MASTER

Modeling, analysis and simulation of a valve made of a shape memory alloy spring

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MODELING, ANALYSIS AND SIMULATION OF A VALVE MADE OF A SHAPE MEMORY ALLOY SPRING

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Abstract

Shape memory alloy (SMA) materials are used in wide areas ranging from micro- and nano- machinery, automotive, aerospace technology, oil exploration, self repairing shielding devices, biomedical implants and medicine. A particularly nice application of shape memory alloys materials is the fabrication of valve mechanisms for rice cooking machines. The situation is rather simple: The valve feels large threshold temperatures and opens. Then the hot vapors escape the boiling room and the local (close to the valve) temperature goes down. Then the valve remembers its initial position, and consequently, it closes. Based on Lagrange and Hamilton principles, we derive a mathematical model for a valve mechanism made of two springs: a shape memory alloy (SMA) spring and a conventional (bias) spring. The resulting shape memory alloy problem is a one-dimensional system of nonlinear hyperbolic-parabolic equations with a free boundary (defining the position of the valve). The research presented in this thesis focusses on the reduced version of the problem the so-called shape memory alloy problem in its fast-temperature-activation limit problem. We review the results concerning the existence and uniqueness of weak solutions and complement these results with new ones on the stability with respect to data and material parameters (especially w.r.t. those entering Falk's model for shape memory alloys). Finally, we approximate numerically the solution of the original shape memory alloy problem using a central finite differences scheme. We illustrate the behavior of the valve and of the displacements in the shape-memory alloy spring for a few sets of parameters.

Keywords: Shape-memory alloys, nonlinear elliptic PDE, stability analysis, finite differences approximation

MSC 2000; 74D99; 74N99; 35J60; 35B35; 65M06
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List of Abbreviations and Shape Memory Alloy Related Symbols

MIS *minimal invasive surgery*

PDE *partial differential equation*

SMA *shape memory alloy*

SME *shape memory effect*

$A_f$ *austenitic finish temperature*

$A_s$ *austenitic start temperature*

$M^d$ *detwinned martensite*

$M_f$ *martensitic finish temperature*

$M_s$ *martensitic start temperature*

$M^t$ *twinned martensite*
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Chapter 1

Introduction

The shape memory alloy (SMA) materials are being used in many fields, one of these fields is as a part of valve mechanism in rice cooking machine. In this thesis, we derive a mathematical model of a valve mechanism made of two springs: a elastic (bias) spring and a shape memory alloy spring based on Lagrange and Hamilton principles. The resulting problem is called shape memory alloy problem and is mathematically quite complicated due to its hidden hyperbolic structure. Therefore, we focus on the reduced version of the problem so called shape memory alloy problem in its fast-temperature-activation limit and complete the well posedness (according to Hadamard's definition) of weak solutions of the reduced problem. The existence and uniqueness of weak solutions to this problem were already shown in [3]. We complete the results reported in [3] with a stability analysis with respect to data and parameters in the analysis part of this thesis. This makes the reduced problem a well-posed problem. In the simulation part of this thesis, using finite difference method, we illustrate some simulation results for the shape memory alloy problem and its sensitivity (stability) analysis with respect to the parameters. Since it is quite complicated to show its sensitivity analysis analytically We show it numerically.

In the next section of this chapter we give a brief back ground of shape memory alloy and then the outline of the thesis in the after coming section.
1.1 Background on Shape Memory Alloys

Metals play an important role in designing structural materials. Techniques of alloying, smelting and forging have been evolving for long time. With the advancements in science and technology, and deeper understanding of the effects of microstructure and processing techniques on the material behavior, the field of material science has radically improved nowadays. The capability to engineer different material properties (mechanical, thermal, electrical, etc.) for a variety of applications has enabled the development of new alloys and composites. The demand for lighter, stronger materials with tailored properties that address both stringent structural requirements and provide additional engineering functionality (e.g., sensing, actuation, electromagnetic shielding) has spawned a new branch of materials called \textit{multifunctional materials}. A specialized subgroup of multifunctional materials exhibiting sensing and actuation capabilities is known as \textit{active materials} [11].

In sensing, a mechanical signal is converted into a non-mechanical output (e.g., voltage), while an actuator converts a non-mechanical input (e.g., electrical power) into a mechanical output. Active materials in general exhibit a mechanical response when subjected to a non-mechanical field (thermal, electrical, magnetic, optical, etc.). The mechanical response of these materials is typically one or more orders of magnitude greater than the response resulting from conventional material behavior such as thermal expansion. Shape memory materials (coupling of thermal with mechanical fields) are active materials.

The shape memory alloys (smart metals, memory alloys, muscle wires, smart alloys) are the unique class of shape memory materials having the ability to recover their shape when the temperature is raised over a threshold level. This unique character of SMAs have made them popular for applications like sensing and actuation, impact absorption and vibration damping applications [11]. As a result of sensitive responses to typically small temperature vibrations, such materials are currently employed in wide areas ranging from micro- and nano-machinery, automotive, aerospace
technology, oil exploration, self repairing shielding devices, biomedical implants [14, 12], medicine [15] and also as joints in various devices. We are interested in understanding simple SMA valve mechanism similar to those used in rice cooking machine[2]. In the case of as valve mechanism in rice cooking machine the valve opens when the temperature is sufficiently high because the property of the shape memory alloy changes and closes when the temperature drops.

The discovery of martensite in steels in the 1890s by Adolf Martens was a major step toward the eventual discovery of the shape memory alloys. The martensitic transformation was perhaps the most widely studied metallurgical phenomenon since the early 1900s. The martensitic transformation is a diffusionless phase transformation in solids, in which atoms move cooperatively and often by a shear-like mechanism [17]. The martensitic transformation, as observed in the Fe-C system, was established as an irreversible process. The concept of thermoelastic martensitic transformation, which explained the reversible transformation of martensite, was introduced in 1949 by Kurdjumov and Khandros, based on experimental observations of the thermally reversible martensitic structure in CuZn and CuAl alloys. By 1953, the occurrence of thermoelastic martensitic transformation was demonstrated in other alloys such as InTi and CuZn.

The reversible martensitic transformation and the alloys that exhibited remained unutilized until 1963. The shape memory effect (shape recovery) was officially reported in 1963 by W. J. Buehler and his co-workers at the US Naval Ordnance Laboratory, Silver Springs, Maryland. They discovered this property in alloy made of Nickel (Ni) and Titanium (Ti) containing equiatomic properties of each element. They named it as NiTiNOL (Ni-Ti-NOL, NOL = Naval Ordnance Laboratory). It was noticed that in addition to its good mechanical properties, comparable to some common engineering metals, the material also possessed a good shape recovery capability. The term shape memory effect was given to the associated shape recovery behavior [11]. Some of the alloys exhibiting the shape memory effect are Cu-Al, Ni-Al, Cu-Al-Ni, Cu-Zn, and Cu-Zn-Al [15].

Shape memory alloys have two phases: martensite and austenite, each with
a different crystal structure and therefore different mechanical properties. Shape memory alloys are mixtures of many martensite and austenite components. The composition of the mixture varies. On top of this the martensite and the austenite transforms into one another [9]. Austenite is the high temperature phase crystal structure, while martensite is the low temperature phase crystal structure. Austenite (generally cubic) has a different crystal structure compared to martensite (tetragonal, orthorhombic or monoclinic). These phase changes can be produced either by thermal or by mechanical actions [9]. The transformation from one structure to the other does not occur by diffusion of atoms, but rather by martensitic transformation [11]. The transformation from the martensites phase to the austenite phase is only dependent on temperature and stress, but not on time, as most phase change situations are, as there is no diffusion involved. The crystal transformation of shape memory alloys is fully reversible. Each martensitic crystal formed can have a different orientation direction, called a variant. The assembly of martenstic variants can exist in two forms: twinned martensite ($M^t$), which is formed by a combination of “self-accommodated” martensitic variants, and detwinned or reoriented martensite in which a specific variant is dominant ($M^d$). The reversible phase transformation from austenite (parent phase) to martensite (product phase) and vice versa forms the basis for unique behavior of SMAs [11].

Upon cooling in the absence of an applied load, the crystal structure changes from austenite to martensite. The phase transition from austenite to martensite is termed as the forward transformation. The transformation results in the formation of several martensitic variants, up to 24 for NiTi, for instance. The arrangement of variants occurring such that the average macroscopic shape change is negligible, results in twinned martensite. When the material is heated from the martensitic phase, the crystal structure transforms back to austenite, and this transition is called reverse transformation, during which is no associated shape change [11].
Figure 1.1: Temperature-induced phase transformation of SMAs without mechanical loading.

A sketch of the crystal structures of twinned martensite and austenite for SMA materials and the transformation between them is shown in Figure 1.1. There are four characteristic temperatures associated with the phase transformation. During the forward transformation, austenite, under zero load, begins to transform to twinned martensite at the martensitic start temperature ($M_s$) and completes transformation to martensite at martensitic finish temperature ($M_f$). At this stage, the transformation is complete and the material is fully in the twinned martensitic phase. Similarly, during heating, the reverse transformation initiates at the austenitic start temperature ($A_s$) and the transformation is completed at the austenitic finish temperature ($A_f$).
Figure 1.2: Picture of the shape memory effect of SMAs showing the detwinning of the material with an applied stress.

If a mechanical load is applied to the material in the twinned martensitic phase (at low temperature), it is possible to detwin the martensite by re-orienting a certain number of variants (see Figure 1.2). The detwinning process results in macroscopic shape change, where the deformed configuration is retained when the load is retained. A subsequent heating of the SMA to a temperature above $A_f$ results in a reverse phase transformation (from detwinned martensite to austenite) and leads to complete shape recovery (see Figure 1.3). Cooling back to a temperature below $M_f$ (forward transformation) leads to the formation of twinned martensite again with no associated shape change that can be observed. The process described
above is referred to as shape memory effect (SME). The load applied must be sufficiently large to start the detwinning process. The minimum stress required for detwinning initiation is termed the detwinning start stress ($\sigma_s$). Sufficiently high load levels typically result in complete detwinning of martensite, where the corresponding stress level is called the detwinning finish stress ($\sigma_f$) [11].

Figure 1.3: Picture of the shape memory effect of SMAs showing the detwinning of the unloading and subsequent heating to austenite under no load condition.

SMAs exhibit the shape memory effect when it is deformed while in the twinned martensitic phase, and then unloaded, at a temperature below
When it is subsequently heated above the temperature level $A_f$, the SMA will regain its original shape by transforming back into the parent austenitic phase.

![Microstructure of shape-memory alloy Ni-Ti](image)

Figure 1.4: Microstructure of shape-memory alloy Ni-Ti (Courtesy of Manufacturing Research Laboratory, Columbia University, Prof. Y. Lawrence Yao).

SMAs have a lot of applications in engineering and medical areas. In the aerospace industry it has spanned the areas of fixed-wing aircraft, rotorcraft, spacecraft and work in all these areas is still progressing. The shape memory and pseudoelastic characteristics coupled with biocompatibility of NiTi make them an attractive candidate for medical applications. The combination of these unique characteristics has led to the development of various applications such as stents, filters, orthodontic wires as well as devices for minimally invasive surgery (MIS). An important requirement for an SMA, or any other material to be used in human body, is that it be biocompatible. Biocompatibility is a property of the material to remain nontoxic through its whole functional period inside the human body. A
biocompatible material should not produce any allergic reaction or inflammatory response in the host. The other requirement for such a material is its biofunctionality. This is the ability to function desirably for its expected service life in the human body environment. These two requirements are crucial for the application of SMAs in the medical industry. The shape memory alloys have been used in automobiles for applications ranging from impact absorption to sensing and actuation. Everyday applications such as coffee makers and rice cookers have also incorporated SMAs. A rice cooker equipped with an SMA valve has the valve actuate when the cooker reaches a certain temperature and releases the excess steam in the chamber [2, 3, 11].

SMA problems are attracted the attention of many mathematicians; see for instance [4] (for detailed account of known mathematical results until 1995) but also [1, 8, 7, 18] and references cited therein.

1.2 Thesis Outline

In the second chapter of this thesis, we derive a mathematical model for a valve made of shape memory alloy material. In deriving the mathematical model for the valve we mainly use the classical principles of Lagrange and of Hamilton. In the third chapter, we recall the concept of a weak solutions to the SMA problem in its fast-temperature-activation limit developed in [3]. We complement these results with a stability analysis with respect to data and parameters (which is the bulk of this thesis). In the fourth chapter, we discuss a finite difference scheme that we have used for sensitivity with respect to parameters. Furthermore, we approximate numerically the solution of the time-dependent SMA problem derived in chapter 2. We also show a couple of simulation examples. We close the thesis with a summary of our results.
Chapter 2

Modeling of Valve Made SMA Spring

In this chapter, we derive a mathematical model for a valve made of a elastic (bias) spring and a spring of a shape memory alloy material (see Figure 2.1 for a description of the valve mechanism). Such a valve made of a elastic spring and a spring of a shape memory alloy material is particularly used in rice cooking machines. We first derive mathematical model for a bias (conventional) spring and then for the spring made of both elastic and shape memory alloy springs. For deriving the mathematical model we rely on the Lagrange’s and Hamilton’s principles from classical mechanics.
2.1 Mathematical Model for the Bias Spring

We consider a bias (conventional) spring (see Figure 2.2) with a weight hangs at $x = 0$. 

Figure 2.1: Valve closes at low temperature while it opens for sufficiently high temperature (greater than a threshold temperature) as a result of SMA effect.
Let $L$ be the original length of the spring (relaxed spring), and $M$ and $m$ be the masses of the spring and of the weight respectively. Let $\ell(t)$ be the length of the spring at time $t > 0$. Here, we divide the spring into $N$ equal parts and denote by $x_i^N(t)$ the position of $i^{th}$ part at $t$. In each part of the spring we have the kinetic\(^1\) energy at time $t$ given by $\frac{1}{2} \frac{M}{N} |\dot{x}_i^N(t)|^2$, the potential\(^2\) energy is $\frac{M}{N} g x_i^N(t)$, while the elastic\(^3\) energy is $\frac{N}{2L} (x_i^N(t) - x_{i-1}^N(t) - \frac{L}{N})^2$, where $\frac{L}{N}$ is the averaging coefficient, $(x_i^N(t) - x_{i-1}^N(t) - \frac{L}{N})$ represents change in length of the partition, $g$ is the gravitational acceleration, $\kappa$ is Hooke's constant and $\dot{x}_i^N(t)$ is time derivative of $x_i^N$. Then the total kinetic energy $U_N(t)$, total potential energy $V_N(t)$ and total elastic energy $E_N(t)$ are as

\(^1\)Kinetic energy is $\frac{1}{2} m \dot{x}^2$ where $m$ is mass and $\dot{x}$ is velocity

\(^2\)Potential energy is $mgh$ where $m$ is mass, $g$ is gravity and $h$ is vertical displacement

\(^3\)Elastic energy is $\frac{1}{2} \kappa x^2$ where $\kappa$ is Hooke's constant and $x$ is displacement

Figure 2.2: Partition of the conventional spring.
follows:

\[ U_N(t) := \sum_{i=1}^{N} \frac{1}{2} \frac{M}{N} |\dddot{x}_i^N(t)|^2 + \frac{1}{2} m |\dot{\ell}(t)|^2, \]  

(2.1)

\[ V_N(t) := -\sum_{i=1}^{N} \frac{M}{N} g x_i^N(t) - m g \ell(t), \]  

and

(2.2)

\[ E_N(t) := \sum_{i=1}^{N} \frac{\kappa N}{2} \left( x_i^N(t) - x_{i-1}^N(t) - \frac{L}{N}\right)^2, \]  

(2.3)

where \( x_0(t) = 0 \) and \( \dot{\ell} \) is time derivative of \( \ell \).

