencies $f_1 = 0.92f_0$ and $f_2 = 1.08f_0$. The active input impedance for both frequencies is shown in figure 3.10 and 3.11 for $f = f_1$ and $f = f_2$, respectively. For $f = f_1$, the scan behaviour is excellent because of the slowly decaying resistance and the relatively small amplitude of the reactance. For $f = f_2$, the scan behaviour is not very good: the resistance becomes very low for $\theta > 40$ degrees and the reactance has a singularity-like behaviour for $\theta = 59$ degrees. This is due to the fact that a grating lobe is coming into real space at $\theta = 59.5$ degrees. When the active input impedance is transformed into an element pattern ($Z_0 = R_{in}(\theta = 0^\circ)$), a blind scan angle occurs at $\theta = 55$ degrees. The normalised element pattern is indicated in figure 3.12 by a solid line.

A prototype of the folded dipole array was also made, in order to validate the software. The array was analysed by measuring the element pattern. The results of the analysis for $f = f_2$ are also shown in figure 3.12 (by hollow bullets). The measurements and theoretic results are in good agreement with one another.
Figure 3.10: Metallic walls parallel to x- and y-axis for $f = f_1$

Figure 3.11: Metallic walls parallel to x- and y-axis for $f = f_2$
Figure 3.12: Normalised element pattern: calculated and measured
Chapter 4

Dual polarization

4.1 Introduction

In the case of radiating elements parallel to the y-axis, as described in the previous chapter, the polarization of the electromagnetic field cannot be controlled. This disadvantage can be circumvented by placing an other radiating element, which is placed parallel to the x-axis, inside the unit cell. Two questions now arise:

1. Does the second radiating element deteriorate the performance of the first element?
2. Can two orthogonal polarization components be excited independent of one another?

The first question can be answered in terms of a scattering matrix and the second one in terms of the axial ratio. For the axial ratio the electric field in the far field region has to be calculated as a function of the currents on the radiating elements. The sources on the elements have equal amplitude and a phase difference of 90 degrees in order to excite a left or right handed circular electromagnetic field. However, in most cases the polarization will not be perfectly circular, i.e. when a right handed circular polarization is wanted and the polarization is not perfectly circular, then there is also a small left handed circular component present. A quantity that defines the purity of the polarization is the axial ratio (AR). It is defined as:

\[
AR = \frac{|E_R| + |E_L|}{|E_R| - |E_L|},
\]

where \(E_R\) is the right handed circular electric field component and \(E_L\) is the left handed circular electric field component. In order find the axial ratio, it is necessary to determine the currents on both fins, which can be done by means of the method of moments.

4.2 Configuration

The configuration of the unit cell, containing two radiating elements perpendicular to one another, is shown in figure 4.1.

![Figure 4.1: Dual polarization](image-url)
The radiating element (fin) parallel to the x-axis will be indicated by $f_x$ and the radiating element parallel to the y-axis will be indicated by $f_y$.

### 4.3 Green’s functions

First, the magnetic vector potentials for an x-directed dipole inside the interior region of the unit cell will be derived. The dipole is placed at $y = y_b$, i.e. on the radiating element parallel to the x-axis. Since the unit cell is a homogeneous region, it is sufficient to use only the x-component of the magnetic vector potential to satisfy all boundary conditions. The magnetic vector potential has to satisfy the homogeneous boundary conditions for the electric field at $z = -d$ and $z = 0$ and the periodicity conditions in the x- and y-direction. Furthermore, the source jump condition at $y = y_b$ has to be satisfied. This set of conditions is formulated in Eq(4.2).

\[
E^{in}_e \times \hat{n} |_{z = -d} = 0, \quad (4.2a)
\]
\[
E^{in}_e \times \hat{n} |_{z = 0} = 0, \quad (4.2b)
\]
\[
A^f_x (0, y, z) = A^f_x (a, y, z) e^{i\Psi_x}, \quad (4.2c)
\]
\[
A^f_x (x, 0, z) = A^f_x (x, b, z) e^{i\Psi_y}, \quad (4.2d)
\]
\[
\nabla^2 A^f_x + k_0^2 A^f_x = -\delta (x - x') \delta (y - y_b) \delta (z - z'), \quad (4.2e)
\]
where $E_e$ is the electric field caused by the x-directed electric dipole and $A^f_x$ is the magnetic vector potential inside the unit cell for the x-directed electric dipole.

The magnetic vector potential is sought in the form

\[
A^f_x = \sum_{m = -\infty}^{\infty} \sum_{p = 1}^{\infty} f_{mp} (y) e^{-j k_x y} \sin \left( \frac{p \pi z}{d} \right), \quad (4.3)
\]
where

\[
k_x^m = \frac{2 \pi m + \Psi_x}{a}. \quad (4.4)
\]

This vector potential satisfies the boundary conditions of Eq(4.2a-d). Substituting Eq(4.3) into Eq(4.2e) yields

\[
\sum_{m = -\infty}^{\infty} \sum_{p = 1}^{\infty} \left( \frac{d^2}{d y^2} f_{mp} (y) + \left( k_x^2 - (k_x^m)^2 - \left( \frac{p \pi}{d} \right)^2 \right) f_{mp} (y) \right) e^{-j k_x y} \sin \left( \frac{p \pi z}{d} \right)
\]

\[
= -\delta (x - x') \delta (y - y_b) \delta (z - z'). \quad (4.5)
\]
This equation can be simplified by applying the orthogonality relations of Eq(4.6).

\[ \int_{0}^{a} e^{-j k_{x}^{p} x} e^{j k_{x}^{p'} x} dx = a \delta_{m m'}, \quad \delta_{p p'}, \quad \delta_{m m'} \text{ are Kronecker deltas.} \]

Using Eq(4.6), Eq(4.5) reduces to the following ordinary differential equation:

\[ \frac{d^2 f_{mp}(\gamma)}{d \gamma^2} + \gamma_{mp}^2 f_{mp}(\gamma) = -F_{mp} \delta (y - y^{b}), \quad \text{(4.7)} \]

where

\[ \gamma_{mp}^2 = \frac{k_{0}^2 - (k_{z}^{m})^2 - \left( \frac{p \pi}{d} \right)^2}{d} \quad \text{Im} (\gamma_{mp}) \leq 0, \quad \text{(4.8a)} \]

\[ F_{mp} = \frac{2}{d} e^{j k_{x}^{p} y_{b}} \left( \frac{p \pi}{d} \right). \quad \text{(4.8b)} \]

The total solution of Eq(4.7) can be written as

\[ f_{mp}(\gamma) = K_{mp}^{+} e^{-j \gamma_{mp}^{y} \gamma} + K_{mp}^{-} e^{j \gamma_{mp}^{y} \gamma} + K_{mp}^{+/-} e^{-j \gamma_{mp}^{y} \gamma}, \quad \text{(4.9)} \]

where

\[ K_{mp}^{+/-} = \frac{F_{mp}}{2j \gamma_{mp}}. \quad \text{(4.10)} \]

The remaining constants can be determined by satisfying the remaining boundary conditions, i.e. the periodicity condition in the y-direction. The magnetic vector potential for an x-directed dipole in the interior region of the unit cell can now be found as

\[ A_{x}^{+/-} = \sum_{m = -\infty}^{\infty} \sum_{p = 1}^{\infty} \left( D_{mp}^{+} e^{-j \gamma_{mp}^{y} \gamma} + D_{mp}^{-} e^{j \gamma_{mp}^{y} \gamma} + D_{mp}^{+/-} e^{-j \gamma_{mp}^{y} \gamma} \right) \]

\[ \cdot e^{-j k_{x}^{m} (x - x')} \sin \left( \frac{p \pi}{d} \right) \sin \left( \frac{p \pi}{d} \right), \quad \text{(4.11)} \]

where

\[ D_{mp}^{+} = \frac{1}{j \gamma_{mp} a d} \cdot \frac{e^{j \gamma_{mp} y_{b}}}{e^{j (\gamma_{mp} y_{b})} - 1}, \quad \text{(4.12a)} \]

\[ D_{mp}^{-} = \frac{1}{j \gamma_{mp} a d} \cdot \frac{e^{-j \gamma_{mp} y_{b}}}{1 - e^{j (\gamma_{mp} y_{b})}}, \quad \text{(4.12b)} \]

\[ D_{mp}^{+/-} = \frac{1}{j \gamma_{mp} a d}. \quad \text{(4.12c)} \]

For a z-directed dipole in the interior region of the unit cell at \( y = y^{b} \), the magnetic vector potential can be found by performing a similar analysis. The magnetic vector potential for a z-
directed dipole can then be written as

\[ A^{lx}_x = \sum_{m = -\infty}^{\infty} \sum_{p = 0}^{\infty} \left( D_{mp} e^{-j\pi m^2} + D_{mp} e^{j\pi m^2} + D_{mp} e^{-j\pi m^2 y - y_0} \right) \]

\[ \cdot e^{-jk_x(x-x')} \cos \left( \frac{p\pi z}{d} \right) \cos \left( \frac{p\pi z'}{d} \right) \frac{1}{e_p}, \]

where \( e_p = 2 \) for \( p = 0 \) and \( e_p = 1 \) for \( p = 1, 2, \ldots \).

The analysis for this vector potential is completely analogous to the analysis of the magnetic vector potential of the \( x \)-directed electric dipole at \( y = y_b \). The magnetic vector potential can now be written as

\[ A^{fy}_y = \sum_{n = -\infty}^{\infty} \sum_{p = 1}^{\infty} f_{np} \left( x \right) e^{-jk_y y} \sin \left( \frac{p\pi z}{d} \right), \]

where

\[ k_y = \frac{2\pi n + \Psi_y}{b}. \]

For a \( y \)-directed electric dipole in the interior region at the radiating element parallel to the \( y \)-axis, the boundary conditions of Eq(4.2) remain valid except for Eq(4.2e), which must be replaced by Eq(4.14). Furthermore, the upper index of the magnetic vector potential is replaced by \( f_y \).

\[ \nabla^2 A^{fy}_y + k_y^2 A^{fy}_y = -\delta(x-x') \delta (y-y') \delta (z-z'), \]

(4.14)

where \( A^{fy}_y \) is the magnetic vector potential for a \( y \)- or \( z \)-directed electric dipole placed at \( x = x_b \).

The magnetic vector potential is sought in the form

\[ A^{fy}_y = \sum_{n = -\infty}^{\infty} \sum_{p = 1}^{\infty} f_{np} \left( x \right) e^{-jk_y y} \sin \left( \frac{p\pi z}{d} \right), \]

(4.15)

where

\[ k_y = \frac{2\pi n + \Psi_y}{b}. \]

(4.16)

The magnetic vector potential can now be written as

\[ A^{fy}_y = \sum_{n = -\infty}^{\infty} \sum_{p = 1}^{\infty} \left( C_{np} e^{-j\beta_{np} x} + C_{np} e^{j\beta_{np} x} + C_{np} e^{-j\beta_{np} x - x_b} \right) \]

\[ \cdot e^{-jk_y y} \sin \left( \frac{p\pi z}{d} \right) \sin \left( \frac{p\pi z'}{d} \right), \]

(4.17)

where

\[ \beta_{np}^2 = k_0^2 - (k_y)^2 - \left( \frac{p\pi}{d} \right)^2 \quad (\text{Im} \left( \beta_{np} \right) \leq 0), \]

(4.18a)

\[ C_{np} = \frac{1}{j\beta_{np} b d} \cdot \frac{e^{j\beta_{np} b}}{1 - e^{j(\psi_x - \beta_{np} a)}}, \]

(4.18b)

\[ C_{np} = \frac{1}{j\beta_{np} b d} \cdot \frac{e^{-j\beta_{np} b}}{1 - e^{j(\psi_x + \beta_{np} a)}}, \]

(4.18c)

\[ C_{np} = \frac{1}{j\beta_{np} b d}. \]

(4.18d)
In the same way, the magnetic vector potential for a z-directed electric dipole can be found as

\[
A_{z} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \left( C_{np} e^{-j\beta_{np}x} + C_{np}^{*} e^{j\beta_{np}x} + C_{np}^{*} e^{-j\beta_{np}x} + C_{np} e^{j\beta_{np}x} \right) 
\cdot e^{-j\kappa_{n}z} \cos \left( \frac{p\pi x}{d} \right) \cos \left( \frac{p\pi z}{d} \right) \frac{1}{E_{p}}. 
\]

(4.19)

For an x-directed magnetic dipole at the aperture plane and in the interior region, the electric vector potential has only an x-directed component because the interior region is homogeneous. Furthermore, the electric vector potential has to obey the boundary conditions of Eq(4.20).

\[
\begin{align*}
E_{m}^{in} \times e_{z} \big|_{z=-d} &= 0, \\
F_{x}^{in} (0, y, z) &= F_{x}^{in} (a, y, z) e^{j\psi_{x}}, \\
F_{x}^{in} (x, 0, z) &= F_{x}^{in} (x, b, z) e^{j\psi_{x}}, \\
m \times e_{z} \big|_{z=0} &= -\partial_{z} F_{x}^{in} \cdot e_{x} \big|_{z=0} = -\delta (x-x') \delta (y-y') \cdot e_{x},
\end{align*}
\]

(4.20a, 4.20b, 4.20c, 4.20d)

where \(E_{in}\) is the electric field caused by the magnetic dipole and \(F_{in}\) is the electric vector potential of the magnetic dipole in the interior region.

Observing the boundary conditions of Eq(4.20a-c), the electric vector potential \(F_{in}\) is sought in the form of Eq(4.21), which satisfies these conditions.

\[
F_{x}^{in} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} B_{mn} e^{-jk_{mn}x} e^{-jk_{mn}y} \cos (k_{mn} (z + d)),
\]

(4.21)

where \(k_{mn}^{2} = k_{0}^{2} - (k_{x})^{2} - (k_{y})^{2} \) (Im \((k_{mn}) \leq 0)), \(k_{mn}^{2} = k_{0}^{2} - (k_{x})^{2} - (k_{y})^{2} \) (Im \((k_{mn}) \leq 0)), \(m = 0, x, y\).

(4.22)

Using the orthogonal relations of the Floquet modes and applying the remaining boundary condition of Eq(4.19d), the electric vector potential of an x-directed magnetic dipole in the interior region is found as

\[
F_{x}^{in} = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-jk_{mn}x} e^{-jk_{mn}y} \cos (k_{mn} (z + d)) \frac{1}{k_{mn} \sin (k_{mn}d)}. 
\]

(4.23)

For a y-directed magnetic dipole in the interior region, the same expression is found:

\[
F_{y}^{in} = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-jk_{mn}x} e^{-jk_{mn}y} \cos (k_{mn} (z + d)) \frac{1}{k_{mn} \sin (k_{mn}d)}. 
\]

(4.24)

For a magnetic dipole in the exterior region, nothing has changed compared to the unit cells in Chapter 3. Therefore, the electric vector potential is given by

\[
E_{i}^{ex} = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{jk_{mn}} e^{-jk_{mn}x} e^{-jk_{mn}y} e^{-jk_{mn}z} \quad i = x, y.
\]

(4.25)

The electric vector potentials in the interior and exterior region are the same as in [8].
4.4 Method of moments

4.4.1 Integral equations

In order to determine the electric and magnetic current densities on both radiating elements and the aperture plane, respectively, three boundary conditions remain to be satisfied: the disappearance of the tangential electric field on both radiating elements and the continuity of the tangential magnetic field over the aperture plane. These conditions are expressed in Eq(4.26).

\[ E^{\text{in}} \times \varepsilon |_{y = y^b} = 0, \quad (4.26a) \]
\[ E^{\text{in}} \times \varepsilon_{\text{ex}} |_{x = x^b} = 0, \quad (4.26b) \]
\[ H^{\text{in}} \times \varepsilon |_{z = 0} = H^{\text{ex}} \times \varepsilon |_{z = 0}, \quad (4.26c) \]

where the indices \(\text{in}\) and \(\text{ex}\) refer to the interior and exterior region, respectively. Eq(4.26) can also be written in tangential components. The tangential components are caused by the electric current densities on the radiating elements parallel to the \(x\)- and \(y\)-axis (indicated by an upper index \(f_x\) and \(f_y\), respectively) and by the magnetic current densities on the aperture plane (indicated by an upper index \(a\)). The tangential electric field components of the radiating element parallel to the \(x\)-axis are indicated by a sub-index \(tx\), the tangential electric field components of the radiating element parallel to the \(y\)-axis are indicated by a sub-index \(ty\) and the tangential magnetic field components of the aperture plane are indicated by a sub-index \(t\). Eq(4.26) can now be written as

\[
(E_{tx}^{\text{in}} + E_{ty}^{\text{in}} + E_{tx}^{\text{in}}) |_{y = y^b} = -E_{tx}^{\text{ex}} |_{y = y^b}, \quad (4.27a)
\]
\[
(E_{ty}^{\text{in}} + E_{ty}^{\text{in}} + E_{ty}^{\text{in}}) |_{x = x^b} = -E_{ty}^{\text{ex}} |_{x = x^b}, \quad (4.27b)
\]
\[
(H_{tx}^{\text{in}} + H_{ty}^{\text{in}} + H_{tx}^{\text{in}} - H_{tx}^{\text{ex}}) |_{z = 0} = 0, \quad (4.27c)
\]

where \(E_y^{\text{ex}}\) is the electric field due to an impressed source at \(x = x^b\) and \(E_x^{\text{ex}}\) is the electric field due to an impressed source at \(y = y^b\). Furthermore, it is assumed that the impressed sources only cause an electric field on the radiating element on which they are placed.

The tangential components can be written in terms of the dyadic Green’s functions on the one hand and the electric current densities on both radiating elements and the magnetic current density on the aperture plane on the other hand. This is done in appendix B1. Recalling the fact that \(M^{ex} = -M^{in}\), to ensure continuity of the electric field over the aperture plane, and substituting the equations of appendix B1 into Eq(4.27) yields Eq(4.28).

\[
\left( \int_{S_{fx}} E_{fx}^{\text{in}} \cdot J_{fx} (x', z') dS_{fx} + \int_{S_{fy}} E_{fy}^{\text{in}} \cdot J_{fy} (y', z') dS_{fy} + \int_{S_{a}} E_{a, in} \cdot M^{\text{in}} (x', y') dS_{a} \right) |_{y = y^b} = -E_{tx}^{\text{ex}} |_{y = y^b}, \quad (4.28a)
\]
\[
\left( \int_{S_{fx}} E_{fx}^{\text{in}} \cdot J_{fx} (x', z') dS_{fx} + \int_{S_{fy}} E_{fy}^{\text{in}} \cdot J_{fy} (y', z') dS_{fy} + \int_{S_{a}} E_{a, in} \cdot M^{\text{in}} (x', y') dS_{a} \right) |_{x = x^b} = -E_{ty}^{\text{ex}} |_{x = x^b}, \quad (4.28b)
\]
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\[
\left( \int_{S_f_x} \frac{d}{a, f_x, dS_{f_x}} \cdot J^x (x', z') \, dS_{f_x} + \int_{S_f_y} \frac{d}{a, f_y, dS_{f_y}} \cdot J^y (y', z') \, dS_{f_y} + \right.
+ \left. \int_{S_a} \frac{d}{a, a, dS_{a}} \cdot M_{m} (x', y') \, dS_{a} \right) \bigg|_{z = 0} = 0 .
\] 

(4.28c)

4.4.2 Matrix equation

In order to find the electric and magnetic current densities on the radiating elements and the aperture plane, respectively, the expansion and testing procedures are applied to Eq(4.28). The electric and magnetic current densities are expanded according to Eq(4.29).

\[
J^x (x', z') = \sum_{q=1}^{N^{fx}_x} j^x_{i,q} J^x_{i,q} (x', z') e_x + \sum_{q=1}^{N^{fx}_z} j^x_{i,q} J^x_{i,q} (x', z') e_z ,
\] 

(4.29a)

\[
J^y (y', z') = \sum_{q=1}^{N^{fy}_y} j^y_{i,q} J^y_{i,q} (y', z') e_y + \sum_{q=1}^{N^{fy}_z} j^y_{i,q} J^y_{i,q} (x', z') e_z ,
\] 

(4.29b)

\[
M_{m} = \sum_{s=1}^{N_{ap}} \left[ m_{x,s} M_{x,s} (x', y') e_x + m_{y,s} M_{y,s} (x', y') e_y \right] ,
\] 

(4.29c)

where \(N^{fx}_x\) and \(N^{fx}_z\) are the number of electric expansion modes in the \(x\)- and \(z\)-direction on the radiating element parallel to the \(x\)-axis, \(N^{fy}_y\) and \(N^{fy}_z\) are the number of electric expansion modes in the \(y\)- and \(z\)-direction on the radiating element parallel to the \(y\)-axis and \(N_{ap}\) is the number of magnetic expansion modes. Furthermore, \(j^x_{i,q}\) and \(j^x_{i,q}\) are the unknown mode coefficients for the electric current densities in the \(x\)- and \(z\)-direction on the radiating element parallel to the \(x\)-axis, \(j^y_{i,q}\) and \(j^y_{i,q}\) are the unknown mode coefficients for the electric current densities on the radiating element parallel to the \(y\)-axis and \(m_{x,s}\) and \(m_{y,s}\) are the unknown mode coefficients for the magnetic current densities on the aperture plane. Substituting Eq(4.29) into Eq(4.28) yields

\[
\left\{ \begin{array}{c}
\sum_{q=1}^{N^{fx}_x} \int_{S_f_x} E^{f_x, f_x, dS_{f_x}} \cdot j^x_{i,q} J^x_{i,q} (x', z') e_x \, dS_{f_x} + \sum_{q=1}^{N^{fx}_z} \int_{S_f_x} E^{f_x, f_x, dS_{f_x}} \cdot j^x_{i,q} J^x_{i,q} (x', z') e_z \, dS_{f_x} + \\
\sum_{q=1}^{N^{fy}_y} \int_{S_f_y} E^{f_y, f_y, dS_{f_y}} \cdot j^y_{i,q} J^y_{i,q} (y', z') e_y \, dS_{f_y} + \sum_{q=1}^{N^{fy}_z} \int_{S_f_y} E^{f_y, f_y, dS_{f_y}} \cdot j^y_{i,q} J^y_{i,q} (x', z') e_z \, dS_{f_y} + \\
\sum_{s=1}^{N_{ap}} \int_{S_a} E^{f_x, a, dS_{a}} \cdot (m_{x,s} M_{x,s} (x', y') e_x + m_{y,s} M_{y,s} (x', y') e_y) \, dS_{a} \right\} \bigg|_{y = y'} = -E^{g_x}_{f_x} \bigg|_{y = y'} ,
\] 

(4.30a)

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\[
\begin{align*}
& \left( \sum_{q=1}^{N_f} \int E_{yq} f_q^x (x', z') \epsilon_x dS_{f_q} + \sum_{q=1}^{N_f} \int E_{yq} f_q^y (x', z') \epsilon_z dS_{f_q} + \\
& \sum_{q=1}^{N_y} \int E_{xq} f_q^y (x', z') \epsilon_y dS_{f_q} + \sum_{q=1}^{N_y} \int E_{xq} f_q^x (x', z') \epsilon_z dS_{f_q} + \\
& \sum_{s=1}^{N_{ap}} \left( \int H_{aq} f_q^x (z', y') \epsilon_x dS_{f_q} + \int H_{aq} f_q^y (z', y') \epsilon_y dS_{f_q} \right) \right|_{x=x^b} = -E_{yq} \quad (4.30b)
\end{align*}
\]

To determine the unknown mode coefficients of the current densities, a total number of \(N_x f_x + N_y f_y + N_z f_y + 2N_{ap}\) equations is needed. These equations are obtained by applying the testing procedure to Eq(4.30). The testing functions have the same form as the expansion function. However, they are defined in unprimed coordinates and have an extra upper index \(w\) (for weighing). The testing functions on the radiating element parallel to the \(x\)-axis are given by

\[
J_{w,x}^q(x, z) = J_{x1}^q(x, z) \epsilon_x, \ldots, J_{z1}^q(x, z) \epsilon_z, \ldots, J_{zN_{ap}}^q(x, z) \epsilon_z,
\]

which are used to test Eq(4.30a). The testing functions on the radiating element parallel to the \(y\)-axis are given by

\[
J_{w,y}^q(y, z) = J_{y1}^q(y, z) \epsilon_y, \ldots, J_{yN_{ap}}^q(y, z) \epsilon_y, \ldots, J_{y1}^q(y, z) \epsilon_y,
\]

which are used to test Eq(4.30b).

Finally, the testing functions on the aperture plane are given by

\[
M_{w}^q(x, y) = M_{x1}^w(x, y) \epsilon_x, \ldots, M_{y1}^w(x, y) \epsilon_y, \ldots, M_{yN_{ap}}^w(x, y) \epsilon_y,
\]

which are used to test Eq(4.30c).
In this way, the following set of equations is obtained:

\[
\left( \begin{array}{c}
\sum_{q=1}^{N_x} j_{xq} \int E_{fx,fx} \, J_{fx,Jw,fx} \, dS_x \, dS_f + \sum_{q=1}^{N_y} j_{yq} \int E_{fy,fy} \, J_{fy,Jw,fy} \, dS_y \, dS_f \\
\sum_{q=1}^{N_x} j_{xq} \int E_{fx,fx} \, J_{fx,Jw,fx} \, dS_x \, dS_f + \sum_{q=1}^{N_y} j_{yq} \int E_{fy,fy} \, J_{fy,Jw,fy} \, dS_y \, dS_f \\
\sum_{s=1}^{N_a} \left[ \int E_{fx,fx} \, J_{fx,Jw,fx} \, dS_x \, dS_f + \int E_{fy,fy} \, J_{fy,Jw,fy} \, dS_y \, dS_f \right] \right)_{y = y^b} \\
\left( \begin{array}{c}
\sum_{q=1}^{N_x} j_{xq} \int E_{fx,fx} \, J_{fx,Jw,fx} \, dS_x \, dS_f + \sum_{q=1}^{N_y} j_{yq} \int E_{fy,fy} \, J_{fy,Jw,fy} \, dS_y \, dS_f \\
\sum_{q=1}^{N_x} j_{xq} \int E_{fx,fx} \, J_{fx,Jw,fx} \, dS_x \, dS_f + \sum_{q=1}^{N_y} j_{yq} \int E_{fy,fy} \, J_{fy,Jw,fy} \, dS_y \, dS_f \\
\sum_{s=1}^{N_a} \left[ \int E_{fx,fx} \, J_{fx,Jw,fx} \, dS_x \, dS_f + \int E_{fy,fy} \, J_{fy,Jw,fy} \, dS_y \, dS_f \right] \right)_{y = y^b} \\
\sum_{s=1}^{N_a} \left[ \int E_{fx,fx} \, J_{fx,Jw,fx} \, dS_x \, dS_f + \int E_{fy,fy} \, J_{fy,Jw,fy} \, dS_y \, dS_f \right] \right)_{x = x^b} \\
\sum_{s=1}^{N_a} \left[ \int E_{fx,fx} \, J_{fx,Jw,fx} \, dS_x \, dS_f + \int E_{fy,fy} \, J_{fy,Jw,fy} \, dS_y \, dS_f \right] \right)_{x = x^b} \end{array} \right)
\]

for \( r = 1, 2, \ldots, N_x \),

for \( r = 1, 2, \ldots, N_y \),
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\[
\left( \sum_{q=1}^{N_{t}} j_{q} f_{x} \int_{S_{x}} E_{f_{x}, f_{x}, \text{in} m_{x}, M_{x}} J_{x} w_{x}, dS_{x} dS_{y} + \sum_{q=1}^{N_{t}} j_{q} f_{y} \int_{S_{x}} E_{f_{y}, f_{y}, \text{in} m_{y}, M_{y}} J_{y} w_{y}, dS_{x} dS_{y} \right) + \left( \sum_{q=1}^{N_{t}} j_{q} f_{x} \int_{S_{x}} E_{f_{x}, f_{x}, \text{in} m_{x}, M_{x}} J_{x} w_{x}, dS_{x} dS_{y} + \sum_{q=1}^{N_{t}} j_{q} f_{y} \int_{S_{x}} E_{f_{y}, f_{y}, \text{in} m_{y}, M_{y}} J_{y} w_{y}, dS_{x} dS_{y} \right)
\]

\[
= - \left[ J_{w_{x}, f_{x}} E_{x_{y}} \cdot e_{x_{y}} dS_{x} \right]_{x=x_{y}}^{x=x_{y}} \tag{4.34d}
\]

for \( r = 1, 2, \ldots, N_{x} \).

\[
\left( \sum_{q=1}^{N_{t}} j_{q} f_{x} \int_{S_{x}} H_{a_{x}, f_{x}, \text{in} m_{x}, M_{x}} J_{x} w_{x}, dS_{x} dS_{a} + \sum_{q=1}^{N_{t}} j_{q} f_{y} \int_{S_{x}} H_{a_{y}, f_{y}, \text{in} m_{y}, M_{y}} J_{y} w_{y}, dS_{x} dS_{a} + \sum_{q=1}^{N_{t}} j_{q} f_{x} \int_{S_{x}} H_{a_{x}, f_{x}, \text{in} m_{x}, M_{x}} J_{x} w_{x}, dS_{x} dS_{a} + \sum_{q=1}^{N_{t}} j_{q} f_{y} \int_{S_{x}} H_{a_{y}, f_{y}, \text{in} m_{y}, M_{y}} J_{y} w_{y}, dS_{x} dS_{a} \right) + \left( \sum_{q=1}^{N_{t}} j_{q} f_{x} \int_{S_{x}} H_{a_{x}, f_{x}, \text{in} m_{x}, M_{x}} J_{x} w_{x}, dS_{x} dS_{a} + \sum_{q=1}^{N_{t}} j_{q} f_{y} \int_{S_{x}} H_{a_{y}, f_{y}, \text{in} m_{y}, M_{y}} J_{y} w_{y}, dS_{x} dS_{a} \right) = 0 \tag{4.34e}
\]

for \( t = 1, 2, \ldots, N_{a_{p}} \).

\[
\left( \sum_{q=1}^{N_{t}} j_{q} f_{x} \int_{S_{x}} H_{a_{x}, f_{x}, \text{in} m_{x}, M_{x}} J_{x} w_{x}, dS_{x} dS_{a} + \sum_{q=1}^{N_{t}} j_{q} f_{y} \int_{S_{x}} H_{a_{y}, f_{y}, \text{in} m_{y}, M_{y}} J_{y} w_{y}, dS_{x} dS_{a} \right) + \left( \sum_{q=1}^{N_{t}} j_{q} f_{x} \int_{S_{x}} H_{a_{x}, f_{x}, \text{in} m_{x}, M_{x}} J_{x} w_{x}, dS_{x} dS_{a} + \sum_{q=1}^{N_{t}} j_{q} f_{y} \int_{S_{x}} H_{a_{y}, f_{y}, \text{in} m_{y}, M_{y}} J_{y} w_{y}, dS_{x} dS_{a} \right) = 0 \tag{4.34f}
\]

for \( t = 1, 2, \ldots, N_{a_{p}} \).
This set of equations can also be written in matrix form, as was done in Eq(4.35).

\[
\begin{bmatrix}
Z_{fx,fx} & Z_{fx,fy} \\
Z_{fx,fx} & Z_{fx,fy}
\end{bmatrix}
\begin{bmatrix}
T_{fx,fx} & T_{fx,fy} \\
T_{fx,fx} & T_{fx,fy}
\end{bmatrix}
\begin{bmatrix}
T_{fx,fx} & T_{fx,ty} \\
T_{fx,ty} & T_{fx,ty}
\end{bmatrix}
\begin{bmatrix}
j_{fx} \\
j_{fy}
\end{bmatrix}
= \begin{bmatrix}
v_{fx,x} \\
v_{fx,y}
\end{bmatrix},
\]

(4.35)

where \(Z_{fx,fx}\) is the self-impedance sub-matrix of the radiating element parallel to the x-axis, \(Z_{fy,fy}\) is the self-impedance sub-matrix of the radiating element parallel to the y-axis, \(Y^{a,a}\) is the self-admittance sub-matrix of the aperture and the \(T\) matrices are coupling sub-matrices between the radiating elements and the aperture plane. The vector on the right-hand side of Eq(4.35) is the source vector, where the first part of the upper index refers to the position of the source (e.g. \(fx\) is on the radiating element parallel to the x-axis), and the second part refers to the direction of the source. The remaining vector contains the unknown mode coefficients of the current densities. The expressions of the matrix elements of Eq(4.35) are given in Appendix B2.

It is noted here that the matrix equation for single polarized arrays, as described in [8], can be obtained by using that part of the matrix equation, which is enclosed by dashed lines.

### 4.4.3 Basis and testing functions

Once a choice has been made for the basis and testing functions, the matrix elements of Eq(4.35) can be calculated. As in the previous chapter, local basis and testing functions will be used on both fins (i.e. for the electric currents) and global basis and testing functions will be used on the aperture plane (i.e. for the magnetic currents). The \(q\)-th basis function for an \(x\)-directed electric current on the fin parallel to the x-axis, is given by Eq(4.36).

\[
J_{fx}(x',z) = \frac{\sin k_x (W_{fx}^x - |x' - x_q|)}{W_{fx}^x \sin (k_x W_{fx}^x)} e^{-z},
\]

(4.36)

\(x_k - W_{fx}^x \leq x' \leq x_k - W_{fx}^x\),

\(z_l - W_{fx}^z \leq z' \leq z_l\),

where

\(q \in \{1, \ldots, N_{fx}^x\}\),

\(k_q \in \{2, \ldots, S_{fx}^x\}\),

(4.37)

\(l_q \in \{2, \ldots, S_{fx}^x + 1\}\).

The \(q'\)-th basis function for a \(z\)-directed electric current on the fin parallel to the x-axis is given...
in Eq(4.38).

\[
J_{\varepsilon q}^{\varepsilon}(x', z') = \frac{\sin k e \left( \frac{W_{\varepsilon}^{\varepsilon} - |z' - z_q|}{W_{\varepsilon}^{\varepsilon} \sin (k e W_{\varepsilon}^{\varepsilon})} \right)}{W_{\varepsilon}^{\varepsilon}} e_{z'},
\]
\begin{align*}
& x_{k_q} - W_{\varepsilon}^{\varepsilon} \leq x' \leq x_{k_q}, \\
& z_{l_q} - W_{\varepsilon}^{\varepsilon} \leq z' \leq z_{l_q} + W_{\varepsilon}^{\varepsilon},
\end{align*}
\tag{4.38}

where
\[
q' \in \{1, ..., N_{\varepsilon}^{\varepsilon} \},
\]
\[k_q \in \{2, ..., S_{\varepsilon}^{\varepsilon} + 1 \},\]
\[l_q \in \{1, ..., S_{\varepsilon}^{\varepsilon} \}.
\tag{4.39}

Furthermore, \( S_{\varepsilon}^{\varepsilon} \) and \( S_{\varepsilon}^{\varepsilon} \) are the number of subdomains in the \( x \)- and \( z \)-direction, respectively, on the fin parallel to the \( x \)-axis.

For the basis functions on the fin parallel to the \( y \)-axis, an analogous definition is used. For a \( y \)-directed current, the \( q \)-th basis function is defined in Eq(4.40). The \( q' \)-th basis function for a \( z \)-directed current is defined in Eq(4.42).

\[
J_{\gamma q}^{\gamma}(y', z') = \frac{\sin k e \left( \frac{W_{\gamma}^{\gamma} - |y' - y_q|}{W_{\gamma}^{\gamma} \sin (k e W_{\gamma}^{\gamma})} \right)}{W_{\gamma}^{\gamma}} e_{y'},
\]
\begin{align*}
& y_{k_q} - W_{\gamma}^{\gamma} \leq y' \leq y_{k_q}, \\
& z_{l_q} - W_{\gamma}^{\gamma} \leq z' \leq z_{l_q} + W_{\gamma}^{\gamma},
\end{align*}
\tag{4.40}

where
\[
q \in \{1, ..., N_{\gamma}^{\gamma} \},
\]
\[k_q \in \{2, ..., S_{\gamma}^{\gamma} \},\]
\[l_q \in \{2, ..., S_{\gamma}^{\gamma} + 1 \},\]
\[l_q \in \{1, ..., S_{\gamma}^{\gamma} \}.
\tag{4.41}

Furthermore, \( S_{\gamma}^{\gamma} \) and \( S_{\gamma}^{\gamma} \) are the number of subdomains in the \( x \)- and \( z \)-direction, respectively, on the fin parallel to the \( x \)-axis.
where
\[ q' \in \{1, \ldots, N_y^f \}, \]
\[ k_q' \in \{2, \ldots, S_y^f + 1\}, \]
\[ l_q' \in \{1, \ldots, S_z^f\}. \]

Furthermore, \( S_y^f \) and \( S_z^f \) are the number of subdomains in the \( y \)- and \( z \)-direction, respectively, on the fin parallel to the \( y \)-axis.

For the testing functions Eq(4.36) to Eq(4.43) remain valid when the primed coordinates are replaced by unprimed coordinates and \( q \) and \( q' \) are replaced by \( r \) and \( r' \) respectively.

The grid definition of the basis and testing functions on both fins is given in figure 4.2.

![Grid definition on both fins](image)

The grid is defined in such a way that there is overlap between two successive basis or testing functions.

For the basis and testing functions on the aperture, Floquet modes are used. Due to orthogonality between these functions and the Green’s functions, the calculation of certain matrix elements is less time consuming. The basis functions are the same as in Eq(3.64)-(3.67). For the testing functions, the complex conjugates of Eq(3.64) and Eq(3.65) are taken, \( s \) is replaced by \( t \) and the primed coordinates are replaced by unprimed coordinates.

The matrix elements of Eq(4.35), using this set of basis and testing functions, are given in appendix B4. In this appendix use has been made of the integral definitions of appendix B3.

### 4.5 Computational and numerical details

Some of the integrals of appendix B3 and matrix elements of appendix B4 need extra attention because they exhibit certain numerical problems. Furthermore, it is possible to reduce the computation time of the matrix by using symmetry properties.

#### 4.5.1 Numerical stability of the matrix elements and integrals

Expressions which contain \( \gamma_{mp}, \beta_{np} \) or \( k_{mn} \) in combination with a sine or cosine function, can
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become unstable, since $\gamma_{mp}$, $\beta_{np}$ and $k_{mn}$ can be real or imaginary.

The first problem of this kind is located within the matrix elements of the self-impedance matrices of the x- and y-fin. The essential parts of these expressions are given in Eq(4.44a) and Eq(4.44b) for the x- and y-fin respectively.

$$D_{mp}^+ e^{-j\gamma_{mp}x_b} + D_{mp}^- e^{j\gamma_{mp}x_b} + D_{mp}^{\pm} = \frac{1}{\gamma_{mp} a d} \cos \left( \frac{\gamma_{mp} b}{d} - \cos \left( \Psi \right) \right).$$  \hspace{1cm} (4.44a)

$$C_{np}^+ e^{-j\beta_{np}x_b} + C_{np}^- e^{j\beta_{np}x_b} + C_{np}^{\pm} = \frac{1}{\beta_{np} b d} \cos \left( \frac{\beta_{np} a}{d} - \cos \left( \Psi \right) \right).$$  \hspace{1cm} (4.44b)

When $\gamma_{mp}$ or $\beta_{np}$ becomes imaginary, both numerator and denominator become very large, which causes inaccuracies. To prevent these problems, asymptotic expressions for Eq(4.44a) and Eq(4.44b) are used when $\text{Im}(\gamma_{mp} b)>1$ or $\text{Im}(\beta_{np} a)>1$, respectively. These asymptotic expressions are given in Eq(4.45).

$$D_{mp}^+ e^{-j\gamma_{mp}x_b} + D_{mp}^- e^{j\gamma_{mp}x_b} + D_{mp}^{\pm} = \frac{1}{\text{Im}(\gamma_{mp}) a d},$$  \hspace{1cm} (4.45a)

$$C_{np}^+ e^{-j\beta_{np}x_b} + C_{np}^- e^{j\beta_{np}x_b} + C_{np}^{\pm} = \frac{1}{\text{Im}(\beta_{np}) b d}.$$  \hspace{1cm} (4.45b)

These expressions can be obtained by dividing both numerator and denominator of Eq(4.44a) and Eq(4.44b) by the dominant term of the denominator and by recalling the fact that the imaginary part of $\gamma_{mp}$ and $\beta_{np}$ is negative or zero.

Other problems, involving $\gamma_{mp}$ and $\beta_{np}$ are located within the integrals $I_{5, x}^x(x_k, n, p)$, $I_{5, y}^x(x_k, n, p)$, $I_{5, y}^y(y_k, m, p)$ and $I_{6, y}^z(y_k, m, p)$. When $\gamma_{mp}$ or $\beta_{np}$ become imaginary, the expressions for these integrals in appendix B3 become unstable. The stability of these expressions can be improved by dividing both numerator and denominator by the dominant term of the denominator (e.g. $e^{-j\beta_{np}a}$ for $I_{5, x}^x(x_k, n, p)$ and $I_{5, y}^y(y_k, m, p)$). The expressions obtained in this way are given in appendix B5.

Numerical problems involving $k_{mn}$ are located within $T^{fx,a}$, $T^{fy,a}$ and $Y^{aa}$. The problems in the coupling-matrices $T^{fx,a}$ and $T^{fy,a}$ are caused by the combination of the parts

$$\frac{1}{\sin \left( k_{mn} d \right)},$$  \hspace{1cm} (4.46)

which can become very small, and

$$\left\{ \begin{array}{ll} I_{5, x}^x(z_p, m, n) & \text{for } T^{fx,a} \text{ and } T^{fx,a} \text{,} \\
I_{5, y}^x(z_p, m, n) & \text{for } T^{fy,a} \text{,} \\
I_{6, x}^a(z_p, m, n) & \text{for } T^{fx,a} \text{,} \end{array} \right.$$  \hspace{1cm} (4.47a)

$$\left\{ \begin{array}{ll} I_{5, y}^y(z_p, m, n) & \text{for } T^{fy,a} \text{ and } T^{fy,a} \text{,} \\
I_{6, y}^a(z_p, m, n) & \text{for } T^{fy,a} \text{,} \end{array} \right.$$  \hspace{1cm} (4.47b)

which can become very large.
The stability of these matrix elements can be improved by combining both factors and dividing both numerator and denominator by the dominant term of the denominator and recalling the fact that the imaginary part of \( k_{mn} \) is negative or zero. In this way, Eq(4.48) is obtained.

\[
\begin{align*}
\frac{A_{x,z}}{\sin(k_{mn}d)} &= \frac{j\kappa}{W_x \sin(k_{e}W_x) (k_x^2 + k_{mn}^2) (1 - e^{-2kd})} \left( e^{k(z_{t} + W_{x}^{(f)})} + e^{k(z_{t} - W_{x}^{(f)})} \\
&+ \frac{e^{-k(z_{t} + 2d - W_{x}^{(f)})} + e^{-k(z_{t} + 2d + W_{x}^{(f)})} - 2\cos(k_{e}W_x) (e^{kz_{t}} + e^{-k(z_{t} + 2d)})}{k (1 - e^{-2kd})} \right),
\end{align*}
\]

where

\[
k = -\text{Im}(k_{mn}) \quad (k \geq 0).
\]

Finally, the part of the self-admittance matrix \( Y_{aa} \) that causes stability problems is given in Eq(4.50).

\[
\cos(k_{mn}d) \quad \sin(k_{mn}d).
\]

4.5.2 Singular points

Most of the closed form expressions for the integrals defined in appendix B3 have singularities. However, most of the singularities are removable. The limits of these expressions in the singular points are listed in appendix B6. This appendix contains only those limits which are likely to occur in the computations.

