Inverse problems in urban traffic flow
optimal arrival rates and traffic light settings at intersections

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Inverse Problems in Urban Traffic Flow

Optimal arrival rates and traffic light settings at intersections

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Abstract

Urban traffic flow affects a large part of our daily life, i.e. when we drive to the office or when we come back home. Traffic flow in cities is often regulated by traffic lights and intersections. Highly undesirable phenomena occurring at heavy traffic, e.g. rush hour, are the so called gridlocks. In this situation, cars block the intersection and are not able to move because cars in front of them are also blocked.

In this master’s thesis, we pose a simple model for capturing the queue lengths of cars waiting at an intersection. Essentially, we control the number of cars in the queue by controlling the rate of cars that arrive at the queue. After we study the behavior of cars leaving the queue at an intersection, we are able to find the optimal arrival functions by solving the associated inverse problem. For this we use the collage method. After an intensive literature study on the collage method, we will use the method to derive recommendations for the arrival rates. These recommended arrival rates will prevent gridlocks at intersections.

In cities, the arrival rate of cars at one intersection often depends on the traffic light settings of the previous intersection. This observation is applied on a real traffic situation in Beek (Limburg, The Netherlands). For this situation recommendations on the traffic light settings are given such that the intersection is prevented from having a gridlock.

Keywords: Traffic flow; Inverse problems; Collage method; Gridlock; Queue lengths; Arrival rate control; Traffic light optimization; Dynamical system

MSC 2010: 15A29; 34L40; 34F05; 45Q05; 60H10;
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Chapter 1

Introduction

Every day while driving our car to work or school, we have to deal with intersections and traffic lights. Traffic flow in a city consists of a complex network of these intersections. Intersections between main roads are often organized by a set of traffic lights. These traffic lights act by a prescribed set of rules, called traffic light settings\(^1\). These traffic light settings differ per intersection, lane, time of the day, etc. Since a lot of research has already been done in the direction of urban traffic and the tuning of traffic lights, the settings are well chosen in many cases. However, in some cases the current traffic light settings lack the possibility to adapt to the way drivers behave on a road.

1.1 Motivation

Drivers these days are more and more in a hurry and pay less attention to what happens around them. This also has the consequence that when cars stand in front of a traffic light, the drivers do not are about what happens in front of them, but simply drive away when the traffic light is switched to green. If the road in front of these cars is crowded, it could happen that the cars cannot pass the intersection before other directions get green light. Because the drivers of these directions also pay less attention to what happens in front of them and only care about the green light, they also start to drive away. If this happens, the whole intersection could be blocked with cars, where in the worst case no cars on the intersection are able to move. This worst case scenario is called \textit{gridlock}. In Figure 1.1 two examples of a gridlock are shown. In both examples it is clear that no movement is possible without strict police intervention.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{gridlock_intersection.png}
\caption{Examples of gridlock at an intersection.}
\end{figure}

\(^{1}\)It is also possible that the intersection is equipped with sensors, but we will not consider this case here.
1.2 Approach

A gridlock is very annoying for drivers. A gridlock at an intersection affects all the cars that are waiting in the queue at that intersection. All these cars are not able to move. The only way to resolve a gridlock is by police intervention. It could take several hours before the traffic at the intersection is back to normal. It is therefore highly desirable if not absolutely necessary to prevent gridlocks at intersections, because when a gridlock occurs, the damage is done and it takes a lot of time to resolve this issue.

Already much research has been done for the prevention of gridlocks in urban traffic flow, see e.g. [12], [25] and [33]. However, all these models have one thing in common: they all consider a network of intersections in order to prevent gridlocks. Other research on urban traffic flow is presented in [9], [13] and [27]. Other models, like the classical Aw-Rascle model [1], describe the behavior of traffic flow on highways. In this thesis we provide a simple model for a single intersection in which we can easily prevent a gridlock.

1.2 Approach

Let us consider two intersections, as sketched in Figure 1.2.

![Figure 1.2: A sketch of two connected intersections.](image)

To prevent a gridlock at intersection 2, we have to prevent the queue of red cars at intersection 1 from getting too large. In other words, we have to control the queue at intersection 1. If we assume that we know the traffic light settings of intersection 1 and also how the cars drive away at intersection 1, then we can control the number of cars in the queue by limiting the amount of cars that arrive at the queue per second (the arrival rate). This leads to the main question of this thesis:

*How should cars arrive at the queue of intersection 1 in order to prevent a gridlock at intersection 2?*

There are two ways to control the way cars arrive at the queue of intersection 1:

1. Control the rate of cars that arrive at queue 1. The rate of cars is the amount of cars that enter the queue per second. If the queue is almost full, the rate should be lowered, possibly to 0. This rate in cars per second can be translated to a desired speed for each car. This would mean that at every moment, a car would have to drive at the prescribed velocity in order to prevent the queue from exceeding a certain maximum length. However, with autonomous cars still futuristic, it is hard to keep drivers at the speed which they should drive. Especially, when this speed is below the maximum speed\(^2\);

\(^2\)This is something which is present at highways. Sometimes in order to prevent a traffic jam on some highway from
1.3. Basic model

2. Control the arrival of cars by smart traffic light settings. Since there are traffic lights placed on intersection 2 as well, these traffic lights should be set in such a way that a gridlock can never occur.

Both ways are investigated in this thesis.

1.3 Basic model

For the derivation of our model, we start with the discrete equations for the number of cars at a queue at a roadblock, first presented in [10]. Since the behaviour of cars in a queue at a roadblock is similar to the behaviour of cars in a queue at traffic lights (see [31]), we can use these discrete equations to derive a model for the queue length at intersections with traffic lights.

Let us consider a small piece of street, see Figure 1.3.

![Figure 1.3: Street with a queue of cars in front of a traffic light.](image)

We only consider what happens before the traffic light and assume what happens after the traffic light as given. We do not distinguish between the cases where the traffic light is part of a roadblock or an intersection with two, three or more traffic lights. We just assume the traffic light is part of some intersection where the light alternates between red and green. This alternating behaviour of the traffic light happens according to some cyclic pattern which is prescribed. The total cycle period is called $T$. Let this cycle start at $t = 0$. For the traffic light shown in Figure 1.3, every cycle consists of three phases:

- Phase I: light is red (from $t = 0$ to $t = T_0$);
- Phase II: light is green (from $t = T_0$ to $t = T_1$);
- Phase III: light is red (from $t = T_1$ to $t = T$).

We know how long the light is green ($T_1 - T_0$) and how long the light is red ($T - T_1 - T_0$). Note that we have some freedom in choosing how to start the cycle. This means that it is possible to let the cycle start at the moment the light switches green ($T_0 = 0$). Then the cycle consists of only two phases: a green phase from $t = 0$ to $t = T_1$ and a red phase from $t = T_1$ to $t = T$ (it is also possible to start with a red phase and end with a green phase). One could argue that it would be easier to assume that there are only two phases, however as we will see in this thesis, we are interested in the arrival rates compared to the traffic light settings in front of the queue. So it is possible that cars should arrive before the light is green or after the light has switched to red again. Therefore it is better to start and end with a red phase.

increasing, cars in front of that piece of highway are required to lower their velocity. However, many drivers do not follow these recommendations. This would be different if cars drove autonomously and their velocity could be controlled from an external place.
1.3. Basic model

1.3.1 Notation

• \( N(t) \) is the number of cars waiting at the traffic light at time \( t > 0 \) (queue length [cars]);
• \( \alpha(t) \) is the arrival rate at time \( t \) in [cars\cdot s\(^{-1}\)];
• \( \beta(t) \) is the passing rate at time \( t \) in [cars\cdot s\(^{-1}\)];
• \( T \) is the total cycle length of the traffic lights;
• \( T_0 \) is the time the light switches to green, \( T_1 \) is the time the light switches to red.

1.3.2 Assumptions

• The passing function \( \beta(t) \) is a positive function which is 0 at \( t = 0 \). Note that this function describes at what rate cars leave the queue.
• The cycle period \( T \) is fixed.
• All cars are assumed to be indistinguishable.
• We assume a heavy traffic scenario (e.g. rush hour). By this we mean that there is a lot of traffic on the road and the queues are nearly full. The queues are never full for a light traffic light scenario, so here a gridlock will not occur.

1.3.3 Discrete equations

Let the cycle start at \( t = 0 \). Since there are three phases, the discrete equations of the queue length at time \( t \) are given by

\[
N(t) = \begin{cases} 
N(0) + \int_0^t \alpha(s)ds & \text{for } 0 \leq t < T_0, \\
N(T_0) + \int_{T_0}^{T_1} \alpha(s)ds - \int_{T_0}^{t} \beta(s - T_0)ds & \text{for } T_0 \leq t < T_1, \\
N(T_1) + \int_{T_1}^{t} \alpha(s)ds & \text{for } T_1 \leq t < T.
\end{cases}
\]

(1.3.1)

The number of cars that have arrived in the interval \([0, t]\) is given by \( \int_0^t \alpha(s)ds \). Note that in our model, we distinguish between the modeling of the arrival and passing of cars. The arrival function \( \alpha(t) \) is simply a function of time which indicates the arrival rate of cars at each time \( t \in [0, T]\) and depends on the time of the day. The passing function \( \beta(t) \) is a function which models how cars leave the queue when the light switches to green. Since cars start to leave the queue at the moment the light switches green (\( \beta(T_0) = 0 \)), we model this behavior by letting \( \beta \) depend on \( T_0 \) as well. We want that \( \beta(t) = 0 \) for \( t = T_0 \). This models the behavior of cars starting to accelerate at the moment the light switches to green. Since there is only green light for \( t \in [T_0, T_1] \), there is only the passing term for this time interval in (1.3.1).

1.3.4 Differential equations

The associated differential equation which we use is given by

\[
\begin{cases} 
\frac{dN}{dt}(t) = \alpha(t) - \beta(t - T_0) \cdot \chi_{[T_0, T_1]}(t) & \text{for } t \in [0, T], \\
N(0) = N_0, 
\end{cases}
\]

(1.3.2)
1.3. Basic model

where \(N_0\) is the initial queue length and \(\chi_{[T_0,T_1]}(t)\) is the characteristic function

\[
\chi_{[T_0,T_1]}(t) = \begin{cases} 
1 & \text{for } t \in [T_0, T_1), \\
0 & \text{for } t \notin [T_0, T_1). 
\end{cases} 
\]  

(1.3.3)

The characteristic function \(\chi\) can be written in terms of the Heaviside function or Heaviside step function. The Heaviside function is defined as

\[
\text{Heaviside}(t) = \begin{cases} 
0 & \text{for } t < 0, \\
\frac{1}{2} & \text{for } t = 0, \\
1 & \text{for } t > 0. 
\end{cases} 
\]  

(1.3.4)

The Heaviside function \(H\) is shown in Figure 1.4.

