MASTER

The compatibility of admissible Minkowski decompositions

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The compatibility of admissible Minkowski decompositions

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Abstract

A recent study in deformation theory concerns toric deformations. This is related to the Minkowski decomposition of the cross cut of a cone that defines a toric variety. Instead of studying the deformations, the approach in this thesis is from a different perspective. Not the deformation theory, but the Minkowski decomposition will play a central role. These Minkowski decompositions will be described explicitly and investigated. The process of finding a decomposition of a given polytope in two summands is not described, as there is already a lot of literature concerning this problem. However, given two such decompositions a question that arises is whether these decompositions are compatible. First this question is answered in the particular case where the decompositions are actually decompositions of the same polytope. Once this problem is tackled, a more complex situation is discussed. This concerns the case where the decompositions come from two different cross cuts of a given cone.
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Chapter 1

Introduction

Consider two polytopes $P$ and $Q$. Their (Minkowski) sum is again a polytope.

$$R = P \oplus Q := \{p + q | p \in P, q \in Q\}$$

We call $P \oplus Q$ a decomposition of $R$. If a polytope $R$ is not decomposable in different summands $P$ and $Q$, then $R$ is said to be indecomposable. Furthermore

$$P \oplus Q = P \oplus Q' \implies Q = Q'.$$

Thus if $R$ and one of the summands is known, the second summand is uniquely determined.

Consider a polytope $P$ with

$$P = P_0 \oplus P_1 = Q_0 \oplus Q_1$$

such that $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are two different decompositions of $P$.

The question that arises is whether the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ have a common refinement. This means that there exist polytopes $S_i$ such that

$$P = \bigoplus_{i=1}^{k} S_i \quad \text{and}$$

$$P_0 = \bigoplus_{i \in I_p} S_i$$

$$Q_0 = \bigoplus_{i \in I_q} S_i$$

with $I_p, I_q \subset \{1, \ldots, k\}$. If such a common refinement exists, the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are said to be compatible. In this thesis a method is given to show whether two decompositions of the same polytope are compatible. The decision problem of whether a lattice polygon admits a Minkowski decomposition is $NP$-complete. Overall,
doing computations with Minkowski sums is very complex, as described in [? ]. Thus instead of computing the refinements of $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$, a different approach is used.

The second part of this thesis concerns a generalization of the concept of compatibility. Consider a cone $\sigma$. The intersection of $\sigma$ and a hyperplane yields a polyhedron. Instead of comparing two decompositions of one polytope, the decompositions of two different polytopes are compared, where both polytopes are constructed as the intersection of $\sigma$ and a hyperplane. Having a common refinement does not make sense for these decompositions. Two such decompositions are said to be compatible if there exists some isomorphism between both decompositions (or their refinement). A method for finding such an isomorphism is given in this thesis.

1.1 Background

An important application connected with this problem is multivariate polynomial factorization. For each multivariate polynomial $f$ there exists a Newton polytope $P_f$. Let $F$ be any field and let $f \in F[X_1, \ldots, X_n]$ be a multivariate polynomial with variables $X_1, \ldots, X_n$.

$$f = \sum c_i \prod_{j=1}^{n} X_i^j$$

with $i = (i_1, \ldots, i_n)$. The corresponding Newton polytope $P_f$ is defined as the convex hull of all the points $i \in \mathbb{Z}^n$ where $c_{i_1, \ldots, i_n} \neq 0$. This polytope has integral vertices and is therefore an integral polytope, or a lattice polytope over lattice $\mathbb{Z}^n$.

$$P_f := \text{Conv} \{ i \in \mathbb{R}^n | c_i \neq 0 \}$$

Example 1.

Consider the polynomial

$$f = 1 + 2X + 3XY^2 + X^3Y$$

The corresponding Newton polygon $P_f$ is given in Figure 1.1.

A first observation about Newton polytopes was made by Ostrowski in 1921 [14], [23].

Theorem 1.1 (Ostrowski). Let $f, g, h \in F[X_1, \ldots, X_n]$ such that $f = gh$. Then

$$P_f = P_g \oplus P_h.$$ 

Proof. First observe that it is obvious that $P_f \subseteq P_g \oplus P_h$. Indeed, if $i$ is a vertex of $P_f$ then $c_i(f) \neq 0$. We know that $f = gh$, thus there exist coefficients $c_j(g)$ and $c_k(h)$ that
are both nonzero with \( j + k = i \).

Conversely let \( i \) be a vertex of \( P_g \oplus P_h \) and let \( j \in P_g \) and \( k \in P_h \) be points such that \( i = j + k \). To prove that this decomposition is unique, suppose that there exist points \( j' \in P_g \) and \( k' \in P_h \) such that

\[
i = j + k = j' + k'.
\]

Then \( i = \frac{1}{2}(j + k) + \frac{1}{2}(j' + k') = \frac{1}{2}(j + k') + \frac{1}{2}(j' + k) \). Hence \( j + k' \) and \( j' + k \in P_g \oplus P_h \) and \( i \) is a vertex of \( P_g \oplus P_h \). This means that \( j + k' \) and \( j' + k \) must be equal. Thus

\[
j + k' = j' + k \quad \text{and} \quad j + k = j' + k'
\]

which means that \( k = k' \) and \( j = j' \). Hence for every vertex \( i \) of \( P_g \oplus P_h \) there exists a unique decomposition \( j + k \) with \( j \in P_g \) and \( k \in P_h \). Furthermore, there is a unique term in the expansion of \( gh \) corresponding to \( i = j + k \). From the fact that \( c_j(g) \neq 0 \) and \( c_k(h) \neq 0 \) one can verify that \( c_i = c_{j+k} \neq 0 \) in \( f \), thus \( i \in P_f \). Hence, \( P_f \supseteq P_g \oplus P_h \) and therefore we can conclude that \( P_f = P_g \oplus P_h \). \( \square \)

**Theorem 1.2.** Let \( f \in F[X_1, \ldots, X_n] \) be a nonzero polynomial not divisible by any \( X_i \). If the Newton polytope is integrally indecomposable then \( f \) is absolutely irreducible.

**Proof.** Let \( f \in F[X_1, \ldots, X_n] \) be a nonzero polynomial not divisible by any \( X_i \), with corresponding Newton polytope \( P_f \). Suppose that \( P_f \) is integrally indecomposable. Moreover, let \( f = gh \) be a nontrivial factorization of \( f \). Since \( f \) is not divisible by any \( X_i \), both factors \( g \) and \( h \) consist of at least two terms. From the previous theorem is known that there exist Newton polytopes \( P_g \) and \( P_h \) such that \( P_f = P_g \oplus P_h \). But this is in contradiction with the assumption that \( P_f \) is integrally indecomposable. Thus, there
does not exist a nontrivial factorization of $f = gh$, which means that $f$ is absolutely irreducible.

The decision problem whether a polytope $P_f$ is integrally decomposable is $NP$-complete, which is shown in [13]. In the literature some pseudo-polynomial time algorithms can be found for decomposing polygons. These algorithms are used for testing and factorizing polynomials via their Newton polynomials.

For a multivariate polynomial $f$ consider the factorization. Let $P_f$ be its Newton polytope and suppose that $P_f = P_{g_0} \oplus P_{h_0} = P_{g_1} \oplus P_{h_1}$. Assume that these two different decompositions are compatible and that $g_0$ and $g_1$ are factors of $f$ and have a degree lower than $h_0$ respectively $h_1$. Now $g_1$ is a factor of $h_0$ and $g_0$ has a common factor with $h_1$.

These different compatible decompositions of a Newton polytope $P_f$ give insight in the factorization of polynomial $f$, which is one of the motivations for this thesis.

1.2 Thesis outline

This thesis is about the compatibility of Minkowski decompositions of polytopes. The next chapter gives an introduction to the mathematics that is used in this thesis. Some definitions relating to polyhedra and geometry are given. Furthermore, the problem description that was the motivation of this thesis will be discussed.

In Chapter 3 the first case is discussed. The question how to determine whether two different decompositions of one polyhedron are compatible plays a central role in this chapter. In the first part this question is narrowed down to specific polytopes. Once a solution is found for this sub category, the main question can be answered.

This is elaborated in Chapter 4 where the definition of compatibility is extended to decompositions of different polytopes. A procedure is discussed how to determine this new notion of compatibility of two decompositions. In general this is very hard to determine. However, a lot of research is done for 2-dimensional case. This research is done on deformation theory, but in this chapter we will approach the topic from a combinatorial point. The fact that thorough research has been done for this topic is the reason that the 2-dimensional case is treated in more detail.
Finally, Chapter 6 concerns the open problems related to this topic. The link to the toric varieties is made in this chapter and some recommendations on some interesting research topics are given.

Some definitions that will be used throughout this thesis and might lead to confusion are listed in the Appendix as a guidance.
Chapter 2

Preliminaries

In this section some definitions and theorems concerning the topic are discussed. These serve as a foundation for this thesis. Furthermore, some basic examples are treated as an introduction to the subject in order to get a good impression of the definitions. There is a lot of theory and background material. For more information, see [15], [16], [20], [25], [26], [27], [? ], [32] or [34].

2.1 Preliminaries

Let \( \mathbb{R}^d \) be the \( d \)-dimensional real vector space. For \( x \in \mathbb{R}^d \) we write \( x = (x_1, \ldots, x_d) \) but we will treat them as column vectors. It will be clear from the context when a vector should be treated as a row vector.

**Definition 2.1.** A set \( S \subseteq \mathbb{R}^d \) is **convex** if for any two points \( x, y \in S \) the line segment

\[
[x, y] := \{ \lambda x + (1 - \lambda)y | 0 \leq \lambda \leq 1 \}
\]

is contained in \( S \).

The intersection of convex sets is again convex. The **convex hull** of a set \( S \) is the smallest convex set containing \( S \) and it is denoted as \( \text{Conv}(S) \). Hence, the line segment \([x, y]\) is the convex hull of \( x \) and \( y \).

**Definition 2.2.** The convex hull of a finite set of points \( S \) is called a **polytope**. A 2-dimensional polytope is sometimes referred to as a **polygon**.

**Definition 2.3.** A **hyperplane** is a set of points satisfying the equation \( a^\top x = z \).

\[
H := \{ x \in \mathbb{R}^d | a^\top x = z \}
\]
for some vector $a \in \mathbb{R}^d$ and $z \in \mathbb{R}$. The vector space $\mathbb{R}^d$ is the union of two closed halfspaces $H^-$ and $H^+$ that are intersecting in $H$.

$$
H^- := \{ x \in \mathbb{R}^d | a^\top x \leq z \}
$$

$$
H^+ := \{ x \in \mathbb{R}^d | a^\top x \geq z \}
$$

The sets satisfying the inequalities $a^\top x < z$ or $a^\top x > z$ are defined as open halfspaces.

Let $H \in \mathbb{R}^d$ be a hyperplane and $Q \in \mathbb{R}^d$ be a polytope. When $Q$ is contained in one of the (closed) halfspaces (i.e. $Q \subseteq H^+$ or $Q \subseteq H^-$), then $H$ is called a supporting hyperplane if

$$
H \cap Q \neq \emptyset.
$$

**Definition 2.4.** A polyhedron $Q \subseteq \mathbb{R}^d$ is the intersection of finitely many closed halfspaces.

$$
Q = \{ x \in \mathbb{R}^d | Ax \leq z \}
$$

The rows $a_i$ of the matrix $A$ correspond to the halfspaces and $Ax \leq z$ means that $a_i x \leq z_i$ for all $i$.

If a polyhedron $Q$ is not bounded then there exist nonzero $y \in \mathbb{R}^d$ such that for all $x \in Q$ the set $\{ x + \lambda y | \lambda \geq 0 \}$ is contained in $Q$. This set of vectors is called a ray of $Q$.

A bounded polyhedron is the convex hull of finitely many points, hence it is a polytope as defined in Definition 2.2.

**Definition 2.5.** Let $Q$ be a polyhedron. A set $F \subseteq Q$ is a face of $Q$ if $F = \emptyset$ or $F = Q$ or if there exists a supporting hyperplane $H$ such that

$$
F = H \cap Q.
$$

A nonempty face has dimension $0 \leq k \leq d$ and is referred to as a $k$-face of $Q$. The vertices of $Q$ are its 0-faces and the set of all the vertices is denoted by $\text{Vert}(Q)$. Moreover, if all vertices of a polyhedron are integral (so belong to lattice $\mathbb{Z}^d$), then it is called an integral or a lattice polyhedron with respect to the lattice $\mathbb{Z}^d$.

The addition of two convex sets can be determined and is defined as the Minkowski sum.

**Definition 2.6.** The Minkowski sum of two polyhedra $P$ and $Q$ is the set

$$
P \oplus Q := \{ x + y | x \in P, y \in Q \}.
$$

Given this definition, the Minkowski decomposition of a polyhedron can be defined. As expected, this is the decomposition of a polyhedron $Q$ in two (smaller) polyhedra $Q_0$
and $Q_1$ such that $Q_0 \oplus Q_1 = Q$. The polyhedra $Q_0$ and $Q_1$ are two summands of $Q$. Note that for a given integral polyhedron $Q$ its summands are not necessarily integral. For example take $Q$ as the unit square, which is the convex hull of the points $(0,0),(1,0), (0,1)$ and $(1,1)$. Now,

$$Q = \frac{1}{2}Q \oplus \frac{1}{2}Q.$$ 

Remark 1. If $Q_0$ and $Q_1$ are two summands of $Q$, then their Minkowski sum is a polyhedron $Q + t \cong Q,$ such that $t$ induces a shift over a vertex. To ensure that $Q$ is equal to $Q_0 \oplus Q_1$ we will embed the summand in such way that $Q_0 = Q_0$ and $Q_1 = Q_1 + t$. Moreover, this means that $0 \in Q_1$. It will be clear from the context if a true summand $Q_i$ or a polyhedron $Q_i + t$ is used and this will not be mentioned explicitly.

Definition 2.7. A Minkowski decomposition $Q = Q_0 \oplus Q_1$ in $\mathbb{R}^d$ is a trivial decomposition when one of the summands $Q_0$ or $Q_1$ is retrieved from $Q$ by a shift over a vertex $x \in \mathbb{R}^d$.

Theorem 2.8. Let $Q_0 \oplus Q_1$ be a Minkowski decomposition of $Q$. Let $F$ be a face of $Q$ that is determined by the supporting hyperplane $H$ of $Q$ with $F \subset H^-$. Let $F_i$ be a face of $Q_i$ with supporting hyperplane $H_i//H$ such that $F_i = H_i \cap Q_i$ for $i = 0, 1$. Then $H = H_0 \oplus H_1$ and $F = F_0 \oplus F_1$. This is the unique decomposition of a face $F$ into parts of $Q_0$ and $Q_1$.

Proof. Let $H$ be a supporting hyperplane of $F$ and $H_i$ the corresponding supporting hyperplanes of $Q_i$ for $i = 0, 1$ such that the hyperplanes $H_0, H_1$ and $H$ are parallel. Thus

$$H = \{x \in \mathbb{R}^d | a^\top x = z\}$$

$$H_i = \{x \in \mathbb{R}^d | a^\top x = z_i\}$$

and $F = Q \cap H$ and $F_i = Q_i \cap H_i$ for $i = 0, 1$. Suppose that $Q \subset H^-$ which means that $a^\top x \geq z$ for all $x \in Q$. Moreover, $a^\top x_i \geq z_i$ for all $x_i \in Q_i$. Note that strict equality only holds if both $x \in F$ and $x_i \in F_i$. Observe that

$$z_0 + z_1 = a^\top y_0 + a^\top y_1$$

for $y_i \in H_i$

$$= a^\top (y_0 + y_1)$$

$$= a^\top y$$

for $y \in H$

$$= z$$
The proof is split in two parts. In the first part we prove $F \subseteq F_0 \oplus F_1$. This is done by means of a proof by contradiction.

Suppose that there exists a point $u \in F$ with $u = u_0 + u_1$ such that $u_i \in Q_i$ but not both $u_i \in F_i$. Without loss of generality, assume that $u_0 \notin F_0$. Thus $z_0 > a^\top u_0$ and $z_1 \geq a^\top u_1$. This gives

$$z = z_0 + z_1 > a^\top u_0 + a^\top u_1 = a^\top u.$$

This implies that $u \notin F$ which is a contradiction. Therefore the assumption was wrong and $F \subseteq F_0 \oplus F_1$.

We now prove that $F \supseteq F_0 \oplus F_1$. Let $u = u_0 + u_1$ such that the point $u_i$ is contained in $F_i$ for $i = 0, 1$. This gives,

$$z = z_0 + z_1 = a^\top u_0 + a^\top u_1 = a^\top (u_0 + u_1) = a^\top u.$$

This implies that $u \in H$ and since $u \in Q$ we have that $u \in F = QQ \cap H$. Thus $F \supseteq F_0 \oplus F_1$.

The combination of the two parts yields that $F = F_0 \oplus F_1$. \qed

In general there are infinitely many Minkowski decompositions $Q_0 \oplus Q_1$ of a polyhedron $Q$, but only finitely many of them are admissible. See [1] or [4] from Altmann about this definition.

**Definition 2.9.** A decomposition $Q = Q_0 \oplus Q_1$ is admissible if and only if for all the faces $F$ of $Q$ at most one of the summand faces $F_i$ does not contain a lattice point.

**Corollary 2.10.** It is enough to check admissibility on the vertices. Furthermore, if $Q$ is a lattice polyhedron and $Q_0 \oplus Q_1$ is an admissible Minkowski decomposition, then the summands $Q_0$ and $Q_1$ are lattice polyhedra.

From now on, each decomposition that is discussed refers to an admissible Minkowski decomposition. When a decomposition is not necessarily admissible, then this will be explicitly made clear in the context.

### 2.2 Admissible Minkowski decompositions

In order to get familiar with the Minkowski addition and decomposition, let $Q \subseteq \mathbb{R}^2$ be a polygon with $n$ edges, which means that it also has $n$ vertices. Thus we may write
$Q = \text{Conv}(v^0, \ldots, v^{n-1})$ where the vertices are numbered counterclockwise. Denote the edges of $Q$ by the vector $d^i = v^{i+1} - v^i$ for all $0 \leq i \leq n - 1$ where $i$ is taken modulo $n$.

**Lemma 2.11.** Let $D$ be the set of all the edge vectors of $Q$. The polygon $Q$ is now completely determined by the (directed) edge set $D$ together with an initial vertex $v$ and initial direction $d \in D$.

**Proof.** To construct the polytope $Q$ from its edge set, simply start in the initial vertex $v$. Take the edges in their cyclic order starting with $d$. This yields the corresponding convex polytope $Q$. □

If a vector $v = (v_0, v_1)$ has $\gcd(v_0, v_1) = 1$ then it is called a primitive vector. Denote the $\gcd(d_0, d_1)$ of an edge $d$ by $\delta$. Now $f = \frac{1}{\delta}d$ is the primitive vector associated with the edge $d$.

Given the edge set $D$, observe that $Q$ is a polygon if and only if

$$\sum_{i=0}^{n-1} d^i = 0.$$  

In general, $Q$ is a lattice polytope if and only if $v^0 \in \mathbb{Z}^2$ and all $d \in D$ are lattice vectors.

**Minkowski addition** Let $P$ and $Q$ be two polygons with $n_P$ respectively $n_Q$ edges (and vertices). The set of the edge vectors of the Minkowski sum $P \oplus Q$ is the union of the edge vectors of $P$ and $Q$ such that the vectors that are a positive multiple of each other are added. Obviously, if $P$ and $Q$ are integral polygons, then their Minkowski sum $P \oplus Q$ is an integral polygon as well.

**Minkowski decomposition** Let $Q$ be a polygon with $n$ edges and consider the edge set $D = \{d^0, \ldots, d^{n-1}\}$. We know that $\sum_{i=1}^{n-1} d^i = 0$. A polygon $Q_0$ is a summand of $Q$ if and only if its edge set is $D_0 = \{\lambda_0 d^0, \ldots, \lambda_{n-1} d^{n-1}\}$ such that $0 \leq \lambda_i \leq 1$ for all $i = 0, \ldots, n - 1$ and

$$\sum_{i=0}^{n-1} \lambda_i d^i = 0.$$  

Trivial summands are singletons and $Q$ itself, with $\lambda = 0$ respectively $\lambda = 1$. Furthermore the Minkowski decomposition $\lambda Q \oplus (1 - \lambda)Q$ is a trivial decomposition of $Q$ for all $0 \leq \lambda \leq 1$. This corresponds to a vector $\lambda = \lambda 1$ in the description above.

**Definition 2.12.** Let $Q \subseteq \mathbb{R}^d$ be a polyhedron for $d \geq 2$. A polyhedron $Q$ is indecomposable if and only if for all the decompositions of $Q = Q_0 \oplus Q_1$ one of the summands equals $\lambda Q$ with $0 \leq \lambda \leq 1$. 

A polytope is *admissibly indecomposable* if it does not have a nontrivial decomposition that is admissible. A lattice polytope is sometimes called *integrally indecomposable* if it does not have a nontrivial admissible decomposition. Throughout this thesis, we will use the term indecomposable when a polytope is admissibly indecomposable, unless stated otherwise.