Although most singularities are removable, there are still three non-removable singularities. These singularities occur when one of the equations in Eq(4.52) holds.

\[
\begin{align*}
\cos(\beta_{np}) &= \cos(\Psi_{x}), \\
\cos(\gamma_{mp}) &= \cos(\Psi_{y}), \\
\sin(k_{mn}d) &= 0.
\end{align*}
\]

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Eq(4.52a) and Eq(4.52b) are located within the self-impedance matrix of the y- and x-fin respectively, and in the coupling-matrices $T_{fx, fy}$ and $T_{fy, fx}$, respectively. When Eq(4.52) is examined more closely, it can be shown that the three equations are equivalent, i.e. the singularities occur for the same set of dimensions of the unit-cell. Eq(4.52) can now be written as:

$$\left(\frac{2\pi m + \Psi_x}{a}\right)^2 + \left(\frac{2\pi n + \Psi_y}{b}\right)^2 + \left(\frac{2\pi d}{d}\right)^2 = k_0^2,$$

where $m$, $n$ and $p$ are integers.

In case the singular points can be avoided by a change of $d$, i.e. by changing the height of the equivalence plane with respect to the ground plane, the height of the equivalence plane with respect to the ground plane is enlarged. This kind of singular points are caused by parallel plate resonance.

In case the singular points can not be avoided by changing $d$, a surface wave propagates over the ground plane.

### 4.5.3 Reduction of computation time

The computation of the matrix elements is rather time consuming. In order to reduce the computation time, use can be made of symmetry properties of certain sub-matrices. The symmetry properties of the self-impedance sub-matrices are given in Eq(4.54) and Eq(4.55) for the x- and y-fin respectively.

Z_{fx, fx}^{xx, rq} = -(Z_{fx, fx}^{xx, qr})^*,

Z_{fx, fx}^{zx, rq} = -(Z_{fx, fx}^{zx, qr})^*,

Z_{fx, fx}^{zz, rq} = -(Z_{fx, fx}^{zz, qr})^*,

Z_{fy, fy}^{yy, rq} = -(Z_{fy, fy}^{yy, qr})^*,

Z_{fy, fy}^{yz, rq} = -(Z_{fy, fy}^{yz, qr})^*,

Z_{fy, fy}^{zz, rq} = -(Z_{fy, fy}^{zz, qr})^*.

A minor reduction in computation time can be obtained by using the symmetry property of the admittance sub-matrix:

$Y_{yx, ts}^{aa} = Y_{yx, ts}^{aa}$

An other way to reduce the computation time is to calculate all integrals with two arguments in advance. The integrals with three arguments are not calculated in advance. However, integrals with three arguments can only change when the next row is calculated. Therefore, these integrals are calculated before the next row is calculated.

### 4.5.4 Source modelling details

In order to model the sources of the fins properly, three aspects need to be considered: the feed-
strip, the current coming out of the ground plane and the source itself. The feedstrip is modelled by using slightly different testing and basis functions at the feedstrip domain. However, using different basis and testing functions implies also different closed form expressions for the integrals \( I_{\text{x}}^{x}(x_k, n, p) \), \( I_{\text{y}}^{y}(x_k, n, p) \), \( I_{\text{x}}^{x}(y_k, n, p) \) and \( I_{\text{y}}^{y}(y_k, n, p) \). Therefore, appendices B3, B5 and B6 also contain expressions for the integrals when \( x_k = x_{c-1}, x_k = x_c \) or \( x_k = x_{c+1} \) and \( y_k = y_{c-1}, y_k = y_c \) or \( y_k = y_{c+1} \), being the left, centre and right feedstrip coordinate, respectively. Furthermore, the width of the feedstrip domains is indicated by an extra sub-index \( f \). This configuration is given in figure 4.3.

**Figure 4.3: Definition of the feedstrip domains**

The basis and testing functions on the feedstrip domains on the fin parallel to the x-axis is given is Eq(4.57) and those on the fin parallel to the y-axis in Eq(4.58).

\[
Jf_x(x, x_{c-1}) = \begin{cases} 
\sin k_e \left( \frac{W_{ SF}^x + x - x_{c-1}}{W_z^x \sin (k_e W_{ SF}^x)} \right) & (x_{c-1} - W_{ SF}^x \leq x \leq x_{c-1}) \\
\sin k_e \left( \frac{W_{ SF}^x - x + x_{c-1}}{W_z^x \sin (k_e W_{ SF}^x)} \right) & (x_{c-1} \leq x \leq x_{c-1} + W_{ SF}^x) 
\end{cases}, \quad (4.57a)
\]

\[
Jf_x(x, x_{c+1}) = \begin{cases} 
\sin k_e \left( \frac{W_{ SF}^x + x - x_{c+1}}{W_z^x \sin (k_e W_{ SF}^x)} \right) & (x_{c+1} - W_{ SF}^x \leq x \leq x_{c+1}) \\
\sin k_e \left( \frac{W_{ SF}^x - x + x_{c+1}}{W_z^x \sin (k_e W_{ SF}^x)} \right) & (x_{c+1} \leq x \leq x_{c+1} + W_{ SF}^x) 
\end{cases}, \quad (4.57b)
\]

\[
Jf_y(y, y_{c-1}) = \begin{cases} 
\sin k_e \left( \frac{W_{ SF}^y + y - y_{c-1}}{W_z^y \sin (k_e W_{ SF}^y)} \right) & (y_{c-1} - W_{ SF}^y \leq y \leq y_{c-1}) \\
\sin k_e \left( \frac{W_{ SF}^y - y + y_{c-1}}{W_z^y \sin (k_e W_{ SF}^y)} \right) & (y_{c-1} \leq y \leq y_{c-1} + W_{ SF}^y) 
\end{cases}, \quad (4.57c)
\]

\[
Jf_y(y, y_{c+1}) = \begin{cases} 
\sin k_e \left( \frac{W_{ SF}^y + y - y_{c+1}}{W_z^y \sin (k_e W_{ SF}^y)} \right) & (y_{c+1} - W_{ SF}^y \leq y \leq y_{c+1}) \\
\sin k_e \left( \frac{W_{ SF}^y - y + y_{c+1}}{W_z^y \sin (k_e W_{ SF}^y)} \right) & (y_{c+1} \leq y \leq y_{c+1} + W_{ SF}^y) 
\end{cases}, \quad (4.58a)
\]

\[
Jf_y(y, y_{c-1}) = \begin{cases} 
\sin k_e \left( \frac{W_{ SF}^y - y + y_{c+1}}{W_z^y \sin (k_e W_{ SF}^y)} \right) & (y_{c+1} - W_{ SF}^y \leq y \leq y_{c+1}) \\
\sin k_e \left( \frac{W_{ SF}^y + y - y_{c+1}}{W_z^y \sin (k_e W_{ SF}^y)} \right) & (y_{c+1} \leq y \leq y_{c+1} + W_{ SF}^y) 
\end{cases}, \quad (4.58b)
\]

\[
Jf_y(y, y_{c+1}) = \begin{cases} 
\sin k_e \left( \frac{W_{ SF}^y + y - y_{c+1}}{W_z^y \sin (k_e W_{ SF}^y)} \right) & (y_{c+1} - W_{ SF}^y \leq y \leq y_{c+1}) \\
\sin k_e \left( \frac{W_{ SF}^y - y + y_{c+1}}{W_z^y \sin (k_e W_{ SF}^y)} \right) & (y_{c+1} \leq y \leq y_{c+1} + W_{ SF}^y) 
\end{cases}, \quad (4.58c)
\]
Due to the different dimension of the feedstrip domain, the integrals $I_{x,x}^k(x_k, n, p)$, $I_{x,y}^k(x_k, n, p)$, $I_{y,x}^k(x_k, n, p)$ and $I_{y,y}^k(x_k, n, p)$ also have to be changed. However, this can easily be done by changing $W_x$ into $W_x'$ and $W_y$ into $W_y'$.

The modelling of the currents coming from behind the ground plane is done by using half modes in the $z$-direction when $z_l = -d$. This means that the basis and testing functions for $z_l = -d$ are defined as:

$$J_{ix}^k(z) = \frac{\sin k_e (W_{ix}^f - z + d)}{W_{ix}^f \sin (k_e W_{ix}^f)} (-d \leq z \leq -d + W_{ix}^f),$$

for the fin parallel to the $x$-axis and

$$J_{iy}^k(z) = \frac{\sin k_e (W_{iy}^f - z + d)}{W_{iy}^f \sin (k_e W_{iy}^f)} (-d \leq z \leq -d + W_{iy}^f),$$

for the fin parallel to the $y$-axis.

It is readily shown that the integrals $I_{x,x}^k(z_l, p)$, $I_{x,y}^k(z_l, m, n)$, $I_{y,x}^k(z_l, p)$ and $I_{y,y}^k(z_l, m, n)$ in appendix B3 should be divided by 2 when $z_l = -d$.

For the sources on the fins, the delta-gap model has been used. For the fin parallel to the $x$-axis, the electric field generated by this source (i.e. $E_{gx}^y$) can be written as in Eq(4.61). The electric field generated by the delta-gap generator on the fin parallel to the $y$-axis is given in Eq(4.62).

$$E_{gx}^y = -V_{gx} \delta (x - x_{gx}) e_x (z_{gx} - W_{gx}^f \leq z \leq z_{gx} + y = y^b),$$

where $V_{gx}$ is the generator voltage on the $x$-fin, $x_{gx}$ and $z_{gx}$ are the grid coordinates of the generator and $y^b$ is the $y$-coordinate of the $x$-fin.

$$E_{gy}^y = -V_{gy} \delta (y - y_{gy}) e_y (z_{gy} - W_{gy}^f \leq z \leq z_{gy} + x = x^b),$$

where $V_{gy}$ is the generator voltage on the $y$-fin, $y_{gy}$ and $z_{gy}$ are the grid coordinates of the generator and $x^b$ is the $x$-coordinate of the $y$-fin. It is noted that the delta-gap generator is always located on the edge of a subdomain.

The source vector, defined in Eq(4.35), now takes the form of Eq(4.63).

$$V_{fx,x} = \begin{cases} V_{gx} & \text{when } x_{k_r} = x_{gx} \\ 0 & \text{else} \end{cases}$$

$$V_{fx,z} = 0$$

$$V_{fy,y} = \begin{cases} V_{gy} & \text{when } y_{k_r} = y_{gy} \\ 0 & \text{else} \end{cases}$$

$$V_{fy,z} = 0$$

$$V_{fx,x} = 0$$

$$V_{fy,y} = 0$$
4.6 Performance parameters for dual polarization

In order to describe the performance of a dual polarized array, an objective measure is needed, which applies to all situations. A measure to describe the coupling between the two orthogonal elements in the unit cell is the scattering matrix, which is described in the next paragraph. A measure to determine the quality of the polarization is the axial ratio, for which the electric field in the far field region is needed. The electric field in this region is examined in paragraph 4.6.2.

4.6.1 Scattering matrix

The coupling between the two elements in the unit cell can be expressed in terms of a scattering matrix. The unit cell is now regarded as a two port. The scattering matrix $S$ of a two port can be calculated from the admittance matrix $Y$ of the same two port, according to the relation [13]

$$ S = (Y_0 U - Y) \cdot (Y_0 U + Y)^{-1}, $$

where $Y_0$ is the characteristic admittance of the transmission line feeding the antenna and $U$ is the unity matrix. The two port, together with the definition of the voltages and currents, is shown in figure 4.4.

![Figure 4.4: Two port definition](image)

The admittance matrix $Y$ for this two port is now defined as:

$$ \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, $$

where $i_1$ is the current in $x$-direction on the $x$-fin located at the delta-gap coordinates on the $x$-fin, $i_2$ is the current in $y$-direction on the $y$-fin located at the delta-gap coordinates on the $y$-fin. $V_1$ and $V_2$ are the delta-gap voltages $V_{g1}$ and $V_{g2}$, respectively, which are defined in Eq(4.61) and Eq(4.62). The current at the delta-gap coordinate is equal to the mode coefficient of the basisfunction at that coordinate. However, it should be noted that the currents, located at the delta-gap, need an extra minus sign to become $i_1$ and $i_2$, since the current flows through the sources.

The relation of the port currents and voltages can also be expressed in terms of the complex power $P$:

$$ I_i = \left( \frac{P_i}{V_i} \right)^* \quad (i = 1, 2). $$

(4.65)
The complex power on port 1 (i.e. on the x-fin) can be written as:

\[ P_1 = -\int_{S_{fx}} E^{gx} \cdot (J_{fx})^* dS, \tag{4.66} \]

where \( J_{fx} \) is the current density on the x-fin. The current density can be written in terms of expansion functions and mode coefficients according to Eq(4.29a). When Eq(4.29a) is sub-

situted into Eq(4.66), Eq(4.67) is obtained.

\[ P_1 = -\int_{S_{fx}} E^{gx} \left( \sum_{q=1}^{N_{fx}} j_{fx, xq} J_{fx, xq} (x', z') \varepsilon_x + \sum_{q=1}^{N_{fx}} j_{fx, zq} J_{fx, zq} (x', z') \varepsilon_z \right) dx' dz'. \tag{4.67} \]

The electric current on port 1 can be found by combining Eq(4.65) and Eq(4.67):

\[ I_1 = \frac{1}{V_1} \int_{S_{fx}} (E^{gx})^* \left( \sum_{q=1}^{N_{fx}} j_{fx, xq} J_{fx, xq} (x', z') \varepsilon_x + \sum_{q=1}^{N_{fx}} j_{fx, zq} J_{fx, zq} (x', z') \varepsilon_z \right) dx' dz'. \tag{4.68} \]

Changing integration and summation and recalling the fact that the basis function \( J \) are real, gives

\[ I_1 = \frac{1}{V_1} \left( \sum_{q=1}^{N_{fx}} j_{fx, xq} \int_{S_{fx}} E^{gx} \cdot J_{fx, xq} (x', z') \varepsilon_x dx' dz' \right)^* + \sum_{q=1}^{N_{fx}} j_{fx, zq} \left( \int_{S_{fx}} E^{gx} \cdot J_{fx, zq} (x', z') \varepsilon_z dx' dz' \right)^*. \tag{4.69} \]

The fact that Galerkin testing is used on the fin (i.e. the basis functions are the same as the test-

ing functions) and the definitions of the excitation vector on the right hand side and the mode coefficient vector on the left hand side of Eq(4.35) can be used to formulate Eq(4.70):

\[ I_1 = \frac{1}{V_1} \left( (V_{r,x})^* \right)^T \left( j_{fx} \right), \tag{4.70} \]

where \( T \) means the transposed of a matrix.

In an analogous way, the current \( I_2 \) can be expressed in terms of the mode coefficients on the y-

fin:

\[ I_2 = \frac{1}{V_2} \left( (V_{r,y})^* \right)^T \left( j_{fy} \right), \tag{4.71} \]

Two vectors are now defined, which have the same the dimensions as the excitation vector on the right hand side of Eq(4.35):

\[ \left[ V_{ex} \right] = \frac{1}{V_1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{4.72a} \]
The source vector $V$ on the right hand side of Eq(4.35) can now be written in terms of $V^{ex}$:

$$ [V] = \begin{bmatrix} V_{ex_x} \\ V_{ex_y} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad (4.73) $$

and Eq(4.70-71) is rewritten as in Eq(4.74).

$$ \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = -\begin{bmatrix} [V_{ex_x}]^* \\ [V_{ex_y}]^* \end{bmatrix}^T \begin{bmatrix} j \xi \xi_x \\ j \xi \xi_q \\ j \xi \eta_x \\ j \xi \eta_q \\ m_{xex} \\ m_{yex} \end{bmatrix}, \quad (4.74) $$

The vector containing the mode coefficients on the right hand side of Eq(4.74) is obtained from the method of moments. If $M$ is the matrix on the left hand side of Eq(4.35), then Eq(4.74) becomes

$$ \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = -\begin{bmatrix} [V_{ex_x}]^* \\ [V_{ex_y}]^* \end{bmatrix}^T \cdot M^{-1} \cdot \begin{bmatrix} V_{ex_x} \\ V_{ex_y} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \quad (4.75) $$

Comparing Eq(4.75) with Eq(4.64), the admittance matrix is given by

$$ Y = -\begin{bmatrix} [V_{ex_x}]^* \\ [V_{ex_y}]^* \end{bmatrix}^T \cdot M^{-1} \cdot \begin{bmatrix} V_{ex_x} \\ V_{ex_y} \end{bmatrix}. \quad (4.76) $$

This relation is also valid when other kind of excitations are used, as long as the sources only have components unequal to zero on the fin on which they are defined and as long as Galerkin testing is done with real functions. This follows from the analysis.

### 4.6.2 The electric field in the far field zone

The polarization characteristics of the array can be studied by analysing the electric field caused by the currents on the fins in the unit cell. The mode coefficients of the current densities on the fins are found by applying the method of moments. Therefore, all that is needed is the electric field in the far field region caused by an $x$, $y$, or $z$-directed piecewise sinusoidal function above a perfectly conducting ground plane, since those are the expansion functions on the
fins. Because the currents on the fins are the same with and without the equivalence plane, the mode coefficients on the aperture plane are not needed.

The ground plane at \( z = -d \) is eliminated from the analysis by using the image principle. An \( x \)-directed electric current density \( J_x(x,y,z) \) above the ground plane has to be replaced by:

\[
I_x^\text{total}(x,y,z) = (J_x(x,y,z) - J_x(x,y,-z-2d)) \varepsilon_x,
\]

(4.77)

when the ground plane is removed. An analogous expression holds for a \( y \)-directed electric current density. For a \( z \)-directed electric current density \( J_z(x,y,z) \) above a ground plane, the ground plane can be removed by replacing the current density by:

\[
I_z^\text{total}(x,y,z) = (J_z(x,y,z) + J_z(x,y,-z-2d)) \varepsilon_z.
\]

(4.78)

The electric field in the far field zone due to an electric current density \( I^\text{total}(r) \) (in cartesian components) without a ground plane, can be written as [9]:

\[
E(\epsilon) = \frac{k_0^2}{j\omega\varepsilon_0 4\pi r} \varepsilon_r \times \varepsilon_r \times \int_V I^\text{total}(\epsilon') e^{jk_0 (\varepsilon_r, \epsilon')} dV,
\]

(4.79)

where \( \varepsilon_r \) is the unit vector in the radial direction, \( V \) is the volume enclosing the electric current density and the primed coordinates are the integration variables. The inner product in the exponent of Eq(4.79) is rewritten in Eq(4.80).

\[
(\varepsilon_r, \epsilon') = \varepsilon' \cos(\phi) \sin(\theta) + y' \sin(\phi) \sin(\theta) + z' \cos(\theta).
\]

(4.80)

For a piecewise sinusoidal current density on the \( x \)-fin in the \( x \)-direction above the ground plane, the current density \( J_x(x,y,z) \varepsilon_x \) is of the form

\[
J_x = \begin{cases} 
\sin(k_x (W_1 + x - x_k)) / W_z \sin(k_e W_1) \delta(y-y_b) & \text{for } (x \in [x_k-W_1, x_k] \wedge z \in [z_l-W_z, z_l]) \\
\sin(k_x (W_2 - x + x_k)) / W_z \sin(k_e W_2) \delta(y-y_b) & \text{for } (x \in [x_k, x_k+W_2] \wedge z \in [z_l-W_z, z_l])
\end{cases}
\]

(4.81)

Combining expression (4.77) and Eq(4.81), the integral on the right hand side of Eq(4.79) becomes

\[
I_x^\text{tot}(x_k, z_l) = \int_V I^\text{total}(\epsilon') e^{jk_0 (\varepsilon_r, \epsilon')} dV = \frac{4jk_0 e^{jk_0 (x_k \cos(\phi) \sin(\theta) + y_b \sin(\theta) \sin(\phi) - d \cos(\theta))}}{(k_0^2 - (k_0 \cos(\phi) \sin(\theta))^2) k_0 W_z \cos(\theta)} \cdot \\
\sin \left( \frac{k_0 W_z \cos(\theta)}{2} \right) \sin \left( k_0 \cos(\theta) \left( z_l + d - \frac{W_z}{2} \right) \right) \cdot \\
\left[ e^{-jk_0 W_1 \cos(\phi) \sin(\theta)} - \cos(k_e W_1) \right] / \sin(k_e W_1) + \\
\left[ e^{jk_0 W_2 \cos(\phi) \sin(\theta)} - \cos(k_e W_2) \right] / \sin(k_e W_2) \cdot \varepsilon_x.
\]

(4.82)

For a piecewise sinusoidal current density on the \( y \)-fin in the \( y \)-direction above the ground
plane, the current density \( J_y(x,y,z) \) is of the form

\[
J_y = \begin{cases} 
\delta(x-x^b) \frac{\sin k_e (W_1 + y-y_k)}{W_z \sin (k_e W_1)} & \text{for } (y \in [y_k-W_1, y_k] \land z \in [z_l-W_z, z_l]) \\
\delta(x-x^b) \frac{\sin k_e (W_2 - y+y_k)}{W_z \sin (k_e W_2)} & \text{for } (y \in [y_k+y_k+W_2, y_k+W_2] \land z \in [z_l-W_z, z_l]) 
\end{cases},
\] (4.83)

Using an analogous expression of Eq(4.77) for a \( y \)-directed current density and Eq(4.83), the integral on the right hand side of Eq(4.71) becomes

\[
I^y_{total}(\epsilon, \epsilon', \epsilon'') = \int_V \int_{\epsilon}^{\epsilon'} e^{j k_0 (x^h \cos (\phi) \sin (\theta) + y \sin (\phi) \sin (\theta) - d \cos (\theta))} \cdot \sin \left( \frac{k_0 W_z \cos (\theta)}{2} \right) \cdot \delta(y-y^b) \cdot e^{j k_0 (x \cos (\phi) \sin (\theta) - y' \sin (\phi) \sin (\theta) - d \cos (\theta))} \cdot \left[ e^{j k_0 (W_z \sin (\phi) \sin (\theta) - \cos (k_e W_1))} + e^{j k_0 (W_z \sin (\phi) \sin (\theta) - \cos (k_e W_2))} \right] \cdot e_y, \] (4.84)

For a piecewise sinusoidal current density on the \( x \)-fin in the \( z \)-direction above the ground plane, the current density \( J_z(x,y,z) \) is of the form

\[
J_z = \frac{\sin k_e (W_z + |z-z^b|)}{W_z \sin (k_e W_z)} \delta(y-y^b) \text{ for } (x \in [x_k-W_x, x_k] \land z \in [z_l-W_z, z_l+W_z]).
\] (4.85)

Combining expression (4.78) and Eq(4.85), the closed form expression of the integral on the right hand side of Eq(4.79) can be written as:

\[
I^z_{total}(\epsilon, \epsilon', \epsilon'') = \int_V \int_{\epsilon}^{\epsilon'} e^{j k_0 (x \cos (\phi) \sin (\theta) + y' \sin (\phi) \sin (\theta) - d \cos (\theta))} \cdot \sin \left( \frac{k_0 W_z \cos (\phi) \sin (\theta)}{2} \right) \cdot \cos (k_0 \cos (\theta) (z_l+d)) \cdot (\cos (k_0 \cos (\theta) W_z) - \cos (k_e W_z)) \cdot e_z \] (4.86)

For a half mode at \( z = -d \), the integral has to be divided by 2.

Finally, for a piecewise sinusoidal current density on the \( y \)-fin in the \( z \)-direction above the ground plane the current density \( J_z(x,y,z) \) is of the form

\[
J_z = \frac{\sin k_e (W_z + |z-z^b|)}{W_y \sin (k_e W_y)} \delta(x-x^b) \text{ for } (y \in [y_k-W_y, y_k] \land z \in [z_l-W_z, z_l+W_z]).
\] (4.87)
The integral on the right hand side of Eq(4.79) can now be written as:

\[
I^{p}(y_{k}, z_{l}) = \int_{V} I^{total}(\ell') e^{jk_{0}(\ell_{0} - \ell')} dV = \frac{8k_{0}}{W_{y} W_{z}} \left( k_{0}^{2} - (k_{0} \cos(\theta))^{2} \right) k_{0} \sin(\varphi) \sin(\theta) \cdot d \cos(\theta) \\
\cdot \sin\left( \frac{k_{0} W_{y} \sin(\varphi) \sin(\theta)}{2} \right) \\
\cdot \cos(k_{0} \cos(\theta)(z_{l} + d)) \cdot \cos(k_{0} \cos(\theta) W_{z}) - \cos(k_{0} W_{z}) \cdot e_{z}.
\] (4.88)

For a half mode at \( z = -d \), the integral has to be divided by 2.

For every basis function on both fins, one of the closed form expressions for the integral in Eq(4.79) holds. When the expressions for the integral are multiplied with the proper mode coefficients of the basis functions, the three cartesian components \((I_{x}, I_{y}, I_{z})\) of the integral in Eq(4.79) can be obtained:

\[
I_{x} = \sum_{q=1}^{N_{x}^{f}} j f_{x}^{q} \cdot I_{x}^{q}(x_{k_{q}}, z_{l_{q}}),
\] (4.89a)

where \( k_{q} \in \{2, \ldots, S_{x}^{f}\} \land l_{q} \in \{2, \ldots, S_{z}^{f} + 1\}, \)

\[
I_{y} = \sum_{q=1}^{N_{y}^{f}} j f_{y}^{q} \cdot I_{y}^{q}(y_{k_{q}}, z_{l_{q}}),
\] (4.89b)

where \( k_{q} \in \{2, \ldots, S_{y}^{f}\} \land l_{q} \in \{2, \ldots, S_{z}^{f} + 1\}, \)

\[
I_{z} = \sum_{q=1}^{N_{z}^{f}} j f_{z}^{q} \cdot I_{z}^{q}(x_{k_{q}}, z_{l_{q}}) + \sum_{q=1}^{N_{y}^{f}} j f_{y}^{q} \cdot I_{z}^{y}(y_{m_{q}}, z_{n_{q}}),
\] (4.89c)

where \( k_{q} \in \{2, \ldots, S_{x}^{f} + 1\} \land l_{q} \in \{1, \ldots, S_{y}^{f}\} \land m_{q} \in \{2, \ldots, S_{y}^{f} + 1\} \land n_{q} \in \{1, \ldots, S_{y}^{f}\} \)

Furthermore, the definition of the mode coefficients of Eq(4.29) has been used.

The electric field in the far field zone has two components in spherical coordinates: one in the \( \theta \)-direction \((E_{\theta})\) and one in the \( \varphi \)-direction \((E_{\varphi})\). These components can be expressed in terms of the cartesian components of the integral: \((I_{x}, I_{y}, I_{z})\). This is done in Eq(4.90).

\[
\begin{bmatrix}
E_{\theta} \\
E_{\varphi}
\end{bmatrix} = \frac{k_{0}^{2}}{j \omega \varepsilon_{0} 4 \pi r} \begin{bmatrix}
\cos \varphi \cos \theta & \sin \varphi \cos \theta & \sin \theta \\
\sin \varphi & \cos \varphi & 0
\end{bmatrix} \begin{bmatrix}
I_{x} \\
I_{y} \\
I_{z}
\end{bmatrix}.
\] (4.90)

The components of the electric field of Eq(4.90) are rewritten in Eq(4.91), in terms of a left handed and a right handed circular component: \(E_{L}\) and \(E_{R}\), respectively [13].

\[
\begin{bmatrix}
E_{L} \\
E_{R}
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & j \end{bmatrix} \begin{bmatrix} E_{\theta} \\
E_{\varphi}
\end{bmatrix}.
\] (4.91)

These components can be used to compute the axial ratio, which has been defined in Eq(4.1).
The axial ratio is calculated for the case
\[ V_{gy} = j V_{gx} \]  \hspace{1cm} (4.92)
i.e. there is a 90 degrees phase shift between the sources on the x- and y-fin, in an attempt to make circular polarization.

4.7 Results

The model presented in this chapter is applied to an array of dipoles and to an array of bunny-ear antennas. For each configuration, scans are performed in the principal plane and in the diagonal plane. The two radiating elements inside the unit cell are chosen identical. Therefore, both E- and H-plane scan behaviour are examined when a scan in one of the principal planes is performed, since this plane is the E-plane for one element and the H-plane for the other. Also, the behaviour of both elements in the diagonal plane should be the same, due to symmetry. This can be used as a software test.

The unit cell configuration of the dipole array is given in figure 4.5. The analysed frequency is 300 MHz, which means that the width of the dipoles is approximately half a wavelength. Both dipoles are divided into 16 subdomains, on which 15 horizontal basis functions are defined, and the sources are placed in the middle of the dipoles. Furthermore, both dipoles are centered with respect to the unit cell. For the aperture, 7 Floquet modes are used in both x- and y-direction. All summations have an upper summation index of 200 and if necessary, a lower summation index of -200 is used.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dipole_array.png}
\caption{Dipole array for dual polarization}
\end{figure}

The simulations for this dipole array are performed in the \( \varphi = 0 \) and \( \varphi = 45 \) degrees planes. In all cases, the characteristic impedance \( Z_0 = 1/Y_0 = 162.4 \, \Omega \), which is the real part of the impedance at broadside.

For \( \varphi = 0 \) degrees, the scattering parameters \( S_{11} \) and \( S_{22} \) are shown in figure 4.6. These parameters are compared with the reflection coefficient \( R \) of an array with the same dimensions but with only one dipole (single polarized) in stead of two (dual polarized) inside the unit cell. Since, for \( \varphi = 0 \), the x-fin is scanning in its E-plane, \( S_{11} \) has to be compared with the reflection coefficient for the array also scanning in the E-plane. For the y-fin, \( \varphi = 0 \) means an H-plane scan is performed. Therefore \( S_{22} \) is compared with the reflection coefficient for the array scan-
Dual polarization

ning in the H-plane. For the H-plane scan \( S_{22} \) and \( R \) are the same, while for the E-plane scan \( S_{11} \) and \( R \) are not the same. This can be explained by recognising that for the dipole scanning in E-plane the other dipole acts as some sort of metallic wall.

The scattering parameters \( S_{12} \) and \( S_{21} \) are equal to zero for the scan in the principal plane.

The axial ratio in the principal planes (either E- or H-plane) is shown in figure 4.7. The axial ratio is compared to the axial ratio in case two half wavelength dipoles (no array environment), which are not influencing one another and are placed over a perfectly conducting ground plane perpendicular to one another. In that case, the current densities on the dipoles can be approximated by one piecewise sinusoidal function on every dipole. The sources of the two dipoles have a phase shift of 90 degrees. The latter case is called ‘without mutual coupling’, while the dual polarized array is indicated by ‘mutual coupling’.

For \( \varphi = 45 \) degrees, the scattering parameters are given in figure 4.8. The parameters \( S_{11} \) and \( S_{22} \) are compared to the reflection coefficient \( R \) of a single polarized array of dipoles, scanning in the diagonal plane. It is observed that \( S_{11} = S_{22} \) (both in modulus and in argument), which is expected because of symmetry. Furthermore, it is seen that \( S_{12} = S_{21} \), since the array is recipro-
The fact that $S_{12}$ and $S_{21}$ are not equal to zero, indicates that a coupling exist between the two elements in the unit cell when the scanning is performed is the diagonal plane.

![Figure 4.8: Scattering parameters in the diagonal plane](image)

In figure 4.9, the axial ratio of the array, scanning in the diagonal plane, is compared to the case of the two isolated dipoles over a ground plane.

![Figure 4.9: Axial ratio in the diagonal plane](image)

For the bunny-ear array, the array design in [8, p. 76] has been used as a starting point. The arrangement of the subdomains is given in figure 4.10, together with the unit cell dimensions (top-view) and the bunny-ear dimensions. The vertical thick white line in the subdomain arrangement indicates a separation between the two sides of the element, which implies that no current can cross this line. The lower side of the bunny is placed 22.9 mm from the ground plane. Due to symmetry, the unit cell in the $x$-direction is chosen the same as in the $y$-direction and thus not the same as in [8]. The array is analysed for the frequency of 1 GHz, which was the centre frequency in [8].

For the aperture plane, 7 Floquet modes are used in both $x$- and $y$-direction and for the summations, upper summation indices of 200 are used to ensure convergence. If needed, lower summation indices of -200 are used.
The bunny-ear array is analysed in the principal ($\phi = 0$ degrees) and diagonal plane ($\phi = 45$ degrees). In all cases, the characteristic impedance $Z_0 = 1/Y_0$ is taken 157 $\Omega$, which is the real part of the impedance at broadside.

For $\phi = 0$ degrees, the scattering parameters $S_{11}$ and $S_{22}$ are given in figure 4.11. The scattering parameters $S_{12}$ and $S_{21}$ were equal to zero in this plane. $S_{11}$ is compared to the active reflection coefficient $R$ of a single polarized bunny-ear array (one bunny-ear in stead of two inside the unit cell) for an E-plane scan. It is seen that $S_{11}$ and $R$ are not the same. This is due to the fact that for dual polarization the perpendicular bunny-ear antenna acts as some sort of metallic wall. $S_{22}$ is compared to the reflection coefficient of a single polarized bunny-ear array scanning in the H-plane. For the latter case, $S_{22}$ and $R$ are the same. The axial ratio in the principal plane is given in figure 4.12.

![Figure 4.10: Bunny-ear configuration for dual polarization](image)

![Figure 4.11: Scattering parameters in the principal plane](image)
For the diagonal plane, the scattering parameters are shown in figure 4.13. It is seen that $|S_{11}| = |S_{22}|$, but it was observed that the phase were also the same. This is due to the symmetry in the diagonal plane. Furthermore, $S_{11}$ and $S_{22}$ are compared to the reflection coefficient $R$ of a single polarized array of bunny-ears. The parameters $S_{12}$ and $S_{21}$ are the same both in modulus and argument, which is expected because the bunny-ear antennas inside the unit cell form a reciprocal network. The fact that $S_{12}$ and $S_{21}$ are not equal to zero indicates that there is coupling between the two elements inside the unit cell. The axial ratio in the diagonal plane is given in figure 4.14.
Although the axial ratio is an important parameter for dual polarized arrays, it is not a design parameter, since it can be influenced by changing the amplitude and phase of the two sources on the fins.

From Eq(4.89-91), it follows that the left-handed and right-handed components of the electric field in the far field zone are linear functions of the electric current mode coefficients on the radiating elements, for a fixed scan angle. Furthermore, from Eq(4.35) it follows that these mode coefficients are linear functions of the source voltages on both fins. Combining these two facts, it follows that the left-handed \( E_L \) and right-handed \( E_R \) electric field components are linear functions of the source voltages, for a fixed scan angle. Therefore, these components can be written as

\[
E_R = R_x(\theta, \varphi)V_x + R_y(\theta, \varphi)V_y, \tag{4.93}
\]

and

\[
E_L = L_x(\theta, \varphi)V_x + L_y(\theta, \varphi)V_y, \tag{4.94}
\]

where \( V_x \) and \( V_y \) are the complex voltages of the sources on the \( x \)- and \( y \)-fin, respectively. In case only the right-handed component is wanted, the left-handed component must be made equal to zero. This can be done by choosing

\[
\frac{V_x}{V_y} = \frac{L_y(\theta, \varphi)}{-L_x(\theta, \varphi)}, \tag{4.95}
\]

Hence, by controlling the (complex) ratio between the source voltages, the axial ratio can always be made equal to 1 (0 dB). It is noted here that in general, Eq(4.95) depends on the scan angle, therefore the choice of using only a 90 degrees phase shift is in general not suited.
Chapter 5

Triangular grid

5.1 Introduction

Grating lobes in an array are often causing problems like blinde scan angles. Therefore, grating lobes should be prevented in general. The presence of grating lobes depends on the unit cell dimensions and the shape of the unit cell. In general, when a grating lobe is entering the array at a certain scan angle \( \theta \), the grating lobe can be removed by decreasing the unit cell dimensions. However, the radiating elements often have a width of \( \lambda/2 \), which means that the elements are practically touching one another if no grating lobes are allowed for all possible scan angles \( \theta \), when a rectangular grid is used.

In [12] it was shown that the distance between two elements can be enlarged somewhat, without the appearance of grating lobes, when a triangular grid is used. In this chapter, the configuration of triangular grid is discussed first. Then the modifications in the model described in [8] are given to bring this kind of grid into the model, after which some simulation results are presented.

5.2 Configuration

The top-view of an array with triangular grid is shown in figure 5.1.

![Figure 5.1: Top-view of triangular grid](image)

The unit cell has now a diamond shape in stead of a rectangular shape. The array can be described in terms of \( \eta_1 \) and \( \eta_2 \) with skewing parameter \( \alpha \). This coordinate system can be
expressed in cartesian coordinates:
\[
\eta_1 = \frac{x}{\sin (\alpha)}, \quad (5.1a)
\]
\[
\eta_2 = y - x \cdot \cot (\alpha). \quad (5.1b)
\]

### 5.3 Periodicity conditions

In a triangular grid, the periodicity conditions are different from the conditions in a rectangular grid. In the parameters \( \eta_1 \) and \( \eta_2 \), these conditions for the magnetic vector potential \( \Delta \) and the electric vector potential \( E \) can be written as

\[
\Delta (\eta_1 + a/\sin (\alpha), \eta_2, z) = \Delta (\eta_1, \eta_2, z) e^{-j\Psi_{\eta_1}}, \quad (5.2a)
\]
\[
\Delta (\eta_1, \eta_2 + b, z) = \Delta (\eta_1, \eta_2, z) e^{-j\Psi_{\eta_2}}, \quad (5.2b)
\]
\[
E (\eta_1 + a/\sin (\alpha), \eta_2, z) = E (\eta_1, \eta_2, z) e^{-j\Psi_{\eta_1}}, \quad (5.2a)
\]
\[
E (\eta_1, \eta_2 + b, z) = E (\eta_1, \eta_2, z) e^{-j\Psi_{\eta_2}}, \quad (5.2b)
\]

where

\[
\Psi_{\eta_1} = \Psi_x + \frac{a}{b \tan (\alpha)}, \quad (5.3a)
\]
\[
\Psi_{\eta_2} = \Psi_y. \quad (5.3b)
\]

Performing a similar analysis as in [8], the Green’s functions in [8] can be transformed into Green’s functions for triangular grid by substituting

\[
\Psi_x \rightarrow \Psi_x - \frac{2\pi n a}{b \tan (\alpha)}. \quad (5.4)
\]

When expression (5.4) is also used for the test and basis functions on the aperture, Eq(5.4) can be used for all expressions of the matrix elements, obtained from the method of moments. However, in certain matrix elements, a specific choice for \( n \) has been made (e.g. \( n = n_1 \)), which means that in expression (5.4) the same choice for \( n \) must be made, due to orthogonality relations used in the analysis.

### 5.4 Test case

Software written by Tangdiongga [14], to analyse an array of monopoles, has been used to validate the model obtained from the combination of [8] and expression (5.4). Three remarks must be made before the results are discussed:

1. Tangdiongga uses filaments while narrow strips are used here,
2. in Tangdiongga \( \alpha \) is defined as the angle between the \( \eta_1 \)-axis and \( \eta_1 \) axis is fixed,
3. the excitation in Tangdiongga is a coaxial cable, while a delta-gap model is used here.

The first remark is taken into account by recognizing that the equivalent radius \( (r_e) \) for a strip of width \( W \) is \( r_e = W/4 \) [1]. The second remark is dealt with by scanning in an other direction in
azimuth:

$$\varphi_T = \frac{\pi}{2} - \varphi_B,$$

(5.5)

where $\varphi_T$ is the azimuth scan coordinate used by Tangdiongga and $\varphi_B$ is the scan coordinate used here.

An array of monopoles has been analysed for two situations: a rectangular grid ($\alpha = 90$ degrees) and a triangular grid ($\alpha = 60$ degrees). A frequency scan is performed from 1.1 to 1.6 GHz at $\theta = 60$ degrees and $\varphi_T = 0$ degrees. The unit cell dimensions are $a = b = 0.115$ m. The monopole is characterised by: height = 40 mm and radius $r = 0.635$ mm or stripwidth $W = 4r$.

For this configuration, a grating lobe is expected to enter the array with a rectangular grid at a frequency $f = 1.397$ GHz. For the array with a triangular grid, no grating lobes are found in this frequency range. The results of the analysis for the rectangular grid and for the triangular grid are shown in figure 5.2 and 5.3, respectively.

**Figure 5.2: Rectangular grid**

**Figure 5.3: Triangular grid**

From figure 5.2 and 5.3 it is seen that the amplitude of the resistance and reactance calculated by means of Tangdiongga's software is larger compared to the model presented in this chapter. This was probably due to the use of two different source models. However, the trend in all figures is the same. Furthermore, it is seen that the effect of the grating lobe in figure 5.2 has dis-
Triangular grid

appeared in figure 5.3.

5.5 Array design

The analysis of a bunny-ear array in [8, p.75] has been used as a starting point to design an array with the following specifications:

1. Operating frequency: 1.2 - 1.6 GHz,
2. Scanning from broadside to $\theta = 70$ degrees for all $\phi$,
3. No grating lobes within the scan-range.

The bunny-ear dimensions and the arrangement of subdomains for the analysis is presented in figure 5.4. The approximate shape is estimated by the solid black curves in figure 5.4. The lower side of the bunny was placed 16.4 mm from the ground plane.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.4.png}
\caption{Subdomain arrangement for a bunny-ear antenna}
\end{figure}

The skewing parameter $\alpha$ was chosen such, that

$$\tan(\alpha) = \frac{b}{2a},$$

where $a$ and $b$ are the unit cell dimensions, defined in figure 5.1. This choice for $\alpha$ means that the elements in a row are shifted half an element compared to the next row. In the simulations, $a$ was taken as the design parameter and $b$ was chosen 79.9 mm.

