Accuracy analysis of the weakly guiding approximation for a single mode cylindrical optical fiber

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Accuracy analysis of the weakly guiding approximation for a single mode cylindrical optical fiber

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Abstract

We compute magnitudes like group slowness $\tau_g$, dispersion $D$ and dispersion slope $S$ for a propagation mode, found by using the weakly guiding approximation of the field vector solutions. Results obtained for these magnitudes are then compared to results obtained by using full-vectorial field vector solutions. It is investigated whether the weakly guiding approximation is accurate enough to produce similar results for $\tau_g$, $D$ and $S$ as acquired by the use of these full-vectorial field vector solutions. In order to achieve this, the difference between the results for $\tau_g$, $D$ and $S$ of both computation methods is analyzed, for several permittivity profiles and values of the relative refractive index difference $\Delta$. 
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Chapter 1

Introduction

The ever increasing demand for bandwidth in optical fibers, puts a severe strain on the design process. Therefore, the need for exact and reliable models is of utmost importance, as they compute pivotal parameters vital for practical design. These parameters include for instance propagation coefficients, group slowness, dispersion and dispersion slope. In [1] results for these parameters are computed using full-vectorial field solutions, for a radially inhomogeneous optical fiber, i.e. a fiber which has an arbitrary core refractive index profile. The problem is formulated by a system of four coupled differential equations, which are solved for the transverse field components. In the core region, these equations are solved either by direct numerical integration, or by substitution of a power series expansion. In the cladding region, the field is derived analytically and described by modified Bessel functions. An equation for the unknown propagation coefficients is then obtained by ensuring that the boundary conditions at the core-cladding interface are satisfied. In [2], a similar numerical method is used to compute the derived fiber parameters, only now the cladding region is considered to be of finite extent. In [3]-[6] on the other hand, propagation coefficients and other parameters are computed using the weakly guiding approximation. This approximation is applicable when the refractive index difference between core and cladding is small. Again, a wave equation has to be solved numerically for the core region, to incorporate arbitrary refractive index profiles. However, this time the wave equation is scalar instead of vectorial.

In optical fiber design, the weak-guidance approximation is often used. However, to our best knowledge no comparison of results has been made between the weakly guiding approximation and the full-vectorial computation of electromagnetic fields. Therefore, we want to investigate the possible difference in results between both computation methods, for several core permittivity profiles and values of the relative refractive index difference $\Delta$. Software which utilizes exact field solutions, i.e. performs a full-vectorial field analysis, to compute propagation coefficients, group slowness $\tau_g$, dispersion $D$ and dispersion slope $S$, is already available. This software is based on the discussion in [1], and will be referred to as the 'f.v. program' from now on. In order to compare results, we have developed a Fortran program which calculates the same fiber parameters as the f.v. program, only now
the weakly guiding approximation is used. This developed program will be referred to as the 'w.g. program'.

In Chapter 2, we first discuss the full-vectorial field analysis of a step-index fiber and an arbitrary-index fiber, to obtain the propagation coefficients for both single as well as multi-mode fibers. In Chapter 3, we apply the weakly guiding approximation to both types of fibers, which yields slightly different propagation coefficients. Chapter 4 treats the computation of \( \tau_g \), \( D \) and \( S \) for a found propagation coefficient, in the case of a single-mode cylindrical optical fiber with an arbitrary permittivity profile. Finally, Chapter 5 gives us an overview of the results obtained by both programs, after which a conclusion is formulated.
Chapter 2

Full-vectorial field analysis

An optical fiber is a cylindrical, dielectric waveguide that transmits light along its axis, by the process of total internal reflection. The fiber consists of a core surrounded by a cladding layer, and a plastic coating in intimate contact with the cladding called the jacket. In our analysis of this optical fiber, we consider the cladding region to be infinite, and thus neglect layers that may possibly surround the cladding. Furthermore, we assume that no bends are present. We use cylindrical coordinates \((r, \phi, z)\) and describe the optical fiber using the model shown in Fig. 2.1.

The core permittivity profile \(\varepsilon_r\) depends on the radial coordinate \(r\) and the frequency \(\omega\). We distinguish between fibers that have a constant permittivity in the core region (step...
index-fibers), and fibers that have a variable permittivity in the core region (arbitrary-index fibers), e.g. a graded profile. The permittivity in the cladding is considered to be constant in the radial direction for all profiles.

In Fig. 2.2, an example of a step and arbitrary refractive index profile is shown, where $n(r) = \sqrt{\mu_r \varepsilon_r(r)}$ is plotted as function of $r$. The relative permeability $\mu_r = 1$, and the variables $n_1$ and $n_2$ denote the refractive index at the core’s center and the cladding region, respectively. For these types of fibers, full-vectorial field solutions are determined in this chapter.

![Diagram](image)

(a) Step profile  
(b) Arbitrary profile

Figure 2.2: Examples of core refractive index profiles, with core radius $a$, and a constant cladding refractive index $n_2$. 

8
2.1 Step-index fiber

The starting point of the full-vectorial field analysis is formed by Maxwell’s equations in cylindrical coordinates \((r, \phi, z)\). This section treats the analytical derivation of solutions of the electromagnetic field components in the core and cladding region, for a step-index fiber. However, these solutions contain some unknown parameters, i.e. the scalar amplitudes and the propagation coefficient. From the continuity conditions of the electric and magnetic fields at the core-cladding boundary, the propagation coefficients and scalar amplitudes of the propagating modes are determined.

2.1.1 Wave equations in cylindrical coordinates

First, we introduce Maxwell’s equations in the frequency domain, assuming a \(e^{j\omega t}\) time dependence

\[
\begin{align*}
-\nabla \times H + j\omega \varepsilon E &= -J, \\
\nabla \times E + j\omega \mu H &= -K,
\end{align*}
\]

(2.1a)

\[
\begin{align*}
\nabla \cdot H &= \sigma_m / \mu, \\
\nabla \cdot E &= \sigma_e / \varepsilon,
\end{align*}
\]

(2.1b)

where \(J\) and \(K\) denote volume densities of electric and magnetic currents, and \(\sigma_e\) and \(\sigma_m\) denote electric and magnetic charge densities, respectively. Eq. (2.1) is used in circular cylindrical coordinates \((r, \phi, z)\), and the following variables are introduced

\[
\begin{align*}
\tilde{\omega} = \frac{\omega a}{c_0}, \quad \rho = \frac{r}{a}, \quad \zeta = \frac{k_z}{k_0}, \quad k_0 = \frac{\tilde{\omega}}{a}, \quad Y_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}}, \quad Z_0 = \frac{1}{Y_0}, \quad n^2 = \varepsilon_r,
\end{align*}
\]

(2.2)

where \(a\) denotes the fiber core radius, \(k_0\) is the free-space wavenumber, \(c_0\) is the speed of light in vacuum, and \(Y_0\) is the free-space wave admittance. We follow [7] and [8], and consider an electromagnetic wave with angular frequency \(\omega\), which propagates along the \(z\) direction with propagation coefficient \(k_z\). We define the electric and magnetic fields in a cylindrical coordinate system \((\rho, \phi, z)\) with the aid of Eq. (2.2) as follows

\[
\begin{align*}
E &= \text{Re} \left\{ E(\rho, \phi) e^{j\tilde{\omega}(\alpha t - \zeta z)} \right\}, \\
H &= \text{Re} \left\{ H(\rho, \phi) e^{j\tilde{\omega}(\alpha t - \zeta z)} \right\}.
\end{align*}
\]

(2.3)

(2.4)

We are interested in the source-free solutions of Eq. (2.1), i.e. the values of \(\zeta\) and the associated modal fields for which \(J = 0\) and \(K = 0\). The equations for the components of \(E\) and \(H\) in circular cylindrical coordinates are given by

\[
\begin{align*}
 j\tilde{\omega} n^2 Y_0 E_\rho &= \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} + j\zeta \tilde{\omega} H_\phi, & j\tilde{\omega} n^2 Y_0 E_\phi &= -j\zeta \tilde{\omega} E_\phi - \frac{1}{\rho} \frac{\partial E_\rho}{\partial \phi}, \\
 j\tilde{\omega} n^2 Y_0 H_\rho &= -j\zeta \tilde{\omega} H_\phi - \frac{\partial H_z}{\partial \rho}, & j\tilde{\omega} n^2 Y_0 H_\phi &= \frac{\partial E_\rho}{\partial \rho} + j\zeta \tilde{\omega} E_\phi, \\
 j\tilde{\omega} n^2 Y_0 E_z &= \frac{1}{\rho} \left[ \frac{\partial (\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi} \right], & j\tilde{\omega} n^2 Y_0 H_z &= \frac{1}{\rho} \left[ \frac{\partial E_\rho}{\partial \phi} - \frac{\partial (\rho E_\phi)}{\partial \rho} \right].
\end{align*}
\]

(2.5a)

(2.5b)

(2.5c)
where we have replaced $\partial_z$ by $-j\omega \zeta / a$. From Eqs. (2.5a) and (2.5b) it follows that the components $E_{\rho}$, $E_{\phi}$, $H_{\rho}$ and $H_{\phi}$ can be expressed in terms of $E_z$ and $H_z$, i.e.

$$E_{\rho} = -\frac{j}{\omega \kappa^2} \left( \zeta \frac{\partial E_z}{\partial \rho} + \frac{Z_0}{\rho} \frac{\partial H_z}{\partial \phi} \right), \quad E_{\phi} = -\frac{j}{\omega \kappa^2} \left( \frac{\zeta}{\rho} \frac{\partial E_z}{\partial \phi} - \frac{Z_0}{\rho} \frac{\partial H_z}{\partial \rho} \right), \quad (2.6a)$$

$$H_{\rho} = -\frac{j}{\omega \kappa^2} \left( \zeta \frac{\partial H_z}{\partial \rho} - \frac{n^2 Y_0}{\rho} \frac{\partial E_z}{\partial \phi} \right), \quad H_{\phi} = -\frac{j}{\omega \kappa^2} \left( \frac{\zeta}{\rho} \frac{\partial H_z}{\partial \phi} + \frac{n^2 Y_0}{\rho} \frac{\partial E_z}{\partial \rho} \right), \quad (2.6b)$$

where

$$\kappa^2 = (n_i^2 - \zeta^2) \quad (2.7)$$

and

$$n_i^2 = \varepsilon_{r_i}. \quad (2.8)$$

In Eqs. (2.7) and (2.8), $i$ indicates the respective region of the fiber, i.e. either the core region ($i = 1$), or the cladding region ($i = 2$) and $n_i$ indicates the refractive index of the respective medium. If we substitute the field components of Eq. (2.6b) into the first expression of Eq. (2.5c) we find

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + (\omega \kappa)^2 E_z = \left[ \nabla_t^2 + (\omega \kappa)^2 \right] E_z = 0. \quad (2.9)$$

Furthermore, if Eq. (2.6a) is substituted into the second expression of Eq. (2.5c), we obtain

$$\frac{\partial^2 H_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial H_z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 H_z}{\partial \phi^2} + (\omega \kappa)^2 H_z = \left[ \nabla_t^2 + (\omega \kappa)^2 \right] H_z = 0. \quad (2.10)$$

Observe that if $E_z$ and $H_z$ are known, all other field components can be evaluated as well. To obtain a solution for $E_z$ and $H_z$ in Eqs. (2.9) and (2.10), we use separation of variables, by expressing the axial electric- and magnetic field component as

$$E_z(\text{core}) \quad (2.13a)$$

$$E_z(\text{cladding}) \quad (2.13b)$$

where $m$ is the azimuthal wave number, $\kappa_1 = \sqrt{n_1^2 - \zeta^2}$ and $\kappa_2 = \sqrt{\zeta^2 - n_2^2}$. In Eq. (2.13) we have defined the square root as $\kappa_i = \sqrt{n_i^2 - \zeta^2}$, where $\text{Im}(\kappa_i) < 0$, and $\text{Re}(\kappa_i) > 0$ when
\[ \text{Im}(\kappa_i) = 0. \]

The solutions of Eqs. (2.12) and (2.13), using [9], are now given by

\[
\Phi_z(\phi) = \begin{cases} 
A \cos(m\phi + \theta) & \text{(core)}, \\
A' \sin(m\phi + \theta) & \text{(cladding)},
\end{cases} \quad (2.14a)
\]

\[
R_z(\rho) = \begin{cases} 
BJ_m(\bar{w}\kappa_1\rho) + B'Y_m(\bar{w}\kappa_1\rho) & \text{(core)}, \\
C\bar{K}_m(\bar{w}\kappa_2\rho) + C'I_m(\bar{w}\kappa_2\rho) & \text{(cladding)},
\end{cases} \quad (2.14b)
\]

respectively, where \( A, A', B, B', C \) and \( C' \) are yet unknown constants, \( J_m, Y_m \) denote the \( m \)th-order Bessel functions of the first and second kind and \( K_m, I_m \) denote the \( m \)th-order modified Bessel functions of the first and second kind. If we observe the first line of Eq. (2.14b), we conclude that solutions proportional to \( Y_m \) cannot be present since \( Y_m \) is singular at \( \rho = 0 \). Therefore, \( B'Y_m(\bar{w}\kappa_1\rho) \) is discarded in Eq. (2.14b). The second line of Eq. (2.14b) can be simplified as well by discarding the term \( C'I_m(\bar{w}\kappa_2\rho) \), as \( I_m \) diverges for \( \rho \to \infty \), which is physically impossible. Furthermore, \( \zeta^2 < n_1^2 \), because if we would take \( \zeta^2 > n_1^2 \) and demand decaying fields in the cladding, the boundary conditions at \( \rho = 1 \) can never be satisfied. Hence, the propagation coefficients for guided modes must lie in the following range

\[ n_2^2 < \zeta^2 \leq n_1^2. \quad (2.15) \]

### 2.1.2 Electromagnetic fields in core and cladding

It can be seen from Eq. (2.14), that for each mode two parameters are needed throughout the two regions, the azimuthal mode number \( m \) and the propagation coefficient \( \zeta \). Electromagnetic waves that propagate in an optical fiber consist of three kinds of modes. In addition to the TE and TM modes found in ordinary metallic waveguides, optical fibers support hybrid modes as well, which have both axial electric- and magnetic fields \( E_z \) and \( H_z \), respectively. The hybrid modes are classified into EH an HE modes. In EH modes the axial electric field \( E_z \) is relatively strong, whereas in HE modes the axial magnetic field \( H_z \) is relatively strong.