Figure 1.4: Plot of the Heaviside function on \([-2, 2]\).

Let us define \(H_c(t) := \text{Heaviside}(t - c)\). The characteristic function can then be written as

\[
\chi_{[T_0,T_1]}(t) = H_{T_0}(t) - H_{T_1}(t). 
\]  

(1.3.5)

How the differential equation in (1.3.2) is derived can be found in Appendix A. The dependence of \(\beta\) on \(T_0\) models the behavior of cars starting to accelerate at the moment the light switches green. While the term characteristic function \(\chi_{[T_0,T_1]}(t)\) makes sure that cars only leave the queue when the light is green.

The differential equation describes how the number of cars in the queue changes depending on the flux of cars entering the queue \((\alpha(t))\) and the flux of cars leaving the queue \((\beta(t))\).

Note that the right-hand side of the ordinary differential equation (ODE), given in (1.3.2), does not depend on \(N(t)\). So if the parameters \(T_0, T_1\) are given, the initial queue length \(N_0\) and the
1.4. Structure of the thesis

arrival and passing functions are known then the ODE can simply be solved by integrating the differential equation. This gives the solution

\[ N(t) = N_0 + \int_0^t \alpha(s) ds - \int_0^t \beta(s - T_0) \cdot \chi_{[T_0,T_1]}(s) ds, \quad \text{for } t \in [0,T]. \]  

(1.3.6)

The characteristic function in the second integral and the use of a substitution make it possible to rewrite (1.3.6) as

\[ N(t) = N_0 + \int_0^t \alpha(s) ds - \int_0^{t-T_0} \beta(s) ds \cdot \chi_{[T_0,T_1]}(t) - \int_{T_0}^{T_1-T_0} \beta(s) ds \cdot H_{T_1}(t), \quad \text{for } t \in [0,T]. \]  

(1.3.7)

(1.3.7) is the solution of the ODE given in (1.3.2). We see that this corresponds with (1.3.1). \( N(t) \) is continuous, but not necessarily continuously differentiable.

1.3.5 Inverse problem

We are interested in controlling the number of cars in the queue \( N(t) \). If the queue becomes too large, then the queue will block the previous intersection. The basic idea is to control the number of cars by limiting the amount of cars per second that arrive at the queue.

Due to the street’s capacity, there is a maximum number of cars allowed in the queue: \( N_{\text{max}} \). We are interested in the rate of cars that arrive, such that the queue is equal to this \( N_{\text{max}} \). In other words, we are interested in the arrival rates \( \alpha(t) \) such that the queue \( N(t) \) is equal (or as close as possible) to the desired number of cars. This is a typical inverse problem. Inverse problems in traffic flow are the main subject of this thesis and are solved via the collage method. As far as we know this thesis is the first time the collage method is used for solving inverse problems in urban traffic flow. The collage method will be explained extensively in Chapter 2. See [23] for a wide variety of applications of this method.

1.4 Structure of the thesis

In this thesis we focus on solving inverse problems in urban traffic flow. For this we use a model for the evolution of queue lengths at traffic lights. To prevent queues from getting too large, we are interested in the optimal way of cars arriving at this intersection. The arriving of cars is given by the so called arrival function. This function gives at each time the rate at which cars arrive at the queue. This rate is in cars per second. The search for the optimal arrival function can be expressed as an inverse problem.

The inverse problems which are presented in this thesis, will be solved via the collage method. This is a method which can solve inverse problems for differential equations of different types, in particular it can treat ordinary differential equations and stochastic differential equations. This method and its mathematical background is explained extensively in Chapter 2. The method is explained in a general setting, for ordinary differential equations without any noise (Section 2.2), ODEs with noise (Section 2.3) and stochastic differential equations (Section 2.4). However, in this thesis, we will restrict the solving of inverse problems in urban traffic flow merely to the ODE case.

Before anything can be said about the optimal arrival of cars at an intersection, first we must know how cars behave when they leave the queue. “Cars leaving the queue” is referred to as the “passing” of cars. In Chapter 3 the passing of cars is investigated based on real traffic data from the intersection of the Onze Lieve Vrouwestraat and the John F. Kennedylaan in Eindhoven. The data is used to derive and compute the passing function. This function is from then on used in this thesis.
1.4. Structure of the thesis

In Chapter 4 the inverse problems for finding the optimal arrival function is presented and solved via the collage method. Two numerical cases are investigated by using different types of function classes. The different function classes are compared.

In urban traffic flow, the arrival of cars at an intersection often depends on how cars have left the previous intersection. In Chapter 5 it is investigated how the traffic lights of the previous intersection affect the queue of the next intersection. Moreover, optimal traffic light settings are found by means of solving the inverse problem, where the arrival function is now a function depending on these traffic light settings. The model is applied to a real traffic situation in Beek (Limburg, The Netherlands).

Finally, in Chapter 6, we review the work that we have done and the results that we have obtained. Moreover, we present our conclusions and give a few suggestions for future work.
Chapter 2

Inverse problem: the collage method

The collage method is a method, which makes use of a nice consequence of the fixed point theorem, see [21, Thm. 1.1]. As we will describe in this chapter, the method can be applied to several type of differential equations.

2.1 General approach

2.1.1 Inverse problems

We consider inverse problems of the following type:

Find the values for the parameters such that a function depending on those parameters is as close as possible to a desired function.

The desired function is often given by some data. We look at specific types of inverse problems, where we are interested in finding unknown parameters in a function which appears in the right-hand side of a differential equation. In other words, we want to find (an approximation of) the function $f$ such that the solution to

$$\begin{cases} \frac{du}{dt}(t) = f(t, u(t)) \quad \text{for } t \in [0, T], \\ u(0) = u_0. \end{cases}$$

is as close as possible to our target solution $\bar{u}(t)$.

2.1.2 Fixed-point theorem

We first give the definition of a contraction mapping and an eventually contractive mapping.

**Definition 2.1.1.** Let $(X,d)$ be a metric space. Then a mapping $T : X \to X$ is called a contraction mapping if there exists a constant $c \in [0, 1)$ such that

$$d(Tu, Tv) \leq cd(u, v),$$

for all $u, v \in X$.

**Definition 2.1.2.** Let $(X,d)$ be a metric space. Then a mapping $T : X \to X$ is called eventually contractive if there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ the mapping $T^n : X \to X$ is a contraction. That is, there exists an $n_0 \in \mathbb{N}$ and $c_n \in [0, 1)$ such that for all $n > n_0$ we have that

$$d(T^n u, T^n v) \leq c_n d(u, v),$$

for all $u, v \in X$. 

2.1. General approach

for all \( u, v \in X \).

Note that any contraction mapping is also eventually contractive. However, the converse is not (always) true. For example, consider the mapping \( T^x = \cos x \) on \( \mathbb{R} \). Then \( T \) is not a contraction, but the iterated function \( \cos (\cos x) \) is. So \( T \) is eventually contractive.

If \((X, d)\) is complete and \( T \) is a contraction, then the following theorem states that there is a unique fixed point and the sequence \( u_0, T(u_0), T(T(u_0)), T(T(T(u_0))), \ldots \) will converge to a unique fixed point for any initial guess \( u_0 \).

**Theorem 2.1.3. [Banach Fixed Point Theorem]** Suppose \((X, d)\) is a complete metric space, and suppose \( T : X \to X \) is a contraction mapping, then there exists a unique fixed point of \( T \). That is, there exists a unique \( u^* \) such that \( Tu^* = u^* \).

In a more general setting, the inverse problem is given by:

Given any target function \( \pi \), find a contraction mapping \( T : X \to X \) with fixed point \( u^* \), such that \( d(\pi, u^*) \) is as small as possible.

Consequences of the Banach Fixed-point theorem are Theorems 2.1.4 and 2.1.5. Theorem 2.1.4 gives a nice result for mappings which are eventually contractive (see Definition 2.1.2), which we will need in Section 2.4.1. Theorem 2.1.5 states that the distance between the target solution \( \pi \) and the fixed point \( u^* \) of \( T \) is bounded from below and above by a constant times the distance between the target solution \( \pi \) and the image of \( \pi \) under \( T \) given by \( T \pi \).

**Theorem 2.1.4. [Generalized Banach Fixed Point Theorem]** Suppose \((X, d)\) is a complete metric space, and suppose \( T : X \to X \) is eventually contractive, then there exists a unique fixed point of \( T \). That is, there exists a unique \( u^* \) such that \( Tu^* = u^* \).

**Proof.** Let \( u_0 \) be arbitrary in \( X \). Define the sequence \( \{u_i\} \) as

\[
u_{i+1} = Tu_i.
\]

Because \( T \) is eventually contractive we have that \( T^n \) is a contraction mapping. Now define the subsequence \( \{u_{n,k}\} \) as follows

\[
u_{n,k+1} = T^n u_{n,k}.
\]

By Theorem 2.1.3 it follows that the subsequence \( \{u_{n,k}\} \), for fixed \( n \), converges to an element \( u^* \in X \), which is a fixed point of \( T^n \). So we have that

\[
\lim_{k \to \infty} T^{n,k} u_0 = u^*,
\]

where \( T^{n,k} u_0 \) means applying the operator \( T^n \) \( k \) successive times to element \( u_0 \in X \). Now we show that \( u^* \) is also a fixed point of \( T \).

Because \( T^n \) is a contraction, we know we can obtain the element \( Tu^* \in X \) by applying \( T^n \) successively to the element \( Tu_0 \). Therefore, we have

\[
Tu^* = T \left[ \lim_{k \to \infty} T^{n,k} u_0 \right] = \lim_{k \to \infty} T^{n,k}[Tu_0].
\]
2.1. General approach

Then, by using (2.1.6) and (2.1.7) we have that
\[ d(u^*, T^* u^*) = d\left( \lim_{k \to \infty} T^{n,k} u_0, \lim_{k \to \infty} T^{n,k} [T u_0] \right) \]
\[ = \lim_{k \to \infty} d(T^{n,k} u_0, T^{n,k}(T u_0)) \]
\[ = \lim_{k \to \infty} d\left( T^n [T^{n,k-1} u_0], T^n [T^{n,k-1}(T u_0)] \right) \]
\[ \leq c_n \lim_{k \to \infty} d\left( T^{n,k-1} u_0, T^{n,k-1}(T u_0) \right) \]
\[ = c_n d(u^*, T u^*), \]
where \( c_n < 1 \) since we have that \( T^n \) is a contraction mapping. Now we have that \( d(u^*, T^* u^*) \leq c_n d(u^*, T u^*) \) for \( c_n < 1 \), so this means that \( d(u^*, T u^*) = 0 \) and hence \( u^* = T u^* \). Consequently, \( u^* \) is a fixed point of \( T \).