\( X(t, x) \) is the position of the point \( x \in (0, L) \) at time \( t \), \( u(t, x) := X(t, x) - x \) for \( x \in (0, L) \) is the displacement, since \( x_i^N(t) - \frac{L}{N} i = u(t, \frac{L}{N} i) \), we have \( x_i^N(t) = u_i(t, \frac{L}{N} i) \). Using these formulas we can rewrite the equations (2.1), (2.2) and (2.3) as follows

\[ U_N(t) = \frac{M}{2L} \sum_{i=1}^{N} \frac{L}{N} |u_i(t, \frac{L}{N} i)|^2 + \frac{1}{2} m |\dot{\ell}(t)|^2, \]

\[ V_N(t) = -\frac{M}{L} \sum_{i=1}^{N} \frac{L}{N} g(u(t, \frac{L}{N} i)) - \frac{L}{N} i - m g \ell(t), \]

and

\[ E_N(t) = \frac{\kappa}{2} \sum_{i=1}^{N} \left( \frac{u(t, \frac{L}{N} i) - u(t, \frac{L}{N} (i-1))}{L/N} \right)^2. \]

Let \( \rho = \frac{N}{L} \). Then, as \( N \to \infty \), we obtain:

\[ U_N(t) \to \rho \int_0^L |u(t, x)|^2 dx + \frac{1}{2} m |\dot{\ell}(t)|^2, \]

\[ V_N(t) \to -\rho \int_0^L g(u(t, x) + x) dx - m g \ell(t) \]

and

\[ E_N(t) \to \frac{\kappa}{2} \int_0^L |u(t, x)|^2 dx. \]

The Lagrangian of our system, \( \mathcal{L}_N(t) \), is defined as the kinetic energy, \( U_N(t) \), of the system minus its potential energy, \( V_N(t) \) and elastic energy, \( E_N(t) \) i.e.

\[ \mathcal{L}_N(t) = U_N(t) - V_N(t) - E_N(t), \quad t \in [0, T]. \]  

(2.4)
We refer the readers to [13] for details on Lagrangian mechanics.
Passing to the limit $N \to \infty$ in (2.4) gives

$$\lim_{N \to \infty} \mathcal{L}_N(t) = \rho \int_0^L |u(t, x)|^2 dx + \frac{m}{2} |\dot{\ell}(t)|^2 + \rho \int_0^L g(u(t, x) + x) dx$$

$$+ mg\ell(t) - \frac{\kappa}{2} \int_0^L |u_x(t, x)|^2 dx := \mathcal{L}(u(t), \ell(t)), \quad t \in [0, T].$$

The Hamilton's principle or the principle of least action (see [13] page 2-4) applied to our setting says: For any $0 \leq t_1 \leq t_2 \leq T$ it holds that

$$\int_{t_1}^{t_2} \mathcal{L}(u(t), \ell(t)) dt \leq \int_{t_1}^{t_2} \mathcal{L}(v(t), s(t)) dt \quad \text{for} \quad (v, s) = (u, \ell) + \delta(\eta, q),$$

for $\delta > 0, \eta$ and $q$ are smooth functions satisfying $\eta(t_1) = \eta(t_2) = 0, q(t_1) = q(t_2) = 0, \eta(t, 0) = 0$ and $\eta(t, L) = q(t)$. This means that the integral $\int_{t_1}^{t_2} \mathcal{L}(u(t), \ell(t)) dt$ which is called action is increased when $(u(t), \ell(t))$ is replaced by any function of the form $(v(t), s(t))$ where $\delta(\eta, q)$ is a pair functions which are small everywhere in the interval time $[t_1, t_2]$; $\delta(\eta, q)$ is called the variation of the pair functions $(u(t), \ell(t))$. Then by Hamilton's principle our equation becomes

$$-\frac{\rho \delta}{2} \int_{t_1}^{t_2} \int_0^L (2u_t + \delta \eta_t) \eta_t dx dt - \frac{m \delta}{2} \int_{t_1}^{t_2} (2\ell' + \delta q') q' dt$$

$$- \rho \delta \int_{t_1}^{t_2} \int_0^L g \eta dx dt - mg \delta \int_{t_1}^{t_2} q dt + \frac{\kappa \delta}{2} \int_{t_1}^{t_2} \int_0^L (2u_x + \delta \eta_x) \eta_x dx dt \leq 0.$$

Dividing by $\delta$ and considering as $\delta \to 0$ we have

$$-\rho \int_{t_1}^{t_2} \int_0^L u_t \eta_t dx dt - m \int_{t_1}^{t_2} \ell' q' dt - \rho \int_{t_1}^{t_2} \int_0^L g \eta dx dt$$

$$- mg \int_{t_1}^{t_2} q dt + \kappa \int_{t_1}^{t_2} \int_0^L u_x \eta_x dx dt = 0.$$

Applying integration by parts we have

$$\rho \int_{t_1}^{t_2} \int_0^L u_t \eta dx dt + m \int_{t_1}^{t_2} \ell' q' dt - \rho \int_{t_1}^{t_2} \int_0^L g \eta dx dt - mg \int_{t_1}^{t_2} q dt$$

$$- \kappa \int_{t_1}^{t_2} \int_0^L u_x \eta dx dt + \kappa \int_{t_1}^{t_2} u_x \eta|_{x=0} dx dt = 0$$

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for any \( \eta \) and \( q \) with \( \eta(t, L) = q(t) \).

For \( q = 0 \), we get \( \rho u_{tt} - \kappa u_{xx} = \rho g \) in \( Q(T) := (0, T) \times (0, L) \). Then, by \( u(t, L) = \ell(t) - L \) we obtain the following system of equations for the conventional spring

\[
\begin{align*}
\rho u_{tt} - \kappa u_{xx} &= \rho g \quad \text{in} \quad Q(T) := (0, T) \times (0, L) \quad (2.5) \\
u(t, 0) &= 0 \quad \text{for} \quad 0 < t < T, \quad (2.6) \\
m\ell''(t) + \kappa u_{x}(t, L) &= mg \quad \text{for} \quad 0 < t < T, \quad (2.7) \\
u(0, x) &= u_{0}(x), \quad u_{t}(0, x) = u_{0}(x) \quad \text{for} \quad 0 \leq x \leq L, \quad (2.8)
\end{align*}
\]

where \( u_{0} \) and \( u_{v} \) are the initial functions of the displacement.

### 2.2 Derivation of a Mathematical Model for a Valve Made of SMA

We model a device made of two kinds of spring: a SMA spring and a bias elastic spring with the temperature field \( \theta \) is given, see Figure 2.3 for the basic geometry we have in mind.

![Figure 2.3: Sketch of the two spring.](image)
Let $L_1$ and $L_2$ be the original lengths of the two springs, $M_1$ and $M_2$ be the masses of two springs, respectively. They are joined and fixed at $x = 0$ and $x = L$, $0 < L < L_1 + L_2 := L_3$. Let $\ell(t)$ be the length of of the length of SMA spring at time $t$, then the length of the bias spring is $L - \ell(t)$. Let $u_1 : (0, T) \times (0, L_1) \rightarrow \mathbb{R}$ and $u_2 : (0, T) \times (L_1, L_3) \rightarrow \mathbb{R}$ be the displacement of the SMA and bias springs respectively. Then the total kinetic energy $U(t)$ of the device is

$$U(t) = \frac{\rho_1}{2} \int_0^{L_1} |u_{1t}(t, x)|^2 \, dx + \frac{\rho_2}{2} \int_{L_1}^{L_3} |u_{2t}(t, x)|^2 \, dx + \frac{m}{2} |\ell(t)|^2,$$

where $\rho_1 = \frac{M_1}{L_1}$ and $\rho_2 = \frac{M_2}{L_2}$, and $m$ is the mass of joint. The total potential energy $V(t)$ is given by

$$V(t) = -\rho_1 \int_0^{L_1} g(u_1(t, \xi) + \xi) \, d\xi - \rho_2 \int_{L_1}^{L_3} g(u_2(t, \xi) + \xi) \, d\xi - mg\ell(t),$$

for $t \in (0, T)$. According to the Falk’s idea the free energy of the SMA is formulated as follows (for more details see [4], chapter 5, [6]): the shear strain $\epsilon$ is defined by

$$\epsilon(t, x) := \frac{\partial u_1}{\partial x}(t, x), \text{ where } (t, x) \in (0, T) \times (0, L_1). \quad (2.9)$$

The local free energy density $F = F(\epsilon, \epsilon_x, \theta)$ in the SMA spring is chosen as

$$F(\epsilon, \epsilon_x, \theta) = F_0(\theta) + \alpha_1 (\theta - \theta_c) \epsilon^2 - \alpha_2 \epsilon^4 + \alpha_3 \epsilon^6 + \gamma \epsilon_x^2,$$

where $\gamma, \alpha_1, \alpha_2$ and $\alpha_3$ are positive constants, $\theta_c$ is some critical temperature, and $F_0$ is a given continuous function of $\theta$. Assume that the elastic energy density function of the bias spring doesn’t depend on $\theta$ and the stress is linear function of the strain. The total free energy of the device $\mathcal{F}(t)$ is given by

$$\mathcal{F}(t) = \int_0^{L_1} F(\epsilon(t, x), \epsilon_x(t, x), \theta) \, dx + \int_{L_1}^{L_3} |u_{2x}(t, x)|^2 \, dx,$$
where $\kappa$ is the Hooke’s constant of the bias spring. Thus, the Lagrangian for the device is given by

$$
\mathcal{L}(u_1, u_2, \ell) = U(t) - V(t) - \mathcal{F}(t)
$$

$$
= \frac{\rho_1}{2} \int_{0}^{L_1} |u_{11}(t, x)|^2 dx + \frac{\rho_2}{2} \int_{L_1}^{L_3} |u_{22}(t, x)|^2 dx + \frac{m}{2} |\ell(t)|^2
$$

$$
+ \rho_1 \int_{0}^{L_1} g(u_1(t, \xi) + \xi) d\xi + \rho_2 \int_{L_1}^{L_3} g(u_2(t, \xi) + \xi) d\xi + mg\ell(t)
$$

$$
- \int_{0}^{L_1} F(\varepsilon(t, x), \varepsilon_a(t, x), \theta) dx - \frac{\kappa}{2} \int_{L_1}^{L_3} |u_{22}(t, x)|^2 dx.
$$

By the Hamilton principle (see [13], page 2-4), for $0 \leq t_1 \leq t_2 \leq T$ we have

$$
\int_{t_1}^{t_2} \mathcal{L}(u_1(t), u_2(t), \ell(t)) dt \leq \int_{t_1}^{t_2} \mathcal{L}(v_1(t), v_2(t), q(t)) dt
$$

for $v_i = u_i + \delta \eta_i \ (i = 1, 2)$ and $q = \ell + \delta s$ with $\eta_1$, $\eta_2$ and $s$ satisfying the property (A0), where

(A0): $\eta_1$ and $\eta_2$ are smooth functions on $[t_1, t_2] \times (0, L_1)$ and $[t_1, t_2] \times (L_1, L_3)$, respectively, with $\eta_i(t_1) = \eta_i(t_2) = 0$, $i = 1, 2$, $s$ is a smooth function on $[t_1, t_2]$ with $s(t_1) = s(t_2) = 0$, $\eta_i(t, 0) = 0$, $\eta_i(t, L_1) = s(t)$, $\eta_2(t, L_1) = s(t)$, $\eta_2(t, L_3) = 0$ and $\delta$ is any positive real number.

This yields:

$$
\int_{t_1}^{t_2} \mathcal{L}(u_1(t), u_2(t), \ell(t)) dt - \int_{t_1}^{t_2} \mathcal{L}(v_1(t), v_2(t), q(t)) dt =
$$

$$
- \frac{\rho_1}{2} \int_{t_1}^{t_2} \int_{0}^{L_1} (2u_{11} + \delta \eta_1) \eta_1 dx dt - \frac{\rho_2}{2} \int_{t_1}^{t_2} \int_{L_1}^{L_3} (2u_{22} + \delta \eta_2) \eta_2 dx dt
$$

$$
- \frac{m}{2} \int_{t_1}^{t_2} \int_{0}^{L_1} (2\ell + \delta s') s' dt - \rho_1 \int_{t_1}^{t_2} \int_{0}^{L_1} \eta_1 \eta_1 dx dt - \rho_2 \int_{t_1}^{t_2} \int_{L_1}^{L_3} \eta_2 \eta_2 dx dt
$$

$$
- mg \int_{t_1}^{t_2} s dt + \frac{\kappa}{2} \int_{t_1}^{t_2} \int_{L_1}^{L_3} (2u_{22} + \delta \eta_2) \eta_2 dx dt +
$$

$$
\int_{t_1}^{t_2} \int_{0}^{L_1} [\alpha_1(\theta - \theta_c)(2\varepsilon + \delta \frac{\partial \eta_1}{\partial x}) \delta \frac{\partial \eta_1}{\partial x} + \alpha_2(\epsilon^4 - (\varepsilon + \delta \frac{\partial \eta_1}{\partial x})^4) +
$$

$$
\alpha_3(-\varepsilon^6 + (\varepsilon + \delta \frac{\partial \eta_1}{\partial x})^6)] dx dt + \int_{t_1}^{t_2} \int_{0}^{L_1} \frac{\gamma}{2} (2\varepsilon_a + \delta \frac{\partial \eta_1}{\partial x}) \delta \frac{\partial \eta_1}{\partial x} dx dt \leq 0.
$$
Dividing by $\delta$ and considering $\delta \to 0$, we have

$$
-\rho_1 \int_{t_1}^{t_2} \int_{L_1}^{L_1} u_{1,\eta_1} \, dx \, dt - \rho_2 \int_{t_1}^{t_2} \int_{L_1}^{L_3} u_{2,\eta_2} \, dx \, dt - m \int_{t_1}^{t_2} 2\ell' s' \, dt
$$

$$
-\rho_1 \int_{t_1}^{t_2} \int_{L_1}^{L_1} g_{1,\eta_1} \, dx \, dt - \rho_2 \int_{t_1}^{t_2} \int_{L_1}^{L_3} g_{2,\eta_2} \, dx \, dt - mg \int_{t_1}^{t_2} s d \, dt
$$

$$
+ \kappa \int_{t_1}^{t_2} \int_{L_1}^{L_3} u_{2x,\eta_2} \, dx \, dt + \int_{t_1}^{t_2} \int_{L_1}^{L_1} \left( \alpha_1 (\theta - \theta_c) 2\varepsilon - 4\alpha_2 \varepsilon^3 + 6\alpha_3 \varepsilon^5 \right) \frac{\partial \eta_1}{\partial x} \, dx \, dt
$$

$$
+ \int_{t_1}^{t_2} \int_{L_1}^{L_1} \gamma \varepsilon_{\eta_1} \frac{\partial^2 \eta_1}{\partial x^2} \, dx \, dt \leq 0. \quad (2.10)
$$