A decomposition \( \bigoplus_{i=0}^{s-1} S_i \) of a polyhedron \( Q \) such that all the summands \( S_i \) of \( Q \) are indecomposable is called a *prime decomposition* of \( Q \). In congruence with Remark 1 we have that \( S_i = S_i + t \) for \( i = 0, \ldots, s - 1 \) to ensure that \( Q = \bigoplus_{i=0}^{s-1} S_i \). Furthermore, each summand \( S_i \) should not be a multiple of \( Q \).

In terms of the edge set \( D \), the polygon \( Q \) is indecomposable if there does not exist any vector \( \lambda = (\lambda_0, \ldots, \lambda_{n-1}) \) such that \( \sum \lambda_i d_i = 0 \) and \( \lambda \neq \lambda 1 \) for \( 0 \leq \lambda \leq 1 \). We will continue with some examples for the Minkowski summands of polygons. Polygons with a relatively small number of vertices \( n \) are treated below.

### 2.2.1 \( n = 3 \)

Let \( Q \in \mathbb{R}^2 \) be a triangle. The edge set of \( Q \) is \( D = \{d^0, d^1, d^2\} \) and \( d^0 + d^1 + d^2 = 0 \). Trivially every triangle is indecomposable, as the equations \( d^0 + d^1 + d^2 = 0 \) and \( \alpha d^0 + \beta d^1 + \gamma d^2 = 0 \) imply that \( \alpha = \beta = \gamma \). Otherwise two of the edges (and hence all three) are pairwise dependent.

### 2.2.2 \( n = 4 \)

Let \( Q \) have four vertices. There are three possibilities.

- **\( d^0 = \alpha d^2 \) and \( d^1 = \beta d^3 \)**
  In this case the polytope \( Q \) is a parallelogram. Hence \( \alpha = \beta = -1 \) and the decomposition consists of two line segments.

![Figure 2.1: The decomposition of a parallelogram.](image)

- **\( d^0 = \alpha d^2 \) and \( d^1 \neq \beta d^3 \)**
  Without loss of generality let \( d^0 = \alpha d^2 \) and \( \alpha < -1 \). Thus \( Q \) is a trapezoid and it can be decomposed in a line segment and a triangle.

\[
(v^0, v^1, v^2, v^3) = (v^0, -v^0) \oplus (v^1, (\alpha + 1)v^2, v^3)
\]
• $d^0 \neq \alpha d^2$ and $d^1 \neq \beta d^3$

In the last case assume that the polytope $Q$ has no parallel edges. Although there does not exist a line segment which is a summand of $Q$, there might be a decomposition, namely that of two triangles. Without loss of generality let $d^0$ be the shortest edge. Now, let $H_0$ be the supporting hyperplane of $d^0$ and $H'_0$ the hyperplane parallel to $H_1$ such that $H'_0 \cap Q$ has the same length as $d^0$, which means that they have the same edge vector. This yields a triangle $Q_0$, which is a summand of $Q$. The other (parts of the) edges of $Q$ that are not used in this triangle form the edge set of another triangle, which is a summand $Q_0$ such that $Q = Q_0 \oplus Q_1$. Figure 2.3 shows how this is done.

Keep in mind that the decompositions that are described above are not always admissible decompositions. When $Q$ is an integral polytope, the decomposition is admissible if and only if the summands $Q_0$ and $Q_1$ are integral polytopes.

**Theorem 2.13.** Let $Q$ be an integral polygon such that $Q_0 \oplus Q_1$ is a decomposition of $Q$. The summand $Q_0$ has an edge set $D_0 = \{\lambda_0 d^0, \ldots, \lambda_{n-1} d^{n-1}\}$ such that $\sum \lambda_i d^i = 0$ and $\lambda \neq \lambda \mathbf{1}$. Now $Q_0 \oplus Q_1$ is an admissible decomposition if and only if $\lambda_i \delta_i \in \mathbb{Z}$ for all $i = 0, \ldots, n - 1$.

**Proof.** By definition $Q_0 \oplus Q_1$ is an admissible decomposition if and only if both summands are integral summands. A summand $Q_0$ is integral if and only if the edges are integral. For each edge we have that $\lambda d = \lambda \delta f$ with $\delta = \gcd(d_0, d_1)$ and $f = \frac{1}{\delta} d$. The vector $f$ is primitive by definition, which means that $\lambda \delta f$ is integral if and only if $\lambda \delta$ is integral. This proves the theorem. \qed
2.2.3 Generalization of decompositions

Polygons with more than four vertices might be decomposable in more than two summands, which is denoted by $\bigoplus Q_i$. Let $Q = \bigoplus_{i=0}^{s-1} S_i$ be a prime decomposition of $Q$. Thus, every summand $S_i$ is an indecomposable summand. Note that this decomposition does not have to be a unique prime decomposition. To see this, consider the (lattice)

![Diagram](image1)

Figure 2.4: Two prime decompositions of a hexagon

hexagon that is shown in Figure 2.4. The hexagon has integral vertices and it is the Minkowski sum of two triangles. But the same hexagon can be decomposed in three line segments. As we saw earlier, every line segment and triangle is an indecomposable polygon. Hence, these two decompositions are different prime decompositions of the same polygon.

**Definition 2.14.** Two decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are compatible if and only if both decompositions can be further decomposed in the same prime decomposition, i.e. if they have a common refinement.

![Diagram](image2)

Figure 2.5: Two decompositions which are not compatible

Figure 2.5 shows an example of two decompositions of a hexagon that are not compatible. The first decomposition is actually a prime decomposition as it consists of two triangles. The second decomposition can be further decomposed in three line segments but there does not exist a refinement of this decomposition that exists of two triangles. Thus,
both decompositions yield the same hexagon but they do not have a common refinement.

To see that there exist many polyhedra that do not have a unique prime decomposition, consider the following situation: Let \( Q \) be a polygon with \( 2n \) edges such that \( P \oplus -P \) is an admissible decomposition of \( Q \) where \( P \) is a polygon with \( n \) edges. The polygon \( Q = \text{Conv}(v^0, \ldots, v^{2n-1}) \) is a centrally symmetric polygon. This means that the polygon \( P \) has \( n \) edges \( d^i \) for \( i = 0, \ldots, n-1 \) and \( -P \) is defined as the mirrored polygon with the edge set \( \{-d^i | i = 0, \ldots, n-1 \} \). The edge set of \( Q \) is the set \( \{d^0, \ldots, d^{n-1}, -d^0, \ldots, -d^{n-1}\} \).

A second prime decomposition can automatically be derived from this edge set:

\[
Q = P \oplus -P = \bigoplus_{i=0}^{n-1} S_i
\]

where the polygons \( S_i \) are the line segments \([v^i, v^{i+1}]\) for \( 0 \leq i \leq n-1 \) counted modulo \( n \). From this can be concluded that if \( P \) is not a centrally symmetric polygon, then \( Q \) has at least two different refinements.

### 2.3 Further definitions

We will continue with some more definitions. More background information can be found in [7], [8], [10], [11], [24], [31], [33] or [34].

**Definition 2.15.** Let \( a^0, \ldots, a^{k-1} \in \mathbb{Q}^d \) be \( k \) pairwise linearly independent vectors. A *convex polyhedral cone* \( \sigma \) is the positive hull of all \( a^i \). 

\[
\sigma := \left\{ \sum_{i=0}^{k-1} \lambda_i a^i | \lambda_i \geq 0 \right\}
\]

Moreover, a cone \( \sigma \) is a *strongly convex polyhedral cone* if \( u \in \sigma \) implies that \( -u \notin \sigma \).

For any two vectors \( u, v \in \sigma \) and scalars \( \alpha, \beta \geq 0 \) the vector \( \{\alpha u + \beta v\} \) is contained in \( \sigma \). Every convex polyhedral cone has a unique minimal set of *generators* \( \{a^0, \ldots, a^{k-1}\} \) (modulo multiplication with a positive scalar). The set that consists of primitive generators is therefore unique. Moreover, the *Hilbert basis* \( E \) of \( \sigma \) is defined as the set of integer vectors such that every generator of \( \sigma \) is a nonnegative combination of these vectors. For strongly convex polyhedral cones this is a finite set and \( a \in E \) for each generator \( a \) of \( \sigma \).

**Example 2.** As an example consider the convex hull of the vectors \((0,1)\) and \((2,1)\). The (primitive) generators are exactly these two vectors, while the Hilbert basis consists of three vectors: \((0,1), (2,1) \) and \((1,1)\).
Throughout this report a \textit{cone} refers to a (finitely generated) rational strongly convex polyhedral cone.

In Definition 2.4 a polyhedron $Q$ is given in terms of inequalities. The vertex representation of $Q$ is the Minkowski sum of the convex hull of the vertex set $\text{Vert}(Q)$, which is a polytope, and the set of the rays of $Q$. This set of rays is the set of primitive generators and it is defined as the \textit{characteristic cone} of the polyhedron $Q$.

\textbf{Definition 2.16.} The characteristic cone of a polyhedron $Q$ is the set of all rays of $Q$.
\[
\text{Char}(Q) := \{ y \in \mathbb{R}^d | x + \lambda y \in Q \text{ for all } x \in Q; \lambda \geq 0 \}.
\]

The vertex representation of $Q$ is denoted by
\[
Q = \text{Conv}(\text{Vert}(Q)) \oplus \text{Char}(Q).
\]

\textbf{Corollary 2.17.} From Definition 2.9 and 2.7 it follows that a nontrivial Minkowski decomposition is admissible if and only if both summands have the same characteristic cone.

\textbf{Definition 2.18.} Let $\sigma$ be a cone that is generated by the vectors $a_i^t$. The dual cone $\sigma^\vee$ of $\sigma$ is the set
\[
\sigma^\vee := \{ r \in \mathbb{Q}^d | \langle \sigma, r \rangle \geq 0 \}.
\]

Observe that the dual cone is indeed a cone. Moreover, $(\sigma^\vee)^\vee = \sigma$. Hence, there are two ways of describing a cone; in terms of its generators and in terms of a set of inequalities.

As before, a polyhedron $Q$ can be described as the intersection of finitely many halfspaces, $Q = \{ x \in \mathbb{R}^d | Ax \leq z \}$. This can be rewritten as $(A^t - z)(\frac{x}{1})^t \leq 0$. Now the same polyhedron can be given as the positive set of vectors $u_0, \ldots, u_{m-1}$.
\[
Q = \left\{ \sum_{i=0}^{m-1} \lambda_i u^t | \lambda_i \geq 0 \right\}
\]

This is a polyhedral cone. Therefore a polyhedron $Q$ can be described as a cone $\sigma(Q) \in \mathbb{R}^{d+1}$. Each vertex $v \in Q$ corresponds to a vector $(v, 1) \in \sigma(Q)$, while each ray $r \in Q$ gives a corresponding vector $(r, 0) \in \sigma(Q)$. The cone $\sigma(Q)$ is generated by the vectors $(v, 1)$ and $(r, 0)$. The initial polyhedron $Q$ is exactly the intersection of $\sigma(Q)$ and the hyperplane $H_1 = \{ x \in \mathbb{R}^{d+1} | (0, 1)^t x = 1 \}$.

The first decomposition of the hexagon that is given in Figure 2.5 is represented by
the columns in Equation (2.1).

\[
\sigma(Q_6) = \begin{pmatrix}
0 & 1 & 2 & 2 & 1 & 0 \\
0 & 0 & 1 & 2 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{pmatrix} \oplus \begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

where \(\sigma(Q_6)\) is a hexagon at height 1. When the polyhedron \(Q\) is bounded (i.e. it is a polytope and \(\text{Char}(Q) = 0\)) then the cone \(\sigma(Q)\) is generated by vectors of the form \((v, 1)\) and the last row is usually omitted in the matrix representation.

Instead of using \(Q^d\) to embed the cones and polyhedra, we are interested in a lattice embedding. This means that not only the rays have primitive generators, but the vertices are all lattice points.

**Definition 2.19.** Let \(b^0, \ldots, b^{n-1} \in \mathbb{R}^n\) be linearly independent vectors. The set

\[
N := \{ \sum_{i=0}^{n-1} \lambda_i b^i | \lambda_i \in \mathbb{Z} \}
\]

is defined as a lattice with basis \(\{b^0, \ldots, b^{n-1}\}\).

The dual lattice is

\[
N^* := M := \{ u \in \mathbb{R}^n | u^T b \in \mathbb{Z}, \forall b \in N \}.
\]

The real vector space from a lattice \(N\) is denoted by \(N_\mathbb{R} := N \otimes \mathbb{Z} \mathbb{R}\). Furthermore \(\mathbb{Z}^n \subset \mathbb{R}^n\) is known to be the *standard lattice*. Given a lattice \(N\), a polytope \(Q\) is called a lattice polytope if and only if all vertices are lattice points. Moreover, we say that the cone is a lattice cone if it is generated by lattice vectors. This is denoted as \(\sigma \subseteq N_\mathbb{R}\). Consequently, we have for the dual cone \(\sigma^\vee \subseteq M_\mathbb{R}\).

As mentioned earlier one can describe a polyhedron \(Q \in \mathbb{R}^d\) in terms of a cone in \(\sigma(Q) \in \mathbb{R}^{d+1}\), where the intersection of \(\sigma(Q)\) and the hyperplane \(H_1\) yields the polyhedron \(Q\). In general, the intersection of \(\sigma\) and a hyperplane yields a polyhedron.

**Definition 2.20.** Let \(\sigma \subseteq N_\mathbb{R}\) be a cone and fix a vector \(r \in M\). Now

\[
H_r := \{ x \in N_\mathbb{R} | r^T x = 1 \}
\]

is the hyperplane from vector \(r\) at height 1. The *cross cut* of \(\sigma\) is the intersection of \(\sigma\) and this hyperplane.

\[
Q_r := \sigma \cap \{ x \in N_\mathbb{R} | r^T x = 1 \}
\]

Whenever the choice of \(r\) is clear from the context, we will use \(Q\) instead of \(Q_r\).
Theorem 2.21. Fix a polyhedral cone $\sigma \subseteq \mathbb{R}^{d+1}$ and a hyperplane $H_r$ as defined in Definition 2.20. Now $Q_r = Q$ is the corresponding cross cut, which is a polyhedron. The cone $\sigma$ can be retrieved from $Q$ if and only if $r \in \sigma^\vee \cap M$.

Proof. For each vector $r \in \sigma^\vee \cap M$ we have that $r^\top b^i \in \mathbb{Z}$ for all lattice generators $b^i$. Moreover, $r^\top a^j \geq 0$ for all generators $a^j$ of $\sigma$. The polyhedron $Q$ can be decomposed as $\text{Conv}(\text{Vert}(Q)) \oplus \text{Char}(Q)$, where $\text{Vert}(Q)$ is the set of vertices of $Q$ and $\text{Char}(Q)$ is the set of its rays. The vertices of $Q$ correspond to the rays $a^j$ such that $r^\top a^j > 0$, which means that $r^\top a^j \geq 1$. Hence, the vertices of $Q$ are $v^j = \frac{a^j}{r^\top a^j}$. Furthermore, the unbounded part of $Q$ is exactly the set of rays for which $r^\top a^j = 0$.

$$\text{Char}(Q) = \sigma \cap \{x \in \mathbb{N}_\mathbb{R}|r^\top x = 0\}$$

The generators $a^j$ such that $r^\top a^j < 0$ do not contribute to any vertex or ray of $Q$. This means that the cone $\sigma$ can only be retrieved from $Q$ if and only if $r \in \sigma^\vee$. \(\square\)

Consider a small example in the 2-dimensional standard lattice $\mathbb{Z}^2$. Let $\sigma = \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix}$ and the dual cone $\sigma^\vee = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}$. This notation means that $\sigma$ is generated by the rays $(1,0)$ and $(-1,4)$. Let $r = (1,1) \in \sigma^\vee$ and $s = (1,0) \notin \sigma^\vee$. The cross cut $Q_r$ is the convex hull of the vertices $(-\frac{1}{3}; \frac{4}{3})$ and $(0,1)$, which is a line segment. The polyhedron $Q_s$ consists of the vertex $(0,1)$ together with the ray $(0,1)$. The ray $(-1,4)$ of $\sigma$ does not come back in $Q_s$ as shown in Figure 2.6.
2.4 Problem description

Define \( N \cong \mathbb{Z}^d \) as a lattice and let \( M \) be its dual lattice. Now \( N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R} \) and \( M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R} \) are real vector spaces of dimension \( d \). A polyhedron \( Q \) is a lattice polytope if all its vertices \( \mathbf{v} \) are lattice points. Throughout this section a polyhedron \( Q \) is considered to be the cross cut of a cone \( \sigma \) and a vector \( \mathbf{r} \in \sigma^\vee \cap M \). Furthermore, the vector \( \mathbf{r} \) is chosen to be a lattice vector such that it has no common divisor, i.e. \( \mathbf{r} \) is a primitive vector. In general \( Q \subseteq N_\mathbb{R} \) is not a lattice polyhedron. However, the corresponding cone \( \sigma \subseteq N_\mathbb{R} \) has only lattice generators and is therefore a lattice cone.

Let \( \sigma \subseteq N_\mathbb{R} \) be a polyhedral cone and fix two vectors \( \mathbf{r} \) and \( \mathbf{s} \) such that they are both contained in \( \sigma^\vee \). Determine the polyhedra \( Q_\mathbf{r} \) and \( Q_\mathbf{s} \) and fix the admissible decompositions \( P_0 \oplus P_1 \) for \( Q_\mathbf{r} \) and \( Q_0 \oplus Q_1 \) for \( Q_\mathbf{s} \). The questions that arise are

\[
\text{Are the decompositions } P_0 \oplus P_1 \text{ and } Q_0 \oplus Q_1 \text{ compatible?}
\]

\[
\text{What does it mean for those decompositions to be compatible?}
\]

In order to answer these questions, it has to be subdivided into smaller problems. Mainly it consist of two cases. The first part treats the case where \( \mathbf{r} = \mathbf{s} \), which means that the two decompositions are decompositions of the same polytope \( Q_\mathbf{r} = Q_\mathbf{s} \). If \( \mathbf{r} \neq \mathbf{s} \) then the problem becomes more complex. The notion of being compatible gets a different meaning, which will be discussed in the Chapter 4. The procedure to check compatibility for \( \mathbf{r} = \mathbf{s} \) is described in the next chapter.
Chapter 3

Equal cross cuts

From now on let $N$ be the standard lattice of degree $d$ which is $N \cong \mathbb{Z}^d$ and therefore $M \cong \mathbb{Z}^d$. This means that $N_\mathbb{R} \cong \mathbb{R}^d$ and the lattice points are integral. Let $\sigma \subseteq N_\mathbb{R}$ be a cone generated by $k$ lattice vectors $a^i \in N$ for all $i = 0, \ldots, k - 1$. Fix a vector $r \in \sigma^\vee \cap M$ and determine the cross cut $Q_r$. This chapter concerns the compatibility of two admissible decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ of the polyhedron $Q_r$. Before a solution is given, some useful sets are defined. Thereafter, the solution will be treated in three parts. In the first part $Q_r$ is assumed to be a lattice polytope. After that a solution for general polytopes will be given. Finally, a general solution will be discussed.

3.1 The cross cut $Q_r$ and its summands

Let $\sigma \subseteq N_\mathbb{R}$ be a cone such that the generators $a^i$ are lattice vectors and fix some primitive vector $r \in \sigma^\vee \cap M$ for which the cross cut $Q_r = Q$ of $\sigma$ is bounded. Let $n$ be the number of edges of $Q$. An edge $[u, v]$ is denoted by the vector $d = v - u \in N$. The edges of $Q$ correspond to vectors $d^i$ for $i = 0, \ldots, n - 1$. Define a sign vector $\varepsilon(F) \in \{-1, 0, 1\}^n$ for every 2-face $F$ of $Q$ such that

$$\varepsilon_i(F) = \begin{cases} 
\pm 1 & \text{if } d^i \text{ is an edge of } F \\
0 & \text{else}
\end{cases}$$

and the edges are oriented as a cycle along the boundary of $F$. There are two such orientations but they are each others negative. Both of them are a suitable choice. If the choice for the face $F$ is obvious from the context, then we will leave this out and simply write $\varepsilon$. 

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Observe that
\[ \sum_{i=0}^{n-1} \varepsilon_i d^i = 0 \]
for every 2-face \( F \) of \( Q \). This property is used to construct the vector space \( V(Q) \).

From [6] we now come to the following definition.

**Definition 3.1.** Let \( Q \) be a polytope that is contained in \( N \). The vector space of the Minkowski summands \( V(Q) \) is defined as
\[ V(Q) = \{ (t_0, \ldots, t_{n-1}) | \sum_{i=0}^{n-1} t_i \varepsilon_i d^i = 0 \text{ for every 2-face } F \text{ of } Q \} \]  

(3.1)

Furthermore, the Minkowski summand cone
\[ C(Q) := V(Q) \cap \mathbb{R}^n_{\geq 0} \]  

(3.2)

is a cone in the vector space \( V(Q) \).