For proving uniqueness, we assume that \( u^*, v^* \) are two fixed points of \( T \), then
\[ d(u^*, v^*) = d(T^* u^*, T^* v^*) \leq c_n d(u^*, v^*), \]
for \( c_n < 1 \). So we have that \( d(u^*, v^*) = 0 \) and hence \( u^* = v^* \). So the \( u^* \) found by successive approximation is a unique fixed point of \( T \).

\[ \text{Theorem 2.1.5. [Collage Method]} \]
\[ \text{Let } (X, d) \text{ be a complete metric space and } T : X \to X \text{ a contraction mapping with contraction factor } c \in [0, 1). \text{ Then for any target } \pi \in X, \]
\[ \frac{1}{1 + c} d(\pi, T \pi) \leq d(\pi, u^*) \leq \frac{1}{1 - c} d(\pi, T \pi), \]
where \( u^* \) is the fixed point of the contraction \( T \).

\[ \text{Proof.} \text{ The first inequality is proven as follows:} \]
\[ d(\pi, T \pi) \leq d(\pi, u^*) + d(u^*, T \pi) \]
\[ = d(\pi, u^*) + d(T u^*, T \pi) \]
\[ \leq d(\pi, u^*) + c d(u^*, \pi) \]
\[ = (1 + c) d(\pi, u^*), \]
where \( \leq \) denotes the triangle inequality. Rewriting (2.1.11) gives
\[ d(\pi, u^*) \leq \frac{1}{1 + c} d(\pi, T \pi). \]

The second inequality is proven similarly:
\[ d(\pi, u^*) \leq d(\pi, T \pi) + d(T \pi, u^*) \]
\[ = d(\pi, T \pi) + d(T \pi, T u^*) \]
\[ \leq d(\pi, T \pi) + c d(u^*, \pi). \]

And so, rewriting (2.1.13) gives
\[ d(\pi, u^*) \geq \frac{1}{1 - c} d(\pi, T \pi). \]

A consequence of this is that whenever the distance between between a target \( \pi \) and its image \( T \pi \) (called the collage distance) is small, also the distance between the target and the fixed point of that
mapping is small. Therefore a small collage distance \( d(\pi, T\pi) \) means that the fixed point of our mapping lies close to our target. Originally, the collage method was designed and used in fractal image coding where given a certain image, one wanted to find the functions that generated that image. This is an inverse problem in fractal image coding.

Our inverse problem can now be reformulated as:

*Given a target function \( u \), find a mapping \( T : X \to X \) such that \( d(u, T\pi) \) is minimal.*

In other words, instead of searching for contraction mappings whose fixed points lie close to a target solution, we seek mappings which send the target solution close to itself.

In many applications, from a family of contraction mappings \( T_\lambda \), where the parameters \( \lambda \) are from a set of allowed parameters \( \Lambda \subset \mathbb{R}^n \), one wants to find the parameters \( \lambda \) such that the distance \( d(\pi, u_\lambda^*) \) between target \( \pi \) and the fixed point \( u_\lambda^* \) of \( T_\lambda \) is small. Due to Theorem 2.1.3, we can also minimize \( d(\pi, T_\lambda\pi) \) over all \( \lambda \in \Lambda \). Note that Theorem 2.1.3 also states that there is a lower bound on the approximation error \( d(\pi, u^*) \).

The collage method is often used in fractal image coding and compression methods [2] and [3], but is also very useful in solving inverse problems for differential equations [14], [15], [16], [17], [22]. It is worth noting that this method can also be extended to differential equations with random coefficients [19] and even stochastic differential equations (SDE), e.g. [18], [29].

In the next sections, the collage method will be explained in detail for different types of differential equations, like ordinary differential equations (ODE, Section 2.2), ODE with random coefficients (Section 2.3) and stochastic differential equations (SDE, Section 2.4). For the application of the collage method to initial boundary value problems we refer to [20] and [21].

### 2.2 Collage method: ODE

#### 2.2.1 The method

Consider the following ODE, with right-hand side function \( f \) and (possibly unknown) initial value \( u_0 \)

\[
\begin{cases}
\frac{du}{dt}(t) = f(t, u(t)) & \text{for } t \in [0, T], \\
u(0) = u_0.
\end{cases}
\] (2.2.1)

By integrating both sides of the differential equation with respect to \( t \), we obtain the corresponding Picard integral operator \( T \)

\[
(Tu)(t) = u_0 + \int_0^t f(s, u(s))ds.
\] (2.2.2)

Under suitable conditions (see Theorem 2.2.1), this mapping \( T \) is a contraction mapping over the space \( C[I] \), where \( I \) is an interval containing \( t = 0 \). Furthermore, the fixed point of this contraction mapping \( T \) is the unique solution of (2.2.1). So instead of solving the ODE given in (2.2.1), we seek the fixed point of (2.2.2).

**Theorem 2.2.1** [17], [21] Let \( f \in L^2([0, T]) \) and \( f(t, u) \) be Lipschitz continuous in the variable \( u \in \mathbb{R} \) with Lipschitz constant \( K \). Then also \( T \) is Lipschitz continuous on the space \( C([-\delta, \delta] \times [-M, M]) \) (with Lipschitz constant \( c = \delta K \)) for some \( \delta, M > 0 \).
Now, let there be a $\delta' > 0$ such that $\delta'K < 1$, then Theorem 2.1.3 states that $T$ is a contraction mapping on $C([-\delta,\delta] \times [-M,M])$. In that case it is possible to solve the inverse problem.

Since the function $f$ is unknown, let us consider it as a function of parameters. We assume that our unknown $f \in L^2([0,T])$, so we write $f$ as a linear combination of basis functions (i.e. we take the $L^2$ expansion of the function $f$)

$$f_\Delta(s,u) = \sum_{i=1}^\infty \lambda_i \phi_i(s,u).$$  \hspace{1cm} (2.2.3)

If the basis functions $\phi_i(s,u)$ are Lipschitz in $u$, then so is $f$ and hence for the appropriate $\delta'$ Theorem 2.2.1 states that $T$ is a contraction. The square of the collage distance $\Delta$ in this case is given by

$$\Delta^2 := \| u - T u \|^2 = \int_{-\delta}^\delta \left( u(t) - u_0 - \int_{0}^t f_\Delta(s,u(s))ds \right)^2 dt.$$  \hspace{1cm} (2.2.4)

By substituting the first $n$ terms of (2.3) in (2.4), we obtain a quadratic form in the parameters $u_0, \lambda_1, \lambda_2, ..., \lambda_n$

$$\Delta^2_n(\Delta) = \int_{-\delta}^\delta \left( u(t) - u_0 - \sum_{i=1}^n \lambda_i \phi_i(s,u(s)) \right)^2 dt.$$  \hspace{1cm} (2.2.5)

If we define $g_i(t) := \int_{0}^t \phi_i(s,u(s))ds$, then we obtain

$$\Delta^2_n(\Delta) = \int_{-\delta}^\delta \left( u(t) - u_0 - \sum_{i=1}^n \lambda_i g_i(t) \right)^2 dt.$$  \hspace{1cm} (2.2.6)

Minimizing $\Delta^2$ with respect to the parameters yields a linear system of $n + 1$ equations

$$\begin{align*}
\frac{\partial \Delta^2}{\partial u_0} &= 0, \\
\frac{\partial \Delta^2}{\partial \lambda_1} &= 0, \\
&\vdots \\
\frac{\partial \Delta^2}{\partial \lambda_n} &= 0.
\end{align*}$$  \hspace{1cm} (2.2.7)

This results in

$$\begin{pmatrix}
\langle 1 \rangle \\
\langle g_1 \rangle \\
\langle g_2 \rangle \\
\vdots \\
\langle g_n \rangle
\end{pmatrix}
\begin{pmatrix}
\langle g_1 \cdot g_1 \rangle \\
\langle g_1 \cdot g_2 \rangle \\
\langle g_2 \cdot g_2 \rangle \\
\vdots \\
\langle g_n \cdot g_n \rangle
\end{pmatrix}
\begin{pmatrix}
u_0 \\
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{pmatrix}
= 
\begin{pmatrix}
\langle u \rangle \\
\langle u \cdot g_1 \rangle \\
\langle u \cdot g_2 \rangle \\
\vdots \\
\langle u \cdot g_n \rangle
\end{pmatrix},
$$  \hspace{1cm} (2.2.8)

where $\langle f \rangle := \int_{-\delta}^\delta f(t)dt$. The solution to this linear system defines then an operator $T_\lambda$.

**Example 2.2.2**

Consider the ODE

$$\begin{align*}
\frac{du}{dt}(t) &= \lambda_1 + \lambda_2 u(t), \quad \text{for } t \in [0,1] \\
u(0) &= u_0.
\end{align*}$$  \hspace{1cm} (2.2.9)
2.2. Collage method: ODE

Our target solution is given by \( \pi(t) = t^2 \). Our parameter vector consists of three unknowns: \( u_0, \lambda_1, \lambda_2 \). As basis functions we take \( \phi_1(s, u(s)) = 1 \) and \( \phi_2(s, u(s)) = \pi(s) = s^2 \). This means that for our functions \( g_i(t), i = 1, 2 \) we obtain

\[
g_1(t) = \int_0^t \phi_1(s, u(s)) ds = t, \tag{2.2.10}
\]
\[
g_2(t) = \int_0^t \phi_2(s, u(s)) ds = \frac{1}{3} t^3.
\]

Our linear system (2.2.8) becomes then

\[
\begin{pmatrix}
1 & 1 & 1 & 12 \\
1 & 2 & 3 & 15 \\
1 & 2 & 1 & 63 \\
12 & 15 & 1 & 63
\end{pmatrix}
\begin{pmatrix}
u_0 \\
\lambda_1 \\
\lambda_2
\end{pmatrix}
= \begin{pmatrix}
1/3 \\
1/4 \\
1/18
\end{pmatrix}
\]

This leads to the values \( u_0 = -\frac{1}{27}, \lambda_1 = \frac{5}{12}, \text{ and } \lambda_2 = \frac{35}{18} \). If we substitute this into (2.2.9) and analytically solve the ODE, then we obtain the solution

\[
u(t) = \frac{67}{378} \exp \left( \frac{35}{18} t \right) - \frac{3}{14}, \text{ for } t \in [0, 1]. \tag{2.2.11}
\]

Plotting this solution together with the target solution \( \pi(t) = t^2 \) yields Figure 2.1.