By integration by parts in (2.10) we have

$$
\rho_1 \int_{t_1}^{t_2} \int_{L_1}^{L_1} u_{1,\eta_1} \, dx \, dt + \rho_2 \int_{t_1}^{t_2} \int_{L_1}^{L_3} u_{2,\eta_2} \, dx \, dt + m \int_{t_1}^{t_2} 2\ell' s' \, dt
$$

$$
- \rho_1 \int_{t_1}^{t_2} \int_{L_1}^{L_1} g_{1,\eta_1} \, dx \, dt - \rho_2 \int_{t_1}^{t_2} \int_{L_1}^{L_3} g_{2,\eta_2} \, dx \, dt - mg \int_{t_1}^{t_2} s d \, dt
$$

$$
- \kappa \int_{t_1}^{t_2} \int_{L_1}^{L_3} u_{2x,\eta_2} \, dx \, dt + \kappa \int_{t_1}^{t_2} \int_{L_1}^{L_3} \eta_2 \, dx \, dt
$$

$$
- \int_{t_1}^{t_2} \int_{L_1}^{L_1} \left( \alpha_1 (\theta - \theta_c) 2\varepsilon - 4\alpha_2 \varepsilon^3 + 6\alpha_3 \varepsilon^5 \right) \eta_1 \, dx \, dt
$$

$$
+ \int_{t_1}^{t_2} \int_{L_1}^{L_1} \gamma \varepsilon_{\eta_1} \frac{\partial \eta_1}{\partial x} \, dx \, dt - \int_{t_1}^{t_2} \gamma \varepsilon_{\eta_1} \eta_1 \, dx \, dt = 0
$$

for any $\eta_1$, $\eta_2$ and $s$ satisfy property (A0). For $s = 0$, thus

$$
\rho_1 \int_{t_1}^{t_2} \int_{L_1}^{L_1} u_{1,\eta_1} \, dx \, dt + \rho_2 \int_{t_1}^{t_2} \int_{L_1}^{L_3} u_{2,\eta_2} \, dx \, dt - \rho_1 \int_{t_1}^{t_2} \int_{L_1}^{L_1} g_{1,\eta_1} \, dx \, dt
$$

$$
- \rho_2 \int_{t_1}^{t_2} \int_{L_1}^{L_1} g_{2,\eta_2} \, dx \, dt - \kappa \int_{t_1}^{t_2} \int_{L_1}^{L_3} u_{2x,\eta_2} \, dx \, dt
$$

$$
- \int_{t_1}^{t_2} \int_{L_1}^{L_1} \left( \alpha_1 (\theta - \theta_c) 2\varepsilon - 4\alpha_2 \varepsilon^3 + 6\alpha_3 \varepsilon^5 \right) \eta_1 \, dx \, dt
$$

$$
+ \int_{t_1}^{t_2} \int_{L_1}^{L_1} \gamma \varepsilon_{\eta_1} \eta_1 \, dx \, dt = 0.
$$
Then replacing the shear strain $\varepsilon$ as defined in (2.9) by $u_{1z}$, we get the following model equations for $(u_1, u_2, \ell)$:

\begin{align*}
\rho_1 u_{1tt} + \gamma u_{1zzz} - (2\alpha_1 \theta u_{1z} + \psi(u_{1z}))_z &= \rho_1 g \text{ in } \Omega \quad (2.11) \\
\rho_2 u_{2tt} - \kappa u_{2xx} &= \rho_2 g \text{ in } (0, T) \times (L_1, L_3) \quad (2.12) \\
m\ell'(t) &= mg + (\gamma u_{1xx} - 2\alpha_1 \theta u_{1z} - \psi(u_{1z}))(t, L_1) + \kappa u_{2x}(t, L_1) \text{ for } 0 < t < T, \quad (2.13) \\
u_1(t, 0) &= 0, u_2(t, L_3) = L - L_3 \text{ for } 0 < t < T, \quad (2.14) \\
u_{1xx}(t, 0) &= u_{1xx}(t, L_1) = 0 \text{ for } 0 < t < T, \quad (2.15) \\
u_1(t, L_1) &= u_2(t, L_1), \ell(t) = u_1(t, L_1) + L_1 \text{ for } 0 < t < T, \quad (2.16) \\
u_1(0, x) &= u_{01}(x), u_1(0, x) = v_{01}(x) \text{ for } 0 \leq x \leq L_1, \quad (2.17) \\
u_2(0, x) &= u_{02}(x), u_2(0, x) = v_{02}(x) \text{ for } L_1 \leq x \leq L_3. \quad (2.18)
\end{align*}

where $\psi(r) = -2\alpha_1 \theta r - 4\alpha_2 r^3 + 6\alpha_3 r^5$, $\Omega := (0, T) \times (0, L_1)$ and $u_{01}$, $u_{02}$, $v_{01}$, and $v_{02}$ are the initial functions of displacements.

### 2.3 Shape Memory Alloy Problem

Considering $\rho_2 = 0$ in (2.11)–(2.18) and replacing $u_1$ by $u$ results in the following system of PDEs, that we refer as shape memory alloy problem. Find the displacement $u(t, x)$ and the valve position $\ell(t)$ such that

\begin{align*}
\rho_1 u_{tt} + \gamma u_{xxx} - (2\alpha_1 \theta u_x + \psi(u_x))_x &= \rho_1 g \text{ in } \Omega \quad (2.19) \\
m\ell'(t) &= mg + (\gamma u_{xxx} - 2\alpha_1 \theta u_x - \psi(u_x))(t, L_1) \text{ for } 0 < t < T, \quad (2.20) \\
u(t, 0) &= 0, \ u_{xx}(t, 0) = u_{xx}(t, L_1) = 0, \text{ for } 0 < t < T, \quad (2.21) \\
u(t, L_1) &= \ell(t) - L_1 \text{ for } 0 < t < T, \quad (2.22) \\
u(0, x) &= u_0(x), u_1(0, x) = v_0(x) \text{ for } 0 \leq x \leq L_1, \quad (2.23)
\end{align*}

where $\psi(r) = -2\alpha_1 \theta r - 4\alpha_2 r^3 + 6\alpha_3 r^5$, $\Omega := (0, T) \times (0, L_1)$ and $u_0$, and $v_0$ are the initial functions of displacement $u$ and $\theta$ is the local temperature. Including given initial conditions for both $u(t, x)$ and $\ell(t)$ we rewrite the
above problem as follows:
Find $u(t, x)$ and $\ell(t)$ such that

$$
\rho_1 u_{tt} + \gamma u_{xxx} - (2\alpha_1 \theta u_x + \psi(u_x))_x = \rho_1 g \quad \text{in} \ \Omega \quad (2.24)
$$

$$
m\ell'(t) = mg + (\gamma u_{xxx} - 2\alpha_1 \theta u_{xx} - \psi(u_x))(t, L_1) \quad (2.25)
$$

for $0 < t < T$, 

$$
u(t, 0) = 0, \quad u_{xx}(t, 0) = u_{xx}(t, L_1) = 0, \quad \text{for} \ 0 < t < T, \quad (2.26)
$$

$$
u(t, L_1) = \ell(t) - L_1 \quad \text{for} \ 0 < t < T, \quad (2.27)
$$

$$
u(0, x) = \frac{\ell(0) - L_1}{L_1} x, \quad u_t(0, x) = 0, \quad \text{for} \ 0 \leq x \leq L_1, \quad (2.28)
$$

$$
\ell(0) = \ell_0, \quad \ell'(0) = 0 \quad \text{for} \ 0 \leq x \leq L_1, \quad (2.29)
$$

where $\psi(r) = -2\alpha_1 \theta \psi - 4\alpha_2 r^3 + 6\alpha_3 r^5$, $\Omega := (0, T) \times (0, L_1)$ and $\theta$ is the local temperature.

### 2.4 Shape Memory Alloy Problem in its Fast-Temperature-Activation Limit

The SMA problem (2.24)–(2.29) is mathematically quite complicated due to its hidden hyperbolic structure [1, 4, 18]. To get a better understanding on the way the SMA effect works, we consider the less complicated problem (2.30)–(2.33) the fast-temperature-activation limit of (2.24)–(2.29) derived in [3]. Specifically, if we let $2\alpha_1 = \alpha$ and consider the limit $\alpha \to \infty$ in (2.24)–(2.29), we have the fast-temperature-activation limit problem (also known as reduced model) which is nonlinear elliptic PDE: For $t > 0$,
find \( u(t, x) \) such that

\[
- (\theta u_x + \frac{1}{\alpha} \psi(u_x))_x = 0 \text{ in } (0, L_1),
\]

\[
\frac{\kappa}{\alpha} \frac{L - (L_1 + L_2)}{L_2} - \left( \frac{\theta u_x + \frac{1}{\alpha} \psi(u_x)}{\alpha} \right) (t, L_1)
\]

\[
+ \frac{\kappa}{L_2} L_1 - \ell(t) = 0,
\]

\[
u(t, 0) = 0, \quad u(t, L_1) = \ell(t) - L_1,
\]

\[
\ell(0) = \ell_0.
\]

Interestingly, the fast-temperature-activation limit problem is very much similar to the SMA problem in the sense that the shape memory effect is still captured at this fast time scale. In Figure 2.4, we see a plot of the position of the valve \( \ell(t) \) as a function of time \( t \). It exhibits a jump for the position of the valve \( \ell(t) \) which indicates the opening of the valve at a moment \( t^* \in (0.2, 0.4) \).

![Graph](image)

Figure 2.4: Position of the valve \( \ell \) as a function of time \( t \). The plot is taken from [3].
Chapter 3

Analysis of Shape Memory Alloy Problem in its Fast-Temperature-Activation Limit

According to Hadamard's definition of well-posedness of a PDEs [10], a problem is well-posed if at least one solution of the problem exists, the solution is unique and it depends continuously on the data and its parameters (stability). In these chapter we consider the well-posedness fast-temperature-activation limit problem (2.30)–(2.33). The existence and uniqueness of weak solutions to this problem were already shown in [3]. Here we complete the results reported in [3] with a stability analysis with respect to data and parameters. Such a stability analysis will make (2.30)–(2.33) well-posed in the sense of Hadamard definition of well-posedness. So the main thing we are going to show in this chapter is the stability of fast-temperature-activation limit problem.
3.1 Function Spaces and Inequalities

In this section, we introduce the function spaces and inequalities we use in the coming section.

\[ L^\infty(0, L_1) := \{ v : \sup_{x \in (0, L_1)} |v| < \infty \} \]

with norm

\[ \|u\|_{L^\infty(0, L_1)} := \sup_{x \in (0, L_1)} |v|. \]

\[ W^{1, \infty}(0, L_1) := \{ v : v \in L^\infty(0, L_1) \text{ and } v' \in L^\infty(0, L_1) \} \]

with norm

\[ \|u\|_{W^{1, \infty}(0, L_1)} := \sup_{x \in (0, L_1)} |u| + \sup_{x \in (0, L_1)} |u'|. \]

\[ H := L^2(0, L_1) := \{ v : \int_0^{L_1} |v|^2 < \infty \} \]

with inner product

\[ (u, v) := \int_0^{L_1} uv dx \]

and norm

\[ \|u\|_{L^2(0, L_1)}^2 := \int_0^{L_1} |u|^2 dx. \]

\[ H^1(0, L_1) := \{ v : v \in L^2(0, L_1) \text{ and } v' \in L^2(0, L_1) \} \]

with inner product

\[ (u, v) := \int_0^{L_1} uv dx + \int_0^{L_1} u'v' dx \]

and norm

\[ \|u\|_{H^1(0, L_1)}^2 := \int_0^{L_1} |u|^2 dx + \int_0^{L_1} |u'|^2 dx. \]

\[ H^2(0, L_1) := \{ v : v \in L^2(0, L_1), v' \in L^2(0, L_1) \text{ and } v'' \in L^2(0, L_1) \} \]

with inner product

\[ (u, v) := \int_0^{L_1} uv dx + \int_0^{L_1} u'v' dx + \int_0^{L_1} u''v'' dx \]
and norm

\[ \|u\|_{H^2(0,L_1)}^2 := \int_0^{L_1} |u|^2 \, dx + \int_0^{L_1} |u'|^2 \, dx + \int_0^{L_1} |u''|^2 \, dx. \]

\[ V := \{ v \in H^1(0,L_1) : v(0) = 0 \}. \]

In the proof of the following theorems we use the following basic inequalities: the Young’s inequality, for any \( \delta \) and \( \beta \)

\[ \delta \beta \leq \frac{1}{2\varepsilon} \delta^2 + \frac{\varepsilon}{2} \beta^2 \text{ for } \varepsilon \in (0,\infty), \]  \( (3.1) \)

the interpolation inequality [19], if \( u \in H^1(0,L_1) \), then there exists a constant \( \tilde{c} \):

\[ |u(L_1)| \leq \tilde{c} \|u\|_{H^1(0,L_1)}^\xi \|u\|_{L^2(0,L_1)}^{1-\xi} \text{ for } \xi \in \left[ \frac{1}{2},1 \right), \]  \( (3.2) \)

the trace inequality, if \( u \in H^1(0,L_1) \) then, there exists a constant \( C_T \):

\[ |u(L_1)| \leq C_T \|u\|_{H^1(0,L_1)}, \]  \( (3.3) \)

and for \( u(0) = 0, \|u\|_{H^1(0,L_1)} \) is equivalent to \( \|u_w\|_{L^2(0,L_1)} \) i.e. there exist two positive constants, \( c_1 \) and \( c_2 \) such that

\[ c_1 \|u\|_{H^1(0,L_1)} \leq \|u_w\|_{L^2(0,L_1)} \leq c_2 \|u\|_{H^1(0,L_1)}. \]

### 3.2 List of Assumptions

(A1): \( \psi(r) = \psi_1(r) + \psi_2(r) \), \( \psi_1(r) \) is a monotone increasing and locally Lipschitz continuous function satisfying \( \psi_1(0) = 0 \) and \( \psi_1'(r) \geq c|r|^q \) for any \( r \in \mathbb{R}, |\psi_1(r) - \psi_1(r')| \leq L(M)|r - r'| \) for \( M > 0, |r|,|r'| \leq M \), where \( q \geq 2 \) and \( c \) is a positive constant, \( L(M) \) is a positive constant depending on \( M \), and \( \psi_2(r) \in C^2(\mathbb{R}) \) and \( \psi_2 \) and \( \psi_2' \) are Lipschitz continuous on \( \mathbb{R} \) satisfying \( \psi_2(|r|) \leq C_{\psi}(|r| + 1), |\psi_2'(r) - \psi_2'(r')| \leq C_{\psi}'|r - r'| \) for \( r, r' \in \mathbb{R} \), where \( C_{\psi} \) is a positive constant.

(A2): \( \theta \in W^{1,\infty}(0,L_1) \cap L^\infty(0,L_1) \) and \( \theta \geq \theta_c \) on \( (0,L_1) \) for some strictly positive constant \( \theta_c \).
(A3): $\alpha, \kappa, L_1, L_2, L$ are real strictly positive constants.

(A4): Assume $\psi(r)$ is Lipschitz continuous function i.e. there exist a constant $K$ such that $|\psi(r) - \psi(r')| \leq K|r - r'|$ with $\psi(0) = 0$ for $r \in \mathbb{R}$.

### 3.3 Review of Known Results

We refer to the system (2.30)-(2.33) as a problem (P).