TM-type and TE-type solutions are present in which

\[
E_z = \begin{cases} 
AJ_m(\bar{w}\kappa_1\rho) \sin(m\phi) & \text{(for } \rho \leq 1), \\
CK_m(\bar{w}\kappa_2\rho) \sin(m\phi) & \text{(for } \rho > 1),
\end{cases} \quad (2.16a)
\]

\[
H_z = 0, \quad (2.16b)
\]

and

\[
H_z = \begin{cases} 
BJ_m(\bar{w}\kappa_1\rho) \cos(m\phi) & \text{(for } \rho \leq 1), \\
DK_m(\bar{w}\kappa_2\rho) \cos(m\phi) & \text{(for } \rho > 1),
\end{cases} \quad (2.17a)
\]

\[
E_z = 0, \quad (2.17b)
\]
where

\[ \kappa_1 = \sqrt{n_1^2 - \zeta^2}, \]  
\[ \kappa_2 = \sqrt{\zeta^2 - n_2^2}, \]

and the subscripts 1 and 2 denote the core and cladding regions, respectively. In a hollow metallic cylindrical waveguide, solutions such as Eq. (2.16) are called TM modes, and solutions such as Eq. (2.17) are called TE modes. In optical fibers these TE and TM modes exist, but only for \( m = 0 \). When \( m \neq 0 \), the boundary conditions at the core-cladding interface can only be satisfied when a linear combination of TE and TM modes in the core and cladding is chosen. Such composite modes are called hybrid modes. The general expressions of the field components incorporating all the TM, TE and hybrid modes are derived by combining Eqs. (2.16) and (2.17) and using Eq. (2.6).

In the core region, the field components are given by

\[ E_z = AJ_m(u\rho)\sin(m\phi), \]  
\[ E_\rho = \left[-A\frac{j\omega c}{u} J'_m(u\rho) + B\frac{j\omega c_0 \mu_0}{u^2} \frac{m}{\rho} J_m(u\rho)\right] \sin(m\phi), \]  
\[ E_\phi = \left[-A\frac{j\omega c}{u^2} \frac{m}{\rho} J_m(u\rho) + B\frac{j\omega c_0 \mu_0}{u} J'_m(u\rho)\right] \cos(m\phi), \]

\[ H_z = BJ_m(u\rho)\cos(m\phi), \]  
\[ H_\rho = \left[A\frac{j\omega c_0 \varepsilon_1}{u^2} \frac{m}{\rho} J_m(u\rho) - B\frac{j\omega c}{u} J'_m(u\rho)\right] \cos(m\phi), \]  
\[ H_\phi = \left[-A\frac{j\omega c_0 \varepsilon_1}{u} J'_m(u\rho) + B\frac{j\omega c}{u^2} \frac{m}{\rho} J_m(u\rho)\right] \sin(m\phi), \]

where

\[ u = \tilde{\omega} \kappa_1 = \tilde{\omega} \sqrt{n_1^2 - \zeta^2}, \]

which denotes the normalized transverse wave number in the core region.

In the cladding region, the field components are given by

\[ E_z = CK_m(w\rho)\sin(m\phi), \]  
\[ E_\rho = \left[C\frac{j\omega c}{w} K'_m(w\rho) - D\frac{j\omega c_0 \mu_0}{w^2} \frac{m}{\rho} K_m(w\rho)\right] \sin(m\phi), \]  
\[ E_\phi = \left[C\frac{j\omega c}{w^2} \frac{m}{\rho} K_m(w\rho) - D\frac{j\omega c_0 \mu_0}{w} K'_m(w\rho)\right] \cos(m\phi), \]
\[ H_z = DK_m(w\rho)\cos(m\phi), \]  
\[ H_\rho = \left[ -C\frac{j\omega\varepsilon_2 m}{w^2} K_m(w\rho) + D\frac{\omega\zeta}{w} K'_m(w\rho) \right] \cos(m\phi), \]  
\[ H_\phi = \left[ C\frac{j\omega\varepsilon_2 m}{w} K_m(w\rho) - D\frac{\omega\zeta m}{(w^2)} K_m(w\rho) \right] \sin(m\phi), \]

where

\[ w = \omega k_2 = \omega \sqrt{\zeta^2 - n_2^2}, \]  

which denotes the normalized transverse wave number in the cladding region.

### 2.1.3 Propagation coefficients

The propagation coefficients are those values for which the boundary conditions (conditions for the continuity of fields) at the core-cladding interface \( \rho = 1 \) are satisfied. These conditions are as follows

\[ E_1^1 = E_2^1, \quad E_1^1 = E_2^2, \]  
\[ H_1^1 = H_2^1, \quad H_1^1 = H_2^2, \]

where the subscripts 1 and 2 again denote the core and cladding regions, respectively.

Next, we substitute Eqs. (2.19)-(2.24) into Eq. (2.25), this leads to

\[ AJ_m(u) - CK_m(w) = 0, \]  
\[ BJ_m(u) - DK_m(w) = 0, \]

\[ A\frac{j\omega\zeta M_m(u)}{u^2} - B\frac{j\omega\varepsilon_0\mu_0 M'_m(u)}{u} + C\frac{j\omega\zeta mK_m(w)}{w^2} - D\frac{j\omega\varepsilon_0\mu_0 K'_m(w)}{w} = 0, \]  
\[ A\frac{j\omega\varepsilon_1 M'_m(u)}{u^2} - B\frac{j\omega\zeta M_m(u)}{u^2} + C\frac{j\omega\varepsilon_2 K'_m(w)}{w} - D\frac{j\omega\zeta mK_m(w)}{w^2} = 0. \]

These equations can be unified in matrix form:

\[ [M] \mathbf{x} = 0, \]

where

\[ [M] = \begin{bmatrix}
J_m(u) & 0 & -K_m(w) & 0 \\
0 & J_m(u) & 0 & K_m(w) \\
-j\omega\zeta M_m(u)/u^2 & -j\omega\varepsilon_0\mu_0 M'_m(u)/u & j\omega\zeta mK_m(w)/w^2 & -j\omega\varepsilon_0\mu_0 K'_m(w)/w \\
j\omega\varepsilon_1 M'_m(u)/u & -j\omega\zeta M_m(u)/u^2 & j\omega\varepsilon_2 K'_m(w)/w & -j\omega\zeta mK_m(w)/w^2 \\
\end{bmatrix}, \]  

(2.28)

(2.29)
and

\[
x = \begin{bmatrix}
A \\
B \\
C \\
D \\
\end{bmatrix}.
\]

(2.30)

The right hand side of Eq. (2.28) is zero, which means that either one of the eigenvalues \( \lambda \) of the matrix \( M \) should be zero, or that \( x = 0 \). However, \( x = 0 \) is a trivial solution and not of interest. To obtain the unknown propagation coefficients \( \zeta \), we have to solve

\[
\text{det} [M] = 0,
\]

(2.31)

using Eqs. (2.21) and (2.24). This yields the following eigenvalue equation

\[
\begin{aligned}
\left[ J'_n(u) + K'_m(w) \right] & \left[ \frac{\varepsilon_1 J'_n(u)}{u J'_n(u)} + \frac{K'_m(w)}{w K'_m(w)} \right] = m^2 \left( \frac{1}{u^2} + \frac{1}{w^2} \right) \left( \frac{\varepsilon_1}{\varepsilon_2 u^2} + \frac{1}{w^2} \right). \\
\end{aligned}
\]

(2.32)

The value of \( \zeta \) for which this eigenvalue equation holds corresponds to the propagation coefficient of a mode.

### 2.1.4 Power density calculation

If the field components of Eqs. (2.19)-(2.24) are known, a computation of the power density \( S \) can be performed, i.e.

\[
S = \frac{1}{2} \text{Re} \{ \mathbf{E} \times \mathbf{H}^* \} \cdot \mathbf{u}_z.
\]

(2.33)

In the previous subsection, we have seen that the propagation coefficients are determined by solving Eq. (2.32). However, this does not complete the field equations needed to determine the power density \( S \), because amplitudes \( A, B, C \) and \( D \) are still unknown. These amplitudes form the eigenvector that corresponds to eigenvalue \( \lambda = 0 \). In combination with the propagation coefficient this eigenvector completes the field solution. For the \((\rho,\phi)\) plane, the results of this computation can be found in section 5.2.

### 2.2 Arbitrary-index fiber

In this section, the theoretical analysis of an arbitrary-index optical fiber is presented. The mathematical basis is again formed by Maxwell's equations, as defined in Eq. (2.1). Since, not all electric and magnetic field components are independent, it is convenient to eliminate the longitudinal components of the field. The remaining field components, which are transverse to the longitudinal axis, form a system of coupled partial differential equations known as the Marcuvitz-Schwinger equations. In the cladding region this system can be solved analytically. For the core region on the other hand, a numerical integration is performed from the core center towards the core-cladding interface. By imposing boundary conditions, the modal propagation coefficients \( \zeta \) are determined.
2.2.1 Marcuvitz-Schwinger equations in cylindrical coordinates

We follow [10], and use a standard cylindrical coordinate system \((x_1, x_2, x_3) = (\rho, \phi, z)\), together with Eq. (2.1), Eq. (2.3) and Eq. (2.4). As state quantities the transverse \((\rho, \phi)\) components of \(E\) and \(H\) are chosen. Thereafter, the longitudinal \((z)\) field component is eliminated. There are two main reasons for this choice. First, the transverse field components fully determine the energy flow through transverse planes. Second, the transverse field components are continuous across transverse planes, which becomes convenient once the more complicated waveguide discontinuity problems are addressed.

In order to eliminate the longitudinal field components, the vectors and operators in Eq. (2.1) are decomposed into their transverse and longitudinal constituents

\[
\begin{align*}
E &= E_t + E_z u_z, \quad H = H_t + H_z u_z, \\
J &= J_t + J_z u_z, \quad K = K_t + K_z u_z, \\
\nabla &= \nabla_t + \frac{e}{\partial z} u_z,
\end{align*}
\]  

(2.34a) \hspace{1cm} (2.34b) \hspace{1cm} (2.34c)

and consequently substituted into Eq. (2.1). With the aid of Eq. (2.2), we find the following general expressions for the source-free (i.e. \(J = 0\) and \(K = 0\)) Marcuvitz-Schwinger equations

\[
\begin{align*}
-\frac{\partial E_t}{\partial z} &= j\omega c_0 a^{-1} \mu \left( [I] + \frac{\nabla_t \nabla_t}{k^2} \right) \cdot (H_t \times u_z), \\
-\frac{\partial H_t}{\partial z} &= j\omega c_0 a^{-1} \varepsilon \left( [I] + \frac{\nabla_t \nabla_t}{k^2} \right) \cdot (u_z \times E_t),
\end{align*}
\]  

(2.35a) \hspace{1cm} (2.35b)

where \([I]\) is the identity matrix. Eq. (2.35) is written as

\[
-\partial_z \begin{pmatrix} E_\rho \\ \rho E_\phi \end{pmatrix} = j\omega a^{-1} \mu_r Z_0 \left[ \begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho \end{pmatrix} + \frac{1}{\omega^2 \varepsilon_r} \begin{pmatrix} \varepsilon_r \partial_\rho (\rho \varepsilon_r)^{-1} \partial_\rho & \varepsilon_r \partial_\rho (\rho \varepsilon_r)^{-1} \partial_\phi \\ \rho^{-1} \partial_\phi \partial_\rho & \rho^{-1} \partial_\phi^2 \end{pmatrix} \right] \begin{pmatrix} \rho H_\phi \\ -H_\rho \end{pmatrix},
\]  

(2.36)

\[
-\partial_z \begin{pmatrix} H_\rho \\ \rho H_\phi \end{pmatrix} = j\omega a^{-1} \varepsilon_r Y_0 \left[ \begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho \end{pmatrix} + \frac{1}{\omega^2 \varepsilon_r} \begin{pmatrix} \partial_\rho \rho^{-1} \partial_\rho & \partial_\rho \rho^{-1} \partial_\phi \\ \rho^{-1} \partial_\phi \partial_\rho & \rho^{-1} \partial_\phi^2 \end{pmatrix} \right] \begin{pmatrix} -\rho E_\phi \\ E_\rho \end{pmatrix},
\]  

(2.37)

where \(\mu_r = 1\) and

\[
\varepsilon_r = \begin{cases} 
\varepsilon_1(\rho, \omega) & \text{for } \rho < 1, \\
\varepsilon_2(\omega) & \text{for } \rho > 1,
\end{cases}
\]  

(2.38)
in which \( \varepsilon_2 \) denotes the permittivity in the homogeneous fiber cladding. Note that \( \varepsilon_r \) is dispersive as it depends on \( \omega \). The fiber only constitutes a waveguide if the condition

\[
\max(\varepsilon_r \mu_r) > \varepsilon_2
\]

(2.39)
is satisfied.

Note that \( \varepsilon_r \) depends on one of the transverse coordinates, viz. \( \rho \). This leads to derivatives of \( \varepsilon_r \) with respect to \( \rho \) in Eqs. (2.36) and (2.37), which is especially problematic if \( \varepsilon_r \) is not known in closed form. To circumvent this problem we change our original longitudinal coordinate \( z \) into \( \rho \), and original transverse coordinates \( \rho \) and \( \phi \) into \( \phi \) and \( z \), respectively. The source-free Marcuvitz-Schwinger equations in the new coordinate system are now given by

\[
-\partial_\rho \left( \frac{\rho a E_\phi}{E_z} \right) = j \omega Z_0 \left[ \begin{pmatrix} \rho a & 0 \\ 0 & (\rho a)^{-1} \end{pmatrix} + \frac{a}{\rho \omega^2 \varepsilon_r} \begin{pmatrix} \partial_z^2 & \partial_\phi \partial_z \\ \partial_\phi \partial_z & \partial_z^2 \end{pmatrix} \right] \begin{pmatrix} H_z \\ - \rho a H_\phi \end{pmatrix},
\]

(2.40a)

\[
-\partial_\rho \left( \frac{\rho a H_\phi}{H_z} \right) = j \omega \frac{Y_0}{\varepsilon_r} \left[ \begin{pmatrix} \rho a & 0 \\ 0 & (\rho a)^{-1} \end{pmatrix} + \frac{a}{\rho \omega^2 \varepsilon_r} \begin{pmatrix} \partial_z^2 & \partial_\phi \partial_z \\ \partial_\phi \partial_z & \partial_z^2 \end{pmatrix} \right] \begin{pmatrix} -E_z \\ \rho a E_\phi \end{pmatrix}.
\]

(2.40b)

These equations are equivalent to Maxwell’s equations, but contain only the transverse field components. The radial components, which are now regarded as the longitudinal components, of the electromagnetic field follow from

\[
E_\rho = \frac{a}{j \omega Y_0 \varepsilon_r} \left( \frac{1}{\rho a} \partial_\phi H_z - \partial_z H_\phi \right),
\]

(2.41a)

\[
H_\rho = \frac{a}{j \omega Z_0} \left( \partial_z E_\phi - \frac{1}{\rho a} \partial_\phi E_z \right).
\]

(2.41b)

The transverse components of the fields can now be written as follows

\[
\begin{pmatrix} E_\phi \\ E_z \end{pmatrix}(\rho, \phi, z) = V(\rho)e(\phi, z) = \begin{pmatrix} V_\phi(\rho) \\ V_z(\rho) \end{pmatrix} e(\phi, z),
\]