![Figure 2.1: Plot of the analytical solution to the inverse problem \( u(t) \) and the target solution \( \pi(t) \).](image)

The value of the collage distance

\[
\Delta = \left( \int_0^1 \left( u(t) - u_0 - \lambda_1 t - \frac{1}{3} \lambda_2 t^3 \right)^2 dt \right)^{\frac{1}{2}} \tag{2.2.12}
\]

equals \( \frac{62113}{5000000} \approx 0.0124 \).
2.2. Collage method: ODE

Example 2.2.3 Traffic flow example
Consider a simplified version 1.3.2 of with fixed arrival and passing rates $\alpha, \beta$

$$\begin{cases}
    \frac{du}{dt}(t) = \alpha - \beta \cdot \chi_{[0,T_1]}(t), & \text{for } t \in [0,T] \\
    u(0) = u_0.
\end{cases}
$$

(2.2.13)

Now let $T_1 = \frac{1}{2}, T = 1, \alpha = \beta = 1$ and $u_0 = \frac{1}{2}$. Directly solving this ODE yields the function

$$u(t) = \begin{cases}
    \frac{1}{2} & \text{for } t \in [0,T_1) \\
    t & \text{for } t \in [T_1,T].
\end{cases}
$$

(2.2.14)

The inverse problem becomes: Given $T_1$, find $u_0, \alpha, \beta$ such that the solution to (2.2.13) is as close as possible to the given solution $u(t)$. The linear system obtained by the collage method is given

![Figure 2.2: Solution $N(t)$ to the direct problem.](image)

by

$$\begin{pmatrix}
    1 & \frac{1}{2} & -\frac{3}{8} \\
    \frac{1}{2} & 1 & -\frac{11}{48} \\
    \frac{3}{8} & \frac{11}{48} & -\frac{1}{6}
\end{pmatrix}
\begin{pmatrix}
    \lambda_0 \\
    \lambda_1 \\
    \lambda_2
\end{pmatrix}
= \begin{pmatrix}
    5 \\
    \frac{5}{8} \\
    \frac{17}{48}
\end{pmatrix}.
$$

(2.2.15)

Solving (2.2.15) yields the solution

$$\begin{pmatrix}
    u_0 \\
    \alpha \\
    \beta
\end{pmatrix}
= \begin{pmatrix}
    \frac{1}{2} \\
    1 \\
    1
\end{pmatrix},
$$

(2.2.16)

which are indeed the values for the parameters we took in our ODE.
2.3 Collage method: ODE with noise

2.3.1 Random fixed points

The deterministic framework of the previous chapter can be extended to problems with parameters with noise. Of course, in reality noise is often present. In what follows, the noise is given by $\omega \in \Omega$. Here $\Omega$ is a sample space. First, we introduce some more notation and background in measure theory [28].

**Definition 2.3.1.** Let $\Omega$ be a (sample) space. $\mathcal{F}$ is called a $\sigma$-algebra on $\Omega$ if it is a family of subsets of $\Omega$ for which the following holds:

- $\Omega \in \mathcal{F}$;
- If $E \in \mathcal{F}$ then $E^c \in \mathcal{F}$ (that is $\Omega \setminus E \in \mathcal{F}$);
- If a countable collection $\{E_i\}$, for $i \in I$, is a subset of $\mathcal{F}$, then also $\bigcup_{i \in I} E_i \in \mathcal{F}$ where $I$ is some index set.

**Definition 2.3.2.** Let $\Omega$ be a sample space and $\mathcal{F}$ be a $\sigma$-algebra. Then $(\Omega, \mathcal{F})$ is called a measurable space. Furthermore $\mu : \mathcal{F} \to [0, 1]$ is called a probability measure if the following holds:

- $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$;
- For all countable collections $E_i$, let $E_i \in \Omega$ be pairwise disjoint sets. Then $\mu(\bigcup_{i \in I} E_i) = \sum_{i \in I} \mu(E_i)$.

**Definition 2.3.3.** If $\Omega$ is a (sample) space, $\mathcal{F}$ a $\sigma$-algebra and $P$ a probability measure, then $(\Omega, \mathcal{F}, P)$ is called a probability space.

**Definition 2.3.4.** Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two measurable spaces. A mapping $T : \Omega \to \Omega'$ is called measurable (or $\mathcal{F}/\mathcal{F}'$-measurable) if the pre-image of a measurable set in $\Omega'$ is a measurable set in $\Omega$:

$$T^{-1}(E') \in \mathcal{F} \quad \text{for all } E' \in \mathcal{F}'.$$  \hfill (2.3.1)

**Definition 2.3.5.** Let $(\Omega, \mathcal{F})$ be a probability space and $(\Omega', \mathcal{F}')$ a measurable space. Then a measurable random variable $X$ is a function $X : \Omega \to \Omega'$ which is $\mathcal{F}/\mathcal{F}'$-measurable.

Usually a random variable is a real number (so $\Omega' = \mathbb{R}$), but as we will see next, it is possible to map $X$ from $\Omega$ to other spaces, like $\Omega' = C([0, T])$.

Now let $X$ be a Polish metric space (complete, separable metric space), equipped with metric $d_X$ and let $T : \Omega \times X \to X$. Then $T$ is a $c(\omega)$-Lipschitz operator if for a.e. $\omega \in \Omega$ we have that

$$d_X(T(\omega, u), T(\omega, v)) \leq c(\omega)d(u, v),$$

where $c(\omega) : \Omega \to X$. If also $0 < c(\omega) \leq c < 1$, then $T$ is a $c(\omega)$-contraction. Furthermore, $u : \Omega \to X$ is called a random fixed point of the random operator $T$ if $u$ is a solution of the random fixed point equation

$$T(\omega, u(\omega)) = u(\omega) \text{ for a.e. } \omega \in \Omega.$$  \hfill (2.3.2)

Inverse problems are now of the type:

Find $\lambda \in \Lambda$ such that $d(T_\lambda(\omega, u(\omega)), \pi(\omega))$ is as small as possible for some target function $\pi(\omega)$. 

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2.3. Collage method: ODE with noise

To study the existence and uniqueness of solutions of equations like in (2.3.2), we consider the space of all measurable functions $u : \Omega \to X$:

$$Y := \{ u \mid u : \Omega \to X \text{ measurable} \}. \quad (2.3.3)$$

Let us define the mapping $\tilde{T} : Y \to Y$ as

$$\tilde{T}(u)(\omega) := T(\omega, u(\omega)). \quad (2.3.4)$$

Now the fixed point problem can be rewritten in the space $Y$:

$$\tilde{T}(u)(\omega) = u(\omega). \quad (2.3.5)$$

Then solutions of the fixed point equation on $Y$, given in (2.3.5) are the solutions of the random fixed point equation given in (2.3.2). Now define the metric on the space $Y$ as follows

$$d_Y(u_1, u_2) := \int_{\Omega} d_X(u_1(\omega), u_2(\omega))dP(\omega) = \mathbb{E}[d_X(u_1(\omega), u_2(\omega))]. \quad (2.3.6)$$

**Theorem 2.3.6.** $(Y, d_Y)$ is a complete metric space. That is $d_Y$ is indeed a metric and every Cauchy sequence in $Y$ converges to a limit also in $Y$.

**Proof.** We follow the line of arguments from [17]:

1. $d_Y$ is a metric.
   
i. Because $d_X$ is a metric, $d_X(\cdot, \cdot) \geq 0$. Now let $u_1, u_2 \in X$. Then
   
   $$\int_{\Omega} d_X(u_1(\omega), u_2(\omega))dP(\omega) \geq 0.$$ 
   
   Note that $u_1 = u_2$ is equivalent with $u_1(\omega) = u_2(\omega)$ a.e. in $\Omega$. So $d_X(u_1(\omega), u_2(\omega)) = 0$ a.e. in $\Omega$ if and only if $d_Y(u_1, u_2) = 0$.

ii. Symmetry is easy to show:

   $$d_Y(u_1, u_2) = \int_{\Omega} d_X(u_1(\omega), u_2(\omega))dP(\omega) = \int_{\Omega} d_X(u_2(\omega), u_1(\omega))dP(\omega) = d_Y(u_2, u_1) \quad (2.3.7)$$

iii. The triangle inequality is also satisfied for $d_Y$, because we know the triangle inequality holds for the metric $d_X$:

   $$d_Y(u_1, u_3) = \int_{\Omega} d_X(u_1(\omega), u_3(\omega))dP(\omega) \leq \int_{\Omega} d_X(u_1(\omega), u_2(\omega)) + d_X(u_2(\omega), u_3(\omega))dP(\omega)$$

   $$= \int_{\Omega} d_X(u_1(\omega), u_2(\omega))dP(\omega) + \int_{\Omega} d_X(u_2(\omega), u_3(\omega))dP(\omega) = d_Y(u_1, u_2) + d_Y(u_2, u_3) \quad (2.3.8)$$
2.3. Collage method: ODE with noise

So, \( d_Y \) is indeed a metric.

II.) \( Y \) is complete.

Let \( u_n \) be a Cauchy sequence in \( Y \). Then

For all \( \varepsilon > 0 \) there exists an \( n_0 > 0 \) such that for all \( n, m \geq n_0 : d_Y(u_n, u_m) < \varepsilon \).

First we show that \( u_n(\omega) \) is a Cauchy sequence in \( X \) a.e. in \( \Omega \) and that \( u_n(\omega) \to u(\omega) \) in \( X \).