Since, in general, the worst scan behaviour occurs for an $E$-plane scan (i.e. $\phi = 90$ degrees), the analysis was focused on the scan behaviour in this plane. Numerical results for a frequency of 1.6 GHz, in the form of the resistance and reactance, are given in figure 5.5 and 5.6, respectively. It is seen that the resistance and reactance exhibit a less extreme behaviour as $a$ is smaller. This has also been observed for even smaller $a$. However, the disadvantage is that when $a$ is smaller, more elements are needed to make a large aperture phased array. Furthermore, a good scan behaviour of the proposed array can only be expected for $\theta < 50$ degrees, which is not far enough according to the second specification. Other attempts to design an array, by varying several antenna structure parameters such as height and width, were unsuccessful, thus far.
Figure 5.5: Resistance as function of $a$ and $\theta$ for $\varphi = 90^\circ$, $f = 1.6$ GHz

Figure 5.5: Reactance as function of $a$ and $\theta$ for $\varphi = 90^\circ$, $f = 1.6$ GHz
Chapter 6

Conclusions and recommendations

In this report, models have been developed to analyse arrays of tapered slot antennas with metallic walls, dual polarization and triangular grid. The use of an equivalence plane has proved to be very useful, since it has simplified the analysis. For arrays of tapered slot antennas without metallic walls, the E-plane scan behaviour is poor, but it has been shown that E-plane scan performance can be improved by using metallic walls between the radiating elements, as long as no grating lobes enter the array. Furthermore, it was found that a grating lobe entering at end-fire can cause a blind scan angle when metallic walls are used. For dual polarization, it was shown that scan behaviour in the principal planes is promising. However, when scanning is performed in the diagonal planes, strong coupling exist between the perpendicular placed radiating elements. The triangular grid model has been used in an attempt to design a wide-band wide-angle bunny-ear array. It was observed that the scan performance of the array improved when the elements were placed closer to one another.

The models, presented in this report, can be used to further study the performance of tapered slot antenna phased arrays. First of all, it should be investigated what the minimum number of Floquet modes is that is needed for a certain unit cell aperture, since this can reduce the computation time in the case of metallic walls. Furthermore, the convergence of the double summation should be examined, since this also reduces the computation time.

The next step in modelling this type of phased arrays is bringing in a dielectric substrate on which the antennas are printed [3]. By applying a dielectric substrate, the dimensions of the radiating elements can become smaller. However, the dielectric can also cause blind scan angles. Also, the effect of a radome covering the array is an important issue, since the radome can be used as a WAIM-sheet (Wide Angle Impedance Matching sheet), to optimise the scan and frequency behaviour of an array. In [2] a method is presented to analyse the effects of a radome. Finally, the design of an “optimal” wide-band wide-angle phased array should be continued.
Appendix A

Metallic walls

A1: Components of the dyadic Green’s functions

The dyadic Green's functions have been defined in Chapter 3 (Eq(3.51)). The components of these functions can be calculated by using the equations for the electric and magnetic field, in terms of the electric and magnetic vector potentials.

Metallic walls parallel to the x-axis

The components of Eq(3.51) for this case are given by

\[ E_{f,n}^{in}(x,y) = \frac{1}{j\omega\varepsilon_0} \sum_{n=p=1}^{\infty} \left( k_0^2 - \left( \frac{n\pi}{b} \right)^2 \right) \left( Q_n^+ e^{-j\beta_{nx}} + Q_n^- e^{j\beta_{nx}} \right) \cos \left( \frac{n\pi y}{b} \right) \cos \left( \frac{n\pi y}{b} \right) \frac{1}{\varepsilon_n} \sin \left( \frac{p\pi z}{a} \right) \sin \left( \frac{p\pi z}{a} \right) \]

(A1)

\[ E_{f,n}^{in}(x,y) = -\frac{-1}{j\omega\varepsilon_0} \sum_{n=p=1}^{\infty} \left( \frac{pn\pi}{d} \right)^2 \left( Q_n^+ e^{-j\beta_{nx}} + Q_n^- e^{j\beta_{nx}} \right) \sin \left( \frac{n\pi y}{b} \right) \cos \left( \frac{n\pi y}{b} \right) \frac{1}{\varepsilon_n} \cos \left( \frac{p\pi z}{a} \right) \cos \left( \frac{p\pi z}{a} \right) \]

(A2)

\[ E_{f,n}^{in}(x,y) = -\frac{-1}{j\omega\varepsilon_0} \sum_{n=p=1}^{\infty} \left( \frac{pn\pi}{d} \right)^2 \left( Q_n^+ e^{-j\beta_{nx}} + Q_n^- e^{j\beta_{nx}} \right) \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{n\pi y}{b} \right) \frac{1}{\varepsilon_n} \cos \left( \frac{p\pi z}{a} \right) \cos \left( \frac{p\pi z}{a} \right) \]

(A3)

\[ E_{f,n}^{in}(x,y) = \frac{1}{j\omega\varepsilon_0} \sum_{n=p=1}^{\infty} \left( k_0^2 - \left( \frac{p\pi}{d} \right)^2 \right) \left( Q_n^+ e^{-j\beta_{nx}} + Q_n^- e^{j\beta_{nx}} \right) \sin \left( \frac{n\pi y}{b} \right) \sin \left( \frac{n\pi y}{b} \right) \frac{1}{\varepsilon_n} \cos \left( \frac{p\pi z}{a} \right) \cos \left( \frac{p\pi z}{a} \right) \]

(A4)

\[ E_{o,n}^{in}(x,y) = -\frac{-2}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} e^{-jkn(x-x)} \cos \left( \frac{n\pi y}{b} \right) \cos \left( \frac{n\pi y}{b} \right) \frac{1}{\varepsilon_n} \sin \left( \frac{kwx}{m} (z + d) \right) \sin \left( \frac{kwx}{m} (z + d) \right) \]

(A5)

\[ E_{o,n}^{in}(x,y) = 0 \]

(A6)

\[ E_{o,n}^{in}(x,y) = -\frac{2}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \left( \frac{n\pi}{b} \right)^2 e^{-jkn(x-x)} \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{n\pi y}{b} \right) \cos \left( \frac{kwx}{m} (z + d) \right) \]

\[ \cos \left( \frac{kwx}{m} (z + d) \right) \]

\[ \frac{kwx}{m} \sin \left( \frac{kwx}{m} (z + d) \right) \]

(A7)

\[ E_{o,n}^{in}(x,y) = -\frac{2}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \left( jk_m(x-x) \right) e^{-jkn(x-x)} \sin \left( \frac{n\pi y}{b} \right) \sin \left( \frac{n\pi y}{b} \right) \cos \left( \frac{kwx}{m} (z + d) \right) \]

\[ \cos \left( \frac{kwx}{m} (z + d) \right) \]

\[ \frac{kwx}{m} \sin \left( \frac{kwx}{m} (z + d) \right) \]

(A8)
Metallic walls

\[ H_{x,y}^{\text{in}} = -\sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \left( \frac{\alpha_{np}}{d} \right) \left( Q_{np}^+ e^{-\beta_{np} y} + Q_{np}^- e^{\beta_{np} y} \right) \cos \left( \frac{n\pi y}{b} \right) \cos \left( \frac{n\pi y}{b} \right) \frac{1}{\varepsilon} \sin \left( \frac{b\pi z}{d} \right) \cos \left( \frac{p\pi z}{d} \right) \] (A9)

\[ H_{z,z}^{\text{in}} = \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \left( \frac{n\pi}{b} \right) \left( Q_{np}^+ e^{-\beta_{np} y} + Q_{np}^- e^{\beta_{np} y} \right) \sin \left( \frac{n\pi y}{b} \right) \cos \left( \frac{n\pi y}{b} \right) \frac{1}{\varepsilon} \cos \left( \frac{p\pi z}{d} \right) \cos \left( \frac{p\pi z}{d} \right) \] (A10)

\[ H_{z,z}^{\text{in}} = 0 \] (A11)

\[ H_{x,z}^{\text{in}} = -\sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \left( -j\beta_{np} Q_{np}^+ e^{-\beta_{np} y} + j\beta_{np} Q_{np}^- e^{\beta_{np} y} \right) \sin \left( \frac{n\pi y}{b} \right) \sin \left( \frac{n\pi y}{b} \right) \frac{1}{\varepsilon} \cos \left( \frac{k_{wn} x}{k_{mn} d} \right) \cos \left( \frac{k_{wn} x}{k_{mn} d} \right) \] (A12)

\[ H_{x,x}^{\text{in}} = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left( \frac{j k_m m \pi}{b} \right) e^{-j k_m z} (x-x') \sin \left( \frac{n\pi y}{b} \right) \cos \left( \frac{n\pi y}{b} \right) \frac{1}{\varepsilon} \cos \left( \frac{k_{wn} x}{k_{mn} d} \right) \cos \left( \frac{k_{wn} x}{k_{mn} d} \right) \] (A13)

\[ H_{y,x}^{\text{in}} = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left( \frac{j k_m m \pi}{b} \right) e^{-j k_m z} (x-x') \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{n\pi y}{b} \right) \frac{1}{\varepsilon} \cos \left( \frac{k_{wn} x}{k_{mn} d} \right) \cos \left( \frac{k_{wn} x}{k_{mn} d} \right) \] (A14)

\[ H_{y,y}^{\text{in}} = -2 \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left( \frac{j k_m m \pi}{b} \right) e^{-j k_m z} (x-x') \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{n\pi y}{b} \right) \frac{1}{\varepsilon} \cos \left( \frac{k_{wn} x}{k_{mn} d} \right) \cos \left( \frac{k_{wn} x}{k_{mn} d} \right) \] (A15)

\[ H_{y,y}^{\text{in}} = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left( \frac{j k_m m \pi}{b} \right) e^{-j k_m z} (x-x') \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{n\pi y}{b} \right) \frac{1}{\varepsilon} \cos \left( \frac{k_{wn} x}{k_{mn} d} \right) \cos \left( \frac{k_{wn} x}{k_{mn} d} \right) \] (A16)

\[ H_{x,x}^{\text{in}} = \frac{1}{j \omega_0 \mu_0 b} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( \frac{k_n^2 - (k_m^2 \omega^2)}{j k_m} \right) e^{-j k_n z} (x-x') e^{-j k_n y} e^{-j k_n z} \] (A17)

\[ H_{x,y}^{\text{in}} = \frac{1}{j \omega_0 \mu_0 b} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( \frac{k_m k_n}{j k_m} \right) e^{-j k_n z} (x-x') e^{-j k_n y} e^{-j k_n z} \] (A18)

\[ H_{y,x}^{\text{in}} = \frac{1}{j \omega_0 \mu_0 b} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( \frac{k_m k_n}{j k_m} \right) e^{-j k_n z} (x-x') e^{-j k_n y} e^{-j k_n z} \] (A19)

\[ H_{y,y}^{\text{in}} = \frac{1}{j \omega_0 \mu_0 b} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( \frac{k_n^2 - (k_m^2 \omega^2)}{j k_m} \right) e^{-j k_n z} (x-x') e^{-j k_n y} e^{-j k_n z} \] (A20)
Metallic walls parallel to the y-axis

The components of Eq(3.51) for this case are given by

\[ E_{f, in}^{x y} = \frac{1}{j \omega \varepsilon_0} \sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} (k_y^2 - (k_x^n)^2) \left( Q_{np}^+ e^{-j \beta_{nr} x} + Q_{np}^- e^{j \beta_{nr} x} + Q_{np}^* e^{-j \beta_{nr} |x|} \right) e^{-j k_x^n (y - y')} \sin \left( \frac{p \pi x}{d} \right) \sin \left( \frac{p \pi y}{d} \right) \]

(A21)

\[ E_{f, in}^{y z} = \frac{1}{j \omega \varepsilon_0} \sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} (j k_y^n \pi d) \left( Q_{np}^+ e^{-j \beta_{nr} x} + Q_{np}^- e^{j \beta_{nr} x} + Q_{np}^* e^{-j \beta_{nr} |x|} \right) e^{-j k_x^n (y - y')} \cos \left( \frac{p \pi x}{d} \right) \sin \left( \frac{p \pi y}{d} \right) \]

(A22)

\[ E_{f, in}^{z z} = \frac{1}{j \omega \varepsilon_0} \sum_{n=-\infty}^{\infty} \sum_{p=0}^{\infty} (k_y^2 - \left( \frac{\pi}{d} \right)^2) \left( Q_{np}^+ e^{-j \beta_{nr} x} + Q_{np}^- e^{j \beta_{nr} x} + Q_{np}^* e^{-j \beta_{nr} |x|} \right) e^{-j k_x^n (y - y')} \cos \left( \frac{p \pi x}{d} \right) \cos \left( \frac{p \pi y}{d} \right) \frac{1}{e_p} \]

(A23)

\[ E_{u, in}^{x x} = \frac{2}{d^2} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{m \pi x}{a} \right) e^{-j k_x^n (y - y')} \frac{\sin (k_{mn}^y (z + d))}{\sin (k_{mn}^y d)} \]

(A25)

(A26)

\[ E_{u, in}^{x z} = \frac{2}{d^2} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} (j k_x^n) \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{m \pi x}{a} \right) e^{-j k_x^n (y - y')} \cos \left( \frac{k_{mn}^y (z + d)}{k_{mn}^y d} \right) \]

(A27)

\[ E_{u, in}^{y y} = \frac{2}{d^2} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \left( \frac{m \pi}{a} \right) \cos \left( \frac{m \pi x}{a} \right) \sin \left( \frac{m \pi x}{a} \right) e^{-j k_x^n (y - y')} \cos \left( \frac{k_{mn}^y (z + d)}{k_{mn}^y d} \right) \]

(A28)

\[ H_{f, in}^{x y} = -\sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} \left( \frac{p \pi}{d} \right) \left( Q_{np}^+ e^{-j \beta_{nr} x} + Q_{np}^- e^{j \beta_{nr} x} + Q_{np}^* e^{-j \beta_{nr} |x|} \right) e^{-j k_x^n (y - y')} \sin \left( \frac{p \pi x}{d} \right) \cos \left( \frac{p \pi y}{d} \right) \]

(A29)

\[ H_{f, in}^{z z} = -\sum_{n=-\infty}^{\infty} \sum_{p=0}^{\infty} (j k_x^n) \left( Q_{np}^+ e^{j \beta_{nr} x} + Q_{np}^- e^{j \beta_{nr} x} + Q_{np}^* e^{-j \beta_{nr} |x|} \right) e^{-j k_x^n (y - y')} \cos \left( \frac{p \pi x}{d} \right) \cos \left( \frac{p \pi y}{d} \right) \frac{1}{e_p} \]

(A30)

\[ H_{f, in}^{y y} = 0 \]

(A31)
Metallic walls parallel to both x- and y-axis

The components of Eq(3.51) for this case are given by

\begin{align*}
E_{yy}^{in} &= \sum_{n=0}^\infty \sum_{p=0}^\infty \left( -j\beta_{np} Q_{np} e^{-j\beta_{np} x} + j\beta_{np} Q_{np} e^{-j\beta_{np} x} - j\beta_{np} Q_{np} e^{-j\beta_{np} x} \right) \\
&\cdot e^{-jk_{mn} \left( y - y' \right)} \cos \left( \frac{n\pi y}{b} \right) \cos \left( \frac{n\pi y}{b} \right) \frac{1}{E_n} (A32)
\end{align*}

\begin{align*}
H_{yy}^{in} &= \sum_{m=0}^\infty \sum_{n=0}^\infty \left( k_0^2 - (k_0)^2 \right) \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{m\pi x}{a} \right) e^{-jk_{mn} \left( y - y' \right)} \\
&\cdot \left( \cos \left( k_{mn} (z + d) \right) \cos \left( k_{mn} (z + d) \right) \right) \frac{k_{mn} \sin \left( k_{mn} d \right)}{k_{mn} \sin \left( k_{mn} d \right)} (A33)
\end{align*}

\begin{align*}
H_{yz}^{in} &= \sum_{m=0}^\infty \sum_{n=0}^\infty \left( k_0^2 - (k_0)^2 \right) \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{m\pi x}{a} \right) e^{-jk_{mn} \left( y - y' \right)} \\
&\cdot \left( \cos \left( k_{mn} (z + d) \right) \cos \left( k_{mn} (z + d) \right) \right) \frac{k_{mn} \sin \left( k_{mn} d \right)}{k_{mn} \sin \left( k_{mn} d \right)} (A34)
\end{align*}

\begin{align*}
H_{zx}^{in} &= \sum_{m=0}^\infty \sum_{n=0}^\infty \left( k_0^2 - (k_0)^2 \right) \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{m\pi x}{a} \right) e^{-jk_{mn} \left( y - y' \right)} \\
&\cdot \left( \cos \left( k_{mn} (z + d) \right) \cos \left( k_{mn} (z + d) \right) \right) \frac{k_{mn} \sin \left( k_{mn} d \right)}{k_{mn} \sin \left( k_{mn} d \right)} (A35)
\end{align*}

\begin{align*}
H_{xy}^{in} &= \sum_{m=0}^\infty \sum_{n=0}^\infty \left( k_0^2 - (k_0)^2 \right) \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{m\pi x}{a} \right) e^{-jk_{mn} \left( y - y' \right)} \\
&\cdot \left( \cos \left( k_{mn} (z + d) \right) \cos \left( k_{mn} (z + d) \right) \right) \frac{k_{mn} \sin \left( k_{mn} d \right)}{k_{mn} \sin \left( k_{mn} d \right)} (A36)
\end{align*}

\begin{align*}
U_{xx}^{in} &= \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(k_0^2 - (k_0)^2)}{jk_{mn}} e^{-jk_{mn} \left( x - x' \right)} e^{-jk_{mn} \left( y - y' \right)} e^{-jk_{mn} \left( z - z' \right)} (A37)
\end{align*}

\begin{align*}
U_{xy}^{in} &= \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(k_0^2 - (k_0)^2)}{jk_{mn}} e^{-jk_{mn} \left( x - x' \right)} e^{-jk_{mn} \left( y - y' \right)} e^{-jk_{mn} \left( z - z' \right)} (A38)
\end{align*}

\begin{align*}
U_{yx}^{in} &= \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(k_0^2 - (k_0)^2)}{jk_{mn}} e^{-jk_{mn} \left( x - x' \right)} e^{-jk_{mn} \left( y - y' \right)} e^{-jk_{mn} \left( z - z' \right)} (A39)
\end{align*}

\begin{align*}
U_{yy}^{in} &= \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(k_0^2 - (k_0)^2)}{jk_{mn}} e^{-jk_{mn} \left( x - x' \right)} e^{-jk_{mn} \left( y - y' \right)} e^{-jk_{mn} \left( z - z' \right)} (A40)
\end{align*}
\[ E_{f,in}^{x} = \frac{-1}{j\omega e_{o,n}} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \left( \frac{p\pi n}{b} \right) \left( Q_{np}^{e^{-j\beta_{np}x}} + Q_{np}^{e^{j\beta_{np}x}} + Q_{np}^{e^{-j\beta_{np}x}} \right) \] 
\[ \cdot \sin\left( \frac{n\pi y}{b} \right) \cos\left( \frac{p\pi x}{d} \right) \sin\left( \frac{p\pi z}{d} \right) \] 
\[ (A42) \]

\[ E_{f,in}^{xy} = \frac{-1}{j\omega e_{o,n}} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \left( \frac{p\pi n}{d} \right) \left( Q_{np}^{e^{-j\beta_{np}x}} + Q_{np}^{e^{j\beta_{np}x}} + Q_{np}^{e^{-j\beta_{np}x}} \right) \] 
\[ \cdot \cos\left( \frac{n\pi y}{b} \right) \sin\left( \frac{n\pi y}{b} \right) \sin\left( \frac{p\pi z}{d} \right) \cos\left( \frac{p\pi z}{d} \right) \] 
\[ (A43) \]

\[ E_{f,in}^{yz} = \frac{1}{j\omega e_{o,n}} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \left( \frac{p\pi n}{a} \right) \left( Q_{np}^{e^{-j\beta_{np}x}} + Q_{np}^{e^{j\beta_{np}x}} + Q_{np}^{e^{-j\beta_{np}x}} \right) \] 
\[ \cdot \sin\left( \frac{n\pi y}{b} \right) \sin\left( \frac{n\pi y}{b} \right) \cos\left( \frac{p\pi z}{d} \right) \cos\left( \frac{p\pi z}{d} \right) \] 
\[ (A44) \]

\[ E_{a,in}^{yx} = \frac{-4}{ab} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left( \frac{m\pi x}{a} \right) \left( \frac{m\pi x}{a} \right) \cos\left( \frac{n\pi y}{b} \right) \cos\left( \frac{n\pi y}{b} \right) \] 
\[ \cdot \sin\left( \frac{k_{mn}w_{xy}(z+d)}{k_{mn}w_{xy}} \right) \sin\left( \frac{k_{mn}w_{xy}(d)}{k_{mn}w_{xy}} \right) \] 
\[ (A45) \]

\[ E_{a,in}^{yy} = 0 \] 
\[ (A46) \]

\[ E_{a,in}^{zx} = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{m\pi x}{a} \right) \cos\left( \frac{m\pi x}{a} \right) \sin\left( \frac{n\pi y}{b} \right) \sin\left( \frac{n\pi y}{b} \right) \] 
\[ \cdot \cos\left( \frac{k_{mn}w_{xy}(z+d)}{k_{mn}w_{xy}} \right) \sin\left( \frac{k_{mn}w_{xy}(d)}{k_{mn}w_{xy}} \right) \] 
\[ (A47) \]

\[ E_{a,in}^{zy} = \frac{-4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{m\pi x}{a} \right) \cos\left( \frac{m\pi x}{a} \right) \sin\left( \frac{n\pi y}{b} \right) \sin\left( \frac{n\pi y}{b} \right) \] 
\[ \cdot \cos\left( \frac{k_{mn}w_{xy}(z+d)}{k_{mn}w_{xy}} \right) \sin\left( \frac{k_{mn}w_{xy}(d)}{k_{mn}w_{xy}} \right) \] 
\[ (A48) \]

\[ H_{f,in}^{xz} = -\sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \left( \frac{p\pi n}{a} \right) \left( Q_{np}^{e^{-j\beta_{np}x}} + Q_{np}^{e^{j\beta_{np}x}} + Q_{np}^{e^{-j\beta_{np}x}} \right) \] 
\[ \cdot \cos\left( \frac{n\pi y}{b} \right) \cos\left( \frac{n\pi y}{b} \right) \sin\left( \frac{p\pi z}{d} \right) \cos\left( \frac{p\pi z}{d} \right) \] 
\[ (A49) \]

\[ H_{f,in}^{xy} = -\sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \left( \frac{n\pi n}{b} \right) \left( Q_{np}^{e^{-j\beta_{np}x}} + Q_{np}^{e^{j\beta_{np}x}} + Q_{np}^{e^{-j\beta_{np}x}} \right) \] 
\[ \cdot \sin\left( \frac{n\pi y}{b} \right) \cos\left( \frac{n\pi y}{b} \right) \cos\left( \frac{p\pi z}{d} \right) \] 
\[ \cdot \cos\left( \frac{p\pi z}{d} \right) \sin\left( \frac{p\pi z}{d} \right) \] 
\[ (A50) \]

\[ H_{f,in}^{yz} = 0 \] 
\[ (A51) \]
Metallic walls

\[ H_{in}^{ex} = \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \left( -j \beta_{np} Q_{np} e^{-j \beta_{np} x} + j \beta_{np} Q_{np} e^{j \beta_{np} x} - j \beta_{np} Q_{np} e^{-j \beta_{np} y} - j \beta_{np} Q_{np} e^{j \beta_{np} y} \right) \frac{1}{k_{mn}^{2}} \sin \left( \frac{n \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \cos \left( \frac{m \pi x}{a} \right) \cos \left( \frac{m \pi y}{b} \right) \frac{\cos \left( k_{wx} (z - d) \right)}{k_{wx} \sin \left( k_{wx} d \right)} \]  

(A52)

\[ H_{in}^{ex} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{i \omega \mu_0 \alpha b} \left( k_{mn}^{2} - \frac{(n \pi b)^2}{a} \right) \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{m \pi y}{a} \right) \frac{\cos \left( n \pi y \right)}{b} \frac{\cos \left( k_{wx} (z + d) \right)}{k_{wx} \sin \left( k_{wx} d \right)} \]  

(A53)

\[ H_{in}^{ex} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{i \omega \mu_0 \alpha b} \left( k_{mn}^{2} - \frac{(n \pi b)^2}{a} \right) \cos \left( \frac{m \pi x}{a} \right) \cos \left( \frac{m \pi y}{a} \right) \frac{\sin \left( n \pi y \right)}{b} \frac{\cos \left( k_{wx} (z + d) \right)}{k_{wx} \sin \left( k_{wx} d \right)} \]  

(A54)

\[ H_{in}^{ex} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{i \omega \mu_0 \alpha b} \left( k_{mn}^{2} - \frac{(n \pi b)^2}{a} \right) \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{m \pi y}{a} \right) \frac{\sin \left( n \pi y \right)}{b} \frac{\cos \left( k_{wx} (z + d) \right)}{k_{wx} \sin \left( k_{wx} d \right)} \]  

(A55)

\[ H_{in}^{ex} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{4}{i \omega \mu_0 \alpha b} \left( k_{mn}^{2} - \frac{(n \pi b)^2}{a} \right) e^{-jk_{mn} (x-x')} e^{-jk_{mn} (y-y')} e^{-jk_{mn} z} \]  

(A56)

\[ H_{ex}^{ex} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(k_{mn}^{2} - k_{mn}^{2}) e^{-jk_{mn} (x-x')} e^{-jk_{mn} (y-y')} e^{-jk_{mn} z}}{jk_{mn}} \]  

(A57)

\[ H_{ex}^{ex} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(k_{mn}^{2} e^{-jk_{mn} (x-x')} e^{-jk_{mn} (y-y')} e^{-jk_{mn} z}}{jk_{mn}} \]  

(A58)

\[ H_{ex}^{ex} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(k_{mn}^{2} e^{-jk_{mn} (x-x')} e^{-jk_{mn} (y-y')} e^{-jk_{mn} z}}{jk_{mn}} \]  

(A59)

\[ H_{ex}^{ex} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(k_{mn}^{2} e^{-jk_{mn} (x-x')} e^{-jk_{mn} (y-y')} e^{-jk_{mn} z}}{jk_{mn}} \]  

(A60)
A2: Expressions for the matrix elements

The matrix elements have been defined in Chapter 3 (Eq(3.59)). The elements can be calculated by using Eq(3.58) and the components of the dyadic Green’s function of the previous paragraph.

Metallic walls parallel to the x-axis

The matrix elements for this case are given by

\[
Z_{yy, rq}^{if} = \frac{1}{j\omega\varepsilon} \sum_{n=0}^{\infty} \frac{1}{\varepsilon_n} \left( \mathcal{O}_{np}^+ + \mathcal{O}_{np}^- \right) \left( \kappa_0^2 - \left( \frac{n\pi}{b} \right)^2 \right) \frac{1}{\varepsilon_n} 
\]

\[
\cdot \int_{S_y} J_y \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{p\pi z}{d} \right) dydz \int_{S_z} J_z \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{p\pi z}{d} \right) dydz
\]

\[
(A61)
\]

\[
Z_{yz, rq}^{if} = -\frac{1}{j\omega\varepsilon} \sum_{n=0}^{\infty} \frac{1}{\varepsilon_n} \left( \mathcal{O}_{np}^+ + \mathcal{O}_{np}^- \right) \left( \frac{n\pi y}{b} \right) \frac{1}{d} 
\]

\[
\cdot \int_{S_y} J_y \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{p\pi z}{d} \right) dydz \int_{S_z} J_z \sin \left( \frac{n\pi y}{b} \right) \cos \left( \frac{p\pi z}{d} \right) dydz
\]

\[
(A62)
\]

\[
Z_{zy, rq}^{if} = -\frac{1}{j\omega\varepsilon} \sum_{n=0}^{\infty} \frac{1}{\varepsilon_n} \left( \mathcal{O}_{np}^+ + \mathcal{O}_{np}^- \right) \left( \frac{n\pi y}{b} \right) \frac{1}{d} 
\]

\[
\cdot \int_{S_y} J_y \sin \left( \frac{n\pi y}{b} \right) \cos \left( \frac{p\pi z}{d} \right) dydz \int_{S_z} J_z \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{p\pi z}{d} \right) dydz
\]

\[
(A63)
\]

\[
Z_{zz, rq}^{if} = \frac{1}{j\omega\varepsilon} \sum_{n=0}^{\infty} \frac{1}{\varepsilon_n} \left( \mathcal{O}_{np}^+ + \mathcal{O}_{np}^- \right) \left( \kappa_0^2 - \left( \frac{n\pi}{b} \right)^2 \right) \frac{1}{\varepsilon_n} 
\]

\[
\cdot \int_{S_y} J_y \cos \left( \frac{n\pi y}{b} \right) \cos \left( \frac{p\pi z}{d} \right) dydz \int_{S_z} J_z \sin \left( \frac{n\pi y}{b} \right) \cos \left( \frac{p\pi z}{d} \right) dydz
\]

\[
(A64)
\]

\[
T_{yz, rs}^{fa} = \frac{2}{ab} \sum_{m=-\infty}^{\infty} \frac{1}{\sin \left( \frac{k_m}{b} \right) \varepsilon_a} 
\]

\[
\cdot \int_{S_y} J_y \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{k_m}{b} \right) \left( z + d \right) dydz \int_{S_z} M_s e^{ik_y x} \cos \left( \frac{n\pi y}{b} \right) dx dy
\]

\[
(A65)
\]

\[
T_{yy, rs}^{fa} = 0
\]

\[
(A66)
\]

\[
T_{xz, rs}^{fa} = \frac{2}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\varepsilon_n} \left( \frac{n\pi}{b} \right) k_m \sin \left( \frac{k_m}{b} \right) 
\]

\[
\cdot \int_{S_y} J_y \sin \left( \frac{n\pi y}{b} \right) \cos \left( \frac{k_m}{b} \right) \left( z + d \right) dydz \int_{S_z} M_s e^{ik_y x} \cos \left( \frac{n\pi y}{b} \right) dx dy
\]

\[
(A67)
\]
Metallic walls

\[ T_{xy, rs} = -\frac{2}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{j k_m^m}{k_{mn}} \] 
\[ \cdot \int_{\mathcal{S}_f} J_{xy} \sin \left( \frac{\pi m y}{b} \right) \cos \left( \frac{\pi n x}{b} (z + d) \right) \, dx \, dy \int_{\mathcal{S}_a} M_{xy} e^{j k_x^m} \sin \left( \frac{\pi n y}{b} \right) \, dx \, dy \]  
(A68)

\[ T_{ys, tq} = -\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{E_p} \] 
\[ \cdot \int_{\mathcal{S}_a} M_{ys}^w \left( Q_{np}^x e^{-j \beta_{np}^x} + Q_{np}^y e^{j \beta_{np}^y} \right) \cos \left( \frac{\pi m y}{b} \right) \, dx \, dy \int_{\mathcal{S}_f} J_{ys} \cos \left( \frac{\pi n x}{b} \right) \left( \frac{\pi x}{d} \right) \, dy \, dz \]  
(A69)

\[ T_{xs, tq} = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{E_p} \] 
\[ \cdot \int_{\mathcal{S}_a} M_{xs}^w \left( Q_{np}^x e^{-j \beta_{np}^x} + Q_{np}^y e^{j \beta_{np}^y} \right) \cos \left( \frac{\pi m y}{b} \right) \, dx \, dy \int_{\mathcal{S}_f} J_{cs} \sin \left( \frac{\pi n x}{b} \right) \cos \left( \frac{\pi x}{d} \right) \, dy \, dz \]  
(A70)

\[ T_{yy, tq} = 0 \]  
(A71)

\[ T_{yz, tq} = -\sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \frac{1}{E_p} \] 
\[ \cdot \int_{\mathcal{S}_a} M_{yz}^w \left( Q_{np}^x e^{-j \beta_{np}^x} + j \beta_{np}^y Q_{np}^y e^{j \beta_{np}^y} \right) \sin \left( \frac{\pi m y}{b} \right) \, dx \, dy \int_{\mathcal{S}_f} J_{yz} \cos \left( \frac{\pi n x}{b} \right) \cos \left( \frac{\pi x}{d} \right) \, dy \, dz \]  
(A72)

\[ Y_{xx, ts} = \frac{2}{j \omega \mu_0 \alpha b} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{E_n} \cos \left( \frac{k_{mn}^w d}{k_{mn}^w} \right) \] 
\[ \cdot \int_{\mathcal{S}_a} M_{xx} e^{-j \beta_{xx}^r} \cos \left( \frac{\pi n y}{b} \right) \, dx \, dy \int_{\mathcal{S}_a} M_{xx} e^{j \beta_{xx}^r} \cos \left( \frac{\pi n y}{b} \right) \, dx \, dy \]  
\[ + \frac{1}{j \omega \mu_0 \alpha b} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( \frac{k_{mn}^w}{k_{mn}^w} \right) \] 
\[ \cdot \int_{\mathcal{S}_a} M_{xx} e^{-j \beta_{xx}^r} e^{-j \beta_{xx}^y} \, dx \, dy \int_{\mathcal{S}_a} M_{xx} e^{j \beta_{xx}^r} e^{j \beta_{xx}^y} \, dx \, dy \]  
(A73)

\[ Y_{xy, ts} = \frac{2}{j \omega \mu_0 \alpha b} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{E_n} \cos \left( \frac{k_{mn}^w d}{k_{mn}^w} \right) \] 
\[ \cdot \int_{\mathcal{S}_a} M_{xy} e^{-j \beta_{xy}^r} \cos \left( \frac{\pi n y}{b} \right) \, dx \, dy \int_{\mathcal{S}_a} M_{xy} e^{j \beta_{xy}^r} \sin \left( \frac{\pi n y}{b} \right) \, dx \, dy \]  
\[- \frac{1}{j \omega \mu_0 \alpha b} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( \frac{k_{mn}^w}{k_{mn}^w} \right) \] 
\[ \cdot \int_{\mathcal{S}_a} M_{xy} e^{-j \beta_{xy}^r} e^{-j \beta_{xy}^y} \, dx \, dy \int_{\mathcal{S}_a} M_{xy} e^{j \beta_{xy}^r} e^{j \beta_{xy}^y} \, dx \, dy \]  
(A74)
Metallic walls parallel to the y-axis

The matrix elements for this case are given by

\[
Z_{yy, rq} = \frac{1}{j\omega \mu_0 b} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( Q_{np} e^{-j\beta_{nr} x} + Q_{np} e^{j\beta_{nr} x} + Q_{np} \right) (k_0^2 - (k_y^*)^2)
\]
\[
\cdot \left[ J_{yr} e^{-j k_y y} \sin \left( \frac{p \pi x}{d} \right) \right] dy dz \left[ J_{qr} e^{j k_y y} \sin \left( \frac{p \pi x}{d} \right) \right] dy' dz'
\]
\[
+ \frac{1}{j\omega \mu_0 b} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (k_0^2 - (k_y^*)^2)
\]
\[
\cdot \left[ J_{yr} e^{-j k_y y} \sin \left( \frac{p \pi x}{d} \right) \right] dy dz \left[ J_{qr} e^{j k_y y} \cos \left( \frac{p \pi x}{d} \right) \right] dy' dz'
\] (A77)

\[
Z_{xz, rq} = \frac{1}{j\omega \mu_0 b} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( Q_{np} e^{-j\beta_{nr} x} + Q_{np} e^{j\beta_{nr} x} + Q_{np} \right) (j k_y^*)
\]
\[
\cdot \left[ J_{xr} e^{-j k_y y} \cos \left( \frac{p \pi x}{d} \right) \right] dy dz \left[ J_{qr} e^{j k_y y} \sin \left( \frac{p \pi x}{d} \right) \right] dy' dz'
\] (A78)

\[
Z_{zz, rq} = \frac{1}{j\omega \mu_0 b} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( Q_{np} e^{-j\beta_{nr} x} + Q_{np} e^{j\beta_{nr} x} + Q_{np} \right) (j k_y^*)
\]
\[
\cdot \left[ J_{xr} e^{-j k_y y} \cos \left( \frac{p \pi x}{d} \right) \right] dy dz \left[ J_{qr} e^{j k_y y} \sin \left( \frac{p \pi x}{d} \right) \right] dy' dz'
\] (A79)

\[
Z_{xt, rq} = \frac{1}{j\omega \mu_0 b} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( Q_{np} e^{-j\beta_{nr} x} + Q_{np} e^{j\beta_{nr} x} + Q_{np} \right) \left(k_0^2 - (p \pi)^2 \right) \frac{1}{k_p}
\]
\[
\cdot \left[ J_{xr} e^{-j k_y y} \cos \left( \frac{p \pi x}{d} \right) \right] dy dz \left[ J_{qr} e^{j k_y y} \cos \left( \frac{p \pi x}{d} \right) \right] dy' dz'
\] (A80)
\[ T_{\gamma_x,rs}^{fa} = \frac{2}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\sin(k_{mn}^x d)} \sin\left(\frac{mp_{x}b}{a}\right) \left( J_{yr} \e^{-jk_{x}y} \sin\left(k_{mn}^y (z + d)\right) \right) dydz \left( M_{ys} \sin\left(\frac{mp_{x}y}{a}\right) \right) e^{jk_{y}y'} dx'dy' \] (A81)

\[ T_{\gamma_y}^{fa} = 0 \] (A82)

\[ T_{\gamma_{z},rs}^{fa} = \frac{2}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{j k_{n}^{z}}{\sin(k_{mn}^y d)} \sin\left(\frac{mp_{x}b}{a}\right) \left( J_{yr} \e^{-jk_{x}y} \cos\left(k_{mn}^y (z + d)\right) \right) dydz \left( M_{ys} \cos\left(\frac{mp_{x}y}{a}\right) \right) e^{jk_{y}y'} dx'dy' \] (A83)

\[ T_{\gamma_{z},ts}^{fa} = \frac{2}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m_{x} \cdot 1}{\sin(k_{mn}^y d)} \sin\left(\frac{mp_{x}b}{a}\right) \left( J_{yr} \e^{-jk_{x}y} \cos\left(k_{mn}^y (z + d)\right) \right) dydz \left( M_{ys} \cos\left(\frac{mp_{x}y}{a}\right) \right) e^{jk_{y}y'} dx'dy' \] (A84)

\[ T_{\gamma_{y},tq}^{af} = - \sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} \left( \frac{p_{x}^{z} \cdot 1}{d} \right) \left( J_{yr} \e^{-jk_{x}y} \sin\left(k_{mn}^y (z + d)\right) \right) dydz \left( J_{yq} \e^{jk_{y}y'} \sin\left(\frac{mp_{x}y}{a}\right) \right) dy'dz' \] (A85)

\[ T_{\gamma_{z},tq}^{af} = - \sum_{n=-\infty}^{\infty} \sum_{p=0}^{\infty} \left( \frac{j k_{n}^{z} \cdot 1}{\sin(k_{mn}^y d)} \right) \left( J_{yr} \e^{-jk_{x}y} \cos\left(k_{mn}^y (z + d)\right) \right) dydz \left( J_{yq} \e^{jk_{y}y'} \cos\left(\frac{mp_{x}y}{a}\right) \right) dy'dz' \] (A86)

\[ T_{\gamma_{y},y}^{af} = 0 \] (A87)

\[ T_{\gamma_{z},tq}^{af} = - \sum_{n=-\infty}^{\infty} \sum_{p=0}^{\infty} \left( \frac{j k_{n}^{z} \cdot 1}{\sin(k_{mn}^y d)} \right) \left( J_{yr} \e^{-jk_{x}y} \cos\left(k_{mn}^y (z + d)\right) \right) dydz \left( J_{yq} \e^{jk_{y}y'} \cos\left(\frac{mp_{x}y}{a}\right) \right) dy'dz' \] (A88)
Metallic walls
Metallic walls parallel to both x- and y-axis

The matrix elements for this case are given by

$$Z_{yy, rq}^{1f} = \frac{1}{j\omega e_0} \sum_{n} \sum_{m} \left( Q_{n} e^{-j\beta_{n} r} + Q_{n} e^{j\beta_{n} r} + Q_{n}^{*} \right) \left( k_0^2 - \left( \frac{n\pi}{b} \right)^2 \right) \frac{1}{e_n} \left[ J_{y, y} \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{p\pi z}{d} \right) dydz \right] \left[ J_{y, -y} \cos \left( \frac{n\pi y}{b} \right) \sin \left( \frac{p\pi z}{d} \right) dydz \right]$$

(A93)

$$Z_{zq, rz}^{1f} = \frac{1}{j\omega e_0} \sum_{n} \sum_{m} \left( Q_{n} e^{-j\beta_{n} r} + Q_{n} e^{j\beta_{n} r} + Q_{n}^{*} \right) \left( \frac{n\pi p r}{b} \right) \left( k_0^2 - \left( \frac{n\pi}{b} \right)^2 \right) \frac{1}{e_p} \left[ J_{z, q} \sin \left( \frac{n\pi y}{b} \right) \cos \left( \frac{p\pi z}{d} \right) dydz \right] \left[ J_{z, -q} \sin \left( \frac{n\pi y}{b} \right) \cos \left( \frac{p\pi z}{d} \right) dydz \right]$$

(A94)

$$Z_{zq, rz}^{1f} = \frac{1}{j\omega e_0} \sum_{n} \sum_{m} \left( Q_{n} e^{-j\beta_{n} r} + Q_{n} e^{j\beta_{n} r} + Q_{n}^{*} \right) \left( k_0^2 - \left( \frac{n\pi}{b} \right)^2 \right) \frac{1}{e_{n}} \left[ J_{z, q} \sin \left( \frac{n\pi y}{b} \right) \cos \left( \frac{p\pi z}{d} \right) dydz \right] \left[ J_{z, -q} \sin \left( \frac{n\pi y}{b} \right) \cos \left( \frac{p\pi z}{d} \right) dydz \right]$$

(A95)

$$T_{y, x, rs}^{1f} = \frac{4}{ab} \sum_{m} \sum_{n} \frac{1}{e_m \sin \left( k_{mn} d \right)} \sin \left( \frac{m\pi x}{a} \right) \left[ J_{y, y} \cos \left( \frac{n\pi y}{b} \right) \sin \left( k_{mn} \left( z + d, z - d \right) \right) dydz \right] \left[ M_{xs} \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) dx dy \right]$$

(A96)

$$T_{y, y, rs}^{1f} = 0$$

(A97)

$$T_{z, x, rs}^{1f} = \frac{4}{ab} \sum_{m} \sum_{n} \frac{1}{k_{mn} \sin \left( k_{mn} d \right)} \sin \left( \frac{m\pi x}{a} \right) \left[ J_{z, y} \sin \left( \frac{n\pi y}{b} \right) \cos \left( k_{mn} \left( z + d, z - d \right) \right) dydz \right] \left[ M_{ys} \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) dx dy \right]$$

(A98)

$$T_{z, y, rs}^{1f} = \frac{4}{ab} \sum_{m} \sum_{n} \frac{1}{k_{mn} \sin \left( k_{mn} d \right)} \sin \left( \frac{m\pi x}{a} \right) \left[ J_{z, y} \sin \left( \frac{n\pi y}{b} \right) \cos \left( k_{mn} \left( z + d, z - d \right) \right) dydz \right] \left[ M_{ys} \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) dx dy \right]$$