**Theorem 3.2.** Every point \( t \in C(Q) \) with \( t_i \leq 1 \) for all \( i \) corresponds to a Minkowski summand \( Q_t \) of \( Q \) which consists of the edges \( t_i d^i \). Moreover, every summand of \( Q \) has a corresponding vector in \( C(Q) \). Thus, \( Q_t \oplus Q_{1-t} = Q \).

**Proof.** Let \( Q \) be a polytope with \( n \) edges such that \( P_0 \oplus P_1 \) is an admissible decomposition. Let \( t \in [0, 1]^n \) be the vector where \( t_i \) is the fraction of the edge \( d^i \) that is used in the summand \( P_0 \). By definition the edges of \( P_0 \) are \( t_i d^i \) and therefore \( \sum \varepsilon_i(F_0) t_i d^i = 0 \) for every 2-face \( F_0 \) of \( P_0 \). This yields that \( \sum \varepsilon_i(F_0) d^i = 0 \) for every 2-face \( F_0 \) of \( P_0 \). But this yields \( \sum \varepsilon_i(F) d^i = 0 \) for every 2-face \( F \) of \( Q \). This is exactly the definition of the elements in \( C(Q) \).

Now let \( t \in C(Q) \) such that \( t_j \leq 1 \ \forall j \). We want to show that there exists a corresponding polytope \( Q_t \) that is a summand of \( Q \).

Without loss of generality one can assume that \( 0 \) is a vertex of \( Q \). Let \( v \neq 0 \) be a vertex of \( Q \). There exists a walk from \( 0 \) to \( v \), using some edges of \( Q \):
\[ v = \sum \lambda_i d^i \]
with \( \lambda_i \in \mathbb{Z} \ \forall i \). The walk from \( 0 \) to \( v \) is a walk through the 1-skeleton of \( Q \), which is the graph consisting of the vertices and edges of \( Q \). Moreover, the polytope is embedded in \( \mathbb{R}^d \) which means that all walks from \( 0 \) to \( v \) are homeomorphic. This means that for any walk from \( 0 \) to \( v \) there exists a continuous deformation to the chosen path. Therefore any walk can be chosen.
The walk can be split in two walks in the following way:

\[ v = \sum \lambda_i d^i \]

\[ = \sum \lambda_i (1 - t_i + t_i) d^i \]

\[ = \sum \lambda_i t_i d^i + \sum \lambda_i (1 - t_i) d^i \]

Define the vertices \( v_t := \sum \lambda_i t_i d^i \) and \( v'_t := \lambda_i (1 - t_i) d^i \). Note that these vertices only depend on the vector \( t \) and not on the particular choice of the walk to \( v \). The edges \( t_i d^i \) are used to reach vertex \( v_t \). Since \( t \) is an element of the cone \( C(Q) \) we know that all its entries are nonnegative and smaller than or equal to 1. Consequently, the vector \( 1 - t \) is also contained in \( C(Q) \). By construction we have that \( v_t + v'_t = v \).

For each vertex \( v \) of \( Q \) such a walk can be constructed, together with the corresponding vertices \( v_t \) and \( v'_t \). A polytope \( Q_t \) can be constructed as the convex hull of all the vertices \( v_t \). Similarly, the convex hull of the vertices \( v'_t \) yields a polytope \( Q'_t \). The polytope \( Q_t \) has exactly all \( t_i d^i \) as its edges, with corresponding vector \( t \). Moreover, \( Q'_t \) corresponds to the vector \( 1 - t \) and \( Q = Q_t \oplus Q'_t \).

All the summands of \( Q \) can be constructed this way and the theorem is proven. \( \square \)

**Remark 2.** The vector \( 1 \in C(Q) \) corresponds to the polytope \( Q \) itself. For each summand \( P \) of \( Q \) there is a corresponding edge vector and this vector will be denoted by \( t_P \in C(Q) \).

This yields the fact that the Minkowski addition now results in a vector summation in \( V(Q) \) or \( C(Q) \). Thus for two summands \( Q_t \) and \( Q_s \) we get

\[ Q_t \oplus Q_s = Q_{t+s}. \]

### 3.2 The compatibility of lattice polytopes that have no interior lattice points on the edges

Let \( \sigma \subseteq N^d \) be a cone such that the cross cut \( Q_r = Q \) is a lattice polytope with \( n \) edges. Moreover, assume that no edge contains interior lattice points, i.e. the edges \( d^i \in \mathbb{Z}^d \) are primitive.

Hence \( Q \) is a lattice polytope and therefore the summands of any admissible decomposition are lattice polytopes. Thus \( t_P \in \{0,1\}^n \) for each admissible summand \( P \) of \( Q_r \).

Consider two decompositions of \( Q \):

\[ Q = P_0 \oplus P_1 \] and

\[ Q = Q_0 \oplus Q_1. \]
Recall that these decompositions are compatible if they have a common refinement. Let $Q$ have a prime decomposition which is denoted as

$$Q = S_0 \oplus \cdots \oplus S_q = \bigoplus_{i=0}^{q-1} S_i$$

where $S_i$ is an indecomposable polytope for $i = 0, \ldots, q - 1$. Construct the vector space $V(Q)$ as in defined in Definition 3.1.

$$V(Q) = \left\{ t \bigg| n - 1 \sum_{i=0}^{n-1} t_i \varepsilon_i d^i = 0 \text{ for every 2-face } F < Q \right\}$$

Construct the cone $C(Q) = V(Q) \cap \mathbb{R}_{\geq 0}^d$. From Theorem 3.2 we know that each ray $t$ of $C(Q)$ is an edge vector of a (multiple of a) Minkowski summand of $Q$.

**Lemma 3.3.** Let $t_S \in C(Q)$ be the edge vector of an indecomposable summand $S$. There does not exist any $t'_S$ and $t''_S$ that are both nonzero and not equal to $t_S$ such that $t_S = t'_S + t''_S$ for $t'_S, t''_S \in C(Q)$.

*Proof.* This directly follows from the fact that the summand $S$ is indecomposable and therefore $t_S$ is a primitive generator of $C(Q)$. \hfill \Box

**Definition 3.4.** The Schur product (or Hadamard or pointwise product) of two vectors of the same length is obtained by the entry-wise multiplication. For two given vectors $x$ and $y$ their Schur product is denoted by $x \circ y = (x_0 y_0, x_1 y_1, \ldots)$.

The edge vector of the polytope $Q$ is by definition the all one vector $1$. Denote the edge vectors of $P_i$ and $Q_i$ by $t_{P_i}$ respectively $t_{Q_i}$. Observe that $t_{P_0} + t_{P_1} = 1$ and $t_{Q_0} + t_{Q_1} = 1$.

**Theorem 3.5.** Two decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ of $Q$ (which is a lattice polytope without lattice points on the edges) have a common refinement if and only if $t_{P_0} \circ t_{Q_0} \in C(Q)$. This means that

$$\sum_{i=0}^{n-1} (t_{P_0})_i (t_{Q_0})_i \varepsilon_i d^i = 0 \quad (3.3)$$

for every 2-face $F$ of $Q$.

One can easily verify that this does not work for a lattice polytope $Q$ that does have interior lattice points on the edges.

**Example 3.** To show that this theorem does not work for polyhedra that have lattice points on the edges, consider the polytope $Q$ with the points $(0, 0), (3, 0), (2, 1)$ and $(0, 1)$. 
This polytope has the edge set

$$D = \{(3, 0), (-1, 1), (-2, 0), (0, -1)\}.$$

\[\text{Figure 3.1: Two decompositions of } Q\]

The polytope has two decompositions, given in figure 3.1. Furthermore, we have

$$t_{P_0} = \left(\frac{1}{3}, 1, 0, 1\right)$$

$$t_{Q_0} = \left(\frac{2}{3}, 1, \frac{1}{2}, 1\right).$$

This leads to the following equation:

$$\sum_{i=0}^{3} (t_{P_0})_i (t_{Q_0})_i \varepsilon_i d^i = \frac{2}{9} (3, 0) + (0, 1) + (-1, -1) \neq 0.$$

The proof of the theorem is given in the following section.

### 3.2.1 Proof of Theorem 3.5

Let $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ be two admissible decompositions of the lattice polytope $Q$. The polytopes $P_0$ and $Q_0$ are lattice polytopes with corresponding edge vectors, and therefore $t_{P_0}, t_{Q_0} \in \{0, 1\}^n$. Consequently, the vector $t_{P_0} \circ t_{Q_0}$ only has binary entries, that are nonzero where $t_{P_0}$ and $t_{Q_0}$ both have nonzero entries.

Remark 3. If Equation (3.3) holds, then it holds for any pair $t_{P_1}$ and $t_{Q_1}$, as the Schur product $t_{P_0} \circ t_{Q_0}$ can be written in terms of any pair $t_{P_1}, t_{Q_1}$.

This is shown by the following computation.

$$t_{P_0} \circ t_{Q_0} = t_{P_0} \circ (1 - t_{Q_1}) = t_{P_0} - t_{P_0} \circ t_{Q_1}$$

$$= 1 - t_{P_1} \circ t_{Q_0} = t_{Q_0} - t_{P_1} \circ t_{Q_0}$$

$$= (1 - t_{P_1}) \circ (1 - t_{Q_1}) = 1 - t_{P_1} - t_{Q_0} - t_{P_0} \circ t_{Q_1}$$
Obviously 1, \(t_{P_0}, t_{P_1}, t_{Q_0}\) and \(t_{Q_1}\) are points in the cone \(C(Q)\). Moreover, \(t_{P_0} - t_{P_0} \circ t_{Q_1}, t_{Q_0} - t_{P_1} \circ t_{Q_0}, 1 - t_{P_1} - t_{Q_0} - t_{P_0} \circ t_{Q_1} \in C(Q)\) if and only if \(t_{P_0} \circ t_{Q_0} \in C(Q)\).

For readability define \(t_P := t_{P_0}\) and \(t_Q := t_{Q_0}\).

Consider a prime decomposition of \(Q\) with \(q\) indecomposable summands.

\[
Q = \bigoplus_{i=0}^{q-1} S_i
\]

Denote the edge vector of an indecomposable summand \(S_i\) by \(t_i\). Now, \(1 = \sum_{i=0}^{q-1} t_i\).

By definition there is a unique summand \(S_j\) that uses edge \(d^i\). Hence for each edge \(d^i\) there exists a unique edge vector \(t_j\) which is nonzero at position \(i\). This yields

\[
t_j \circ t_j = \begin{cases} t_i & \text{if } i = j \\ 0 & \text{else.} \end{cases} \quad (3.4)
\]

This follows from the fact that \(t_S \in \{0,1\}^n\) for every summand \(S\).

### 3.2.1.1 A common refinement implies \(t_P \circ t_Q \in C(Q)\)

The first thing to prove is that if \(P_0 \oplus P_1\) and \(Q_0 \oplus Q_1\) have a common refinement, then \(t_P \circ t_Q \in C(Q)\). Assume that \(P_0 \oplus P_1\) and \(Q_0 \oplus Q_1\) have a common refinement \(\bigoplus_{i=0}^{q-1} S_i\) such that every \(S_i\) is indecomposable. Define \(I\) as the subset of \(\{0, \ldots, q - 1\}\) such that \(S_i\) is a summand of \(P_0\) for \(i \in I\). Equivalently, define subset \(J\) for polytope \(Q_0\).

\[
P_0 = \bigoplus_{i \in I} S_i \quad (3.5)
\]

\[
Q_0 = \bigoplus_{j \in J} S_j \quad (3.6)
\]

The edge vectors \(t_P = \sum_{i \in I} t_i\) and \(t_Q = \sum_{j \in J} t_j\) are contained in \(C(Q)\). From Equation (3.4) we can compute the Schur product \(t_P \circ t_Q\) in terms of the indecomposable summands.

\[
t_P \circ t_Q = \sum_{i \in I} t_i \circ \sum_{j \in J} t_j = \sum_{i \in I \cap J} t_i \quad (3.7)
\]

Hence \(t_P \circ t_Q\) is the sum of the edge vectors \(t_i\) with \(i \in I \cap J\). These vectors correspond to the primitive summands that are contained in both summands \(P_0\) and \(Q_0\). The vector \(t_i\) is contained in \(C(Q)\) for all \(i\) and therefore the vector \(\sum_{i \in I \cap J} t_i\) is contained in \(C(Q)\) as well. This proves the first part of Theorem 3.5.
3.2.1.2 $t_P \circ t_Q \in C(Q)$ implies compatibility

The next part is to show that if $t_P \circ t_Q \in C(Q)$ then the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ of $Q$ are compatible. Let $t_P \circ t_Q \in C(Q)$. From theorem 3.2 we know that this is equal to the edge vector of a summand $Z_0$ of $Q$. Moreover, as $t_P \circ t_Q$ is a strict subvector of both $t_P$ and $t_Q$, the polytope $Z_0$ is a summand of both $P_0$ and $Q_0$.

$$P_0 = Z_0 \oplus Z_0^P$$
$$Q_0 = Z_0 \oplus Z_0^Q$$

We know that the polytopes $Z_0^P$ and $Z_0^Q$ do not have any edge in common. Similarly, we can construct a polytope $Z_1$ which is a summand of $P_1$ and $Q_1$. Now,

$$Q = Z_0 \oplus Z_0^P \oplus Z_1 \oplus Z_1^P$$
$$= Z_0 \oplus Z_0^Q \oplus Z_1 \oplus Z_1^Q.$$  

An immediate result is that $Z_0^P = Z_1^Q$ and $Z_0^Q = Z_1^P$. Consequently, the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ share a common refinement.

**Example 4.**

Let $Q_6$ be the hexagon that is discussed in the previous chapter. It is a polygon, hence it is its unique 2-face. Fix the vector $\varepsilon$ such that the edges are oriented counterclockwise. The matrix $D_6$ lists the directed edges $\varepsilon_i d_i^j$ as columns.

$$D_6 = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix}$$

Figure 3.2 shows three decompositions of $Q_6$. The edge vectors of the summands $P_0$, 

![Figure 3.2: Three decompositions $P_0 \oplus P_1$, $Q_0 \oplus Q_1$ and $R_0 \oplus R_1$ of $Q_6$.](image)
$Q_0$ and $R_0$ are given in (3.9), (3.10) and (3.11).

$$t_P = (0,1,1,0,1,1)$$  \hspace{1cm} (3.9)

$$t_Q = (1,0,1,1,0,1)$$  \hspace{1cm} (3.10)

$$t_R = (0,1,0,1,0,1)$$  \hspace{1cm} (3.11)

Observe that the vector $1 - t_P$ is indeed the edge vector of the summand $P_1$ just as $1 - t_Q$ is for $Q_1$ and $1 - t_R$ is for $R_1$. The common refinement of the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ is the second prime decomposition in Figure 2.4 and consists of three line segments. The Schur product $t_P \circ t_Q = (0,0,1,0,0,1)$ is the edge vector of a line segment, which is a summand of $Q$. Therefore, $t_P \circ t_Q$ is contained in $C(Q)$. On the other hand, the decomposition $R_0 \oplus R_1$ is a prime decomposition itself, consisting of two triangles. Thus, $R_0 \oplus R_1$ is not compatible to $P_0 \oplus P_1$ or $Q_0 \oplus Q_1$. We will compute $t_P \circ t_R$ and $t_R \circ t_Q$ to verify this.

$$t_P \circ t_R = (0,1,0,0,0,1)$$

$$t_R \circ t_Q = (0,0,0,1,0,1)$$

These vectors do not correspond to a polytope, hence they are not vectors of $C(Q)$. Moreover, Equation (3.3) gives

$$\sum_{i=0}^{5} (t_P)_i (t_Q)_i \varepsilon_i d^i = (0,1) + (0,-1) = 0$$

$$\sum_{i=0}^{5} (t_P)_i (t_R)_i \varepsilon_i d^i = (1,0) + (0,-1) = (1,-1)$$

$$\sum_{i=0}^{5} (t_R)_i (t_Q)_i \varepsilon_i d^i = (-1,0) + (0,-1) = (-1,-1).$$

### 3.3 The compatibility of general lattice polytopes

Let $Q$ be the cross cut of a cone $\sigma \subseteq N_R$ such that $Q$ is a lattice polytope. Now the edges are allowed to have interior lattice points and therefore an edge $d^i$ is not necessarily primitive. Define $f^i \in \mathbb{Z}^d$ as the corresponding primitive edge and let $\delta_i \in \mathbb{Z}$ be the nonnegative integer such that $d^i = \delta_i f^i$.

The entries of the edge vector $t \in \mathbb{Q}^n$ are not binary anymore, but $t \in [0,1]^n$, so $0 \leq t_i \leq 1$ for each $i$. Let $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ be two admissible decompositions of $Q$. 
The edge vectors $t_P$ and $t_P$ are not necessarily disjoint. Therefore, the Schur product $t_{P_0} \circ t_{Q_0}$ will not directly show whether the decompositions have a common refinement. To be able to determine if a common refinement of $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ exists, define a vector $\tilde{t}$.

**Definition 3.6.** Let $t \in Q^n$ be the edge vector of a given polytope $Q$. For each edge $d^i$ let $f^i$ be the corresponding primitive vector and $\delta_i \in \mathbb{Z}$ such that $d^i = \delta_i f^i$. Furthermore, let $\tilde{n} = \sum_{i=0}^{n-1} i \delta_i$. Now each edge $t_i d^i$ of $Q$ can be written as $t_i \delta_i f^i$. Define

$$\tilde{t} \in \mathbb{Z}^{\tilde{n}}$$

as the **primitive edge vector** of length $\tilde{n}$ such that $Q$ is built from (primitive) edges $t_i f^i$ for $0 \leq i \leq \tilde{n} - 1$. Every edge $d^i$ is split into $\delta_i$ primitive edges $f^i$.

Let $\mathcal{C}(Q)$ be the cone as defined in Definition 3.1. This cone is generated by the primitive vectors $t_i$, which correspond to the edge vectors of the primitive summands of $Q$. Now define $\mathcal{C}(Q)$ as the cone that is generated by the rays $\tilde{t}_i$ which are the primitive edge vectors of the indecomposable summands of $Q$.

**Theorem 3.7.** Every summand $P$ of $Q$ has a unique primitive edge vector $\tilde{t}$ such that $\tilde{t} \in \mathcal{C}(Q)$.

**Proof.** Let $Q$ be a polytope such that $\bigoplus S_i$ is a prime decomposition. From Definition 2.8 we know that for each face $F$ of $Q$ there exists a unique decomposition $F = \bigoplus_{i=0}^{q-1} F_{S_i}$, such that $F_{S_i}$ is a face of the indecomposable summand $S_i$ of $Q$. The edges of $Q$ are its 1-faces, which means that there exists a unique prime decomposition for each edge. Moreover, $Q$ is a lattice polytope and therefore the summands are lattice polytopes. Hence, every lattice segment of an edge of $Q$ is a lattice segment of a uniquely determined indecomposable summand $S_i$. Thus there exists a unique primitive edge vector $\tilde{t}_i$ corresponding to a summand $S_i$. Every summand $Q_t$ of $Q$ is the sum of indecomposable summands and therefore the primitive edge vector $\tilde{t}$ of a summand $Q_t$ is uniquely determined. \qed

**Corollary 3.8.** All the primitive edge vectors $\tilde{t}$ have binary entries.

Determine the primitive edge vectors $\tilde{t}_P$ and $\tilde{t}_Q$ and compute

$$\sum_{i=0}^{n-1} \sum_{j=0}^{\delta_i - 1} (\tilde{t}_P)_j (\tilde{t}_Q)_j \epsilon_i d^i$$

(3.12)

for each 2-face of $Q$. According to Theorem 3.5 the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are compatible if and only if (3.12) evaluates to zero for every 2-face of $Q$. 

Example 5.

Consider the polytope $Q_5$ which is the convex hull of the vertices of the matrix $Q_5$. It is a polygon and therefore it is its unique 2-face.

$$Q_5 = \begin{pmatrix} 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 2 \end{pmatrix}$$

$Q_5$ is a lattice polytope with edges that have interior lattice points. Moreover, it has two prime decompositions and both are given in Figure 3.3. Furthermore Figure 3.4 shows three different decompositions of $Q_5$. The primitive edge vectors $\tilde{t}$ of the first summand of each decomposition are given below.

$$\tilde{t}_P = (1, 0, 0, 1, 1, 0, 1, 1)$$
$$\tilde{t}_Q = (0, 1, 1, 1, 0, 1, 1, 0)$$
$$\tilde{t}_R = (0, 0, 1, 1, 0, 0, 1, 1)$$
The Schur products of these vectors can be computed.

\[ \tilde{t}_P \circ \tilde{t}_Q = (0, 0, 0, 1, 0, 1, 0) \]
\[ \tilde{t}_Q \circ \tilde{t}_R = (0, 0, 0, 1, 0, 1, 1) \]
\[ \tilde{t}_P \circ \tilde{t}_R = (0, 0, 1, 1, 0, 0, 1, 0) \]

For these vectors the sum in (3.12) evaluate to (0, 1) + (0, -1) = 0, (0, 1) + (0, -1) + (0, -1) = (0, -1) respectively (0, 1) + (0, 1) + (0, -1) = (0, 1). From this can be concluded that the decompositions \( P_0 \oplus P_1 \) and \( Q_0 \oplus Q_1 \) are compatible but the decompositions \( P_0 \oplus P_1 \) and \( R_0 \oplus R_1 \) and the decompositions \( Q_0 \oplus Q_1 \) and \( R_0 \oplus R_1 \) do not have a common refinement.