(2.42a)

\[
\begin{pmatrix} H_\phi \\ H_z \end{pmatrix}(\rho, \phi, z) = I(\rho)h(\phi, z) = \begin{pmatrix} I_\phi(\rho) \\ I_z(\rho) \end{pmatrix} h(\phi, z).
\]

(2.42b)

If we want to solve the vectorial eigenproblem of Eq. (2.40), it is necessary to determine the vectorial modal amplitudes \( V(\rho) \) and \( I(\rho) \) and also the scalar mode functions \( e(\phi, z) \) and \( h(\phi, z) \). These mode functions are equivalent to Eqs. (2.3) and (2.4) are described by

\[
e(\phi, z) = N_E e^{-j m \phi - j \frac{\omega}{\varepsilon_0 c} z},
\]

(2.43a)

\[
h(\phi, z) = N_H e^{-j m \phi - j \frac{\omega}{\varepsilon_0 c} z}.
\]

(2.43b)
If we substitute Eqs. (2.42) and (2.43) into (2.40), and use the fact that \( \varepsilon_r = n^2 \), we find

\[
-\partial_\rho \left( \begin{array}{c} \rho V_\phi \\ V_z \end{array} \right) = j\tilde{\omega}Z_0 \left[ \begin{array}{cc} \rho & 0 \\ 0 & \rho^{-1} \end{array} \right] - \frac{1}{\rho\tilde{\omega}n^2} \left( \begin{array}{cc} m_2^2 & m\tilde{\omega}\zeta \\ m\tilde{\omega}\zeta & \tilde{\omega}^2\zeta^2 \end{array} \right) \left( \begin{array}{c} I_z \\ -\rho I_\phi \end{array} \right),
\]

(2.44)

and

\[
-\partial_\rho \left( \begin{array}{c} \rho I_\phi \\ I_z \end{array} \right) = j\tilde{\omega}n^2Y_0 \left[ \begin{array}{cc} \rho & 0 \\ 0 & \rho^{-1} \end{array} \right] - \frac{1}{\rho\tilde{\omega}n^2} \left( \begin{array}{cc} m_2^2 & m\tilde{\omega}\zeta \\ m\tilde{\omega}\zeta & \tilde{\omega}^2\zeta^2 \end{array} \right) \left( \begin{array}{c} -V_z \\ \rho V_\phi \end{array} \right),
\]

(2.45)
after dividing by \( e^{-j\omega_0-\frac{j\tilde{\omega}z}{\varepsilon_r}} \) on both sides.

### 2.2.2 Analysis of vectorial modal amplitudes

Besides the mode functions in Eq. (2.43), the vectorial amplitudes \( V(\rho) \) and \( I(\rho) \) need to be determined. First, we recall the variables of Eq. (2.2) and express the vectorial amplitudes in scaled field vectors \( f_E \) and \( f_H \) as follows

\[
V = -jC_0\rho|\rho| \left( \begin{array}{c} \rho^{-1} \\ 0 \\ 1 \end{array} \right) f_E,
\]

(2.46a)

\[
I = C_0Y_0\rho|\rho| \left( \begin{array}{c} \rho^{-1} \\ 0 \\ 1 \end{array} \right) f_H,
\]

(2.46b)

where \( C_0 \) is an arbitrary amplitude factor. Next, we substitute Eq. (2.46) into Eq. (2.44) and Eq. (2.45). This leads to the following system of differential equations for the scaled field vectors

\[
\partial_\rho f = \rho^{-1}[A(\rho, \zeta)]f,
\]

(2.47)

where

\[
f = \left( \begin{array}{c} f_E \\ f_H \end{array} \right),
\]

(2.48)

and

\[
[A(\rho, \zeta)] = \begin{pmatrix}
-|m| & 0 & m\zeta/\varepsilon_r & (\tilde{\omega}\rho^2n^2 - m^2/\omega^2)/\varepsilon_r \\
0 & -|m| & \tilde{\omega}(\zeta^2 - n^2)/\varepsilon_r & m\zeta/\varepsilon_r \\
m\zeta/\mu_r & (\tilde{\omega}\rho^2n^2 - m^2/\omega^2)/\mu_r & -|m| & 0 \\
\tilde{\omega}(\zeta^2 - n^2)/\mu_r & m\zeta/\mu_r & 0 & -|m|
\end{pmatrix}.
\]

(2.49)

### 2.2.3 Solution of the initial-value problem

#### Solution strategy

The differential equation to be solved is described by Eq. (2.47). The solution procedure used for this problem involves the numerical integration of

\[
\partial_\rho f = \rho^{-1}[A(\rho, \zeta)]f
\]

(2.50)
in the core region. The differential equation is repeatedly integrated from the core's center, where \( \rho \) is small, towards the core-cladding interface, where \( \rho = 1 \). The general solution to this equation may be expressed as the linear combination of four independent solutions denoted as \( f_1, f_2, f_3 \) and \( f_4 \) at \( \rho = 1 \).

The solutions that remain bounded in the cladding, say \( f_3 \) and \( f_4 \), can be determined analytically. To compute the solution vectors in the core (\( f_1 \) and \( f_2 \)), the mentioned numerical integration method is used. The known analytical solutions at the core's center are integrated towards the core-cladding interface. By imposing boundary conditions, the modal propagation coefficients \( \zeta \) can be determined by solving the characteristic equation

\[
C(\zeta) = \det[f_1(\rho), f_2(\rho), f_3(\rho), f_4(\rho)]|_{\rho=1} = 0.
\]

**Field solutions in the core region**

We are interested in finding the field vector solutions \( f_1, f_2, f_3 \) and \( f_4 \). The independent solutions \( f_1(\rho) \) and \( f_2(\rho) \) for the core region can be obtained by integrating the initial value problem

\[
\partial_\rho f = \frac{1}{\rho}[A(\rho, \zeta)]f, \tag{2.52}
\]

from \( \rho \) close to zero to \( \rho = 1 \) (core-cladding interface), starting with the respective approximate initial field vectors \( f_1(0) \) and \( f_2(0) \). These initial field vectors are determined writing \( f \) and \( [A(\rho, \zeta)] \) as follows for \( \rho = 0 \)

\[
f = f(0) + \rho f'(0) + O(\rho^2), \quad [A] = [A(0)] + \rho[A'(0)] + O(\rho^2). \tag{2.53a}
\]

After combining Eqs. (2.52) and (2.53), we find

\[
f'(0) + O(\rho) = \rho^{-1}[A(0)]f(0) + [A(0)]f'(0) + [A'(0)]f(0) + O(\rho), \tag{2.54}
\]

and hence

\[
[A(0)]f(0) = 0, \quad f'(0) = [I - A(0)]^{-1}[A'(0)]f(0). \tag{2.55a}
\]

From Eq. (2.55) two independent solutions \( f_1(0) \) and \( f_2(0) \) are found, which turn out to be

\[
(f_1(0) \quad f_2(0)) = 
\begin{pmatrix}
\varepsilon_r(0)^{-1}m\zeta & |m| \\
\varepsilon_r(0)^{-1}\mu(\zeta^2 - n_0^2) & 0 \\
|m| & \mu_r(0)^{-1}m\zeta \\
0 & \mu_r(0)^{-1}\mu(\zeta^2 - n_0^2)
\end{pmatrix}. \tag{2.56}
\]

We assume further that the permittivity profile starts horizontally at \( \rho = 0 \). However, in practical cases there can be a dip of a certain shape in the profile at this point. This assumption leads to

\[
\varepsilon_r'(0) = (\partial_\rho \varepsilon_r)|_{\rho=0} = 0 \quad \text{and also} \quad \mu_r'(0) = (\partial_\rho \mu_r)|_{\rho=0} = 0, \tag{2.57}
\]

which implies that \( [A'(0)] \) and hence \( f_1'(0) \) and \( f_2'(0) \) vanish.
Field solutions in the cladding region

In the homogeneous cladding the field vectors $f_3$ and $f_4$ that remain bounded in the cladding follow from the fields given by Eqs. (2.22a), (2.22c), (2.23a) and (2.23c). Consequently, we use Eqs. (2.40) and (2.46), which eventually leads to

\[
(f_3(\rho) \quad f_4(\rho)) = \rho^{-|m|} \begin{pmatrix}
(\bar{\omega} \kappa^2)^{-1} m \zeta K_{|m|} & \kappa^{-1} \rho \mu_r K'_{|m|} \\
K_{|m|} & 0 \\
\kappa^{-1} \rho \varepsilon_r K'_{|m|} & (\bar{\omega} \kappa^2)^{-1} m \zeta K_{|m|} \\
0 & K_{|m|}
\end{pmatrix}, \tag{2.58}
\]

in which $K_{|m|} = K_{|m|}(\bar{\omega} \kappa \rho)$, $K'_{|m|} = K'_{|m|}(\bar{\omega} \kappa \rho)$ and $\kappa = \sqrt{\zeta^2 - n^2_2}$. We can also obtain field vectors $f_3$ and $f_4$ starting from Eqs. (2.44) and (2.45), respectively. For one field solution vector this is shown in Appendix A. Once the four independent solution vectors for $\rho = 1$ have been derived, the characteristic function $C$ is known. By solving the characteristic equation given by Eq. (2.51), we obtain the normalized propagation coefficient $\zeta$. 
Chapter 3

Field analysis using weakly guiding approximation

In this chapter, a mathematical derivation of the fields in the weakly guiding case will be treated. Again, a step-index fiber and an arbitrary-index fiber will be discussed.

3.1 Step-index fiber

The assumption of weak guidance comprises that $\varepsilon_1 \simeq \varepsilon_2$. The starting point of our analysis is Eq. (2.32), see [7], which can be rewritten as

\[
\begin{align*}
\frac{J_0'(u)}{uJ_0(u)} + \frac{K_0'}{wK_0} &= 0 \quad &\text{(for } m = 0), \\
\frac{J_m'(u)}{uJ_m(u)} + \frac{K_m'}{wK_m} &= \pm m \left( \frac{1}{u^2} + \frac{1}{w^2} \right) \quad &\text{(for } m \geq 0),
\end{align*}
\]  

(3.1a)  
(3.1b)

if we assume weak guidance. Observe that Eq. (3.1b) gives two sets of solutions, one for the positive sign and one for the negative sign. These different signs follow from taking the square root on both sides of Eq. (2.32), after using $\frac{\varepsilon_1}{\varepsilon_2} = 1$. Eq. (3.1a) holds for TE-TM type solutions only, whereas hybrid modes are described by Eq. (3.1b). We start from Eq. (3.1b), first the expression for the positive sign is derived, with the aid of

\[
\frac{J_m'(u)}{uJ_m(u)} = -\frac{J_{m+1}(u)}{uJ_m(u)} + \frac{m}{u^2},
\]  

(3.2)

and

\[
\frac{K_m'(w)}{wK_m(w)} = -\frac{K_{m+1}(w)}{wK_m(w)} + \frac{m}{w^2},
\]  

(3.3)

we find

\[
\frac{J_{m+1}(u)}{uJ_m(u)} - \frac{K_{m+1}(w)}{wK_m} = 0.
\]  

(3.4)
For the negative sign, we use again Eqs. (3.2) and (3.3) together with Eq. (3.1b), this leads to

\[ \frac{J_{m-1}(u)}{u J_m(u)} - \frac{K_{m-1}(w)}{w K_m} = 0. \]  

(3.5)

The modes whose propagation coefficients are given as the solution to Eq. (3.1a) are TE and TM modes, Eq. (3.4) holds for EH modes and Eq. (3.5) corresponds to HE modes.

Under the assumption of weak guidance, a unified expression for the propagation coefficients can be derived from Eqs. (3.1a), (3.4) and (3.5). This is done by using Eq. (3.5) in combination with the following recurrence relations

\[ J_{m+1}(u) + J_{m-1}(u) = \frac{2m}{u} J_m(u), \]  

(3.6a)

\[ K_{m+1}(u) - K_{m-1}(u) = \frac{2m}{w} K_m(w), \]  

(3.6b)

which leads to

\[ \frac{J_{m-1}(u)}{u J_{m-2}(u)} = -\frac{K_{m-1}(w)}{w K_{m-2}(w)}. \]  

(3.7)

Next, a new parameter \( l \) is defined as follows:

\[ l = \begin{cases} 
1 & \text{(For TE and TM modes i.e. } m = 0), \\
m + 1 & \text{(For EH modes i.e. } m \geq 1), \\
m - 1 & \text{(For HE modes i.e. } m \geq 1). 
\end{cases} \]  

(3.8)

With this parameter Eqs. (3.1a), (3.4) and (3.5) can be written in the unified form

\[ \frac{J_l(u)}{u J_{l-1}(u)} = -\frac{K_l(w)}{K_{l-1}(w)}. \]  

(3.9)

If we use this parameter \( l \), various modes are analyzed in a unified manner to bring forth the concept of so called linearly polarized (LP) modes. The fundamental mode HE_{11} corresponds here to LP_{01}, for instance.

### 3.2 Arbitrary-index fiber

It will be shown in section 3.2.1, that to obtain the field solutions of an arbitrary-index fiber in the weakly guiding situation, \( E_z \) and \( H_z \) are assumed to be negligible with respect to the transverse field components. This assumption leads to a simplification of Eq. (2.51) in section 2.2.3, i.e.

\[ C(\zeta) = \det[f_1(\rho), f_2(\rho)]|_{\rho=1} = 0, \]  

(3.10)

in which \( f_1(\rho) \) denotes a field vector in the core region and \( f_2(\rho) \) denotes a field vector in the cladding region. Both field vectors now consist of two components, \( E_\phi \) and \( H_\phi \), instead of
four. From Eq. (3.10) it follows that only one solution vector in the core, and one solution vector in the cladding is needed. In order to find these solution vectors, a similar solution method is used as described in section 2.2.3. These solution vectors are derived in section 3.2.2 and 3.2.3, respectively.

3.2.1 Approximation of \( E_z \) and \( H_z \) by zero

To show that \( E_z \) and \( H_z \) are negligibly small, we follow [11] and [13], and recall that under the weakly guiding approximation the following equation holds

\[
(\varepsilon_1 - \varepsilon_2)/\varepsilon_1 \ll 1,
\]

which implies that \( \varepsilon_1 \approx \varepsilon_2 \).