(A99)
\[ T_{xy,tq} = -\sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \left( \frac{p \pi \nu}{d} \right) \int_{S_\nu} M_{x_1}^n \left[ Q_{xy} e^{-i\beta_{xy}^x} + Q_{xy} e^{i\beta_{xy}^y} + Q_{xy}^* e^{-i\beta_{xy}^x} + Q_{xy}^* e^{i\beta_{xy}^y} \right] \cos \left( \frac{n \pi y}{b} \right) \, dx \, dy \]
\[ + \int_{S_\nu} J_{tq} \cos \left( \frac{n \pi y}{b} \right) \sin \left( \frac{p \pi \nu}{d} \right) \, dy \, dz' \]  
\hspace{1cm} (A101)

\[ T_{yx,tq} = \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \left( \frac{n \pi \nu}{b} \right) \int_{S_\nu} M_{x_1}^n \left[ -j \beta_{xy} e^{-i\beta_{xy}^x} + j \beta_{xy} e^{i\beta_{xy}^y} - j \beta_{xy} e^{-i\beta_{xy}^x} + j \beta_{xy} e^{i\beta_{xy}^y} \right] \sin \left( \frac{n \pi y}{b} \right) \, dx \, dy \]
\[ + \int_{S_\nu} J_{tq} \sin \left( \frac{n \pi y}{b} \right) \cos \left( \frac{p \pi \nu}{d} \right) \, dy \, dz' \]  
\hspace{1cm} (A102)

\[ T_{yy,tq} = 0 \]  
\hspace{1cm} (A103)

\[ T_{xy,ty} = -\sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \left( \frac{p \pi \nu}{d} \right) \int_{S_\nu} M_{x_1}^n \left[ -j \beta_{xy} e^{-i\beta_{xy}^x} + j \beta_{xy} e^{i\beta_{xy}^y} - j \beta_{xy} e^{-i\beta_{xy}^x} + j \beta_{xy} e^{i\beta_{xy}^y} \right] \sin \left( \frac{n \pi y}{b} \right) \, dx \, dy \]
\[ + \int_{S_\nu} J_{tq} \sin \left( \frac{n \pi y}{b} \right) \cos \left( \frac{p \pi \nu}{d} \right) \, dy \, dz' \]  
\hspace{1cm} (A104)

\[ \gamma_{xx,tx} = -4j \omega_{ab} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{\cos \left( k w x y d \right)}{k w x y a} \left( k_0^2 - \left( \frac{m \pi}{a} \right)^2 \right) \]
\[ \cdot \int_{S_\nu} M_{x_1}^n \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \, dx \, dy \int_{S_\nu} M_{x_1}^n \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \, dx' \, dy' \]
\[ + \frac{1}{j \omega_{ab}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(k_0^2 - (k_n^2)^2)}{j k_{mn}} \]
\[ \cdot \int_{S_\nu} M_{x_1}^n e^{-j k_{x} x} e^{-j k_{y} y} \, dx \, dy \int_{S_\nu} M_{x_1}^n e^{j k_{x} x} e^{j k_{y} y} \, dx' \, dy' \]  
\hspace{1cm} (A105)

\[ \gamma_{yy,ty} = -4j \omega_{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\cos \left( k w x y d \right)}{k w x y a} \left( \frac{m \pi n \pi}{a} \right) \]
\[ \cdot \int_{S_\nu} M_{x_1}^n \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \, dx \, dy \int_{S_\nu} M_{y_1}^n \cos \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \, dx' \, dy' \]
\[ - \frac{1}{j \omega_{ab}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(k_m^2 k_n^2)}{j k_{mn}} \]
\[ \cdot \int_{S_\nu} M_{x_1}^n e^{-j k_{x} x} e^{-j k_{y} y} \, dx \, dy \int_{S_\nu} M_{y_1}^n e^{j k_{x} x} e^{j k_{y} y} \, dx' \, dy' \]  
\hspace{1cm} (A106)
Metallic walls

\[
\gamma_{yy,ts} = \frac{4}{j\omega \mu_0 ab} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \cos \left(\frac{k_{mn} \pi}{a} y \right) \sin \left(\frac{\pi a}{b} x \right) \int_{S_s} \left[ M_y \cos \left(\frac{m \pi x}{a}\right) \sin \left(\frac{n \pi y}{b}\right) \right] dx dy
\]

\[
- \frac{1}{j\omega \mu_0 ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left(\frac{k_m^2 - (\frac{n \pi}{b})^2}{jk_{mn}}\right) \cdot \int_{S_s} \left[ M_y e^{-j k_m^2 x} e^{-j k_n^2 y} \right] dx dy
\]

(A107)

\[
\gamma_{yy,ts} = \frac{4}{j\omega \mu_0 ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos \left(\frac{k_{mn} \pi}{a} y \right) \sin \left(\frac{\pi a}{b} x \right) \int_{S_s} \left[ M_y \cos \left(\frac{m \pi x}{a}\right) \sin \left(\frac{n \pi y}{b}\right) \right] dx dy
\]

\[
+ \frac{1}{j\omega \mu_0 ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left(\frac{k_m^2 - (\frac{n \pi}{b})^2}{jk_{mn}}\right) \cdot \int_{S_s} \left[ M_y e^{-j k_m^2 x} e^{-j k_n^2 y} \right] dx dy
\]

(A108)
A3: Closed form expressions for the integrals

The integrals obtained from the method of moments have been calculated in closed form and can be written as a combination of the integrals derived in this paragraph. The integrals have lower indices consisting of a ranking number and the integration variable, separated by a comma. Most of the integrals also have upper indices, which describe the situation in which they are applied: wy means that there are walls parallel to the y-axis (and possibly to the x-axis as well), nwy means there are no walls parallel to the y-axis, wx means there are walls parallel to the x-axis (and possibly also walls parallel to the y-axis), nwx means there are no walls parallel to the x-axis and wxy means there are walls parallel to both x- and y-axis. When no upper index is present, the integral applies to all situations.

\[ I_{1,x}^{wy}(m, m) = \int_0^a \left( \frac{m\pi x}{a} \right) e^{j(\frac{m\pi x}{a})} \frac{(-1)^m e^{j\pi y} - 1}{(k_x^m)^2 - (\frac{m\pi}{a})^2} \, dx \quad (A109) \]

\[ I_{1,y}^{wy}(m, m) = \int_0^a \left( \frac{m\pi x}{a} \right) e^{j(\frac{m\pi x}{a})} \frac{(-1)^m e^{j\pi y} - 1}{(k_x^m)^2 - (\frac{m\pi}{a})^2} \, dx \quad (A110) \]

\[ I_{2,x}^{wy}(m, m) = \int_0^a \left( \frac{m\pi x}{a} \right) e^{j(\frac{m\pi x}{a})} \frac{-jk_x^m(-1)^m e^{j\pi y} - 1}{(k_x^m)^2 - (\frac{m\pi}{a})^2} \, dx \quad (A111) \]

\[ I_{2,y}^{wy}(m, m) = \int_0^a \left( \frac{m\pi x}{a} \right) e^{j(\frac{m\pi x}{a})} \frac{jk_x^m(-1)^m e^{j\pi y} - 1}{(k_x^m)^2 - (\frac{m\pi}{a})^2} \, dx \quad (A112) \]

\[ I_{3,x}^{wy}(m, n, p) = \int_0^a \left( Q_{np} e^{-j\beta_{np} x} + Q_{np} e^{j\beta_{np} x} \right) e^{j(\frac{m\pi x}{a})} \, dx = \frac{-4}{bd(\beta_{np}^2 - (k_x^m)^2)} \quad (A113) \]

\[ I_{3,y}^{wy}(m, n, p) = \int_0^a \left( Q_{np} e^{-j\beta_{np} x} + Q_{np} e^{j\beta_{np} x} \right) e^{j(\frac{m\pi x}{a})} \, dx \]

\[ = \frac{2(e^{j\pi y} \sin(\beta_{np} x) - \sin(\beta_{np} x))}{bd \sin(\beta_{np} a)(\beta_{np}^2 - (k_x^m)^2)} \cdot \begin{cases} 1 & \text{for walls parallel to the y-axis} \\ 2 & \text{for walls parallel to x- and y-axis} \end{cases} \quad (A114) \]

\[ I_{4,x}^{wy}(m, n, p) = \int_0^a \left( -j\beta_{np} Q_{np} e^{-j\beta_{np} x} + j\beta_{np} Q_{np} e^{j\beta_{np} x} \right) e^{j(\frac{m\pi x}{a})} \, dx = \frac{4jk_x^m}{bd(\beta_{np}^2 - (k_x^m)^2)} \quad (A115) \]
Metallic walls

\[ I_{\alpha \beta}^{\gamma}(m, n, p) = \int \left[ -j\beta_{\alpha} e^{j\beta_{\alpha} x} + j\beta_{\beta} e^{j\beta_{\beta} x} - j\beta_{\alpha} e^{j\beta_{\alpha} x} - j\beta_{\beta} e^{j\beta_{\beta} x} \right] e^{j \frac{\pi}{b} (x - n^2) \text{sgn}(x - x^b)} e^{j \frac{\pi}{b} (x - x^b)} dx \]

\[ = -2j \beta_{\alpha} \left( e^{j \frac{\pi}{b} (x - x^b)} - e^{-j \frac{\pi}{b} (x - x^b)} \right) \]

\[ \frac{b \sin (\beta_{\alpha} a)}{\beta_{\alpha} a} \left[ \beta_{\alpha} a \left( \frac{\beta_{\alpha} a}{k_{\alpha}} \right)^2 \right] \]

\[ I_{\gamma}^{\gamma}(x, W, y) = \int \cos \left( \frac{n \pi y}{b} \right) \sin \left( \frac{n \pi y}{b} \right) \frac{1}{W \sin (k_{\gamma} W)} dy \]

\[ = \frac{2k_{\gamma} \cos \left( \frac{n \pi y}{b} \right) \cos \left( \frac{n \pi y}{b} \right) - \cos \left( k_{\gamma} W \right)}{W \sin (k_{\gamma} W)} \]

(A116)

(A117)

\[ I_{\gamma}^{\gamma}(x, W, y) = \int \cos \left( \frac{n \pi y}{b} \right) \sin \left( \frac{n \pi y}{b} \right) \frac{1}{W \sin (k_{\gamma} W)} dy + \int \cos \left( \frac{n \pi y}{b} \right) \sin \left( \frac{n \pi y}{b} \right) \frac{1}{W \sin (k_{\gamma} W)} dy \]

\[ = \frac{1}{W \sin (k_{\gamma} W)} \left[ \cos \left( \frac{n \pi y}{b} \right) \cos \left( \frac{n \pi y}{b} \right) \right] \]

\[ + \frac{1}{W \sin (k_{\gamma} W)} \left[ \cos \left( \frac{n \pi y}{b} \right) \cos \left( \frac{n \pi y}{b} \right) \right] \]

(A118)

(A119)

\[ I_{\gamma}^{\gamma}(x, W, y) = \int \cos \left( \frac{n \pi y}{b} \right) \sin \left( \frac{n \pi y}{b} \right) \frac{1}{W \sin (k_{\gamma} W)} dy + \int \cos \left( \frac{n \pi y}{b} \right) \sin \left( \frac{n \pi y}{b} \right) \frac{1}{W \sin (k_{\gamma} W)} dy \]

\[ = \frac{1}{W \sin (k_{\gamma} W)} \left[ \cos \left( \frac{n \pi y}{b} \right) \cos \left( \frac{n \pi y}{b} \right) \right] \]

\[ + \frac{1}{W \sin (k_{\gamma} W)} \left[ \cos \left( \frac{n \pi y}{b} \right) \cos \left( \frac{n \pi y}{b} \right) \right] \]

(A120)

(A121)

(A122)
Metallic walls

\[ I_{1, y}^{Wx} (y_e, n) = \int_{y_e - W_{y}}^{y_e + W_{y}} e^{jk_{y}y} \frac{\sin k_{x}(W_{y} - |y - y_{e}|)}{W_{y} \sin (k_{x} W_{y})} \sin (\frac{\pi n}{b}) \sin (\frac{\pi n y}{b}) \]  

\[ I_{2, y}^{Wx} (y_{e+1}, n) = \int_{y_{e+1} - W_{y}}^{y_{e+1} + W_{y}} e^{jk_{y}y} \frac{\sin k_{x}(W_{y} - |y - y_{e+1}|)}{W_{y} \sin (k_{x} W_{y})} \sin (\frac{\pi n}{b}) \sin (\frac{\pi n y}{b}) \]  

\[ I_{3a, y}^{Wx} (n, n) = \int_{0}^{b} \sin (\frac{\pi n y}{b}) e^{jk_{y}y} \sin (\frac{\pi n}{b}) \sin (\frac{\pi n y}{b}) \]  

\[ I_{3b, y}^{Wx} (n, n) = \int_{0}^{b} \sin (\frac{\pi n y}{b}) e^{-jk_{y}y} \sin (\frac{\pi n}{b}) \sin (\frac{\pi n y}{b}) \]  

\[ I_{4a, y}^{Wx} (n, n) = \int_{0}^{b} \cos (\frac{\pi n y}{b}) e^{jk_{y}y} \sin (\frac{\pi n}{b}) \sin (\frac{\pi n y}{b}) \]  

\[ I_{4b, y}^{Wx} (n, n) = \int_{0}^{b} \cos (\frac{\pi n y}{b}) e^{-jk_{y}y} \sin (\frac{\pi n}{b}) \sin (\frac{\pi n y}{b}) \]  

\[ I_{1, y} (z_{p}, p) = \int_{z_{1} - W_{z}}^{z_{1} + W_{z}} \cos \left( \frac{p \pi z}{d} \right) \sin k_{x}(W_{z} - |z - z_{1}|) \sin (\frac{\pi n}{d}) \sin (\frac{\pi n y}{d}) \]  

\[ I_{2, y} (z_{p}, p) = \int_{z_{1} - W_{z}}^{z_{1} + W_{z}} \sin \left( \frac{p \pi z}{d} \right) \sin (\frac{\pi n}{d}) \sin (\frac{\pi n y}{d}) \]
Furthermore, use can be made of the orthogonal relations for the Floquet modes:

\[ a \int_{-\infty}^{\infty} e^{j k_{x} x} e^{-j k'_{x} x} dx = \alpha \delta_{m m'} \]  
\[ b \int_{-\infty}^{\infty} e^{j k_{y} y} e^{-j k'_{y} y} dy = b \delta_{n n'} \]
A4: Expressions for the matrix elements in closed form

The matrix elements of appendix A2 can be written in the closed form by using the integrals defined in the previous paragraph.

Metallic walls parallel to the x-axis

The matrix elements for this case are given by

\[ Z_{yy, rq} = \frac{1}{j\omega} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \left( \Omega_{np}^+ + \Omega_{np}^- \right) \left( k_0^2 - \left( \frac{n\pi}{b} \right)^2 \right) \frac{1}{E_n} t_{1, y}^{zw}(y_k, n) l_{2, z}(z_l, p) l_{1, y}^{zw}(y_k, n) l_{2, z}(z_l, p) \]  
(A141)

\[ Z_{yx, rq} = \frac{1}{j\omega} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \left( \Omega_{np}^+ + \Omega_{np}^- \right) \left( \frac{n\pi p \pi}{b} \right) \frac{1}{E_p} t_{1, y}^{zw}(y_k, n) l_{2, z}(z_l, p) l_{1, y}^{zw}(y_k, n) l_{2, z}(z_l, p) \]  
(A142)

\[ Z_{zx, rq} = \frac{1}{j\omega} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \left( \Omega_{np}^+ + \Omega_{np}^- \right) \left( \frac{np \pi \pi}{b} \right) \frac{1}{E_p} t_{1, y}^{zw}(y_k, n) l_{2, z}(z_l, p) l_{1, y}^{zw}(y_k, n) l_{2, z}(z_l, p) \]  
(A143)

\[ T_{xa, s} = \sum_{n=0}^{\infty} \frac{1}{\sin \left( k_{m_n} d \right)} \frac{1}{E_n} t_{1, y}^{ws}(y_k, n) l_{2, y}^{ws}(z_l, m_n, n) l_{4, y}^{ws}(n, n, n) \]  
(A145)

\[ T_{xa, s} = 0 \]  
(A146)

\[ T_{ya, s} = 0 \]  
(A147)

\[ T_{ya, s} = 0 \]  
(A148)

\[ T_{xa, s} = 0 \]  
(A149)

\[ T_{ya, s} = 0 \]  
(A150)

\[ T_{xa, s} = 0 \]  
(A151)
\[ \tau_{y_{z, t}} = - \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \frac{1}{e^{n, p}} f^{nw}(n_r, n_\nu) f^{wz}_{3a, y}(n_r, n) I_{2, y}^{x}(y_{k, q}, n) I_{1, a}^{x}(z_{l, p}) \]  
(A152)

\[ y_{x_{s, t}} = \frac{2a}{jw \mu_0 b} \sum_{n=0}^{\infty} \frac{1}{e^{n, p}} \cos \left( \frac{k^{w_{x}}}{m_r} d \right) \left( k_x^2 - \left( \frac{k_y^m}{m_r} \right)^2 \right) \delta_{m_r, m_{a, y}} f^{w_{x}}(n_r, n) f^{w_{y}}(n_s, n) \]  
\[ + \frac{ab}{jw \mu_0} \frac{k_y^m}{m_r} \delta_{m_r, m_{n, n}} \]  
(A153)

\[ y_{x_{s, t}} = \frac{2a}{jw \mu_0 b} \sum_{n=1}^{\infty} \frac{1}{e^{n, p}} \sin \left( \frac{k^{w_{x}}}{m_r} d \right) \left( jk_x m_{n, n} \frac{n \pi}{b} \right) \delta_{m_r, m_{a, y}} f^{w_{x}}(n_r, n) f^{w_{y}}(n_s, n) \]  
(A154)

\[ y_{y_{s, t}} = - \frac{2a}{jw \mu_0 b} \sum_{n=1}^{\infty} \frac{1}{e^{n, p}} \sin \left( \frac{k^{w_{x}}}{m_r} d \right) \left( jk_x m_{n, n} \frac{n \pi}{b} \right) \delta_{m_r, m_{a, y}} f^{w_{x}}(n_r, n) f^{w_{y}}(n_s, n) \]  
(A155)

\[ y_{y_{s, t}} = \frac{2a}{jw \mu_0 b} \sum_{n=1}^{\infty} \frac{1}{e^{n, p}} \sin \left( \frac{k^{w_{x}}}{m_r} d \right) \left( k_x^2 - \left( \frac{n \pi}{b} \right)^2 \right) \delta_{m_r, m_{a, y}} f^{w_{x}}(n_r, n) f^{w_{y}}(n_s, n) \]  
\[ + \frac{ab}{jw \mu_0} \frac{k_y^m}{m_r} \delta_{m_r, m_{n, n}} \]  
(A156)
Metallic walls parallel to the y-axis

The matrix elements for this case are given by

\[ Z_{yy, rq} = \frac{1}{j_0 e_0} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \left( Q_{np} e^{-j \beta_{np} x_p} + Q_{np} e^{j \beta_{np} x_p} + Q_{np}^{sc} \right) \left( k_y^2 - \left( k_x^2 \right)^2 \right) \]
\[ \cdot \left( I_{1, y}(y_k, n) \right)^* I_{2, z}(z_{l_q}, p) I_{1, y}(y_k, n) I_{2, z}(z_{l_q}, p) \]  
\[ (A157) \]

\[ Z_{yz, rq} = \frac{1}{j_0 e_0} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \left( Q_{np} e^{-j \beta_{np} x_p} + Q_{np} e^{j \beta_{np} x_p} + Q_{np}^{sc} \right) \left( j k_y^m \right) \]
\[ \cdot \left( I_{1, y}(y_k, n) \right)^* I_{2, z}(z_{l_q}, p) I_{1, y}(y_k, n) I_{2, z}(z_{l_q}, p) \]  
\[ (A158) \]

\[ Z_{zy, rq} = \frac{1}{j_0 e_0} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \left( Q_{np} e^{-j \beta_{np} x_p} + Q_{np} e^{j \beta_{np} x_p} + Q_{np}^{sc} \right) \left( j k_y^m \right) \]
\[ \cdot \left( I_{2, y}(y_k, n) \right)^* I_{1, z}(z_{l_q}, p) I_{2, y}(y_k, n) I_{1, z}(z_{l_q}, p) \]  
\[ (A159) \]

\[ Z_{zz, rq} = \frac{1}{j_0 e_0} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left( Q_{np} e^{-j \beta_{np} x_p} + Q_{np} e^{j \beta_{np} x_p} + Q_{np}^{sc} \right) \left( k_y^2 - \left( k_x^2 \right)^2 \right) \frac{1}{k_y^p} \]
\[ \cdot \left( I_{2, y}(y_k, n) \right)^* I_{2, z}(z_{l_q}, p) I_{2, y}(y_k, n) I_{2, z}(z_{l_q}, p) \]  
\[ (A160) \]

\[ T_{yy}^a = -\frac{2}{a} \sum_{m=1}^{\infty} \frac{1}{\sin \left( k_y^m a \right)} \sin \left( \frac{m \pi x_b}{a} \right) \left( I_{1, y}(y_k, n) \right)^* I_{1, z}(z_{l_q}, m, n) I_{1, y}(y_k, n) I_{1, z}(z_{l_q}, m, n) \]  
\[ (A161) \]

\[ T_{yy}^f = 0 \]  
\[ (A162) \]

\[ T_{yz}^a = -\frac{2}{a} \sum_{m=1}^{\infty} \frac{k_y^m}{\sin \left( k_y^m a \right)} \sin \left( \frac{m \pi x_b}{a} \right) \left( I_{1, y}(y_k, n) \right)^* I_{2, z}(z_{l_q}, m, n) I_{1, y}(y_k, n) I_{2, z}(z_{l_q}, m, n) \]  
\[ (A163) \]

\[ T_{zy}^a = -\frac{2}{a} \sum_{m=1}^{\infty} \frac{k_y^m}{\sin \left( k_y^m a \right)} \sin \left( \frac{m \pi x_b}{a} \right) \left( I_{2, y}(y_k, n) \right)^* I_{1, z}(z_{l_q}, m, n) I_{2, y}(y_k, n) I_{1, z}(z_{l_q}, m, n) \]  
\[ (A164) \]

\[ T_{xy}^f = -\sum_{p=1}^{\infty} \left( \frac{p \pi}{d} \right) b \cdot I_{1, y}(m, n, p) I_{1, y}(y_k, n) I_{1, z}(z_{l_q}, p) \]  
\[ (A165) \]

\[ T_{xy}^f = -\sum_{p=0}^{\infty} j k_y^{1-p} \frac{1}{p} \cdot b \cdot I_{1, y}(m, n, p) I_{2, y}(y_k, n) I_{1, z}(z_{l_q}, p) \]  
\[ (A166) \]
Metallic walls

\[ T_{yy}^{gf} = 0 \quad \text{(A167)} \]

\[ T_{yy, tq}^{gf} = - \sum_{p=0}^{\infty} \frac{1}{p} I_{m}^{yw}(m, n, p) I_{m}^{rw}(y, \zeta, q, p) \]

\[ Y_{xx, st}^{aa} = - \frac{2b}{j \omega \mu_0} \sum_{m=1}^{\infty} \frac{\cos(k_{m}^{yw} a)}{k_{m}^{yw} \sin(k_{m}^{yw} a)} \left( k_{0}^{2} - \left( \frac{m \pi}{a} \right)^{2} \right) \delta_{n, n_{l}} I_{1}^{yw}(m, n, m) I_{1}^{yw}(m, n, m) \]

\[ + \frac{ab}{j \omega \mu_0} \left( \frac{k_{n}^{2} - \left( \frac{k_{n}}{k_{m}} \right)^{2}}{k_{m}^{2}} \right) \delta_{m, m_{l}} \delta_{n, n_{l}} \quad \text{(A169)} \]

\[ Y_{sy, ts}^{aa} = - \frac{2b}{j \omega \mu_0} \sum_{m=1}^{\infty} \frac{\cos(k_{m}^{yw} a)}{k_{m}^{yw} \sin(k_{m}^{yw} a)} \left( jk_{y}^{n} m \pi a \right) \delta_{n, n_{l}} I_{2}^{yw}(m, n, m) I_{2}^{yw}(m, n, m) \]

\[ - \frac{ab}{j \omega \mu_0} \left( \frac{k_{n}^{2} k_{y}^{n}}{k_{y}^{2}} \right) \delta_{m, m_{l}} \delta_{n, n_{l}} \quad \text{(A170)} \]

\[ Y_{yx, tx}^{aa} = \frac{2b}{j \omega \mu_0} \sum_{m=1}^{\infty} \frac{\cos(k_{m}^{yw} a)}{k_{m}^{yw} \sin(k_{m}^{yw} a)} \left( jk_{n}^{y} m \pi a \right) \delta_{n, n_{l}} I_{1}^{yw}(m, n, m) I_{1}^{yw}(m, n, m) \]

\[ - \frac{ab}{j \omega \mu_0} \left( \frac{k_{n}^{2} k_{y}^{n}}{k_{y}^{2}} \right) \delta_{m, m_{l}} \delta_{n, n_{l}} \quad \text{(A171)} \]

\[ Y_{yy, ts}^{aa} = - \frac{2b}{j \omega \mu_0} \sum_{m=0}^{\infty} \frac{\cos(k_{m}^{yw} a)}{k_{m}^{yw} \sin(k_{m}^{yw} a)} \left( k_{0}^{2} - \left( \frac{k_{n}}{k_{m}} \right)^{2} \right) \delta_{n, n_{l}} I_{2}^{yw}(m, n, m) I_{2}^{yw}(m, n, m) \]

\[ + \frac{ab}{j \omega \mu_0} \left( \frac{k_{n}^{2} k_{y}^{n}}{k_{y}^{2}} \right) \delta_{m, m_{l}} \delta_{n, n_{l}} \quad \text{(A172)} \]
Metallic walls parallel to both x- and y-axis

The matrix elements for this case are given by

\[
Z_{yw, rq} = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \left( Q_{np} e^{-j\beta_{np} x} + Q_{np} e^{j\beta_{np} x} + Q_{np}^{*} \left( k_{0}^{2} - \left( \frac{m \pi}{b} \right)^{2} \right) \right) \frac{1}{E_{n}}
\]

\[
\cdot I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, p) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, p)
\]

(A173)

\[
Z_{zw, rq} = - \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \left( Q_{np} e^{-j\beta_{np} x} + Q_{np} e^{j\beta_{np} x} + Q_{np}^{*} \right) \left( \frac{m \pi p \pi}{b d} \right)
\]

\[
\cdot I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, p) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, p)
\]

(A174)

\[
Z_{zw, rq} = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \left( Q_{np} e^{-j\beta_{np} x} + Q_{np} e^{j\beta_{np} x} + Q_{np}^{*} \right) \left( \frac{m \pi p \pi}{b d} \right)
\]

\[
\cdot I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, p) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, p)
\]

(A175)

\[
Z_{zw, rq} = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \left( Q_{np} e^{-j\beta_{np} x} + Q_{np} e^{j\beta_{np} x} + Q_{np}^{*} \right) \left( k_{0}^{2} - \left( \frac{m \pi}{b} \right)^{2} \right) \frac{1}{E_{n}}
\]

\[
\cdot I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, p) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, p)
\]

(A176)

\[
T_{x, y, rs}^{a} = - \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{E_{n} \sin(k_{mn} \alpha)} \sin \left( \frac{m \pi x}{a} \right) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, m, n) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, m, n)
\]

(A177)

\[
T_{y, y, rs}^{a} = 0
\]

(A178)

\[
T_{x, z, rs}^{a} = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{E_{n} \sin(k_{mn} \alpha)} \sin \left( \frac{m \pi x}{a} \right) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, m, n) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, m, n)
\]

(A179)

\[
T_{x, z, rs}^{a} = - \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{E_{n} \sin(k_{mn} \alpha)} \sin \left( \frac{m \pi x}{a} \right) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, m, n) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, m, n)
\]

(A180)

\[
T_{x, y, ts}^{a} = - \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{1}{E_{p}} \sin \left( \frac{m \pi x}{a} \right) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, m, n) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, m, n)
\]

(A181)

\[
T_{x, z, ts}^{a} = \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{1}{E_{p}} \sin \left( \frac{m \pi x}{a} \right) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, m, n) I_{1, y}^{w}(y_{k}, n) I_{2, z}(z_{l}, m, n)
\]

(A182)
Metallic walls

\[ T_{yy}^{af} = 0 \]  
(A183)

\[ T_{zz \text{ or } q}^{af} = -\sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \frac{1}{e_{p}} (y_{q}^{w \text{ or } z}(m, n, p)) (y_{z}^{w \text{ or } q}(n, m, p)) \delta_{m, n, p} \]  
(A184)

\[ \gamma_{aa}^{xx, ts} = -\frac{4}{j\omega\mu_{0}a} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{e_{n}} \cos\left(\frac{k_{n}^{w \text{ or } x}d}{a b}\right) \gamma_{n, m, x}^{w \text{ or } x}(m, n, p) \gamma_{m, y}^{w \text{ or } y}(n, m, p) \delta_{m, n, a} \]  
(A185)

\[ y_{aa}^{xx, ts} = \frac{4}{j\omega\mu_{0}a} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{e_{n}} \cos\left(\frac{k_{n}^{w \text{ or } x}d}{a b}\right) \gamma_{n, m, x}^{w \text{ or } x}(m, n, p) \gamma_{m, y}^{w \text{ or } y}(n, m, p) \delta_{m, n, a} \]  
(A186)

\[ y_{aa}^{yy, ts} = \frac{4}{j\omega\mu_{0}a} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{e_{n}} \cos\left(\frac{k_{n}^{w \text{ or } x}d}{a b}\right) \gamma_{n, m, y}^{w \text{ or } y}(m, n, p) \gamma_{m, y}^{w \text{ or } y}(n, m, p) \delta_{m, n, a} \]  
(A187)

\[ y_{aa}^{yy, ts} = -\frac{4}{j\omega\mu_{0}a} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{e_{n}} \cos\left(\frac{k_{n}^{w \text{ or } y}d}{a b}\right) \gamma_{n, m, y}^{w \text{ or } y}(m, n, p) \gamma_{m, y}^{w \text{ or } y}(n, m, p) \delta_{m, n, a} \]  
(A188)
A5: Removable singularities

Most of the integrals, defined in appendix A3, have removable singularities. The limits, if they exist, of these expressions in the singular points are given here.

\[
\lim_{k_y^{(m)}} \int_{k_x^{(m)}}^{\infty} (m, n, p) = \left\{ \begin{array}{ll}
\frac{a}{2j} & (m \neq 0) \\
0 & (m = 0)
\end{array} \right. 
\]  

(A189)

\[
\lim_{k_y^{(m)}} \int_{k_x^{(m)}}^{\infty} (m, n, p) = \left\{ \begin{array}{ll}
\frac{a}{2j} & (m \neq 0) \\
0 & (m = 0)
\end{array} \right. 
\]  

(A190)

\[
\lim_{k_y^{(m)}} \int_{k_x^{(m)}}^{\infty} (m, n, p) = \left\{ \begin{array}{ll}
\frac{a}{2j} & (m \neq 0) \\
0 & (m = 0)
\end{array} \right. 
\]  

(A191)

\[
\lim_{k_y^{(m)}} \int_{k_x^{(m)}}^{\infty} (m, n, p) = \left\{ \begin{array}{ll}
\frac{a}{2j} & (m \neq 0) \\
0 & (m = 0)
\end{array} \right. 
\]  

(A192)

\[
\lim_{k_y^{(m)}} \int_{k_x^{(m)}}^{\infty} (m, n, p) = \left\{ \begin{array}{ll}
\frac{a}{2j} & (m \neq 0) \\
0 & (m = 0)
\end{array} \right. 
\]  

(A193)

\[
\lim_{k_y^{(m)}} \int_{k_x^{(m)}}^{\infty} (m, n, p) = \left\{ \begin{array}{ll}
\frac{a}{2j} & (m \neq 0) \\
0 & (m = 0)
\end{array} \right. 
\]  

(A194)

\[
\lim_{\beta_{x}^{(m)}} \int_{k_x^{(m)}}^{\infty} (m, n, p) = \frac{1}{jk_x^{(m)}} \left( \sin(k_x^{(m)}(x - a)) + (a - x) \sin(k_x^{(m)}(a + x)) \right) 
\]  

(A195)

\[
\lim_{\beta_{x}^{(m)}} \int_{k_x^{(m)}}^{\infty} (m, n, p) = \frac{1}{jk_x^{(m)}} \left( \sin(k_x^{(m)}(x - a)) + (a - x) \sin(k_x^{(m)}(a + x)) \right) 
\]  

(A196)

\[
\lim_{\beta_{x}^{(m)}} \int_{k_x^{(m)}}^{\infty} (m, n, p) = \frac{-1}{bd} \left( \sin(k_x^{(m)}(x - a)) + (a - x) \sin(k_x^{(m)}(a + x)) \right) 
\]  

(A197)

\[
\lim_{\beta_{x}^{(m)}} \int_{k_x^{(m)}}^{\infty} (m, n, p) = \frac{-1}{bd} \left( \sin(k_x^{(m)}(x - a)) + (a - x) \sin(k_x^{(m)}(a + x)) \right) 
\]  

(A198)
Metallic walls

\[
\lim_{k_x \to \pm \frac{n\pi}{b}} I_{1, y}^{w_x}(y_k, n) = \frac{W_x}{W_z} \cos \left( \frac{n\pi y_k}{b} \right) \tag{A199}
\]

\[
\lim_{k_x \to \pm \frac{n\pi}{b}} I_{1, y}^{w_x}(y_{c-1}, n) = -\frac{W_y \sin \left( \frac{n\pi}{b} (y_{c-1} - W_y) \right)}{2W_z \sin \left( \frac{n\pi}{b} W_y \right)} + \frac{W_y \sin \left( \frac{n\pi}{b} (y_{c-1} + W_y) \right)}{2W_z \sin \left( \frac{n\pi}{b} W_y \right)} \tag{A200}
\]

\[
\lim_{k_x \to \pm \frac{n\pi}{b}} I_{1, y}^{w_x}(y_c, n) = \frac{W_y}{W_z} \cos \left( \frac{n\pi y_c}{b} \right) \tag{A201}
\]

\[
\lim_{k_x \to \pm \frac{n\pi}{b}} I_{1, y}^{w_x}(y_{c+1}, n) = -\frac{W_y \sin \left( \frac{n\pi}{b} (y_{c+1} - W_y) \right)}{2W_z \sin \left( \frac{n\pi}{b} W_y \right)} + \frac{W_y \sin \left( \frac{n\pi}{b} (y_{c+1} + W_y) \right)}{2W_z \sin \left( \frac{n\pi}{b} W_y \right)} \tag{A202}
\]

\[
\lim_{k_x \to \pm \frac{n\pi}{b}} I_{1, y}^{w_x}(y_k, n) = \frac{W_y}{W_z} e^{jk_y y_k} \tag{A203}
\]

\[
\lim_{k_x \to \pm \frac{n\pi}{b}} I_{1, y}^{w_x}(y_{c-1}, n) = \frac{e^{jk_y y_{c-1}}}{2jW_z} \left( \frac{W_y e^{-jk_y W_y}}{\sin (k_y W_y)} + \frac{W_y e^{jk_y W_y}}{\sin (k_y W_y)} \right) \tag{A204}
\]

\[
\lim_{k_x \to \pm \frac{n\pi}{b}} I_{1, y}^{w_x}(y_c, n) = \frac{W_y}{W_z} e^{jk_y y_c} \tag{A205}
\]

\[
\lim_{k_x \to \pm \frac{n\pi}{b}} I_{1, y}^{w_x}(y_{c+1}, n) = \frac{e^{jk_y y_{c+1}}}{2jW_z} \left( \frac{W_y e^{-jk_y W_y}}{\sin (k_y W_y)} + \frac{W_y e^{jk_y W_y}}{\sin (k_y W_y)} \right) \tag{A206}
\]

\[
\lim_{n \to 0} I_{1, y}^{w_x}(y_k, n) = 0 \tag{A207}
\]

\[
\lim_{k_y \to \pm \frac{n\pi}{b}} I_{1, y}^{w_x}(y_k, n) = W_y \tag{A208}
\]

\[
\lim_{k_y \to \pm \frac{n\pi}{b} \setminus 0} \frac{b}{2j} (n \neq 0) \tag{A209}
\]

\[
\lim_{k_y \to \pm \frac{n\pi}{b} \setminus 0} \frac{b}{2j} (n \neq 0) \tag{A210}
\]

\[
\lim_{k_y \to \pm \frac{n\pi}{b} \setminus 0} \frac{b}{2j} (n \neq 0) \tag{A211}
\]
Metallic walls

\[
\lim_{k'_{i} \to \left( \frac{n\pi}{b} \right)} I_{3i}(n, n, n) = \begin{cases} 
-\frac{b}{2j} & (n \neq 0) \\
0 & (n = 0)
\end{cases}
\]  
(A212)

\[
\lim_{k'_{i} \to \left( \frac{n\pi}{b} \right)} I_{4i}(n, n, n) = \begin{cases} 
\frac{b}{2} & (n \neq 0) \\
0 & (n = 0)
\end{cases}
\]  
(A213)

\[
\lim_{k'_{i} \to \left( \frac{n\pi}{a} \right)} I_{5i}(n, n, n) = \begin{cases} 
\frac{b}{2} & (n \neq 0) \\
0 & (n = 0)
\end{cases}
\]  
(A214)

\[
\lim_{k_{i} \to \left( \frac{n\pi}{d} \right)} I_{1i}(z, p, p) = \frac{W}{W_{y}} \cos \left( \frac{p\pi z_{i}}{d} \right)
\]  
(A215)

\[
\lim_{p \to 0} I_{2i}(z, p, p) = 0
\]  
(A216)

\[
\lim_{k_{mn}^{z} \to 0 \sin (k_{mn}^{x}d)} I_{3i}(z, p, m, n) = \frac{2\left(z_{i} - \frac{W_{x}}{2} + d\right)\left(\frac{W_{x}}{2}\right)}{d}
\]  
(A217)

\[
\lim_{k_{mn}^{z} \to 0 \sin (k_{mn}^{x}d)} I_{5i}(z, p, m, n) = \frac{2\left(z_{i} - \frac{W_{x}}{2} + d\right)\left(\frac{W_{x}}{2}\right)}{d}
\]  
(A218)

\[
\lim_{k_{mn}^{z} \to 0 \sin (k_{mn}^{x}d)} I_{5i}(z, p, m, n) = \frac{2\left(z_{i} - \frac{W_{x}}{2} + d\right)\left(\frac{W_{x}}{2}\right)}{d}
\]  
(A219)

\[
\lim_{k_{mn}^{z} \to 2k_{x}} I_{3i}(z, p, m, n) = \frac{W}{W_{y}} \cos \left( k_{x} (z_{i} + d) \right)
\]  
(A220)

\[
\lim_{k_{mn}^{z} \to 2k_{x}} I_{4i}(z, p, m, n) = \frac{W}{W_{y}} \cos \left( k_{x} (z_{i} + d) \right)
\]  
(A221)

\[
\lim_{k_{mn}^{z} \to 2k_{x}} I_{5i}(z, p, m, n) = \frac{W}{W_{y}} \cos \left( k_{x} (z_{i} + d) \right)
\]  
(A222)
Appendix B

Dual polarization

B1: Components of the dyadic Green's functions

The tangential electric fields on the fins and the tangential magnetic fields on the aperture plane can be expressed in terms of the dyadic Green’s functions and the current distributions:

\[ E_{fx, in}^{ex} = \int_{S_f} E_{fx, in}^{ex} \cdot \eta^x(x', z') \, dS_f \]  \hspace{1cm} (B1)

\[ E_{fy, in}^{ex} = \int_{S_f} E_{fy, in}^{ex} \cdot \eta^y(y', z') \, dS_f \]  \hspace{1cm} (B2)

\[ E_{fx, in}^{au} = \int_{S_a} E_{fx, au}^{in} \cdot \eta^{in}(x', y') \, dS_a \]  \hspace{1cm} (B3)

\[ E_{fy, in}^{au} = \int_{S_a} E_{fy, au}^{in} \cdot \eta^{in}(x', y') \, dS_a \]  \hspace{1cm} (B4)

\[ E_{fx, in}^{uy} = \int_{S_a} E_{fx, uy}^{in} \cdot \eta^{uy}(x', y') \, dS_a \]  \hspace{1cm} (B5)

\[ E_{fy, in}^{uy} = \int_{S_a} E_{fy, uy}^{in} \cdot \eta^{uy}(x', y') \, dS_a \]  \hspace{1cm} (B6)

\[ H_{fx, in}^{au} = \int_{S_f} H_{fx, au}^{in} \cdot \eta^{in}(x', z') \, dS_f \]  \hspace{1cm} (B7)

\[ H_{fy, in}^{au} = \int_{S_f} H_{fy, au}^{in} \cdot \eta^{in}(x', z') \, dS_f \]  \hspace{1cm} (B8)

\[ H_{fx, in}^{uy} = \int_{S_a} H_{fx, uy}^{in} \cdot \eta^{uy}(x', z') \, dS_a \]  \hspace{1cm} (B9)

\[ H_{fy, in}^{uy} = \int_{S_a} H_{fy, uy}^{in} \cdot \eta^{uy}(x', y') \, dS_a \]  \hspace{1cm} (B10)

where the primed coordinates refer to the source coordinates. Furthermore, the first part of the upper indices of the dyadic Green’s functions refers to the location of the tangential component, the second part refers to the location of the current density causing the field and the third
part refers to the region (i.e. interior of exterior). Finally, \(S_{fx}, S_{fy}\) and \(S_a\) are the surfaces of the radiating element parallel to the x-axis, the radiating element parallel to the y-axis and the aperture plane, respectively. The dyadic Green’s functions are defined as

\[
\begin{align*}
\underline{E}_{fx,fx, in} & = \begin{bmatrix} EF_{fx,fx, in} & EF_{fx,fx, in} \\ EF_{fx,fx, in} & EF_{fx,fx, in} \end{bmatrix} \\
\underline{E}_{fy,fx, in} & = \begin{bmatrix} EF_{fy,fx, in} & EF_{fy,fx, in} \\ EF_{fy,fx, in} & EF_{fy,fx, in} \end{bmatrix} \\
\underline{E}_{fx,a, in} & = \begin{bmatrix} EF_{fx,a, in} & EF_{fx,a, in} \\ EF_{fx,a, in} & EF_{fx,a, in} \end{bmatrix} \\
\underline{E}_{fy,a, in} & = \begin{bmatrix} EF_{fy,a, in} & EF_{fy,a, in} \\ EF_{fy,a, in} & EF_{fy,a, in} \end{bmatrix} \\
\underline{H}_{a,fx, in} & = \begin{bmatrix} HA_{a,fx, in} & HA_{a,fx, in} \\ HA_{a,fx, in} & HA_{a,fx, in} \end{bmatrix} \\
\underline{H}_{a,fy, in} & = \begin{bmatrix} HA_{a,fy, in} & HA_{a,fy, in} \\ HA_{a,fy, in} & HA_{a,fy, in} \end{bmatrix} \\
\underline{H}_{a,a, in} & = \begin{bmatrix} HA_{a,a, in} & HA_{a,a, in} \\ HA_{a,a, in} & HA_{a,a, in} \end{bmatrix} \\
\underline{H}_{a,a, ex} & = \begin{bmatrix} HA_{a,a, ex} & HA_{a,a, ex} \\ HA_{a,a, ex} & HA_{a,a, ex} \end{bmatrix}
\end{align*}
\]
The dyadic Green's functions have been defined above. The components of these functions can be calculated by using the equations for the electric and magnetic field, in terms of the electric and magnetic vector potentials.