**Definition 1.** The weak formulation of the problem (P) is defined as below:
Find $u \in V \cap H^2(0, L_1)$ such that

$$
(\theta u_x + \frac{1}{\alpha} \psi(u_x), \eta_x) + \frac{\kappa}{\alpha L_2} u(L_1) \eta(L_1) = \frac{\kappa}{\alpha L_2} (L - L_1 - L_2) \eta(L_1), \text{ for all } \eta \in V.
$$

(3.4)

**Theorem 3.1.** Assume (A1)-(A3) to be fulfilled. If $\alpha$ is sufficiently large, then there exist at least a weak solution $u \in V \cap H^2(0, L_1)$ to the problem (P).

**Theorem 3.2.** Assume (A1)-(A3) to be fulfilled. Additionally $\alpha > \frac{c_0}{\kappa \delta_x}$. Then problem (P) has at most a weak solution.

So if the assumptions (A1)-(A3) holds for the problem (P), then there exist a unique weak solution with

$$
\|u\|_{H^2(0, L_1)} \leq M
$$

(3.5)

(for the proofs and details we refer the reader to [3] pp. 4-7).

### 3.4 A Stability Result

We denote by $\mathcal{M}$ and $\mathcal{N}$ the following compact sets: $\mathcal{M} := [\alpha_{\min}, \alpha_{\max}]$ and $\mathcal{N} := [\kappa_{\min}, \kappa_{\max}]$ for $\alpha_{\min}, \alpha_{\max}, \kappa_{\min}, \kappa_{\max} \in (0, \infty)$. 


Theorem 3.3. Let $u_1$ and $u_2$ be two arbitrary weak solutions to the system (2.30)–(2.33). Assume (A1)-(A3). If (A4) also holds, then there exist a constant $C$ such that

$$
\|u_1 - u_2\|^2_{H^1(0,L_1)} \leq C \left( \|\theta_2 - \theta_1\|^2_{L^\infty(0,L_1)} + \max_{\alpha_1 \in M} |\alpha_1 - \alpha_2|^2 + \max_{\kappa_1, \kappa_2 \in \mathcal{M}} \|\kappa_1 - \kappa_2\|^2 + \|f_1 - f_2\|^2_{L^2(0,L_1)} \right). \tag{3.6}
$$

Furthermore,

$$
|\ell_1(t) - \ell_2(t)|^2 \leq \hat{c}C \left( \|\theta_2 - \theta_1\|^2_{L^\infty(0,L_1)} + \max_{\alpha_1 \in M} |\alpha_1 - \alpha_2|^2 + \max_{\kappa_1, \kappa_2 \in \mathcal{M}} \|\kappa_1 - \kappa_2\|^2 + \|f_1 - f_2\|^2_{L^2(0,L_1)} \right), \tag{3.7}
$$

where $C$ is the constant entering (3.6) and $\hat{c}$ is the constant entering the interpolation inequality (3.2).

Proof: Since $u_1$ and $u_2$ are weak solution in the sense of the Definition 1, they satisfy

$$
(\theta_1 u_{1z} + \frac{1}{\alpha_1} \psi(u_{1z}), \eta_x) + \frac{\kappa_1}{\alpha_1 L_2} u_1(L_1) \eta(L_1) = \frac{\kappa_1}{\alpha_1 L_2} (L - L_1 - L_2) \eta(L_1) + (f_1, \eta), \quad \text{for all } \eta \in V \tag{3.8}
$$

and

$$
(\theta_2 u_{2z} + \frac{1}{\alpha_2} \psi(u_{2z}), \eta_x) + \frac{\kappa_2}{\alpha_2 L_2} u_2(L_1) \eta(L_1) = \frac{\kappa_2}{\alpha_2 L_2} (L - L_1 - L_2) \eta(L_1) + (f_2, \eta), \quad \text{for all } \eta \in V. \tag{3.9}
$$

Subtracting (3.9) from (3.8) leads to

$$
(\theta_1 u_{1z} - \theta_2 u_{2z} + \frac{1}{\alpha_1} \psi(u_{1z}) - \frac{1}{\alpha_2} \psi(u_{2z}), \eta_x) + \left( \frac{\kappa_1}{\alpha_1 L_2} u_1(L_1) - \frac{\kappa_2}{\alpha_2 L_2} u_2(L_1) \right) \eta(L_1) = \left( \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right) (L - L_1 - L_2) \eta(L_1) + (f_1 - f_2, \eta).
$$

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Hence, we have

$$
\theta_1(u_{1x} - u_{2x}, \eta_x) = (\theta_2 - \theta_1)(u_{2x}, \eta_x) + \left( \frac{1}{\alpha_2} \psi(u_{2x}) - \frac{1}{\alpha_1} \psi(u_{1x}) \right)(\eta_x)
+ \left( \frac{\kappa_2}{\alpha_2 L_2} - \frac{\kappa_1}{\alpha_1 L_2} \right) u_{2x}(L_1) \eta(L_1)
+ \left( \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right) (L - L_1 - L_2) \eta(L_1) + (f_1 - f_2, \eta).
$$

Now, choosing $\eta := u_1 - u_2$ we have

$$
|\theta_2 - \theta_1(u_{2x}, \eta_x)| = |(\theta_2 - \theta_1)(u_{2x}, u_{1x} - u_{2x})|
\leq |\theta_2 - \theta_1|_{L^\infty(0, L_1)} |u_{2x}|_{L^2(0, L_1)} |u_{1x} - u_{2x}|_{L^2(0, L_1)}
\leq M |\theta_2 - \theta_1|_{L^\infty(0, L_1)} |u_{1x} - u_{2x}|_{L^2(0, L_1)}
\quad \text{(for $M$ as in (3.5))}
\leq \frac{M^2}{2\varepsilon} |\theta_2 - \theta_1|_{L^2(0, L_1)}^2 + \frac{\varepsilon}{2} |u_{1x} - u_{2x}|_{L^2(0, L_1)}^2,
\quad \text{(by (3.1))}
$$

$$
\left| \left( \frac{1}{\alpha_2} \psi(u_{2x}) - \frac{1}{\alpha_1} \psi(u_{1x}) \right)(\eta_x) \right|
= \left| \left( \frac{1}{\alpha_2} \psi(u_{2x}) - \frac{1}{\alpha_1} \psi(u_{1x}) \right), u_{1x} - u_{2x} \right|
\leq \left\| \left( \frac{1}{\alpha_2} \psi(u_{2x}) - \psi(u_{1x}) \right) \right\|_{L^2(0, L_1)}
+ \left\| \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \psi(u_{1x}) \right\|_{L^2(0, L_1)}
\quad |u_{1x} - u_{2x}|_{L^2(0, L_1)}
\leq K \frac{1}{\alpha_2} |u_{1x} - u_{2x}|_{L^2(0, L_1)}
+ \left\| \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \psi(u_{1x}) \right\|_{L^2(0, L_1)}
\quad \left( u_{1x} - u_{2x} \right)_{L^2(0, L_1)}.
$$

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\[
\left| \frac{1}{\alpha_2} \psi(u_{2e}) - \frac{1}{\alpha_1} \psi(u_{1e}), \eta_e \right| \leq K \frac{1}{\alpha_2} \| u_{1e} - u_{2e} \|_{L^2(0,L_1)}^2
\]
\[+ \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \left| K \| u_{1e} \|_{L^2(0,L_1)} \right| \| u_{1e} - u_{2e} \|_{L^2(0,L_1)} \quad \text{(because of (A4))} \]
\[\leq K \frac{1}{\alpha_2} \| u_{1e} - u_{2e} \|_{L^2(0,L_1)}^2 \]
\[+ KM \left| \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right| \| u_{1e} - u_{2e} \|_{L^2(0,L_1)} \]
\[\leq K \frac{1}{\alpha_2} \| u_{1e} - u_{2e} \|_{L^2(0,L_1)} \]
\[+ \frac{\varepsilon}{2} \| u_{1e} - u_{2e} \|_{L^2(0,L_1)}^2 + \frac{(KM)^2}{2\varepsilon} \left| \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right|^2 \]
\[= (K \frac{1}{\alpha_2} + \frac{\varepsilon}{2}) \| u_{1e} - u_{2e} \|_{L^2(0,L_1)}^2 \]
\[+ \frac{(KM)^2}{2\varepsilon} \left| \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right|^2.
\]

\[
\left| \left( \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right) (L - L_1 - L_2) \eta(L_1) \right| = \left| \left( \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right) (L - L_1 - L_2) \right|
\]
\[\left| (u_1(L_1) - u_2(L_1)) \right| \]
\[\leq \left| \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right| \| L - L_1 - L_2 \|
\]
\[\| u_1(L_1) - u_2(L_1) \|, \]
\[\left| \left( \frac{\kappa_2}{\alpha_2 L_2} u_2(L_1) - \frac{\kappa_1}{\alpha_1 L_2} u_1(L_1) \right) \eta(L_1) \right| = \left| \left( \frac{\kappa_2}{\alpha_2 L_2} u_2(L_1) - \frac{\kappa_1}{\alpha_1 L_2} u_1(L_1) \right) \right|
\]
\[\left| (u_1(L_1) - u_2(L_1)) \right| \]
\[\leq \left| \frac{\kappa_2}{\alpha_2 L_2} u_2(L_1) - \frac{\kappa_1}{\alpha_1 L_2} u_1(L_1) \right| \| u_1(L_1) - u_2(L_1) \| \]
\[\leq \frac{\kappa_2}{\alpha_2 L_2} \| u_1(L_1) - u_2(L_1) \|^2 + \left| \frac{\kappa_2}{\alpha_2 L_2} - \frac{\kappa_1}{\alpha_1 L_2} \right| \| u_1(L_1) \| \| u_1(L_1) - u_2(L_1) \|,
\]
\[(f_1 - f_2, \eta) = (f_1 - f_2, u_1 - u_2)\]
\[\leq \| f_1 - f_2 \|_{L^2(a, L_1)} \| u_{1e} - u_{2e} \|_{L^2(0, L_1)}\]
\[\leq \frac{1}{2\varepsilon} \| f_1 - f_2 \|_{L^2(a, L_1)}^2 + \frac{\varepsilon}{2} \| u_{1e} - u_{2e} \|_{L^2(0, L_1)}^2,
\]
which implies
\[\theta_1 \| u_{1e} - u_{2e} \|_{L^2(0, L_1)}^2 \leq \frac{M^2}{2\varepsilon} \| \theta_2 - \theta_1 \|_{L^2(0, L_1)}^2 + \frac{\varepsilon}{2} \| u_{1e} - u_{2e} \|_{L^2(0, L_1)}^2\]
\[+ \frac{K}{\alpha_2} \frac{\varepsilon}{2} \| u_{1e} - u_{2e} \|_{L^2(0, L_1)}^2 + \frac{(KM)^2}{2\varepsilon} \left( \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right)^2\]
\[+ \frac{\kappa_2}{\alpha_2 L_2} \| u_{1e} - u_{2e} \|_{L^2(0, L_1)}^2\]
\[+ \frac{\kappa_2}{\alpha_2 L_2} \frac{\kappa_1}{\alpha_1 L_2} \| u_{1e} - u_{2e} \|_{L^2(0, L_1)}^2\]
\[+ \frac{\kappa_1}{\alpha_1 L_2} \| u_{1e} - u_{2e} \|_{L^2(0, L_1)}^2\]
\[+ \frac{1}{2\varepsilon} \| f_1 - f_2 \|_{L^2(0, L_1)}^2 + \frac{\varepsilon}{2} \| u_{1e} - u_{2e} \|_{L^2(0, L_1)}^2\]
\[= \left( \frac{K}{\alpha_2} + \frac{3\varepsilon}{2} \right) \| u_{1e} - u_{2e} \|_{L^2(0, L_1)}^2 + \frac{M^2}{2\varepsilon} \| \theta_2 - \theta_1 \|_{L^2(0, L_1)}^2\]
\[+ \frac{(KM)^2}{2\varepsilon} \left( \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right)^2\]
\[+ \frac{\kappa_2}{\alpha_2 L_2} \frac{\kappa_1}{\alpha_1 L_2} \| u_{1e} - u_{2e} \|_{L^2(0, L_1)}^2\]
\[+ \frac{\kappa_1}{\alpha_1 L_2} \| u_{1e} - u_{2e} \|_{L^2(0, L_1)}^2\]
\[+ \frac{1}{2\varepsilon} \| f_1 - f_2 \|_{L^2(0, L_1)}^2\]

By applying the trace inequality, we get

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\[
\theta_1 \| u_{1x} - u_{2x} \|_{L^2(0, L)}^2 \leq \left( \frac{K}{\alpha_2} + \frac{3\varepsilon}{2} \right) \| u_{1x} - u_{2x} \|_{L^2(0, L)}^2 + \frac{M^2}{2\varepsilon} \| \theta_2 - \theta_1 \|_{L^\infty(0, L)}^2 \\
+ \frac{(KM)^2}{2\varepsilon} \left( \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right)^2 + C_2^2 \frac{\kappa_2}{\alpha_2 L_2} \| u_1 - u_2 \|_{H^1(0, L)}^2 \\
+ C_2^2 \frac{\kappa_2 - \kappa_1}{\alpha_2 L_2} \left| \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right| \| u_1 \|_{H^1(0, L)} \| u_1 - u_2 \|_{H^1(0, L)} \\
+ C_2 \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \left| L - L_1 - L_2 \right| \| u_1 - u_2 \|_{H^1(0, L)} \\
\left\| f_1 - f_2 \right\|_{L^2(0, L)}^2 \leq \left( \frac{K}{\alpha_2} + \frac{3\varepsilon}{2} \right) \| u_{1x} - u_{2x} \|_{L^2(0, L)}^2 + \frac{M^2}{2\varepsilon} \| \theta_2 - \theta_1 \|_{L^\infty(0, L)}^2 \\
+ \frac{(KM)^2}{2\varepsilon} \left( \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right)^2 + C_2^2 \frac{\kappa_2}{\alpha_2 L_2} \| u_1 - u_2 \|_{H^1(0, L)}^2 \\
+ M C_2 \left| \frac{\kappa_2}{\alpha_2 L_2} - \frac{\kappa_1}{\alpha_1 L_2} \right| \| u_1 \|_{H^1(0, L)} \| u_1 - u_2 \|_{H^1(0, L)} \\
+ C_2 \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \left| L - L_1 - L_2 \right| \| u_1 - u_2 \|_{H^1(0, L)} \\
\left\| f_1 - f_2 \right\|_{L^2(0, L)}^2 \leq \left( \frac{K}{\alpha_2} + \frac{3\varepsilon}{2} \right) \| u_{1x} - u_{2x} \|_{L^2(0, L)}^2 + \frac{M^2}{2\varepsilon} \| \theta_2 - \theta_1 \|_{L^\infty(0, L)}^2 \\
+ \frac{(KM)^2}{2\varepsilon} \left( \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right)^2 + C_2^2 \frac{\kappa_2}{\alpha_2 L_2} \| u_1 - u_2 \|_{H^1(0, L)}^2 \\
+ \frac{(MC_2)^2}{2\varepsilon} \left( \frac{\kappa_2 - \kappa_1}{\alpha_2 L_2} \right)^2 + 2 \| u_1 - u_2 \|_{H^1(0, L)}^2 \\
+ C_2 \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \left| L - L_1 - L_2 \right|^2 \\
+ \frac{\varepsilon}{2} \| u_1 - u_2 \|_{H^1(0, L)}^2 + \frac{1}{2\varepsilon} \left\| f_1 - f_2 \right\|_{L^2(0, L)}^2,
\]
\[
\theta_1 \| u_{1x} - u_{2x} \|^2_{L^2(0,L_1)} \leq \left( \frac{K}{\alpha_2} + \frac{3\varepsilon}{2} \right) \| u_{1x} - u_{2x} \|^2_{L^2(0,L_1)} + \frac{M^2}{2\varepsilon} \| \theta_2 - \theta_1 \|^2_{L^\infty(0,L_1)} + \frac{(KM)^2}{2\varepsilon} \left| \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right|^2 + \frac{(MC_T^2)^2}{2\varepsilon} \left| \frac{\kappa_2}{\alpha_2 L_2} - \frac{\kappa_1}{\alpha_1 L_2} \right|^2 \\
+ \frac{C_T^2}{2\varepsilon} \left( \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right) \| L - L_1 - L_2 \|^2 + \frac{1}{2\varepsilon} \| f_1 - f_2 \|^2_{L^2(0,L_1)} + \left( \frac{C_T^2 \kappa_2}{\alpha_2 L_2} + \varepsilon \right) \| u_1 - u_2 \|^2_{H^1(0,L_1)}.
\]