In accordance to the discussion in sections 2.1.1 and 2.2.1, the electric and magnetic fields of a mode of a cylindrical waveguide are expressible in the following form

\[
E(\rho, \phi, z) = E(\rho, \phi)e^{-j\frac{\phi\rho^2}{2}} = (E_t + E_z u_z) e^{-j\frac{\phi\rho^2}{2}}, \tag{3.12a}
\]

\[
H(\rho, \phi, z) = H(\rho, \phi)e^{-j\frac{\phi\rho^2}{2}} = (H_t + H_z u_z) e^{-j\frac{\phi\rho^2}{2}}, \tag{3.12b}
\]

where \( u_z \) is the unit vector parallel to the fiber axis and subscripts \( t \) and \( z \) denote transverse and longitudinal components, respectively. Furthermore, we define the relative refractive index difference

\[
\Delta = \frac{n_1^2 - n_2^2}{2n_1^2}, \tag{3.13}
\]

and the normalized frequency

\[
V = \tilde{\omega} \sqrt{n_1^2 - n_2^2} = \tilde{\omega} n_1 \sqrt{2\Delta}. \tag{3.14}
\]

As \( \Delta \to 0 \), \( V \) must remain finite, which is the second condition for weak guidance. Consequently it is required that \( \tilde{\omega} n_1 \sim \Delta^{-1/2} \).

To derive expressions for \( E_z \) and \( H_z \), we start from Eq. (2.1), where \( J = 0, \sigma_\phi = 0, K = 0 \) and \( \sigma_m = 0 \). In combination with Eq. (2.2), \( \nabla_n = a\nabla \) and \( \nabla_{tn} = a\nabla_t \), we find

\[
\nabla_n \times E = -jZ_0 \tilde{\omega} H, \tag{3.15a}
\]

\[
\nabla_n \times H = jY_0 \tilde{\omega} n^2 E. \tag{3.15b}
\]

Next, we take the curl on both sides of Eq. (3.15a). In the resulting equation we substitute Eq. (3.15b), and apply the vector identity \( \nabla_n \times \nabla_n \times E = \nabla_n (\nabla_n \cdot E) - \nabla_n^2 E \), which yields

\[
\nabla_n^2 + \tilde{\omega}^2 n^2 = \nabla_n (\nabla_n \cdot E). \tag{3.16}
\]

Since \( \nabla_n \cdot D = \nabla_n \cdot (n^2 E) = 0 \), we can use

\[
\nabla_n \cdot (n^{-2} n^2 E) = n^{-2} \nabla_n \cdot (n^2 E) + n^2 E \cdot \nabla_n n^{-2}, \tag{3.17}
\]

22
to state that
\[ \nabla_n \cdot \mathbf{E} = n^2 \mathbf{E} \cdot \nabla_n n^{-2}. \]  
(3.18)
If we substitute Eqs. (3.18) and (3.12) into Eq. (3.16), we obtain
\[ \left[ \nabla_{tn}^2 + \tilde{\omega}^2 (n^2 - \zeta^2) \right] \mathbf{E}_t = -\nabla_{tn} \left( n^2 \mathbf{E}_t \cdot \nabla_{tn} n^{-2} \right), \]  
(3.19)
As the refractive index difference has little variation in the weak-guidance situation, i.e. \( \Delta \to 0 \), we can write
\[ n = n_1 [1 + \Delta f(\rho)]. \]  
(3.20)
The right hand side of Eq. (3.19) is negligibly small under the weak-guidance assumption. We can elucidate this by substituting Eq. (3.20) into Eq. (3.19), which leads to
\[ \left\{ \nabla_{tn}^2 + n_1^2 \tilde{\omega}^2 [1 + \Delta f(\rho)]^2 - \langle \tilde{\omega} \psi \rangle^2 \right\} \mathbf{E}_t = \nabla_{tn} \left\{ n_1^2 [1 + \Delta f(\rho)]^2 \mathbf{E}_t \cdot \nabla_{tn} [n_1^2 (1 + \Delta f(\rho))^2] \right\}. \]  
(3.21)
Neglecting terms of order \( \Delta^2 \), we arrive at
\[ \left[ \nabla_{tn}^2 + V^2 f(\rho) + \tilde{\omega}^2 (n_1^2 - \zeta^2) \right] \mathbf{E}_t = -2 \Delta \nabla_{tn} [\mathbf{E} \cdot \nabla_{tn} f(\rho)], \]  
(3.22)
from which it is clear that the right-hand side can be approximated by zero, if \( \Delta \to 0 \). From Eq. (3.22), we determine \( \mathbf{E}_t \). To derive an expression for the axial component \( E_z \), we use
\[ \nabla_n \cdot (n^2 \mathbf{E}) = \nabla_{tn} \cdot (n^2 \mathbf{E}_t) + n^2 \frac{\partial}{\partial z} E_z. \]  
(3.23)
Since \( \nabla_n \cdot (n^2 \mathbf{E}) = 0 \), Eq. (3.23) relates \( E_z \) to \( \mathbf{E}_t \) as follows
\[ E_z = -\frac{j}{\tilde{\omega} n_1^2} \nabla_{tn} \cdot (n^2 \mathbf{E}_t). \]  
(3.24)
Furthermore, we note that
\[ \tilde{\omega} \psi = \tilde{\omega} n_1 [1 + O(\Delta)] \approx \tilde{\omega} n_1 \quad \text{(for \( \Delta \to 0 \)).} \]  
(3.25)
When we substitute Eqs. (3.20) and (3.25) into Eq. (3.24), we obtain
\[ E_z = \frac{j}{\tilde{\omega} n_1} \nabla_{tn} \cdot \mathbf{E}_t, \]  
(3.26)
which can be rewritten, with the aid of Eq. (3.14), as follows
\[ E_z = j \frac{(2\Delta)^{1/2}}{V} (\nabla_{tn} \cdot \mathbf{E}_t). \]  
(3.27)
A similar analysis can be performed for \( H_z \), which eventually leads to
\[ H_z = -j \frac{(2\Delta)^{1/2}}{V} (\gamma_0)^{1/2} n_1 \mathbf{u}_z \cdot (\nabla_{tn} \times \mathbf{E}_t). \]  
(3.28)
Hence, the longitudinal components of Eqs. (3.27) and (3.28) reduce to zero in the limit \( \Delta \to 0 \).
3.2.2 Field solution in the core region

In order to determine the initial core solution vector \( f_1(0) \) for weakly guiding fibers, we use Eq. (3.19) with \( \Delta \to 0 \). This equation contains two transverse field components \( E_x \) and \( E_y \) or \( E_y \) and \( E_y \). Eq. (3.19) can be written as a scalar wave equation by separating the transverse components \( E_x \) and \( E_y \) as follows

\[
\begin{align*}
\left[ \nabla^2_{tn} + \alpha^2(n^2 - \zeta^2) \right] E_x &= 0, \quad (3.29a) \\
\left[ \nabla^2_{tn} + \alpha^2(n^2 - \zeta^2) \right] E_y &= 0. \quad (3.29b)
\end{align*}
\]

Eq. (3.29) is derived in cylindrical coordinates, i.e.

\[
E_x(\text{or } E_y) = R(\rho)e^{-jm\phi}.
\]

which leads, after a multiplication by \( \rho^2 \), to the following scalar wave equation

\[
\rho \partial_\rho \left( \rho \partial_\rho R \right) - \{ m^2 - \alpha^2[n^2(\rho) - \zeta^2]\rho^2 \} R = 0.
\]

Hence, the vectorial wave equation becomes a scalar wave equation in the weakly guiding approximation. If we use \( \Psi = \rho \partial_\rho R \), and also the previously defined variables of Eq. (2.2), we derive the following system

\[
\begin{align*}
\left\{ \begin{array}{l}
\Psi = \rho \partial_\rho R \\
\rho \partial_\rho \Psi = \{ m^2 - \alpha^2[n^2(\rho) - \zeta^2]\rho^2 \} R
\end{array} \right.
\end{align*}
\]

(3.32)

where \( R \) is equivalent to \( \rho^{lm}f_E \) in Eq. (A.9), \( \Psi \) is equivalent to \( \rho^{lm}f_H \) in Eq. (A.9) and \( \alpha \) is the core radius. If we write Eq. (3.32) in matrix form, we find

\[
\partial_\rho \begin{pmatrix} \rho^{lm}f_E \\ \rho^{lm}f_H \end{pmatrix} = \rho^{-1} \begin{pmatrix} 0 & 1 \\ m^2 - \alpha^2[n^2(\rho) - \zeta^2]\rho^2 & 0 \end{pmatrix} \begin{pmatrix} \rho^{lm}f_E \\ \rho^{lm}f_H \end{pmatrix}.
\]

(3.33)

which is the equivalent of Eq. (A.7), only now in the weakly guiding case (\( f_E \) and \( f_H \) are no vectors here).

If we want to define an initial core vector, to be able to use Adam’s method, it is necessary to modify the system of Eq. (3.33). We start with

\[
\partial_\rho (\rho^{lm}f_E) = \rho^{-1}(\rho^{lm}f_H), \quad (3.34)
\]

which can be written as

\[
|m|\rho^{lm}f_E + \rho^{lm}\partial_\rho f_H = \rho^{-1}(\rho^{lm}f_H). \quad (3.35)
\]

Next, we write Eq. (3.35) as

\[
\partial_\rho f_E = \rho^{-1}(-|m|f_E + f_H), \quad (3.36)
\]
and do the same for $\partial_\rho (\rho^{\text{iml}} f_H)$, which gives us
\[
\partial_\rho f_H = \rho^{-1} \{ m^2 - \tilde{\omega}^2 \rho^2 [n^2(\rho) - \zeta^2] f_E - |m| f_H \}
\] (3.37)

Now, we use Eqs. (3.36) and (3.37) to construct the following matrix
\[
\partial_\rho \begin{pmatrix} f_E \\ f_H \end{pmatrix} = \rho^{-1} \begin{pmatrix} -|m| & 1 \\ m^2 & -|m| \end{pmatrix} \begin{pmatrix} m^2 - \tilde{\omega}^2 [n^2(\rho) - \zeta^2] \rho^2 & 1 \\ -|m| & 0 \end{pmatrix} \begin{pmatrix} f_E \\ f_H \end{pmatrix}.
\] (3.38)

The initial core vector is found after we use the boundary condition for $\rho = 0$, $A(0)f(0) = 0$. From
\[
\begin{pmatrix} -|m| & 1 \\ m^2 & -|m| \end{pmatrix} \begin{pmatrix} f_E \\ f_H \end{pmatrix} = 0,
\] (3.39)

it follows that
\[
f_1(0) = \begin{pmatrix} 1 \\ |m| \end{pmatrix}.
\] (3.40)

### 3.2.3 Field solution in the cladding region

To derive a field solution for the cladding region, we use Eq. (3.33)
\[
\partial_\rho \begin{pmatrix} \rho^{\text{iml}} f_E \\ \rho^{\text{iml}} f_H \end{pmatrix} = \rho^{-1} \begin{pmatrix} 0 \\ m^2 - \tilde{\omega}^2 [n^2(\rho) - \zeta^2] \rho^2 \\ 0 \end{pmatrix} \begin{pmatrix} \rho^{\text{iml}} f_E \\ \rho^{\text{iml}} f_H \end{pmatrix}.
\] (3.41)

with a constant refractive index $n_2$. As already discussed in section 2.2.3, we work towards Bessel’s differential equation in cylindrical coordinates to find the eventual solution $f_2(1)$. First, we take $\partial_\rho \rho \partial_\rho$ on both sides of the first line of Eq. (3.41) in combination with $R = \rho^{\text{iml}} f_E$ and $\Psi = \rho^{\text{iml}} f_H$, this gives us
\[
\rho \partial_\rho \rho \partial_\rho (\rho^{\text{iml}} f_E) = \rho \partial_\rho (\rho^{\text{iml}} f_H).
\] (3.42)

Next, we substitute the second line of Eq. (3.41) into Eq. (3.42), this leads to
\[
\rho \partial_\rho \rho \partial_\rho (\rho^{\text{iml}} f_E) = \{ m^2 - \tilde{\omega}^2 [n^2(\rho) - \zeta^2] \} (\rho^{\text{iml}} f_E),
\] (3.43)

After some manipulations, which are shown in Appendix A for the full-vectorial case, we arrive at
\[
\rho \partial_\rho \rho \partial_\rho (\rho^{\text{iml}} f_E) - \{ m^2 + x^2 \} (\rho^{\text{iml}} f_E) = 0,
\] (3.44)

where $x^2 = \tilde{\omega}^2 \rho^2 (\zeta^2 - n_2^2)$. This is again Bessel’s differential equation in cylindrical coordinates. If we recall that we need Bessel functions of the form $K_m$ in the cladding region it follows that
\[
\rho^{\text{iml}} f_E = AK_m(x),
\] (3.45)

and from $\partial_\rho (\rho^{\text{iml}} f_E) = \rho^{-1} (\rho^{\text{iml}} f_H)$ we derive
\[
\rho^{\text{iml}} f_H = AK_m'(x)x,
\] (3.46)

which finally gives us $f_2(1)$
\[
\rho^{\text{iml}} f_2 = A \begin{pmatrix} K_m(x) \\ K_m'(x)x \end{pmatrix}
\] (3.47)
Chapter 4

Dispersion

In this chapter, several types of dispersion will be treated. In section 4.2.2, equations for the group slowness $T_g$, dispersion $D$ and dispersion slope $S$ are derived. These equations are implemented in the w.g. program, which is discussed in Appendix C.

In general, we distinguish between two types of dispersion in optical fibers, i.e. intramodal- and intermodal (or modal) dispersion. Intramodal dispersion, occurs in all types of fibers, whereas intermodal dispersion only occurs in multimode fibers. Both types lead to a certain amount of pulse spreading, which means that the wave is separated into spectral components with different wavelengths, due to a dependence of the wave's speed on its wavelength. This is an undesirable effect as it limits the information capacity of the fiber.

4.1 Intermodal dispersion

If we excite an optical fiber with a light pulse, for instance by using a laser, this pulse is made up of a group of modes. Since each mode has its own propagation coefficient, different modes travel different distances over the same time period. This causes the light pulse to spread, which is called modal dispersion. As the length of the fiber increases, modal dispersion increases as well. However, by choosing a proper refractive index profile, e.g. a graded profile, this type of dispersion can be reduced considerably. Still, modal dispersion is the dominant source of dispersion in multimode fibers. For single-mode fibers on the other hand, only the fundamental mode exists, and consequently modal dispersion is absent. Therefore, single-mode fibers have the lowest amount of total dispersion.

4.2 Intramodal dispersion

Intramodal dispersion occurs because different colors of light travel through different media at different speeds. It depends primarily on fiber materials and consists of two types, i.e. waveguide dispersion and material dispersion. Waveguide dispersion occurs because the propagation coefficient $\zeta(\lambda)$ is a function of the wavelength of operation. Material dispersion is caused by the fact that optical fibers are composed of dispersive materials,
which leads to a dependence of the refractive index profile on the wavelength and radial position, i.e. \( n = n(\rho, \lambda) \) for circular fibers.