\[
E_{fx,fx,\text{in}} = \frac{1}{j \mu_0 \varepsilon_0} \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} (k_0^2 - (k_m^2 + k_p^2)^2) \left( D_{mp}^* e^{-j\gamma_{mp}y} + D_{mp}^* e^{j\gamma_{mp}y} + D_{mp}^* e^{-j\gamma_{mp}y} - y\right) \\
\cdot e^{-j k_0^2 (x-x')} \sin \left( \frac{p \pi x}{d} \right) \sin \left( \frac{p \pi z}{d} \right) \tag{B21}
\]

\[
E_{fx,fx,\text{in}} = \frac{1}{j \mu_0 \varepsilon_0} \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} \left( \frac{p \pi x}{d} \right)^2 \left( D_{mp}^* e^{-j\gamma_{mp}y} + D_{mp}^* e^{j\gamma_{mp}y} + D_{mp}^* e^{-j\gamma_{mp}y} - y\right) \\
\cdot e^{-j k_0^2 (x-x')} \sin \left( \frac{p \pi x}{d} \right) \sin \left( \frac{p \pi z}{d} \right) \tag{B22}
\]

\[
E_{fx,fx,\text{in}} = -\frac{1}{j \mu_0 \varepsilon_0} \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} \left( \frac{p \pi x}{d} \right)^2 \left( D_{mp}^* e^{-j\gamma_{mp}y} + D_{mp}^* e^{j\gamma_{mp}y} + D_{mp}^* e^{-j\gamma_{mp}y} - y\right) \\
\cdot e^{-j k_0^2 (x-x')} \cos \left( \frac{p \pi x}{d} \right) \cos \left( \frac{p \pi z}{d} \right) \tag{B23}
\]

\[
E_{fx,fx,\text{in}} = \frac{1}{j \mu_0 \varepsilon_0} \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} \left( k_0^2 - (k_m^2 + k_p^2)^2 \right) \left( D_{mp}^* e^{-j\gamma_{mp}y} + D_{mp}^* e^{j\gamma_{mp}y} + D_{mp}^* e^{-j\gamma_{mp}y} - y\right) \\
\cdot e^{-j k_0^2 (x-x')} \cos \left( \frac{p \pi x}{d} \right) \cos \left( \frac{p \pi z}{d} \right) \tag{B24}
\]

\[
E_{fx,xy,\text{in}} = -\frac{1}{j \mu_0 \varepsilon_0} \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} \left( \frac{p \pi x}{d} \right) \left( -j \beta_{np} C_{np} e^{-j\beta_{np}x} + j \beta_{np} C_{np} e^{j\beta_{np}x} - j \beta_{np} C_{np} e^{-j\beta_{np}x} - y\right) \\
\cdot e^{-j k_0^2 (y-y')} \sin \left( \frac{p \pi x}{d} \right) \sin \left( \frac{p \pi z}{d} \right) \tag{B25}
\]

\[
E_{fx,xy,\text{in}} = -\frac{1}{j \mu_0 \varepsilon_0} \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} \left( \frac{p \pi x}{d} \right) \left( -j \beta_{np} C_{np} e^{-j\beta_{np}x} + j \beta_{np} C_{np} e^{j\beta_{np}x} - j \beta_{np} C_{np} e^{-j\beta_{np}x} - y\right) \\
\cdot e^{-j k_0^2 (y-y')} \cos \left( \frac{p \pi x}{d} \right) \sin \left( \frac{p \pi z}{d} \right) \tag{B26}
\]

\[
E_{fx,yz,\text{in}} = -\frac{1}{j \mu_0 \varepsilon_0} \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} \left( \frac{p \pi y}{d} \right) \left( C_{np} e^{-j\beta_{np}x} + C_{np} e^{j\beta_{np}x} + C_{np} e^{-j\beta_{np}x} - y\right) \\
\cdot e^{-j k_0^2 (y-y')} \sin \left( \frac{p \pi x}{d} \right) \cos \left( \frac{p \pi z}{d} \right) \tag{B27}
\]

\[
E_{fx,zz,\text{in}} = -\frac{1}{j \mu_0 \varepsilon_0} \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} \left( k_0^2 - (k_m^2 + k_p^2)^2 \right) \left( C_{np} e^{-j\beta_{np}x} + C_{np} e^{j\beta_{np}x} + C_{np} e^{-j\beta_{np}x} - y\right) \\
\cdot e^{-j k_0^2 (y-y')} \cos \left( \frac{p \pi x}{d} \right) \cos \left( \frac{p \pi z}{d} \right) \tag{B28}
\]

\[
E_{fx,xx,\text{in}} = 0 \tag{B29}
\]
\[ E_{xy, in} = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-j k_x^m (x-x')} e^{-j k_y^m (y-y')} \frac{\sin (k_{mn} (z+d))}{k_{mn} \sin (k_{mn} d)} \]  
(B30)

\[ E_{zx, in} = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} j k_y^m e^{-j k_x^m (x-x')} e^{-j k_y^m (y-y')} \frac{\cos (k_{mn} (z+d))}{k_{mn} \sin (k_{mn} d)} \]  
(B31)

\[ E_{zy, in} = -\frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} j k_x^m e^{-j k_x^m (x-x')} e^{-j k_y^m (y-y')} \frac{\cos (k_{mn} (z+d))}{k_{mn} \sin (k_{mn} d)} \]  
(B32)

\[ E_{yx, in} = -\frac{1}{j ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (j k_x^m) \left( -j \gamma_{mp} D^+ e^{-j \gamma_{mp} y} + j \gamma_{mp} D^- e^{j \gamma_{mp} y} -j \gamma_{mp} D^+ e^{-j \gamma_{mp} y} -j \gamma_{mp} D^- e^{j \gamma_{mp} y} \right) \ \text{sgn} (y-y') \sin \left( \frac{p \pi z}{d} \right) \sin \left( \frac{p \pi z}{d} \right) \]  
(B33)

\[ E_{yz, in} = -\frac{1}{j ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( \frac{p \pi}{d} \right) \left( D^+ e^{-j \gamma_{mp} y} + D^- e^{j \gamma_{mp} y} -j \gamma_{mp} D^+ e^{-j \gamma_{mp} y} -j \gamma_{mp} D^- e^{j \gamma_{mp} y} \right) \ \text{sgn} (y-y') \cos \left( \frac{p \pi z}{d} \right) \cos \left( \frac{p \pi z}{d} \right) \]  
(B34)

\[ E_{zx, in} = -\frac{1}{j ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (j k_x^m) \left( D^+ e^{-j \gamma_{mp} y} + D^- e^{j \gamma_{mp} y} + D^+ e^{-j \gamma_{mp} y} + D^- e^{j \gamma_{mp} y} \right) \ \text{cos} \left( \frac{p \pi z}{d} \right) \cos \left( \frac{p \pi z}{d} \right) \]  
(B35)

\[ E_{zy, in} = \frac{1}{j ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( k_0^m - (k_0^m) \right) \left( C_{np} e^{-j \beta_{np} x} + C_{np} e^{j \beta_{np} x} + C_{np} e^{-j \beta_{np} x} + C_{np} e^{j \beta_{np} x} \right) \ \text{sin} \left( \frac{p \pi z}{d} \right) \sin \left( \frac{p \pi z}{d} \right) \]  
(B36)

\[ E_{yy, in} = \frac{1}{j ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( j k_n^m \right) \left( C_{np} e^{j \beta_{np} x} + C_{np} e^{-j \beta_{np} x} + C_{np} e^{j \beta_{np} x} + C_{np} e^{-j \beta_{np} x} \right) \ \text{cos} \left( \frac{p \pi z}{d} \right) \cos \left( \frac{p \pi z}{d} \right) \]  
(B37)

\[ E_{yz, in} = \frac{1}{j ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( j k_n^m \right) \left( C_{np} e^{-j \beta_{np} x} + C_{np} e^{j \beta_{np} x} + C_{np} e^{j \beta_{np} x} + C_{np} e^{-j \beta_{np} x} \right) \ \text{cos} \left( \frac{p \pi z}{d} \right) \cos \left( \frac{p \pi z}{d} \right) \]  
(B38)

\[ E_{yy, in} = \frac{1}{j ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( j k_n^m \right) \left( C_{np} e^{-j \beta_{np} x} + C_{np} e^{j \beta_{np} x} + C_{np} e^{j \beta_{np} x} + C_{np} e^{-j \beta_{np} x} \right) \ \text{sin} \left( \frac{p \pi z}{d} \right) \cos \left( \frac{p \pi z}{d} \right) \]  
(B39)
\[
E_{yy,z}^{in} = \frac{1}{j\omega \varepsilon_0}\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( k_n^2 - \left( \frac{p\pi}{d} \right)^2 \right) \left( C_{np} e^{-j\beta_{np} x} + C_{np} e^{j\beta_{np} x} + C_{np} e^{-j\beta_{np} x} + C_{np} e^{j\beta_{np} x} \right) e^{-jk_n^y (y-y')} \cos \left( \frac{p\pi z}{d} \right) \cos \left( \frac{p\pi z}{d} \right) \frac{1}{\epsilon_p} \]  \tag{B40}

\[
E_{yx,z}^{in} = -\frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-j\beta_{mn}^z (x-x')} e^{-jk_n^y (y-y')} \sin \left( \frac{k_{mn}^z (z+d)}{k_{mn}^z d} \right) \sin \left( \frac{k_{mn}^z d}{k_{mn}^z d} \right) \]  \tag{B41}

\[
E_{yx,z}^{in} = 0 \]  \tag{B42}

\[
E_{yz,z}^{in} = \frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{jk_n^y e^{-j\beta_{mn}^y (x-x')} e^{-jk_n^y (y-y')} \cos \left( \frac{k_{mn}^y (z+d)}{k_{mn}^y d} \right)}{k_{mn}^y \sin \left( \frac{k_{mn}^y d}{k_{mn}^y d} \right)} \]  \tag{B43}

\[
E_{yz,z}^{in} = -\frac{1}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{jk_n^y e^{-j\beta_{mn}^y (x-x')} e^{-jk_n^y (y-y')} \cos \left( \frac{k_{mn}^y (z+d)}{k_{mn}^y d} \right)}{k_{mn}^y \sin \left( \frac{k_{mn}^y d}{k_{mn}^y d} \right)} \]  \tag{B44}

\[
H_{hx,z}^{in} = 0 \]  \tag{B45}

\[
H_{hx,z}^{in} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( -j\gamma_{mp} D_{mp}^+ e^{-j\gamma_{mn} y} + j\gamma_{mp} D_{mp}^- e^{j\gamma_{mn} y} -j\gamma_{mp} e^{-j\gamma_{mn} y} \right) \]  \tag{B46}

\[
H_{hx,y}^{in} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( \frac{p\pi}{d} \right) \left( D_{mp}^+ e^{-j\gamma_{mn} y} + D_{mp}^- e^{j\gamma_{mn} y} + D_{mp}^+ e^{-j\gamma_{mn} y} \right) \]  \tag{B47}

\[
H_{hx,y}^{in} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( jk_n^x \right) \left( D_{mp} e^{-j\gamma_{mn} y} + D_{mp} e^{j\gamma_{mn} y} + D_{mp} e^{-j\gamma_{mn} y} \right) \]  \tag{B48}

\[
H_{hx,y}^{in} = -\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( \frac{p\pi}{d} \right) \left( C_{np} e^{-j\beta_{np} x} + C_{np} e^{j\beta_{np} x} + C_{np} e^{-j\beta_{np} x} \right) \]  \tag{B49}

\[
H_{hx,y}^{in} = -\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( jk_n^y \right) \left( C_{np} e^{-j\beta_{np} x} + C_{np} e^{j\beta_{np} x} + C_{np} e^{-j\beta_{np} x} \right) \]  \tag{B50}
Dual polarization

\[ H^{a,fy,in}_{yy} = 0 \]  
\[ H^{a,fx, in}_{yz} = -\sum_{n=-\infty}^{\infty} \sum_{p=0}^{\infty} \left( -j\beta_{np} C^{*}_{np} e^{j\beta_{np}x} + j\beta_{np} C_{np} e^{j\beta_{np}x} - j\beta_{np} C^{*}_{np} e^{j\beta_{np}x} - j\beta_{np} C_{np} e^{j\beta_{np}x} \right) \cdot e^{-jk_m^y (y-y')} e^{jk_m^x (x-x')} \frac{1}{k_m^x d} \frac{1}{k_m^y d} \]  
\[ H^{a,a, in}_{xx} = \frac{1}{j\omega \mu_0 \epsilon_0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( k_m^2 - (k_m^n)^2 \right) e^{-jk_m^x (x-x')} e^{-jk_m^y (y-y')} \cos \left( \frac{p\pi^2}{d^2} \right) \cos \left( \frac{q\pi^2}{d^2} \right) \frac{1}{k_m^x d} \frac{1}{k_m^y d} \]  
\[ H^{a,a, in}_{xy} = \frac{1}{j\omega \mu_0 \epsilon_0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( k_m^2 - (k_m^n)^2 \right) e^{-jk_m^x (x-x')} e^{-jk_m^y (y-y')} \cos \left( \frac{p\pi^2}{d^2} \right) \cos \left( \frac{q\pi^2}{d^2} \right) \frac{1}{k_m^x d} \frac{1}{k_m^y d} \]  
\[ H^{a,a, in}_{yx} = \frac{1}{j\omega \mu_0 \epsilon_0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( k_m^2 - (k_m^n)^2 \right) e^{-jk_m^x (x-x')} e^{-jk_m^y (y-y')} \cos \left( \frac{p\pi^2}{d^2} \right) \cos \left( \frac{q\pi^2}{d^2} \right) \frac{1}{k_m^x d} \frac{1}{k_m^y d} \]  
\[ H^{a,a, ex}_{xx} = \frac{1}{j\omega \mu_0 \epsilon_0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( k_m^2 - (k_m^n)^2 \right) e^{-jk_m^x (x-x')} e^{-jk_m^y (y-y')} e^{-jk_m \phi} \]  
\[ H^{a,a, ex}_{xy} = \frac{1}{j\omega \mu_0 \epsilon_0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( k_m^2 - (k_m^n)^2 \right) e^{-jk_m^x (x-x')} e^{-jk_m^y (y-y')} e^{-jk_m \phi} \]  
\[ H^{a,a, ex}_{yx} = \frac{1}{j\omega \mu_0 \epsilon_0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( k_m^2 - (k_m^n)^2 \right) e^{-jk_m^x (x-x')} e^{-jk_m^y (y-y')} e^{-jk_m \phi} \]  
\[ H^{a,a, ex}_{yy} = \frac{1}{j\omega \mu_0 \epsilon_0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( k_m^2 - (k_m^n)^2 \right) e^{-jk_m^x (x-x')} e^{-jk_m^y (y-y')} e^{-jk_m \phi} \]
B2: Expressions for the matrix elements

The matrix elements have been defined in Chapter 4 (Eq.(4.35)). The elements can be calculated by using Eq.(4.34) and the components of the dyadic Green’s functions of appendix B1.

\[
Z_{fs,fx}^{\text{xx},rq} = \frac{1}{j\omega e_0} \sum_{m = -p}^{\infty} \sum_{n = -p}^{\infty} \left( k_0^2 - (k_x^m)^2 \right) \left( D_{mp}^+ e^{-j\nu_{mp}^x} + D_{mp}^- e^{j\nu_{mp}^x} + D_{mp}^* \right) \nonumber
\]

\[
\cdot \int_{S_{\mu}} \left( J_{fx}^{x} e^{-jk_x^m z} \sin \left( \frac{p \pi z}{d} \right) \right) dx dz \int_{S_{\mu}} J_{fx}^{x} e^{jk_x^m z} \cos \left( \frac{p \pi z}{d} \right) dz' dx' \tag{B61}
\]

\[
Z_{fs,fx}^{\text{zx},rq} = \frac{1}{j\omega e_0} \sum_{m = -p}^{\infty} \sum_{n = -p}^{\infty} \left( k_0^2 - (k_x^m)^2 \right) \left( D_{mp}^+ e^{-j\nu_{mp}^x} + D_{mp}^- e^{j\nu_{mp}^x} + D_{mp}^* \right) \nonumber
\]

\[
\cdot \int_{S_{\mu}} J_{fx}^{x} e^{-jk_x^m z} \sin \left( \frac{p \pi z}{d} \right) dz dx \int_{S_{\mu}} J_{fx}^{x} e^{jk_x^m z} \cos \left( \frac{p \pi z}{d} \right) dz' dx' \tag{B62}
\]

\[
Z_{fs,fx}^{\text{zz},rq} = \frac{1}{j\omega e_0} \sum_{m = -p}^{\infty} \sum_{n = -p}^{\infty} \left( k_0^2 - (k_x^m)^2 \right) \left( D_{mp}^+ e^{-j\nu_{mp}^x} + D_{mp}^- e^{j\nu_{mp}^x} + D_{mp}^* \right) \nonumber
\]

\[
\cdot \int_{S_{\mu}} J_{fx}^{x} e^{-jk_x^m z} \cos \left( \frac{p \pi z}{d} \right) dz dx \int_{S_{\mu}} J_{fx}^{x} e^{jk_x^m z} \cos \left( \frac{p \pi z}{d} \right) dz' dx' \tag{B63}
\]

\[
Z_{fs,fx}^{\text{xy},rq} = \frac{1}{j\omega e_0} \sum_{m = -p}^{\infty} \sum_{n = -p}^{\infty} \left( k_0^2 - (k_x^m)^2 \right) \left( D_{mp}^+ e^{-j\nu_{mp}^x} + D_{mp}^- e^{j\nu_{mp}^x} + D_{mp}^* \right) \nonumber
\]

\[
\cdot \int_{S_{\mu}} J_{fx}^{x} e^{-jk_x^m z} \cos \left( \frac{p \pi z}{d} \right) dz dx \int_{S_{\mu}} J_{fx}^{x} e^{jk_x^m z} \cos \left( \frac{p \pi z}{d} \right) dz' dx' \tag{B64}
\]

\[
T_{fx,fx}^{\text{yx},rq} = \frac{1}{j\omega e_0} \sum_{m = -p}^{\infty} \sum_{n = -p}^{\infty} \left( k_0^2 - (k_y^m)^2 \right) \left( D_{mp}^+ e^{-j\nu_{mp}^y} + D_{mp}^- e^{j\nu_{mp}^y} + D_{mp}^* \right) \nonumber
\]

\[
\cdot \int_{S_{\mu}} J_{fx}^{x} e^{-jk_y^m y} \sin \left( \frac{p \pi y}{d} \right) dy dx \int_{S_{\mu}} J_{fx}^{x} e^{jk_y^m y} \sin \left( \frac{p \pi y}{d} \right) dy' dx' \tag{B65}
\]

\[
T_{fx,fx}^{\text{zy},rq} = \frac{1}{j\omega e_0} \sum_{m = -p}^{\infty} \sum_{n = -p}^{\infty} \left( k_0^2 - (k_y^m)^2 \right) \left( D_{mp}^+ e^{-j\nu_{mp}^y} + D_{mp}^- e^{j\nu_{mp}^y} + D_{mp}^* \right) \nonumber
\]

\[
\cdot \int_{S_{\mu}} J_{fx}^{x} e^{-jk_y^m y} \cos \left( \frac{p \pi y}{d} \right) dy dx \int_{S_{\mu}} J_{fx}^{x} e^{jk_y^m y} \cos \left( \frac{p \pi y}{d} \right) dy' dx' \tag{B66}
\]
\[
\mathbf{T}_{fx, fy} = \frac{1}{j \omega e_0} \sum_{m = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} \left( j k_n^e \frac{p \mathbf{R}}{d} e^{-j k_n^e z} \right) \int_{S_f} f_x (C_{np} e^{-j \mathbf{b}_{np} x} + C_{np} e^{-j \mathbf{b}_{np} x}) \cos \left( \frac{p \mathbf{R} z}{d} \right) dx dz \]
\[
\mathbf{T}_{fz, rz} = \frac{1}{j \omega e_0} \sum_{m = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} \left( j k_n^e \frac{p \mathbf{R}}{d} e^{-j k_n^e z} \right) \int_{S_f} f_z (C_{np} e^{-j \mathbf{b}_{np} x} + C_{np} e^{-j \mathbf{b}_{np} x}) \cos \left( \frac{p \mathbf{R} z}{d} \right) dx dz \]
\[
\mathbf{T}_{fx, rz} = 0 \quad (B69)
\]
\[
\mathbf{T}_{fx, ar} = \frac{1}{ab} \sum_{m = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} \frac{e^{-j k_n^e z}}{k_{mn} \sin (k_{mn} d)} \int_{S_f} f_x e^{-j k_n^e z} \sin \left( k_{mn} (z + d) \right) dx dz \int M_{ys} e^{i k_n^e y} e^{i k_n^e y} dx dy \quad (B70)
\]
\[
\mathbf{T}_{fz, ar} = \frac{1}{ab} \sum_{m = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} \frac{j k_n^e e^{-j k_n^e z}}{k_{mn} \sin (k_{mn} d)} \int_{S_f} f_z e^{-j k_n^e z} \cos \left( k_{mn} (z + d) \right) dx dz \int M_{ys} e^{i k_n^e y} e^{i k_n^e y} dx dy \quad (B71)
\]
\[
\mathbf{T}_{fx, rz} = -\frac{1}{ab} \sum_{m = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} \frac{j k_n^e e^{-j k_n^e z}}{k_{mn} \sin (k_{mn} d)} \int_{S_f} f_x e^{-j k_n^e z} \cos \left( k_{mn} (z + d) \right) dx dz \int M_{ys} e^{i k_n^e y} e^{i k_n^e y} dx dy \quad (B72)
\]
\[
\mathbf{T}_{fy, rz} = \frac{1}{j \omega e_0} \sum_{m = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} \left( j k_n^e \frac{p \mathbf{R}}{d} e^{-j k_n^e z} \right) \int_{S_f} f_y (\mathbf{J}_{mp} D_{mp}^+ + \mathbf{J}_{mp} D_{mp}^- + \mathbf{J}_{mp} D_{mp}^+ - \mathbf{J}_{mp} D_{mp}) e^{-j \mathbf{b}_{mp} (y - y') \sgn (y - y')} \sin \left( \frac{p \mathbf{R} z}{d} \right) dy dz \quad (B73)
\]
\[
\mathbf{T}_{fy, rz} = \frac{1}{j \omega e_0} \sum_{m = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} \left( j k_n^e \frac{p \mathbf{R}}{d} e^{-j k_n^e z} \right) \int_{S_f} f_y (\mathbf{J}_{mp} D_{mp}^+ + \mathbf{J}_{mp} D_{mp}^- + \mathbf{J}_{mp} D_{mp}^+ - \mathbf{J}_{mp} D_{mp}) e^{-j \mathbf{b}_{mp} (y - y') \sgn (y - y')} \sin \left( \frac{p \mathbf{R} z}{d} \right) dy dz \quad (B74)
\]
\[ T_{f_{y}, \delta_{x, \rho}} = \frac{1}{j \omega e_{0}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{k_{0}^{2}}{d} \right) e^{-jk_{0} z} 
abla_{x, \rho} \cdot \int \nabla_{z, \rho} e^{ik_{x} x} \sin \left( \frac{p_{n} z}{d} \right) dx \left( \frac{p_{n} z}{d} \right) dy \right] \quad (B75) \]

\[ T_{f_{y}, \delta_{z, \rho}} = \frac{1}{j \omega e_{0}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{k_{0}^{2}}{d} \right) e^{-jk_{0} z} \nabla_{z, \rho} \cdot \int \nabla_{x, \rho} e^{ik_{y} y} \cos \left( \frac{p_{n} z}{d} \right) dy \right] \quad (B76) \]

\[ Z_{f_{y}, \delta_{x, \rho}} = \frac{1}{j \omega e_{0}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{k_{0}^{2}}{d} \right) e^{-jk_{0} z} \nabla_{x, \rho} \cdot \int \nabla_{z, \rho} e^{ik_{y} y} \sin \left( \frac{p_{n} z}{d} \right) dy \right] \quad (B77) \]

\[ Z_{f_{y}, \delta_{z, \rho}} = \frac{1}{j \omega e_{0}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{k_{0}^{2}}{d} \right) e^{-jk_{0} z} \nabla_{z, \rho} \cdot \int \nabla_{x, \rho} e^{ik_{y} y} \cos \left( \frac{p_{n} z}{d} \right) dy \right] \quad (B78) \]

\[ Z_{f_{y}, \delta_{x, \sigma}} = \frac{1}{j \omega e_{0}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{k_{0}^{2}}{d} \right) e^{-jk_{0} z} \nabla_{x, \sigma} \cdot \int \nabla_{z, \sigma} e^{ik_{y} y} \sin \left( \frac{p_{n} z}{d} \right) dy \right] \quad (B79) \]

\[ Z_{f_{y}, \delta_{z, \sigma}} = \frac{1}{j \omega e_{0}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{k_{0}^{2}}{d} \right) e^{-jk_{0} z} \nabla_{z, \sigma} \cdot \int \nabla_{x, \sigma} e^{ik_{y} y} \cos \left( \frac{p_{n} z}{d} \right) dy \right] \quad (B80) \]

\[ T_{f_{y}, \delta_{x, \sigma}} = \frac{1}{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{e^{-jk_{0} z}}{\sin \left( \frac{p_{n}}{d} \right)} \nabla_{x, \sigma} \cdot \int \nabla_{z, \sigma} e^{ik_{y} y} \sin \left( \frac{p_{n}}{d} \right) dy \right] \quad (B81) \]

\[ T_{f_{y}, \delta_{y, \sigma}} = 0 \quad (B82) \]
\[ T_{ff_{, ij}} = \frac{1}{ab} \sum_{m=-n}^{\infty} \sum_{n=-m}^{\infty} \frac{j k_m e^{-j k_m z}}{k_{mn} \sin (k_{mn} d)} \cdot \int_{S_f} J_{ij} e^{-j k_{mn} \cos (k_{mn} z + d)} dydz \int_{S_s} M_{s2} e^{j k_m x} e^{j k_m y} dx' dy' \]  
(B83)

\[ T_{ff_{, ij}} = -\frac{1}{ab} \sum_{m=-n}^{\infty} \sum_{n=-m}^{\infty} \frac{j k_m e^{-j k_m z}}{k_{mn} \sin (k_{mn} d)} \cdot \int_{S_f} J_{ij} e^{-j k_{mn} \cos (k_{mn} z + d)} dydz \int_{S_s} M_{s2} e^{j k_m x} e^{j k_m y} dx' dy' \]  
(B84)

\[ T_{a, f_{s, ij}} = 0 \]  
(B85)

\[ T_{a, f_{s, ij}} = \sum_{m=-n}^{\infty} \sum_{n=-m}^{\infty} \frac{1}{d} \int_{S_s} M_{s2} e^{-j k_m z} \left( -j \gamma_{mp} D^+ e^{-j \gamma_{mp} y} + j \gamma_{mp} D^- e^{j \gamma_{mp} y} + D^+ e^{-j \gamma_{mp} y} + D^- e^{j \gamma_{mp} y} \right) ds \]  
(B86)

\[ T_{a, f_{s, ij}} = \sum_{m=-n}^{\infty} \sum_{n=-m}^{\infty} \frac{1}{d} \left( \frac{\partial \Pi}{\partial x} \right) \int_{S_s} M_{s2} e^{j k_m z} \left( D^+ e^{-j \gamma_{mp} y} + D^- e^{j \gamma_{mp} y} + D^+ e^{-j \gamma_{mp} y} + D^- e^{j \gamma_{mp} y} \right) ds \]  
(B87)

\[ T_{a, f_{s, ij}} = \sum_{m=-n}^{\infty} \sum_{n=-m}^{\infty} \frac{(j k_m)}{d} \int_{S_s} M_{s2} e^{-j k_m z} \left( D^+ e^{-j \gamma_{mp} y} + D^- e^{j \gamma_{mp} y} + D^+ e^{-j \gamma_{mp} y} + D^- e^{j \gamma_{mp} y} \right) ds \]  
(B88)

\[ T_{a, f_{s, ij}} = -\sum_{n=-m}^{\infty} \sum_{m=-n}^{\infty} \left( \frac{\partial \Pi}{\partial x} \right) \int_{S_s} M_{s2} e^{j k_m z} \left( C^+ e^{-j \beta_{np} x} + C^- e^{j \beta_{np} x} + C^+ e^{-j \beta_{np} x} + C^- e^{j \beta_{np} x} \right) e^{j k_m y} ds \]  
(B89)

\[ T_{a, f_{s, ij}} = -\sum_{n=-m}^{\infty} \sum_{m=-n}^{\infty} \left( \frac{\partial \Pi}{\partial x} \right) \int_{S_s} M_{s2} e^{j k_m z} \left( C^+ e^{-j \beta_{np} x} + C^- e^{j \beta_{np} x} + C^+ e^{-j \beta_{np} x} + C^- e^{j \beta_{np} x} \right) e^{-j k_m y} ds \]  
(B90)

\[ T_{a, f_{s, ij}} = 0 \]  
(B91)
\[ T_{y_a,y_b,\ell q} = \sum_{n=-\infty}^{\infty} \sum_{p=0}^{\infty} \frac{1}{S_a} \int M_{y_a} \left( -j\beta_{np} C_{np} e^{-j(k^2_d)} + j\beta_{np} C_{np} e^{-j(k^2_d)} \right) e^{-j(k^2_d) sgn (x - x')} e^{-j(k^2_d) dy'dc'} \]

\[ y_{u,a,xx,ts} = \frac{1}{j\omega_\mu_0 ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(k_0^2 - k^2_m)}{k_m} \left( \frac{\cos (k_m x)}{\sin (k_m x)} + j \right) \int M_{xx} e^{-j(k^2_d) e^{-j(k^2_d)} dy'dy'} \]

\[ y_{u,a,xy,ts} = \frac{1}{j\omega_\mu_0 ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(k_0^2 - k^2_m)}{k_m} \left( \frac{\cos (k_m x)}{\sin (k_m x)} + j \right) \int M_{xx} e^{-j(k^2_d) e^{-j(k^2_d)} dy'dy'} \]

\[ y_{u,a,xs,ts} = \frac{1}{j\omega_\mu_0 ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(k_0^2 - k^2_m)}{k_m} \left( \frac{\cos (k_m x)}{\sin (k_m x)} + j \right) \int M_{xx} e^{-j(k^2_d) e^{-j(k^2_d)} dy'dy'} \]

\[ y_{u,a,ys,ts} = \frac{1}{j\omega_\mu_0 ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(k_0^2 - k^2_m)}{k_m} \left( \frac{\cos (k_m x)}{\sin (k_m x)} + j \right) \int M_{xx} e^{-j(k^2_d) e^{-j(k^2_d)} dy'dy'} \]
B3: Closed form expressions for the integrals

The integrals obtained form the method of moments have been calculated in closed form and can be written as a combination of the integrals derived in this paragraph. The integrals have lower indices consisting of a ranking number and the integration variable, separated by a comma. The integrals also have an upper index, which describes the location of the basis or testing function, i.e. on the fin parallel to the x-axis (indicated by $f_x$), on the fin parallel to the y-axis (indicated by $f_y$) or on the aperture plane (indicated by $a$).

\[ I_{1, x}^s(m_r, n, p) = \int_0^{x_k} e^{jk_x x} \left( C_{np} e^{-j\beta_np x} + C_{np} e^{j\beta_np x} + C_{np}^* e^{-j\beta_np x} - x_l \right) dx = \frac{2ek_x x_k}{bd \left( k_x^2 - \beta_{np}^2 \right)} \]  

(B97)

\[ I_{2, x}^s(m_r, n, p) = \int_0^{x_k} e^{jk_x x} \left(-j\beta_np e^{-j\beta_np x} + j\beta_np C_{np} e^{j\beta_np x} - j\beta_np C_{np} e^{-j\beta_np x} - x_l \right) dx = \frac{-2j\beta_np e^{jk_x x}}{bd \left( k_x^2 - \beta_{np}^2 \right)} \]  

(B98)

\[ I_{3, x}^s(x_c, m) = \int_{x_k - W_l}^{x_k + W_l} \left( \frac{\sin k (W_z - |x - x_c|)}{W_z \sin (k e W_z)} \right) e^{-jke_{W_z} x} dx = \frac{2ke^{-jke_{W_z} x}}{W_z \sin (k e W_z)} \frac{\cos (k m W_{z/2}) - \cos (k e W_z)}{k^2 - (k^2)^2} \]  

(B99)

For the feedstrip coordinates, a slightly different definition is used for the basis and testing function. For the left feedstrip coordinate $x_{c-1}$, Eq(B100) obtained:

\[ I_{3, x}^s(x_c, m) = \int_{x_k - W_l}^{x_k + W_l} \left( \frac{\sin k (W_z - |x - x_c|)}{W_z \sin (k e W_z)} \right) e^{-jke_{W_z} x} dx = \frac{2ke^{-jke_{W_z} x}}{W_z \sin (k e W_z)} \frac{\cos (k m W_{z/2}) - \cos (k e W_z)}{k^2 - (k^2)^2} \]  

(B100)

For the centre feedstrip coordinate $x_c$:

\[ I_{3, x}^s(x_c, m) = \int_{x_k - W_l}^{x_k + W_l} \left( \frac{\sin k (W_z - |x - x_c|)}{W_z \sin (k e W_z)} \right) e^{-jke_{W_z} x} dx = \frac{2ke^{-jke_{W_z} x}}{W_z \sin (k e W_z)} \frac{\cos (k m W_{z/2}) - \cos (k e W_z)}{k^2 - (k^2)^2} \]  

(B101)

and for the right feedstrip coordinate, $x_{c+1}$:

\[ I_{3, x}^s(x_c, m) = \int_{x_k - W_l}^{x_k + W_l} \left( \frac{\sin k (W_z - |x - x_c + 1|)}{W_z \sin (k e W_z)} \right) e^{-jke_{W_z} x} dx + \int_{x_k + W_l}^{x_k + W_l} \left( \frac{\sin k (W_z - |x - x_c|)}{W_z \sin (k e W_z)} \right) e^{-jke_{W_z} x} dx = \frac{2ke^{-jke_{W_z} x}}{W_z \sin (k e W_z)} \frac{\cos (k m W_{z/2}) - \cos (k e W_z)}{k^2 - (k^2)^2} \]  

(B102)
\[ \begin{align*} 
I_{x,x}(x_k, n, p) &= \int_{x_k - W_x^p}^{x_k + W_x^p} \frac{\sin k_x (W_x^p - |x - x_k|)}{W_x^p \sin (k_x W_x^p)} \left( -j\beta_{np} C_{np} e^{-\beta_{np} x} + j\beta_{np} C_{np} e^{\beta_{np} x} - j\beta_{np} C_{np} e^{-\beta_{np} x} + j\beta_{np} C_{np} e^{\beta_{np} x} \right) e^{-j\beta_{np} e^{\beta_{ps} r}} \left( x - x_b \right) \text{sgn} \left( x - x_b \right) dx 
\end{align*} \] (B104)

\[ \begin{align*} 
I_{x,x}(x_k, n, p) \bigg|_{x_b \leq x_k + W_x^p} &= \frac{2k_x}{bd \left( k_x^2 - \beta_{np}^2 \right)} \left( \cos (\beta_{np} (x_k - x_b + a)) - e^{i\Psi_x} \cos (\beta_{np} (x_k - x_b)) \right) \left( \cos (\beta_{np} W_x^p - \cos (k_x W_x^p)) \right) 
\end{align*} \] (B105)

\[ \begin{align*} 
I_{x,x}(x_k, n, p) \bigg|_{x_k \leq x_b \leq x_k + W_x^p} &= \frac{k_x}{bd \left( k_x^2 - \beta_{np}^2 \right) \cos (\beta_{np} a) - 2 \cos (k_x W_x^p) \cos (\beta_{np} (x_k - x_b))} 
\end{align*} \] (B106)

\[ \begin{align*} 
I_{x,x}(x_k, n, p) \bigg|_{x_k - W_x^p \leq x_b \leq x_k} &= \frac{k_x}{bd \left( k_x^2 - \beta_{np}^2 \right) \cos (\beta_{np} a) - 2 \cos (k_x W_x^p) \cos (\beta_{np} (x_k - x_b))} 
\end{align*} \] (B107)

\[ \begin{align*} 
I_{x,x}(x_k, n, p) \bigg|_{x_b \leq x_k - W_x^p} &= \frac{2k_x}{bd \left( k_x^2 - \beta_{np}^2 \right) \cos (\beta_{np} a) - 2 \cos (k_x W_x^p) \cos (\beta_{np} (x_k - x_b))} 
\end{align*} \] (B108)
Due to the different dimensions of the feedstrip, compared to the dimensions of the other sub-domains on the fin, different definitions are used for $I_{3,x}$ for the feedstrip coordinates $x_{c-1}$, $x_c$ and $x_{c+1}$. For the left feedstrip domain, the definition of Eq(B109) is used:

$$I_{3,x}(x_{c-1}, n, p)$$

$$= \int_{x_{c-1}}^{x_{c-1} - \frac{W_f}{2}} \frac{\sin k_x (W_f x - |x - x_{c-1}|)}{W_z \sin (k_x W_z^f)} \left( -j \beta_{np} C_n e^{j \beta_{np} x} + j \beta_{np} C_n e^{-j \beta_{np} x} - j \beta_{np} C_n e^{j \beta_{np} x} - j \beta_{np} C_n e^{-j \beta_{np} x} \right) sgn (x - x^b) \ dx$$

$$+ \int_{x_{c-1} + \frac{W_f}{2}}^{x_{c-1} + W^f} \frac{\sin k_x (W_f x - |x - x_{c-1}|)}{W_z \sin (k_x W_z^f)} \left( -j \beta_{np} C_n e^{j \beta_{np} x} + j \beta_{np} C_n e^{-j \beta_{np} x} - j \beta_{np} C_n e^{j \beta_{np} x} - j \beta_{np} C_n e^{-j \beta_{np} x} \right) sgn (x - x^b) \ dx \quad \text{(B109)}$$

$$H_{3,x}(x_{c-1}, n, p)\big|_{x^b \geq x_{c-1} + \frac{W_f}{2}} = \frac{k_x}{W_z \sin (k_x W_z^f)} \left( e^{-j \Psi_3} \left( \cos (\beta_{np} (x_{c-1} - W_f x - x^b)) - \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} - x^b)) \right) + \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} - x^b)) \right)$$

$$+ e^{-j \Psi_4} \left( \cos (\beta_{np} (x_{c-1} + W_f x - x^b)) - \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} - x^b)) \right) + \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} - x^b))$$

$$+ \cos (\beta_{np} (x_{c-1} - a + W_f x^b)) \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} + a - x^b)) - \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} + a - x^b))$$

$$- \cos (\beta_{np} (x_{c-1} + a + W_f x^b)) \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} + a - x^b)) \quad \text{(B110)}$$

$$H_{3,x}(x_{c-1}, n, p)\big|_{x_{c-1} - \frac{W_f}{2} \leq x \leq x_{c-1} + \frac{W_f}{2} + W_f} = \frac{k_x}{W_z \sin (k_x W_z^f)} \left( e^{-j \Psi_3} \left( \cos (\beta_{np} (x_{c-1} - W_f x - x^b)) - \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} - x^b)) \right) + \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} - x^b)) \right)$$

$$+ e^{-j \Psi_4} \left( \cos (\beta_{np} (x_{c-1} + W_f x - x^b)) - \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} - x^b)) \right) + \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} - x^b))$$

$$+ \cos (\beta_{np} (x_{c-1} - a + W_f x^b)) \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} + a - x^b)) - \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} + a - x^b))$$

$$+ \cos (\beta_{np} (x_{c-1} - a + W_f x^b)) \cos (k_x W_z^f) \cos (\beta_{np} (x_{c-1} + a - x^b)) \quad \text{(B111)}$$
\[ I_{31}^{\pm}(x_{c} - 1, n, p) \bigg|_{x_{c-1} - W_{sf} - x} = \frac{k_{e}}{bd(k_{e}^{2} - \beta_{np}^{2}) (\cos (\beta_{np} a) - \cos (\Psi_{x}))} \]

\[ + e^{-\Psi_{x}} \left( \cos \left( k_{e} (x_{c-1} - W_{sf} - x) \right) - \cos \left( k_{e} W_{sf} \right) \cos (\beta_{np} (x_{c-1} - x)) \right) \frac{W_{sf}^{\pm} \sin (k_{e} W_{sf}^{\pm})}{W_{sf}^{\pm} \sin (k_{e} W_{sf}^{\pm})} \]

\[ + \frac{\cos (\beta_{np} (x_{c-1} - a - W_{sf} - x)) - 2 \cos (k_{e} (x_{c-1} - W_{sf} - x)) \cos (\beta_{np} a)}{W_{sf}^{\pm} \sin (k_{e} W_{sf}^{\pm})} \]

\[ I_{31}^{0}(x_{c} - 1, n, p) \bigg|_{x_{c-1} - W_{sf} - x} = \frac{k_{e}}{bd(k_{e}^{2} - \beta_{np}^{2}) (\cos (\beta_{np} a) - \cos (\Psi_{x}))} \]

\[ + e^{-\Psi_{x}} \left( \cos \left( k_{e} (x_{c-1} - W_{sf} - x) \right) - \cos \left( k_{e} W_{sf} \right) \cos (\beta_{np} (x_{c-1} - x)) \right) \frac{W_{sf}^{\pm} \sin (k_{e} W_{sf}^{\pm})}{W_{sf}^{\pm} \sin (k_{e} W_{sf}^{\pm})} \]

\[ + \frac{\cos (\beta_{np} (x_{c-1} - a - W_{sf} - x)) - \cos (k_{e} W_{sf}) \cos (\beta_{np} (x_{c-1} - a - x))}{W_{sf}^{\pm} \sin (k_{e} W_{sf}^{\pm})} \]