Since \( u_1(0) = 0 \) and \( u_2(0) = 0 \), \( \| u_1 - u_2 \|_{H^1(0,L_1)} \) is equivalent to \( \| u_{1x} - u_{2x} \|_{L^2(0,L_1)} \) i.e. there exist two positive constants \( c_1 \) and \( c_2 \) such that

\[
c_1 \| u_1 - u_2 \|_{H^1(0,L_1)} \leq \| u_{1x} - u_{2x} \|_{L^2(0,L_1)} \leq c_2 \| u_1 - u_2 \|_{H^1(0,L_1)}.
\]

This gives,

\[
\theta_1 c_1^2 \| u_1 - u_2 \|^2_{H^1(0,L_1)} \leq \left( \frac{K}{\alpha_2} + \frac{3\varepsilon}{2} \right) c_2^2 \| u_1 - u_2 \|^2_{H^1(0,L_1)} + \frac{M^2}{2\varepsilon} \| \theta_2 - \theta_1 \|^2_{L^\infty(0,L_1)} + \frac{(KM)^2}{2\varepsilon} \left| \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right|^2 + \frac{(MC_T^2)^2}{2\varepsilon} \left| \frac{\kappa_2}{\alpha_2 L_2} - \frac{\kappa_1}{\alpha_1 L_2} \right|^2 \\
+ \frac{C_T^2}{2\varepsilon} \left( \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right) \| L - L_1 - L_2 \|^2 + \frac{1}{2\varepsilon} \| f_1 - f_2 \|^2_{L^2(0,L_1)} + \left( \frac{C_T^2 \kappa_2}{\alpha_2 L_2} + \varepsilon \right) \| u_1 - u_2 \|^2_{H^1(0,L_1)}.
\]

Which implies

\[
\left( \theta_1 c_1^2 - \left( \frac{K}{\alpha_2} + \frac{3\varepsilon}{2} \right) c_2^2 - \left( \frac{C_T^2 \kappa_2}{\alpha_2 L_2} + \varepsilon \right) \right) \| u_1 - u_2 \|^2_{H^1(0,L_1)} \leq \frac{M^2}{2\varepsilon} \| \theta_2 - \theta_1 \|^2_{L^\infty(0,L_1)} + \frac{(KM)^2}{2\varepsilon} \left| \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right|^2 + \frac{(MC_T^2)^2}{2\varepsilon} \left| \frac{\kappa_2}{\alpha_2 L_2} - \frac{\kappa_1}{\alpha_1 L_2} \right|^2 \\
+ \frac{C_T^2}{2\varepsilon} \left( \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right) \| L - L_1 - L_2 \|^2 + \frac{1}{2\varepsilon} \| f_1 - f_2 \|^2_{L^2(0,L_1)}.
\]

Let us choose

\[
\varepsilon \in \left( 0, \frac{1}{2c_2^2 + 1} \left( \theta_1 c_1^2 - \frac{c_2^2 K}{\alpha_2} - \frac{C_T^2 \kappa_2}{\alpha_2 L_2} \right) \right)
\]
for $\theta_1 > \frac{1}{c_1^2} \left( \frac{c_2^2 \kappa_1}{\alpha_2} + \frac{C_2^2 \kappa_2}{\alpha_2 L_2} \right)$ and

$$A := \left( \theta_1 c_1^2 - \left( \frac{K}{\alpha_2} + \frac{3\varepsilon}{2} \right) c_2^2 - \left( \frac{C_2^2 \kappa_2}{\alpha_2 L_2} + \varepsilon \right) \right).$$

Then we get the estimate

$$\|u_1 - u_2\|^2_{H^1([0,L_1])} \leq C \left( \|\theta_2 - \theta_1\|^2_{L^\infty([0,L_1])} + \max_{\alpha_1,\alpha_2 \in \mathcal{N}} |\alpha_1 - \alpha_2|^2 \right.$$

$$+ \max_{\kappa_1,\kappa_2 \in \mathcal{M}} |\kappa_1 - \kappa_2|^2 + \|f_1 - f_2\|^2_{L^2([0,L_1])} \right),$$

where

$$C = \max_{\alpha_1,\alpha_2 \in \mathcal{N}, \kappa_1, \kappa_2 \in \mathcal{M}} \left\{ \frac{M^2 (MC^2_{\mathcal{T}})^2 + C_2^2 (L - L_1 - L_2)^2}{2\varepsilon A} \frac{2\kappa_1 + (KM)^2}{2\varepsilon \alpha_1 \alpha_2 L_2 A} \frac{(MC^2_{\mathcal{T}})^2 + C_2^2 (L - L_1 - L_2)^2}{2\varepsilon \alpha_1 \alpha_2 L_2 A} \frac{1}{2\varepsilon A} \right\}.$$

$$|\ell_1(t) - \ell_2(t)|^2 = |u_1(t,L_1) - u_2(t,L_1)|^2$$

$$\leq \hat{c} \left( \|u_1 - u_2\|^2_{H^1([0,L_1])} \|u_1 - u_2\|^2_{L^2([0,L_1])} \right)$$

(by 3.2)

$$\leq \hat{c} \left( \|u_1 - u_2\|^2_{H^1([0,L_1])} \|u_1 - u_2\|^2_{L^2([0,L_1])} \right)$$

(by 3.1)

$$\leq \hat{c} \left( \|u_1 - u_2\|^2_{H^1([0,L_1])} \|u_1 - u_2\|^2_{L^2([0,L_1])} \right)$$

(choose $\xi = \frac{1}{2}$)

$$\leq \hat{c} \left( \|u_1 - u_2\|^2_{H^1([0,L_1])} \|u_1 - u_2\|^2_{L^2([0,L_1])} \right)$$

$$\leq \hat{c} C \left( \|\theta_2 - \theta_1\|^2_{L^\infty([0,L_1])} + \max_{\alpha_1,\alpha_2 \in \mathcal{N}} |\alpha_1 - \alpha_2|^2 \right.$$

$$+ \max_{\kappa_1,\kappa_2 \in \mathcal{M}} |\kappa_1 - \kappa_2|^2 + \|f_1 - f_2\|^2_{L^2([0,L_1])} \right).$$

The above theorem shows the stability of the fast-temperature-activation limit problem when the assumption (A4) is taken into account.
Theorem 3.4. Let $u_1$ and $u_2$ be arbitrary weak solutions to the system (2.30)–(2.33). Assume that (A1)–(A3) hold. Then there exist a constant $C = C(\psi_1, \psi_2)$ such that

$$\|u_1 - u_2\|^2_{H^1(0,L_1)} \leq C \left( \|\theta_2 - \theta_1\|^2_{L^\infty(0,L_1)} + \max_{\alpha_1, \alpha_2 \in \mathcal{N}} |\alpha_1 - \alpha_2|^2 \\
+ \max_{\kappa_1, \kappa_2 \in \mathcal{M}} |\kappa_1 - \kappa_2|^2 + \|f_1 - f_2\|^2_{L^2(0,L_1)} \right). \quad (3.10)$$

Furthermore,

$$|\ell_1(t) - \ell_2(t)|^2 \leq \hat{c}C \left( \|\theta_2 - \theta_1\|^2_{L^\infty(0,L_1)} + \max_{\alpha_1, \alpha_2 \in \mathcal{N}} |\alpha_1 - \alpha_2|^2 \\
+ \max_{\kappa_1, \kappa_2 \in \mathcal{M}} |\kappa_1 - \kappa_2|^2 + \|f_1 - f_2\|^2_{L^2(0,L_1)} \right), \quad (3.11)$$

where $C$ is the constant entering (3.10) and $\hat{c}$ is the constant entering the interpolation inequality (3.2).

Proof: Since $u_1$ and $u_2$ are weak solution in the sense of the Definition 1, they satisfy

$$(\theta_1 u_{1x} + \frac{1}{\alpha_1} \psi(u_{1x}), \eta_x) + \frac{\kappa_1}{\alpha_1 L_2} u_1(L_1) \eta(L_1) = \frac{\kappa_1}{\alpha_1 L_2} (L - L_1 - L_2) \eta(L_1) + (f_1, \eta), \quad \forall \eta \in V \quad (3.12)$$

and

$$(\theta_2 u_{2x} + \frac{1}{\alpha_2} \psi(u_{2x}), \eta_x) + \frac{\kappa_2}{\alpha_2 L_2} u_2(L_1) \eta(L_1) = \frac{\kappa_2}{\alpha_2 L_2} (L - L_1 - L_2) \eta(L_1) + (f_2, \eta), \quad \forall \eta \in V. \quad (3.13)$$

Subtracting (3.13) from (3.12) leads to

$$(\theta_1 u_{1x} - \theta_2 u_{2x} + \frac{1}{\alpha_1} \psi(u_{1x}) - \frac{1}{\alpha_2} \psi(u_{2x}), \eta_x) \\
+ (\frac{\kappa_1}{\alpha_1 L_2} u_1(L_1) - \frac{\kappa_2}{\alpha_2 L_2} u_2(L_1)) \eta(L_1) \\
= (\frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2}) (L - L_1 - L_2) \eta(L_1) + (f_1 - f_2, \eta).$$
Which results in
\[
\theta_1(u_{1x} - u_{2x}, \eta) = (\theta_2 - \theta_1)(u_{2x}, \eta) + \left( \frac{1}{\alpha_2} \psi(u_{2x}) - \frac{1}{\alpha_1} \psi(u_{1x}) \right) \\
+ \left( \frac{\kappa_2}{\alpha_2 L_2} u_2(L_1) - \frac{\kappa_1}{\alpha_1 L_2} u_1(L_1) \right) \eta(L_1) \\
+ \left( \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right) (L - L_1 - L_2) \eta(L_1) + (f_1 - f_2, \eta).
\]

Now, choosing \( \eta := u_1 - u_2 \) we have
\[
\left| (\theta_2 - \theta_1)(u_{2x}, \eta) \right| = \left| (\theta_2 - \theta_1)(u_{2x}, u_{1x} - u_{2x}) \right| \\
\leq \| \theta_2 - \theta_1 \|_{L^\infty([0, l_1])} \| u_{2x} - L_1 \|_{L^2([0, l_1])} \| u_{1x} - u_{2x} \|_{L^2([0, l_1])} \\
\leq M \| \theta_2 - \theta_1 \|_{L^\infty([0, l_1])} \| u_{1x} - u_{2x} \|_{L^2([0, l_1])} \\
\text{ (for } M \text{ as in (3.5))} \\
\leq \frac{M^2}{2\varepsilon} \| \theta_2 - \theta_1 \|_{L^\infty([0, l_1])}^2 + \frac{\varepsilon}{2} \| u_{1x} - u_{2x} \|_{L^2([0, l_1])}^2,
\]
(by (3.1))
\[
\left| \left( \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right) (L - L_1 - L_2) \eta(L_1) \right| = \left| \left( \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right) (L - L_1 - L_2) \right| \\
\leq \left| \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right| \| L - L_1 - L_2 \| \\
= \left| \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right| \| u_1(L_1) - u_2(L_1) \| \]
\[
\left| \left( \frac{\kappa_2}{\alpha_2 L_2} u_2(L_1) - \frac{\kappa_1}{\alpha_1 L_2} u_1(L_1) \right) \eta(L_1) \right| = \left| \left( \frac{\kappa_2}{\alpha_2 L_2} u_2(L_1) - \frac{\kappa_1}{\alpha_1 L_2} u_1(L_1) \right) \| u_1(L_1) - u_2(L_1) \| \right| \\
\leq \frac{\kappa_2}{\alpha_2 L_2} \| u_1(L_1) - u_2(L_1) \|^2 + \left| \frac{\kappa_2}{\alpha_2 L_2} - \frac{\kappa_1}{\alpha_1 L_2} \right| \| u_1(L_1) \| \| u_1(L_1) - u_2(L_1) \|,
\]
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\[
\left| \left( \frac{1}{\alpha_2} \psi(u_{2x}) - \frac{1}{\alpha_1} \psi(u_{1x}), \eta_x \right) \right| \\
\leq \left( \frac{1}{\alpha_2} \left| \psi(u_{2x}) - \psi(u_{1x}) \right| \right)_{L^2(0,L_1)} \\
+ \left| \frac{1}{\alpha_2} \left| \psi(u_{1x}) \right| \right|_{L^2(0,L_1)} \\
\left| u_{1x} - u_{2x} \right|_{L^2(0,L_1)} \\
\leq \left( \frac{1}{\alpha_2} \left( \left| \psi(u_{1x}) - \psi(u_{2x}) \right| \right)_{L^2(0,L_1)} \\
+ \left| \psi(u_{1x}) \right| \left| \psi(u_{2x}) \right|_{L^2(0,L_1)} \\
+ \left| \psi(u_{1x}) \right|_{L^2(0,L_1)} \left| u_{1x} - u_{2x} \right|_{L^2(0,L_1)} \\
(\text{because of (A1)}) \\
\leq \left( \frac{C_{\psi_1} + C_{\psi_2}}{\alpha_2} \left| u_{1x} - u_{2x} \right|_{L^2(0,L_1)}^2 \\
+ \left( (C_{\psi_1} + C_{\psi_2}) M + C_{\psi_2} \right) \\
\left| \frac{1}{\alpha_2} \left| \psi(u_{1x}) \right| - \left| \psi(u_{2x}) \right|_{L^2(0,L_1)} \right| \\
\leq \left( C_{\psi_1} + C_{\psi_2} \right) \frac{1}{\alpha_2} \left| u_{1x} - u_{2x} \right|_{L^2(0,L_1)} \\
+ \frac{\varepsilon}{2} \left| u_{1x} - u_{2x} \right|_{L^2(0,L_1)} \\
+ \left( (C_{\psi_1} + C_{\psi_2}) M + C_{\psi_2} \right) \left| \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right|^2 \\
= \left( C_{\psi_1} + C_{\psi_2} \right) \frac{1}{\alpha_2} + \frac{\varepsilon}{2} \left| u_{1x} - u_{2x} \right|_{L^2(0,L_1)} \\
+ \left( (C_{\psi_1} + C_{\psi_2}) M + C_{\psi_2} \right) \left| \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right|^2.
\]
\[(f_1 - f_2, \eta) \leq \|(f_1 - f_2, u_1 - u_2)\| \leq \|f_1 - f_2\|_{L^2(0, L_1)}\|u_1 - u_2\|_{L^2(0, L_1)} \leq \frac{1}{2\varepsilon}\|f_1 - f_2\|^2_{L^2(0, L_1)} + \frac{\varepsilon}{2}\|u_1 - u_2\|^2_{L^2(0, L_1)},\]