### 4.2.1 Theoretical description of material dispersion

Light sources which are nominally monochromatic, i.e. consisting of one wavelength, usually have a small spread \( \delta \lambda \) associated with them. Since the refractive index is wavelength dependent, modes which are excited at slightly different wavelengths and follow the same path, will propagate at different speeds. The fact that these modes will propagate at different speeds can be explained by introducing the phase velocity \( v_p \), which is defined as the velocity at which the phase of any one frequency component of the wave will propagate. For a wave described by Eqs. (2.3) and (2.4), the phase velocity is given by

\[
v_p = \frac{\omega}{k_z} = \frac{c_0}{\zeta(\lambda)} = \frac{c_0}{n(\lambda)},
\]

which makes clear that \( v_p \) depends on \( \lambda \). This leads to a form of pulse spreading quite different from the modal dispersion described in section 4.1. The wavelength dependency of \( n \) is quantified by an empirical formula, the Sellmeier equation. This Sellmeier equation for a composite PCVD Silica system has the following form \([12]\)

\[
n^2 = \varepsilon_r(\lambda) = 1 + \sum_{i=1}^{3} \frac{a_i \lambda^2}{\lambda^2 - \lambda_i^2},
\]

where

\[
a_i(X^{Ge},X^{F}) = a_i^0 + X^{Ge} d a_i^{Ge} + X^{F} d a_i^{F}, \quad (4.3a)
\]
\[
\lambda_i(X^{Ge},X^{F}) = \lambda_i^0 + X^{Ge} d \lambda_i^{Ge} + X^{F} d \lambda_i^{F}. \quad (4.3b)
\]

In Eq. (4.3), \( a_i(X^{Ge},X^{F}) \) and \( \lambda_i(X^{Ge},X^{F}) \) are the Sellmeier parameters, \( X^{Ge} \) and \( X^{F} \) are concentrations of germanium and fluorine respectively, \( a_i^0 \) and \( \lambda_i^0 \) are the Sellmeier parameters for pure Silica, and finally \( d a_i^{Ge}, d a_i^{F}, d \lambda_i^{Ge}, d \lambda_i^{F} \) are the material specific variation terms of the first order.

### 4.2.2 Group slowness, dispersion and dispersion slope

The group velocity is defined as the rate at which changes in amplitude will propagate, and it is often thought of as the velocity at which energy or information is conveyed along the wave. The group slowness is the reciprocal value of the group velocity, and using Eq. (2.2) is given by

\[
\tau_g = \frac{1}{v_g} = \frac{d k_z}{d \omega} = \frac{1}{c_0} \left\{ \dot{\omega} d \omega \zeta + \zeta \right\}.
\]

Furthermore, we derive an expression for the dispersion, i.e.

\[
D = \frac{d \tau_g}{d \lambda} = -\frac{\omega^2}{2\pi c_0} \frac{d^2 k_z}{d \omega^2} = -\frac{\dot{\omega}}{2\pi a c_0} \left\{ 2 d \omega \zeta + \ddot{\omega} d^2 \zeta \right\},
\]

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and the dispersion slope
\[
S = \frac{dD}{d\lambda} = \frac{\bar{\omega}^3}{(2\pi\alpha)^2 c_0} \{4d_\omega\zeta + 5\bar{\omega}d_\omega^2\zeta + \bar{\omega}^2d_\omega^3\zeta\}.
\] (4.6)

From Eqs. (4.4)-(4.6), it follows that we need to compute the first, second and third order derivatives of the propagation coefficient \(\zeta\) with respect to \(\bar{\omega}\). As an example, we derive the first order derivative \(d_\varnothing\zeta\). The higher order derivatives \(d_\varnothing^2\zeta\) and \(d_\varnothing^3\zeta\), are obtained in a similar way.

To compute \(d_\varnothing\zeta\), we differentiate the characteristic equation of Eq. (3.10) with respect to \(\bar{\omega}\) as follows
\[
\frac{dC}{d\bar{\omega}} = 0,
\] (4.7)
which leads to
\[
\frac{\partial C}{\partial \omega}\bigg|_{\zeta = \text{const}} + \frac{\partial C}{\partial \zeta} \frac{d\zeta}{d\bar{\omega}} = 0.
\] (4.8)
Consequently, it follows that
\[
\frac{d\zeta}{d\bar{\omega}} = -\frac{\partial C/\partial \omega}{\partial C/\partial \zeta} = -\frac{\partial_\varnothing C}{\partial_\varnothing C}.
\] (4.9)

Hence, we require derivatives of the characteristic function \(C\). Therefore, we recall Eq. (3.10) for the weakly guiding situation. The necessary derivatives follow from
\[
\partial_\varnothing C = \det[\partial_{\varnothing}\mathbf{f}_1, \partial_{\varnothing}\mathbf{f}_2]|_{\rho = 1} + \det[\partial_{\varnothing}\mathbf{f}_1, \partial_\varnothing\mathbf{f}_2]|_{\rho = 1},
\] (4.11)
\[
\partial_\omega C = \det[\partial_\omega\mathbf{f}_1, \partial_\omega\mathbf{f}_2]|_{\rho = 1} + \det[\partial_\omega\mathbf{f}_1, \partial_\omega\mathbf{f}_2]|_{\rho = 1}.
\] (4.12)
Thus, we need \(\partial_\varnothing\mathbf{f}_1, \partial_\varnothing\mathbf{f}_2, \partial_\omega\mathbf{f}_1\) and \(\partial_\omega\mathbf{f}_2\) in addition to \(\mathbf{f}_1\) and \(\mathbf{f}_2\) at \(\rho = 1\). To obtain the core solution vectors \(\mathbf{f}_1, \partial_\varnothing\mathbf{f}_1\) and \(\partial_\omega\mathbf{f}_1\), the system defined in Eq. (2.47) is expanded as follows
\[
\partial_\rho \mathbf{f} = \rho^{-1}[A] \mathbf{f},
\] (4.13a)
\[
\partial_\varnothing \partial_\rho \mathbf{f} = \rho^{-1}[(\partial_\varnothing[A])\mathbf{f} + [A](\partial_\rho \mathbf{f})],
\] (4.13b)
\[
\partial_\omega \partial_\rho \mathbf{f} = \rho^{-1}[(\partial_\omega[A])\mathbf{f} + [A](\partial_\omega \mathbf{f})],
\] (4.13c)
which can be written in matrix form
\[
\partial_\rho \begin{bmatrix} \mathbf{f} \\ \partial_\varnothing \mathbf{f} \\ \partial_\omega \mathbf{f} \end{bmatrix} = \rho^{-1} \begin{bmatrix} A & 0 & 0 \\ \partial_\varnothing A & A & 0 \\ \partial_\omega A & 0 & A \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \partial_\varnothing \mathbf{f} \\ \partial_\omega \mathbf{f} \end{bmatrix},
\] (4.14)
or equivalently
\[ \partial_\rho g = \frac{1}{\rho}[B]g. \]  
(4.15)

This is basically the same type of system we had in Eq. (2.47). The only difference is that the solution vector \( g \) and solution matrix \([B]\) have grown in size. The cladding solution vectors \( f_2, \partial_\zeta f_2 \) and \( \partial_\omega f_2 \) are obtained analytically by differentiating \( f_2 \) to \( \zeta \) or \( \omega \), respectively. If both solution vectors are known, we can determine \( d_\omega \zeta \), and consequently \( \tau_\phi \) in Eq. (4.4).

To compute the dispersion of Eq. (4.5) and dispersion slope of Eq. (4.6), we need \( d_\omega^2 \zeta \) and \( d_\phi^2 \zeta \) as well. Therefore, we require higher order derivatives of the characteristic function \( C \), and consequently higher order derivatives of the solution vectors \( f_1 \) and \( f_2 \) are required.

Repeated differentiation of Eq. (4.7) to \( \omega \) gives us the following relationships

\[ \frac{d^2 \zeta}{d\omega^2} = -\left( \frac{\partial^2_C (\partial_\zeta C)^2 - 2\partial_\omega C \partial_\zeta C \partial_\omega \partial_\zeta C + (\partial_\omega C)^2 \partial^2_\zeta C}{(\partial_\zeta C)^3} \right), \]  
(4.16)

and

\[ \frac{d^3 \zeta}{d\omega^3} = -\left( \frac{\partial^3_C + (d_\omega \zeta)^3 \partial^3_\zeta C + 3(d_\omega \zeta) \partial_\zeta \partial^2_\zeta C + 3(d_\omega \zeta)^2 \partial_\omega \partial^2_\zeta C}{\partial_\zeta C} \right) + \frac{3(d_\zeta \zeta) \partial^2_\zeta C + 3(d_\omega \zeta) (d^2_\zeta \zeta) \partial^2_\zeta C}{\partial_\zeta C}. \]  
(4.17)

Observe that higher order derivatives of the characteristic function are needed to compute the second and third order derivatives of the propagation coefficient to \( \omega \). To compute these higher order derivatives of \( C \), it is necessary to calculate the higher order derivatives of the according solution vectors for both core \( (f_1) \) and cladding \( (f_2) \) as well. In order to calculate the necessary derivatives of the core solution vector \( f_1 \), the system in Eq. (4.14) is expanded for each higher order derivative of \( C \). The derivatives of the cladding solution vector \( f_2 \) are derived analytically. The complete system to be solved is shown in Appendix B.

The computation of Eqs. (4.4)-(4.6), described in this section, is implemented in the w.g. program, which is treated in Appendix C.3.
Chapter 5

Simulation results

In this chapter simulation results of the w.g. program are set against results obtained by the use of the f.v. program. Our main goal is to analyze in which cases the weakly guiding approximation provides an accurate representation of the full-vectorial computation method. In section 5.1, we compare the results for the computation of propagation coefficients produced by the f.v. and w.g. programs in the case of a step-index refractive index profile. The propagation coefficients are computed for two situations, namely $\varepsilon_1 \approx \varepsilon_2$ and $\varepsilon_1 \gg \varepsilon_2$. The f.v. program used in section 5.1 is based on the discussion in section 2.1, whereas the w.g. program is based on the discussion in section 3.1. In section 5.2 the power density of the fundamental mode is computed using both programs, also in the case of a step-index profile. Finally, in section 5.3 the results obtained for the group slowness, dispersion and dispersion slope, using the w.g. program for an arbitrary-index fiber, are compared to the results produced by the f.v. program.

5.1 Propagation coefficient computation for step-index fiber

In this section, we will set the computed normalized propagation coefficients $\zeta$ against the normalized frequency $V$ of Eq. (3.14) for two situations. First, we use a small refractive index difference $\Delta = 0.21\%$ and compute the propagation coefficients using both programs. Consequently, we do the same for a large refractive index difference $\Delta = 12.9\%$.

Let us start with $\Delta = 0.21\%$. Results for the computed propagation coefficients by both programs are shown in Fig. 5.1(a). Here, $\lambda = 1300\text{nm}$, and the core radius is varied between approximately $3\mu\text{m}$ and $30\mu\text{m}$. The solid lines correspond to results computed by the w.g. program, whereas the dashed lines are computed using the f.v. program. We observe that one solid line coincides with several dashed lines, which illustrates the LP modi concept mentioned in section 3.1.
Figure 5.1: Propagation modes produced by both basic programs with (a) $\Delta \approx 0.21\%$, and (b) $\Delta \approx 12.9\%$. The solid lines are the approximated results, and the dashed lines constitute exact results for the propagation modes.

If we increase $\Delta$, we observe a difference between the exact and LP modes. This is shown in Fig. 5.1(b), where $\Delta = 12.9\%$. This value for $\Delta$ is of little practical use, although these contrasts do occur in semiconductor waveguides for laser applications [13]. Thus, for a larger refractive index difference the weak-guidance approximation is less successful.
5.2 Power density computation for step-index fiber

In this section, it is investigated if there is a noticeable difference between the computed power density $S = \frac{1}{2} \text{Re} \{ \mathbf{E} \times \mathbf{H}^* \} \cdot \mathbf{u}_z$, of the fundamental mode $\text{HE}_{11}$ and the $\text{LP}_{01}$. Therefore, we consider a single-mode step-index fiber with $\Delta = 0.3\%$, $\lambda = 1550\text{nm}$, core radius of $5\mu\text{m}$ and $V = 2.24$. We compute the difference in $S$ between the f.v. and w.g. programs for two different values of $\Delta$. The results for $S_{\text{diff}} = S_{\text{exact}} - S_{\text{wg}}$ for $\Delta = 0.3\%$ are shown in Fig 5.2.

![Graphs showing the difference in power density](image)

(a) $S_{\text{diff}}$ for $\Delta = 0.3\%$

(b) $S_{\text{diff}}$ for $\Delta = 0.3\%$ with $\phi$ constant

Figure 5.2: $S_{\text{diff}}$ plots of the fundamental mode in a single-mode fiber, with $\Delta = 0.3\%$, where (a) is $S_{\text{diff}}$ in the horizontal plane, and (b) is $S_{\text{diff}}$ vs $\rho$ in the vertical plane with constant $\phi$.

In Fig. (5.2) the dark areas in the left picture indicate where the difference in $S$ computed by both programs is the largest. It is noted that these areas are situated near $\rho = 1$, i.e. the core-cladding interface. Furthermore, both plots show that $S_{\text{diff}}$ also depends on the value of $\phi$, for $\phi = \pi/2$, the difference in $S$ is much larger than for $\phi = 0$. 

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Fig. 5.3 shows the results of a similar computation of $S_{\text{diff}}$ using both programs, only now for $\Delta = 5\%$.

Figure 5.3: $S_{\text{diff}}$ plots of the fundamental mode in a single-mode fiber, with $\Delta = 5\%$, where (a) is $S_{\text{diff}}$ in the horizontal plane, and (b) is $S_{\text{diff}}$ vs $\rho$ in the vertical plane with constant $\phi$.

Again, we observe that $S_{\text{diff}}$ is the largest near the core-cladding interface for $\phi = \pi/2$, and the increase in $\Delta$ causes the overall $S_{\text{diff}}$ to increase as well.

We have seen that for both values of $\Delta$, the largest difference in $S$ is located near $\rho = 1$ and for $\phi = \pi/2$. This can possibly be explained by the fact that the weakly-guiding approximation assumes the fields to be uniform in those areas, which is not the case if exact fields are considered.
5.3 Comparison of the group slowness, dispersion and dispersion slope

In sections 5.3.1-5.3.3, simulations are performed to compute the group slowness, dispersion and dispersion slope for three different core permittivity profiles, using the w.g. and f.v. programs. The used profiles are shown in Fig. 5.4. In order to analyze the single mode behaviour, the core radius $a$ is chosen to be $4.1 \, \mu m$, the azimuthal mode number $m$ is zero, $\Delta = 0.27\%$ and the observed wavelength range is $1450\,nm < \lambda < 1650\,nm$.

![Figure 5.4: Refractive index profiles, i.e. (a) step profile, (b) graded profile and (c) arbitrary profile, with core radius $a = 4.1 \mu m$ and refractive index difference $\Delta = 0.27\%$.](image)
5.3.1 Group slowness simulation results

Let us start with the computation of the group slowness $\tau_g$ as function of $\lambda$ in the range $1450\text{nm} < \lambda < 1650\text{nm}$. In Fig. 5.5, the group slowness for the three profiles of Fig. 5.4 is shown for the w.g. and f.v. program.