For the centre feedstrip domain, definition (B114) is used:

\[ I_{31}^{\pm}(x_{c} - 1, n, p) \]

\[ x_{c} + W_{sf} \sin k_{e} (W_{sf}^{\pm} |x - x_{c}| - j\beta_{np} C_{np} e^{-j\beta_{np} x} + j\beta_{np} C_{np} e^{j\beta_{np} x} - j\beta_{np} C_{np} e^{-j\beta_{np} x} - j\beta_{np} C_{np} e^{j\beta_{np} x} \operatorname{sgn} (x - x_{c}) \right) dx \] (B114)

\[ I_{31}^{0}(x_{c} - 1, n, p) \bigg|_{x_{c-1} - W_{sf} - x} = 2k_{e} \left( \cos (\beta_{np} (x_{c} - x)) - e^{j\Psi_{x}} \cos (\beta_{np} (x_{c} - x)) \right) \left( \cos (\beta_{np} W_{sf}) - \cos (k_{e} W_{sf}) \right) \]

\[ \frac{bd(k_{e}^{2} - \beta_{np}^{2}) (\cos (\beta_{np} a) - \cos (\Psi_{x}))}{W_{sf}^{\pm} \sin (k_{e} W_{sf}^{\pm})} \] (B115)
Dual polarization

\[
\begin{align*}
I_{\ell z}(x_c, n, p)_{|x_z - w_{yz} \leq x_z \leq x_c + w_{yz}} &= \frac{k_e}{bd \left(k_e^2 - \beta_{np}^2\right) \left(\cos(\beta_{np} a) - \cos(\Psi_x)\right) W_{z_f} \sin (k_e W_{z_f})} \\
& \cdot \left(-e^{j\Psi_x} \left[\cos(\beta_{np} (x_c - W_{z_f} x^b)) + \cos(k_e (x_c + W_{z_f} x^b)) - 2 \cos(k_e W_{z_f}) \cos(\beta_{np} (x_c - x^b))\right]\right) \\
& + e^{j\Psi_x} \left[\cos(\beta_{np} (x_c + W_{z_f} x^b)) - \cos(k_e (x_c + W_{z_f} x^b))\right] \\
& + \cos(\beta_{np} (x_c + a - W_{z_f} x^b)) + 2 \cos(k_e (x_c + W_{z_f} x^b)) \cos(\beta_{np} a) - 2 \cos(k_e W_{z_f}) \cos(\beta_{np} (x_c + a - x^b)) \\
& - \cos(\beta_{np} (x_c^z - W_{z_f} x^b - a)) \right) \\
\end{align*}
\] (B116)

\[
\begin{align*}
I_{\ell z}(x_c, n, p)_{|x_c - w_{yz} \leq x_z \leq x_c + w_{yz}} &= \frac{k_e}{bd \left(k_e^2 - \beta_{np}^2\right) \left(\cos(\beta_{np} a) - \cos(\Psi_x)\right) W_{z_f} \sin (k_e W_{z_f})} \\
& \cdot \left(-e^{j\Psi_x} \left[\cos(\beta_{np} (x_c - W_{z_f} x^b)) + \cos(k_e (x_c - W_{z_f} x^b)) - 2 \cos(k_e W_{z_f}) \cos(\beta_{np} (x_c - x^b))\right]\right) \\
& + e^{j\Psi_x} \left[\cos(\beta_{np} (x_c - W_{z_f} x^b)) - \cos(k_e (x_c - W_{z_f} x^b))\right] \\
& + \cos(\beta_{np} (x_c + a - W_{z_f} x^b)) - 2 \cos(k_e (x_c - W_{z_f} x^b)) \cos(\beta_{np} a) + 2 \cos(k_e W_{z_f}) \cos(\beta_{np} (x_c - a - x^b)) \\
& - \cos(\beta_{np} (x_c + W_{z_f} x^b - a)) \right) \\
\end{align*}
\] (B117)

For the right feedstrip coordinate, the definition (B119) is used:

\[
\begin{align*}
I_{\ell z}(x_{c+1}, n, p) &= \int_{|x_{c+1} - w_{yz} |}^{x_{c+1} + w_{yz}} \frac{\sin k_e (W_{z_f} - |x - x_{c+1}|)}{W_{z_f} \sin (k_e W_{z_f})} \left(-j \beta_{np} C_{np} e^{-j k_e x} + j \beta_{np} C_{np} e^{j k_e x} - j \beta_{np} C_{np} e^{-j k_e x} - j \beta_{np} C_{np} e^{j k_e x} e^{-j k_e x} e^{j k_e x} e^{-j k_e x} sgn (x - x^b)\right) dx \\
& + \int_{|x_{c+1} - w_{yz} |}^{x_{c+1} + w_{yz}} \frac{\sin k_e (W_{z_f} - |x - x_{c+1}|)}{W_{z_f} \sin (k_e W_{z_f})} \left(-j \beta_{np} C_{np} e^{-j k_e x} + j \beta_{np} C_{np} e^{j k_e x} - j \beta_{np} C_{np} e^{-j k_e x} - j \beta_{np} C_{np} e^{j k_e x} e^{-j k_e x} e^{j k_e x} e^{-j k_e x} sgn (x - x^b)\right) dx \\
\end{align*}
\] (B119)

\[
\begin{align*}
I_{\ell z}(x_{c+1}, n, p)_{|x_{c+1} + a + W_{z_f} x^b} &= \frac{k_e}{bd \left(k_e^2 - \beta_{np}^2\right) \left(\cos(\beta_{np} a) - \cos(\Psi_x)\right) W_{z_f} \sin (k_e W_{z_f})} \\
& \cdot \left(-j\Psi_x \left[\cos(\beta_{np} (x_{c+1} + a - W_{z_f} x^b)) - \cos(k_e W_{z_f}) \cos(\beta_{np} (x_{c+1} - x^b))\right]\right) \\
& + \cos(\beta_{np} (x_{c+1} + a - W_{z_f} x^b)) - \cos(k_e W_{z_f}) \cos(\beta_{np} (x_{c+1} + a - x^b)) \\
& + \cos(\beta_{np} (x_{c+1} + a + W_{z_f} x^b)) - \cos(k_e W_{z_f}) \cos(\beta_{np} (x_{c+1} + a - x^b)) \right) \\
\end{align*}
\] (B120)
\[ I_{5,s}^{F}(x_{c+1}, n, p)_{x_{c+1} \leq x \leq x_{c+1} + W_{s}^{F}} = k_e \left\{ \begin{array}{l} \frac{bd}{W_{s}^{F} \sin(k_e W_{s}^{F})} \left[ \cos(\beta_{np}(x_{c+1} - W_{s}^{F} - x_{c}^{b})) - \cos(k_e W_{s}^{F}) \cos(\beta_{np}(x_{c+1} - x_{c}^{b})) \right] \\
+ \frac{1}{W_{s}^{F} \sin(k_e W_{s}^{F})} \left[ \cos(\beta_{np}(x_{c+1} + W_{s}^{F} - x_{c}^{b})) - \cos(k_e W_{s}^{F}) \cos(\beta_{np}(x_{c+1} - x_{c}^{b})) \right] \\
+ e^{-j\psi_{s}} \left[ \frac{1}{W_{s}^{F} \sin(k_e W_{s}^{F})} \left( \cos(\beta_{np}(x_{c+1} + W_{s}^{F} - x_{c}^{b})) - \cos(\beta_{np}(x_{c+1} - x_{c}^{b})) \right) \right] \\
+ e^{j\psi_{s}} \left[ \frac{1}{W_{s}^{F} \sin(k_e W_{s}^{F})} \left( \cos(\beta_{np}(x_{c+1} + W_{s}^{F} - x_{c}^{b})) - \cos(\beta_{np}(x_{c+1} - x_{c}^{b})) \right) \right] \right\} (B121) \]

\[ I_{5,s}^{F}(x_{c+1}, n, p)_{x_{c+1} \leq x \leq x_{c+1} + W_{s}^{F}} = k_e \left\{ \begin{array}{l} \frac{bd}{W_{s}^{F} \sin(k_e W_{s}^{F})} \left[ \cos(\beta_{np}(x_{c+1} - W_{s}^{F} - x_{c}^{b})) - \cos(k_e W_{s}^{F}) \cos(\beta_{np}(x_{c+1} - x_{c}^{b})) \right] \\
+ \frac{1}{W_{s}^{F} \sin(k_e W_{s}^{F})} \left[ \cos(\beta_{np}(x_{c+1} + W_{s}^{F} - x_{c}^{b})) - \cos(k_e W_{s}^{F}) \cos(\beta_{np}(x_{c+1} - x_{c}^{b})) \right] \\
+ e^{-j\psi_{s}} \left[ \frac{1}{W_{s}^{F} \sin(k_e W_{s}^{F})} \left( \cos(\beta_{np}(x_{c+1} + W_{s}^{F} - x_{c}^{b})) - \cos(k_e W_{s}^{F}) \cos(\beta_{np}(x_{c+1} - x_{c}^{b})) \right) \right] \\
+ e^{j\psi_{s}} \left[ \frac{1}{W_{s}^{F} \sin(k_e W_{s}^{F})} \left( \cos(\beta_{np}(x_{c+1} + W_{s}^{F} - x_{c}^{b})) - \cos(k_e W_{s}^{F}) \cos(\beta_{np}(x_{c+1} - x_{c}^{b})) \right) \right] \right\} (B122) \]

\[ I_{5,s}^{F}(x_{c+1}, n, p)_{x_{c+1} \leq x \leq x_{c+1} + W_{s}^{F}} = k_e \left\{ \begin{array}{l} \frac{bd}{W_{s}^{F} \sin(k_e W_{s}^{F})} \left[ \cos(\beta_{np}(x_{c+1} - W_{s}^{F} - x_{c}^{b})) - \cos(k_e W_{s}^{F}) \cos(\beta_{np}(x_{c+1} - x_{c}^{b})) \right] \\
+ \frac{1}{W_{s}^{F} \sin(k_e W_{s}^{F})} \left[ \cos(\beta_{np}(x_{c+1} + W_{s}^{F} - x_{c}^{b})) - \cos(k_e W_{s}^{F}) \cos(\beta_{np}(x_{c+1} - x_{c}^{b})) \right] \\
+ e^{-j\psi_{s}} \left[ \frac{1}{W_{s}^{F} \sin(k_e W_{s}^{F})} \left( \cos(\beta_{np}(x_{c+1} + W_{s}^{F} - x_{c}^{b})) - \cos(k_e W_{s}^{F}) \cos(\beta_{np}(x_{c+1} - x_{c}^{b})) \right) \right] \\
+ e^{j\psi_{s}} \left[ \frac{1}{W_{s}^{F} \sin(k_e W_{s}^{F})} \left( \cos(\beta_{np}(x_{c+1} + W_{s}^{F} - x_{c}^{b})) - \cos(k_e W_{s}^{F}) \cos(\beta_{np}(x_{c+1} - x_{c}^{b})) \right) \right] \right\} (B123) \]

\[ I_{4,s}^{F}(x_{c+1}, n, p) = \int_{x_{c+1} - W_{s}^{F}}^{x_{c+1}} \left( C_{n} e^{j\beta_{np} x} + C^{*} e^{-j\beta_{np} x} \right) dx \] (B124)
\[ I_{x}^{f}(x, n, p) \bigg|_{x \leq x_k} = -2\sin\left(\beta_{np}\frac{W_{Pz}}{2}\right)\left(e^{i\Psi}\sin\beta_{np}(x_k - x^b - \frac{W_{Pz}}{2}) - \sin\beta_{np}(a + x_k - x^b - \frac{W_{Pz}}{2})\right) \frac{\beta_{np}^2 bd (\cos (\beta_{np} a) - \cos (\Psi))}{\beta_{np}^2 bd (\cos (\beta_{np} a) - \cos (\Psi))} \]

\[ I_{x}^{f}(x, n, p) \bigg|_{x_k - W_{Pz} \leq x^* \leq x_k} = \frac{\cos\beta_{np}(a + x_k - W_{Pz} - x^b) - \cos (\beta_{np} a) - e^{-i\Psi}(\cos\beta_{np}(x_k - W_{Pz} - x^b) - 1)}{\beta_{np}^2 bd (\cos (\beta_{np} a) - \cos (\Psi))} \]

\[ I_{x}^{f}(x, n, p) \bigg|_{x_k - a - x^b} = \frac{2\sin\left(\beta_{np}\frac{W_{Pz}}{2}\right)\left(-e^{i\Psi}\sin\beta_{np}(x_k - x^b - \frac{W_{Pz}}{2}) - \sin\beta_{np}(x_k - a - x^b - \frac{W_{Pz}}{2})\right)}{\beta_{np}^2 bd (\cos (\beta_{np} a) - \cos (\Psi))} \]

\[ I_{x, y}^{f}(n, m, p) = \int_{0}^{b} e^{jk_{n}y} \left( D_{m}^{e^{-j\gamma_{mp}y}} + D_{m}^{e^{-j\gamma_{mp}y}} + D_{m}^{e^{-j\gamma_{mp}y}} + D_{m}^{e^{-j\gamma_{mp}y}} \right) dy = \frac{2e^{jk_{n}y \theta}}{ad\left(k_{n}^{2} - y_{mp}^{2}\right)} \]

\[ I_{x, y}^{f}(n, m, p) = \int_{0}^{b} e^{jk_{n}y} \left( D_{m}^{e^{-j\gamma_{mp}y}} + D_{m}^{e^{-j\gamma_{mp}y}} + D_{m}^{e^{-j\gamma_{mp}y}} + D_{m}^{e^{-j\gamma_{mp}y}} \right) dy = \frac{2e^{jk_{n}y \theta}}{ad\left(k_{n}^{2} - y_{mp}^{2}\right)} \]

For the feedstrip coordinates \( y_{c-1}, y_{c} \) and \( y_{c+1} \), a slightly different definition is used.

\[ I_{y}^{f}(y_{c-1}, n) = \int \sin k_{e}(W_{Pz} - |y - y_{c-1}|) e^{-jk_{n}y} dy = \frac{2e^{jk_{n}y \theta}}{k_{z}^{2} - (k_{n}^{2})} \]

\[ I_{y}^{f}(y_{c}, n) = \int \sin k_{e}(W_{Pz} - |y - y_{c}|) e^{-jk_{n}y} dy = \frac{2e^{jk_{n}y \theta}}{k_{z}^{2} - (k_{n}^{2})} \]

\[ I_{y}^{f}(y_{c+1}, n) = \int \sin k_{e}(W_{Pz} - |y - y_{c+1}|) e^{-jk_{n}y} dy = \frac{2e^{jk_{n}y \theta}}{k_{z}^{2} - (k_{n}^{2})} \]
\begin{align}
I_{Y_j}^{y}(y_k, n) &= \frac{\gamma_k}{\gamma_k - w_{\gamma}} \int e^{-j k_{\gamma}^2 y} dy = \frac{2}{k_{\gamma}^2} e^{-j k_{\gamma}^2 (y_k - w_{\gamma}/2)} \sin \left( \frac{k_{\gamma}^2 w_{\gamma}}{2} \right) \quad (B134) \\
I_{Y_j}^{y}(y_k, m, p) &= \int_{y_k + w_{\gamma}}^{y_k + w_{\gamma} + w_{\gamma}} \frac{\sin k_{\gamma} (W_{\gamma}/y - |y - y_k|)}{W_{\gamma} \sin (k_{\gamma} W_{\gamma}/y)} \left( - j \gamma_{mp} D_{mp}^e e^{-j \gamma_{mp} y} + j \gamma_{mp} D_{mp}^e e^{j \gamma_{mp} y} - j \gamma_{mp} D_{mp}^e e^{-j \gamma_{mp} y} - j \gamma_{mp} D_{mp}^e e^{j \gamma_{mp} y} \right) \left( y^2 \sin (y - y_k) \right) dy \quad (B135) \\
I_{Y_j}^{y}(y_k, m, p) &\mid_{y_k + w_{\gamma} \leq y_k + w_{\gamma}} = \frac{k_{\gamma}}{W_{\gamma} \sin (k_{\gamma} W_{\gamma}/y)} \left( - j \gamma_{mp} \cos (\gamma_{mp} (y_k + b - y_{\gamma})) + j \gamma_{mp} \cos (\gamma_{mp} (y_k - y_{\gamma})) + \cos (\gamma_{mp} (y_k + W_{\gamma}/y - y_{\gamma})) + \cos (\gamma_{mp} (y_k - W_{\gamma}/y - y_{\gamma})) - 2 \cos (k_{\gamma} W_{\gamma}/y) \cos (\gamma_{mp} (y_k - y_{\gamma})) \right) + \left( \cos (\gamma_{mp} (y_k + b - W_{\gamma}/y - y_{\gamma})) + 2 \cos (k_{\gamma} (y_k + W_{\gamma}/y - y_{\gamma})) \cos (\gamma_{mp} (y_k + b - y_{\gamma})) \right) - \cos (\gamma_{mp} (y_k + W_{\gamma}/y - y_{\gamma} - b)) \right) \quad (B136) \\
I_{Y_j}^{y}(y_k, m, p) &\mid_{y_k - w_{\gamma} \leq y_k + w_{\gamma}} = \frac{k_{\gamma}}{W_{\gamma} \sin (k_{\gamma} W_{\gamma}/y)} \left( - j \gamma_{mp} \cos (\gamma_{mp} (y_k - y_{\gamma})) - j \gamma_{mp} \cos (\gamma_{mp} (y_k + b - y_{\gamma})) + \cos (\gamma_{mp} (y_k - W_{\gamma}/y - y_{\gamma})) - \cos (\gamma_{mp} (y_k - W_{\gamma}/y - y_{\gamma} - b)) \right) - \cos (\gamma_{mp} (y_k + W_{\gamma}/y - y_{\gamma} - b)) \right) \quad (B137) \\
I_{Y_j}^{y}(y_k, m, p) &\mid_{y_k - w_{\gamma} \leq y_k - w_{\gamma}} = \frac{k_{\gamma}}{W_{\gamma} \sin (k_{\gamma} W_{\gamma}/y)} \left( - j \gamma_{mp} \cos (\gamma_{mp} (y_k - y_{\gamma})) - j \gamma_{mp} \cos (\gamma_{mp} (y_k + b - y_{\gamma})) - 2 \cos (k_{\gamma} (y_k - W_{\gamma}/y - y_{\gamma})) \cos (\gamma_{mp} (y_k - b - y_{\gamma})) \right) - \cos (\gamma_{mp} (y_k + W_{\gamma}/y - y_{\gamma} - b)) \right) \quad (B138) \\
I_{Y_j}^{y}(y_k, m, p) &\mid_{y_k - w_{\gamma} \leq y_k - w_{\gamma}} = \frac{2 k_{\gamma}}{W_{\gamma} \sin (k_{\gamma} W_{\gamma}/y)} \left( - j \gamma_{mp} \cos (\gamma_{mp} (y_k - y_{\gamma})) - j \gamma_{mp} \cos (\gamma_{mp} (y_k + b - y_{\gamma})) + \cos (\gamma_{mp} (y_k - W_{\gamma}/y - y_{\gamma})) - \cos (\gamma_{mp} (y_k - W_{\gamma}/y - y_{\gamma} - b)) \right) \quad (B139) 
\end{align}
Due to the different dimensions of the feedstrip domains, a different definition is used for $I_{g, y}$.

For the left feedstrip coordinate, $y_{c-I}$, definition (B140) is used:

$$
I_{g, y}^{L}(y_{c-I}, m, p) = \begin{cases} 
\frac{\sin k_{e}(W_{g}^{L} - |y - y_{c-I}|)}{W_{g}^{L} \sin (k_{e} W_{g}^{L})} & \text{for } y_{c-1} - W_{g}^{L} \\
+ \frac{\sin k_{e}(W_{g}^{L} - |y - y_{c-I}|)}{W_{g}^{L} \sin (k_{e} W_{g}^{L})} & \text{for } y_{c-1} + W_{g}^{L} 
\end{cases}
$$

$$
\int_{y_{c-1} - W_{g}^{L}}^{y_{c-1} + W_{g}^{L}} \left( -jY_{mp} D_{mp} e^{i\Phi_{mp}} + jY_{mp} D_{mp}^{*} e^{-i\Phi_{mp}} - jY_{mp} D_{mp}^{*} e^{i\Phi_{mp}} + jY_{mp} D_{mp} e^{-i\Phi_{mp}} \right) \text{sgn} (y - y_{b}) \, dy
$$

(B140)

$$
I_{g, y}^{L}(y_{c-I}, m, p) \bigg|_{y_{c-1} + W_{g}^{L}} = \frac{k_{e}}{ad \left( k_{e}^{2} - \gamma_{mp}^{2} \right)} \left( \cos \left( \gamma_{mp} (y_{c-I} - W_{g}^{L} - y_{b}) \right) - \cos \left( k_{e} W_{g}^{L} \cos \left( \gamma_{mp} (y_{c-I} - y_{b}) \right) \right) \right)
$$

$$
+ \frac{\cos \left( \gamma_{mp} (y_{c-I} + W_{g}^{L} - y_{b}) \right) - \cos \left( k_{e} W_{g}^{L} \cos \left( \gamma_{mp} (y_{c-I} - y_{b}) \right) \right) \cos \left( \gamma_{mp} (y_{c-I} + b - y_{b}) \right) - \cos \left( k_{e} W_{g}^{L} \cos \left( \gamma_{mp} (y_{c-I} + b - y_{b}) \right) \right) \cos \left( \gamma_{mp} (y_{c-I} + b - y_{b}) \right) - \cos \left( k_{e} W_{g}^{L} \cos \left( \gamma_{mp} (y_{c-I} + b - y_{b}) \right) \right) \right) \} \right)
$$

(B141)

$$
I_{g, y}^{L}(y_{c-I}, m, p) \bigg|_{y_{c-1} + W_{g}^{L}} = \frac{k_{e}}{ad \left( k_{e}^{2} - \gamma_{mp}^{2} \right)} \left( \cos \left( \gamma_{mp} (y_{c-I} - W_{g}^{L} - y_{b}) \right) - \cos \left( k_{e} W_{g}^{L} \cos \left( \gamma_{mp} (y_{c-I} - y_{b}) \right) \right) \right)
$$

$$
+ \frac{\cos \left( \gamma_{mp} (y_{c-I} + W_{g}^{L} - y_{b}) \right) - \cos \left( k_{e} W_{g}^{L} \cos \left( \gamma_{mp} (y_{c-I} - y_{b}) \right) \right) \cos \left( \gamma_{mp} (y_{c-I} + b - y_{b}) \right) - \cos \left( k_{e} W_{g}^{L} \cos \left( \gamma_{mp} (y_{c-I} + b - y_{b}) \right) \right) \cos \left( \gamma_{mp} (y_{c-I} + b - y_{b}) \right) - \cos \left( k_{e} W_{g}^{L} \cos \left( \gamma_{mp} (y_{c-I} + b - y_{b}) \right) \right) \right) \} \right)
$$

(B142)
\[
I_{b,y}(y_{e-1}, m, p) \Big|_{y_{e-1} - W_{y} \leq y' \leq y_{e-1}} = \frac{k_e}{ad(k_e^2 - \gamma_{mp}^2)} \left( \cos(\gamma_{mp}b) - \cos(\Psi_{y}) \right)
\]

\[
e^{-\Psi_{y}} \left( \frac{\cos(k_e(y_{e-1} - W_{y} - y')) - \cos(k_e W_{y}) \cos(\gamma_{mp}(y_{e-1} - y'))}{W'_{y} \sin(k_e W_{y})} \right)
\]

\[
= \frac{\cos(\gamma_{mp}(y_{e-1} + W_{y} - y')) - \cos(k_e W_{y}) \cos(\gamma_{mp}(y_{e-1} - y'))}{W'_{y} \sin(k_e W_{y})}
\]

\[
+ \frac{\cos(\gamma_{mp}(y_{c-1} - b - W_{y} - y')) - 2\cos(k_e(y_{e-1} - W_{y} - y')) \cos(\gamma_{mp}b)}{W'_{y} \sin(k_e W_{y})}
\]

\[
= \frac{\cos(\gamma_{mp}(y_{e-1} - b + W_{y} - y')) - \cos(k_e W_{y}) \cos(\gamma_{mp}(y_{e-1} - b - y'))}{W'_{y} \sin(k_e W_{y})}
\]

\[
= \frac{\cos(\gamma_{mp}(y_{e-1} - W_{y} - y')) - \cos(k_e W_{y}) \cos(\gamma_{mp}(y_{e-1} - y'))}{W'_{y} \sin(k_e W_{y})}
\]

\[\text{(B143)}\]

\[
I_{b,y}(y_{e-1}, m, p) \Big|_{y_{e-1} - W_{y} \leq y' \leq y_{e-1}} = \frac{k_e}{ad(k_e^2 - \gamma_{mp}^2)} \left( \cos(\gamma_{mp}b) - \cos(\Psi_{y}) \right)
\]

\[
e^{-\Psi_{y}} \left( \frac{\cos(\gamma_{mp}(y_{e-1} - W_{y} - y')) - \cos(k_e W_{y}) \cos(\gamma_{mp}(y_{e-1} - y'))}{W'_{y} \sin(k_e W_{y})} \right)
\]

\[
= \frac{\cos(\gamma_{mp}(y_{e-1} - b - W_{y} - y')) - \cos(k_e W_{y}) \cos(\gamma_{mp}(y_{e-1} - b - y'))}{W'_{y} \sin(k_e W_{y})}
\]

\[
+ \frac{\cos(\gamma_{mp}(y_{c-1} - W_{y} - y')) - 2\cos(k_e(y_{e-1} - W_{y} - y')) \cos(\gamma_{mp}b)}{W'_{y} \sin(k_e W_{y})}
\]

\[
= \frac{\cos(\gamma_{mp}(y_{e-1} - b + W_{y} - y')) - \cos(k_e W_{y}) \cos(\gamma_{mp}(y_{e-1} - b - y'))}{W'_{y} \sin(k_e W_{y})}
\]

\[\text{(B144)}\]

For the centre feedstrip coordinate, \(y_{c}\), definition (B145) is used

\[
I_{b,y}(y_{e}, m, p)
\]

\[
= \int_{y_{e} - W_{y}}^{y_{e} + W_{y}} \frac{\sin(k_e(W_{y} - y'))}{W'_{y} \sin(k_e W_{y})} \left( -j\gamma_{mp}D_{mp}e^{-j\omega_{y}y} + j\gamma_{mp}D_{mp}e^{j\omega_{y}y} - j\gamma_{mp}D_{mp}^{*}e^{-j\omega_{y}y} - j\gamma_{mp}D_{mp}^{*}e^{j\omega_{y}y} \text{sgn}(y - b) \right) dy
\]

\[\text{(B145)}\]

\[
I_{b,y}(y_{e}, m, p) \Big|_{y' \geq y_{e} + W_{y}} = \frac{2k_e}{ad(k_e^2 - \gamma_{mp}^2)} \left( \cos(\gamma_{mp}(y_{e} - b - y')) - \cos(\gamma_{mp}(y_{e} - y')) \right)
\]

\[
= \frac{\cos(\gamma_{mp}(y_{e} - b - y')) - \cos(\gamma_{mp}(y_{e} - y'))}{W'_{y} \sin(k_e W_{y})}
\]

\[\text{(B146)}\]
\[
\begin{align*}
H_{ij}^c(y_c, m, p) &= \frac{k_e}{y_c s_y s_{y_c} + W_{ij}} \\
&= \frac{a d (k_e^2 - \gamma_{mp}^2)}{W_{ij}^c \sin (k_e W_{sf}^j)} \\
&\cdot \left( \frac{k_e}{-e^{j\psi_Y} \left[ \cos (\gamma_{mp} (y_c - W_{sf}^j - y^b)) + \cos (k_e (y_c + W_{sf}^j - y^b)) - 2 \cos (k_e W_{sf}^j) \cos (\gamma_{mp} (y_c - y^b)) \right]}
+ \frac{k_e}{e^{-j\psi_Y} \left[ \cos (\gamma_{mp} (y_c + W_{sf}^j - y^b)) - \cos (k_e (y_c + W_{sf}^j - y^b)) \right]}
+ \frac{k_e}{\cos (\gamma_{mp} (y_c + b - W_{sf}^j - y^b)) + 2 \cos (k_e (y_c + W_{sf}^j - y^b)) \cos (\gamma_{mp} (y_c + b - y^b)) - \cos (\gamma_{mp} (y_c + W_{sf}^j - y^b))} \right)
\end{align*}
\]

For the right feedstrip coordinate, \( y_{c+1} \), definition (B150) is used.
\[
\begin{align*}
H_{ij}^c(y_{c+1}, m, p) &= 2k_e \left( \frac{e^{j\psi_Y} \cos (\gamma_{mp} (y_c + y^b)) - \cos (\gamma_{mp} (y_c + b - y^b))}{a d (k_e^2 - \gamma_{mp}^2)} \right) \left( \frac{\cos (\gamma_{mp} W_{sf}^j) - \cos (k_e W_{sf}^j)}{W_{sf}^j \sin (k_e W_{pf}^j)} \right) \\
&= \int_{y_{c+1} - W_{sf}^j}^{y_{c+1} - W_{sf}^j} \frac{\sin k_e (W_{sf}^j - (y - y_{c+1}))}{W_{sf}^j \sin (k_e W_{sf}^j)} \left( -j \gamma_{mp} D_{mp}^+ e^{-j\gamma_{mp} y} + j \gamma_{mp} D_{mp}^- e^{j\gamma_{mp} y} - j \gamma_{mp} D_{mp}^+ e^{-j\gamma_{mp} y} - j \gamma_{mp} D_{mp}^- e^{j\gamma_{mp} y} - \sgn (y - y^b) \right) dy \\
&+ \int_{x_{c+1}}^{x_{c+1} - W_{sf}^j} \frac{\sin k_e (W_{sf}^j - (x - x_{c+1}))}{W_{sf}^j \sin (k_e W_{sf}^j)} \left( -j \gamma_{mp} D_{mp}^+ e^{-j\gamma_{mp} x} + j \gamma_{mp} D_{mp}^- e^{j\gamma_{mp} x} - j \gamma_{mp} D_{mp}^+ e^{-j\gamma_{mp} x} - j \gamma_{mp} D_{mp}^- e^{j\gamma_{mp} x} - \sgn (y - y^b) \right) dx
\end{align*}
\]
\[ \begin{align*}
&\mu_s^p(y_{c+1}, m, p)_{|y_{c+1} \geq y_s, p} = \frac{k_e}{W_z^p \sin (k_e W_{yz}^p)} \\
&\quad \cdot \left( -e^{j\zeta'}, \cos (\gamma_{mp}(y_{c+1} - W_{yf}^p - y_p)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} - y_b)) \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + y_b))}{W_z^p \sin (k_e W_{yz}^p)} \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + b - W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + b - y_b))}{W_z^p \sin (k_e W_{yz}^p)} \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + b + W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + b + y_b))}{W_z^p \sin (k_e W_{yz}^p)} \right) \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + b + W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + b + y_b))}{W_z^p \sin (k_e W_{yz}^p)} \right) \\
&\quad \cdot \left( -e^{j\zeta'}, \cos (\gamma_{mp}(y_{c+1} - W_{yf}^p - y_p)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} - y_b)) \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + y_b))}{W_z^p \sin (k_e W_{yz}^p)} \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + b - W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + b - y_b))}{W_z^p \sin (k_e W_{yz}^p)} \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + b + W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + b + y_b))}{W_z^p \sin (k_e W_{yz}^p)} \right) \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + b + W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + b + y_b))}{W_z^p \sin (k_e W_{yz}^p)} \\
&\quad \cdot \left( -e^{j\zeta'}, \cos (\gamma_{mp}(y_{c+1} - W_{yf}^p - y_p)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} - y_b)) \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + y_b))}{W_z^p \sin (k_e W_{yz}^p)} \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + b - W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + b - y_b))}{W_z^p \sin (k_e W_{yz}^p)} \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + b + W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + b + y_b))}{W_z^p \sin (k_e W_{yz}^p)} \right) \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + b + W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + b + y_b))}{W_z^p \sin (k_e W_{yz}^p)} \\
&\quad \cdot \left( -e^{j\zeta'}, \cos (\gamma_{mp}(y_{c+1} - W_{yf}^p - y_p)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} - y_b)) \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + y_b))}{W_z^p \sin (k_e W_{yz}^p)} \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + b - W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + b - y_b))}{W_z^p \sin (k_e W_{yz}^p)} \\
&\quad + \frac{\cos (\gamma_{mp}(y_{c+1} + b + W_{yf}^p - y_b)) - \cos (k_e W_{yz}^p \cos (\gamma_{mp}(y_{c+1} + b + y_b))}{W_z^p \sin (k_e W_{yz}^p)} \right) \right)
\end{align*} \]
\[ V_{y}^{y}(y_{c+1}, m, p) \big|_{y_{c+1} - W_{f}^{j}}^{y_{c+1} - W_{f}^{j}} = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]

\[ = \frac{k_{e}}{W_{f}^{j} \sin(k_{e} W_{f}^{j})} \left[ \cos\left(\gamma_{mp}(y_{c+1} - b) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b))\right) - \cos(k_{e} W_{f}^{j})\cos(\gamma_{mp}(y_{c+1} - b)) \right] \]
\[ I_{x}^{x}(z, m, n) = \int_{z_{l} - W_{y}}^{z_{l}} \sin \left( k_{m n} (z + d) \right) \cos \left( \frac{p \pi z}{d} \right) \frac{k_{e} \cos \left( \frac{p \pi z}{d} \right)}{k_{e} - \left( \frac{p \pi z}{d} \right)^2} \sin \left( \frac{k_{m n} W_{y}}{2} \right) \text{d}z \]

\[ I_{y}^{x}(z, p) = \int_{z_{l} - W_{y}}^{z_{l}} \frac{W_{y}}{W_{y}^{2} \sin \left( k_{e} W_{y}^{2} \right)} \cos \left( \frac{p \pi z}{d} \right) \frac{2 k_{e} \cos \left( \frac{p \pi z}{d} \right)}{W_{y}^{2} \sin \left( k_{e} W_{y}^{2} \right)} \sin \left( \frac{k_{m n} W_{y}}{2} \right) \text{d}z \]

\[ I_{y}^{y}(z, p) = \int_{z_{l} - W_{y}}^{z_{l}} \sin \left( \frac{p \pi z}{d} \right) \text{d}z = \frac{2 d}{p \pi} \sin \left( \frac{p \pi}{d} \left( z_{l} - \frac{W_{y}}{2} \right) \right) \sin \left( \frac{p \pi W_{y}}{2d} \right) \]

\[ I_{x}^{y}(z, m, n) = \int_{z_{l} - W_{y}}^{z_{l}} \sin \left( k_{m n} (z + d) \right) \cos \left( k_{m n} \left( z_{l} - \frac{W_{y}}{2} + d \right) \right) \sin \left( \frac{k_{m n} W_{y}}{2} \right) \text{d}z \]

\[ I_{y}^{y}(z, p) = \int_{z_{l} - W_{y}}^{z_{l}} \sin \left( k_{m n} \left( z_{l} - \frac{W_{y}}{2} + d \right) \right) \sin \left( \frac{k_{m n} W_{y}}{2} \right) \text{d}z \]
B4: Expressions for the matrix elements in closed form

The matrix elements of appendix B2 can be written in the closed form by using the integrals defined in the previous paragraph.

\[
Z_{zx, rq}^{fx, ft} = \frac{1}{j0 \omega E_0} \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \left( k_0^2 - (k_m^2) \right) \left( D_{mp}^+ e^{-j \omega_0 p} + D_{mp}^- e^{j \omega_0 p} + D_{mp}^{*+} \right) \\
\cdot \left[ H_{1,x}(x_k, m) H_{2,z}(z_t, p) \left( H_{1,z}(x_k, m) ^* H_{2,z}(z_t, p) \right) \right] 
\]

(B167)

\[
Z_{zl, rz}^{fx, ft} = \frac{1}{j0 \omega E_0} \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \left( j k_m^x \frac{d}{d \rho} \right) \left( D_{mp}^+ e^{-j \omega_0 p} + D_{mp}^- e^{j \omega_0 p} + D_{mp}^{*+} \right) \\
\cdot \left[ H_{2,x}(x_k, m) H_{1,z}(z_t, p) \left( H_{2,z}(x_k, m) ^* H_{1,z}(z_t, p) \right) \right] 
\]

(B168)

\[
Z_{zz, rz}^{fx, ft} = \frac{1}{j0 \omega E_0} \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \left( j k_m^x \frac{d}{d \rho} \right) \left( D_{mp}^+ e^{-j \omega_0 p} + D_{mp}^- e^{j \omega_0 p} + D_{mp}^{*+} \right) \\
\cdot \left[ H_{2,z}(x_k, m) H_{1,z}(z_t, p) \left( H_{2,z}(x_k, m) ^* H_{1,z}(z_t, p) \right) \right] 
\]

(B169)

\[
Z_{zx, rz}^{fx, ft} = \frac{1}{j0 \omega E_0} \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \left( k_0^2 - \frac{p \rho}{d} \right)^2 \left( D_{mp}^+ e^{-j \omega_0 p} + D_{mp}^- e^{j \omega_0 p} + D_{mp}^{*+} \right) \\
\cdot \left[ H_{2,z}(x_k, m) H_{1,z}(z_t, p) \left( H_{2,z}(x_k, m) ^* H_{1,z}(z_t, p) \right) \right] 
\]

(B170)

\[
\tau_{xy, ry}^{fx, fy} = \frac{1}{j0 \omega E_0} \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} j k_n^{y} e^{-j k_n^{y} \rho} \\
\cdot \left[ H_{3,x}(x_k, n, p) H_{2,z}(z_t, p) \left( H_{2,y}(y_k, n) ^* H_{2,z}(z_t, p) \right) \right] 
\]

(B171)

\[
\tau_{xz, rz}^{fx, fy} = \frac{1}{j0 \omega E_0} \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \left( \frac{p \rho}{d} \right) e^{-j k_n^{y} \rho} \\
\cdot \left[ H_{3,x}(x_k, n, p) H_{2,z}(z_t, p) \left( H_{2,y}(y_k, n) ^* H_{2,z}(z_t, p) \right) \right] 
\]

(B172)

\[
\tau_{zy, rz}^{fx, fy} = \frac{1}{j0 \omega E_0} \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \left( j k_n^{y} \frac{d}{d \rho} \right) e^{-j k_n^{y} \rho} \\
\cdot \left[ H_{3,x}(x_k, n, p) H_{2,z}(z_t, p) \left( H_{2,y}(y_k, n) ^* H_{2,z}(z_t, p) \right) \right] 
\]

(B173)
\[ T_{xx, rq}^{f_x, f_y} = \frac{1}{j\omega \varepsilon_0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( \frac{k_0^2 - \left( \frac{p \pi}{d} \right)^2}{\varepsilon_p} \right) e^{-jk_0^2 \phi} \cdot \mathcal{H}_{x, x}(k^2, m, n) \mathcal{H}_{x, x}^*(\xi, \zeta, p) \] (B174)

\[ T_{xx, rq}^{f_x, a} = 0 \] (B175)

\[ T_{xy, rs}^{f_x, a} = \frac{\varepsilon_j^y e^{-jk_0^2 \phi}}{\sin \left( \frac{k_m n_j}{d} \right)} \mathcal{H}_{x, x}(s, y, m, n) \mathcal{H}_{x, x}^*(\xi, \zeta, p) \] (B176)

\[ T_{xy, us}^{f_x, a} = \frac{jk_y e^{-jk_0^2 \phi}}{k_m n_j} \mathcal{H}_{x, x}(s, y, m, n) \mathcal{H}_{x, x}^*(\xi, \zeta, p) \] (B177)

\[ T_{xy, as}^{f_x, a} = \frac{-jk_y e^{-jk_0^2 \phi}}{k_m n_j} \mathcal{H}_{x, x}(s, y, m, n) \mathcal{H}_{x, x}^*(\xi, \zeta, p) \] (B178)

\[ T_{yz, rs}^{f_x, f_y} = \frac{1}{j\omega \varepsilon_0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( \frac{p \pi}{d} \right) e^{-jk_0^2 \phi} \cdot \mathcal{H}_{y, y}(y, m, n) \mathcal{H}_{y, x}^*(\xi, \zeta, p) \] (B179)

\[ T_{yz, rq}^{f_x, f_y} = \frac{1}{j\omega \varepsilon_0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( \frac{p \pi}{d} \right) e^{-jk_0^2 \phi} \cdot \mathcal{H}_{y, y}(y, m, n) \mathcal{H}_{y, x}^*(\xi, \zeta, p) \] (B180)

\[ T_{yz, rs}^{f_x, f_y} = \frac{1}{j\omega \varepsilon_0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( \frac{p \pi}{d} \right) e^{-jk_0^2 \phi} \cdot \mathcal{H}_{y, y}(y, m, n) \mathcal{H}_{y, x}^*(\xi, \zeta, p) \] (B181)

\[ T_{yz, rq}^{f_x, f_y} = \frac{1}{j\omega \varepsilon_0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( \frac{p \pi}{d} \right) e^{-jk_0^2 \phi} \cdot \mathcal{H}_{y, y}(y, m, n) \mathcal{H}_{y, x}^*(\xi, \zeta, p) \] (B182)

\[ Z_{yy, rq}^{f_y, f_y} = \frac{1}{j\omega \varepsilon_0} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( k_0^2 - (k_y^2) \right) \left( C_{np} e^{-j\beta_{np} \phi} + C_{np} e^{j\beta_{np} \phi} + C_{np}^* \right) \cdot \mathcal{H}_{y, y}(y, m, n) \mathcal{H}_{y, y}^*(\xi, \zeta, p) \] (B183)
\[
Z_{fy_{yz}, rq} = \frac{1}{j\omega e_0} \sum_{n=0}^{\infty} \sum_{p=0}^{n} \left( jk_{x,y}^{2} \right) \left( C_{n_p}^+ e^{-j\beta_{np} x^b} + C_{n_p}^- e^{j\beta_{np} x^b} + e^{j\beta_{np} x^b} \right)
\cdot \overline{H^y_y(z_{q}, p)} H^y_y(z_{q}, p)
\]  
(B184)

\[
Z_{fy_{zz}, rq} = \frac{1}{j\omega e_0} \sum_{n=0}^{\infty} \sum_{p=0}^{n} \left( jk_{y,y}^{2} \right) \left( C_{n_p}^+ e^{-j\beta_{np} x^b} + C_{n_p}^- e^{j\beta_{np} x^b} + e^{j\beta_{np} x^b} \right)
\cdot \overline{H^y_y(z_{q}, p)} H^y_y(z_{q}, p)
\]  
(B185)

\[
Z_{fy_{zy}, rq} = \frac{1}{j\omega e_0} \sum_{n=0}^{\infty} \sum_{p=0}^{n} \left( \frac{k_{z}^2 - \left( \frac{\pi}{d} \right)^2}{e_p} \right) \left( C_{n_p}^+ e^{-j\beta_{np} x^b} + C_{n_p}^- e^{j\beta_{np} x^b} + e^{j\beta_{np} x^b} \right)
\cdot \overline{H^y_y(z_{q}, p)} H^y_y(z_{q}, p)
\]  
(B186)

\[
T_{f_{y} y_{z}, rs} = \frac{-e^{-jk_{y}^{*} e_{y}}}{\sin (k_{m_p} d)} H^y_{i_{z}, y_{y} (n_{p}, m_{p}, n_{p})} H^y_{i_{z}, z_{y} (z_{q}, m_{p}, n_{p})}
\]  
(B187)

\[
T_{f_{y} y_{z}, rs} = 0
\]  
(B188)

\[
T_{f_{y} y_{z}, rs} = \frac{jk_{y}^{*} e_{y}}{k_{m_{p}} \sin (k_{m_p} d)} H^y_{i_{z}, y_{y} (n_{p}, m_{p}, n_{p})} H^y_{i_{z}, z_{y} (z_{q}, m_{p}, n_{p})}
\]  
(B189)

\[
T_{f_{y} y_{z}, rs} = \frac{-jk_{y}^{*} e_{y}}{k_{m_{p}} \sin (k_{m_p} d)} H^y_{i_{z}, y_{y} (n_{p}, m_{p}, n_{p})} H^y_{i_{z}, z_{y} (z_{q}, m_{p}, n_{p})}
\]  
(B190)

\[
T_{a_{y} x_{z}, q} = 0
\]  
(B191)

\[
T_{a_{y} x_{z}, q} = a \sum_{p=0}^{\infty} \frac{1}{j\omega e_0} I^z_{i_{z}, y (n_{p}, m_{p}, p)} H^x_{i_{z}, x (z_{q}, m_{p})} H^x_{i_{z}, z (z_{q}, p)}
\]  
(B192)

\[
T_{a_{y} x_{z}, q} = \sum_{p=1}^{\infty} \left( \frac{\pi e_p}{d} \right) I^z_{i_{z}, y (n_{p}, m_{p}, p)} H^x_{i_{z}, x (z_{q}, m_{p})} H^x_{i_{z}, z (z_{q}, p)}
\]  
(B193)

\[
T_{a_{y} x_{z}, q} = a \sum_{p=0}^{\infty} \left( jk_{x}^{2} \right) I^z_{i_{z}, y (n_{p}, m_{p}, p)} H^x_{i_{z}, x (z_{q}, m_{p})} H^x_{i_{z}, z (z_{q}, p)}
\]  
(B194)
\[ T_{xy, iq}^{\alpha, \beta} = -b \sum_{p=1}^{\infty} \left( \frac{2\pi}{d} \right)^2 I_{1, x}^{\rho}(m, n, p) I_{1, y}^{\rho}(y, n, p) I_{1, z}^{\rho}(z, p) \] (B195)

\[ T_{xz, iq}^{\alpha, \beta} = -b \sum_{p=0}^{\infty} \left( \frac{j \rho(p)}{d} \right)^2 I_{2, x}^{\rho}(m, n, p) I_{2, z}^{\rho}(y, n, p) I_{1, z}^{\rho}(z, p) \] (B196)

\[ T_{yz, iq}^{\alpha, \beta} = 0 \] (B197)

\[ T_{yx, iq}^{\alpha, \beta} = -b \sum_{p=0}^{\infty} \frac{1}{d} I_{2, x}^{\rho}(m, n, p) I_{1, y}^{\rho}(y, n, p) I_{1, z}^{\rho}(z, p) \] (B198)

\[ y_{xx, is}^{\alpha, \beta} = -ab \left( \frac{k_0^2 - \left( \frac{k_m d}{k_m n} \right)^2}{k_m n} \right) \left( \frac{k_m d}{n m} \right) \delta_{m_m, n_n} \] (B199)

\[ y_{yx, is}^{\alpha, \beta} = ab \left( \frac{k_m k_n}{k_m n} \right) \left( \frac{k_m d}{n m} \right) \delta_{m_m, n_n} \] (B200)

\[ y_{yx, is}^{\alpha, \beta} = ab \left( \frac{k_m k_n}{k_m n} \right) \left( \frac{k_m d}{n m} \right) \delta_{m_m, n_n} \] (B201)

\[ y_{yy, is}^{\alpha, \beta} = -ab \left( \frac{k_0^2 - \left( \frac{k_m d}{k_m n} \right)^2}{k_m n} \right) \left( \frac{k_m d}{n m} \right) \delta_{m_m, n_n} \] (B202)
Appendix B5: Stable expressions for $I_{3x}$, $I_{4x}$, $I_{3y}$ and $I_{4y}$