Let \( k > 0 \) and \( \varepsilon = 3^{-k} \). Then there exists an increasing sequence \( n_k \) such that

\[
d_Y(u_n, u_{n_k}) < 3^{-k}. \tag{2.3.9}
\]

Choosing \( n = n_{k+1} \) yields \( d_Y(u_{n_{k+1}}, u_{n_k}) < 3^{-k} \). Now define

\[
A_k := \left\{ \omega \in \Omega \left| d_X(u_{n_{k+1}}(\omega), u_{n_k}(\omega) > 2^{-k} \right) \right\}. \tag{2.3.10}
\]

Then on one hand we have

\[
\int_{A_k} d_X(u_{n_{k+1}}(\omega), u_{n_k}(\omega))dP(\omega) \leq \int_{\Omega} d_X(u_{n_{k+1}}(\omega), u_{n_k}(\omega))dP(\omega)
< \int_{\Omega} 3^{-k}dP(\omega)
= 3^{-k} \int_{\Omega} dP(\omega)
= 3^{-k}. \tag{2.3.11}
\]

On the other hand we have

\[
\int_{A_k} d_X(u_{n_{k+1}}(\omega), u_{n_k}(\omega))dP(\omega) > \int_{A_k} 2^{-k}dP(\omega)
> 2^{-k} \int_{A_k} dP(\omega)
= 2^{-k} P(A_k). \tag{2.3.12}
\]

So combining (2.3.11) and (2.3.12) we obtain the strict inequality

\[
P(A_k) < \left(\frac{2}{3}\right)^k \tag{2.3.13}
\]

Now let \( A := \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k \), then

\[
P\left( \bigcup_{k=m}^{\infty} A_k \right) \leq \sum_{k=m}^{\infty} P(A_k)
< \sum_{k=m}^{\infty} \left(\frac{2}{3}\right)^k
= \left(\frac{2}{3}\right)^m \frac{1 - \frac{2}{3}^m}{1 - \frac{2}{3}}
= 3 \left(\frac{2}{3}\right)^m, \tag{2.3.14}
\]
2.3. Collage method: ODE with noise

and so \( P(A) < 3 \left( \frac{2}{3} \right)^m \) for all \( m \), which implies \( P(A) = 0 \). This means that for all \( \omega \not\in \Omega \setminus A \) there exists an \( m_0(\omega) \) such that for all \( m \geq m_0(\omega) \) we have that \( \omega \not\in A_m \) and thus \( d_X(u_{n,m+1}(\omega), u_{n,m}(\omega)) < 2^{-m} \). This implies that \( u_{n,m}(\omega) \) is Cauchy for all \( \omega \not\in \Omega \setminus A \). Because the space \( X \) is complete, every Cauchy sequence in \( X \) converges in \( X \), so \( u_{n,m}(\omega) \to u(\omega) \).

Now we show that \( u_n \to u \) in \( Y \).

\[
d_Y(u_n, u) = \int \Omega d_X(u_{n,k}(\omega), u(\omega))dP(\omega)
\]

\[
= \int \Omega \lim_{i \to \infty} d_X(u_{n,k}(\omega), u_{n,i}(\omega))dP(\omega)
\]

\[
\leq \lim \inf_{i \to \infty} \int \Omega d_X(u_{n,k}(\omega), u_{n,i}(\omega))dP(\omega)
\]

\[
= \lim \inf_{i \to \infty} d_Y(u_{n,k}, u_{n,i}) \leq 3^{-k},
\]

for all \( k > 0 \). So we have that

\[
\lim_{k \to \infty} d_Y(u_{n,k}, u) = 0.
\]

We write

\[
d_Y(u_n, u) \leq d_Y(u_n, u_{n,k}) + d_Y(u_{n,k}, u).
\]

The first term on the right-hand side of (2.3.17) goes to zero by (2.3.9) while the second term of (2.3.17) goes to zero by (2.3.16) as \( k \to \infty \). So we have that \( u_n \to u \) in \( Y \). This completes the proof.

Similar to the deterministic case, we have the following version of the collage theorem.

**Theorem 2.3.7.** Let \( (Y, d_Y) \) be a complete, separable, metric space and \( \tilde{T} : \Omega \times X \to X \) a random \( c(\omega) \)-contraction mapping with \( c(\omega) < c \in [0, 1) \) a.e. \( \omega \in \Omega \). Then for any \( u \in X \)

\[
\frac{1}{1+c} d_Y(u, \tilde{T} u) \leq d_Y(u, \tilbar{u}) \leq \frac{1}{1-c} d_Y(u, \tilbar{T} u),
\]

where \( \tilbar{u} \) is the fixed point of the random contraction \( \tilbar{T} \).

**2.3.2 Random integral operator**

Consider the following ordinary random differential equation (where \( \omega \in \Omega \) indicates the randomness)

\[
\begin{cases}
\frac{d}{dt}(\omega, t) = f(t, \omega, u(t, \omega)), \\
u(0, \omega) = u_0(\omega).
\end{cases}
\]

Here, both the vector field \( f \) and the initial condition \( u_0 \) are random variables defined on an appropriate probability space \( (\Omega, \mathcal{F}, P) \). The corresponding integral operator form is given by

\[
(Tu)(t, \omega) = u_0(\omega) + \int_0^t f(s, \omega, u(s, \omega))ds.
\]

Solutions to (2.3.19) are fixed points of (2.3.20) and vice versa.

In practice, \( \Omega \) usually consists of some realizations of data (say \( \omega_j \) for \( j = 1, 2, ..., N \)). This means
2.3. Collage method: ODE with noise

that the metric \( d_Y(u_1, u_2) \) is nothing more than taking the average over the metrics
\( d_X(u_1(\omega_j), u_2(\omega_j)) \) for \( j = 1, 2, \ldots, N \), where \( d_Y, d_X \) are the metrics defined as before. Furthermore, instead of finding the values of the parameters, it is better to find the mean value and the variance of the parameters we are interested in since we are dealing with random variables here.

Example 2.3.8
Suppose \( f(t, \omega, u) \) in (2.3.19) is a polynomial in \( t \) and \( u \) with random variables \( u_0, a_i \) on the same probability space \( (\Omega, F, P) \)

\[
f(t, \omega, u) = a_0(\omega) + a_1(\omega)t + a_2(\omega)u + a_3(\omega)t^2 + a_4(\omega)u^2 + a_5(\omega)tu + \ldots
\]  

(2.3.21)

Now assume data is given in the form of realizations \( u(t, \omega_j) \) for \( j = 1, 2, \ldots, N \). The goal is to find the means and variances of \( u_0, a_i \). We have a list of target functions \( u(t, \omega_j) \). Each realization is the solution of a fixed point equation of the form

\[
u(t, \omega_j) = u_0(\omega_j) + \int_0^t f(s, \omega_j, u(s, \omega_j)) ds.
\]  

(2.3.22)

For each realization \( j \) we can use the target function \( u(t, \omega_j) \) in order to find realizations for \( u_0(\omega_j), a_i(\omega_j) \) via the collage method for polynomial deterministic integral equations. We then can construct the approximations for mean and variance

\[
\mu \approx \mu_N = \frac{1}{N} \sum_{j=1}^{N} u_0(\omega_j), \quad \nu_i \approx (\nu_i)_N = \frac{1}{N} \sum_{j=1}^{N} a_i(\omega_j),
\]  

\[
\sigma^2 \approx \sigma^2_N = \frac{1}{N} \sum_{j=1}^{N} (u_0(\omega_j) - \mu_N)^2, \quad \sigma_i^2 \approx (\sigma_i^2)_N = \frac{1}{N} \sum_{j=1}^{N} (a_i(\omega_j) - (\nu_i)_N)^2.
\]  

(2.3.23)

Example 2.3.9
Let us consider the following random differential equation:

\[
\begin{aligned}
\frac{du}{dt}(\omega, t) &= A_0 + A_1 u(\omega, t) + A_2 u(\omega, t)^2, \\
\ \ \ \ u(0, \omega) &= u_0(\omega),
\end{aligned}
\]  

(2.3.24)

where \( u_0, A_i, i = 0, 1, 2 \) are real valued random variables on the same probability space. The parameters are normally distributed with the following mean and variance:

\[
\begin{aligned}
u_0 &\sim \mathcal{N}(0.5, 0.01), \\
u_0 &\sim \mathcal{N}(0.7, 0.01), \\
u_0 &\sim \mathcal{N}(0.6, 0.01), \\
u_0 &\sim \mathcal{N}(-0.4, 0.01).
\end{aligned}
\]  

(2.3.25)

We make \( M \) realizations of the ODE by generating the parameters from the given distributions for \( M = 10, 100, 500, 1000 \). For each realization, the ODE is solved numerically. The solution is then sampled at 11 uniformly distributed points and a fifth degree polynomial \( u(t, \omega_j) \) is fitted through these points. Then for each fitted polynomial, the collage method as explained in previous sections is used to approximate the mean and variance of the parameters. The results are listed in Table 2.1. We see that the mean and variance of our parameters converge to the values given in (2.3.25). If we do the same, but with larger variances, say

\[
\begin{aligned}
u_0 &\sim \mathcal{N}(0.5, 0.04), \\
u_0 &\sim \mathcal{N}(0.7, 0.04), \\
u_0 &\sim \mathcal{N}(0.6, 0.04), \\
u_0 &\sim \mathcal{N}(-0.4, 0.04).
\end{aligned}
\]  

(2.3.26)
2.4. Collage method: SDE

Table 2.1: Results collage method for an ODE with noise (small variance).

<table>
<thead>
<tr>
<th>M</th>
<th>$u_0$</th>
<th>$A_0$</th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$\mathcal{N}(0.4879, 0.01748)$</td>
<td>$\mathcal{N}(0.6381, 0.01318)$</td>
<td>$\mathcal{N}(0.6034, 0.01045)$</td>
<td>$\mathcal{N}(-0.4252, 0.01323)$</td>
</tr>
<tr>
<td>100</td>
<td>$\mathcal{N}(0.5047, 0.01266)$</td>
<td>$\mathcal{N}(0.7055, 0.01155)$</td>
<td>$\mathcal{N}(0.5855, 0.00841)$</td>
<td>$\mathcal{N}(-0.4055, 0.00952)$</td>
</tr>
<tr>
<td>500</td>
<td>$\mathcal{N}(0.5041, 0.00971)$</td>
<td>$\mathcal{N}(0.7034, 0.01097)$</td>
<td>$\mathcal{N}(0.5940, 0.01068)$</td>
<td>$\mathcal{N}(-0.3929, 0.01043)$</td>
</tr>
<tr>
<td>1000</td>
<td>$\mathcal{N}(0.5009, 0.01022)$</td>
<td>$\mathcal{N}(0.6987, 0.00973)$</td>
<td>$\mathcal{N}(0.5965, 0.01013)$</td>
<td>$\mathcal{N}(-0.3991, 0.01038)$</td>
</tr>
</tbody>
</table>

Table 2.2: Results collage method for an ODE with noise (larger variance).