### 3.4 The compatibility of general lattice polyhedra

Let \( Q \) be a polyhedron with \( n \) bounded edges. This polyhedron is the Minkowski sum of a polytope and its characteristic cone.

\[ Q = \text{Conv}(\text{Vert}(Q)) \oplus \text{Char}(Q) \]

A decomposition \( P_0 \oplus P_1 \) of \( Q \) is admissible if and only if for every face \( F \) at most one of its summands \( F_i \) does not contain a lattice point. This means that if \( Q \) is not bounded, then both summands should have the same characteristic cone. Observe that \( \text{Conv}(\text{Vert}(P_0)) \oplus \text{Conv}(\text{Vert}(P_1)) \) is not necessarily equal to \( \text{Conv}(\text{Vert}(Q)) \) for an admissible decomposition of \( Q = P_0 \oplus P_1 \). An example is given in Figure 3.5.

![Figure 3.5: An example of a polyhedron Q and an admissible decomposition P_0 \oplus P_1 such that Conv(Vert(Q)) \neq Conv(Vert(P_0)).](image-url)

If \( \text{Conv}(\text{Vert}(P_0)) \oplus \text{Conv}(\text{Vert}(P_1)) = \text{Conv}(\text{Vert}(Q_0)) \oplus \text{Conv}(\text{Vert}(Q_1)) = \text{Conv}(\text{Vert}(Q)) \) then it automatically follows that the decompositions \( P_0 \oplus P_1 \) and \( Q_0 \oplus Q_1 \) are compatible if and only if the decompositions \( \text{Conv}(\text{Vert}(P_0)) \oplus \text{Conv}(\text{Vert}(P_1)) \) and \( \text{Conv}(\text{Vert}(Q_0)) \oplus \text{Conv}(\text{Vert}(Q_1)) \) are compatible. These decompositions are decompositions of the polytope \( \text{Conv}(\text{Vert}(Q)) \), which means that the procedure that is described in the previous sections can be used to determine whether these decompositions are compatible.
From now on assume that at least one of the polytopes \( \text{Conv}(\text{Vert}(P_0)) \oplus \text{Conv}(\text{Vert}(P_1)) \) and \( \text{Conv}(\text{Vert}(Q_0)) \oplus \text{Conv}(\text{Vert}(Q_1)) \) is not equal to the polytope \( \text{Conv}(\text{Vert}(Q)) \). Let \( F \) be an unbounded 2-face of \( Q \) that has bounded edges \( d^i \) for \( i \in I_F \) where \( I_F \) is a subset of \( \{0, \ldots, n-1\} \). Observe that every decomposition of \( F \) has \( \bigoplus d^i (\oplus \text{Char}(Q)) \) as a refinement. Therefore we can say that every pair of decompositions have a common refinement regarding the unbounded 2-faces of \( Q \). Furthermore define \( \varepsilon \) as the sign vector of a bounded 2-face \( B \), such that \( \varepsilon_i = \pm 1 \) if the edge \( d^i \) is an edge of \( B \) and 0 otherwise. Now \( \sum \varepsilon_i d^i = 0 \) for every bounded 2-face \( B \). Equivalent to the previous section, the vector space \( \mathcal{V}(Q) \) and the corresponding cone \( \mathcal{C}(Q) \) can be constructed. The elements \( t \in \mathcal{C}(Q) \) correspond to the summands \( Q_t \) of \( Q \) such that it has bounded edges \( t_i d^i \). Every summands has the same characteristic cone, and therefore a summand \( Q_t \) does not have any bounded edges that is not a bounded edge of \( Q \).

The bounded edge vector of a decomposition \( P_0 \oplus P_1 \) is denoted by \( t_P \). This is the edge vector of the summand \( P_0 \) and \( 1 - t_P \) is the bounded edge vector of the summand \( P_1 \).

**Theorem 3.9.** The decompositions \( P_0 \oplus P_1 \) and \( Q_0 \oplus Q_1 \) have a common refinement if and only if \( t_P \circ t_Q \in \mathcal{C}(Q) \). This means that

\[
\sum_{i=0}^{n-1} (t_P)_i (t_Q)_i b^i = 0
\]  

(3.13)

for every bounded 2-face \( B \) of \( Q \).

The proof for this theorem is similar to the proof of Theorem 3.5.

This concludes the proof of this section. Now we have shown how to determine whether two decompositions of a polyhedron have a common refinement. One can generalize the theorem for polyhedra, combining the given proofs. The proof for this is not given explicitely, but the reader can see that a generalization is easily made.
Chapter 4

Unequal cross cuts

In the previous chapter the decompositions \( P_0 \oplus P_1 \) and \( Q_0 \oplus Q_1 \) are decompositions of the polyhedron \( Q_r \). If the two decompositions have a common refinement, then the decompositions are compatible. This section concerns the case where the decompositions sum up to different, but related polyhedra. Given a cone \( \sigma \subseteq N_{\mathbb{R}} \) and two vectors \( r, s \in \sigma^\vee \), one can determine the polyhedra \( Q_r \) and \( Q_s \). The question that plays a central role in this chapter is whether two given decompositions of \( Q_r \) and \( Q_s \) are compatible. An answer can not be given directly. The summands should be embedded in a higher dimensional lattice in order to determine if two decompositions are compatible. The first step is to adjust the notion of compatibility for this particular case. Some basic steps are given which explain the approach that is used in this chapter. The procedure will be elaborated for the special case where \( \sigma \) is a 2-dimensional cone.

In [18] Nathan Ilten describes this problem for \( \sigma \) being a 2-dimensional cone in terms of Deformation Theory of toric varieties. The main objective of this chapter is to understand this combinatorically. One can see [1], [2], [4] from Altmann, or [12] [17] for more information about this Deformation theory and toric varieties.

4.0.1 The basic principle

To be able to understand the procedure that is used in this chapter, we show how the original polytope \( Q_r \) can be obtained from a higher dimensional polytope which is created from the summands \( P_0 \) and \( P_1 \) such that \( Q_r = P_0 \oplus P_1 \). This is done by means of an example. Consider the hexagon \( Q_6 = P_0 \oplus P_1 \) and its decomposition in two triangles as in Figure 2.5. This decomposition is admissible and described in Equation (2.1). The 2-dimensional summands are embedded at different heights in the extended lattice \( N \times \mathbb{Z} \). In general \( P_0 \) is embedded with \( z = 1 \) and \( P_1 \) at height \( z = 0 \), i.e. the
vertices \( v \) of \( P_0 \) become \((v, 1)\) and the vertices of \( P_1 \) are now \((v, 0)\). This is in line with Remark 1 where we consider \( Q_6 \) as the polyhedral cone \( \sigma(Q_6) \) such that the polyhedron \( Q_6 \) is embedded at height 1.

Consider the convex hull of these vertices. This yields a polytope with 6 vertices and 8 2-faces living in \( N \times \mathbb{Z} \cong \mathbb{R}^3 \). Its vertices are given below.

\[
\text{Conv}(P_0, P_1) = \begin{pmatrix}
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

The original polytope \( Q_6 \) is now obtained by intersecting this polytope at \( z = \frac{1}{2} \) as illustrated in Figure 4.1.

**Figure 4.1:** The convex hull of \( P_0 \) and \( P_1 \) with the intersection which yields \( \frac{1}{2} Q_6 \).

\[
H_6 := \{ a \in \mathbb{R}^3 | a^\top (0, 0, 1) = \frac{1}{2} \}
\]

\[
\frac{1}{2} Q_6 = \text{Conv}(P_0, P_1) \cap H_6
\]

Instead of determining the polytope \( \text{Conv}(P_0, P_1) \) one can consider a cone \( \sigma_r \) that is embedded in \( N \times \mathbb{Z}^2 \). Its rays are constructed from the summands \( P_0 \) and \( P_1 \) such that the rays are \((v_0, 1, 0)\) and \((v_1, 0, 1)\) for the vertices \( v_i \) of \( P_i \). Intersecting this cone yields the original cone \( \sigma \). This principle of first lifting the summands and then intersecting the obtained polytope is used to determine whether cross cuts \( Q_r \) and \( Q_s \) are compatible if \( r \) is not equal to \( s \).

### 4.1 The procedure for \( r \neq s \)

As before, let \( N \cong \mathbb{Z}^d \) be a lattice with \( M \) its dual lattice. Note that as \( N \cong \mathbb{Z}^d \) we also have that \( M \cong \mathbb{Z}^d \). Let \( \sigma \subseteq N_\mathbb{R} \cong \mathbb{R}^d \) be a cone that is generated by \( k \) lattice vectors \( a^i \). Let \( \sigma^\vee \) be the dual cone and fix two primitive vectors \( r, s \in \sigma^\vee \cap M \). As defined in
Definition 2.20, the cross cut of σ and the hyperplane \( H_r \) gives the polyhedron

\[ Q_r = \sigma \cap \{ x \in N_{\mathbb{R}} | r^\top x = 1 \}. \]

Similarly, the cross cut of the cone σ and the hyperplane \( H_s \) yields the polyhedron \( Q_s \).

Fix two admissible decompositions \( Q_r = P_0 \oplus P_1 \) and \( Q_s = Q_0 \oplus Q_1 \).

The polytopes \( Q_r \) and \( Q_s \) are not equivalent for \( r \neq s \) and therefore it is not straightforward to see what is meant by being compatible. This cannot be determined directly from the polytopes \( Q_r \) and \( Q_s \) and their decompositions \( P_0 \oplus P_1 \) and \( Q_0 \oplus Q_1 \). Instead of comparing the decompositions \( P_0 \oplus P_1 \) and \( Q_0 \oplus Q_1 \), two new cones \( \sigma_{r,s} \) and \( \sigma_{s,r} \) are constructed in a higher dimensional lattice. These cones depend on the choice of the decompositions of the polytopes. If these cones satisfy some predefined conditions (which will be given later on), then the decompositions \( P_0 \oplus P_1 \) and \( Q_0 \oplus Q_1 \) are compatible. The question why the compatibility can be derived from these two cones, will become clear when the construction of these cones is given.

Given the two decompositions \( Q_r = P_0 \oplus P_1 \) and \( Q_s = Q_0 \oplus Q_1 \), the cones \( \sigma_r \) and \( \sigma_s \) can be constructed such that the rays of these cones correspond to the vertices of the summands of \( Q_r \) respectively \( Q_s \). Observe that the cone \( \sigma \) can be obtained from \( \sigma_r \) and \( \sigma_s \) by taking \( \sigma_r \cap N \) or \( \sigma_s \cap N \) respectively.

Two vectors \( \tilde{s} \in \sigma_r^\vee \) and \( \tilde{r} \in \sigma_s^\vee \) can be constructed from the cones \( \sigma_r \) and \( \sigma_s \) such that they relate to the vectors \( r \) and \( s \). Now the cross cut of \( \sigma_r \) and \( H_{\tilde{s}} = \{ x \in N_{\mathbb{R}} | \tilde{s}^\top x = 1 \} \) yields the polyhedron \( Q_{r,s} = \sigma_r \cap H_{\tilde{s}} \), where \( N_{\mathbb{R}} \cong \mathbb{R}d + 2 \). This polyhedron depends on both vectors \( r \) and \( s \). Similarly, the polyhedron \( Q_{s,r} \) is constructed.

Fix an admissible decomposition of the polyhedron \( Q_{r,s} \) and one for \( Q_{s,r} \). As before, two new cones \( \sigma_{r,s} \) and \( \sigma_{s,r} \) can be constructed such that the rays of these cones correspond to the vertices of the summands of \( Q_{r,s} \) respectively \( Q_{s,r} \). The polyhedra \( Q_{r,s} \) and \( Q_{s,r} \) are not equivalent and therefore their decompositions will not be comparable. However, if the original decompositions \( P_0 \oplus P_1 \) and \( Q_0 \oplus Q_1 \) are compatible, then the lifted cones \( \sigma_{r,s} \) and \( \sigma_{s,r} \) will have dual cones that are similar. This means that there exists a mapping between the cones \( \sigma_{r,s} \) and \( \sigma_{s,r} \).

**Definition 4.1.** If the dual cones \( \sigma_{r,s}^\vee \) and \( \sigma_{s,r}^\vee \) are row equivalent then the original decompositions of \( Q_r \) and \( Q_s \) are compatible.

We will clarify this definition with an example. Let \( \sigma \) be the cone that is generated by the rays \((0, 0, 1), (0, 1, 1), (1, 0, 1) \) and \((1, 1, 1) \). We compute three cross cuts of this cone, together with a decomposition and show whether they are compatible. The vectors \( r = (0, 0, 1), s = (1, 0, 1) \) and \( t = (0, 1, 1) \) are all elements in \( \sigma^\vee \) and they are used to
determine the polyhedra $Q_r$, $Q_s$ and $Q_t$. This results in the three polytopes given in

\[
\begin{align*}
(1,1,1) \\
(1,0,1) \\
(0,0,1) \\
(0,1,1) \\
(\frac{1}{2},\frac{1}{2},\frac{1}{2})
\end{align*}
\]

\[
\begin{align*}
(1,0,1) \\
(0,0,1) \\
(0,\frac{1}{2},\frac{1}{2}) \\
(\frac{1}{2},\frac{1}{2},\frac{1}{2}) \\
(\frac{1}{2},0,\frac{1}{2})
\end{align*}
\]

\[
\begin{align*}
(0,1,1)
\end{align*}
\]

Figure 4.2: The polytopes $Q_r$, $Q_s$ and $Q_t$.

Figure 4.2. The decompositions of these polytopes are given below. Remark that these decompositions are actually not admissible decompositions, but this will not affect our example. Intuitively, one can see that the decompositions of $Q_s$ and $Q_t$ are compatible,

\[
\begin{align*}
\begin{array}{c}
\framebox{ } - \\
\framebox{ } - \\
\framebox{ } - \\
\framebox{ } - \\
\framebox{ } -
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\oplus \\
\oplus \\
\oplus \\
\oplus \\
\oplus
\end{array}
\end{align*}
\]

while both of these decompositions are not compatible with the decomposition of $Q_r$.

In this chapter we will elaborate on the construction of the cones $\sigma_{r,s}^\vee$ and $\sigma_{s,r}^\vee$ which shows the compatibility of the decompositions of $Q_r$ and $Q_s$.

4.1.1 Lifting the summands of $Q_r$

Fix an admissible decomposition of the polyhedron $Q_r = P_0 \oplus P_1$. As before the polytope $Q_r$ has $n$ edges. Define the extended lattice $\mathbb{N} = N \times \mathbb{Z}^2 \cong \mathbb{Z}^{d+2}$ from $N$ and write $\overline{\mathbb{M}}$ as its dual lattice. A new cone can be constructed in this extended lattice. This cone is constructed from the decomposition of $Q_r$.

In line with Remark 1 we will embed the summand $P_0$ in $H_r^0 = \{x \in \mathbb{N} | r^\top x = 1\}$ and the summand $P_1$ in $H_r^1 = \{x \in \mathbb{N} | r^\top x = 0\}$. From this embedding we are able to construct the lifted cone $\sigma_r$. 
**Definition 4.2.** Let $Q_r = P_0 \oplus P_1$ be an admissible decomposition. The cone $\sigma_r \subseteq \mathbb{N}$ is generated by the rays $(v_0, 1, 0), (v_1, 0, 1)$ and $(z, 0, 0)$ such that each vertex $v_0$ of $P_0$ corresponds to a ray $(v_0, 1, 0)$ of $\sigma_r$, each vertex $v_1$ of $P_1$ corresponds to a ray $(v_1, 0, 1)$ and each $z \in \text{Char}(Q)$ corresponds to a ray $(z, 0, 0)$. Let $k_i$ be the number of vertices of the summand $P_i$. Moreover let $l$ be the number of rays of $Q$. The cone $\sigma_r$ is generated by $k_0 + k_1 + l$ extremal rays.

$$
\sigma_r := \text{Cone}\left\{ (v_0, 1, 0), (v_1, 0, 1), (z, 0, 0) \mid v_0 \in P_0, v_1 \in P_1, z \in \text{Char}(Q) \right\}
$$

**Remark 4.** The original cone can be obtained from $\sigma_r$ by $\sigma = \sigma_r \cap \mathbb{N}_R$.

The decompositions given in Figure 4.3 result in the following cones $\sigma_r, \sigma_s$ and $\sigma_t$.

$$
\sigma_r = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix},
\quad
\sigma_s = \begin{pmatrix}
0 & 0 & 1/2 & 0 & 0 \\
0 & 1/2 & 0 & 0 & 1/2 \\
1 & 1 & 1/2 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix},
\quad
\sigma_t = \begin{pmatrix}
0 & 1/2 & 0 & 0 & 1/2 \\
0 & 0 & 1/2 & 0 & 0 \\
1 & 1 & 1/2 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}
$$

If the role of the summands $P_0$ and $P_1$ are swapped, then the resulting cone $\sigma_r$ is different. However, there will exist an equivalency between the two cones $\sigma_{r,s}$ and $\sigma_{s,r}$.

The reader can check that the decomposition of $Q_r$ does not give a satisfying lifted cone $\sigma_{r,s}$ or $\sigma_{r,t}$, which means that they are not compatible.

### 4.1.2 Determining the cross cut of $\sigma_r$ and $\tilde{s}$

The cones $\sigma_r$ and $\sigma_s$ can be computed as explained in the previous section. The next step is to compute the lifted polyhedra $Q_{r,s}$ and $Q_{s,r}$. The polyhedron $Q_{r,s}$ is defined as the cross cut of the cone $\sigma_r$ and a vector $\tilde{s} \in \sigma_r^\vee$ that corresponds to the vector $s \in \sigma^\vee$. Define a surjection $\phi_r : \mathbb{N} \to \mathbb{N}$ such that $\phi(\tilde{s}) = s$.

$$
\phi_r := \left( I_d | r \right)
$$

with $I_d$ the $d \times d$ identity matrix.

Moreover the matrix $\phi_r^\top$ has the nice property that $\phi_r^\top \sigma$ results in the cone $\sigma_r \cap \mathbb{N}$.

**Theorem 4.3.** Let $\sigma_r \subseteq \mathbb{N}_R$ be the lifted cone and $r, s \in \sigma^\vee$. There exists a vector $\tilde{s} \in \sigma_r^\vee \cap \mathbb{M}$ such that $\phi_r(\tilde{s}) = s$. Moreover, if $r \neq s$ then the cross cut $Q_{r,s}$ is uniquely determined.
Proof. The vectors $(r, -1, 0)$ and $(0, 1, -1)$ span the nullspace of $\phi_r$.