When $\gamma_{mp}$ or $\beta_{mp}$ become imaginary, the expressions of $I_{3x}$, $I_{4x}$, $I_{3y}$ and $I_{4y}$ in appendix B3 become unstable. These expressions can be rewritten in a stable expression.

\[
H^k_{3x}(x_k, n, p) \bigg| \begin{array}{c}
|x_k + W^x_{3x} = b d (k_e - \beta_{3x}) \left( 1 + e^{-2b_{3x}\alpha^2 - 2e^{-b_{3x}\alpha^2 \cos(\Psi_3)} W^x_{3x} \sin(k_e W^x_{3x}) \right) - e^{-\Psi_3} \left( e^{i\beta_{3x}} (x_k + W^x_{3x} - x^k) + e^{j\beta_{3x}} (x_k + W^x_{3x} + x^k) \right)
\end{array}
\]

\[
+ \left( e^{i\beta_{3x}} (x_k - W^x_{3x} - x^k) + e^{j\beta_{3x}} (x_k + W^x_{3x} - x^k) \right) \right) \right)
\]

\[
H^k_{4x}(x_k, n, p) \bigg| \begin{array}{c}
|x_k - x^k \leq x_k + W^x_{4x} = b d (k_e - \beta_{4x}) \left( 1 + e^{-2b_{4x}\alpha^2 - 2e^{-b_{4x}\alpha^2 \cos(\Psi_4)} W^x_{4x} \sin(k_e W^x_{4x}) \right) - e^{-\Psi_4} \left( e^{i\beta_{4x}} (x_k + W^x_{4x} - x^k) + e^{j\beta_{4x}} (x_k - W^x_{4x} + x^k) \right)
\end{array}
\]

\[
+ \left( e^{i\beta_{4x}} (x_k - W^x_{4x} - x^k) + e^{j\beta_{4x}} (x_k + W^x_{4x} - x^k) \right) \right) \right)
\]

\[
H^k_{3y}(x_k, n, p) \bigg| \begin{array}{c}
|x_k - W^x_{3y} \leq x_k = b d (k_e - \beta_{3y}) \left( 1 + e^{-2b_{3y}\alpha^2 - 2e^{-b_{3y}\alpha^2 \cos(\Psi_3)} W^x_{3y} \sin(k_e W^x_{3y}) \right) - e^{-\Psi_3} \left( e^{i\beta_{3y}} (x_k + W^x_{3y} - x^k) + e^{j\beta_{3y}} (x_k - W^x_{3y} + x^k) \right)
\end{array}
\]

\[
+ \left( e^{i\beta_{3y}} (x_k - W^x_{3y} - x^k) + e^{j\beta_{3y}} (x_k + W^x_{3y} - x^k) \right) \right) \right)
\]

\[
H^k_{4y}(x_k, n, p) \bigg| \begin{array}{c}
|x_k - W^x_{4y} \leq x_k = b d (k_e - \beta_{4y}) \left( 1 + e^{-2b_{4y}\alpha^2 - 2e^{-b_{4y}\alpha^2 \cos(\Psi_4)} W^x_{4y} \sin(k_e W^x_{4y}) \right) - e^{-\Psi_4} \left( e^{i\beta_{4y}} (x_k + W^x_{4y} - x^k) + e^{j\beta_{4y}} (x_k - W^x_{4y} + x^k) \right)
\end{array}
\]

\[
+ \left( e^{i\beta_{4y}} (x_k - W^x_{4y} - x^k) + e^{j\beta_{4y}} (x_k + W^x_{4y} - x^k) \right) \right) \right)
\]
\[
I_{S_{r}}^{(x_{s}, n, p)}(x_{s}, x_{s-1} + W_{s}^{2}) = \frac{k_{r}}{\sin (k_{r} W_{s}^{2})} \left[ e^{-j\Psi_{s}} \left( e^{j\beta_{s}}(x_{s} - W_{s}^{2} - x_{s}) + e^{-j\beta_{s}}(x_{s} - W_{s}^{2} + x_{s}) - \cos (k_{r} W_{s}^{2}) (e^{j\beta_{s}}(x_{s} - x_{s}) + e^{-j\beta_{s}}(x_{s} + x_{s})) \right) + e^{j\beta_{s}}(x_{s} + W_{s}^{2} - x_{s}) + e^{-j\beta_{s}}(x_{s} + W_{s}^{2} + x_{s}) \right) \right] \]

(B206)

\[
I_{S_{s}}^{(x_{s}, n, p)}(x_{s}, x_{s-1} + W_{s}^{2}) = \frac{k_{s}}{\sin (k_{s} W_{s}^{2})} \left[ e^{-j\Psi_{s}} \left( e^{j\beta_{s}}(x_{s} - W_{s}^{2} - x_{s}) + e^{-j\beta_{s}}(x_{s} - W_{s}^{2} + x_{s}) - \cos (k_{s} W_{s}^{2}) (e^{j\beta_{s}}(x_{s} - x_{s}) + e^{-j\beta_{s}}(x_{s} + x_{s})) \right) + e^{j\beta_{s}}(x_{s} + W_{s}^{2} - x_{s}) + e^{-j\beta_{s}}(x_{s} + W_{s}^{2} + x_{s}) \right) \right] \]

(B207)

\[
I_{S_{s}}^{(x_{s}, n, p)}(x_{s}, x_{s-1} + W_{s}^{2}) = \frac{k_{s}}{\sin (k_{s} W_{s}^{2})} \left[ e^{-j\Psi_{s}} \left( e^{j\beta_{s}}(x_{s} - W_{s}^{2} - x_{s}) + e^{-j\beta_{s}}(x_{s} - W_{s}^{2} + x_{s}) - \cos (k_{s} W_{s}^{2}) (e^{j\beta_{s}}(x_{s} - x_{s}) + e^{-j\beta_{s}}(x_{s} + x_{s})) \right) + e^{j\beta_{s}}(x_{s} + W_{s}^{2} - x_{s}) + e^{-j\beta_{s}}(x_{s} + W_{s}^{2} + x_{s}) \right) \right] \]

(B208)
\[ H_{L,A}(x_c-1, n, p) \big|_{x_c-1 - W_{\sigma} \leq x \leq x_c} = \frac{k_e}{b d (k_e^2 - \beta_{np}^2) \left( 1 + e^{-2i\beta_{np}} - 2e^{-i\beta_{np}} \cos \left( \Psi_1 \right) \right) W_{Lx} \sin (k_e W_{Lx})} \]

\[ \cdot \left( e^{-i\Psi_1} \left( 2e^{-i\beta_{np}} \cos \left( k_e (x_c-1 - W_{Lx} - x^b) \right) - \cos \left( k_e W_{Lx} \right) \left( e^{i\beta_{np}} (x_c-1 - x^b + a) + e^{-i\beta_{np}} (x_c-1 - x^b - a) \right) \right) + \cos \left( k_e W_{Lx} \right) \left( 2e^{-i\beta_{np}} \cos \left( k_e (x_c-1 - W_{Lx} - x^b) \right) - \cos \left( k_e W_{Lx} \right) \left( e^{i\beta_{np}} (x_c-1 - x^b + a) + e^{-i\beta_{np}} (x_c-1 - x^b - a) \right) \right) \right) \]

\[ - e^{-i\Psi_1} \left( \frac{\left( e^{i\beta_{np}} (x_c-1 - W_{Lx} - x^b) + e^{-i\beta_{np}} (x_c-1 - W_{Lx} + x^b) \right) - \cos \left( k_e W_{Lx} \right) \left( e^{i\beta_{np}} (x_c-1 - x^b + a) + e^{-i\beta_{np}} (x_c-1 - x^b - a) \right)}{W_{Lx} \sin (k_e W_{Lx})} \right) + \]

\[ \cos \left( k_e W_{Lx} \right) \left( e^{i\beta_{np}} (x_c-1 + 2a - W_{Lx} - x^b) + e^{-i\beta_{np}} (x_c-1 + W_{Lx} - x^b) \right) \]

\[ \frac{\left( e^{i\beta_{np}} (x_c-1 - 2a - W_{Lx} - x^b) + e^{-i\beta_{np}} (x_c-1 - W_{Lx} - x^b) \right) - \cos \left( k_e W_{Lx} \right) \left( e^{i\beta_{np}} (x_c-1 - 2a - x^b) + e^{-i\beta_{np}} (x_c-1 - x^b) \right)}{W_{Lx} \sin (k_e W_{Lx})} \right) \right) \right) \]  

(B209)

\[ H_{L,A}(x_c-1, n, p) \big|_{x_c-1 - W_{\sigma} \leq x \leq x_c} = \frac{k_e}{b d (k_e^2 - \beta_{np}^2) \left( 1 + e^{-2i\beta_{np}} - 2e^{-i\beta_{np}} \cos \left( \Psi_1 \right) \right) W_{Lz} \sin (k_e W_{Lz})} \]

\[ \cdot \left( e^{-i\Psi_1} \left( \frac{\left( e^{i\beta_{np}} (x_c-1 - W_{Lz} - x^a) + e^{-i\beta_{np}} (x_c-1 - W_{Lz} + x^a) \right) - \cos \left( k_e W_{Lz} \right) \left( e^{i\beta_{np}} (x_c-1 - x^a + a) + e^{-i\beta_{np}} (x_c-1 - x^a - a) \right)}{W_{Lz} \sin (k_e W_{Lz})} \right) + \cos \left( k_e W_{Lz} \right) \left( e^{i\beta_{np}} (x_c-1 - 2a - W_{Lz} - x^b) + e^{-i\beta_{np}} (x_c-1 - W_{Lz} - x^b) \right) \]

\[ \frac{\left( e^{i\beta_{np}} (x_c-1 - 2a - W_{Lz} - x^b) + e^{-i\beta_{np}} (x_c-1 + W_{Lz} - x^b) \right) - \cos \left( k_e W_{Lz} \right) \left( e^{i\beta_{np}} (x_c-1 - 2a - x^b) + e^{-i\beta_{np}} (x_c-1 - x^b) \right)}{W_{Lz} \sin (k_e W_{Lz})} \right) \right) \right) \right) \]  

(B210)

\[ H_{L,A}(x_c, n, p) \big|_{x = x_c + W_{\sigma}} = \frac{k_e}{b d (k_e^2 - \beta_{np}^2) \left( 1 + e^{-2i\beta_{np}} - 2e^{-i\beta_{np}} \cos \left( \Psi_1 \right) \right) W_{Lx} \sin (k_e W_{Lx})} \]

\[ \cdot \left( e^{-i\Psi_1} \left( \frac{\left( e^{i\beta_{np}} (x_c - W_{Lx} - x^b - a) + e^{-i\beta_{np}} (x_c - W_{Lx} - x^b + a) \right) - \cos \left( k_e W_{Lx} \right) \left( e^{i\beta_{np}} (x_c - x^b + a) + e^{-i\beta_{np}} (x_c - x^b - a) \right)}{W_{Lx} \sin (k_e W_{Lx})} \right) + e^{i\beta_{np}} (x_c + W_{Lx} - x^b + a) \right) \]

\[ + \left( e^{i\beta_{np}} (x_c - W_{Lz} - x^b) + e^{-i\beta_{np}} (x_c + 2a - W_{Lz} - x^b) - 2cos \left( k_e W_{Lz} \right) \left( e^{i\beta_{np}} (x_c - x^b) + e^{-i\beta_{np}} (x_c + x^b + 2a) \right) + e^{i\beta_{np}} (x_c + 2a + W_{Lz} - x^b) \right) \]  

(B211)
\[
I_s^e(x, n, \rho) \left|_{s_x \leq x \leq s_y + W_{q-y}} = \frac{k_e}{b d (k_e^2 - \beta_{np}^2)} \left( 1 + e^{-2i\beta_{np}a - 2e^{-j\beta_{np}a}\cos \left( \Psi \right)} \right) W_{q-y} \sin (k_e W_{q-y}) \right.
\]
\[\left. \cdot \left( e^{i\beta_{np}(x - s_y - x^0 + a)} + e^{-i\beta_{np}(x - s_y - x^0 - a)} - 2\cos (k_e W_{q-y}) \left( e^{i\beta_{np}(x - x^0 - a)} + e^{-i\beta_{np}(x - x^0 + a)} \right) + 2e^{-j\beta_{np}a}\cos (k_e (x_c + W_{q-y} - x^0)) \right) \right) \]
\[+ e^{-i\Psi} \cdot \left( e^{i\beta_{np}(x + W_{q-y} - x^0 - a)} + e^{-i\beta_{np}(x + W_{q-y} - x^0 + a)} - 2e^{-j\beta_{np}a}\cos (k_e (x_c + W_{q-y} - x^0)) \right) \]
\[+ \left( e^{i\beta_{np}(x - W_{q-y} - x^0)} + e^{-i\beta_{np}(x + 2a - W_{q-y} - x^0)} - 2\cos (k_e W_{q-y}) \left( e^{i\beta_{np}(x - x^0 - a)} + e^{-i\beta_{np}(x - x^0 + a)} \right) \right) \]
\[+ 2\cos (k_e (x_c + W_{q-y} - x^0)) \left( 1 + e^{-2i\beta_{np}a} \right) - \left( e^{i\beta_{np}(x - 2a + W_{q-y} - x^0)} + e^{-i\beta_{np}(x + W_{q-y} - x^0)} \right) \right) \right) \ (B212)
\]

\[
I_s^e(x, n, \rho) \left|_{s_x \leq W_{q-y} - x^0 \leq s_x} = \frac{k_e}{b d (k_e^2 - \beta_{np}^2)} \left( 1 + e^{-2i\beta_{np}a - 2e^{-j\beta_{np}a}\cos \left( \Psi \right)} \right) W_{q-y} \sin (k_e W_{q-y}) \right.
\]
\[\left. \cdot \left( e^{i\beta_{np}(x_c - W_{q-y} - x^0 - a)} + e^{-i\beta_{np}(x_c - W_{q-y} - x^0 + a)} - 2e^{-j\beta_{np}a}\cos (k_e (x_c - W_{q-y} - x^0)) \right) \right) \]
\[+ e^{i\beta_{np}(x_c + W_{q-y} - x^0 - a)} + e^{-i\beta_{np}(x_c + W_{q-y} - x^0 + a)} \right) \]
\[\left. - e^{i\Psi} \cdot \left( e^{i\beta_{np}(x_c - W_{q-y} - x^0 - a)} + e^{-i\beta_{np}(x_c - W_{q-y} - x^0 + a)} - 2e^{-j\beta_{np}a}\cos (k_e (x_c - W_{q-y} - x^0)) \right) \right) \]
\[+ \left( e^{i\beta_{np}(x_c - W_{q-y} - x^0)} + e^{-i\beta_{np}(x_c + 2a - W_{q-y} - x^0)} - 2\cos (k_e (x_c - W_{q-y} - x^0)) \left( 1 + e^{-2i\beta_{np}a} \right) \right) \]
\[+ 2\cos (k_e W_{q-y}) \left( e^{i\beta_{np}(x_c - 2a - x^0)} + e^{-i\beta_{np}(x_c - x^0)} \right) \right) \left( e^{i\beta_{np}(x_c + W_{q-y} - x^0)} + e^{-i\beta_{np}(x_c + W_{q-y} - x^0)} \right) \right) \ (B213)
\]

\[
I_s^e(x, n, \rho) \left|_{s_x - W_{q-y} \leq x \leq s_x} = \frac{k_e}{b d (k_e^2 - \beta_{np}^2)} \left( 1 + e^{-2i\beta_{np}a - 2e^{-j\beta_{np}a}\cos \left( \Psi \right)} \right) W_{q-y} \sin (k_e W_{q-y}) \right.
\]
\[\left. \cdot \left( e^{i\beta_{np}(x - W_{q-y} - x^0 - a)} + e^{-i\beta_{np}(x - W_{q-y} - x^0 + a)} - 2\cos (k_e W_{q-y}) \left( e^{i\beta_{np}(x - x^0 - a)} + e^{-i\beta_{np}(x - x^0 + a)} \right) \right) \right) \]
\[+ e^{i\beta_{np}(x_c + W_{q-y} - x^0 - a)} + e^{-i\beta_{np}(x_c + W_{q-y} - x^0 + a)} \right) \]
\[\left. - \left( e^{i\beta_{np}(x_c - 2a - W_{q-y} - x^0)} + e^{-i\beta_{np}(x_c - x^0)} \right) \left( e^{i\beta_{np}(x_c + W_{q-y} - x^0)} + e^{-i\beta_{np}(x_c + W_{q-y} - x^0)} \right) \right) \right) \ (B214)
\]
\[
H_{x}(x_{c+1}, n, p) = \begin{cases} \frac{k_{e}}{\varepsilon_{0}} \left[ \frac{b d (k_{e}^{2} - \beta_{n p}^{2})}{1 + e^{-2j\beta_{n p}a} - 2e^{-j\beta_{n p}}\cos(\Psi_{x})} \right] \\
\left( -e^{j\Psi_{x}} \left[ \frac{e^{j\beta_{n p}a}(x_{c+1} - W_{x}^{2} - x^{2} - a) + e^{-j\beta_{n p}a}(x_{c+1} - W_{x}^{2} + x^{2} + a) - \cos(k_{e}W_{x}^{2}) (e^{j\beta_{n p}a}(x_{c+1} - x^{2} - a) + e^{-j\beta_{n p}a}(x_{c+1} - x^{2} + a))}{W_{x}^{2}\sin(k_{e}W_{x}^{2})} \right. \\
+ \left. \frac{e^{j\beta_{n p}a}(x_{c+1} + W_{x}^{2} - x^{2} - a) + e^{-j\beta_{n p}a}(x_{c+1} + W_{x}^{2} - x^{2} + a)}{W_{x}^{2}\sin(k_{e}W_{x}^{2})} \right) \end{cases}
\] (B215)

\[
H_{x}(x_{c+1}, n, p) = \begin{cases} \frac{k_{e}}{\varepsilon_{0}} \left[ \frac{b d (k_{e}^{2} - \beta_{n p}^{2})}{1 + e^{-2j\beta_{n p}a} - 2e^{-j\beta_{n p}}\cos(\Psi_{x})} \right] \\
\left( -e^{j\Psi_{x}} \left[ \frac{e^{j\beta_{n p}a}(x_{c+1} - W_{x}^{2} - x^{2} - a) + e^{-j\beta_{n p}a}(x_{c+1} - W_{x}^{2} + x^{2} + a) - \cos(k_{e}W_{x}^{2}) (e^{j\beta_{n p}a}(x_{c+1} - x^{2} - a) + e^{-j\beta_{n p}a}(x_{c+1} - x^{2} + a))}{W_{x}^{2}\sin(k_{e}W_{x}^{2})} \right. \\
+ \left. \frac{e^{j\beta_{n p}a}(x_{c+1} + W_{x}^{2} - x^{2} - a) + e^{-j\beta_{n p}a}(x_{c+1} + W_{x}^{2} - x^{2} + a)}{W_{x}^{2}\sin(k_{e}W_{x}^{2})} \right) \end{cases}
\] (B216)

\[
H_{x}(x_{c+1}, n, p) = \begin{cases} \frac{k_{e}}{\varepsilon_{0}} \left[ \frac{b d (k_{e}^{2} - \beta_{n p}^{2})}{1 + e^{-2j\beta_{n p}a} - 2e^{-j\beta_{n p}}\cos(\Psi_{x})} \right] \\
\left( -e^{j\Psi_{x}} \left[ \frac{e^{j\beta_{n p}a}(x_{c+1} - W_{x}^{2} - x^{2} - a) + e^{-j\beta_{n p}a}(x_{c+1} - W_{x}^{2} + x^{2} + a) - \cos(k_{e}W_{x}^{2}) (e^{j\beta_{n p}a}(x_{c+1} - x^{2} - a) + e^{-j\beta_{n p}a}(x_{c+1} - x^{2} + a))}{W_{x}^{2}\sin(k_{e}W_{x}^{2})} \right. \\
+ \left. \frac{e^{j\beta_{n p}a}(x_{c+1} + W_{x}^{2} - x^{2} - a) + e^{-j\beta_{n p}a}(x_{c+1} + W_{x}^{2} - x^{2} + a)}{W_{x}^{2}\sin(k_{e}W_{x}^{2})} \right) \end{cases}
\] (B217)
\[ \begin{align*}
I_{x,x}^z(x_c + n, p) & \Big|_{x^s \leq x_s^z \leq w^z} = \frac{k_e}{\beta_{np}^2 b d} \frac{1}{1 + e^{-2j \beta_{np} a} - 2e^{-j \beta_{np} a} \cos(\Psi_x)} \left( e^{-j \Psi_x} \left( e^{j \beta_{np} (x_s - x^s - 2a) + j \beta_{np} x_s - x^s + a) - \cos(k_e W_{x}^z) \left( e^{j \beta_{np} (x_s - x^s + 2a) + e^{-j \beta_{np} (x_s - x^s - 2a)} \right) + e^{j \beta_{np} (x_s + 2a) + e^{-j \beta_{np} (x_s - 2a)} \right) \right) \right) \left( e^{j \beta_{np} (x_s + 2a) + e^{-j \beta_{np} (x_s - 2a)} \right)
\end{align*} \]

\[ \begin{align*}
I_{z,x}^z(x_c + n, p) & \Big|_{x^s \leq x_s^z \leq w^z} = \frac{k_e}{\beta_{np}^2 b d} \frac{1}{1 + e^{-2j \beta_{np} a} - 2e^{-j \beta_{np} a} \cos(\Psi_x)} \left( e^{-j \Psi_x} \left( e^{j \beta_{np} (x_s - x^s - 2a) + j \beta_{np} x_s - x^s + a) - \cos(k_e W_{x}^z) \left( e^{j \beta_{np} (x_s - x^s + 2a) + e^{-j \beta_{np} (x_s - x^s - 2a)} \right) + e^{j \beta_{np} (x_s + 2a) + e^{-j \beta_{np} (x_s - 2a)} \right) \right) \right) \left( e^{j \beta_{np} (x_s + 2a) + e^{-j \beta_{np} (x_s - 2a)} \right)
\end{align*} \]

\[ \begin{align*}
I_{x,y}^x(y_c + m, p) & \Big|_{y^p \leq y_s^x + w^x} = \frac{k_e}{\beta_{np}^2 b d} \frac{1}{1 + e^{-2j \beta_{np} b} - 2e^{-j \beta_{np} b} \cos(\Psi_y)} \left( e^{-j \Psi_y} \left( e^{j \beta_{np} y_s - y^p - b) + j \beta_{np} y_s - y^p + b) - \cos(k_e W_{x}^z) \left( e^{j \beta_{np} y_s - y^p + 2b) + e^{-j \beta_{np} y_s - y^p - 2b) \right) + e^{j \beta_{np} y_s + 2b) + e^{-j \beta_{np} y_s - 2b) \right) \right) \right) \left( e^{j \beta_{np} y_s + 2b) + e^{-j \beta_{np} y_s - 2b) \right)
\end{align*} \]
\[ H_{3, y} (\mathbf{y}, m, p) \bigg|_{y_y \leq y_y \leq y_y + W_y} = \frac{k_e}{ad (k_e^2 - \gamma_{mp}^2) \left( 1 + e^{-2j\Psi_{mp}} - 2e^{-j\Psi_{mp}} \cos (\Psi_{mp}) \right) W_y^2 \sin (k_e W_y^y)} \]
\[ \left( -e^{-j\Psi_{mp}} (e^{j\Psi_{mp}} (y_y - W_y^y - y^y - b) + e^{-j\Psi_{mp}} (y_y - W_y^y - y^y + b)) - 2cos (k_e W_y^y) (e^{j\Psi_{mp}} (y_y - y^y - b) + e^{-j\Psi_{mp}} (y_y - y^y + b)) + 2e^{-j\Psi_{mp}} \cos (k_e (y_y + W_y^y - y^y)) \right) \]
\[ + e^{-j\Psi_{mp}} (e^{j\Psi_{mp}} (y_y + W_y^y - y^y - b) + e^{-j\Psi_{mp}} (y_y + W_y^y - y^y + b)) - 2e^{-j\Psi_{mp}} \cos (k_e (y_y + W_y^y - y^y)) \]
\[ + \left( e^{j\Psi_{mp}} (y_y - W_y^y - y^y) + e^{-j\Psi_{mp}} (y_y + 2b - W_y^y - y^y) - 2 \cos (k_e W_y^y) (e^{j\Psi_{mp}} (y_y - y^y) + e^{-j\Psi_{mp}} (y_y - y^y + 2b)) + 2 \cos (k_e (y_y + W_y^y - y^y)) (1 + e^{-2j\Psi_{mp}}) - (e^{j\Psi_{mp}} (y_y - 2b + W_y^y - y^y) + e^{-j\Psi_{mp}} (y_y + W_y^y - y^y)) \right) \right) \tag{B223} \]

\[ H_{3, y} (\mathbf{y}, m, p) \bigg|_{y_y - W_y \leq y_y \leq y_y} = \frac{k_e}{ad (k_e^2 - \gamma_{mp}^2) \left( 1 + e^{-2j\Psi_{mp}} - 2e^{-j\Psi_{mp}} \cos (\Psi_{mp}) \right) W_y^2 \sin (k_e W_y^y)} \]
\[ \left( -e^{-j\Psi_{mp}} (2e^{-j\Psi_{mp}} \cos (k_e (y_y - W_y^y - y^y)) - 2 \cos (k_e W_y^y) (e^{j\Psi_{mp}} (y_y - y^y - b) + e^{-j\Psi_{mp}} (y_y - y^y + b)) \right) \]
\[ + e^{j\Psi_{mp}} (y_y + W_y^y - y^y - b) + e^{-j\Psi_{mp}} (y_y + W_y^y - y^y + b)) - 2e^{-j\Psi_{mp}} \cos (k_e (y_y - W_y^y - y^y)) \]
\[ + \left( e^{j\Psi_{mp}} (y_y - W_y^y - y^y) + e^{-j\Psi_{mp}} (y_y + 2b - W_y^y - y^y) - 2 \cos (k_e (y_y - W_y^y - y^y)) (1 + e^{-2j\Psi_{mp}}) - (e^{j\Psi_{mp}} (y_y - 2b + W_y^y - y^y) + e^{-j\Psi_{mp}} (y_y + W_y^y - y^y)) \right) \right) \tag{B224} \]

\[ H_{3, y} (\mathbf{y}, m, p) \bigg|_{y_y \leq y_y - W_y} = \frac{k_e}{ad (k_e^2 - \gamma_{mp}^2) \left( 1 + e^{-2j\Psi_{mp}} - 2e^{-j\Psi_{mp}} \cos (\Psi_{mp}) \right) W_y^2 \sin (k_e W_y^y)} \]
\[ \left( -e^{-j\Psi_{mp}} (e^{j\Psi_{mp}} (y_y - W_y^y - y^y - b) + e^{-j\Psi_{mp}} (y_y - W_y^y - y^y + b)) - 2 \cos (k_e W_y^y) (e^{j\Psi_{mp}} (y_y - y^y - b) + e^{-j\Psi_{mp}} (y_y - y^y + b)) \right) \]
\[ + e^{j\Psi_{mp}} (y_y + W_y^y - y^y - b) + e^{-j\Psi_{mp}} (y_y + W_y^y - y^y + b)) - 2e^{-j\Psi_{mp}} \cos (k_e (y_y + W_y^y - y^y)) \]
\[ - \left( e^{j\Psi_{mp}} (y_y - 2b + W_y^y - y^y) + e^{-j\Psi_{mp}} (y_y - 2b - W_y^y - y^y) - 2 \cos (k_e W_y^y) (e^{j\Psi_{mp}} (y_y - y^y - b) + e^{-j\Psi_{mp}} (y_y - y^y + b)) \right) \right) \tag{B225} \]
\[ I^\nu_y(y_c^{-1}, m, p) |_{y_c \geq y_{c1} + W_\nu}^{y_c \geq y_{c1} + W_y} = \frac{k_e}{W_\nu \sin (k_e W_\nu)} \left( \begin{array}{c}
abla \nu_\nu (y_{c1} - W_\nu - y^b) + e^{-\nu_\nu} (y_{c1} - W_y - y^b)
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu_\nu (y_{c1} - W_\nu - y^b) + e^{-\nu_\nu} (y_{c1} - W_y - y^b)
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu_\nu (y_{c1} - W_\nu - y^b) + e^{-\nu_\nu} (y_{c1} - W_y - y^b)
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu_\nu (y_{c1} - W_\nu - y^b) + e^{-\nu_\nu} (y_{c1} - W_y - y^b)
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu_\nu (y_{c1} - W_\nu - y^b) + e^{-\nu_\nu} (y_{c1} - W_y - y^b)
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu_\nu (y_{c1} - W_\nu - y^b) + e^{-\nu_\nu} (y_{c1} - W_y - y^b)
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
\end{array} \right) \] (B226)

\[ I^\nu_y(y_c^{-1}, m, p) |_{y_{c1} \leq y^b \leq y_{c1} + W_\nu}^{y_{c1} + W_\nu} = \frac{k_e}{W_\nu \sin (k_e W_\nu)} \left( \begin{array}{c}
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu_\nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
\end{array} \right) \] (B227)

\[ I^\nu_y(y_c^{-1}, m, p) |_{y_{c1} - W_\nu \leq y^b \leq y_{c1}}^{y_{c1} - W_y} = \frac{k_e}{W_\nu \sin (k_e W_\nu)} \left( \begin{array}{c}
abla \nu (y_{c1} - W_\nu - y^b) + e^{-\nu} (y_{c1} - W_y - y^b)
abla \nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu (y_{c1} - W_\nu - y^b) + e^{-\nu} (y_{c1} - W_y - y^b)
abla \nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu (y_{c1} - W_\nu - y^b) + e^{-\nu} (y_{c1} - W_y - y^b)
abla \nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu (y_{c1} - W_\nu - y^b) + e^{-\nu} (y_{c1} - W_y - y^b)
abla \nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu (y_{c1} - W_\nu - y^b) + e^{-\nu} (y_{c1} - W_y - y^b)
abla \nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
abla \nu (y_{c1} - W_\nu - y^b) + e^{-\nu} (y_{c1} - W_y - y^b)
abla \nu (y_{c1} - y^b) + W_\nu \sin (k_e W_\nu)
\end{array} \right) \] (B228)
\[ H_{\nu, \lambda}(y, m, p) |_{y \leq y_{\nu, m, p} < \lambda} = \frac{k_{\nu}}{\sin(k_{\nu} W_{\nu}^{\text{eff}})} \left( 1 + e^{-2i\gamma_{\nu}} \right) \]
\[ \left( e^{i\gamma_{\nu}}(y - W_{\nu}^{\text{eff}} - y_{\nu} - b) + e^{-i\gamma_{\nu}}(y_{\nu} + W_{\nu}^{\text{eff}} - y + b) - 2e^{-i\gamma_{\nu}} \cos(k_{\nu} W_{\nu}^{\text{eff}}) \right) \]
\[ + 2e^{-2i\gamma_{\nu}} \cos(k_{\nu} (y_{\nu} + W_{\nu}^{\text{eff}} - y_{\nu} - b)) \]
\[ + e^{-i\gamma_{\nu}} \left( e^{i\gamma_{\nu}}(y_{\nu} + W_{\nu}^{\text{eff}} - y_{\nu} - b) + e^{-i\gamma_{\nu}}(y_{\nu} + W_{\nu}^{\text{eff}} - y_{\nu} + b) - 2e^{-i\gamma_{\nu}} \cos(k_{\nu} W_{\nu}^{\text{eff}}) \right) \]
\[ + 2e^{-2i\gamma_{\nu}} \cos(k_{\nu} (y_{\nu} + W_{\nu}^{\text{eff}} - y_{\nu} - b)) \]
\[ + e^{-i\gamma_{\nu}} \left( e^{i\gamma_{\nu}}(y_{\nu} + W_{\nu}^{\text{eff}} - y_{\nu} - b) + e^{-i\gamma_{\nu}}(y_{\nu} + W_{\nu}^{\text{eff}} - y_{\nu} + b) - 2e^{-i\gamma_{\nu}} \cos(k_{\nu} W_{\nu}^{\text{eff}}) \right) \]
\[ + 2e^{-2i\gamma_{\nu}} \cos(k_{\nu} (y_{\nu} + W_{\nu}^{\text{eff}} - y_{\nu} - b)) \left( 1 + e^{-2i\gamma_{\nu}} \right) \]
\[ \left( e^{i\gamma_{\nu}}(y_{\nu} + W_{\nu}^{\text{eff}} - y_{\nu} - b) + e^{-i\gamma_{\nu}}(y_{\nu} + W_{\nu}^{\text{eff}} - y_{\nu} + b) - 2e^{-i\gamma_{\nu}} \cos(k_{\nu} W_{\nu}^{\text{eff}}) \right) \]
\[ + 2e^{-2i\gamma_{\nu}} \cos(k_{\nu} (y_{\nu} + W_{\nu}^{\text{eff}} - y_{\nu} - b)) \left( 1 + e^{-2i\gamma_{\nu}} \right) \]

\[ \left( e^{i\gamma_{\nu}}(y_{\nu} - W_{\nu}^{\text{eff}} - y_{\nu} - b) + e^{-i\gamma_{\nu}}(y_{\nu} - W_{\nu}^{\text{eff}} - y + b) - 2e^{-i\gamma_{\nu}} \cos(k_{\nu} W_{\nu}^{\text{eff}}) \right) \]
\[ + 2e^{-2i\gamma_{\nu}} \cos(k_{\nu} (y_{\nu} - W_{\nu}^{\text{eff}} - y_{\nu} - b)) \left( 1 + e^{-2i\gamma_{\nu}} \right) \]

\[ (B229) \]

\[ (B230) \]

\[ (B231) \]
\[ H^\prime_2(y, m, p) \mid y - W_{y}^p \leq y^p \leq y_c = \frac{k_e}{ad(k^2_e - \gamma^2_{mp}) \left( 1 + e^{-2\gamma_{mp}b} - 2e^{-\gamma_{mp}b} \cos(\Psi_y) \right) W_{y}^p \sin(k_e W_{y}^p \gamma)} \]

\[ \times \left( e^{-\gamma_{mp}b} \left( 2e^{-\gamma_{mp}b} \cos(k_e (y_c - W_{y}^p - y^p)) - 2 \cos(k_e W_{y}^p) (e^{\gamma_{mp}b} (y_c - y^p - b) + e^{-\gamma_{mp}b} (y_c - y^p + b)) \right) ight. \\

\left. + e^{\gamma_{mp}b} (y_c + W_{y}^p - y^p) + e^{-\gamma_{mp}b} (y_c + W_{y}^p - y^p) \right) \\

- e^{-\gamma_{mp}b} \left( e^{\gamma_{mp}b} (y_c - W_{y}^p - y^p) + e^{-\gamma_{mp}b} (y_c - W_{y}^p - y^p) - 2e^{-\gamma_{mp}b} \cos(k_e (y_c - W_{y}^p - y^p)) \right) \\

\left. + \left( e^{\gamma_{mp}b} (y_c - W_{y}^p - y^p) + e^{-\gamma_{mp}b} (y_c + 2b - W_{y}^p - y^p) - 2 \cos(k_e (y_c - W_{y}^p - y^p)) \right) (1 + e^{-2\gamma_{mp}b}) \\

\right) \\

+ 2 \cos(k_e W_{y}^p) (e^{\gamma_{mp}b} (y_c - 2b - y^p) + e^{-\gamma_{mp}b} (y_c - y^p)) \\

\left. - (e^{\gamma_{mp}b} (y_c - 2b + W_{y}^p - y^p) + e^{-\gamma_{mp}b} (y_c + W_{y}^p - y^p)) \right) \] (B232)

\[ H^\prime_3(y, m, p) \mid y^p \leq y_c - W_{y}^p = \frac{k_e}{ad(k^2_e - \gamma^2_{mp}) \left( 1 + e^{-2\gamma_{mp}b} - 2e^{-\gamma_{mp}b} \cos(\Psi_y) \right) W_{y}^p \sin(k_e W_{y}^p \gamma)} \]

\[ \times \left( e^{-\gamma_{mp}b} \left( 2e^{-\gamma_{mp}b} \cos(k_e (y_c - W_{y}^p - y^p)) - 2 \cos(k_e W_{y}^p) (e^{\gamma_{mp}b} (y_c - y^p - b) + e^{-\gamma_{mp}b} (y_c - y^p + b)) \right) \\

\left. + e^{\gamma_{mp}b} (y_c + W_{y}^p - y^p) + e^{-\gamma_{mp}b} (y_c + W_{y}^p - y^p) \right) \\

- \left( e^{\gamma_{mp}b} (y_c - 2b - W_{y}^p - y^p) + e^{-\gamma_{mp}b} (y_c - W_{y}^p - y^p) - 2 \cos(k_e W_{y}^p) (e^{\gamma_{mp}b} (y_c - 2b - y^p) + e^{-\gamma_{mp}b} (y_c - y^p)) \\

\right) \\

+ e^{\gamma_{mp}b} (y_c - 2b + W_{y}^p - y^p) + e^{-\gamma_{mp}b} (y_c + W_{y}^p - y^p) \right) \] (B233)

\[ H^\prime_3(y, m + 1, p) \mid y^p \geq y_{c+1} + W_{y}^p = \frac{k_e}{ad(k^2_e - \gamma^2_{mp}) \left( 1 + e^{-2\gamma_{mp}b} - 2e^{-\gamma_{mp}b} \cos(\Psi_y) \right) W_{y}^p \sin(k_e W_{y}^p \gamma)} \]

\[ \times \left( e^{-\gamma_{mp}b} \left( 2e^{-\gamma_{mp}b} \cos(k_e (y_c - W_{y}^p - y^p)) - 2 \cos(k_e W_{y}^p) (e^{\gamma_{mp}b} (y_c - y^p - b) + e^{-\gamma_{mp}b} (y_c - y^p + b)) \right) \\

\left. + e^{\gamma_{mp}b} (y_c + W_{y}^p - y^p) + e^{-\gamma_{mp}b} (y_c + W_{y}^p - y^p) \right) \\

- \left( e^{\gamma_{mp}b} (y_c + 2b - W_{y}^p - y^p) - \cos(k_e W_{y}^p) (e^{\gamma_{mp}b} (y_c + 2b - y^p) + e^{-\gamma_{mp}b} (y_c - y^p)) \\