<table>
<thead>
<tr>
<th>M</th>
<th>$u_0$</th>
<th>$A_0$</th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$\mathcal{N}(0.4657, 0.06317)$</td>
<td>$\mathcal{N}(0.6579, 0.05599)$</td>
<td>$\mathcal{N}(0.6768, 0.03416)$</td>
<td>$\mathcal{N}(-0.4126, 0.04428)$</td>
</tr>
<tr>
<td>100</td>
<td>$\mathcal{N}(0.5198, 0.03670)$</td>
<td>$\mathcal{N}(0.7123, 0.03601)$</td>
<td>$\mathcal{N}(0.5847, 0.04151)$</td>
<td>$\mathcal{N}(-0.3819, 0.03996)$</td>
</tr>
<tr>
<td>500</td>
<td>$\mathcal{N}(0.5110, 0.03919)$</td>
<td>$\mathcal{N}(0.6936, 0.03931)$</td>
<td>$\mathcal{N}(0.5924, 0.03831)$</td>
<td>$\mathcal{N}(-0.3960, 0.04057)$</td>
</tr>
<tr>
<td>1000</td>
<td>$\mathcal{N}(0.4920, 0.03984)$</td>
<td>$\mathcal{N}(0.6939, 0.03764)$</td>
<td>$\mathcal{N}(0.6038, 0.04067)$</td>
<td>$\mathcal{N}(-0.4089, 0.04103)$</td>
</tr>
</tbody>
</table>

we obtain the results in Table 2.2. We see that the results in Table 2.2 converge to the values in (2.3.26).

2.4 Collage method: SDE

2.4.1 Stochastic differential equations

Before we present the collage method for the case of a stochastic differential equation, we first introduce some more notation.

**Definition 2.4.1.** Let $(\Omega, \mathcal{F}, P)$ be a probability space (see Definition 3.3). A Brownian motion $\{B_t\}_{t \geq 0}$, with values in $\mathbb{R}$, is a continuous-time stochastic process which satisfies the following properties:

- $B_0 = 0$;
- $B_t$ is almost surely continuous: There exists $\Omega_1 \subset \Omega$, such that $P(\Omega_1) = 1$, and for every $\omega \in \Omega_1$, $B_t(\omega)$ is a continuous function of $t$;
- $B_t$ has independent increments: For all $0 \leq s_1 < t_1 \leq s_2 < t_2 < \infty$ we have that $B_{t_1} - B_{t_2}$ and $B_{t_2} - B_{t_2}$ are independent;
- $B_t - B_s \sim \mathcal{N}(0, t - s)$ for $0 \leq s \leq t$ (the increments are normally distributed with mean 0 and variance $t - s$).

**Definition 2.4.2.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X_1 : \Omega \to \Omega'$, then the law of $X_1$ is a measure $\mu_1$, defined by

$$\mu_1 := P(X_1^{-1}).$$

In other words, $\mu_1$ in (2.4.1) is called the pushforward measure or image measure of $P$. If $X_1$ is a continuous process, this is also known as the probability distribution function of the variable $X_1$.

Now let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{B_t\}_{t \geq 0}$ be a Brownian motion, $X_0$ be a measurable
random variable, \( g \) a bounded function \( g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) Lipschitz in both arguments with respectively constants \( K_1, K_2 \). We consider the following differential equation:

\[
\begin{align*}
    dX_t &= \int_{\mathbb{R}} g(X_t, y) d\mu_t(y) dt + dB_t, \quad \text{for } t \in [0, T] \\
    X_{t=0} &= X_0.
\end{align*}
\]

(2.4.2)

Let \( T > 0 \). Let us consider the complete metric space \( (C([0, T]), d_{\infty}) \), where \( C([0, T]) \) is the space of continuous functions on the interval \([0, T]\) and \( d_{\infty}(x, y) := \sup_{t \in [0, T]} |x(t) - y(t)| \). Furthermore, let \( M(C([0, T])) \) be the space of probability measures on \( C([0, T]) \). We can look at (2.4.2) in two ways. If we fix a time \( t \in [0, T] \), then \( X_t(\cdot) \) becomes a random variable in \( \mathbb{R} \).

If we fix \( \omega \in \Omega \), that is we take one realization of the path \( X_t \) which is notated by \( X_t(\omega) \), then \( X_t(\omega) \) becomes a random variable in the space \( C([0, T]) \). This is equivalent with saying that for every process \( \{X_t\} \), one can define a random variable from \( \Omega \) to \( C([0, T]) \). This induces a probability measure \( \tilde{\mu} \) on \( M(C([0, T])) \), via

\[
    \tilde{\mu}(\cdot) = P(X_t^{-1}(\cdot)).
\]

(2.4.3)

Define a mapping \( \Phi : M(C([0, T])) \to M(C([0, T])) \) which associates to \( \mu \in M(C([0, T])) \) the law of the process of

\[
    X_t = X_0 + B_t + \int_0^t \int_{C([0, T])} g(X_s, w) d\mu_s(w) ds.
\]

(2.4.4)

In other words, \( (\Phi(\mu))(\cdot) := P(X_t^{-1}(\cdot)) \). This is the push-forward measure of \( \mu \). If \( X_t \) is a solution of (2.4.2), then its law on \( C([0, T]) \) is a fixed point of \( \Phi \). This mapping \( \Phi \) will become our contraction mapping as we will see in Theorem 4.11. Let us introduce firstly some more mathematical background.

**Definition 2.4.3.** Let \( (X, d) \) be a Polish space (complete, separable metric space). Let \( M(X) \) be the set of probability measures on \( X \). Let \( \mu_1, \mu_2 \in M(X) \) be two probability measures. The joint representation (or “coupling” or “transference plan”) of \( \mu_1, \mu_2 \) is a measure \( P \) on \( X \times X \) such that for all bounded, measurable functions \( f : X \to \mathbb{R} \) we have that

\[
\begin{align*}
    \int_{X \times X} f(x) dP(x, y) &= \int_X f(x) d\mu_1(x), \\
    \int_{X \times X} f(y) dP(x, y) &= \int_X f(y) d\mu_2(y).
\end{align*}
\]

(2.4.5)

The space \( C(\mu_1, \mu_2) \) is the set of all joint representations of \( \mu_1, \mu_2 \). The probability measures \( P \) that satisfy (2.4.5) are said to have marginals \( \mu_1 \) and \( \mu_2 \).

**Definition 2.4.4.** (See [11]) The \( k \)-Wasserstein distance between probability measures \( \mu_1, \mu_2 \) is defined as

\[
    W_k(\mu_1, \mu_2) := \left( \inf_{P \in C(\mu_1, \mu_2)} \int_{X \times X} (d(x, y))^k dP(x, y) \right)^{\frac{1}{k}},
\]

(2.4.6)

where \( P \) is a coupling between \( \mu_1, \mu_2 \). This defines a distance in the space of probability measures. From now on, \( k = 1 \) is used, so \( W(\mu_1, \mu_2) := W_1(\mu_1, \mu_2) \). This is also called the Kantorovich-Rubinstein metric. In practice, it is hard to compute this distance. However, there are examples for which it is fairly easy to compute the Wasserstein distance, see e.g. [32], [26] and [24].

Before we prove that \( W(\mu_1, \mu_2) \) is a metric on \( M(X) \), we need a special lemma, called the gluing
Let $\mu_1, \mu_2, \mu_3$ be three probability measures, supported in the Polish spaces $X_1, X_2, X_3$ respectively. Let $P_{1,2} \in C(\mu_1, \mu_2)$ and $P_{2,3} \in C(\mu_2, \mu_3)$ be two couplings. Then there exists a probability measure $P \in M(X_1 \times X_2 \times X_3)$ with marginals $P_{1,2} \in M(X_1 \times X_2)$ and $P_{2,3} \in M(X_2 \times X_3)$.

**Proof.** (See [30])

Here we need the concept of *disintegration of measure*. When $X$ and $Y$ are Polish spaces, the disintegration of measure theorem allows one to write any probability measure on $X \times Y$ as an average of probability measures on $\{x\} \times Y$, for $x \in X$. In particular, if $\pi$ is a probability measure on $X \times Y$, with marginal $\mu$ on $X$, then there exists a measurable mapping $x \mapsto \pi_x$, from $X$ into $M(Y)$ such that

$$
\pi = \int_X (\delta_x \otimes \pi_x) d\mu(x),
$$

(2.4.7)

where $\otimes$ denotes the product between the two measures. This notation can be explained as follows. Let $A \in X, B \in Y$, then $A \times B \in X \times Y$. Since $\pi$ is a probability measure on $X \times Y$, we can use (2.4.7) and write

$$
\pi(A \times B) = \int_A \pi_x(B) d\mu(x).
$$

(2.4.8)

The Dirac measure $\delta_x$ makes sure the integral is only taken over the elements $x$ that are in $A$. This means that for all $u \in C_b(X \times Y)$ (that means that for all $u$ bounded and continuous on $X_1 \times X_2$) we have that

$$
\int_{X \times Y} u(x, y) d\pi(x, y) = \int_X \left( \int_Y u(x, y) d\pi_x(y) \right) d\mu(x).
$$

(2.4.9)

This shows us that if the probability measure $\pi$ on $X \times Y$ exists and has a marginal $\mu$ on $X$ there exists a marginal on $Y$ given by $\pi_x$.