Figure 5.5: Computed group slowness results, for a (a) step profile, (b) graded profile and (c) arbitrary profile. Solid lines indicate the results computed by the w.g. program, whereas the dashed lines are generated by the f.v. program. Furthermore, $a = 4.1\mu\text{m}$ and $\Delta = 0.27\%$.

In the case of $\Delta = 0.27\%$, both programs are very close to each other. The weakly guiding approximation turns out to be an accurate representation of the results produced by the f.v. program for small $\Delta$. However, it is not visible how close the obtained results exactly
are to each other. Therefore, we compute the relative error

\[
\text{rel. error}(\%) = \left(\frac{\text{result 'w.g. program'} - \text{result 'f.v. program'}}{\text{result 'f.v. program'}}\right) \times 100\%.
\] (5.1)

Results for the computation of this relative error are shown in Fig. 5.6, where we compute the relative error given by Eq. (5.1) as function of the wavelength \(\lambda\), for the three profiles in Fig. 5.4. These results are intended to illustrate the difference between both programs in a more detailed manner.

![Figure 5.6: Relative error group slowness for various profiles, with \(a = 4.1\mu m\). The solid line is the relative error when a step profile is used, the dotted line is the error for a graded input profile, and the dashed line is the relative error for an arbitrary profile.](image)

We observe that the relative error of \(\tau_g\) versus \(\lambda\) for all profiles is negligible, i.e. the rel. error \(\tau_g < 4.0 \cdot 10^{-4}\%\). This shows that for small \(\Delta\), the weakly guiding approximation is valid to compute \(\tau_g\).

Now, let us increase the relative refractive index difference \(\Delta\). In Fig. 5.7, we have computed the relative error for several values of the relative refractive index difference \(\Delta\) in the case of a step-index profile and a graded-index profile, using a fixed \(\lambda = 1550\text{nm}\) and a constant core radius \(a = 4.1\mu m\).
We observe that, the rel. error in $\tau_g < 9.0 \cdot 10^{-3}\%$, for $\Delta = 5\%$. Although this value for $\Delta$ is not of any practical use, the relative error is still negligible.

Next, we compute the relative error as function of $\Delta$ for $\lambda = 1550\text{nm}$, however this time with a fixed value for the normalized frequency $V = 2.405$. To clarify the reason for including this simulation, we observe Fig. 5.1. If we compute the relative error as function of $\Delta$ for a fixed core radius $a$, the increase of $\Delta$ causes $V$ to increase as well. This means that we compute the relative error as function of $\Delta$ for different propagation coefficients and move to the right in Fig. 5.1, where the graphs of the modes seem to be closer to each other anyway. This could, and probably will, affect the computed relative error as function of $\Delta$ as shown in Fig. 5.7. The results of this extra simulation are shown in Fig. 5.8.

In order to keep $V$ constant for different values of $\Delta$, it is necessary to adapt the core radius $a$ accordingly. In other words, for a larger $\Delta$ a smaller core radius is needed to ensure that $V = 2.405$. The reason for choosing $V = 2.405$, is explained if we look at Fig. 5.1. At this normalized frequency, single mode behaviour is just guaranteed, and thus we have a propagation coefficient which is as close as possible to $n_1$. This has several practical advantages, as this mode travels close to the core's center. The mode will be confined in the core even in the case of small bends in the fiber.
Figure 5.8: Relative error of $\tau_g$, with $\lambda = 1550\text{nm}$ and constant $V = 2.405$.

Immediately, it is clear that the absolute value of the calculated relative error of $\tau_g$ is much larger. However, the maximum error is still smaller than approximately 0.036%, and shows similar behaviour in relation to a larger $\Delta$ as seen in Fig. 5.7, i.e. the error grows for larger $\Delta$. According to [14], actual fibers can have a deviation in the core radius of 0.8%. Computation of the relative error in the group slowness due to this deviation in the core radius for $\Delta = 2\%$ and a graded-index profile gives us a value of 0.011%. The computed relative error in $\tau_g$ if we use the weakly guiding approximation is 0.0054%, which is well below the error caused by the manufacturing process.
5.3.2 Dispersion simulation results

In this section, simulation results of the dispersion $D$, for the various profiles of Fig. 5.4, are shown. The same simulations have been performed for each profile as in the case of the group slowness. In Fig. 5.9, the computed dispersion is set against the wavelength for each profile.

![Graphs showing dispersion results for different profiles](image)

Figure 5.9: Computed dispersion results, for (a) step profile, (b) graded profile and (c) arbitrary profile. Solid lines indicate the results produced by the w.g. program and dashed lines are produced by the f.v. program. Furthermore, $a = 4.1\mu m$, $\Delta = 0.27\%$ and $1450nm < \lambda < 1650nm$.

It can be seen that similar results are acquired for the computed dispersion using both programs. Again, to get more insight, the relative error given by Eq. (5.1) is plotted in Fig. 5.10.
We observe that the relative error in the dispersion between the w.g. program and the f.v. program is smaller than 0.25%. Again, the approximation produces similar results as the exact computation. However, it is prominent that the relative error of the dispersion $D$ for the step profile is larger than the one of the other two profiles. Furthermore, it is noted that the relative error grows as $\lambda$ increases. This can possibly be explained with the aid of Fig. 5.11, where a schematic cross section is shown, showing the transverse electric field vectors for the fundamental mode $HE_{11}$. Fig. 5.11(a) illustrates how the transverse electric field actually is, and Fig. 5.11(b) shows the electric field as assumed by the weakly guiding approximation. If the step-index profile is used, the difference in the electric field near the cladding as shown in Fig. 5.11, has a larger effect on the computed dispersion than in the case of a graded-index profile, where the field near the cladding is more suppressed. In other words, the relative error is larger when a step-index profile is used, because this profile does not suppress the field near the cladding. Near the cladding, the electric field is assumed to be uniform by the approximation, which is not the case in the exact situation.

Figure 5.10: Relative error dispersion for various profiles, with $a = 4.1\mu m$. 
Figure 5.11: Cross section of a fiber: (a) exact transverse E-field vectors of fundamental mode (HE_{11}), and (b) the E-field vectors under the weakly guiding approximation (LP_{01}).

Next, we compute again the relative error as function of the relative refractive index difference $\Delta$, for the step profile and the graded profile with a non-variable core radius $a$. The results are shown in Fig. 5.12.

Figure 5.12: Relative error of $D$ versus $\Delta$, with $\lambda = 1550\mu\text{m}$ and a constant core radius $a = 4.1\mu\text{m}$. The solid line indicates the error for the step profile, and the dotted line is the relative error for the graded profile.

The results produced by the w.g. program are very close to the results obtained by using the f.v. program. It is seen that even in the case of a large $\Delta$ of 5%, the error is only
approximately 2.5%.

For reasons explained in section 5.3.1, we compute the relative error as function of $\Delta$ for the step- and graded profile, only now with a constant $V$ of 2.405, instead of a constant core radius $a$.

![Graph](image)

Figure 5.13: Relative error of $D$, with $\lambda = 1550\text{nm}$ and constant $V = 2.405$.

Fig. 5.13 makes clear that the relative error of $D$ is significantly larger than in the previous simulation and again grows as $\Delta$ increases, also the slope increases for larger $\Delta$. However, besides this general increase in the value of the computed relative error of $D$, the simulation does not show any different behaviour compared to previous simulation in this section. The error still increases for larger $\Delta$, only now it is more evident that the approximation is not accurate anymore for larger $\Delta$. We observe that for $\Delta = 2\%$, the relative error in $D$ is approximately 4% for a graded-index profile. Again, we compare this value to the relative error in $D$ caused by the manufacturing process. The error in $D$ caused by a 0.8% deviation in the core radius, see [14], turned out to be approximately 3.26%. Thus, for values $\Delta < 2\%$ the weak-guidance approximation should provide an accurate representation of the full-vectorial computation method.
5.3.3 Dispersion slope simulation results

We start again with the computation of the dispersion slope as function of the wavelength $\lambda$, using the w.g. and f.v. programs. The results are shown in Fig. 5.14.

Figure 5.14: Computed dispersion slope results versus $\lambda$, for a (a) step profile, (b) graded profile and (c) arbitrary profile. Solid lines indicate the results produced by the w.g. program and dashed lines are produced by the f.v. program. Furthermore, $a = 4.1\mu$m, $\Delta = 0.27\%$ and $1450\text{nm} < \lambda < 1650\text{nm}$.

We observe that similar results are acquired for the computed dispersion slope $S$ by both programs. To get a better view of the difference between these results, we again analyze the relative error versus several useful variables. In order to achieve this, we first compute the relative error of the dispersion slope as function of the wavelength for the three profiles, this is shown in Fig. 5.15.
Again, the results for the three different profiles are similar and the error does not exceed 0.1% for the complete wavelength range. Furthermore, it is seen that the graph of the step profile is a slowly descending continuous line, contrary to the graphs of the other two profiles. This may be caused by the fact that in the computation of the relative error, we subtract two numbers which are almost equal and are only meaningful up to a certain amount of digits. For the used profiles, which all have $\Delta = 0.27\%$, the small value for the relative error of $S$ indicates that the w.g program provides an accurate reproduction of the results obtained with the f.v. program.
In accordance to the simulations performed in sections 5.3.1 and 5.3.2, we now increase \( \Delta \) and compute the relative error. Results of the computation of the relative error of \( S \) as function of the relative refractive index difference \( \Delta \) for the step profile and the graded profile, are shown in Fig. 5.16.

![Figure 5.16: Relative error of \( S \) versus \( \Delta \), with \( \lambda = 1550\mu m \) and a constant core radius \( a = 4.1\mu m \).](image)

The maximum value of the relative error in \( S \) is approximately 0.55\%, when \( \Delta \) is increased while the core radius \( a \) is kept constant at 4.1\( \mu m \). Note that the behaviour is similar as in the case of \( \tau_g \) and \( D \) in Figs. 5.7 and 5.12, respectively, where the respective relative error also increases for larger \( \Delta \).

Lastly, we compute the relative error as function of \( \Delta \) for the step- and graded profile, with \( V = 2.405 \) constant. Results are depicted in Fig. 5.17.
Fig. 5.17 makes clear that the absolute value of the relative error of $S$ is significantly larger than in the previous simulation as $\Delta$ increases, and reaches those larger values for more modest magnitudes of $\Delta$. Besides this increase in the computed relative error of $S$, the simulation does not show any different behaviour compared to previous simulations. The relative error in $S$ by using the weak-guidance approximation for $\Delta = 2\%$ and a graded-index profile is 1.18%. The deviation in the core radius of actual fibers [14] leads to a relative error in $S$ of 0.86%. So, for $\Delta < 2\%$ the weak-guidance approximation should be usable.
5.3.4 Computation time comparison

In this section, results for the computation time are compared between the f.v. program for arbitrary-index fibers (section 2.2), the w.g. program for arbitrary-index fibers (section 3.2), and the w.g. program for step-index fibers, based on the discussion in section 3.1.

We compute the propagation coefficient \( \zeta \) using the programs mentioned, with a step-index permittivity profile, \( \Delta = 0.3\% \), \( \lambda = 1550\text{nm} \), and core radius \( a = 5\mu\text{m} \). The results are shown in Table 5.1. We observe that the w.g. program only suitable for step-index fibers,

<table>
<thead>
<tr>
<th>Used program</th>
<th>( \zeta )</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>f.v. program (arb.)</td>
<td>1.4462752</td>
<td>0.9125 s</td>
</tr>
<tr>
<td>w.g. program (arb.)</td>
<td>1.4462783</td>
<td>0.5409 s</td>
</tr>
<tr>
<td>w.g. program (step)</td>
<td>1.4462783</td>
<td>0.01543 s</td>
</tr>
</tbody>
</table>

Table 5.1: Computation time results for a \( \zeta \) search, using a step-index permittivity profile, \( \Delta = 0.3\% \), \( \lambda = 1550\text{nm} \) and \( a = 5\mu\text{m} \). Computations are performed on an AMD Athlon64 3200+.

is much faster than the other two programs. This is caused by the fact that no numerical integration method is used in this program. However, this program is of limited use, because the only possible refractive index profiles are of the step-index type. The other two programs incorporate arbitrary profiles as well, which is of course more useful in optical fiber design. Furthermore, it is noted that the developed w.g. program is faster than the f.v. program.

We have also analyzed the complete computation time of the f.v. program and the w.g. program in the case of a graded-index input permittivity profile, using the same values for \( \Delta \), \( \lambda \) and \( a \) as for the step-index profile. In Table 5.2, the computation time of both programs is shown, together with the obtained values for the propagation coefficient \( \zeta \), group slowness \( \tau_g \), dispersion \( D \) and dispersion slope \( S \). Again, we see that the developed w.g. program

<table>
<thead>
<tr>
<th>Used program</th>
<th>( \zeta )</th>
<th>( \tau_g )</th>
<th>( D )</th>
<th>( S )</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>f.v. program (arb.)</td>
<td>1.4450836</td>
<td>1.46609</td>
<td>14.8003</td>
<td>0.0619799</td>
<td>1.05 s</td>
</tr>
<tr>
<td>w.g. program (arb.)</td>
<td>1.4450858</td>
<td>1.46609</td>
<td>14.9128</td>
<td>0.0619779</td>
<td>0.6733 s</td>
</tr>
</tbody>
</table>

Table 5.2: Complete computation time results for a graded-index permittivity profile, with \( \Delta = 0.3\% \), \( \lambda = 1550\text{nm} \) and \( a = 5\mu\text{m} \).

is faster than the f.v program. This is to be expected, because the system of Eq. (2.47) is significantly simplified when the weakly guiding approximation is used. We have seen in section 3.2 that only two field solution vectors are needed instead of four, and that these vectors consist of two components instead of four, which reduces the size of matrix \([B]\) equivalently.
Conclusion

The work on hand determines in which cases the weakly guiding approximation of the field solutions is accurate enough to be used instead of the full-vectorial field solutions for a single-mode circularly cylindrical optical fiber. The accuracy of the approximation is analyzed by the computation of the group slowness $\tau_g$, dispersion $D$ and dispersion slope $S$. Based on the obtained simulation results, we can conclude that the results for the group slowness, dispersion and dispersion slope computed by the developed w.g. program approximate the corresponding values produced by the f.v. program very well, for a relative refractive index difference smaller than 2% and several permittivity profiles. Also, we have seen that the developed w.g. program computes these values faster, which is especially beneficial if a great number of modes is analyzed. For future research, it is useful to analyze the accuracy of the weakly guiding approximation for multi-mode fibers as well, because the f.v. program and the developed w.g. program already provide the possibility to calculate multiple propagation coefficients and according values for $\tau_g$, $D$ and $S$. If more modes are present, the dispersion between modes (inter-modal dispersion) can be investigated as well, instead of focusing mainly on material and waveguide dispersion.
Appendix A