By definition $\phi_r((s,0,0)) = s$. This means that $\tilde{s}$ can be written as

$$\tilde{s} = (s,0,0) + \alpha_0(r, -1, 0) + \alpha_1(0,1,-1)$$

for some integers $\alpha_0, \alpha_1 \in \mathbb{Z}$. We show existence of a $\tilde{s} \in \sigma_r^\vee \cap \mathcal{M}$ by constructing such a vector. This vector $\tilde{s}$ is an element of $\sigma_r^\vee$ and therefore

$$\tilde{s}^\top a \geq 0 \text{ for all } a \in \sigma_r.$$ 

Recall that the extremal rays of $\sigma_r$ are constructed from the Minkowski summands of $Q_r$. Moreover, the summands are embedded such that $P_0 \subset H_r$ and $P_1 \subset H_r^0$. Therefore,

$$r^\top v_0 = 1 \quad \text{and} \quad r^\top v_1 = 0$$

where $v = v_0 + v_1$ and $v_0 \in P_0, v_1 \in P_1$. The rays $z$ of $Q_r$ are exactly the generators of $\sigma$ such that $r^\top z = 0$. Thus

$$(r,-1,0)^\top a = \begin{cases} (r,-1,0)^\top (v_0,1,0) \\ (r,-1,0)^\top (v_1,0,1) \\ (r,-1,0)^\top (z,0,0) \end{cases}$$

all evaluate to 0. This means that the vector $(r,-1,0)$ does not contribute to a different cross cut $Q_{r,s}$. From now on, assume that $\alpha_0 = 0$ and $\tilde{s} = (s,0,0) + \alpha_1(0,1,-1)$

Consider $\alpha_1(0,1,-1)^\top a$ for all extremal rays $a \in \sigma_r$.

$$(0,1,-1)^\top a = \begin{cases} (0,1,-1)^\top (v_0,1,0) = 1 \\ (0,1,-1)^\top (v_1,0,1) = -1 \\ (0,1,-1)^\top (z,0,0) = 0 \end{cases}$$

Thus $(0,1,-1) \notin \sigma_r^\vee$. From the fact that $s \in \sigma_r^\vee$ we know that $s^\top v \geq 0$ for each vertex $v$ of $Q$. This yields $s^\top v = s^\top v_0 + s^\top v_1 \geq 0$ with $v_0, v_1$ a vertex of $P_0$ respectively $P_1$. Thus $(s,0,0)^\top (v_1,0,1) \geq -(s,0,0)^\top (v_0,1,0)$, which proves that there exists an integer $\alpha_1$ such that the vector $\tilde{s} = (s,0,0) + \alpha_1(0,1,-1) \in \sigma_r^\vee$. \(\square\)

To continue with our example, we will construct the lifted vector $\tilde{t}$ which is an element in $\sigma_s^\vee$. We have $\alpha = 0$ as $\sigma_s$ only has nonnegative entries. Hence, $\tilde{t} = (0,1,1,0,0)$. Similarly, the vector $\tilde{s} = (1,0,1,0,0) \in \sigma_t^\vee$ is constructed.
4.1.3 Lifting the summands of $Q_{r,s}$

The next step is to determine the lifted polyhedron $Q_{r,s}$ which is the cross cut of $\sigma_r$ and the hyperplane $H_{\tilde{s}} = \{x \in \mathbb{N}_R | \tilde{s}^T x = 1\}$.

$$Q_{r,s} = \sigma_r \cap H_{\tilde{s}}$$

Fix an admissible decomposition of this polyhedron.

$$Q_{r,s} = \tilde{Q}_0 \oplus \tilde{Q}_1$$

These summands are lifted to a higher dimensional lattice $\overline{\mathbb{N}} := \mathbb{N} \times \mathbb{Z}^2 \cong \mathbb{Z}^{d+4}$, yielding the cone $\sigma_{r,s}$. The summands of $Q_{r,s}$ are embedded at different heights, which results in a second lifted cone $\sigma_{r,s} \subseteq \overline{\mathbb{N}}_R$. Each vertex $\tilde{v}_0$ of the summand $\tilde{Q}_0$ corresponds to a ray $(\tilde{v}_0, 0, 1)$ of $\sigma_{r,s}$. Similarly, each vertex $\tilde{v}_1$ of $\tilde{Q}_1$ corresponds to a ray $(\tilde{v}_1, 1, 0)$. Moreover, the rays $\tilde{z}$ of $Q_r$ correspond to the rays $(\tilde{z}, 0, 0)$. There exists a surjection from $\sigma_{r,s}$ to $\sigma$ and is induced by the following matrix.

$$\phi_{r,s} = \begin{pmatrix} I_d & r & r & s & s \end{pmatrix}$$

(4.2)

where $I_d$ is the $d$-dimensional identity matrix.

The cross cuts $\sigma_s \cap H_t$ and $\sigma_t \cap H_{\tilde{s}}$ yield the polyhedra $Q_{s,t}$ and $Q_{t,s}$. Observe that they both have four vertices and one ray, which is indicated by the last row.

$$Q_{s,t} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 1 & 0 \\ 1 & \frac{2}{3} & 1 & 0 & 0 \\ 1 & \frac{2}{3} & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$Q_{t,s} = \begin{pmatrix} 0 & \frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & \frac{2}{3} & 1 & 0 & 0 \\ 1 & \frac{2}{3} & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

These polyhedra have a lattice shift as an admissible decomposition. These decompositions are fixed, which yields a lattice shift over the point $(0, 0, 1, 1, 0)$.

4.1.4 Comparing the cones $\sigma_{r,s}$ and $\sigma_{s,r}$

If the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ of $Q_r$ and $Q_s$ respectively are compatible, then there exist cones $\sigma_{r,s}^\vee$ and $\sigma_{s,r}^\vee$ that are row equivalent. These cones depend on the choice of the decomposition of $Q_{r,s}$ and $Q_{s,r}$. The decompositions of $Q_{r,s}$ and $Q_{s,r}$ should be chosen correctly in order for the equivalence to exist. If a suitable admissible
decomposition of these lifted polyhedra does not exist, then the initial decompositions \( P_0 \oplus P_1 \) and \( Q_0 \oplus Q_1 \) are not compatible.

In our example the lifted cone \( \sigma_{s,t} \) is generated by six rays. Its dual cone \( \sigma_{s,t}^\vee \) is generated by six rays as well. The rays of both cones are given below.

\[
\sigma_{s,t} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 1 & 0 \\ 1 & 0 & -\frac{1}{3} & 0 & -1 & 0 \\ 1 & 0 & -\frac{1}{3} & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \sigma_{s,t}^\vee = \begin{pmatrix} 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}
\]

From the construction of the polyhedra \( Q_{s,t} \) and \( Q_{t,s} \) can easily be seen that the cones \( \sigma_{s,t} \) and \( \sigma_{t,s} \) are generated by the same rays, except that the first two rows are swapped. Thus, the cones \( \sigma_{t,s}^\vee \) and \( \sigma_{s,t}^\vee \) are the same up to interchanging the first two rows, which means that these cones are indeed row equivalent. Therefore the decompositions of \( Q_s \) and \( Q_t \) are compatible.

### 4.2 The procedure for \( \sigma \subseteq \mathbb{Z}^2 \)

In this section we will elaborate the specific case where \( d = 2 \). This is the specific case where Ilten elaborates on in [18]. The result that is described in this section is in line with the results in this article, although derived with a different approach. Being a 2-dimensional cone, \( \sigma \) is embedded in the lattice \( N \cong \mathbb{Z}^2 \) and the polyhedra \( Q_r \) and \( Q_s \) are line segments. An admissible decomposition is defined according to Definition 2.9.

However, a decomposition of the form \( Q_r = \lambda Q_r \oplus (1 - \lambda) Q_r \) such that \( 0 < \lambda < 1 \) is said to be a nontrivial admissible decomposition as well.

In order for a line segment to have an admissible decomposition, it has to contain at least one lattice point. Therefore, the choice for the vectors \( r \) for the polytope \( Q_r \) is limited. The line segment \( \sigma \cap \{ x \in \mathbb{R}^2 | r^\top x = 1 \} \) contains a lattice point if and only if \( r \) is a (primitive) vector that is an element of the Hilbert basis of the 2-dimensional cone \( \sigma^\vee \). Thus, from now on we will assume that \( r \in E \) where \( E \) is the Hilbert basis of \( \sigma^\vee \).

As one can see, the cross cut is a line segment that is embedded in a 2-dimensional lattice. There exists a projection such that this line segment is embedded in a 1-dimensional lattice. As each cross cut contains at least one interior lattice point (as \( r \) is a primitive vector and an element in the Hilbert basis), one can take this lattice point as the origin.
of this projected lattice. The notion of being compatible does not change due to this projection, it will only simplify the computations.

4.2.1 Constructing the cone $\sigma_h$

**Theorem 4.4.** Let $N \cong \mathbb{Z}^2$ be the 2-dimensional standard lattice and let $M \cong \mathbb{Z}^2$ be its dual lattice. Let $\sigma \subseteq N_\mathbb{R}$ be a 2-dimensional cone that is generated by two (primitive) vectors $u$ and $v$. Without loss of generality one can say that

$$\sigma = \begin{pmatrix} 1 & -q \\ 0 & n \end{pmatrix}$$

where $\det(u, v) = n$ and $q, n \in \mathbb{Z}$ are relatively prime with $0 \leq q < n$.

**Proof.** For all primitive generators $u = (u_1, u_2)$ and $v = (v_1, v_2)$ there exists a lattice automorphism mapping $u$ to $(1, 0)$ and $v$ to $(-q, n)$. If the vectors $u$ and $v$ are independent, then the matrix $\sigma$ is invertible. The group $GL(2, N)$ is exactly the set of $2 \times 2$ invertible matrices over the lattice $N$. Therefore there exists a (not necessarily unique) lattice automorphism $\phi : N \to N$ such that $\phi(u) = (1, 0)$ and $\phi(u) = (-q, n)$, where $q$ and $n$ with $0 \leq q < n$ are relatively prime.

From now on assume that $u = (1, 0)$ and $v = (-q, n)$. The dual cone $\sigma^\vee$ is generated by two rays.

$$\sigma^\vee = \begin{pmatrix} 0 & n \\ 1 & q \end{pmatrix}$$

Furthermore, since the vectors $r$ and $s$ are elements of the Hilbert basis of $\sigma^\vee$ we are interested in this Hilbert basis. In the 2-dimensional case the Hilbert basis of $\sigma^\vee$ is defined as the set of vectors such that every ray in $\sigma^\vee$ is the sum of a nonnegative multiple of these vectors. Let $E = \{w^0, \ldots, w^{g+1}\}$ be the Hilbert basis of $\sigma^\vee$ such that $w^0 = (1, 0)$ and $w^{g+1} = (n, q)$. The vectors $w^i$ with $1 \leq i \leq g$ can be determined from the (negative) continued fraction expansion:

$$\frac{n}{n-q} = a_1 - \frac{1}{a_2 - \frac{1}{\ddots - \frac{1}{a_g}}}.$$ 

For more information about this continued fraction expansion, see [2] and [29].

The elements of the Hilbert basis are now obtained from $w^{i-1} + w^{i+1} = a_i w^i$ for $i = 1, \ldots, g$. 
Remark 5. The cross cut \( Q_r \) contains at least one interior lattice point and is bounded if and only if \( r = w^h = (w^h_1, w^h_2) \) for some \( 1 \leq h \leq g \). This line segment will be denoted by \( Q_h \).

\[
Q_r := Q_h = \left[ \left( \frac{-q}{nw^h_2 - qw^h_1}, \frac{1}{nw^h_2 - qw^h_1} \right) \right]
\]

**Theorem 4.5.** Let \( Q_h \) be the intersection of \( \sigma \) and \( w^h \). If \( 1 < h < g \) then \( Q_h \) does not have lattice endpoints. Moreover, if \( h \in \{1, g\} \) then \( Q_h \) has exactly one lattice endpoint.

**Proof.** If \( \frac{1}{w^h_1} \in N \) then we must have \( w^h_1 = 1 \). This means that \( w^h = (w^h_1, w^h_2) = (1, 1) = w^1 \) since \( 0 < w^h_2 \leq w^h_1 \). This gives

\[
\left( \frac{-q}{nw^h_2 - qw^h_1}, \frac{1}{nw^h_2 - qw^h_1} \right) \notin N.
\]

Moreover, \( \frac{1}{nw^h_2 - qw^h_1} \in N \) means that \( nw^h_2 - qw^h_1 = 1 \) since \( n \) and \( q \) are relatively prime. This implies that \( w^h = w^g \). Thus, \( Q_h \) has exactly one lattice endpoint when \( h \in \{1, g\} \) and no lattice endpoints otherwise. \( \square \)

The polyhedron \( Q \) is embedded in \( N \cong \mathbb{Z}^2 \) and it always has an interior lattice point, say \( v \in N \). Therefore we can create the projected polyhedron \( Q'_h \) which is embedded in a lattice of one dimension less, which is referred to as \( N^- (\cong \mathbb{Z}) \). Every point \( x \in Q_h \) is projected to a specific \( x' \in Q'_h \):

\[
x = v + x r
\]

where \( r \) is the vector that is perpendicular to \( w^h \) (i.e. \( r = (w^h_2, -w^h_1) \)). In general this can be captured in the following theorem.

**Theorem 4.6.** Let \( Q \) be a polyhedron that is embedded in an \( d \)-dimensional lattice \( N \).

Let \( v \in Q \) be a lattice vertex. Moreover, let \( g^i \in N \) for \( i = 1, \ldots, d - 1 \) be vectors such that \( v, g^1, \ldots, g^{d-1} \) is a generating set of \( N \). Now \( Q' \) is the projected polyhedron of \( Q \) if for every \( x \in Q \) there exists a \( x' \in Q' \) such that

\[
x = v + \sum_{i=1}^{d-1} x^i g^i.
\]

**Remark 6.** The vertex \( v \in Q \) is projected to the origin of the lattice \( (d - 1) \)-dimensional lattice \( N^- \) and every lattice point \( u \in N \) corresponds to a lattice point \( u' \in N^- \).

Furthermore, if \( Q \) is a polyhedron of dimension \( d \), then there is at least one \( x \in Q \) such that Equation 4.3 does not hold true.

As \( Q_h \) is a line segment and \( N \cong \mathbb{Z}^2 \) there exists a projected polyhedron \( Q'_h \) in \( N^- \cong \mathbb{Z} \). From now on, we will simply denote this polyhedron as \( Q_h \) for readability. Thus \( Q_h \) is
seen as the line segment in \( \mathbb{Z} \) and say \( Q_h = [a, b] \) where \( a, b \in \mathbb{Q} \) and \( a < 0 < b \). The end points \( a \) and \( b \) can be computed explicitly. The projection yields

\[
\frac{1}{nw_2^h - qw_1^h} (-q, n) = v + a(w_2^h, -w_1^h)
\]

\[
a = \frac{-q}{nw_2^h - qw_1^h} - v_1
\]

\[
= \frac{-nv_1 + qv_2}{nw_2^h - qw_1^h}
\]

(4.4)

\[
\frac{1}{w_1^h}, 0 = v + b(w_2^h, -w_1^h)
\]

\[
b = \frac{1}{w_1^h} - v_1
\]

\[
= \frac{v_2}{w_1^h}
\]

(4.5)

**Lemma 4.7.** Let

\[
Q_h = [a, b] = [a, v] \oplus [0, b - v]
\]

be a decomposition of \( Q_h \) such that \( v < b \). It is an admissible decomposition if and only if

1. \( v \in N^- \) and \( v \neq a \), or

2. \( v \notin N^- \) but \( (b - v) \in N^- \)

In the second case \( v \) might be equal to \( a \). Note that \( v = 0 \) is one of the admissible decompositions of \( Q_h \) by construction.

These are all possible admissible decompositions, up to lattice shifts. Although these lattice shifts are admissible decompositions they will not be taken into consideration in this chapter, as their decompositions do not differ from the decompositions given here.

Define the extended lattice \( N^+ \cong \mathbb{Z}^3 \) and fix an admissible decomposition \([a, v] \oplus [0, b - v]\). A new cone \( \sigma_h \subseteq N^+_h \) is now constructed.

\[
\sigma_h = \begin{pmatrix} a & v & 0 & b - v \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}
\]
If \( v = a \) then the first two columns are identical, which means that the cone \( \sigma_h \) has three extremal rays. The dual cone \( \sigma^\vee_h \) is generated by the following vectors.

\[
\sigma^\vee_h = \begin{pmatrix}
1 & -1 & 0 & 0 \\
-a & v & 1 & 0 \\
0 & b - v & 0 & 1
\end{pmatrix} = \begin{pmatrix}
w_2^h - qw_1^h & -w_1^h & 0 & 0 \\
v_1 + qv_2 & vw_1 & 1 & 0 \\
0 & v_2 - vw_1 & 0 & 1
\end{pmatrix} \tag{4.6}
\]

If \( v = a \) then the ray \((0, 0, 1)\) is not an extremal ray of \( \sigma^\vee_h \). It is however an element of the Hilbert basis of \( \sigma^\vee_h \).

The Hilbert basis \( E_h = \{ w_0^h, \ldots, w_{g+1}^h, \tilde{w}_h \} \) of \( \sigma^\vee_h \) consists of \( g + 3 \) vectors, with \( w_0^h = (-w_1^h, vw_1^h, v_2 - vw_1^h) \), \( w_{g+1}^h = (nw_2^h - qw_1^h, nv_1 + qv_2, 0) \) and \( \tilde{w}_h = (0, 0, 1) \). There exists a surjection \( \phi_h \) such that \( \phi_h(w_i^h) = w_i \) for \( i = 0, \ldots, g + 1 \) and \( \phi_h(\tilde{w}_h) = w_h \). This mapping is induced by the following matrix:

\[
\phi_h = \begin{pmatrix}
v_2 & w_1^h & w_1^h \\
v_1 & w_2^h & w_2^h
\end{pmatrix} \tag{4.7}
\]

where \( v = (v_1, v_2) \in N \) is the lattice point that is projected to the origin.

**Example.**

The statements in this section will be illustrated with an example. Let \( n = 4 \) and \( q = 1 \) and determine the corresponding cones \( \sigma \) and \( \sigma^\vee \).

\[
\sigma = \begin{pmatrix}
1 & -1 \\
0 & 4
\end{pmatrix} \quad \quad \sigma^\vee = \begin{pmatrix}
0 & 4 \\
1 & 1
\end{pmatrix}
\]

The continued fraction expansion for \( \frac{n}{n-q} \) gives us the factors \( \{a_1, \ldots, a_g\} \) which are used to determine the Hilbert basis for \( \sigma^\vee \).

\[
\frac{n}{n-q} = \frac{4}{3} = 2 - \frac{1}{2 - \frac{1}{2}}
\]

Therefore \( \{a_1, a_2, a_3\} = \{2, 2, 2\} \) and

\[
2w^1 = (0, 1) + w^2 \\
2w^2 = w^1 + w^3 \\
2w^3 = w^2 + (4, 1)
\]

which results in \( \{w^0, w^1, w^2, w^3, w^4\} = \{(0, 1), (1, 1), (2, 1), (3, 1), (4, 1)\} \). This is indeed the Hilbert basis of \( \sigma^\vee \). Fix \( r = w^1 = (1, 1) \). This results in the cross cut \( Q_h = Q_1 \):

\[
Q_1 = \sigma \cap H_1 = \left[ \left( \frac{-1}{3}, \frac{4}{3} \right), (1, 0) \right] \rightarrow \left[ -\frac{1}{3}, 1 \right]
\]
This line segment has a unique admissible decomposition.

\[ Q_1 = [-\frac{1}{3}, 0] \oplus [0, 1] \]  

(4.8)

This leads to the cone \( \sigma_1 \) and its dual cone \( \sigma_1^\vee \).

\[
\begin{align*}
\sigma_1 &= \begin{pmatrix}
\frac{-1}{3} & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix} \\
\sigma_1^\vee &= \begin{pmatrix}
3 & -1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix}
\end{align*}
\]  

(4.9)

The Hilbert basis of \( \sigma_1^\vee \) is given by the columns of \( E_1 \).

\[ E_1 = \begin{pmatrix}
-1 & 0 & 1 & 2 & 3 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \]

The matrix \( \phi_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \) now induces the surjection \( E_1 \rightarrow E : \phi_1(\bar{w}) = w \) where \( E \) is the Hilbert basis of \( \sigma^\vee \). Moreover, Figure 4.6 shows how to obtain the original cone \( \sigma \) from \( \sigma_1 \).

### 4.2.2 Determining the cross cut \( Q_{hk} \)

In order to construct the lifted polyhedron \( Q_{r,s} = Q_{hk} \) we are interested in a vector \( \bar{w}^k \in E_h \) such that \( \phi_h(\bar{w}^k) = \bar{w}^k \) where \( E_h \) is the Hilbert basis of the cone \( \sigma_h^\vee \).
From Theorem 4.3 we know that there exists a vector \( \phi(\sigma) \in \sigma_h \cap M^+ \). To see that this vector yields a unique cross cut \( Q_{hk} \), we will construct this vector. The matrix \( \phi_h \) has \((0, 1, -1)\) as its nullspace. Furthermore, for the vector \((w_1^k w_2^h - w_2^k w_1^h, v_1 w_1^k + v_2 w_2^k, 0)\) we have

\[
\phi_h((w_1^k w_2^h - w_2^k w_1^h, v_1 w_1^k + v_2 w_2^k, 0)) = w^k.
\]

Thus,

\[
\tilde{w}^k = (w_1^k w_2^h - w_2^k w_1^h, v_1 w_1^k + v_2 w_2^k, 0) + \alpha(0, 1, -1)
\]

for some \( \alpha \in \mathbb{Z} \).

Now distinguish two cases, namely \( h < k \) and \( k < h \). First, let \( h < k \). The cone is generated by the rays described in 4.6. Observe that

\[
\phi_h((0, 1, 0)) = w^h \quad \text{and} \quad \phi_h((nw_2^h - qw_1^h, nv_1 + qv_2, 0)) = w^{q+1}
\]

where \( \phi_h \) is as described in 4.7. An immediate result is that \((w_1^k w_2^h - w_2^k w_1^h, v_1 w_1^k + v_2 w_2^k, 0) \notin \sigma_h^\chi\), which means that \( \alpha = 0 \).

Consider the situation where \( h > k \) which means that \((w_1^k w_2^h - w_2^k w_1^h, v_1 w_1^k + v_2 w_2^k, 0) \notin \sigma_h^\chi\) is not an element of the Hilbert basis \( E_h \). For the generators of \( \sigma_h \) we get

\[
(0, 0, 1)^\top \tilde{w}^k = -\alpha \\
(b - v, 0, 1)^\top \tilde{w}^k = (\frac{w_1^k}{w_2^h} - v)(w_1^k w_2^h - w_2^k w_1^h) - \alpha \\
= -v(w_1^k w_2^h - w_2^k w_1^h) + \frac{w_1^k}{w_2^h} v_1 w_1^k + v_2 w_2^h - \alpha \\
= -v(w_1^k w_2^h - w_2^k w_1^h) - (v_1 w_1^k + v_2 w_2^k) + \frac{w_1^k}{w_2^h} - \alpha \\
\geq 0
\]

\[
(v, 1, 0)^\top \tilde{v} = v(w_1^k w_2^h - w_2^k w_1^h) + (v_1 w_1^k + v_2 w_2^k) + \alpha \\
\geq 0.
\]
This yields
\[-v(w_1^k w_2^h - w_2^k w_1^h) - (v_1 w_1^k + v_2 w_2^h) \leq \alpha \leq -v(w_1^k w_2^h - w_2^k w_1^h) - (v_1 w_1^k + v_2 w_2^h) + \frac{w_1^h}{w_1^k}.