Now consider $P_{1,2}$ and $P_{2,3}$ as stated in the lemma and disintegrate both measures with respect to their common marginal $\mu_2$. This means there exist measurable mappings $P_{1,2,2}$ and $P_{2,3,2}$, from $X_2$ into $M(X_1), M(X_3)$ respectively such that

$$
P_{1,2} = \int_{X_2} (P_{1,2,2} \otimes \delta_{x_2}) d\mu_2(x_2),
$$

$$
P_{2,3} = \int_{X_2} (P_{2,3,2} \otimes \delta_{x_2}) d\mu_2(x_2).
$$

(2.4.10)

This means that for all $u \in C_b(X_1 \times X_2)$ and for all $v \in C_b(X_2 \times X_3)$ we have that

$$
\int_{X_1 \times X_2} u(x_1, x_2) dP_{1,2}(x_1, x_2) = \int_{X_2} \left( \int_{X_1} u(x_1, x_2) dP_{1,2,2}(x_1) \right) d\mu_2(x_2),
$$

$$
\int_{X_2 \times X_3} v(x_2, x_3) dP_{2,3}(x_2, x_3) = \int_{X_2} \left( \int_{X_3} v(x_2, x_3) dP_{2,3,2}(x_3) \right) d\mu_2(x_2).
$$

(2.4.11)

Then $P \in M(X_1 \times X_2 \times X_3)$ can be constructed as

$$
P = \int_{X_2} (P_{1,2,2} \otimes \delta_{x_2} \otimes P_{2,3,2}) d\mu_2(x_2).
$$

(2.4.12)
2.4. Collage method: SDE

And this can then be interpreted as for all \( w \in C_b(X_1 \times X_2 \times X_3) \) we have that

\[
\int_{X_1 \times X_2 \times X_3} w(x_1, x_2, x_3) dP(x_1, x_2, x_3) = \int_{X_2} \int_{X_1} \int_{X_3} w(x_1, x_2, x_3) dP_{2,3;2}(x_3) dP_{1,2;2}(x_1) d\mu_2(x_2).
\]  

(2.4.13)

**Theorem 2.4.6.** \( W(\mu_1, \mu_2) \) is a metric on \( M(X) \).

**Proof.** This proof is in the line of [30]:

i. \( W(\mu_1, \mu_2) \geq 0 \):

\[
d(x, y) \geq 0all x, y \in X \text{ because } d \text{ is a metric itself. } P(x, y) \geq 0 \text{ because it is a measure.}
\]

So therefore we also have

\[
\inf_{P \in C(\mu_1, \mu_2)} \int_{X \times X} d(x, y) dP(x, y) \geq 0.
\]  

(2.4.14)

ii. \( W(\mu_1, \mu_2) = 0 \) if and only if \( \mu_1 = \mu_2 \):

Let \( \mu_1, \mu_2 \in M(X) \) such that \( W(\mu_1, \mu_2) = 0 \). Let \( P \in C(\mu_1, \mu_2) \). It is clear that \( P \) is supported (non-zero) on the diagonal \( y = x \), because \( d(x, y) \) has to be zero. Then \( \forall f \in C_b(X) \) (a test function \( f \) bounded and continuous on \( X \)), such that

\[
\int_X f d\mu_1 = \int_X f(x) d\mu_1(x)
\]

\[
= \int_{X \times X} f(x) dP(x, y)
\]

\[
= \int_{X \times X} f(y) dP(x, y)
\]

(2.4.15)

Now let \( E \in X \) and \( f_n : X \rightarrow \mathbb{R} \) such that \( f_n \rightarrow \chi_E \) for \( n \rightarrow \infty \). Then \( \int_E d\mu_1 = \int_E d\mu_2 \) and hence \( \mu_1 = \mu_2 \).

Now let \( \mu_1 = \mu_2 \). Then by definition (2.4.5) we have that

\[
\int_{X \times X} f(x) dP(x, y) = \int_X f(x) d\mu_1(x)
\]

\[
= \int_X f(x) d\mu_1(y)
\]

\[
\int_{X \times X} f(y) dP(x, y) = \int_X f(y) d\mu_2(y) = \int_X f(y) d\mu_1(y) = \int_X f(x) d\mu_1(x)
\]

(2.4.16)

We can subtract both equations and obtain

\[
\int_{X \times X} f(x) - f(y) dP(x, y) = \int_X f(x) - f(x) d\mu_1(x) = 0
\]  

(2.4.17)
2.4. Collage method: SDE

This can only be true if the measure $P$ is identically zero or all the mass is concentrated on the diagonal $x = y$, since this holds for all test functions $f$. Hence, $W(\mu_1, \mu_1) = 0$.

iii. $W(\mu_1, \mu_2) = W(\mu_2, \mu_1)$: Since we have that the joint representation of $\mu_1$ and $\mu_2$ is symmetric (see (2.4.5)) and $d(x, y)$ is symmetric because it is a metric, we also have that the metric $W$ is symmetric.

iv. Triangle inequality: $W(\mu_1, \mu_3) \leq W(\mu_1, \mu_2) + W(\mu_2, \mu_3)$:

For this we need the gluing lemma, which states that if $\mu_1, \mu_2, \mu_3 \in M(X)$ and if $P_{1,2} \in \mathcal{C}(\mu_1, \mu_2)$ and $P_{2,3} \in \mathcal{C}(\mu_2, \mu_3)$ are joint representations, then there exists a $P \in M(X \times X \times X)$ which glues together marginals $P_{1,2}$ and $P_{2,3}$. Then the other marginal of $P$ is the joint representation $P_{1,3}$. We obtain:

$$W(\mu_1, \mu_3) = \int_{X \times X} d(x, z) dP_{1,3}(x, z)$$

$$= \int_{X \times X \times X} d(x, z) dP(x, y, z)$$

$$\leq \int_{X \times X \times X} (d(x, y) + d(x, z)) dP(x, y, z)$$

$$= \int_{X \times X \times X} d(x, y) dP(x, y, z) + \int_{X \times X \times X} d(x, z) dP(x, y, z)$$

$$= \int_{X \times X} d(x, y) dP_{1,2}(x, y) + \int_{X \times X \times X} d(y, z) dP_{2,3}(y, z)$$

$$= W(\mu_1, \mu_2) + W(\mu_2, \mu_3)$$

So $W(\mu_1, \mu_2)$ is a metric.

Because we are dealing with stochastic differential equations, the space $X$ is equal to $C([0, T])$. Then $x, y \in X$ means that $x, y$ are continuous functions on the interval $[0, T]$; $x(t), y(t) \in C([0, T])$. This also means that the metric $d_{\infty}$ on $C([0, T])$ is defined as

$$d_{\infty}(x, y) = d_T(x(\cdot), y(\cdot)) := \sup_{s \leq T} |x(s) - y(s)|.$$  \hspace{1cm} (2.4.19)

The subscript $T$ is added to the metric to indicate that the supremum has to be taken over all $s \leq T$. Therefore we also introduce this notation for the Wasserstein metric $W_t$, which will now be as follows

$$W_T(\mu_1, \mu_2) := \inf_{P \in \mathcal{C}(\mu_1, \mu_2)} \int_{X \times X} d_T(x, y) dP(x, y).$$  \hspace{1cm} (2.4.20)

Before we prove the completeness of the space of probability measures together with the Wasserstein metric, we need the following theorem by Prokhorov.

**Theorem 2.4.7. [Prokhorov’s theorem]** Let $(X, d)$ be a complete separable metric space. Let $\mu_n$ be a sequence in $M(X)$ which is tight. That is, a set of probability measures $\Gamma$ on $X$ is tight if and only if for all $\varepsilon > 0$ there exists a compact subset $K$ of $X$ such that $\mu(K) \geq 1 - \varepsilon$ for all $\mu \in \Gamma$.

\footnote{Next, we will see that we also consider the distance $d_T(x_t, y_t)$, where $x_t, y_t$ are processes, hence the subscript $t$.}
2.4. Collage method: SDE

Then there exists a subsequence \( \mu_{n_k} \) and a measure \( \mu \in M(X) \) such that \( \mu_{n_k} \) converges weakly to \( \mu \). The proof is given in [8, Thm. 11.5.4].

**Theorem 2.4.8.** \((M(X), W_T)\) is complete.

**Proof.** This proof is in the spirit of [5] and [4]:

Let \( \mu_n \) be a Cauchy sequence, that is

\[
\forall \varepsilon > 0 \ \exists n_0 > 0 \ \forall n, m \geq n_0 : W_T(\mu_n, \mu_m) < \frac{\varepsilon}{2}.
\]  

(2.4.21)

Because \( \mu_n \) is Cauchy, the set of \( \mu_n \)'s is tight (see [4]). Now Prokhorov’s theorem states that since \( \mu_n \) is tight, there exists a measure \( \mu \in M(X) \) and a subsequence \( \mu_{n_k} \) such that \( \mu_{n_k} \) converges weakly to \( \mu \) as \( k \to \infty \). That is for all \( f \in C_b(X) \)

\[
\int_X f(\cdot) d\mu_{n_k} \xrightarrow{k \to \infty} \int_X f(\cdot) d\mu.
\]  

(2.4.22)

Now let \( P'_{n_k, m_k} \in C(\mu_{n_k}, \mu_{m_k}) \) be the optimal coupling, that is

\[
\int_{X \times X} d_T(x_1, x_2) dP'_{n_k, m_k} = \inf_{P_{n_k, m_k} \in C(\mu_{n_k}, \mu_{m_k})} \int_{X \times X} d_T(x_1, x_2) dP_{n_k, m_k} = W_T(\mu_{n_k}, \mu_{m_k})
\]  

(2.4.23)

The sequence \( \mu_{n_k} \) is uniformly tight, which means that the sequence \( P'_{n_k, m_k} \) is also uniformly tight for given \( m_k \). Then again by Prokhorov’s theorem, there exists a subsequence \( (P'_{n_{k_i}, m_k})_l \) of \( P'_{n_k, m_k} \) converging to \( P'_{m_k} \) on \( X \times X \) in the weak topology. Then by semi-continuity we have

\[
\int_{X \times X} d_T(x_1, x_2) dP'_{m_k} \leq \liminf_{l \to \infty} \int_{X \times X} d_T(x_1, x_2) dP'_{n_{k_i}, m_k} = \liminf_{l \to \infty} W_T(\mu_{n_{k_i}}, \mu_{m_k}).
\]  

(2.4.24)

\( P'_{n_{k_i}, m_k} \) has marginals \( \mu_{n_{k_i}}, \mu_{m_k} \) and in the limit, \( P'_{m_k} \) has marginals \( \mu, \mu_{m_k} \). So

\[
W_T(\mu, \mu_{m_k}) \leq \int_{X \times X} d_T(x_1, x_2) dP'_{m_k},
\]  

(2.4.25)

for all \( m_k \). Since \( \mu_n \) is a Cauchy sequence, also \( \mu_{n_k} \) is a Cauchy sequence. We have that for \( k_1, k_2 \geq n_0 \):

\[
W_T(\mu_{n_{k_1}}, \mu_{n_{k_2}}) \leq W_T(\mu_{n_1}, \mu_{n_2}) < \frac{\varepsilon}{2}.
\]  

(2.4.26)

Then by combining (2.4.24), (2.4.25) and (2.4.26) we obtain the inequality

\[
W_T(\mu, \mu_{m_k}) < \frac{\varepsilon}{2},
\]  

(2.4.27)

for \( m_k \) sufficiently large. Then for sufficiently large \( n, n_k \) by the triangle inequality we have that

\[
W_T(\mu, \mu_n) \leq \underbrace{W_T(\mu, \mu_{n_k})}_{< \frac{\varepsilon}{2} \text{ by (2.4.27)}} + W_T(\mu_{n_k}, \mu_n) \leq \frac{\varepsilon}{2} < \varepsilon.
\]  

(2.4.28)

Therefore, \( \mu_n \) converges to \( \mu \in M(X) \) in the Wasserstein metric.

\[ \square \]

We now present the theorem that states the existence and uniqueness of solutions to the stochastic differential equation.
2.4. Collage method: SDE

**Theorem 2.4.9. [Existence and Uniqueness]** Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{B_t\}_{t \geq 0}$ be a classical Brownian motion, $X_0$ be a measurable random variable, $g$ a bounded, Lipschitz continuous function $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Note that $g$ is Lipschitz continuous in both arguments with respectively constant $K_1, K_2$. We consider the following differential equation:

\[
\begin{cases}
    dX_t = \int_{\mathbb{R}} g(X_t, y) d\mu_t(y) dt + dB_t, & \text{for } t \in [0, T] \\
    X_{t=0} = X_0.
\end{cases}
\]

Then there exists a unique solution to (2.4.29)

**Proof.** Argueing as in [29] and [6]:

We have seen that if $X_t$ for $t \leq T$ is a solution to (2.4.29), then its law on $C([0, T])$ is a fixed point of $\Phi$, given in (2.4.4). Conversely, if $\mu$ is such a fixed point, (2.4.4) defines a solution up to time $T$. Our problem is now translated into a fixed point problem for $\Phi$. First we prove the following contraction lemma.