which implies

\[
\theta_1\|u_1 - u_2\|^2_{L^2(0, L_1)} \leq \frac{M^2}{2\varepsilon}\|\theta_2 - \theta_1\|^2_{L^2(0, L_1)} + \frac{\varepsilon}{2}\|u_1 - u_2\|^2_{L^2(0, L_1)} + \left(\frac{C_1 + C_2}{\alpha_2} + \frac{\varepsilon}{2}\right)\|u_1 - u_2\|^2_{L^2(0, L_1)} + \frac{(C_1 + C_2)M + C_2}{\alpha_1}\left|\frac{1}{\alpha_2} - \frac{1}{\alpha_1}\right|^2\]

\[
+ \frac{\kappa_2}{\alpha_2 L_2}|u_1(L_1) - u_2(L_1)|^2 + \frac{\kappa_2}{\alpha_2 L_2} - \frac{\kappa_1}{\alpha_1 L_2}|u_1(L_1)||u_1(L_1) - u_2(L_1)| + \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2}|L - L_1 - L_2||u_1(L_1) - u_2(L_1)| + \frac{1}{2\varepsilon}\|f_1 - f_2\|^2_{L^2(0, L_1)} + \frac{\varepsilon}{2}\|u_1 - u_2\|^2_{L^2(0, L_1)} \]

\[
\frac{C_1 + C_2}{\alpha_2} + \frac{3\varepsilon}{2}\|u_1 - u_2\|^2_{L^2(0, L_1)} + \frac{M^2}{2\varepsilon}\|\theta_2 - \theta_1\|^2_{L^2(0, L_1)} + \frac{(C_1 + C_2)M + C_2}{\alpha_1}\left|\frac{1}{\alpha_2} - \frac{1}{\alpha_1}\right|^2\]

\[
+ \frac{\kappa_2}{\alpha_2 L_2}|u_1(L_1) - u_2(L_1)|^2 + \frac{\kappa_2}{\alpha_2 L_2} - \frac{\kappa_1}{\alpha_1 L_2}|u_1(L_1)||u_1(L_1) - u_2(L_1)| + \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2}|L - L_1 - L_2||u_1(L_1) - u_2(L_1)| + \frac{1}{2\varepsilon}\|f_1 - f_2\|^2_{L^2(0, L_1)}.
\]

By applying the trace inequality, we get
\[ \begin{align*}
\theta_1 |u_{1x} - u_{2x}|^2_{L^2(0,L_1)} & \leq \left( \frac{C_{\psi_1} + C_{\psi_2}}{\alpha_2} + \frac{3\varepsilon}{2} \right) |u_{1x} - u_{2x}|^2_{L^2(0,L_1)} \\
& + \frac{M^2}{2\varepsilon} |\theta_2 - \theta_1|^2_{L^\infty(0,L_1)} \\
& + \frac{((C_{\psi_1} + C_{\psi_2})M + C_{\psi_2})^2}{2\varepsilon} \left\| \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right\|^2 \\
& + C_T^2 \frac{\kappa_2}{\alpha_2 L_2} |u_1 - u_2|^2_{H^1(0,L_1)} \\
& + C_T^2 \frac{\kappa_2}{\alpha_2 L_2} - \frac{\kappa_1}{\alpha_1 L_2} \left\| u_1 \right\|_{H^1(0,L_1)} |u_1 - u_2|_{H^1(0,L_1)} \\
& + C_T \left| \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right| |L - L_1 - L_2||u_1 - u_2|_{H^1(0,L_1)} \\
& + \frac{1}{2\varepsilon} \left\| f_1 - f_2 \right\|^2_{L^2(0,L_1)} \\
& \end{align*} \]
+ \frac{(MC_T^2)^2}{2\varepsilon} \left| \frac{\kappa_2}{\alpha_2 L_2} - \frac{\kappa_1}{\alpha_1 L_2} \right|^2 + \frac{\varepsilon}{2} \|u_1 - u_2\|^2_{H^1(0,L_1)}
+ \frac{C_T^2}{2\varepsilon} \left( \left| \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right| \left| L - L_1 - L_2 \right|^2 \right)
+ \frac{\varepsilon}{2} \|u_1 - u_2\|^2_{H^1(0,L_1)} + \frac{1}{2\varepsilon} ||f_1 - f_2||^2_{L^2(0,L_1)}.

Hence,

$$\theta_1 \|u_1 - u_2\|^2_{L^2(0,L_1)} \leq \left( \frac{C_{\psi_1} + C_{\psi_2}}{\alpha_2} + \frac{3\varepsilon}{2} \right) \|u_1 - u_2\|^2_{L^2(0,L_1)}
+ \frac{M^2}{2\varepsilon} \|\theta_2 - \theta_1\|^2_{L^\infty(0,L_1)}
+ \left( (C_{\psi_1} + C_{\psi_2} M + C_{\psi_2})^2 \right) \frac{1}{\alpha_2 - 1} \left| \frac{1}{\alpha_1} \right|^2
+ \frac{(MC_T^2)^2}{2\varepsilon} \left| \frac{\kappa_2}{\alpha_2 L_2} - \frac{\kappa_1}{\alpha_1 L_2} \right|^2
+ \frac{C_T^2}{2\varepsilon} \left( \left| \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right| \left| L - L_1 - L_2 \right|^2 \right)
+ \frac{1}{2\varepsilon} ||f_1 - f_2||^2_{L^2(0,L_1)}
+ \left( \frac{C_T^2 \kappa_2}{\alpha_2 L_2} + \varepsilon \right) \|u_1 - u_2\|^2_{H^1(0,L_1)}.$$

Since $u_1(0) = 0$ and $u_2(0) = 0$, $\|u_1 - u_2\|_{H^1(0,L_1)}$ is equivalent to $\|u_{1x} - u_{2x}\|_{L^2(0,L_1)}$ i.e. there exist two positive constants $c_1$ and $c_2$ such that

$$c_1 \|u_1 - u_2\|_{H^1(0,L_1)} \leq \|u_{1x} - u_{2x}\|_{L^2(0,L_1)} \leq c_2 \|u_1 - u_2\|_{H^1(0,L_1)}.$$
This gives,
\[
\theta_1 c_1^2 \| u_{1x} - u_{2x} \|^2_{H^1(0,L_1)} \leq \left( \frac{C_{\psi_1} + C_{\psi_2}}{\alpha_2} + \frac{3\varepsilon}{2} \right) c_2^2 \| u_{1x} - u_{2x} \|^2_{H^1(0,L_1)} \\
+ \frac{M^2}{2\varepsilon} \| \theta_2 - \theta_1 \|^2_{L^\infty(0,L_1)} \\
+ \left( \frac{(C_{\psi_1} + C_{\psi_2}) M + C_{\psi_2}}{2\varepsilon} \right) \left\| \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right\|^2 \leq \\
+ \frac{MC^2}{2\varepsilon} \left( \frac{\kappa_2}{\alpha_2 L_2} - \frac{\kappa_1}{\alpha_1 L_2} \right)^2 \\
+ \frac{C_\varepsilon}{2\varepsilon} \left( \frac{\kappa_1}{\alpha_1 L_2} \right) \left( \frac{\kappa_2}{\alpha_2 L_2} \right) \left\| L - L_1 - L_2 \right\|^2 \\
+ \frac{1}{2\varepsilon} \| f_1 - f_2 \|^2_{L^2(0,L_1)} \\
+ \left( \frac{C_\varepsilon^2 \kappa_2}{\alpha_2 L_2} + \varepsilon \right) \| u_1 - u_2 \|^2_{H^1(0,L_1)}.
\]
This implies,
\[
\left( \theta_1 c_1^2 - \left( \frac{C_{\psi_1} + C_{\psi_2}}{\alpha_2} + \frac{3\varepsilon}{2} \right) c_2^2 - \left( \frac{C_\varepsilon^2 \kappa_2}{\alpha_2 L_2} + \varepsilon \right) \right) \| u_{1x} - u_{2x} \|^2_{H^1(0,L_1)} \leq \\
\frac{M^2}{2\varepsilon} \| \theta_2 - \theta_1 \|^2_{L^\infty(0,L_1)} + \left( \frac{(C_{\psi_1} + C_{\psi_2}) M + C_{\psi_2}}{2\varepsilon} \right) \left\| \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right\|^2 \\
+ \frac{(MC^2)^2 + C_\varepsilon^2 |L - L_1 - L_2|^2}{2\varepsilon} \left( \frac{\kappa_1}{\alpha_1 L_2} - \frac{\kappa_2}{\alpha_2 L_2} \right)^2 + \frac{1}{2\varepsilon} \| f_1 - f_2 \|^2_{L^2(0,L_1)}.
\]
Let us choose now
\[
\varepsilon \in \left( 0, \frac{1}{3 \varepsilon c_2^2} + 1 \left( \theta_1 c_1^2 - \left( \frac{C_{\psi_1} + C_{\psi_2}}{\alpha_2} \right) c_2^2 - \left( \frac{C_\varepsilon^2 \kappa_2}{\alpha_2 L_2} \right) \right) \right)
\]
for \( \theta_1 > \frac{1}{c_1} \left( \left( \frac{C_{\psi_1} + C_{\psi_2}}{\alpha_2} \right) c_2^2 + \frac{C_\varepsilon^2 \kappa_2}{\alpha_2 L_2} \right) \) and
\[
B := \left( \theta_1 c_1^2 - \left( \frac{C_{\psi_1} + C_{\psi_2}}{\alpha_2} + \frac{3\varepsilon}{2} \right) c_2^2 - \left( \frac{C_\varepsilon^2 \kappa_2}{\alpha_2 L_2} + \varepsilon \right) \right) \right).
\]

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Then we get the estimate

\[ ||u_1 - u_2||_{H^1(0,L_1)} \leq C \left( ||\theta_2 - \theta_1||_{L^\infty(0,L_1)} + \max_{\alpha_1, \alpha_2 \in \mathbb{N}} |\alpha_1 - \alpha_2|^2 + \max_{\kappa_1, \kappa_2 \in \mathcal{M}} |\kappa_1 - \kappa_2|^2 + ||f_1 - f_2||_{L^2(0,L_1)} \right). \]

where

\[
C = \max_{\alpha_1, \alpha_2 \in \mathbb{N}, \kappa_1, \kappa_2 \in \mathcal{M}} \left\{ \frac{M^2}{2\epsilon B'} \frac{(MC_T^2)^2 + C_T^2(L - L_1 - L_2)^2}{2\epsilon \alpha_1 \alpha_2 L_2 B} 2\kappa_1 \right. \\
+ \left. \frac{((C_{\psi_1} + C_{\psi_2})M + C_{\psi_2})^2}{2\epsilon \alpha_1 \alpha_2 B} \right. \\
+ \left. \frac{(MC_T^2)^2 + C_T^2(L - L_1 - L_2)^2}{2\epsilon \alpha_1 \alpha_2 L_2 B} 2\alpha_1, \frac{1}{2\epsilon B} \right\}.
\]

The proof of (3.11) goes in the same way as for (3.7).

\[\square\]

Summarizing, Theorem 3.3 shows the stability of shape memory alloy problem in its fast-temperature-activation limit under the assumption that \( \psi \) is Lipshitz continuous. In Theorem 3.4 we improve this result. Namely, Theorem 3.4 shows the stability of shape memory alloy problem in its fast-temperature-activation limit without \( \psi \) satisfying \( (A4) \). Both stability analyses are done with respect to the SMA-related parameter \( \alpha \), the Hooke’s constant \( \kappa \), the local temperature profile \( \theta \), and the right hand side \( f \).
Chapter 4

Numerical Scheme and Simulation Results

In this chapter we first apply a finite difference based numerical scheme to the linear part (vibrating beam problem) of the shape memory alloy problem (4.1)–(4.6). This is the so-called “vibrating beam problem” [5]. Then we adopt the scheme to the shape memory alloy problem (4.1)–(4.6) derived in chapter 2. We apply the numerical scheme to the linear part of the shape memory alloy material to see how the numerical scheme works in case of linear problem. We wish to solve numerically the problem: Find $u(t, x)$ and $\ell(t)$ such that

$$\rho_1 u_{tt} + \gamma u_{xxxx} - (2\alpha_1 \theta u_x + \psi(u_x))_x = \rho_1 g \text{ in } \Omega$$  
(4.1)

$$m\ell'(t) = mg + (\gamma u_{xxx} - 2\alpha_1 \theta u_{xx} - \psi(u_x))(t, L_1)$$  
(4.2)

for $0 < t < T$,

$$u(t, 0) = 0, \quad u_{xx}(t, 0) = u_{xx}(t, L_1) = 0, \quad \text{for } 0 < t < T,$$  
(4.3)

$$u(t, L_1) = \ell(t) - L_1 \text{ for } 0 < t < T,$$  
(4.4)

$$u(0, x) = \frac{\ell(0) - L_1}{L_1} x, \quad u_t(0, x) = 0, \quad \text{for } 0 \leq x \leq L_1,$$  
(4.5)

$$\ell(0) = \ell_0, \quad \ell'(0) = 0 \text{ for } 0 \leq x \leq L_1,$$  
(4.6)

where $\psi(r) = -2\alpha_1 \theta \dot{r} - 4\alpha_2 \ddot{r}^3 + 6\alpha_3 \ddot{r}^5$, $\Omega := (0, T) \times (0, L_1)$ and $\theta$ is the local temperature. We close the chapter with a stability analysis with
4.1 Vibrating Beam Problem

In this section we present a finite difference approximation of a solution to the following problem: Find $u$ such that

\[
\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = f(x, t) \quad \text{in} \quad (0, T) \times (0, 1),
\]
\[
u(t, 0) = g_0(t), \quad u(t, 1) = g_1(t) \quad \text{for} \quad t \in (0, T),
\]
\[
u_x(t, 0) = h_0(t), \quad u_x(t, 1) = h_1(t) \quad \text{for} \quad t \in (0, T),
\]
\[
u_{xx}(t, 0) = w_0(t), \quad u_{xx}(t, 1) = w_1(t) \quad \text{for} \quad t \in (0, T),
\]
\[
u(0, x) = u_0(x), \quad u_t(0, x) = v_0(x) \quad \text{for} \quad x \in (0, 1).
\]

This equation is called homogeneous vibrating beam problem for $f(t, x) = 0$. For $f(t, x) \neq 0$ we call it inhomogeneous vibrating beam problem which is the linear part of shape memory alloy problem. The general solution for the vibrating beam problem can be find using the method of separation of variables.