For \(h > k\) the fraction \(\frac{w_1^h}{w_1^k}\) is less than 1 and therefore there is a unique \(\alpha\) such that it is a lattice point. This \(\alpha \in N^-\) is defined as

\[
\alpha = \begin{cases} 
0 & \text{if } h < k \\
-v(w_1^k w_2^h - w_2^k w_1^h) - (v_1 w_1^k + v_2 w_2^h) & \text{if } h > k \text{ and } v \in N^- \\
w_1^h - v(w_1^k w_2^h - w_2^k w_1^h) - (v_1 w_1^k + v_2 w_2^h) & \text{if } h > k \text{ and } v \notin N^- 
\end{cases}
\]

and Theorem 4.8 is proven.

The polyhedron \(Q_{hk}\) is determined from the cone \(\sigma_h\) and \(\tilde{\omega}^k \in \sigma_h^\vee \cap M^+\).

\[
Q_{hk} = \sigma_h \cap H_{\tilde{\omega}^k} = \left( \begin{array}{ccc} a & v & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \cap \{x \in N_+^3 | \tilde{\omega}^k \top x = 1\}
\]

We have \(\tilde{\omega}^k = (w_1^k w_2^h - w_2^k w_1^h, v_1 w_1^k + v_2 w_2^h, 0) + \alpha(0,1,-1)\) with \(\alpha \in N^-\) as defined in Theorem 4.8.

For readability we will denote

\[
\beta = (w_1^k w_2^h - w_2^k w_1^h).
\]

The construction of this projection is described in the following section. The polyhedron \(Q_{hk}\) is embedded in the lattice \(N^+\). However, there exists a projection polyhedron \(Q'_{hk}\) in \(N\) as described in Theorem 4.6.

### 4.2.2.1 Determining \(Q_{hk}\) for \(h < k\).

First consider the case where \(h < k\) which means that \(\alpha = 0\), and therefore

\[
\tilde{\omega}^k = (\beta, v_1 w_1^k + v_2 w_2^h, 0).
\]

In this case \((0,0,1) \top \tilde{\omega}^k = 0\) while \(a \top \tilde{\omega}^k \neq 0\) for the generators \(a \neq (1,0,0)\) of \(Q_h\). Hence \(Q_{hk}\) is a polyhedron built from one ray and three vertices. The vertices of \(Q_{hk}\)
are $\frac{a}{\tilde{w}^k} \cdot a$ for the generators $a \neq (1,0,0)$ of $\sigma_h$.

$$Q_{hk} = \left( \begin{array}{ccc} \frac{-nv_1 + qv_2}{nw_2^h - qw_1^h} & v & \frac{v}{\beta} \\
 & \frac{v}{\beta} + v_1 w_1^k + v_2 w_2^k & 0 \\
0 & 0 & -1 \\
1 & 1 & 1 \\
\end{array} \right)$$ (4.10)

The last column represents the ray of the polyhedron. The polyhedron consists of two bounded edges combined with one ray. Moreover, the lattice vertex $u = \left( \frac{1 - (v_1 w_1^k + v_2 w_2^k)}{\beta}, 1, 0 \right)$ is a vertex on the boundary of the polyhedron. This vertex is always a lattice point by construction, and from the following computation we know that it lies on the boundary of the polyhedron.

$$(\frac{1 - (v_1 w_1^k + v_2 w_2^k)}{\beta}, 1, 0) \cdot \tilde{w}^k = (1 - (v_1 w_1^k + v_2 w_2^k)) + v_1 w_1^k + v_2 w_2^k$$

$$= 1$$

where $\tilde{w}^k$ is taken as above. Taking the lattice vertex $u$ as the projection vertex and the vectors $(v_1 w_1^k + v_2 w_2^k, -\beta, 0)$ and $(0, 0, 1)$ as generators yields the polyhedron $Q'_{hk}$ in $\mathbb{Z}^2$.

This gives

$$\left( \begin{array}{ccc} \frac{-nv_1 + qv_2}{nw_2^h - qw_1^h} & 1 - (v_1 w_1^k + v_2 w_2^k) \\
 & \frac{v}{\beta} + v_1 w_1^k + v_2 w_2^k & 0 \\
0 & 0 & 1 \\
\end{array} \right) = x_1 \left( \begin{array}{ccc} v_1 w_1^k + v_2 w_2^k \\
-\beta \\
0 \\
\end{array} \right) + x_2 \left( \begin{array}{ccc} 0 \\
0 \\
1 \\
\end{array} \right)$$

for the first vertex of $Q_{hk}$. From this equation we see that this vertex is projected to $x = (x_1, x_2)$ with

$$x_2 = 0$$ and

$$x_1 = \frac{1}{\beta} (1 - \frac{nw_2^h - qw_1^h}{nw_2^h - qw_1^h}).$$

For the second vertex we can derive the projection as well:

$$\left( \begin{array}{ccc} \frac{v}{\beta + v_1 w_1^k + v_2 w_2^k} & 1 - (v_1 w_1^k + v_2 w_2^k) \\
 & \frac{v}{\beta + v_1 w_1^k + v_2 w_2^k} & 0 \\
0 & 0 & 1 \\
\end{array} \right) = x_1 \left( \begin{array}{ccc} v_1 w_1^k + v_2 w_2^k \\
-\beta \\
0 \\
\end{array} \right) + x_2 \left( \begin{array}{ccc} 0 \\
0 \\
1 \\
\end{array} \right)$$

thus

$$x_2 = 0$$ and

$$x_1 = \frac{1}{\beta} (1 - \frac{v}{\beta + v_1 w_1^k + v_2 w_2^k}).$$
The third vertex becomes
\[
\begin{pmatrix}
\frac{1}{\beta} - \frac{1}{\beta} \left( \frac{v_1 w_1^k + v_2 w_2^k}{v_1 w_1^k + v_2 w_2^k} \right) - 1 \\
- \frac{1}{\beta} \left( v - \frac{v_1}{w_1} \right)
\end{pmatrix} = x_1 \begin{pmatrix} v_1 w_1^k + v_2 w_2^k + \beta \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
thus
\[x_2 = - \frac{1}{\beta(v - \frac{v_1}{w_1})} \]
and
\[x_1 = \frac{1}{\beta}.\]

Thus, the projected polyhedron is described as:
\[
Q'_{hk} = \begin{pmatrix}
\frac{1}{\beta} (1 - \frac{vw_1^h - qw_1^h}{vw_2^h - qw_1^h}) & \frac{1}{\beta} (1 - \frac{1}{vw_1^h + vw_2^h}) & \frac{1}{\beta} & 0 \\
0 & 0 & - \frac{1}{(v - \frac{v_1}{w_1})} & 1
\end{pmatrix}. \quad (4.11)
\]

4.2.2.2 Determining \(Q_{hk}\) for \(h > k\).

Now determine \(Q_{hk}\) for \(k < h\). There are two options for \(\alpha \neq 0\). First consider the decomposition of \(Q_h\) as a decomposition of the first category as described in Lemma 4.7 which means that \(v \in N\). Thus \(\alpha = -v \beta - (v_1 w_1^k + v_2 w_2^k) \in \mathbb{Z}\). The cross cut \(Q_{hk}\) is computed as the intersection of \(\sigma_h\) and
\[
\tilde{w}^k = (\beta, v_1 w_1^k + v_2 w_2^k, 0) + \alpha(0, 1, -1) \quad (4.12)
\]
with \(\alpha = -v \beta - (v_1 w_1^k + v_2 w_2^k)\) and \(v \in N\).

Now \(\tilde{w}^k \top a > 0\) for the rays of \(\sigma_h\) that are not equal to \((v, 1, 0)\) and \(\tilde{w}^k \top (v, 1, 0) = 0\). Hence, again \(Q_{hk}\) consists of three vertices \(\tilde{w}^k \top a\) and one ray. Like the previous polyhedron, it consists of two bounded edges and one ray.

\[
Q_{hk} = \begin{pmatrix}
\frac{a}{\beta(a-v)} & 0 & (b-v) \frac{w_1^k}{w_1^h} & v \\
\frac{1}{\beta(a-v)} & 0 & 0 & 1 \\
0 & \frac{1}{v^2 + v_1 w_1^h + v_2 w_2^h} & \frac{w_1^h}{w_1^2} & 0 \\
1 & 1 & 1 & 0
\end{pmatrix}. \quad (4.13)
\]

Here, \(a\) and \(b\) are as defined in Equations (4.4) and (4.5). If \(\alpha\) is equal to 1, then the center vertex \((0,0,-\frac{1}{\alpha})\) is a lattice point, hence it can be taken as the vertex \(u\) in Equation 4.7. Otherwise there exists a \(\lambda\) such that \(\lambda \alpha \in N\) and \(-\frac{1}{\alpha} + \lambda \beta \in N\). This gives a lattice point
\[u = (0,0,-\frac{1}{\alpha}) + \lambda(\alpha,0,\beta)\]
that is contained in a bounded edge of the polyhedron. Taking \( g^1 = (-\alpha, 0, -\beta) \) and
\( g^2 = (v, 1, 0) \) as generators, yields the projection of the vertices.

The first vertex is determined by the following.

\[
\begin{pmatrix}
\frac{a}{\beta(a-v)} - \lambda \alpha \\
\frac{1}{\beta(a-v)} + \frac{1}{\alpha - \lambda \beta}
\end{pmatrix} = x_1 \begin{pmatrix} -\alpha \\ \beta \end{pmatrix} + x_2 \begin{pmatrix} v \\ 1 \end{pmatrix}
\]

\[
x_2 = \frac{1}{\beta(a-v)}
\]

\[
x_1 = \lambda - \frac{1}{\alpha \beta}
\]

The second vertex is retrieved by:

\[
\begin{pmatrix}
-\lambda \alpha \\
0
\end{pmatrix} + \frac{1}{\alpha - \lambda \beta}
\]

\[
x_2 = 0
\]

\[
x_1 = \lambda
\]

The third vertex is obtained from the following description.

\[
\begin{pmatrix}
(b - v) \frac{w_h^k}{w_1^h} - \lambda \alpha \\
0
\end{pmatrix} = x_1 \begin{pmatrix} -\alpha \\ \beta \end{pmatrix} + x_2 \begin{pmatrix} v \\ 1 \end{pmatrix}
\]

\[
x_2 = 0
\]

\[
x_1 = \lambda - \frac{1}{\beta (\frac{w_h^k}{w_1^h} + \frac{1}{\alpha})}
\]

Therefore, the polyhedron is projected to the polyhedron:

\[
Q'_{hk} = \begin{pmatrix}
\lambda - \frac{1}{\alpha \beta} \\
\frac{1}{\beta(a-v)} \\
1
\end{pmatrix} \begin{pmatrix} \lambda \\ \lambda - \frac{1}{\beta (\frac{w_h^k}{w_1^h} + \frac{1}{\alpha})} \\
0 \\
1
\end{pmatrix}.
\] (4.14)

Finally the third option for the polyhedron \( Q_{hk} \) has again \( k < h \) but now \( \alpha = -v \beta - (v_1 w_1^k + v_2 w_2^k) + \frac{w_h^k}{w_1^h} \). This gives

\[
\tilde{w}^k = (\beta, -v \beta + \frac{w_h^k}{w_1^h}, v \beta + v_1 w_1^k + v_2 w_2^k - \frac{w_h^k}{w_1^h})
\] (4.15)
The polyhedron $Q_{hk}$ consists of two bounded edges (three vertices) and one ray:

$$Q_{hk} = \begin{pmatrix}
\frac{a}{\beta(a-v)+\frac{v^h}{w^1}} & -\frac{w^h}{w^1} & 0 & b-v \\
\frac{1}{\beta(a-v)+\frac{v^h}{w^1}} & \frac{w^h}{w^1} & 0 & 0 \\
0 & 0 & -\frac{1}{a} & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}.$$  \hspace{1cm} (4.16)

Observe that $v=a$ indeed yields that the first vertex is the same as the second vertex of $Q_{hk}$. The vertex $(v^{w^h}_{w^1}, 0)$ is a lattice vertex by definition, and therefore it can be taken as $u$ for the projection. Together with generators $(v - \beta\frac{w^h}{w^1}, 1, 0)$ and $(b-v, 0, 1)$ we can compute the new vertices. The first vertex gives:

$$x_1 = \frac{1}{\beta(a-v)+\frac{v^h}{w^1}} - \frac{w^h}{w^1},$$

$$x_2 = 0,$$

$$x_1 = \frac{1}{\beta(a-v)+\frac{v^h}{w^1}} - \frac{w^h}{w^1}.$$

By definition, the second vertex is mapped to 0. The third vertex is mapped to:

$$x_2 = \frac{1}{\alpha},$$

$$x_1 = -\frac{w^h}{w^1}.$$

This yields the following polyhedron.

$$Q'_{hk} = \begin{pmatrix}
\frac{1}{\beta(a-v)+\frac{v^h}{w^1}} & -\frac{w^h}{w^1} & 0 & -\frac{w^h}{w^1} & 0 \\
0 & 0 & -\frac{1}{a} & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}.$$  \hspace{1cm} (4.17)

Example.

To continue with Example 4.2.1 fix $w^2 = (2, 1)$. From $h = 1 < 2 = k$ we know

$$\tilde{w}^2 = (\beta, v_1w^k_1 + v_2w^k_2, 0) = (1, 1, 0).$$
The intersection of $\sigma_1 = \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and $H_{\tilde{w}^2}$ results in $Q_{hk} = Q_{12}$.

$$Q_{12} = \sigma_1 \cap H_{\tilde{w}^2} = \begin{pmatrix} -\frac{1}{2} & 0 & 1 & 0 \\ \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

In order to construct the projected polyhedron $Q'_h$, consider the lattice point $u = (1 - (v_1 w^k_1 + v_2 w^k_2), 1, 0) = (0, 1, 0)$. This is the center vertex of the polyhedron $Q_{12}$. The projection can now be derived from the generators $(1, -1, 0)$ and $(0, 0, 1)$.

$$Q'_{12} = \begin{pmatrix} -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad (4.18)$$

The next step is to compute the polyhedron $Q'_{21}$. First determine $Q'_2$ and all its admissible decompositions.

$$Q_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 2 \end{pmatrix} \rightarrow [-\frac{1}{2}, \frac{1}{2}] \quad (4.19)$$

$$= [-\frac{1}{2}, 0] \oplus [0, \frac{1}{2}] \quad (4.20)$$

$$= \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} \oplus [0, 1] \quad (4.21)$$

Figure 4.7: The admissible Minkowski decompositions of $Q_2$.

Consider the decomposition given in 4.20. The cone corresponding to this decomposition is denoted by $\sigma^1_2$.

$$\sigma^1_2 = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and $\sigma^{1\gamma}_2 = \begin{pmatrix} 2 & 0 & 0 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
Furthermore,

\[ \alpha = v\beta - (v_1 w_1^k + v_2 w_2^k) \]
\[ = -1 \]

and therefore

\[ \tilde{w}^1 = (\beta, v_1 w_1^k + v_2 w_2^k, 0) + \alpha(0, 1, -1) \]
\[ = (-1, 1, 0) - 1(0, 1, -1) \]
\[ = (-1, 0, 1). \]

The polyhedron \( Q_{21}^1 \) is the intersection of \( \sigma_2^1 \) and the hyperplane corresponding to this \( \tilde{w}^1 \).

\[
Q_{21}^1 = \begin{pmatrix}
-1 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 \\
0 & 1 & 2 & 0 \\
1 & 1 & 1 & 0
\end{pmatrix}
\]

By taking

\[ u = (0, 0, -\frac{1}{\alpha}) + \lambda(\alpha, 0, 1) = (0, 0, 1) \]

The polyhedron \( Q_{21}^1 \) is the intersection of \( \sigma_2^1 \) and the hyperplane corresponding to this \( \tilde{w}^1 \).

\[ Q_{21}^1 = \begin{pmatrix}
-1 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}
\] \quad (4.22)

### 4.2.3 Constructing the cone \( \sigma_{hk} \) and check compatibility

If there exists an admissible decomposition of the polyhedron \( Q_{hk} \in N \) then its summands can be embedded in \( N^+_R \cong \mathbb{R}^3 \).

The cone \( \sigma_{hk} \subseteq N^+_R \) can be constructed from this embedding. The polyhedron \( Q_{hk} \) is built from three vertices and one ray.

\[ Q_{hk} = \begin{pmatrix}
v^0 & v^1 & v^2 & r \\
1 & 1 & 1 & 0
\end{pmatrix}
\]

Assume that \( v^1 \) is the center vertex, so \([v^0, v^1] \) and \([v^1, v^2] \) are the bounded edges of \( Q_{hk} \). The vertex \( v^i \) might be, but not necessarily is, a lattice point.

There are three different descriptions for \( Q_{hk} \) possible, which all yield a different description of the cone \( \sigma_{hk} \).
First consider the description given in (4.10), where \( h < k \) and \( \alpha = 0 \). The vertices are

\[
\begin{align*}
v^0 &= \left( \frac{1}{\beta} \left( 1 - \frac{nw_h}{nw_2 - qw_1}, 0 \right) \right) \\
v^1 &= \left( \frac{1}{\beta} \left( 1 - \frac{1}{v_2 + v_1w_1 + v_2w_2^k}, 0 \right) \right) \\
v^2 &= \left( \frac{1}{\beta} \left( v - \frac{v_2}{w_1^k} \right) \right)
\end{align*}
\]

The fact that \( h < k \) implies that \( \beta > 0 \) and therefore the edge \([v^1, v^2]\) does not have interior lattice points. This directly follows from the first entry of both vertices. Thus, an admissible decomposition of \( Q_{hk} \) can be described as

\[
Q_{hk} = \begin{pmatrix} v^0 & u & r \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & v^1 - u & -v^2 + u & r \\ 1 & 1 & 1 & 0 \end{pmatrix}
\]

(4.23)

where \( u \in (v^0, v^1) \) is a lattice point. If the center vertex \( v^1 \) is a lattice point and there exists a lattice point \( u \neq v^1 \) then there is another admissible decomposition:

\[
Q_{hk} = \begin{pmatrix} v^0 & u & u - v^1 + v^2 & r \\ 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & v^1 - u & u & r \\ 1 & 1 & 1 & 0 \end{pmatrix}
\]

(4.24)

For the other two descriptions of \( Q_{hk} \) that are given in (4.13) and (4.16) we have that \( h > k \) and \( \alpha \neq 0 \). Take \( \alpha = -(v_1w_1^k + v_2w_2^k) \) and consider the bounded edges of the polyhedron, that can be described by the following vertices:

\[
\begin{align*}
v^0 &= \left( \frac{1}{\alpha \beta} \frac{1}{\beta(a - v)} \right) \\
v^1 &= \left( -\lambda, 0 \right) \\
v^2 &= \left( \frac{1}{\beta} \frac{w_1^k}{w_1^k} + \frac{1}{\alpha} - \lambda, 0 \right)
\end{align*}
\]

The center vertex is \( v^1 \) and the bounded edge \([v^0, v^1]\) does not have an interior lattice point. This can be seen from the first entries of \( v^0 \) and \( v^1 \) and the fact that \( 0 > \alpha \in \mathbb{Z} \). Therefore any admissible Minkowski decomposition of \( Q_{hk} \) is induced by a lattice point \( u \in [v^1, v^2) \).

\[
Q_{hk} = \begin{pmatrix} v^0 & v^1 & u & r \\ 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & v^2 - u & -v^1 & r \\ 1 & 1 & 1 & 0 \end{pmatrix}
\]

(4.25)

Furthermore, if the vertex \( v^1 \) is a lattice point and there exists a lattice point \( u \neq v^1 \) then there is another admissible decomposition:

\[
Q_{hk} = \begin{pmatrix} v^0 & v^1 & v^1 - u + v^2 & r \\ 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & u - v^1 & -v^1 & r \\ 1 & 1 & 1 & 0 \end{pmatrix}
\]

(4.26)
For the polyhedron $Q_{hk}$ with $h > k$ and $\alpha = \frac{w_k}{w_1} - (v_1 w_k^1 + v_2 w_2^1) \in \mathbb{Z}$ the center vertex is $v^1 = (0,0)$. Furthermore,

$$v^0 = \left(\frac{1}{\beta(a-v) + \frac{w_k}{w_1}}, 0\right)$$

$$v^2 = \left(-\frac{w_k}{w_1}, -\frac{1}{\alpha}\right)$$

As $h > k$ we know that $\frac{w_k}{w_1} < 1$ which tells us that the edge $[v^1, v^2]$ does not have an interior lattice point. An admissible decomposition for this description of $Q_{hk}$ is therefore induced by a lattice point $u \in (v^0, v^1]$. The decomposition and the cone $\sigma_{hk}$ is equivalent to the description in 4.23 and when applicable 4.24.