\right) \\

+ \left( e^{\gamma_{mp}b} (y_c + 2b + W_{y}^p - y^p) - \cos(k_e W_{y}^p) (e^{\gamma_{mp}b} (y_c + 2b - y^p) + e^{-\gamma_{mp}b} (y_c - y^p)) \right) \right) \] (B234)
\[\begin{align*}
I_{xy}(y_{c+1}, m, p) &\bigg|_{y_{c+1} \leq y \leq y_{c+1} + \Delta y} = \frac{k_e}{W_L \sin (k_e W_L^y)} \left( 
\alpha d (k_e^2 - \gamma_{mp}^2) \left( 1 + e^{-2\gamma_{mp}^b} - 2e^{-}\gamma_{mp}^b \cos (\Psi_y) \right) 
+ e^{-}\gamma_{mp}^b \cos (k_e (y_{c+1} + W_L^y - y^b) - \cos (k_e W_L^y) \left( e^{\gamma_{mp}^b (y_{c+1} - y^b)} + e^{-}\gamma_{mp}^b (y_{c+1} - y^b) \right) 
+ e^{-}\gamma_{mp}^b \cos (k_e (y_{c+1} + W_L^y - y^b) - \cos (k_e W_L^y) \left( e^{\gamma_{mp}^b (y_{c+1} - y^b)} + e^{-}\gamma_{mp}^b (y_{c+1} - y^b) \right) 
+ e^{-}\gamma_{mp}^b \cos (k_e (y_{c+1} + W_L^y - y^b) - \cos (k_e W_L^y) \left( e^{\gamma_{mp}^b (y_{c+1} - y^b)} + e^{-}\gamma_{mp}^b (y_{c+1} - y^b) \right) 
+ e^{-}\gamma_{mp}^b \cos (k_e (y_{c+1} + W_L^y - y^b) - \cos (k_e W_L^y) \left( e^{\gamma_{mp}^b (y_{c+1} - y^b)} + e^{-}\gamma_{mp}^b (y_{c+1} - y^b) \right) 
+ e^{-}\gamma_{mp}^b \cos (k_e (y_{c+1} + W_L^y - y^b) - \cos (k_e W_L^y) \left( e^{\gamma_{mp}^b (y_{c+1} - y^b)} + e^{-}\gamma_{mp}^b (y_{c+1} - y^b) \right) 
+ e^{-}\gamma_{mp}^b \cos (k_e (y_{c+1} + W_L^y - y^b) - \cos (k_e W_L^y) \left( e^{\gamma_{mp}^b (y_{c+1} - y^b)} + e^{-}\gamma_{mp}^b (y_{c+1} - y^b) \right) 
+ e^{-}\gamma_{mp}^b \cos (k_e (y_{c+1} + W_L^y - y^b) - \cos (k_e W_L^y) \left( e^{\gamma_{mp}^b (y_{c+1} - y^b)} + e^{-}\gamma_{mp}^b (y_{c+1} - y^b) \right) 
+ e^{-}\gamma_{mp}^b \cos (k_e (y_{c+1} + W_L^y - y^b) - \cos (k_e W_L^y) \left( e^{\gamma_{mp}^b (y_{c+1} - y^b)} + e^{-}\gamma_{mp}^b (y_{c+1} - y^b) \right) 
+ e^{-}\gamma_{mp}^b \cos (k_e (y_{c+1} + W_L^y - y^b) - \cos (k_e W_L^y) \left( e^{\gamma_{mp}^b (y_{c+1} - y^b)} + e^{-}\gamma_{mp}^b (y_{c+1} - y^b) \right) \right) \right). 
\end{align*}\]
\[ I_{k'}^0(y_k', m, p) \mid y'_k \geq y_k = \frac{1}{\gamma_{mp}^2 a d} \left( 1 + e^{-2i\gamma_{mp} b} - 2e^{-j\gamma_{mp}^b \cos (\Psi_y)} \right) \]
\[ \cdot \left[ e^{-j\Psi_y} (e^{i\gamma_{mp} (y_k' - y' - W_y' - b)} + e^{i\gamma_{mp} (y_k' - y' - W_y' + b)} - e^{j\gamma_{mp} (y_k' - y' + b)} \right) \]
\[ + (e^{j\gamma_{mp} (y_k' - y' - W_y')} + e^{-j\gamma_{mp} (y_k' - y' - W_y' + 2b)} - e^{j\gamma_{mp} (y_k' - y' + 2b)}) \] \]  (B238)

\[ I_{k'}^0(y_k', m, p) \mid y_k - W_y' \leq y'_k \leq y_k = \frac{1}{\gamma_{mp}^2 a d} \left( 1 + e^{-2i\gamma_{mp} b} - 2e^{-j\gamma_{mp}^b \cos (\Psi_y)} \right) \]
\[ \cdot \left[ e^{j\Psi_y} (e^{i\gamma_{mp} (y_k - W_y' - y')} + e^{i\gamma_{mp} (2b + y_k - W_y' - y')} + 2 (1 + e^{-2j\gamma_{mp} b}) \right) \]
\[ - e^{-j\Psi_y} (e^{j\gamma_{mp} (y_k - W_y' - y' - b)} + e^{-j\gamma_{mp} (y_k - W_y' - y' + b)} - 2e^{-j\gamma_{mp} b}) \]
\[ + e^{j\gamma_{mp} (y_k - 2b - y')} + e^{-j\gamma_{mp} (y_k - y')} - e^{-j\Psi_y} (e^{j\gamma_{mp} (y_k - y' - b)} + e^{-j\gamma_{mp} (y_k - y' + b)} - 2e^{-j\gamma_{mp} b}) \] \]  (B239)

\[ I_{k'}^0(y_k', m, p) \mid y'_k \leq y_k - W_y' = \frac{1}{\gamma_{mp}^2 a d} \left( 1 + e^{-2i\gamma_{mp} b} - 2e^{-j\gamma_{mp}^b \cos (\Psi_y)} \right) \]
\[ \cdot \left[ e^{-j\Psi_y} (e^{i\gamma_{mp} (y_k - y' - W_y' - b)} + e^{i\gamma_{mp} (y_k - y' - W_y' + b)} - e^{j\gamma_{mp} (y_k - y' + b)} \right) \]
\[ - (e^{j\gamma_{mp} (y_k - y' - W_y')} + e^{-j\gamma_{mp} (y_k - y' - 2b)} - e^{j\gamma_{mp} (y_k - y' + 2b)} \right) \] \]  (B240)
## B6 Removable singularities

Most of the integrals, defined in appendix B3, have removable singularities. The limits, if they exist, of these expressions in the singular points are given here.

\[
\lim_{k_x \to 2k_x^n} \frac{\mathcal{I}_{1, x}^f (x_m, m)}{W_{f x}^z} = \frac{e^{-j\kappa_z x}}{W_{f x}^z}
\]  
\text{(B241)}

\[
\lim_{k_x \to 2k_x^n} \frac{\mathcal{I}_{1, x}^f (x_{n-1}, m)}{2jW_{f x}^z} = \frac{W_{f x}^z e^{j\kappa_x W_{f x}^z x} - W_{f x}^z e^{-j\kappa_x W_{f x}^z x}}{\sin (k_x W_{f x}^z) - \sin (k_x W_{f x}^z)}
\]  
\text{(B242)}

\[
\lim_{k_x \to 2k_x^n} \frac{\mathcal{I}_{1, x}^f (x_n, m)}{W_{f x}^z} = \frac{e^{j\kappa_x W_{f x}^z x}}{W_{f x}^z}
\]  
\text{(B243)}

\[
\lim_{k_x \to 2k_x^n} \frac{\mathcal{I}_{1, x}^f (x_{n+1}, m)}{2jW_{f x}^z} = \frac{W_{f x}^z e^{-j\kappa_x W_{f x}^z x} - W_{f x}^z e^{j\kappa_x W_{f x}^z x}}{\sin (k_x W_{f x}^z) - \sin (k_x W_{f x}^z)}
\]  
\text{(B244)}

\[
\lim_{k_x \to 0, x} \frac{\mathcal{I}_{1, x}^f (x_m, m)}{W_{f x}^z} = W_{f x}^z
\]  
\text{(B245)}

\[
\lim_{\beta_{x_0} \to k_x^n} \mathcal{I}_{3, x}^f (x_{k_x}, n, p) \big|_{x^+ = x^- = x_0} = \frac{\left( \cos k_x (x_k + a - x^b) - e^{j\Psi_x} \cos k_x (x_k - x^b) \right) W_{f x}^z}{W_{f x}^z \sin (k_x W_{f x}^z) - \sin (k_x W_{f x}^z)}
\]  
\text{(B246)}

\[
\lim_{\beta_{x_0} \to k_x^n} \mathcal{I}_{3, x}^f (x_{k_x}, n, p) \big|_{x_k \leq x \leq x_k + W_{f x}^z} = \frac{1}{2W_{f x}^z \sin (k_x W_{f x}^z) \sin (k_x W_{f x}^z) - \sin (k_x W_{f x}^z)}
\]
\[
\cdot \left[ e^{j\Psi_x} (x_k + W_{f x}^z - x_k) - e^{-j\Psi_x} (x_k - x_k) \right] \sin k_x (W_{f x}^z + x_k - x^b) - e^{j\Psi_x} W_{f x}^z \sin k_x (W_{f x}^z - x_k + x^b)
\]
\[
+ W_{f x}^z \sin k_x (W_{f x}^z - x_k + x^b - a) + (x_k - x_k) \sin k_x (W_{f x}^z + x_k - x^b + a)
\]
\[
- (x_k + W_{f x}^z - x^b) \sin k_x (W_{f x}^z + x_k - x^b - a)
\]  
\text{(B247)}

\[
\lim_{\beta_{x_0} \to k_x^n} \mathcal{I}_{3, x}^f (x_{k_x}, n, p) \big|_{x_k - W_{f x}^z \leq x \leq x_k} = \frac{1}{2W_{f x}^z \sin (k_x W_{f x}^z) \sin (k_x W_{f x}^z) - \sin (k_x W_{f x}^z)}
\]
\[
\cdot \left[ -e^{j\Psi_x} (x_k + W_{f x}^z - x_k) + e^{-j\Psi_x} (x_k - x^b) \right] \sin k_x (W_{f x}^z - x_k + x^b) - e^{-j\Psi_x} W_{f x}^z \sin k_x (W_{f x}^z - x_k - x^b)
\]
\[
+ (x_k + W_{f x}^z - x_k) \sin k_x (W_{f x}^z - x_k - x^b - a) - (x_k - x^b) \sin k_x (W_{f x}^z - x_k + x^b + a)
\]
\[
- W_{f x}^z \sin k_x (W_{f x}^z + x_k - x^b - a)
\]  
\text{(B248)}

\[
\lim_{\beta_{x_0} \to k_x^n} \mathcal{I}_{3, x}^f (x_{k_x}, n, p) \big|_{x \leq x_k - W_{f x}^z} = \frac{e^{j\Psi_x} \cos k_x (x_k - x^b) - \cos k_x (x_k - a - x^b) W_{f x}^z}{W_{f x}^z \sin (k_x W_{f x}^z) - \sin (k_x W_{f x}^z)}
\]  
\text{(B249)}

\[
\begin{align*}
\lim_{\beta_{np} \to k_{3,3} (x_{c-1}, n, p)} \left[ x_{c-1} = x_{c-1} + w_{0} \right] & = \frac{1}{2 W_{0}^f b d (\cos (k_{e} a) - \cos (\Psi_{f})} \\
\left( e^{-j \Psi_{f}} \left[ W_{f}^f \sin k_{e} (W_{f}^f + x_{c-1} - x_{b}) \right] + W_{s}^f \frac{\sin k_{e} (W_{s}^f + x_{c-1} - x_{b})}{\sin (k_{e} W_{s}^f)} \right) \\
\left( W_{s}^f \sin k_{e} (W_{s}^f + x_{c-1} - a + x_{b}) \right) \left( W_{s}^f \sin k_{e} (W_{s}^f + x_{c-1} + a - x_{b}) \right) \right)
\end{align*}
\] (B250)

\[
\begin{align*}
\lim_{\beta_{np} \to k_{3,3} (x_{c-1}, n, p)} \left[ x_{c-1} + w_{0} \right] & = \frac{1}{2 W_{0}^f b d (\cos (k_{e} a) - \cos (\Psi_{f})} \\
\left( e^{-j \Psi_{f}} \left[ W_{f}^f \sin k_{e} (W_{f}^f + x_{c-1} - x_{b}) \right] + W_{s}^f \frac{\sin k_{e} (W_{s}^f + x_{c-1} - x_{b})}{\sin (k_{e} W_{s}^f)} \right) \\
\left( W_{s}^f \sin k_{e} (W_{s}^f + x_{c-1} + x_{b} - a) \right) \left( W_{s}^f \sin k_{e} (W_{s}^f + x_{c-1} - x_{b} - a) \right) \right)
\end{align*}
\] (B251)

\[
\begin{align*}
\lim_{\beta_{np} \to k_{3,3} (x_{c-1}, n, p)} \left[ x_{c-1} - w_{0} \right] & = \frac{1}{2 W_{0}^f b d (\cos (k_{e} a) - \cos (\Psi_{f})} \\
\left( e^{-j \Psi_{f}} \left[ W_{f}^f \sin k_{e} (W_{f}^f + x_{c-1} - x_{b}) \right] + W_{s}^f \frac{\sin k_{e} (W_{s}^f + x_{c-1} - x_{b})}{\sin (k_{e} W_{s}^f)} \right) \\
\left( W_{s}^f \sin k_{e} (W_{s}^f + x_{c-1} - x_{b} + a) \right) \left( W_{s}^f \sin k_{e} (W_{s}^f + x_{c-1} - x_{b} - a) \right) \right)
\end{align*}
\] (B252)

\[
\begin{align*}
\lim_{\beta_{np} \to k_{3,3} (x_{c-1}, n, p)} \left[ x_{c-1} \right] & = \frac{1}{2 W_{0}^f b d (\cos (k_{e} a) - \cos (\Psi_{f})} \\
\left( e^{-j \Psi_{f}} \left[ W_{f}^f \sin k_{e} (W_{f}^f + x_{c-1} - x_{b}) \right] + W_{s}^f \frac{\sin k_{e} (W_{s}^f + x_{c-1} - x_{b})}{\sin (k_{e} W_{s}^f)} \right) \\
\left( W_{s}^f \sin k_{e} (W_{s}^f + x_{c-1} - a + x_{b}) \right) \left( W_{s}^f \sin k_{e} (W_{s}^f + x_{c-1} - a - x_{b}) \right) \right)
\end{align*}
\] (B253)

\[
\begin{align*}
\lim_{\beta_{np} \to k_{3,3} (x_{c}, n, p)} \left[ x_{c} + w_{0} \right] & = \frac{1}{2 W_{0}^f b d (\cos (k_{e} a) - \cos (\Psi_{f})} W_{f}^f \\
\left( e^{-j \Psi_{f}} \left[ W_{f}^f \sin k_{e} (W_{f}^f + x_{c} - x_{b}) \right] + W_{s}^f \frac{\sin k_{e} (W_{s}^f + x_{c} - x_{b})}{\sin (k_{e} W_{s}^f)} \right) \\
\left( W_{s}^f \sin k_{e} (W_{s}^f + x_{c} - x_{b} - a) \right) \left( W_{s}^f \sin k_{e} (W_{s}^f + x_{c} - x_{b} + a) \right) \right)
\end{align*}
\] (B254)

\[
\begin{align*}
\lim_{\beta_{np} \to k_{3,3} (x_{c}, n, p)} \left[ x_{c} \right] & = \frac{1}{2 W_{0}^f b d (\cos (k_{e} a) - \cos (\Psi_{f})} W_{f}^f \\
\left( e^{-j \Psi_{f}} \left[ W_{f}^f \sin k_{e} (W_{f}^f + x_{c} - x_{b}) \right] + W_{s}^f \frac{\sin k_{e} (W_{s}^f + x_{c} - x_{b})}{\sin (k_{e} W_{s}^f)} \right) + (x_{c} - x_{c}) \sin k_{e} (W_{s}^f + x_{c} - x_{b} - a) \right)
\end{align*}
\] (B255)
\[
\begin{align*}
\lim_{{\beta_{np} \to \beta_{np}}^i} H_{\beta_{np}}(x, n, p) & = \frac{1}{2W_{f}^2 \sin (k_e W_{f}^2) b (\cos (k_e a) - \cos (\Psi_x))} \\
& \cdot \left[ -e^{j\Psi_x} (x^b + W_{f}^2 x_c + x^b) + e^{-j\Psi_x} (x_c - x^b) \right] \\
& \cdot \left[ \sin k_e (W_{f}^2 x_c + x^b) + e^{-j\Psi_x} \cdot W_{f}^2 \sin \left( W_{f}^2 x_c - x^b \right) \right] \\
& \cdot (x^b + W_{f}^2 x_c - x^b) \sin k_e (W_{f}^2 x_c + x^b - a) - (x_c - x^b) \sin \left( W_{f}^2 x_c + x^b + a \right) \\
& - W_{f}^2 \sin \left( W_{f}^2 x_c + x^b + b \right) \\
\end{align*}
\]
(B256)

\[
\begin{align*}
\lim_{{\beta_{np} \to \beta_{np}}^i} H_{\beta_{np}}(x, n, p) & = \frac{1}{2W_{f}^2 \sin (k_e W_{f}^2) b (\cos (k_e a) - \cos (\Psi_x))} \\
& \cdot \left[ e^{j\Psi_x} (x^b + W_{f}^2 x_c + x^b) - e^{-j\Psi_x} (x_c - x^b) \right] \\
& \cdot \left[ \sin k_e (W_{f}^2 x_c + x^b) + e^{-j\Psi_x} \cdot W_{f}^2 \sin \left( W_{f}^2 x_c - x^b \right) \right] \\
& \cdot (x^b - x_c + 1) \sin k_e (W_{f}^2 x_c + x^b + a) - (x_c + W_{f}^2 x_c - x^b) \sin k_e (W_{f}^2 x_c + x^b - a) \\
\end{align*}
\]
(B257)

\[
\begin{align*}
\lim_{{\beta_{np} \to \beta_{np}}^i} H_{\beta_{np}}(x, n, p) & = \frac{1}{2W_{f}^2 \sin (k_e W_{f}^2) b (\cos (k_e a) - \cos (\Psi_x))} \\
& \cdot \left[ -e^{j\Psi_x} (x^b + W_{f}^2 x_c + x^b) + e^{-j\Psi_x} (x_c - x^b) \right] \\
& \cdot \left[ \sin k_e (W_{f}^2 x_c + x^b) + e^{-j\Psi_x} \cdot W_{f}^2 \sin \left( W_{f}^2 x_c - x^b \right) \right] \\
& \cdot (x^b - x_c + 1) \sin k_e (W_{f}^2 x_c + x^b + a) - (x_c + W_{f}^2 x_c - x^b) \sin k_e (W_{f}^2 x_c + x^b - a) \\
\end{align*}
\]
(B258)

\[
\begin{align*}
\lim_{{\beta_{np} \to \beta_{np}}^i} H_{\beta_{np}}(x, n, p) & = \frac{1}{2W_{f}^2 \sin (k_e W_{f}^2) b (\cos (k_e a) - \cos (\Psi_x))} \\
& \cdot \left[ e^{j\Psi_x} (x^b + W_{f}^2 x_c + x^b) - e^{-j\Psi_x} (x_c - x^b) \right] \\
& \cdot \left[ \sin k_e (W_{f}^2 x_c + x^b) + e^{-j\Psi_x} \cdot W_{f}^2 \sin \left( W_{f}^2 x_c - x^b \right) \right] \\
& \cdot (x^b - x_c + 1) \sin k_e (W_{f}^2 x_c + x^b + a) - (x_c + W_{f}^2 x_c - x^b) \sin k_e (W_{f}^2 x_c + x^b - a) \\
\end{align*}
\]
(B259)

\[
\begin{align*}
\lim_{{\beta_{np} \to \beta_{np}}^i} H_{\beta_{np}}(x, n, p) & = \frac{1}{2W_{f}^2 \sin (k_e W_{f}^2) b (\cos (k_e a) - \cos (\Psi_x))} \\
& \cdot \left[ -e^{j\Psi_x} (x^b + W_{f}^2 x_c + x^b) + e^{-j\Psi_x} (x_c - x^b) \right] \\
& \cdot \left[ \sin k_e (W_{f}^2 x_c + x^b) + e^{-j\Psi_x} \cdot W_{f}^2 \sin \left( W_{f}^2 x_c - x^b \right) \right] \\
& \cdot (x^b + W_{f}^2 x_c - x^b) \sin k_e (W_{f}^2 x_c + x^b + a) - (x_c + W_{f}^2 x_c + x^b) \sin k_e (W_{f}^2 x_c + x^b - a) \\
\end{align*}
\]
(B260)

\[
\begin{align*}
\lim_{{\beta_{np} \to \beta_{np}}^i} H_{\beta_{np}}(x, n, p) & = \frac{1}{2W_{f}^2 \sin (k_e W_{f}^2) b (\cos (k_e a) - \cos (\Psi_x))} \\
& \cdot \left[ e^{j\Psi_x} (x^b + W_{f}^2 x_c + x^b) - e^{-j\Psi_x} (x_c - x^b) \right] \\
& \cdot \left[ \sin k_e (W_{f}^2 x_c + x^b) + e^{-j\Psi_x} \cdot W_{f}^2 \sin \left( W_{f}^2 x_c - x^b \right) \right] \\
& \cdot (x^b + W_{f}^2 x_c - x^b) \sin k_e (W_{f}^2 x_c + x^b + a) - (x_c + W_{f}^2 x_c + x^b) \sin k_e (W_{f}^2 x_c + x^b - a) \\
\end{align*}
\]
(B261)
Dual polarization

\[
\lim_{\beta \to 0} \mathcal{B}^\zeta_{\nu, x}(x', n, p) \bigg|_{x' \approx x_k} = \frac{-W^\zeta_x \left( e^{i\Psi_y} \left( x_k - x^b - \frac{W^\zeta_x}{2} \right) - \left( x_k + a - x^b - \frac{W^\zeta_x}{2} \right) \right)}{bd \left( 1 - \cos(\Psi_y) \right)} \quad (B262)
\]

\[
\lim_{\beta \to 0} \mathcal{B}^\zeta_{\nu, x}(x', n, p) \bigg|_{x' \approx x_k} = \frac{-W^\zeta_x \left( e^{i\Psi_y} \left( x_k - x^b - \frac{W^\zeta_x}{2} \right) - \left( x_k + a - x^b - \frac{W^\zeta_x}{2} \right) \right)}{bd \left( 1 - \cos(\Psi_y) \right)} \quad (B263)
\]

\[
\lim_{\beta \to 0} \mathcal{B}^\zeta_{\nu, x}(x', n, p) \bigg|_{x' \approx x_k - W^\zeta_z x} = \frac{W^\zeta_z \left( e^{i\Psi_y} \left( x_k - x^b - \frac{W^\zeta_x}{2} \right) - \left( x_k + a - x^b - \frac{W^\zeta_x}{2} \right) \right)}{bd \left( 1 - \cos(\Psi_y) \right)} \quad (B264)
\]

\[
k_y \to 2k^\zeta_1, y(y_k, n) = \frac{W^\zeta_y}{W^\zeta_z} e^{-i\Psi_y} \quad (B265)
\]

\[
k_y \to 2k^\zeta_y, y(y_{c-1}, n) = \frac{e^{-i\Psi_y} - i}{2jW^\zeta_y} \left( \frac{W^\zeta_y e^{i\Psi_y}}{\sin(k^\zeta_y W^\zeta_y)} - \frac{W^\zeta_y e^{-i\Psi_y}}{\sin(k^\zeta_y W^\zeta_y)} \right) \quad (B266)
\]

\[
k_y \to 2k^\zeta_y, y(y_{c+1}, n) = \frac{e^{-i\Psi_y} + i}{2jW^\zeta_y} \left( \frac{W^\zeta_y e^{i\Psi_y}}{\sin(k^\zeta_y W^\zeta_y)} - \frac{W^\zeta_y e^{-i\Psi_y}}{\sin(k^\zeta_y W^\zeta_y)} \right) \quad (B267)
\]

\[
k_y \to 2k^\zeta_y, y(y', n) = W^\zeta_y \quad (B268)
\]

\[
\gamma_{m} \to k_x, y(y, m, p) \bigg|_{y \approx y_k + W^\zeta_y} = \frac{\gamma_{m} \left( \cos k_x (y_k + b - y^b) - e^{i\Psi_y} \cos k_x (y_k - y^b) \right)}{W^\zeta_y a d \left( \cos(k, b) - \cos(\Psi_y) \right)} \quad (B270)
\]

\[
\gamma_{m} \to k_x, y(y, m, p) \bigg|_{y \approx y_k + W^\zeta_y} = \frac{1}{2 W^\zeta_y \sin(k_x W^\zeta_y) a d \left( \cos(k, b) - \cos(\Psi_y) \right)} \cdot \left[ e^{-i\Psi_y} (y_k + W^\zeta_y - y^b) - e^{i\Psi_y} (y^b - y_k) \right] \sin k_x (W^\zeta_y + y_k - y^b) - e^{i\Psi_y} \cdot W^\zeta_y \sin k_x (W^\zeta_y - y_k + y^b) + W^\zeta_y \sin k_x (W^\zeta_y - y_k + y^b - b) + (y^b - y_k) \sin k_x (W^\zeta_y + y_k - y^b + b) - (y_k + W^\zeta_y - y^b) \sin k_x (W^\zeta_y + y_k - y^b - b) \quad (B271)
\]
\[
\lim_{\gamma_{sp} \to k \gamma_{sp}} \frac{I^{y}_{y_{sp}}(y_{sp}, m, p)}{y_{sp} \leq y_{sp} \leq y_{sp} + \tilde{d}_{sp}} = \frac{1}{2W^{y}_{sp} \sin (k_{y_{sp}} \tilde{d}_{sp}) \cos (\Psi_{y_{sp}}) - \cos (\Psi_{y_{sp}})}
\]

\[
\left[ -e^{-j\Psi_{y_{sp}}}(y_{sp} + W^{y}_{sp} - y_{sp}) + e^{j\Psi_{y_{sp}}}(y_{sp} - y_{sp}) \right] \sin k_{y_{sp}}(W^{y}_{sp} - y_{sp} + y_{sp}) + e^{-j\Psi_{y_{sp}}}. W^{y}_{sp} \sin k_{y_{sp}}(W^{y}_{sp} + y_{sp} - y_{sp}) + (y_{sp} + W^{y}_{sp} - y_{sp}) \sin k_{y_{sp}}(W^{y}_{sp} - y_{sp} + y_{sp} + b) - (y_{sp} - y_{sp}) \sin k_{y_{sp}}(W^{y}_{sp} - y_{sp} + y_{sp} + b) - W^{y}_{sp} \sin k_{y_{sp}}(W^{y}_{sp} + y_{sp} - y_{sp} - b) \right) \]

\[
\lim_{\gamma_{sp} \to k \gamma_{sp}} \frac{I^{y}_{y_{sp}}(y_{sp}, m, p)}{y_{sp} \leq y_{sp} \leq y_{sp} + \tilde{d}_{sp}} = \left( \frac{e^{-j\Psi_{y_{sp}}} \cos k_{y_{sp}}(y_{sp} - y_{sp}) - \cos k_{y_{sp}}(y_{sp} - b - y_{sp})}{W^{y}_{sp} \sin (k_{y_{sp}} \tilde{d}_{sp}) \cos (\Psi_{y_{sp}}) - \cos (\Psi_{y_{sp}})} \right) \]

\[
\lim_{\gamma_{sp} \to k \gamma_{sp}} \frac{I^{y}_{y_{sp}}(y_{sp}, m, p)}{y_{sp} \leq y_{sp} \leq y_{sp} + \tilde{d}_{sp}} = \frac{1}{2W^{y}_{sp} \sin (k_{y_{sp}} \tilde{d}_{sp}) \cos (\Psi_{y_{sp}}) - \cos (\Psi_{y_{sp}})}
\]

\[
\left[ -e^{-j\Psi_{y_{sp}}}(y_{sp} + W^{y}_{sp} - y_{sp} - b_{sp}) + e^{j\Psi_{y_{sp}}}(y_{sp} - y_{sp} - b_{sp}) \right] \sin k_{y_{sp}}(W^{y}_{sp} - y_{sp} + y_{sp} - b_{sp}) + e^{-j\Psi_{y_{sp}}}. W^{y}_{sp} \sin k_{y_{sp}}(W^{y}_{sp} - y_{sp} + y_{sp} + b_{sp}) + (y_{sp} + W^{y}_{sp} - y_{sp} - b_{sp}) \sin k_{y_{sp}}(W^{y}_{sp} - y_{sp} + y_{sp} + b_{sp}) - (y_{sp} - y_{sp} - b_{sp}) \sin k_{y_{sp}}(W^{y}_{sp} - y_{sp} + y_{sp} + b_{sp}) - W^{y}_{sp} \sin k_{y_{sp}}(W^{y}_{sp} + y_{sp} - y_{sp} - b_{sp}) \right) \]

\[
\lim_{\gamma_{sp} \to k \gamma_{sp}} \frac{I^{y}_{y_{sp}}(y_{sp}, m, p)}{y_{sp} \leq y_{sp} \leq y_{sp} + \tilde{d}_{sp}} = \left( \frac{e^{-j\Psi_{y_{sp}}} \cos k_{y_{sp}}(y_{sp} - y_{sp}) - \cos k_{y_{sp}}(y_{sp} - b - y_{sp})}{W^{y}_{sp} \sin (k_{y_{sp}} \tilde{d}_{sp}) \cos (\Psi_{y_{sp}}) - \cos (\Psi_{y_{sp}})} \right) \]

\[
\lim_{\gamma_{sp} \to k \gamma_{sp}} \frac{I^{y}_{y_{sp}}(y_{sp}, m, p)}{y_{sp} \leq y_{sp} \leq y_{sp} + \tilde{d}_{sp}} = \frac{1}{2W^{y}_{sp} \sin (k_{y_{sp}} \tilde{d}_{sp}) \cos (\Psi_{y_{sp}}) - \cos (\Psi_{y_{sp}})}
\]

\[
\left[ -e^{-j\Psi_{y_{sp}}}(y_{sp} + W^{y}_{sp} - y_{sp} - b_{sp} - y_{sp}) + e^{j\Psi_{y_{sp}}}(y_{sp} - y_{sp} - b_{sp} - y_{sp}) \right] \sin k_{y_{sp}}(W^{y}_{sp} - y_{sp} + y_{sp} + b_{sp} + y_{sp}) + e^{-j\Psi_{y_{sp}}}. W^{y}_{sp} \sin k_{y_{sp}}(W^{y}_{sp} - y_{sp} + y_{sp} + b_{sp} + y_{sp}) + (y_{sp} + W^{y}_{sp} - y_{sp} - b_{sp} - y_{sp}) \sin k_{y_{sp}}(W^{y}_{sp} - y_{sp} + y_{sp} + b_{sp} + y_{sp}) - (y_{sp} - y_{sp} - b_{sp} - y_{sp}) \sin k_{y_{sp}}(W^{y}_{sp} - y_{sp} + y_{sp} + b_{sp} + y_{sp}) - W^{y}_{sp} \sin k_{y_{sp}}(W^{y}_{sp} + y_{sp} - y_{sp} - b_{sp} - y_{sp}) \right) \]

\[
\lim_{\gamma_{sp} \to k \gamma_{sp}} \frac{I^{y}_{y_{sp}}(y_{sp}, m, p)}{y_{sp} \leq y_{sp} \leq y_{sp} + \tilde{d}_{sp}} = \left( \frac{e^{-j\Psi_{y_{sp}}} \cos k_{y_{sp}}(y_{sp} - y_{sp}) - \cos k_{y_{sp}}(y_{sp} - b - y_{sp})}{W^{y}_{sp} \sin (k_{y_{sp}} \tilde{d}_{sp}) \cos (\Psi_{y_{sp}}) - \cos (\Psi_{y_{sp}})} \right) \]

\[
\lim_{\gamma_{sp} \to k \gamma_{sp}} \frac{I^{y}_{y_{sp}}(y_{sp}, m, p)}{y_{sp} \leq y_{sp} \leq y_{sp} + \tilde{d}_{sp}} = \frac{1}{2W^{y}_{sp} \sin (k_{y_{sp}} \tilde{d}_{sp}) \cos (\Psi_{y_{sp}}) - \cos (\Psi_{y_{sp}})}
\]
\[
\lim_{y_{m} \to k_{e}} \frac{I_{y}^{\gamma}(y_{e}, m, p)}{w_{y}^{2}} \bigg|_{y_{e} \leq y \leq y_{e} + w_{y}^{2}} = \left( \frac{\cos k_{e} (y_{e} + b - y^{b}) - e^{j\Psi} \cos k_{e} (y_{e} - y^{b})}{W_{\gamma}^{y} \sin (k_{e} b)} \right) \left( \frac{W_{\gamma}^{y}}{\cos (k_{e} b) - \cos (\Psi_{y})} \right)
\]

(B278)

\[
\lim_{y_{m} \to k_{e}} \frac{I_{y}^{\gamma}(y_{e}, m, p)}{w_{y}^{2}} \bigg|_{y_{e} \leq y \leq y_{e} + w_{y}^{2}} = \left( \frac{1}{2W_{\gamma}^{y} \sin (k_{e} W_{\gamma}^{y}) \cos (k_{e} b) - \cos (\Psi_{y})} \right)
\cdot \left( e^{-j\Psi_{y}} (y_{e} + W_{\gamma}^{y} - y^{b}) - e^{j\Psi_{y}} (y^{b} - y_{e}) \right) \sin k_{e} \left( W_{\gamma}^{y} + y_{e} - y^{b} \right) - e^{j\Psi_{y}} \cdot W_{\gamma}^{y} \sin k_{e} \left( W_{\gamma}^{y} - y_{e} + y^{b} \right)
\]
\[+ W_{\gamma}^{y} \sin k_{e} \left( W_{\gamma}^{y} - y_{e} + y^{b} + b \right) \] (B279)

\[
\lim_{y_{m} \to k_{e}} \frac{I_{y}^{\gamma}(y_{e}, m, p)}{w_{y}^{2}} \bigg|_{y_{e} \leq y \leq y_{e} + w_{y}^{2}} = \left( \frac{1}{2W_{\gamma}^{y} \sin (k_{e} W_{\gamma}^{y}) \cos (k_{e} b) - \cos (\Psi_{y})} \right)
\cdot \left( e^{-j\Psi_{y}} (y^{b} + W_{\gamma}^{y} - y_{e}) + e^{j\Psi_{y}} (y_{e} - y^{b}) \right) \sin k_{e} \left( W_{\gamma}^{y} - y_{e} - y^{b} \right) + e^{j\Psi_{y}} \cdot W_{\gamma}^{y} \sin k_{e} \left( W_{\gamma}^{y} + y_{e} - y^{b} \right)
\]
\[+ W_{\gamma}^{y} \sin k_{e} \left( W_{\gamma}^{y} + y_{e} - y^{b} + b \right) \] (B280)

\[
\lim_{y_{m} \to k_{e}} \frac{I_{y}^{\gamma}(y_{e}, m, p)}{w_{y}^{2}} \bigg|_{y_{e} \leq y \leq y_{e} + w_{y}^{2}} = \left( \frac{e^{-j\Psi_{y}} \cos k_{e} (y_{e} - y^{b}) - \cos k_{e} (y_{e} - b - y^{b})}{W_{\gamma}^{y} \sin (k_{e} b)} \right) \left( \frac{W_{\gamma}^{y}}{\cos (k_{e} b) - \cos (\Psi_{y})} \right)
\]

(B281)

\[
\lim_{y_{m} \to k_{e}} \frac{I_{y}^{\gamma}(y_{e} + 1, m, p)}{w_{y}^{2}} \bigg|_{y_{e} + 1 \leq y \leq y_{e} + 1 + w_{y}^{2}} = \left( \frac{1}{2W_{\gamma}^{y} \sin (k_{e} W_{\gamma}^{y}) \cos (k_{e} b) - \cos (\Psi_{y})} \right)
\cdot \left( e^{-j\Psi_{y}} \left( W_{\gamma}^{y} \sin k_{e} \left( W_{\gamma}^{y} - y_{e} + 1 + y^{b} \right) + \frac{W_{\gamma}^{y} \sin k_{e} \left( W_{\gamma}^{y} + y_{e} + 1 - y^{b} \right)}{\sin (k_{e} W_{\gamma}^{y})} \right) \right)
\]
\[+ \left( \frac{W_{\gamma}^{y} \sin k_{e} \left( W_{\gamma}^{y} - y_{e} + 1 + b \right)}{\sin (k_{e} W_{\gamma}^{y})} + \frac{W_{\gamma}^{y} \sin k_{e} \left( W_{\gamma}^{y} + y_{e} + 1 + b - y^{b} \right)}{\sin (k_{e} W_{\gamma}^{y})} \right) \] (B282)

\[
\lim_{y_{m} \to k_{e}} \frac{I_{y}^{\gamma}(y_{e} + 1, m, p)}{w_{y}^{2}} \bigg|_{y_{e} + 1 \leq y \leq y_{e} + b + w_{y}^{2}} = \left( \frac{1}{2W_{\gamma}^{y} \sin (k_{e} b)} \right)
\cdot \left( e^{-j\Psi_{y}} \left( \frac{\sin k_{e} \left( W_{\gamma}^{y} + y_{e} + 1 - y^{b} \right)}{\sin (k_{e} W_{\gamma}^{y})} \right) + \frac{W_{\gamma}^{y} \sin k_{e} \left( W_{\gamma}^{y} - y_{e} + b \right)}{\sin (k_{e} W_{\gamma}^{y})} \right)
\]
\[+ \left( \frac{W_{\gamma}^{y} \sin k_{e} \left( W_{\gamma}^{y} - y_{e} + 1 + b \right)}{\sin (k_{e} W_{\gamma}^{y})} + \frac{\left( W_{\gamma}^{y} + y_{e} + 1 - y^{b} \right) \sin k_{e} \left( W_{\gamma}^{y} + y_{e} + 1 - y^{b} \right)}{\sin (k_{e} W_{\gamma}^{y})} \right) \] (B283)
\begin{align}
\lim_{\gamma \to k_x} H_{x,y}^{(1)}(y_{c+1}, m, p) |_{y_{c+1} - W_{y}^{f} \leq y \leq y_{c+1}} &= \frac{1}{2 W_{y}^{ad} (\cos (k_x b) - \cos (\Psi_y))}
\cdot \left[ e^{j \Psi_y} (y^b + W_{y}^{f} - y_{c+1}) + e^{-j \Psi_y} (y_{c+1}^b - y^b) \right] \frac{\sin k_x (W_{y}^{f} - y_{c+1}^b + y^b)}{\sin (k_x W_{y}^{f})}
\cdot e^{-j \Psi_y} \cdot \frac{W_{y}^{f} \sin k_x (W_{y}^{f} + y_{c+1}^b - y^b)}{\sin (k_x W_{y}^{f})}
\cdot \left( y_{c+1}^b - y^b \right) \sin k_x (W_{y}^{f} - y_{c+1}^b + y^b - b) - \frac{W_{y}^{f} \sin k_x (W_{y}^{f} + y_{c+1}^b - y^b)}{\sin (k_x W_{y}^{f})}
\right)
\end{align}

(B284)

\begin{align}
\lim_{\gamma \to k_x} H_{x,y}^{(1)}(y_{c+1}, m, p) |_{y_{c+1} - W_{y}^{f} \leq y \leq y_{c+1}} &= \frac{1}{2 W_{y}^{ad} (\cos (k_x b) - \cos (\Psi_y))}
\cdot \left[ e^{j \Psi_y} (y^b + W_{y}^{f} - y_{c+1}) + e^{-j \Psi_y} (y_{c+1}^b - y^b) \right] \frac{\sin k_x (W_{y}^{f} + y_{c+1}^b - y^b)}{\sin (k_x W_{y}^{f})}
\cdot \left( y_{c+1}^b - y^b \right) \sin k_x (W_{y}^{f} - y_{c+1}^b + y^b - b) - \frac{W_{y}^{f} \sin k_x (W_{y}^{f} + y_{c+1}^b - y^b)}{\sin (k_x W_{y}^{f})}
\right)
\end{align}

(B285)

\begin{align}
\lim_{\gamma \to 0} H_{x,y}^{(1)}(y_{k}, m, p) |_{y_{k} - W_{y}^{f} \leq y \leq y_{k}} &= \frac{W_{y}^{f} \left( e^{j \Psi_y} \left( y_{k} - y^b - \frac{W_{y}^{f}}{2} \right) - \left( y_{k} + b - y^b - \frac{W_{y}^{f}}{2} \right) \right)}{ad (1 - \cos (\Psi_y))}
\end{align}

(B286)

\begin{align}
\lim_{\gamma \to 0} H_{x,y}^{(1)}(y_{k}, m, p) |_{y_{k} - W_{y}^{f} \leq y \leq y_{k}} &= \frac{W_{y}^{f} \left( e^{-j \Psi_y} \left( y_{k} - y^b - \frac{W_{y}^{f}}{2} \right) - \left( y_{k} + b - y^b - \frac{W_{y}^{f}}{2} \right) \right)}{ad (1 - \cos (\Psi_y))}
\end{align}

(B287)

\begin{align}
\lim_{k \to p \frac{\pi}{d}} H_{x,z}^{(3)}(z_p, p) &= \frac{W_{x}^{f} \cos \left( \frac{p \pi z_i}{d} \right)}{W_{x}^{f}}
\end{align}

(B288)

\begin{align}
\lim_{p \to 0} H_{x, z}^{(3)}(z_p, p) = 0
\end{align}

(B289)

\begin{align}
\lim_{k \to e_{3,d}^{p}} H_{x, z}^{(3)}(z_p, m, n) &= \frac{W_{x}^{f} \cos (k_x (z_i + d))}{W_{x}^{f}}
\end{align}

(B290)

\begin{align}
\lim_{k \to 0 \sin (k_{mn} d)} H_{x, z}^{(1)}(z_p, m, n) &= \frac{W_{x}^{f} \left( z_i - \frac{z_i}{2} + d \right)}{d}
\end{align}

(B291)

\begin{align}
\lim_{k \to 0 \frac{\pi}{d}} H_{x, z}^{(3)}(z_p, p) &= \frac{W_{x}^{f} \cos \left( \frac{p \pi z_i}{d} \right)}{d}
\end{align}

(B292)
\[
\lim_{p \to 0} f_\ell^\ell (z, p) = 0
\]  \hspace{1cm} (B294)

\[
\lim_{k_{m,n} \to k_{\ell}} f_\ell^\ell (z, p, m, n) = \frac{Wf_\ell^\ell}{Wf_\ell^\ell} \cos (k_{\ell} (z_\ell + d))
\]  \hspace{1cm} (B295)

\[
\lim_{k_{m,n} \to 0} f_\ell^\ell (z, p, m, n) = \frac{Wf_\ell^\ell}{Wf_\ell^\ell} \left( z_\ell - \frac{Wf_\ell^\ell}{2} + d \right) \frac{d}{d}
\]  \hspace{1cm} (B296)
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