4.1.1 Numerical Scheme (4.7)–(4.11)

The central finite difference approximation scheme for $\frac{\partial^4 u}{\partial x^4}$ is

\[
\frac{d^4 u}{dx^4} \bigg|_{x=x_i} \approx \frac{u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}}{\Delta x^4}
\]

and for $\frac{\partial^2 u}{\partial t^2}$ is

\[
\frac{\partial^2 u}{\partial t^2} \bigg|_{t=t_n} \approx \frac{u^{n-1} - 2u^n - u^{n+1}}{\Delta t^2}.
\]

Correspondingly, the central finite difference approximation for (4.7) in space and time is

\[
\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = f(t_n, x_i) - \frac{u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}}{\Delta x^4}
\]
where \( x_i = (i - 1)\Delta x \) for \( i = 1, 2, \ldots, N + 1 \) with \( N\Delta x = 1 \), \( t_n = n\Delta t \) for \( n = 0, 1, 2, \ldots \), and

\[
u^n_i \approx u(x_i, t_n).
\]

Finally, the central finite difference method numerical scheme is

\[
u^n_i + 1 = 2u^n_i - u^n_{i-1} + \Delta t^2 f(t_n, x_i) - \frac{\Delta t^2}{\Delta x^4} \left( u^n_{i-2} - 4u^n_{i-1} + 6u^n_i - 4u^n_{i+1} + u^n_{i+2} \right).
\]

The numerical scheme (4.15) is explicit, two step scheme with second order accuracy in both \( \Delta x \) and \( \Delta t \) (since we expect classical solutions to (4.7)–(4.11) to exist). This scheme needs one extra initial condition \( u^n_i = u(\Delta t, x) \) because of the term \( u^{n-1}_i \). Using Taylor expansion \( u(\Delta t, x) \) can be expressed as follows

\[
u(\Delta t, x) = u(0, x) + \Delta t u_t(0, x) + \frac{1}{2} \Delta t^2 u_{tt}(0, x) + \mathcal{O}(\Delta t^3)
\]

\[
u(0, x) + \Delta t u_t(0, x) + \frac{1}{2} \Delta t^2 (f(0, x) - \frac{\partial^4 u}{\partial x^4}(0, x)) + \mathcal{O}(\Delta t^3).
\]

So, we can now approximate the extra initial condition \( u^n_1 \) numerically as

\[
u^n_1 \approx u(0, x_1) + \Delta t u_t(0, x_1) + \frac{1}{2} \Delta t^2 (f(0, x_1) - \frac{\partial^4 u}{\partial x^4}(0, x_1)).
\]

Because of the occurrence of the terms \( u^n_{i-2} \) and \( u^n_{i+2} \), we also need to approximate \( u^n_2 \) and \( u^n_{N-1} \). To approximate the latter two terms we used central finite difference method for the second order derivative at the boundary by approximating \( u_{xx}(x_1) \approx u_{xx}(x_2) \) and \( u_{xx}(x_{N+1}) \approx u_{xx}(x_N) \) which give us

\[
u^n_2 \approx \frac{-u_{xx}(t_n, 0) \Delta x^2 + u^n_1 + u^n_3}{2}
\]

and

\[
u^n_{N} \approx \frac{-u_{xx}(t_n, 1) \Delta x^2 + u^n_{N-1} + u^n_{N+1}}{2}.
\]

### 4.1.2 Numerical Results for Selected Vibrating Beam Problem

We implement the scheme (4.15) for selected test case in which the exact solutions are known. Now we present the cases and the plots of the errors
$u_{i}^{n} - u(t_{n}, x_{i})$ at time $t$ below. In each case the of problems we used a space step $\Delta x = 0.01$ and time step $\Delta t = 0.001 \Delta x$. Here, we expect that $\Delta t = o(\Delta x^{3})$ because of the fact that the central finite difference approximation for the time and the space should coincide which keeps the order of convergence.

Case 1: Find $u(t, x)$ such that

$$\begin{align*}
\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} &= 0 \quad \text{in} \quad (0, T) \times (0, 1), \\
u(t, 0) &= \cos t, \quad u(t, 1) = e \cos t \quad \text{for} \quad t \in (0, T), \\
\frac{\partial u}{\partial x}(t, 0) &= \cos t, \quad \frac{\partial u}{\partial x}(t, 1) = e \cos t \quad \text{for} \quad t \in (0, T), \\
\frac{\partial^2 u}{\partial x^2}(t, 0) &= \cos t, \quad \frac{\partial^2 u}{\partial x^2}(t, 1) = e \cos t \quad \text{for} \quad t \in (0, T), \\
u(0, x) &= e^x, \quad \frac{\partial u}{\partial x}(0, x) = 0 \quad \text{for} \quad x \in (0, 1).
\end{align*}$$

The exact solution for Case 1 is $u(t, x) = e^x \cos t$ and the plot of the error $u_{i}^{n} - u(t_{n}, x_{i})$ is given in Figure 4.1 below, where $u_{i}^{n}$ is given by (4.28).

![Figure 4.1: Plot of the error $u_{i}^{n} - u(t_{n}, x_{i})$ for Case 1.](image-url)
Case 2: Find $u(t, x)$ such that

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 26 \quad \text{in} \quad (0, T) \times (0, 1),$$

$$u(t, 0) = t^2, \quad u(t, 1) = t^2 + 1 \quad \text{for} \quad t \in (0, T),$$

$$u_x(t, 0) = 0, \quad u_x(t, 1) = 4 \quad \text{for} \quad t \in (0, T),$$

$$u_{xx}(t, 0) = 0, \quad u_{xx}(t, 1) = 12 \quad \text{for} \quad t \in (0, T),$$

$$u(0, x) = x^4, \quad u_t(0, x) = 0 \quad \text{for} \quad x \in (0, 1).$$

The exact solution for Case 2 is $u(t, x) = t^2 + x^4$ and the plot of the error $u^n_i - u(t^n, x_i)$ is given in Figure 4.2 below, where $u^n_i$ is given by (4.28).

![Plot of the error $u^n_i - u(t^n, x_i)$ for Case 2.](image)

Figure 4.2: Plot of the error $u^n_i - u(t^n, x_i)$ for Case 2.
Case 3: Find $u(t, x)$ such that

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 2 \quad \text{in} \quad (0, T) \times (0, 1),$$

$$u(t, 0) = t^2, \quad u(t, 1) = t^2 + 1 \quad \text{for} \quad t \in (0, T),$$

$$u_x(t, 0) = 0, \quad u_x(t, 1) = 2 \quad \text{for} \quad t \in (0, T),$$

$$u_{xx}(t, 0) = 2, \quad u_{xx}(t, 1) = 2 \quad \text{for} \quad t \in (0, T),$$

$$u(0, x) = x^2, \quad u_t(0, x) = 0 \quad \text{for} \quad x \in (0, 1).$$

The exact solution for Case 3 is $u(t, x) = t^2 + x^2$ and the plot of the error $u^n_i - u(t_n, x_i)$ is given in Figure 4.3 below, where $u^n_i$ is given by (4.28).

![Plot of the error $u^n_i - u(t_n, x_i)$ for Case 3.](image)

Figure 4.3: Plot of the error $u^n_i - u(t_n, x_i)$ for Case 3.
Case 4: Find $u(t, x)$ such that

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = (t^4 - x^2) \cos x t \quad \text{in} \quad (0, T) \times (0, 1),$$

$u(t, 0) = 1, \quad u(t, 1) = \cos t \quad \text{for} \quad t \in (0, T),$

$u_x(t, 0) = 0, \quad u_x(t, 1) = -t \sin t \quad \text{for} \quad t \in (0, T),$

$u_{xx}(t, 0) = -t^2, \quad u_{xx}(t, 1) = -t^2 \cos t \quad \text{for} \quad t \in (0, T),$

$u(0, x) = 1, \quad u_x(0, x) = 0 \quad \text{for} \quad x \in (0, 1).$

The exact solution for Case 4 is $u(t, x) = \cos(t x)$ and the plot of the error $u^n_i - u(t_n, x_i)$ is given in Figure 4.4 below, where $u^n_i$ is given by (4.28).

Figure 4.4: Plot of the error $u^n_i - u(t_n, x_i)$ for Case 4.
Case 5: Find \( u(t, x) \) such that

\[
\frac{\partial^2 u}{\partial t^2} + 4 \frac{\partial^4 u}{\partial x^4} = 3 \cos x \sin t \quad \text{in} \quad (0,T) \times (0,1),
\]

\[
\begin{align*}
    u(t,0) &= \sin t, & u(t,1) &= \cos 1 \sin t & \text{for } t \in (0,T), \\
    u_x(t,0) &= 0, & u_x(t,1) &= -\sin 1 \sin t & \text{for } t \in (0,T), \\
    u_{xx}(t,0) &= -\sin t, & u_{xx}(t,1) &= -\cos 1 \sin t & \text{for } t \in (0,T), \\
    u(0,x) &= 0, & u_t(0,x) &= \cos x & \text{for } x \in (0,1).
\end{align*}
\]

The exact solution for Case 5 is \( u(t, x) = \cos x \sin t \) and the plot of the error \( u^n_i - u(t_n, x_i) \) is given in Figure 4.5 below, where \( u^n_i \) is given by (4.28).

![Figure 4.5: Plot of the error \( u^n_i - u(t_n, x_i) \) for Case 5.](image)

In all the above cases the order of error is higher than that of the numerical scheme. The errors at the boundaries are larger than the errors in the domain because of the fact that we used lower order of accuracy at the boundaries.
4.2 The Time Dependent Shape Memory Alloy Problem

Let us recall that the shape memory alloy problem is to find the displacement $u(t,x)$ and the valve position $\ell(t)$ such that

$$\rho_1 u_{tt} + \gamma u_{xxx} - (2\alpha_1 \theta u_x + \psi(u_x))_x = \rho_1 g \quad \text{in } \Omega$$  \hspace{1cm} (4.19)

$$m\ell'(t) = mg + (\gamma u_{1xxx} - 2\alpha_1 \theta u_{1x} - \psi(u_x))(t, L_1)$$  \hspace{1cm} (4.20)

for $0 < t < T$, $u(t,0) = 0$, $u_x(t,0) = u_x(t, L_1) = 0$, for $0 < t < T$,  \hspace{1cm} (4.21)

$u(t,L_1) = \ell(t) - L_1$ for $0 < t < T$,  \hspace{1cm} (4.22)

$u(0,x) = \frac{\ell(0) - L_1}{L_1} x$, $u_x(0,x) = 0$, for $0 \leq x \leq L_1$,  \hspace{1cm} (4.23)

$\ell(0) = \ell_0$, $\ell'(0) = 0$ for $0 \leq x \leq L_1$,  \hspace{1cm} (4.24)

where $\psi(r) = -2\alpha_1 \theta^r - 4\alpha_2 r^3 + 6\alpha_3 r^5$, $\Omega := (0,T) \times (0,L_1)$ and $\theta$ is the local temperature.

4.2.1 Numerical Scheme for (4.19)–(4.24)

The central finite difference scheme of the shape memory alloy problem (4.19)–(4.24), which we propose here, is:

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + \Delta t^2 g - \frac{\gamma \Delta t^2}{\rho_1 \Delta x^4} (u_{i-2}^n - 4u_{i-1}^n + 6u_i^n - 4u_{i+1}^n + u_{i+2}^n)$$

$$+ \frac{2\Delta t^2 \alpha_1 \theta(t_n, x_i)}{\rho_1 \Delta x^2} (u_{i-1}^n - 2u_i^n - u_{i+1}^n)$$

$$+ \frac{\Delta t^2}{\rho_1 \Delta x^2} \psi'(u_{i+1}^n - u_{i-1}^n)(u_{i-1}^n - 2u_i^n - u_{i+1}^n),$$  \hspace{1cm} (4.25)

$$\ell^{n+1} = 2\ell^n - \ell^{n-1} + \Delta t^2 mg + \Delta t^2 \gamma \frac{-u_{N-2}^n + 2u_{N-1}^n - u_N^n}{2 \Delta x^3}$$

$$- \Delta t^2 \alpha_1 \theta(t) \frac{u_{N+1}^n - u_{N-1}^n}{\Delta x} - \Delta t^2 \psi(\frac{u_{N+1}^n - u_{N-1}^n}{2 \Delta x}),$$  \hspace{1cm} (4.26)

where $x_i = (i-1)\Delta x$ for $i = 1, 2, \ldots, N+1$ with $N \Delta x = L_1$, $t_n = n \Delta t$ for $n = 0, 1, 2, \ldots$, $\ell^n \approx \ell(t_n)$,  \hspace{1cm} (4.27)
and
\[ u^n_i \approx u(x_i, t_n). \] (4.28)

From the boundary condition \( u(t, 0) = 0 \) and \( u(t, L_1) = \ell(t) - L_1 \), we have
\[ u^n_1 = 0, \quad u^n_{N+1} = \ell^n - L_1, \] (4.29)

and also from \( u_{xx}(t, 0) = u_{xx}(t, L_1) = 0 \) we can approximate
\[ u^n_2 \approx \frac{u^n_1 + u^n_3}{2} \quad \text{and} \quad u^n_N \approx \frac{u^n_{N+1} + u^n_{N-1}}{2}. \] (4.30)

by considering the second order derivative of \( u \) with respect to space at \( x_1 \) and \( x_{N+1} \) are equal to the second order derivative of \( u \) with respect to space at \( x_2 \) and \( x_N \) respectively. In the above scheme we also need the estimates for \( u^n_i \) and \( \ell(\Delta t) \). In this case using the Taylor’s expansion (assuming more regularity for the solution than this actually has) we have
\[ u(\Delta t, x) = u(0, x) + \Delta tu_i(0, x) + \frac{1}{2}\Delta t^2 u_{ii}(0, x) + O(\Delta t^3). \] (4.31)

and
\[ \ell(\Delta t) = \ell(0) + \Delta t\ell'(0) + \frac{\Delta t^2}{2}\ell''(0) + O(\Delta t^3). \] (4.32)

Substituting the initial conditions we have in the shape memory alloy problem (4.19)–(4.24) we can approximate the above term as follows:
\[ u(\Delta t, x) \approx \frac{\ell(0) - L_1 x}{L_1} + \frac{1}{2}\Delta t^2 g \] (4.33)

and
\[ \ell(\Delta t) \approx \ell(0) + \frac{\Delta t^2}{2} g. \] (4.34)

4.2.2 Numerical Illustration for (4.19)–(4.24)

Implementing the above scheme for \( N = 40 \), \( \Delta t = 0.0001 \Delta x \), and the parameter values \( \alpha_1 = 1/5 \), \( \alpha_2 = 2 \), \( \alpha_3 = 1 \), \( g = 9.8 \), \( m = 10^{-3} \), \( \rho_1 = 0.834 \),
$L_1 = L_2 = 0.01$, $L = 0.015$, $\ell_0 = -2.5 \times 10^{-3}$, $\theta(t) = 80t + 20$, $\theta_c = 1$, and $\gamma = 0.75$, we get the results shown in Figure 4.6 and Figure 4.7.

**Figure 4.6:** Displacement $u$ as function of the spatial coordinate at given time $t$.

**Figure 4.7:** Position of the valve $\ell$ as a function of time $t$. 

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In Figure 4.6 we see that the displacement \( u \) is linear with respect to changes in the space variable \( x \). Figure 4.7 indicates that the valve position \( \ell \) might have two “critical” points (critical point in the sense that the function changes its monotonicity property at this point) for \( t \in [0, 0.18] \). Are these points corresponding to opening or closing of the valve? Unfortunately, we cannot answer this question for the moment.