Field vector solution in the cladding region

The field vector \( \mathbf{f}_3 \) can be obtained by using Eq. (2.44). The other solution vector \( \mathbf{f}_4 \) is can be derived in a similar way, using Eq. (2.45). Starting from Eq. (2.44), we use

\[
V_\phi = -j \rho^{-1} P_\phi \rho |m|, \quad V_z = -j P_z \rho |m|,
\]

\[
I_\phi = Y_0 \rho^{-1} Q_\phi \rho |m|, \quad I_z = Y_0 Q_z \rho |m|
\]

and write down the expression for \( \rho V_\phi \) in Eq. (2.44)

\[
\partial_\rho (\rho |m| P_\phi) = \rho^{-1} n^{-2} \left\{ m \zeta \rho |m| Q_\phi + \frac{\bar{\omega}^2 n^2 \rho^2 - m^2}{\bar{\omega}} \rho |m| Q_z \right\}. \tag{A.2}
\]

The same is done for \( V_z \) in Eq. (2.44), which leads to

\[
\partial_\rho (\rho |m| P_z) = \rho^{-1} n^{-2} \left\{ \bar{\omega} (\zeta^2 - n^2) \rho |m| Q_\phi - m \zeta \rho |m| Q_z \right\}. \tag{A.3}
\]

Eqs. (A.2) and (A.3) result in

\[
\partial_\rho \left( \frac{\rho |m| P_\phi}{\rho |m| P_z} \right) = \rho^{-1} n^{-2} \left( \begin{array}{c} m \zeta \\ \bar{\omega} (\zeta^2 - n^2) \end{array} \right) \left( \begin{array}{c} \bar{\omega}^2 n^2 \rho^2 - m^2 \\ -m \zeta \end{array} \right) \left( \begin{array}{c} \rho |m| Q_\phi \\ \rho |m| Q_z \end{array} \right). \tag{A.4}
\]

If we keep in mind that \( \mathbf{f}_E = \left( \begin{array}{c} P_\phi \\ P_z \end{array} \right) \) and \( \mathbf{f}_H = \left( \begin{array}{c} Q_\phi \\ Q_z \end{array} \right) \) and furthermore

\[
[A_i] = \left( \begin{array}{cc} \bar{\omega}^2 n^2 \rho^2 - m^2 & -m \zeta \\ -m \zeta & -\bar{\omega} (\zeta^2 - n^2) \end{array} \right), \tag{A.5}
\]

\[
[J_i] = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \tag{A.6}
\]
the complete system becomes
\[
\partial_\rho \begin{pmatrix} \rho^{m|l}\mathbf{f}_E \\ \rho^{m|l}\mathbf{f}_H \end{pmatrix} = \rho^{-1} \begin{pmatrix} 0 & n^{-2}[A_4][J_i] \\ [A_4][J_i] & 0 \end{pmatrix} \begin{pmatrix} \rho^{m|l}\mathbf{f}_E \\ \rho^{m|l}\mathbf{f}_H \end{pmatrix}.
\] (A.7)

We want to work towards Bessel’s differential equation in cylindrical coordinates, because the solution to this equation is known. Therefore, we multiply the expressions in Eq. (A.7) with \( \rho \) as follows
\[
\begin{align*}
\rho \partial_\rho (\rho^{m|l}\mathbf{f}_E) &= n^{-2}[A_4][J_i](\rho^{m|l}\mathbf{f}_H), \\
\rho \partial_\rho (\rho^{m|l}\mathbf{f}_H) &= [A_4][J_i](\rho^{m|l}\mathbf{f}_E),
\end{align*}
\] (A.8a)
(A.8b)

and take \( \rho \partial_\rho \) on both sides of Eq. (A.8a), this leads to
\[
\rho \partial_\rho \partial_\rho (\rho^{m|l}\mathbf{f}_E) = n^{-2}\rho[[A_4][J_i]\partial_\rho(\rho^{m|l}\mathbf{f}_H) + (\rho^{m|l}\mathbf{f}_H)\partial_\rho([A_4][J_i])].
\] (A.9)

In Eq. (A.9), the term \( \partial_\rho([A_4][J_i]) \) produces a matrix which only has one non zero element \( X \) given by
\[
\partial_\rho([A_4][J_i]) = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.
\] (A.10)

It is convenient when the term \( \rho n^{-2}(\rho^{m|l}\mathbf{f}_H)\partial_\rho([A_4][J_i]) \) in Eq. (A.9) reduces to zero. If we recall that \( \mathbf{f}_H = \begin{pmatrix} \phi \\ \varphi \end{pmatrix} \), it becomes clear that choosing \( Q_\phi = 0 \) leads to the wanted result.

Next, after substitution of Eq. (A.8b) into Eq. (A.9) we obtain
\[
\rho \partial_\rho \rho \partial_\rho (\rho^{m|l}\mathbf{f}_E) = n^{-2}[A_4][J_i][A_4][J_i](\rho^{m|l}\mathbf{f}_E),
\] (A.11)
in which
\[
[A_4][J_i][A_4][J_i] = \{n^2[x^2 + m^2][I_i]\},
\] (A.12)
where
\[
[I_i] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\] (A.13)
and
\[
x^2 = \tilde{\omega}^2 \rho^2(\zeta^2 - n^2).
\] (A.14)

If we combine Eqs. (A.11)-(A.14), we obtain the following equation
\[
\rho \partial_\rho \rho \partial_\rho (\rho^{m|l}\mathbf{f}_E) - (x^2 + m^2)[I_i](\rho^{m|l}\mathbf{f}_E) = 0.
\] (A.15)

The solution of Eq. (A.15) is given by
\[
\rho^{m|l}\mathbf{f}_E = \rho^{m|l}\begin{pmatrix} P_\phi \\ P_z \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} K_m(x).
\] (A.16)
The next task is to determine the relationship between the constants $A$ and $B$ and to find $f_H = \begin{pmatrix} Q_\phi \\ Q_z \end{pmatrix}$.

Recalling Eq. (A.8a), we can write

$$\partial_\rho (\rho^{\text{rel}} f_E) = \rho^{-1} n^{-2} [A_t][J_t] (\rho^{\text{rel}} f_H),$$

(A.17)

which leads to

$$\rho^{\text{rel}} f_H = \rho \partial_\rho (\rho^{\text{rel}} f_E) n^2 ([A_t][J_t])^{-1},$$

(A.18)

where

$$([A_t][J_t])^{-1} = \frac{1}{(m^2 + x^2) n^2 [A_t][J_t]}.$$  

(A.19)

Eq. (A.18) is written as follows

$$\rho^{\text{rel}} \begin{pmatrix} Q_\phi \\ Q_z \end{pmatrix} = \rho \frac{1}{(m^2 + x^2) [A_t][J_t]} \partial_\rho \rho^{\text{rel}} \begin{pmatrix} P_\phi \\ P_z \end{pmatrix},$$

(A.20)

If we use $Q_z = 0$, it follows from Eq. (A.20) that

$$A = \frac{m \zeta}{\omega (\zeta^2 - n^2)} B$$

(A.21)

and the expression for $\rho^{\text{rel}} Q_\phi$ becomes

$$\rho^{\text{rel}} Q_\phi = B \frac{\rho \varepsilon_r}{\omega (\zeta^2 - n^2)} K'_m(x)x'.$$  

(A.22)

Finally, the before mentioned equations result in the following solution vector

$$\rho^{\text{rel}} f_3 = \rho^{\text{rel}} \begin{pmatrix} f_E \\ f_H \end{pmatrix} = B \begin{pmatrix} m \zeta \omega \frac{K_m(x)}{\omega (\zeta^2 - n^2) K_m(x)x'} \\ K_m(x) \\ \omega \frac{pn^2}{\omega (\zeta^2 - n^2)} K'_m(x)x' \end{pmatrix}.$$  

(A.23)
Appendix B

Used system of differential equations

Consider the system of Eq. (4.15)

\[ \partial_\rho g = \frac{1}{\rho} [B] g. \quad (B.1) \]

In order to compute all derivatives of the characteristic equation \( C \) we require all necessary derivatives of the core solution vector \( f_1 \), which are present in \( g \). These are found by integrating Eq. (B.1) towards \( \rho = 1 \). The complete system of differential equations consists of

\[
\begin{align*}
\partial_\rho f & = \rho^{-1} Af, \\
\partial_\xi \partial_\rho f & = \rho^{-1} [ (\partial_\xi A)f + A(\partial_\xi f) ], \\
\partial_\omega \partial_\rho f & = \rho^{-1} [ (\partial_\omega A)f + A(\partial_\omega f) ], \\
\partial^2_\rho \partial_\rho f & = \rho^{-1} [ (\partial^2_\rho A)f + (2 \partial_\xi A)(\partial_\rho f) + A(\partial^2_\rho f) ], \\
\partial^2_\rho \partial_\rho f & = \rho^{-1} [ (\partial^2_\rho A)f + (2 \partial_\omega A)(\partial_\rho f) + A(\partial^2_\rho f) ], \\
\partial_\xi \partial_\xi \partial_\rho f & = \rho^{-1} [ (\partial_\xi \partial_\xi A)f + (\partial_\xi A)(\partial_\xi f) + (\partial_\omega A)(\partial_\xi f) + A(\partial_\xi \partial_\xi f) ], \\
\partial^2_\xi \partial_\rho f & = \rho^{-1} [ (\partial^2_\xi A)f + (3 \partial_\xi A)(\partial^2_\xi f) + (3 \partial^2_\omega A)(\partial_\xi f) + A(\partial^2_\xi f) ], \\
\partial^2_\omega \partial_\rho f & = \rho^{-1} [ (\partial^2_\omega A)f + (3 \partial_\omega A)(\partial^2_\rho f) + (3 \partial^2_\xi A)(\partial_\rho f) + A(\partial^2_\omega f) ], \\
\partial_\xi \partial^2_\xi \partial_\rho f & = \rho^{-1} [ (\partial_\xi \partial^2_\xi A)f + (\partial^2_\xi A)(\partial_\xi f) + (2 \partial_\xi \partial_\omega A)(\partial_\xi f) + (2 \partial_\omega A)(\partial_\xi \partial_\xi f) + (2 \partial_\xi A)(\partial_\xi \partial_\xi f) + A(\partial_\xi \partial^2_\xi f) ], \\
\partial_\omega \partial^2_\xi \partial_\rho f & = \rho^{-1} [ (\partial_\omega \partial^2_\xi A)f + (\partial^2_\xi A)(\partial_\xi f) + (2 \partial_\xi \partial_\omega A)(\partial_\xi f) + (2 \partial_\xi A)(\partial_\omega \partial_\xi f) + A(\partial_\xi \partial^2_\xi f) ], \\
\partial_\xi \partial^2_\omega \partial_\rho f & = \rho^{-1} [ (\partial_\xi \partial^2_\omega A)f + (\partial^2_\omega A)(\partial_\rho f) + (3 \partial_\xi \partial_\omega A)(\partial_\xi f) + (3 \partial^2_\omega A)(\partial_\xi f) + (2 \partial_\xi A)(\partial_\xi \partial^2_\xi f) + (3 \partial^2_\omega A)(\partial_\xi f) + A(\partial_\xi \partial^2_\omega f) ] ,
\end{align*}
\]

where \( \partial_\xi \partial_\xi f = \partial_\omega \partial_\xi f \).
The system of Eq. (B.1) consists of the following components

\[
\mathbf{g} = \begin{bmatrix} f \\ \partial_{\xi} f \\ \partial_{\tau} f \\ \partial_{\tau}^2 f \\ \partial_{\tau}^3 f \\ \partial_{\tau} \partial_{\xi} f \\ \partial_{\tau}^2 \partial_{\xi} f \\ \partial_{\tau} \partial_{\xi}^2 f \\ \partial_{\tau}^2 \partial_{\xi}^2 f \end{bmatrix},
\]

(B.3)

and

\[
[B] = \begin{bmatrix}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial_{\xi} A & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial_{\tau} A & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial_{\tau}^2 A & 2 \partial_{\xi} A & 0 & A & 0 & 0 & 0 & 0 & 0 \\
\partial_{\tau}^3 A & 0 & 2 \partial_{\tau} A & 0 & A & 0 & 0 & 0 & 0 \\
\partial_{\tau} \partial_{\xi} A & \partial_{\tau} A & \partial_{\xi} A & 0 & 0 & A & 0 & 0 & 0 \\
\partial_{\tau}^2 \partial_{\xi} A & 3 \partial_{\tau}^2 A & 0 & 3 \partial_{\tau} A & 0 & 0 & A & 0 & 0 \\
\partial_{\tau}^3 \partial_{\xi} A & 0 & 3 \partial_{\tau}^2 A & 0 & 3 \partial_{\tau} A & 0 & 0 & A & 0 \\
\partial_{\tau} \partial_{\xi}^2 A & \partial_{\tau}^2 A & 2 \partial_{\tau} \partial_{\xi} A & 0 & \partial_{\xi} A & 2 \partial_{\tau} A & 0 & 0 & A \\
\partial_{\tau}^2 \partial_{\xi}^2 A & 2 \partial_{\tau} \partial_{\xi} A & \partial_{\xi}^2 A & \partial_{\tau} A & 0 & 2 \partial_{\tau} A & 0 & 0 & 0 \\
\end{bmatrix}.
\]

(B.4)
Appendix C

Fortran program code description

In this appendix, a discussion of the developed program code can be found. In section C.1 a basic program is discussed, which computes the propagation coefficients for step-index fibers with the addition of a power density computation if a mode is found. In section C.2 this basic program is expanded to be able to compute the propagation coefficients of arbitrary-index fibers. Finally, for the determined propagation coefficients using the program code discussed in section C.2.2, the final addition to form the complete program is discussed in section C.3. This program code is used for the computation of the group slowness, dispersion and dispersion slope for a found mode in the case of weakly guiding fibers with an arbitrary permittivity profile.

C.1 Fortran code description for finding the propagation coefficients of a step-index fiber

C.1.1 Basic program

This code consists of a main program 'propagation.f95' and two modules called by this main program, 'eigenvalues.f95' and 'dereigenvalues.f95'. In this section, these program parts will be described separately and how they interact with each other.

'eigenvalues.f95'

This is a module called by the main program 'propagation.f95'. The module contains an implementation of Eq. (2.32), called 'equation' in the program code. Available NAG20 routines 'S17DEF' and 'S18DCF' are used for determining the Bessel functions \( J_m(u) \) and \( K_m(u) \) as found in Eq. (2.32). When called by the main program, the module takes a found propagation coefficient 'zeta' as input and returns the value of 'equation' for that particular 'zeta'.
This is another module called by the main program, and it contains the derivative of Eq. (2.32), 'derequation'. It works in a similar way as 'eigenvalues.f95'. When called by 'propagation.f95' it also takes 'zeta' as input and returns in this case the value of the derivative of Eq. (2.32). In the description of the main program 'propagation.f95' it will become clear why this derivative is needed to be able to find the propagation coefficients for a certain mode.