If for the decomposition of $Q_h$ we have that $v = a$ then the decomposition of $Q_{hk}$ consists of one bounded edge, as this results in $v^0 = v^1$. This bounded edge does not have any interior lattice points, which confirms the fact that there does not exist an admissible decomposition for this polyhedron for $h \neq k$.

**Theorem 4.9.** A polytope $Q_h$ has an admissible decomposition with $v = a$ if and only if $w_h^1 = nw_h^2 - qw_h^1$. Moreover, if such a decomposition exists, then there does not exist a $w_k$ with $k \neq h$ with $0 < k < g$ such that $Q_k$ has an admissible decomposition with $v = a$.

**Proof.** In order for $a \oplus [0, b - a]$ to be an admissible decomposition, the line segment $[a, b]$ has to have integral length. Thus,

$$\frac{1}{w_1} + \frac{q}{nw_2^h - qw_1^h} = \frac{nw_2^h}{w_1(nw_2^h - qw_1^h)} \in \mathbb{Z} \quad \text{and} \quad \frac{n}{nw_2^h - qw_1^h} \in \mathbb{Z}.$$  

An immediate result is that $w_1^h | n$. Moreover, it means that either $nw_2^h - qw_1^h | w_1^h$ or $w_1^h | nw_2^h - qw_1^h$. Let $x \in \mathbb{Z}$ be an integer such that $xw_1^h = n$. The first case yields

$$nw_2^h - qw_1^h \mid w_1^h$$

$$xw_1^h w_2^h - qw_1^h \mid w_1^h$$

$$(xw_2^h - q)w_1^h \mid w_1^h$$

which implies that $nw_2^h - qw_1^h = w_1^h$.

In the second case we have that $w_1^h | (xw_2^h - q)w_1^h$. Let $y \geq 1$ be an integer such that
nw_2^h - qw_1^h = yw_1^h. This yields

\[\begin{align*}
  xw_2^h - q &= y \\
  xw_2^h &= q + y.
\end{align*}\]

Thus, \(x|n\) and \(x|q + y\). From the fact that \(\frac{n}{nw_2^h - qw_1^h} \in \mathbb{Z}\) we know that \(\frac{n}{yw_1^h} = \frac{x}{y}\) is a lattice point, and therefore \(y|x\). This yields \(y|n\) and \(y|q + y\), and therefore \(y|q\). The fact that \(q\) and \(n\) are relatively prime yields \(y = 1\).

This means that \(Q_h\) has integral length if and only if \(nw_2^h - qw_1^h = w_1^h\). The next thing to prove is that if there exists a \(w^h\) such that \(nw_2^h - qw_1^h = w_1^h\), then this \(w^h\) is unique in the Hilbert basis of \(\sigma^\vee\). To see this, let \(w^h\) be the smallest element of \(E\) such that \(w_1^h = nw_2^h - qw_1^h\). For every \(w^k\) with \(k > h\) we know that \(w_1^k > w_1^h\) and \(w_2^k \geq w_2^h\). Moreover, \(nw_2^h - qw_1^h > nw_2^k - qw_1^k\). Thus,

\[nw_2^h - qw_1^h < nw_2^k - qw_1^k = w_1^h < w_1^k.\]

This shows that if there exists a \(w^h\) with \(0 < h < g\) such that \(w_1^h = nw_2^h - qw_1^h\), then this vector is unique. Therefore we can conclude that an admissible decomposition of \(Q_h\) such that \(v = a\) is not compatible with any other admissible decomposition of \(Q_k\) with \(k \neq h\).

**Example.**

To continue with the example, decompose the polyhedra \(Q_{12}\) and \(Q_{21}^1\). First consider

\[Q_{12} = \begin{pmatrix}
  -\frac{1}{2} & 0 & 1 & 0 \\
  0 & 0 & 1 & 1 \\
  1 & 1 & 1 & 0
\end{pmatrix}.
\]

Its bounded edges do not have any interior lattice points. Hence there is only one admissible decomposition with \(u = v^1 = (0, 0)\).

\[Q_{12} = \begin{pmatrix}
  -\frac{1}{2} & 0 & 0 \\
  0 & 0 & 1 \\
  1 & 1 & 0
\end{pmatrix} \oplus \begin{pmatrix}
  0 & 1 & 0 \\
  0 & 1 & 1 \\
  1 & 1 & 0
\end{pmatrix}
\]

Now consider \(Q_{21}^1\). It is a polyhedron of the second form with \(\alpha = -(v_1w_1^h + v_2w_2^h) =
\]

\[\begin{array}{c}
\text{Figure 4.8: Minkowski decomposition of } Q_{12}.
\end{array}\]
The bounded edges do not have interior vertices, which yields a unique admissible decomposition with $u = u^1 = (0, 0)$.

$$Q_{21}^1 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$= \left( \begin{array}{c} -1 \\ 2 \\ 1 \end{array} \right) \oplus \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right)$$

The third polyhedron $Q_{21}^2$ is a line segment with no interior lattice points combined with one ray. According to the definition of admissibility, there does not exist an admissible decomposition of $Q_{21}^2$, which means that it can not be lifted in order to obtain cone $\sigma_{21}^2$.

The computations of the admissible decompositions result in the cones $\sigma_{12}, \sigma_{12}^\vee$.

$$\sigma_{12} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\sigma_{12}^\vee = \begin{pmatrix} -1 & 2 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Similarly, the cones $\sigma_{21}^1$ and $\sigma_{21}^1^\vee$ can be computed from the polyhedron $Q_{21}^1$.

$$\sigma_{21}^1 = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\sigma_{21}^1^\vee = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

There exists a morphism between the cones $\sigma_{12}$ and $\sigma_{21}^1$. The matrix $\phi_{12}$ defines this morphism such that the matrix $\sigma_{21}^1^\vee$ can be written as $\phi_{12} \sigma_{12}^\vee$.

$$\phi_{12} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(4.27)
This means that there exists a common refinement for the cross cuts $Q_1$ and $Q_2$ with the corresponding decompositions.

4.3 More results for the example with $(n, q) = (4, 1)$

In the example of the previous section we showed that the first decomposition of cross cut $Q_2$ is compatible to the unique decomposition of $Q_1$. In this section all decompositions of the cross cuts $Q_1, Q_2$ and $Q_3$ are compared. This is done via the procedure which is described in the previous section.

4.3.1 Are the decompositions of $Q_1$ and $Q_3$ compatible?

4.3.1.1 Decomposing and lifting $Q_{13}$

In this case $w^h = (1, 1)$ and $w^k = (3, 1)$. This gives $\alpha = 0$ and $\tilde{w}^3 = (2, 1, 0)$. The polytope $Q_1$ has a unique decomposition and the corresponding cone $\sigma_1$ is computed in the previous section and described in (4.9). The intersection of $\sigma_1$ and $H_{\tilde{w}^3} = \{ x \in \mathbb{R}^3 | \tilde{w}^3 \top x = 1 \}$ yields the polyhedron $Q_{13}$.

$$Q_{13} = \sigma_1 \cap H_{\tilde{w}^3}$$

$$= \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cap H_{\tilde{w}^3}$$

$$= \begin{pmatrix} -1 & 0 & \frac{1}{2} & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

where the generators $(1, -2, 0)$ and $(0, 0, 1)$ impose the projection. There is one admissible decomposition of $Q_{13}$, as the edges do not have any interior lattice points. This decomposition is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$
The polytope $Q_{13}$ gives the $\sigma_{13}$.

$$
\sigma_{13} = \begin{pmatrix}
-1 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2} & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0
\end{pmatrix}
$$

$\sigma_{13}^\vee = \begin{pmatrix}
-1 & 1 & -2 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}$

### 4.3.1.2 Decomposing and lifting $Q_{31}$

Now consider $w^3 = (3,1)$ and determine the polyhedron which is the intersection of the hyperplane from $w^3$ and $\sigma$. This yields

$$
Q_3 = \begin{pmatrix}
\frac{1}{3} & -1 \\
0 & 1
\end{pmatrix} \rightarrow [-1, \frac{1}{3}]
$$

$$
= [-1,0] \oplus [0, \frac{1}{3}]. \tag{4.28}
$$

The projection is imposed by the generators $(0,1)$ and $(1,-3)$. The given decomposition is the only admissible decomposition of $Q_3$. These summands yield the cone $\sigma_3$.

$$
\sigma_3 = \begin{pmatrix}
-1 & 0 & 0 & \frac{1}{3} \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix}
$$

The decomposition is of the first category and therefore

$$
\tilde{w}^1 = (\beta, -v\beta, v\beta + v_1w_1^k + v_2w_2^k)
$$

$$
= (w_1^w^3 - w_2^1w_1^3, 0, v_1w_1^1 + v_2w_2^1)
$$

$$
= (1 - 3, 0, 1) = (-2, 0, 1).
$$

The intersection with $\sigma_3$ and $\tilde{w}^1$ results in polyhedron $Q_{31}$. It has a unique admissible decomposition.

$$
Q_{31} = \begin{pmatrix}
-\frac{1}{2} & 0 & 1 & 0 \\
\frac{1}{2} & 0 & 0 & 1 \\
0 & 1 & 3 & 0 \\
1 & 1 & 1 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
-\frac{1}{2} & 0 & 1 & 0 \\
\frac{1}{2} & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}
$$

$$
= \begin{pmatrix}
-\frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & 1 \\
1 & 1 & 0
\end{pmatrix} \oplus \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
$$
The vectors \((1,0,2)\) and \((0,1,0)\) are the generators for the projection. This unique decomposition gives the following cone \(\sigma_{31}\).

\[
\sigma_{31} = \begin{pmatrix}
-\frac{1}{2} & 0 & 0 & 1 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0
\end{pmatrix}
\]

\[
\sigma_{31}^\lor = \begin{pmatrix}
-1 & 1 & 2 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

### 4.3.1.3 Comparing the cones \(\sigma_{13}\) and \(\sigma_{31}\)

There exists a matrix \(\phi_{13}\) which maps the generators of \(\sigma_{13}\) to the generators of \(\sigma_{31}\).

\[
\phi = \begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

The existence of such a matrix shows that the two original decompositions of \(Q_1\) and \(Q_3\) are compatible. This matrix is defined by the mapping matrices that maps the vectors of \(\sigma_{hk}\) to \(\sigma\):

\[
\begin{pmatrix}
(1,0) \\
\mathbf{w}^h \\
\mathbf{w}^k
\end{pmatrix}
\]

where \((1,0)\) is a vector in the Nullspace of \(\mathbf{v}\) which is the vertex of \(Q_h \in N\) that represents the origin in the projection of the polyhedron. One can see that \(\phi_{13}\) defines a mapping between the two matrices.

\[
\begin{pmatrix}
1 & 3 & 1 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 3 & 3 \\
0 & 1 & 1 & 1
\end{pmatrix}
\]

Hence, the equivalency of the cones \(\sigma_{13}^\lor\) and \(\sigma_{31}^\lor\) confirms the compatibility of the (unique) decompositions of \(Q_1\) and \(Q_3\).

### 4.3.2 Are the decompositions of \(Q_2\) and \(Q_3\) compatible?

#### 4.3.2.1 Decomposing and lifting \(Q_{23}\)

The polyhedron \(Q_2\) has two admissible decompositions. The second decomposition is of the last category with \(v = a\). According to Section 4.2.3 this leads to a polyhedron \(Q_{23}\) which does not have an admissible decomposition. Therefore the only decomposition of \(Q_2\) that will be treated is given in 4.20.
The cone $\sigma_2$ is determined in the previous section. Furthermore $\alpha = 0$ yields $\tilde{w}^3 = (1, 1, 0)$. The cross cut of $\sigma_2$ and $\tilde{w}^3$ yields the polyhedron $Q_{23}$ which has a unique admissible decomposition.

\[
Q_{23} = \left(\begin{array}{cccc}
-\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{array}\right) \cap H_{\tilde{w}^3}
\]

\[
= \left(\begin{array}{cccc}
-1 & 0 & 1 & 0 \\
2 & 1 & 0 & 0 \\
0 & 0 & 2 & 1 \\
1 & 1 & 1 & 0
\end{array}\right) \rightarrow \left(\begin{array}{cccc}
-1 & 0 & 1 & 0 \\
0 & 0 & 2 & 1 \\
1 & 1 & 1 & 0
\end{array}\right)
\]

\[
= \left(\begin{array}{cccc}
-1 & 0 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{array}\right) \oplus \left(\begin{array}{cccc}
0 & 1 & 0 \\
0 & 2 & 1 \\
1 & 1 & 0
\end{array}\right)
\]

The lifted cone $\sigma^1_{23}$ is generated by the following rays.

\[
\sigma^1_{23} = \left(\begin{array}{cccc}
-1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0
\end{array}\right)
\]

\[
\sigma^1_{23} = \left(\begin{array}{cccc}
-1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0
\end{array}\right)
\]

\[
4.3.2.2 \text{ Decomposing and lifting } Q_{32}
\]

The next step is to follow the same procedure for $Q_s = Q_3$. The cone $\sigma_3$ is computed in Section 4.3.1.2. From $\alpha = -v\beta - (v_1 w_1^k + v_2 w_2^k) = -1$ one can compute $\tilde{w}^1 = (-1, 0, 1)$. The cross cut yields the polyhedron $Q_{32}$ which has a unique admissible decomposition.

\[
Q_{32} = \left(\begin{array}{cccc}
-1 & 0 & 0 & \frac{1}{3} \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{array}\right) \cap H_{\tilde{w}^1}
\]

\[
= \left(\begin{array}{cccc}
-1 & 0 & \frac{1}{2} & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & \frac{3}{2} & 0 \\
1 & 1 & 1 & 0
\end{array}\right) \rightarrow \left(\begin{array}{cccc}
-1 & 0 & \frac{1}{2} & 0 \\
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{array}\right)
\]

\[
= \left(\begin{array}{cccc}
-1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{array}\right) \oplus \left(\begin{array}{cccc}
0 & \frac{1}{2} & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{array}\right)
\]
This determines cone $\sigma_{32}$.

$$\sigma_{32} = \begin{pmatrix} -1 & 0 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\sigma_{32}^\lor = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4.3.2.3 Comparing the cones $\sigma_{23}^1$ and $\sigma_{32}$

The same conclusion can be drawn as for the cones $\sigma_{21}^1$ and $\sigma_{12}$. The first decomposition of $Q_2$ is compatible to the unique decomposition of $Q_3$, but the second decomposition of $Q_2$ is not. The mapping matrix is described as below:

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

4.3.3 Conclusion

To validate these answers, we go back to the original starting point, where this problem is described with Deformation theory and toric varieties. One can vary that a different approach does not lead to different results, as we see from [18]. This gives more insight in the problem that can be used in further research.
Chapter 5

**example** \((-q, n) = (-3, 8)\)

### 5.1 The construction of \(\sigma\)

Let \(N, M \cong \mathbb{Z}^2\) be dual lattices. Now consider the example with \((n, q) = (8, 3)\). Consider the cones that are generated by the following matrices.

\[
\sigma = \begin{pmatrix} 1 & -3 \\ 0 & 8 \end{pmatrix}, \quad \sigma^\vee = \begin{pmatrix} 0 & 8 \\ 1 & 3 \end{pmatrix}
\]

The Hilbert basis of \(\sigma^\vee\) is computed from the continued fraction expansion of \(\frac{n}{n-q}\).

\[
\{w^0, \ldots, w^{g+1}\} = \{(0), (\frac{1}{1}), (\frac{2}{5}), (\frac{5}{8})\}
\]

Each cross cut \(Q_i = \sigma \cap H_i\) yields a polytope for \(i = 1, \ldots, 3\). This polytope is a line segment embedded in \(N\). In this section the compatibility of all possible decompositions of polytopes \(Q_i\) is verified, using the procedure described earlier.

### 5.2 The cross cuts and their decompositions

![Cone σ and the intersections Q1, Q2 and Q3](image.png)

**Figure 5.1:** Cone \(\sigma\) and the intersections \(Q_1, Q_2\) and \(Q_3\)
First determine the cross cuts $Q_1, Q_2$ and $Q_3$. Figure 5.1 shows the cone $\sigma$ and these intersections. The line segment $Q_1$ has a unique decomposition.

\[
Q_1 = \begin{pmatrix} 1 & -\frac{3}{5} \\ 0 & \frac{8}{5} \end{pmatrix} \rightarrow [-\frac{3}{5}, 1]
= [-\frac{3}{5}, 0] \oplus [0, 1]
\] (5.1)

\[= \bullet \oplus \]

Figure 5.2: Decomposition of $Q_1$

Now consider the polytope derived from $\mathbf{w}^2 = (2, 1)$. It does not have a unique admissible decomposition, but four of them which are given below.

\[
Q_2 = \begin{pmatrix} -\frac{3}{4} & \frac{1}{5} \\ 0 & \frac{6}{5} \end{pmatrix} \rightarrow [-\frac{3}{2}, \frac{1}{2}]
= [-\frac{3}{2}, -\frac{1}{2}] \oplus [0, 1]
= [-\frac{3}{2}, 0] \oplus [0, \frac{1}{2}]
= [-\frac{3}{2}, -1] \oplus [0, \frac{3}{2}]
= \{-\frac{3}{2}\} \oplus [0, 2]
\] (5.2) (5.3) (5.4) (5.5)

\[= \bullet \oplus \]

\[= \bullet \oplus \]

\[= \bullet \oplus \]

\[= \bullet \oplus \]

Figure 5.3: the decompositions of $Q_2$

Similarly, the unique decomposition of $Q_3$ can be determined.

\[
Q_3 = \begin{pmatrix} \frac{1}{5} & -\frac{3}{5} \\ 0 & \frac{8}{5} \end{pmatrix} \rightarrow [-1, \frac{3}{5}]
= [-1, 0] \oplus [0, \frac{3}{5}]
\] (5.6)
5.3 Are the decompositions of $Q_1$ and $Q_2$ compatible?

The first decompositions that are treated are those of polytopes $Q_1$ and $Q_2$. The first polytope has a unique decomposition which is given in Equation 5.2. This decomposition is lifted and embedded in $\mathcal{N} = N \times \mathbb{Z}^2$ yielding the cone $\sigma_{12} \subset \mathcal{N}$ and its dual cone $\sigma_{12}^\vee \subset \mathcal{M}$. On the other hand, $Q_2$ has more than one admissible decomposition. These decompositions are treated separately. Finally the cone $\sigma_{12}^\vee$ is compared with the cones $\sigma_{21}^\vee$ to verify the compatibility.