### 4.2.3 Sensitivity Analysis with Respect to the Parameters \( \alpha_1, \alpha_2, \alpha_3, \gamma, \theta_c \) and \( \rho_1 \)

The stability analysis performed in chapter 3 cannot be done here in a straightforward manner. This is the reason why we choose to approach numerically the stability issue with respect to parameters of (4.19)–(4.24). Implementing the above scheme for \( N = 40, \Delta t = 0.01 \Delta x \), and the parameter values \( \alpha_1 = 1/5, \alpha_2 = 2, \alpha_3 = 1, g = 9.8, m = 10^{-3}, \rho_1 = 0.834, L_1 = L_2 = 1, L = 1.5, L_0 = -2.5.10^{-3}, \theta(t) = 80t + 20, \theta_c = 1, \) and \( \gamma = 0.75 \) we get the following results (see Figure 4.8 and 4.9).

![Figure 4.8: Displacement \( u \) as function of the spatial coordinate at given time \( t \).](image)
Figure 4.9: Position of the valve \( \ell \) as a function of time \( t \).

From Figure 4.8 and Figure 4.9 we see that the displacement \( u \) is a linear function while the valve position \( \ell \) has two “critical points”. By considering the above values of the parameters as standard parameter values next we see the sensitivity of the valve position \( \ell \) with respect to each parameters.

Using now \( N = 40 \), \( \Delta t = 0.01 \Delta x \), and the parameter values \( \alpha_1 = 5 \), \( \alpha_2 = 2 \), \( \alpha_3 = 1 \), \( \kappa = 0.8 \cdot 10^9 \), \( g = 9.8 \), \( m = 10^{-3} \), \( \rho_1 = 0.834 \), \( L_1 = L_2 = 1 \), \( L = 1.5 \), \( \ell_0 = -2.5 \cdot 10^{-3} \), \( \theta(t) = 80t + 20 \), \( \theta_c = 1 \), and \( \gamma = 0.75 \), we get the results shown in Figure 4.10 and Figure 4.11.
Figure 4.10: Displacement $u$ as function of the spatial coordinate at given time $t$.

Figure 4.11: Position of the valve $\ell$ as a function of time $t$.

In Figure 4.10 and Figure 4.11 we can see that for $\alpha_1 = 5$ the displacement $u$ preserves its linearity, while the valve position $\ell$ exhibits sort of almost periodic behavior with respect to the time $t$. 
Figure 4.12: Position of the valve $\ell$ as a function of time $t$ for different values of $\alpha_1$.

Figure 4.12 shows the valve position $\ell$ obtained for $N = 40$, $\Delta t = 0.01\Delta x$, and the parameter values $\alpha_1 = 1, 2, 3, 4, 5, 6$, $\alpha_2 = 2$, $\alpha_3 = 1$, $g = 9.8$, $m = 10^{-3}$, $\rho_1 = 0.834$, $L_1 = L_2 = 1$, $L = 1.5$, $\ell_0 = -2.5.10^{-3}$, $\theta(t) = 80t + 20$, $\theta_c = 1$, and $\gamma = 0.75$. Interestingly, we see that as the values of $\alpha_1$ increases
the valve position $l(t)$ approaches an almost periodic behavior in time.

Figure 4.13: Position of the valve $l$ as a function of time $t$ for different values of $\alpha_2$.

Figure 4.13 shows the valve positions $l$ obtained when changing $\alpha_2$, while keeping $N = 40$, $\Delta t = 0.01 \Delta x$, and all other parameters fixed (i.e. $\alpha_1 = 1/5$, $\alpha_2 = 1, 2, 3, 4, 5, 6$, $\alpha_3 = 1$, $g = 9.8$, $m = 10^{-3}$, $\rho_1 = 0.834$, $L_1 = L_2 = 1$, $\dot{L} = 1.5$, $\ell_0 = -2.5 \cdot 10^{-3}$, $\theta(t) = 80t + 20$, $\theta_c = 1$, and $\gamma = 0.75$. We see that the parameter $\alpha_2$ affects drastically the valve position $l$. In principle,
more “critical” points can appear.

Figure 4.14: Position of the valve $\ell$ as a function of time $t$ for different values of $\alpha_3$.

Figure 4.14 shows valve position $\ell$ obtained when gradually changing $\alpha_3$, while keeping $N = 40$, $\Delta t = 0.01\Delta x$, and the other parameter values fixed (i.e. $\alpha_1 = 1/5$, $\alpha_2 = 2$, $\alpha_3 = 1, 2, 3, 4, 5, 6$, $g = 9.8$, $m = 10^{-3}$, $\rho_1 = 0.834$, $L_1 = L_2 = 1$, $L = 1.5$, $\ell_0 = -2.5 \times 10^{-3}$, $\theta(t) = 80t + 20$, $\theta_c = 1$, and $\gamma = 0.75$). Apparently, as the higher $\alpha_3$, the higher the peak value of the
valve position $\ell$.

Figure 4.15: Position of the valve $\ell$ as a function of time $t$ for different values of $\gamma$.

Figure 4.15 points out the valve position $\ell$ obtained for $N = 40$, $\Delta t = 0.01\Delta x$, and the parameter values $\alpha_1 = 1/5$, $\alpha_2 = 2$, $\alpha_3 = 1$, $g = 9.8$, $m =$

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$10^{-3}$, $\rho_1 = 0.834$, $L_1 = L_2 = 1$, $L = 1.5$, $\ell_0 = -2.5 \times 10^{-3}$, $\theta(t) = 80t + 20$, and $\theta_c = 1$. Here we modify $\gamma$ as follows: $\gamma = 0.2, 0.4, 0.6, 0.8, 1.0, 1.2$. We note that the valve position $\ell$ preserves its shape but shows a very little change at the peak point for $\gamma \in [0.2, 1.2]$.

Figure 4.16: Position of the valve $\ell$ as a function of time $t$ for different values of $\theta_c$. 

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In Figure 4.16 we show again the valve position $\ell$, for $N = 40$, $\Delta t = 0.01\Delta x$, and the parameter values $\alpha_1 = 1/5$, $\alpha_2 = 2$, $\alpha_3 = 1$, $g = 9.8$, $m = 10^{-3}$, $\rho_1 = 0.834$, $L_1 = L_2 = 1$, $L = 1.5$, $\ell_0 = -2.5 \cdot 10^{-3}$, $\theta(t) = 80t + 20$, and $\theta_c$ where $\theta_c = 0, 20, 40, 60, 80, 100$, and $\gamma = 0.75$. It seems that the critical temperature $\theta_c$ really affects the shape of the valve position $\ell(t)$. Again, more “critical” points seem to appear.

![Graphs showing valve position as a function of time for different $\rho_1$ values.](image)

**Figure 4.17:** Position of the valve $\ell$ as a function of time $t$ for different values of $\rho_1$. 

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Finally, in Figure 4.17 we show the valve position $\ell$ calculated for $N = 40, \Delta t = 0.01\Delta x$, and the parameter values $\alpha_1 = 1/5, \alpha_2 = 2, \alpha_3 = 1, g = 9.8, m = 10^{-3}, \rho_1 = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, L_1 = L_2 = 1, L = 1.5, \ell_0 = -2.5 \times 10^{-3}, \theta(t) = 80t + 20, \theta_c = 1$, and $\gamma = 0.75$. We note that for different values of $\rho_1$ the valve position $\ell$ preserves its shape and shows very slight difference at its peak point. The model seems to be insensitive with respect to $\rho_1$.

From all the above Figures 4.12, 4.13, 4.12, 4.15, 4.16 and 4.17 we can see that the valve position $\ell$ is sensitive to the SMA dependent parameters $\alpha_1, \alpha_2, \alpha_3, \gamma$, critical temperature $\theta_c$ and density $\rho_1$. Considering all these one can say that the SMA problem is stable problem with respect to its parameters.
Chapter 5

Conclusions

In this thesis, we have modeled a valve mechanism made of a shape memory alloy spring and an elastic spring. The result is a nonlinear parabolic-hyperbolic PDE system that we refer to as SMA problem. We reduced this to a simpler version called the shape memory alloy problem in its fast-temperature activation limit. We showed the stability of weak solutions with respect to data and parameters completing on this way the well-posedness of reduced problem. We have proposed a finite difference scheme to approximate the SMA problem numerically. We have also done the sensitivity analysis of SMA problem with respect to physically relevant parameters numerically.

Our numerical sensitivity analysis shows that in most of the cases some "critical" points appear. We do not have answers to the open questions:

- What do they actually represent?
- Do they indicate the opening or closing of the valve? What is the definition of opening and closing of the valve?
- How do the parameters affect there “critical” points?

The time-dependent problem (SMA problem) is not fully understood mathematically. It would be interesting to see to which extent the analysis done
in chapter 3 and the sensitivity analysis from section 4.2.3 apply to this more general situation. It would also be interesting to find out the connection between as \( \alpha \to \infty \) and \( \rho_1 \to 0 \) in the SMA problem.
Bibliography


% This program solves the differential equation of the
% form \( d^2u/dt^2 + a d^4u/dx^4 = 3 \cos(x) \sin(t) \) with
% boundary condition \( u(0,t) = \sin(t) \), \( u_x(0,t) = 0 \),
% \( u_{xx}(0,t) = -\sin(t) \) and \( u(1,t) = \cos(1) \sin(t) \),
% \( u_x(1,t) = -\sin(1) \sin(t) \), \( u_{xx}(1,t) = -\cos(1) \sin(t) \) and
% initial condition \( u(x,0) = 0 \) and \( u_t(x,0) = \cos(x) \)

clear all;
close all;

Nx = 100;
dx = 1 / Nx;
x = 0 : dx : 1;
a = 4;
dt = 0.001*dx;

f=@(x,t) 3*cos(x).*sin(t);

% initial condition
u1 = zeros(size(x));
ut = cos(x);

u=0+dt*ut+1/2*dt^2*(f(x,0)-0); % Taylor expansion: u(x,0)

for j = 2 : 1000
    t=j*dt;
    u2=u1;
    u1=u;

    % solution at grid points
    u(3:Nx−1)=dt^2*f(x(3:Nx−1),t)+2*u1(3:Nx−1)−... u2(3:Nx−1)−a*(dt^2/dx^4)*(u1(1:Nx−3)−... 4*u1(2:Nx−2)+6*u1(3:Nx−1)−4*u1(4:Nx)... +u1(5:Nx+1));

    % boundary condition
    u(1)=sin(t); u(Nx+1)=cos(1)*sin(t);
    uxxleft=−sin(t);uxxright=−cos(1)*sin(t);

    u(2)=−uxxleft*dx^2+u(1)+u(3))/2;
    u(Nx)=−uxxright*dx^2+u(Nx−1)+u(Nx+1))/2;
    uexact=cos(x).*sin(t);

    % plot(x,u,'rx',x,uexact)
    plot(x,u−uexact)
    title(sprintf('t=%g', t))
drawnow
end

% This program solves the differential equation of the
% To apply this program for different vibrating beam
% problem we need to change only the boundary
% conditions, initial conditions, the coefficient 'a'
% and the right hand side.
Appendix B

MATLAB Code for SMA Problem

% This program solves the SMA problem
clear all;
close all;

%%% parameters
alpha1 = 5; alpha2=2; alpha3 = 1;
alpha = alpha1;
g = 9.8; rho = 0.834;
kappa = 0.8e9;
m = 1e-3;
L1 = 1; % L1 = 1e-2;
L2 = 1; % L2 = 1e-2;
L = 1.5; % L = 1.5e-2;
gamma = 0.75;
theta_critical = 1;

%%% psi and theta
psi=@(r)-2*theta_critical*alpha1.*r-4*alpha2.*r.^3 ... 
+6*alpha3.*r.^5;
psider=@(r)-2*theta_critical*alpha1-12*alpha2.*r.^2 ... 
+30*alpha3.*r.^4;
\[ \theta = \Theta(t) (80t + 20); \]

\[
\text{%% Grid}
\]
\[
\text{Nx} = 40;
\]
\[
dx = L_1 / \text{Nx};
\]
\[
x = 0 : dx : L_1;
\]
\[
dt = 0.01 * dx;
\]
\[
\text{Nt} = 1 / dt;
\]

\[
\text{%% initial condition}
\]
\[
10 = -2.5e-3;
\]
\[
u_1 = (10 - L_1) * x / L_1;
\]
\[
l(1) = 10;
\]
\[
ut = \text{zeros(size} (u_1));
\]
\[
t = 0;
\]

\[
\text{%% u(x, dt) and l(dt) using Taylor expansion}
\]
\[
u = u_1 + dt^2 / 2 * g;
\]
\[
l(2) = 10 + dt * l1 + dt^2 / 2 * g;
\]

\[
\text{Ntt} = \text{Nt};
\]
\[
\text{for} \ j = 2 : \text{Ntt}
\]
\[
t = j * dt;
\]
\[
u_2 = u_1;
\]
\[
u_1 = u;
\]

\[
\text{% the solution of at grid points 3:Nx-1}
\]
\[
u(3: \text{Nx}-1) = 2 * u_1(3: \text{Nx}-1) - u_2(3: \text{Nx}-1) + g * dt^2 \ldots
\]
\[
- (\text{gamma} / \text{rho}) * (dt^2 / dx^4) * (u_1(1: \text{Nx}-3) - \ldots
\]
\[
4 * u_1(2: \text{Nx}-2) + 6 * u_1(3: \text{Nx}-1) - 4 * u_1(4: \text{Nx}) + \ldots
\]
\[
\text{u_1(5: Nx+1)} + (2 * \text{alpha} / \text{rho}) * \text{theta(t)} \ldots
\]
\[
(dt^2 / dx^2) * (u_1(2: \text{Nx}-2) - 2 * u_1(3: \text{Nx}-1) + \ldots
\]
\[
\text{u_1(4: Nx)} + (1 / \text{rho}) * (dt^2 / dx^2) * \ldots
\]
\[
\text{psider}((u_1(4: Nx) - u_1(2: \text{Nx}-2)) / (2 * dx)) \ldots
\]
\[(u_1(2:Nx-2) - 2u_1(3:Nx-1) + u_1(4:Nx))\];

% boundary conditions
uxxleft = 0; uxxright = 0;
u(1) = 0;
\(u(2) = (-uxxleft \cdot dx^2 + u(1) + u(3)) / 2;\)
u(Nx+1) = l(j) - L1;
u(Nx) = (-uxxright \cdot dx^2 + u(Nx-1) + u(Nx+1)) / 2;

% 1(t)
l(j+1) = 2*l(j) - l(j-1) + \(dt^2 \cdot m \cdot g + dt^2 \cdot 2 \cdot gamma \cdot \ldots\)
\((-u(Nx-2) + 2*u(Nx-1) - u(Nx)) / (2 * dx^3) - dt^2 \cdot \ldots\)
alpha \cdot theta(t) \cdot (u(Nx+1) - u(Nx-1)) / (dx) - \ldots
dt^2 \cdot psi((u(Nx+1) - u(Nx-1)) / (2 * dx));

% figure of u
if mod(j,100) == 0;
    plot(x,u, 'Color', 'red', 'LineWidth', 2)
xlabel('x')
ylabel('u')
grid on
title(sprintf('t =%g', t))
drawnow
end
hold on
end

% figure of l
figure
\(tt = 0:dt:Ntt*dt;\)
plot(tt, l, 'LineWidth', 2)
xlabel('t')
ylabel('l(t)')
grid on