This is the main program. The goal of this program is to calculate all necessary data to form a graph of propagation coefficients $\zeta (n_1 < \zeta < n_2)$ versus the normalized frequency $V$. This main program consists of a couple of loop structures. The first loop (the DO-loop changing $m$) runs through a prior chosen amount of azimuthal mode numbers. For each different $n$ the Bessel functions present in both 'equation' and 'derequation' change accordingly. After this a DO-loop for 'p' is started, this changes the core radius and thus the normalized frequency $V$. For one azimuthal mode number $m$, and one normalized frequency $V$, the values for 'equation' and 'derequation' for the complete $\zeta$ interval (returned after calling 'eigval.f95' and 'dereigval.f95') are stored in the arrays 'fval' and 'fvald'. Next, these function values (present in the array 'fval') are used to check for zero-crossings, that is, to try and find $\zeta$ values for which Eq. (2.32) holds. When a change in the sign of two successive function values is detected, the NAG20 routine 'C05ADF' is used to search for a zero crossing of 'equation' between these two function values. Found values are stored in an output file 'zeros.dat'. After this the program searches for three function values in the constructed array 'fval' indicating a minimum close to the horizontal axis. In order to find the exact location of this minimum, again the zero-search routine 'C05ADF' is used, only now for the derivative of 'equation'. The found $\zeta$ ('zetaz' in the program) of the minimum is used to determine the function value, which is stored in the temporary variable 'test'. When this function value appears to be negative, there are two zero-crossings of the function 'eigval' and in this case again the 'C05ADF' routine is used to determine and store the found $\zeta$ values. The same is done for a possible maximum beneath the horizontal axis. After finishing this search for zeros, the same is done for a different normalized frequency $V$. When for every desired frequency (depending on the value of 'p') zeros have been searched, the same is done for the next value of $m$.

### C.1.2 Program code for weakly guiding approximation

In order to obtain a program which computes propagation coefficients within the weakly guiding approximation, the discussed program mentioned above is modified. Instead of using Eq. (2.32), Eq. (3.9) and its derivative to zeta are implemented in the new program. This is rather straightforward, but in order to get correct results it is necessary to modify
Eq. (3.9). This is done by writing Eq. (3.9) as

\[
\frac{uK_{l+1} - wK_{l+1}J_l}{wuK_{l+1}J_l} = 0, \quad (C.1)
\]

and only using the nominator in the program code. This is possible here because the denominator is always finite. This way certain asymptotic behaviour causing the program to find false propagation coefficients is circumvented.

C.1.3 Program code for power density calculation

The power density \( S \) is computed using the module 'powerdensitycalc.f95'. The module is called by the main program 'propagation.f95' after finding a zero of Eq. (2.32) for a fixed frequency. In this module the eigenvalues and according eigenvectors of matrix \( M \) as shown in Eq. (2.29), are determined using internal NAG routine 'F02GBF'. As mentioned in section 2.1.3, the eigenvalue \( \lambda = 0 \) is of interest. So, in order to determine the scalar amplitudes \( A, B, C \) and \( D \) the program computes the four eigenvectors and four eigenvalues first. After this, the eigenvector which belongs to eigenvalue \( \lambda = 0 \) is extracted and stored. Then a loop structure over \( \phi \) and \( r \) is used to calculate \( E_{\phi}, E_r, H_{\phi} \) and \( H_r \) for a certain found \( \zeta \) of choice inside the fiber core. Finally, the following equation is used to compute the power density in the fiber core

\[
S = \frac{1}{2} \text{Re} \{ E \times H^* \} \cdot u_z \quad (C.2)
\]

leading to

\[
S = \frac{1}{2} \text{Re} \{ (E_rH_{\phi}^* - E_{\phi}H_r^*) \} \quad (C.3)
\]

C.2 Fortran code description for finding the propagation coefficients of arbitrary-index fiber

It is important to emphasize that the basic program described in section C.2.1 serves only as a starting point for the developed program in sections C.2.2 and C.3. This means that the dependence of the relative core permittivity \( \varepsilon_{r1} \) on the frequency \( \tilde{\omega} \) (or wavelength \( \lambda \)) and the radial position \( \rho \), has not been incorporated yet. A program for the computation of propagation coefficients and several important parameters (group slowness, dispersion and dispersion slope) using the exact field vector solutions is already available [1], and we are not interested in developing a copy of this program. The development of an equivalent program which uses the weakly guiding approximation is of main importance. The discussion of this developed program is split into two parts. In section C.2.2 it is described how the propagation coefficients for arbitrary-index fibers are found and in section C.3 the computation of the group slowness, dispersion and dispersion slope is treated. These quantities are only computed for the found propagation coefficients.
C.2.1 Basic program

In section 2.2 we discussed how to compute the propagation coefficients $\zeta$ and field components of arbitrary-index fibers. The program in this section calculates these $\zeta$ values in a way described in section 2.2. This program is then used for the final program discussed in sections C.2.2 and C.3. In short, this program builds further upon parts of the program code discussed in section C.1.1, the main difference is the way the determinants are calculated. The Fortran code consists of a main program 'propagationarb.f95' and two modules called by this main program, 'eigenvaluesarb.f95' and 'dereigenvaluesarb.f95'. However, to find the propagation coefficients for an arbitrary-index fiber, a different approach is needed to calculate the solution matrix 'C'. As we have seen in section 2.2, this solution matrix contains the field vector solutions $f_1, f_2, f_3, f_4$, and is found if we start with the system of differential equations in Eq. (2.42). From the boundary condition at $\rho = 0$ two starting vectors $f_1(0)$ and $f_2(0)$ are found. Next, the solutions $f_1(1)$ and $f_2(1)$ are computed using Adam's method (internal NAG routine 'D02CJF'). To use the routine 'D02CJF' we also need to specify a subroutine 'FCN'. Together with the analytically derived solution vectors $f_3(1)$ and $f_4(1)$, the solution matrix is completed. Finally, in order to find the propagation coefficients it is checked where the determinant of this solution matrix is zero. This was also done in the case of the step-index fiber discussed in section C.1, only the way the solution matrix is constructed is different.

'propagationarb.f95'

This part of the program is identical to 'Propagation.f95' with the exception of a couple of changed names.

'eigenvaluesarb.f95'

This is a module called by the main program 'Propagation.f95'. For a certain 'zeta' value the determinant of the matrix 'C' is computed in a way described above.

'dereigenvaluesarb.f95'

This is another module called by the main program and it computes the derivative of the determinant of the matrix 'C'. In order to find the derivative of the determinant we compute the derivatives of the starting vectors $f_1(0)$ and $f_2(0)$, and use them as input to the numerical integration to obtain the derivatives of $f_1(1)$ and $f_2(1)$. Also, the derivatives of the solution vectors $f_3(1)$ and $f_4(1)$ are computed and thus the new solution matrix 'C' is completed. Now, we only need to compute the determinant of this matrix using internal NAG routine 'F03AAF'.
C.2.2 Program code for weakly guiding approximation

In accordance to the discussion in section 3.2 the basic program of section C.2.1 is adapted for the weakly guiding situation. Besides the change in the number of solution vectors (now two instead of four) and the change in size of the solution matrix as already mentioned in section 3.2, the main difference is the fact that in this program the $\omega$ dependency of the core permittivity $\varepsilon_{r1}$ is included. In order to find the propagation coefficients $\zeta$ we only need the characteristic equation $C$ and it's derivative to $\zeta$, $\partial_C C$, so we do not need derivatives of $\varepsilon_{r1}$ with respect to $\omega$ yet. The $\omega$ dependence of the permittivity is included by using the three term Sellmeier equation discussed in section 4.2.1, this Sellmeier equation is implemented by use of the function 'epsr sm', which requires several other functions and subroutines in order to operate properly. The essential functions and subroutines to be able to use 'epsr sm', are taken from the existing software which uses the exact computation method, and modified if necessary. The most important program parts will be described briefly here.

'rd input.f'

This subroutine reads the input file 'fib mmf.in', which contains several important parameters. For instance the shape of the permittivity profile, the core radius and parameters describing the geometry of the fiber.

'check.f'

This subroutine checks the input parameters read by 'rd input.f'.

'doping.f'

This subroutine returns the doping concentrations of fluorine and germanium for a certain radius $\rho$ in the core. First the fluorine concentration is determined, then 'german0.f' is used to determine the germanium concentration for the Sellmeier equation, to obtain the specified permittivity (at a certain wavelength). The permittivity in our case is an artificial profile, which we specified in the input file 'fib mmf.in'.

'epsr sm.f'

This function returns the permittivity at a specified frequency $\omega$ according to a three term Sellmeier equation for doped SiO2. It takes the fluorine and germanium concentrations determined by 'doping.f' as input.

'propagationarb2.f95'

This is the main program, and is generally identical to 'propagationarb.f95' with the exception of the use of 'epsr sm' to determine the interval in which the zeros of the characteristic
equation (ζ's) are searched for.

'eigenvaluesarb2.f95'

This function returns the value of the characteristic equation \( C \) for a certain input \( ζ \) and is called from 'propagationarb2.f95' to determine where \( C = 0 \). The function is identical to 'eigenvaluesarb.f95', only the specified start vector \( f_1(0) \) and the solution vector \( f_2(1) \) have different sizes, due to the fact that we have used the weakly guiding approximation.

'dzetaeigenvaluesarb2.f95'

This function returns the value of \( \partial_ζ C \) for a certain \( ζ \) and is also used to find where \( C = 0 \). That is, besides searching for zero crossings of \( C \), it is also checked if \( C \) has either a maximum just beneath the horizontal axis, or a minimum just above the horizontal axis.

C.3 Fortran code description of developed program for \( τ_γ \), \( D \) and \( S \) calculation

Together with the program code discussed in section C.2.2, the program code discussed in this section forms the developed w.g. program for arbitrary-index fibers. This program computes the group slowness \( τ_γ \), dispersion \( D \) and dispersion slope \( S \) for a found mode. In short the program discussed in section C.2.2 is expanded by use of the subroutine 'grpCE.f'. The main program 'propagationarb2.f95' in section C.2.2 is modified and renamed to 'Dispersion.f95', and the subroutine 'grpCE.f' is called from this new main file, after a mode \( ζ \) is found. In the subroutine 'grpCE.f' Eqs. (4.4)-(4.6) are computed with the aid of the derivatives of \( C \). The main program components of the w.g. program will be discussed briefly here.

'Dispersion.f95'

This is roughly the same program code as 'propagationarb2.f95' in section C.2.2. It calls 'eigvalarb2.f95' and 'dzetaeigvalarb2.f95' to find the propagation coefficients, but now also calls 'grpCE.f' to compute \( τ_γ \), \( D \) and \( S \). This program also uses the other functions and subroutines already discussed in section C.2.2, therefore these will not be listed here.

'grpCE.f'

This part of the program contains an implementation of Eqs. (4.4)-(4.6). For each found \( ζ \), given a certain mode number \( m \) and frequency \( \bar{ω} \), the quantities \( τ_γ \), \( D \) and \( S \) are computed. In order to compute these values, the derivatives of the characteristic equation \( C \) are needed and therefore called from 'grpCE.f'. We want to analyze single mode behaviour,
therefore \( m = 0 \) is used in combination with a small core radius, so that only one \( \zeta \) is found for each frequency/wavelength. After values for the group slowness \( \tau_g \), dispersion \( D \) and dispersion slope \( S \) are computed, three output files are written. For each wavelength \( \lambda \) and found \( \zeta \) the value of the group slowness is written to 'fib mmf.g00', the computed dispersion is written to 'fib mmf.d00' and the dispersion slope is written to 'fib mmf.s00'.

The functions which compute the necessary derivatives of \( C \) in a way described in section 4.2.2 are listed next. They work generally in a similar way like 'dereigenvaluesarb.f95' and 'dzetaeigenvaluesarb2.f95' discussed in previous sections, only now the used \( f \) vectors and matrix \( [B] \) are larger.

'eigvalarb2.f95'

This function computes the characteristic equation \( C \) for an input \( \zeta \). The function is used to compute the propagation coefficients and is called from 'Dispersion.f95'.

'dzetaeigvalarb2.f95'

This function is also called from 'Dispersion.f95', and computes \( \partial \zeta C \) for an input \( \zeta \). The function is used to compute the propagation coefficients and also \( \tau_g, D \) and \( S \).

'domegaeigvalarb2.f95'

This part of the program computes \( \partial \omega C \) for an input \( \zeta \), and is used for the computation of \( \tau_g, D \) and \( S \). Because of the \( \omega \) dependence of the permittivity \( \varepsilon_r \), it contains also derivatives of this permittivity to \( \omega \). The first derivative of the permittivity to \( \omega \) \( (\partial_\omega \varepsilon_r) \) is computed by 'dedw sm.f'. This function is the first derivative to \( \omega \) of the Sellmeier equation shown in section 4.2.2. For the higher order derivatives \( (\partial^2_\omega \varepsilon_r) \) and \( (\partial^3_\omega \varepsilon_r) \) the functions 'dedw2 sm.f' and 'dedw3 sm.f' are used, which are the second and third derivative of Eq. (4.2) to \( \omega \), respectively. However, these are used in the higher order derivatives of \( C \) with respect to \( \omega \).

'dzeta2eigvalarb2.f95'

This part of the program determines \( \partial^2 \zeta C \) for an input \( \zeta \), and is used to compute \( D \) and \( S \).

'domega2eigvalarb2.f95'

\( \partial^2 \zeta C \) is computed for an input \( \zeta \), and is used to compute \( D \) and \( S \). Furthermore, it should be noted that this file uses the earlier mentioned 'dedw2 sm.f' function in addition to 'epsr sm.f' and 'dedw sm.f'.
'domegadzetaeigvalarb2.f95'
This function computes $\partial_\omega \partial_\zeta C$ for an input $\zeta$ and is used to compute $D$ and $S$.

'dzeta3eigvalarb2.f95'
This part of the program computes $\partial_\zeta^2 C$ for an input $\zeta$. The function is used to compute $S$ only.

'domega3eigvalarb2.f95'
This part of the program computes $\partial_\omega^3 C$ for an input $\omega$, and is used to compute $S$. Furthermore, it should be noted that this file uses the earlier mentioned 'dedw3 sm.f' function in addition to 'epsr sm.f', 'dedw sm.f' and 'dedw2 sm.f'.

'dzetadomega2eigvalarb2.f95'
This function computes $\partial_\zeta \partial_\omega^2 C$ for an input $\zeta$, and is used to compute $S$ only.

'domegadzeta2eigvalarb2.f95'
This function determines $\partial_\omega \partial_\zeta^2 C$ for an input $\zeta$, it is used to compute $S$. 
Bibliography