5.3.1 Compute $Q_{12}$

The decomposition of $Q_1$ is given in Equation 5.2. The lifted cone $\sigma_1$ can be determined from this decomposition, together with its dual cone $\sigma_1^\vee$.

$$\sigma_1 = \begin{pmatrix} -\frac{3}{5} & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \sigma_1^\vee = \begin{pmatrix} 5 & 0 & -1 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Now $\phi_1$ induces a mapping such that $\phi_1(\mathbf{w}_i) = \mathbf{w}_i^i$ for all $i = 0, \ldots, 4$ where $\mathbf{w}_i \in \sigma_1^\vee$ is an element of the Hilbert basis $E_1$ and $\mathbf{w}_i^i$ is an element of the Hilbert basis of $\sigma^\vee$.

$$\phi_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Moreover, the Hilbert basis of $\sigma_1^\vee$ consists of the columns $(\mathbf{w}_1, \mathbf{w}_4, \mathbf{w}_2)$ of matrix $E_1$ such that $\tilde{\mathbf{w}}_1^i$ maps to $\mathbf{w}_i$.

$$E_1 = \begin{pmatrix} -1 & 0 & 1 & 3 & 5 & 0 \\ 0 & 1 & 1 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
The cross cut of $\sigma_1$ with $w_1^2 = (1, 1, 0)$ yields the following polyhedron and the corresponding decompositions. The first two decompositions are as described in 4.23 and the latter one is admissible as the vertex $v_1 = (0, 0, 1)$ is a lattice point.

\[
Q_{12} = \begin{pmatrix}
-\frac{3}{2} & 0 & 1 & 0 \\
\frac{5}{2} & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
-\frac{3}{2} & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix} 
\]

\[= \begin{pmatrix}
-\frac{3}{2} & 0 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix} \oplus \begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{pmatrix} \quad (5.7)
\]

\[= \begin{pmatrix}
-\frac{3}{2} & -1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix} \oplus \begin{pmatrix}
0 & 1 & 2 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix} \quad (5.8)
\]

\[= \begin{pmatrix}
-\frac{3}{2} & -1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix} \oplus \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix} \quad (5.9)
\]
There exist three admissible decompositions of polyhedron $Q_{12}$ which are given above. These decompositions lead to three different cones $\sigma_{12}$.

$$
\sigma_{12}^1 = \begin{pmatrix}
-\frac{3}{2} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0
\end{pmatrix},
\sigma_{12}^{1\vee} = \begin{pmatrix}
-1 & 2 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 3 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix},
\sigma_{12}^2 = \begin{pmatrix}
-\frac{3}{2} & -1 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0
\end{pmatrix},
\sigma_{12}^{2\vee} = \begin{pmatrix}
-1 & 2 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 3 & -1 & 0 & 1 & 0 \\
1 & 0 & 2 & 0 & 0 & 1
\end{pmatrix},
\sigma_{12}^3 = \begin{pmatrix}
-\frac{3}{2} & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix},
\sigma_{12}^{3\vee} = \begin{pmatrix}
-1 & 2 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 3 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}
$$

The dual cones $\sigma_{12}^{i\vee}$ all have their Hilbert basis $E_{12}^i$ which maps to $E$ using the matrix $\phi_{12}$, which does not depend on the decomposition.

$$
\phi_{12} = \begin{pmatrix}
1 & 1 & 2 & 2 \\
0 & 1 & 1 & 1
\end{pmatrix}
$$

### 5.3.2 Compute $Q_{21}$

The same procedure is followed to compute $\sigma_{21}$. The polytope $Q_2$ has more than one admissible decomposition. A first remark is that the fourth decomposition is a decomposition such that $v = a$ and therefore it is not compatible with the decomposition of $Q_1$. For the other decompositions it is not straightforward to see whether they are compatible to $Q_1$, thus the cones $\sigma_{21}$ are computed for the other three admissible decompositions of $Q_2$. The morphisms $\phi_2$ and $\phi_{21}$ do not depend on the choice of the decomposition, but are the same for all decompositions.

$$
\phi_2 = \begin{pmatrix}
1 & 2 & 2 \\
0 & 1 & 1
\end{pmatrix},
\phi_{21} = \begin{pmatrix}
1 & 2 & 1 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}
$$
First consider the decomposition described in Equation 5.2. It yields cone $\sigma_2^1$, where the upper indicator stands for the choice of the decomposition.

$$
\sigma_2^1 = \begin{pmatrix}
-\frac{3}{2} & -\frac{1}{2} & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix}
\quad
\sigma_2^{1\lor} = \begin{pmatrix}
-1 & -2 & 3 & 0 \\
0 & -1 & 2 & 0 \\
1 & 2 & 0 & 1
\end{pmatrix}
$$

The vertex $v = \left( -\frac{1}{2}, 2 \right)$ is not a lattice point, thus $\alpha = \frac{w_1^k}{w_1^l} - (v_1w_1^k + v_2w_2^k) = \frac{1}{2} - \left( -\frac{3}{2} \right) = -1$. This yields $\tilde{w}_1 = (-1,1,0) - (0,1,-1) = (-1,0,1)$. The intersection of $\tilde{w}_1$ and $\sigma_2^1$ gives the polyhedron $Q_{21}^1$. This polyhedron has a unique admissible decomposition which is given below.

$$
Q_{21}^1 = \begin{pmatrix}
-1 & -1 & 0 & 1 \\
\frac{2}{3} & 2 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-\frac{1}{3} & 1 & -1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}
$$

$$
= \begin{pmatrix}
-1 & -\frac{1}{3} & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}
\oplus
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
$$

The cone $\sigma_{21}^1$ can be constructed from this decomposition.

$$
\sigma_{21}^1 = \begin{pmatrix}
-1 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\quad
\sigma_{21}^{1\lor} = \begin{pmatrix}
-1 & 3 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

The second decomposition as described in Equation 5.3, gives cone $\sigma_2^2$. From the vertex $v = (0,1)$ we get $\alpha = -(v_1w_1^k + v_2w_2^k) = -1$. This gives $\tilde{w}_1 = (-1,1,0) - (0,1,-1) = (-1,0,1)$ which results in the cross cut $Q_{21}^2$. It has a unique admissible decomposition,
which is given below.

\[
\sigma^2_2 = \begin{pmatrix}
-\frac{3}{2} & 0 & 0 & \frac{1}{2} \\
4 & 1 & 0 & -1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

\[
Q^2_{21} = \begin{pmatrix}
-1 & 0 & 1 & 0 \\
\frac{2}{3} & 0 & 0 & 1 \\
0 & 1 & 2 & 0 \\
1 & 1 & 1 & 0 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
-1 & 0 & 1 & 0 \\
\frac{2}{3} & 0 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-1 & 0 & 0 \\
\frac{2}{3} & 0 & 1 \\
1 & 1 & 0 \\
\end{pmatrix} \oplus \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{pmatrix}
\]

\[
\sigma^2_{21} = \begin{pmatrix}
\frac{2}{3} & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}
\]

The third decomposition is described by Equation 5.4. Now \(\alpha = -(v_1w_1^k + v_2w_2^k) = -2\) yields \(\tilde{\omega}_1 = (-1, 1, 0) - 2(0, 1, -1) = (-1, -1, 2)\). The cross cut of \(\sigma^3_2\) yields the polyhedron \(Q^3_{21}\).

\[
Q^3_{21} = \begin{pmatrix}
-1 & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 0 & 1 \\
0 & \frac{1}{2} & 2 & 0 \\
1 & 1 & 1 & 0 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
-1 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-1 & \frac{1}{2} & 0 & 0 \\
2 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{pmatrix} \oplus \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{pmatrix}
\]

This unique decomposition leads to the cone \(\sigma^3_{21}\) and its dual cone.

\[
\sigma^3_{21} = \begin{pmatrix}
-1 & -\frac{1}{2} & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}
\]

\[
\sigma^{3\vee}_{21} = \begin{pmatrix}
-1 & 4 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
5.3.3 Conclusion

Although there exists one decomposition for $Q_1$, the lifted polyhedron $Q_{12}$ has three admissible decompositions. These decompositions lead to three different cones $\sigma_{12}$ and their dual cones $\sigma_{12}^\vee$. The first decomposition of $Q_{12}$ yields the cone $\sigma_{12}^1$. This cone can be mapped to the cone $\sigma_{21}^2$, which is induced by the second decomposition of $Q_2$.

There exists a similar morphism between the cones $\sigma_{12}^2$ and $\sigma_{21}^3$, which shows compatibility of the second decomposition of $Q_2$ and the unique admissible decomposition of $Q_1$.

Moreover, the first decomposition of $Q_2$ is compatible with the unique admissible decomposition of $Q_1$. This is shown by the cones $\sigma_{12}^3$ and $\sigma_{21}^1$.

\[
\begin{align*}
\pi_1 &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\pi_1 \sigma_{12}^1 &= \sigma_{21}^2 \\
\pi_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\
\end{pmatrix} \\
\pi_2 \sigma_{12}^2 &= \sigma_{21}^3 \\
\pi_3 &= \begin{pmatrix} 2 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\
\end{pmatrix} \\
\pi_3 \sigma_{12}^3 &= \sigma_{21}^1 
\end{align*}
\]

5.4 Are the decompositions of $Q_1$ and $Q_3$ compatible?

Consider the cross cuts that are retrieved from the rays $w^1 = (1, 1)$ and $w^3 = (5, 2)$. The polyhedra $Q_1$ and $Q_3$ have one admissible decomposition which are given in Equations 5.2 and 5.6.
5.4.1 Compute $Q_{13}$

The decomposition of $Q_1$ is induced by $a = (0, 1)$. Moreover, $\alpha = 0$ and $w_1^2 = (3, 2, 0)$. The polyhedron $Q_{13}$ is computed, together with its decomposition.

\[
Q_{13} = \begin{pmatrix}
-3 & 0 & \frac{1}{3} & 0 \\
5 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 1 \\
1 & 1 & 1 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
-1 & \frac{1}{2} & \frac{2}{3} & 0 \\
0 & 0 & \frac{1}{3} & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix} \oplus \begin{pmatrix}
0 & \frac{1}{2} & \frac{2}{3} & 0 \\
0 & 0 & \frac{1}{3} & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}
\]

Embedding these summands at different heights yields the cone $\sigma_{13}$.

\[
\sigma_{13} = \begin{pmatrix}
-1 & 0 & 0 & \frac{1}{2} & \frac{2}{3} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3} & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}
\]

\[
\sigma_{13}^\vee = \begin{pmatrix}
-2 & 1 & -3 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 2 & 0 & 0 & 1
\end{pmatrix}
\]

The Hilbert basis of $\sigma_{13}^\vee$ to the elements of $E$ by matrix $\phi_{13} = \begin{pmatrix} 3 & 1 & 5 & 5 \\ 1 & 1 & 2 & 2 \end{pmatrix}$

5.4.2 Compute $Q_{31}$

Polytope $Q_3$ has a unique decomposition which is given in Equation 5.6. This yields the following cones.

\[
\sigma_3 = \begin{pmatrix}
-1 & 0 & 0 & \frac{3}{5} \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]

\[
\sigma_3^\vee = \begin{pmatrix}
-5 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
3 & 0 & 0 & 1
\end{pmatrix}
\]

Furthermore, $u = (-1, 3)$ and $\alpha = -(v_1 w_1^k + v_2 w_2^k) = -(-1 + 3) = -2$ determine $\bar{w}_1 = (-3, 2, 0) - 2(0, 1, -1) = (-3, 0, 2)$. The cross cut of $\sigma_3$ and $w_3^2$ yields the polyhedron
Q₃₁.

\[
\begin{pmatrix}
-\frac{1}{3} & 0 & 3 & 0 \\
\frac{1}{3} & 0 & 0 & 1 \\
0 & \frac{1}{2} & 5 & 0 \\
1 & 1 & 1 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
-\frac{2}{3} & -\frac{1}{2} & 1 & 0 \\
\frac{1}{3} & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
-\frac{2}{3} & -\frac{1}{2} & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

This is a unique admissible decomposition. The next step is to compute cone σ₃₁.

\[
\sigma_{31} = \begin{pmatrix}
-\frac{2}{3} & -\frac{1}{2} & 0 & 0 & 1 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\]

\[
\sigma_{31}^\vee
\]

5.4.3 Conclusion

To check compatibility of the unique decompositions of Q₁ and Q₃, cones σ₁₃ and σ₃₁ are compared. The given matrix π₁₃ describes the mapping that verifies the compatibility of the two decompositions.

\[
\pi_{13} = \begin{pmatrix}
2 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

\[
\pi_{13}\sigma_{13} = \sigma_{31}
\]

5.5 Are the decompositions of Q₂ and Q₃ compatible?

Consider the decompositions of Q₂ and Q₃. As discussed earlier, the fourth decomposition of Q₂ is induced by v = a. From Theorem 4.9 we know that this decomposition is not compatible to any other decomposition of Qₖ with k ≠ h. Thus to check compatibility, only the first three decompositions of Q₂ are discussed.
5.5.1 Compute $Q_{23}$

There are three polyhedra $Q_{21}$ that can be computed. These will be discussed successively. Matrix $\phi_{23}$ does not depend on the choice of decomposition, but solely on $w^2$ and $w^3$. They induce the mapping to $\sigma^\vee$.

\[
\phi_{21} = \begin{pmatrix} 1 & 2 & 5 & 5 \\ 0 & 1 & 2 & 2 \end{pmatrix}
\]

The cones $\sigma_2$ are computed in Section 5.3.2. As $2 > 1$ have $\alpha = 0$ and $\tilde{w}^3 = (1, 2, 0)$ for all its decompositions. The first decomposition of $Q_2$ as described in Equation 5.2 gives the polyhedron $Q_{23}^1$.

\[
Q_{23}^1 = \sigma_{23}^1 \cap H_{\tilde{w}^3} = \begin{pmatrix} -3 & -1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \end{pmatrix}
\]

(5.10)

This is the unique decomposition of $Q_{23}^1$ which gives cone $\sigma_{23}^1$ and its dual.

\[
\sigma_{23}^1 = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \frac{2}{3} & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \sigma_{23}^1\vee = \begin{pmatrix} -2 & 1 & -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}
\]

The second decomposition is induced by $v = (0, 1)$ and is given in Equation 5.3. The intersection of $\sigma_2^2$ and $\tilde{w}^3 = (1, 2, 0)$ gives the polyhedron $Q_{23}^2$ which has a unique admissible decomposition. The cross cut of the cone $\sigma_2^2$ yields $Q_{23}^2$.

\[
Q_{23}^2 = \begin{pmatrix} -3 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \frac{1}{2} & 1 \end{pmatrix}
\]
Cone $\sigma_{23}^2$ is constructed from polyhedron $Q_{23}^2$.

$$\sigma_{23}^2 = \begin{pmatrix} -1 & 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_{23}^{2\vee} = \begin{pmatrix} -4 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Now consider the third decomposition described in Equation 5.4. Cross cut $Q_{23}^3 = \sigma_{23}^2 \cap H_{\bar{w}^3}$ has a unique admissible decomposition.

$$Q_{23}^3 = \begin{pmatrix} -3 & -\frac{1}{3} & 1 & 0 \\ 2 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_{23}^{3\vee} = \begin{pmatrix} -1 & \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

This leads to cone $\sigma_{23}^3$.

$$\sigma_{23}^3 = \begin{pmatrix} -1 & 0 & 0 & \frac{1}{3} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_{23}^{3\vee} = \begin{pmatrix} -3 & 1 & -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

### 5.5.2 Compute $Q_{32}$

The next step is to compute $\sigma_{32}$. The cone $\sigma_3$ is computed in Section 5.4.2. Moreover, $\alpha = -(v_1 w_1^k + v_2 w_2^k) = -(-2 + 3) = -1$ gives $\bar{w}^2 = (-1, 2, 0) - 2(0, 1, -1) = (-1, 0, 1)$. This results in the polyhedron $Q_{32}$.

$$Q_{32} = \begin{pmatrix} -1 & 0 & 0 & \frac{3}{2} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & 0 \\ 1 & 1 & \frac{3}{2} & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_{32}^{3\vee} = \begin{pmatrix} -1 & 0 & \frac{3}{2} & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$
This polyhedron has three admissible decomposition that all result in a different cone $\sigma_{32}$.

\[
Q_{32} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}
\]

$\sigma_{32}^1 = \begin{pmatrix} -1 & 0 & 0 & \frac{3}{2} & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

$\sigma_{32}^1 \lor = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Another admissible decomposition of $Q_{32}$ yields the cone $\sigma_{32}^2$.

\[
Q_{32} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}
\]

$\sigma_{32}^2 = \begin{pmatrix} -1 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

$\sigma_{32}^2 \lor = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
Finally, the last decomposition of $Q_{32}$ gives the cone $\sigma_{32}^3$.

$$Q_{32} = \begin{pmatrix}
-1 & 0 & \frac{1}{2} & 0 \\
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{pmatrix} \oplus \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{pmatrix}$$

$$\sigma_{32}^3 = \begin{pmatrix}
-1 & 0 & \frac{1}{2} & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}$$

$$\sigma_{32}^{3\vee} = \begin{pmatrix}
-2 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}$$

### 5.5.3 Conclusion

Consider the mapping matrix

$$\pi_{23} = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}$$ (5.12)
and the matrix descriptions of the cones $\sigma_{23}$ and $\sigma_{32}$. We now have the following equations.

\[
\begin{align*}
\pi_{23}^1 \sigma_{23}^1 &= \sigma_{32}^1 \\
\left(\begin{array}{cccc}
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\end{array}\right) \left(\begin{array}{cccc}
-2 & 1 & -1 & 0 & 0 & 0 \\
3 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{array}\right) &= \left(\begin{array}{cccc}
-2 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 \\
\end{array}\right)
\end{align*}
\]

\[
\begin{align*}
\pi_{23}^2 \sigma_{23}^2 &= \sigma_{32}^2 \\
\left(\begin{array}{cccc}
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\end{array}\right) \left(\begin{array}{cccc}
-4 & 1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
2 & 0 & 1 & 0 & 0 & 1 \\
\end{array}\right) &= \left(\begin{array}{cccc}
-2 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
2 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
\end{array}\right)
\end{align*}
\]

\[
\begin{align*}
\pi_{23}^3 \sigma_{23}^3 &= \sigma_{32}^3 \\
\left(\begin{array}{cccc}
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\end{array}\right) \left(\begin{array}{cccc}
-3 & 1 & -1 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
\end{array}\right) &= \left(\begin{array}{cccc}
-2 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 \\
\end{array}\right)
\end{align*}
\]

These show that all the decomposition of $Q_2$ are compatible with the unique decomposition of $Q_3$.

### 5.6 Conclusion

We now have verified that all the admissible decompositions are compatible, except for the decomposition that is constructed from $a = v$. This decomposition is however compatible with another decomposition of $Q_2$, namely the decomposition as described in 5.2. This decomposition is of the second type, where $v \not\in N$ and $v = -\frac{1}{2}$. This decomposition is not discussed in this chapter, as these decompositions are of the same polytope. Although the compatibility can be verified equivalently with the process that is described in this chapter, it can also be derived as described in the previous chapter. As that works more efficiently, the decompositions that come from the same polytope are not verified in this chapter. However, the reader can easily verify which decompositions are compatible.

Overall, the results that come from the computations in this thesis are according to the results in [18]. Thus as a conclusion, the combinatorial approach gives equivalent results for these decompositions, which strengthen our results.


Chapter 6

Discussion

The previous chapter shows an example of how the compatibility of two decompositions is determined for degree = 2. The case where the decompositions come from the same line segment is not treated in that chapter, as it is more convenient to use the other procedure for this. It is possible to determine this with the two computed cones \( \sigma_{hk} \) as well, but as one can see that is much more complex. As mentioned earlier, this example is treated in [18] with respect to toric varieties. It is interesting to see that two different approaches lead to the same result. A possible next step is to combine both approaches that might lead to new results.

Another possible opportunity for further research is to generalize the mapping between the two cones \( \sigma_{hk} \) and \( \sigma_{kh} \). This thesis ended by showing the morphism, but one can describe this in terms of the given factors (\( w_k, w_h, \alpha \) etcetera).

When such a description of \( \pi \) is found, then it is possible to extend the example towards a \( d \)-dimensional solution. Although from examples we have seen that such a solution most likely does exist, we did not succeed to give a generic description for the higher dimensional lattices. However, this thesis gives a good starting point for research.

The computational approach that is taken in this thesis might help the notion of compatibility in other research areas. Many of the available literature concerning this topics make the link to toric varieties and deformation theory. It would be interesting to combine these results in order to find a generic description. Although several steps were taken in this direction, we were not able to combine the results of the two research area. However, with this thesis we were able to determine some promising descriptions.

We will not elaborate on the deformation theory in detail, as that would require a whole new set of definitions and prerequisites. Refer to the existing literature like [3], [2] and [5] as background information.
In terms of versal deformations, one of the main objectives of this thesis was to combine the versal deformation in degree $-R$ and $-S$ to a deformation that is versal in both degrees, in which we succeeded by taking a different approach. [6] describes the problem description with respect to the deformation theory.
Appendix A

Admissible Minkowski decomposition

The decomposition of a polyhedron

\[ Q = \bigoplus_{i=0}^{q-1} Q_i = Q_0 \oplus Q_1 \oplus \ldots \oplus Q_{q-1} \]

is admissible if and only if for all the faces \( F \) of \( Q \) at most one of the summand faces \( F_i \) does not contain a lattice point.

Compatible decompositions

Two admissible Minkowski decompositions are compatible when they can be further decomposed in the same prime decomposition. This is defined as the common refinement of the decompositions.

Cross cut

The cross cut is the intersection of a cone \( \sigma \) and a hyperplane \( H \). The resulting intersection is a polyhedron.

\[ Q := \sigma \cap H. \]

In this thesis the hyperplane \( H \) is taken as the set \( \{ x \in \mathbb{N}^\mathbb{R} | r^\top x = 1 \} \) for a given vector \( r \). Therefore the obtained polytope is sometimes referred to as \( Q_r \).

Edge vector

The edge vector \( t \in \mathbb{Q}^n \) of a polyhedron \( Q \) is a vector such that \( 0 \leq t_i \leq 1 \) for each entry \( t_i \) \((i = 0, \ldots, n - 1)\) where \( n \) is the size of the edge set of \( Q \). This edge set is the set of primitive vectors that relate to the edges and therefore \( t_i d_i \) is an edge of \( Q \) for all \( i \).

Indecomposable polyhedron

When a polyhedron \( Q \) is called indecomposable, this means that the polyhedron is not decomposable in a nontrivial way. This means that every decomposition of \( Q \) is a trivial one. Thus, the decomposition is of the type \( Q = \lambda Q \oplus (1 - \lambda)Q \) for \( 0 \leq \lambda \leq 1. \)
Prime decomposition

A prime decomposition is an admissible Minkowski decomposition of a polyhedron $Q$ such that all its summands are admissibly indecomposable polyhedra.

Polyhedral cone

A cone in this thesis is actually a strongly convex polyhedral cone that is convex and finitely generated. This means that it is the positive hull of a finite set of generators, and if $u$ is such a generator, then by definition $-u$ is not. Moreover, these generators are rational vectors, i.e. $u \in \mathbb{Q}^d$. 
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