**Lemma 2.4.10. [Contraction lemma]**

For $t \leq T$,

\[
W_t(\Phi(\mu_1), \Phi(\mu_2)) \leq c_T \int_0^t W_s(\mu_1, \mu_2) ds, \quad \mu_1, \mu_2 \in M(C([0, T])),
\]

(2.4.30)

where $c_T$ is a constant and $W_s(\mu_1, \mu_2)$ is the distance between the images of $\mu_1, \mu_2$ on $C([0, s])$.

**Proof.** As in [6],

For $t \leq T$, let $\Phi(\mu_1), \Phi(\mu_2)$ be two mappings from $M(C([0, T]))$ to itself. Let $\mu_1, \mu_2 \in M(C([0, T]))$ be two probability measures with corresponding processes $\{X^k_t\}_{t \in [0, T]}, k = 1, 2$ given by

\[
X^1_t = X_0 + B_t + \int_0^t \int g(X^1_s, w_s) d\mu_1(w) ds,
\]

\[
X^2_t = X_0 + B_t + \int_0^t \int g(X^2_s, w_s) d\mu_2(w) ds,
\]

(2.4.31)

where $X_0, B_t$ are the same for both processes. The domain of the second integral $X = C([0, T])$
is omitted. We then can compare the two processes \( \{X^1_t\}_{t \in [0,T]} \) and \( \{X^2_t\}_{t \in [0,T]} \):

\[
d_s(X^1(\cdot), X^2(\cdot)) \leq \int_0^t \left| \int g(X^1_{s}, w_s) d\mu_1(w) - \int g(X^2_{s}, w_s) d\mu_2(w) \right| ds
\]

(applying the triangle inequality yields)

\[
\leq \int_0^t \left| \int g(X^1_{s}, w_s) d\mu_1(w) - \int g(X^1_{s}, w_s) d\mu_2(w) \right| ds + \int_0^t \left| \int g(X^1_{s}, w_s) - g(X^2_{s}, w_s) d\mu_2(w) \right| ds
\]

(by Definition 4.3, for all joint representations \( P \in C(\mu_1, \mu_2) \) we have that)

\[
= \int_0^t \left| \int g(X^1_{s}, w_s) dP(w^1, w^2) - \int g(X^1_{s}, w_s) dP(w^1, w^2) \right| ds + \int_0^t \left| \int g(X^1_{s}, w_s) - g(X^1_{s}, w_s) d\mu_2(w) \right| ds
\]

(put the absolute signs inside the inner integral)

\[
\leq \int_0^t \left| \int g(X^1_{s}, w_s) - g(X^1_{s}, w_s) \right| dP(w^1, w^2) ds + \int_0^t \left| \int g(X^1_{s}, w_s) - g(X^1_{s}, w_s) \right| d\mu_2(w) ds
\]

\((g\text{ is Lipschitz continuous})\)

\[
\leq \int_0^t K_2 |w^1_s - w^2_s| dP(w^1, w^2) ds + \int_0^t K_1 |X^1_s - X^2_s| d\mu_2(w) ds
\]

(constants in front of the integral)

\[
\leq K_2 \int_0^t |w^1_s - w^2_s| dP(w^1, w^2) ds + K_1 \int_0^t |X^1_s - X^2_s| ds \int d\mu_2(w)
\]

\((\mu_2\text{ is a probability measure so } \int d\mu_2(w) = 1 \text{ yields})\)

\[
\leq K_2 \int_0^t |w^1_s - w^2_s| dP(w^1, w^2) ds + K_1 \int_0^t |X^1_s - X^2_s| ds
\]

(because this holds for all \( P \in C(\mu_1, \mu_2) \), it also holds in the limit for the infimum)

\[
\leq K_2 \int_0^t W_s(\mu_1, \mu_2) ds + K_1 \int_0^t |X^1_s - X^2_s| ds
\]

\((\text{because } |X^1_s - X^2_s| \leq \sup_{u \leq s} |X^1_u - X^2_u| = d_s(X^1(\cdot), X^2(\cdot)) \text{ we have that})\)

\[
\leq K_2 \int_0^t W_s(\mu_1, \mu_2) ds + K_1 \int_0^t d_s(X^1(\cdot), X^2(\cdot)) ds
\]

(2.4.32)
2.4. Collage method: SDE

So we end up with the inequality
\[ d_t(X^1(\cdot), X^2(\cdot)) \leq K_2 \int_0^t W_s(\mu_1, \mu_2) ds + \int_0^t K_1 d_s(X^1(\cdot), X^2(\cdot)) ds. \] (2.4.33)

Now we present the following lemma:

**Lemma 2.4.11. [Gronwall’s lemma]** Let \( a(t), b(t), \phi(t) \) be continuous functions. If \( b(t) \) is non-negative, \( a(t) \) non-decreasing and

\[ \phi(t) \leq a(t) + \int_0^t b(s)\phi(s) ds, \] (2.4.34)

then the following inequality holds

\[ \phi(t) \leq a(t) \exp \left( \int_0^t b(s) ds \right). \] (2.4.35)

Let us define

\[ a(t) := K_2 \int_0^t W_s(\mu_1, \mu_2) ds, \]
\[ b(s) := K_1, \] (2.4.36)
\[ \phi(t) := d_t(X^1(\cdot), X^2(\cdot)). \]

Then, by (2.4.35), we obtain

\[
\begin{align*}
    d_t(X^1(\cdot), X^2(\cdot)) & \leq K_2 \int_0^t W_s(\mu_1, \mu_2) ds \cdot \exp \left( \int_0^t K_1 ds \right), \\
    & \leq K_2 \int_0^t W_s(\mu_1, \mu_2) ds \cdot \exp \left( \int_0^T K_1 ds \right), \\
    & \leq K_2 \exp (K_1 T) \int_0^t W_s(\mu_1, \mu_2) ds, \\
    & \leq c_T \int_0^t W_s(\mu_1, \mu_2) ds,
\end{align*}
\] (2.4.37)

where \( c_T := K_2 \exp (K_1 T) \). Given a measure \( \mu \), the mapping \( \Phi(\cdot) \) associates which each \( \mu \) the law of the corresponding process \( \{ X_t \}_{t \in [0, T]} \). Then the Wasserstein distance between \( \Phi(\mu_1) \) and \( \Phi(\mu_2) \) is given by:

\[ W_t(\Phi(\mu_1), \Phi(\mu_2)) = \inf_{\tilde{P} \in C(\Phi(\mu_1), \Phi(\mu_2))} \int_{X \times X} d_t(X^1(\cdot), X^2(\cdot)) d\tilde{P}(X^1(\cdot), X^2(\cdot)). \] (2.4.38)

Now let \( P^* \in C(\Phi(\mu_1), \Phi(\mu_2)) \), not necessarily the one which minimizes the distance between
2.4. Collage method: SDE

\(\mu_1, \mu_2\). Then the Wasserstein distance can be bounded from above by

\[
W_t(\Phi(\mu_1), \Phi(\mu_2)) \leq \int_{X \times X} d_t(X^1(\cdot), X^2(\cdot)) dP^*(X^1(\cdot), X^2(\cdot))
\]

\[
\leq \int_{X \times X} \left( c_T \int_0^t W_s(\mu_1, \mu_2) ds \right) dP^*(X^1(\cdot), X^2(\cdot))
\]

\[
= c_T \int_0^t W_s(\mu_1, \mu_2) ds \left( \int_{X \times X} dP^*(X^1(\cdot), X^2(\cdot)) \right)
\]

\[
= c_T \int_0^t W_s(\mu_1, \mu_2) ds,
\]

because \(P^*\) is a probability measure on \(X \times X\) and therefore \(\int_{X \times X} dP^*(X^1(\cdot), X^2(\cdot)) = 1\). So we have proven that

\[
W_t(\Phi(\mu_1), \Phi(\mu_2)) \leq c_T \int_0^t W_s(\mu_1, \mu_2) ds. \quad (2.4.39)
\]

Let \(t = T\) and \(\mu_1, \mu_2 \in M(C([0, T]))\). If we iterate this lemma, we obtain

\[
W_T(\Phi^k(\mu_1), \Phi^k(\mu_2)) = W_T(\Phi(\Phi^{k-1}(\mu_1)), \Phi(\Phi^{k-1}(\mu_2)))
\]

\[
\leq c_T \int_0^T W_s(\Phi^{k-1}(\mu_1), \Phi^{k-1}(\mu_2)) ds
\]

\[
\leq c_T \int_0^T \left( c_T \int_0^s W_r(\Phi^{k-2}(\mu_1), \Phi^{k-2}(\mu_2)) dr \right) ds
\]

\[
= c_T^2 \int_0^T \left( \int_0^s W_r(\Phi^{k-2}(\mu_1), \Phi^{k-2}(\mu_2)) dr \right) ds
\]

\[
\vdots
\]

\[
\leq c_T^k \int_0^T \int_0^s \cdots \int_0^b W_a(\mu_1, \mu_2) da \ldots dr ds
\]

\[
\leq c_T^k \int_0^T \int_0^s \cdots \int_0^b W_T(\mu_1, \mu_2) da \ldots dr ds
\]

\[
= c_T^k W_T(\mu_1, \mu_2) \int_0^T \int_0^s \cdots \int_0^b da \ldots dr ds
\]

\[
= \frac{c_T^k T^k}{k!} W_T(\mu_1, \mu_2).
\]

Then, for sufficient large \(k\), we have that \(c_T, k := \frac{c_T^k T^k}{k!} < 1\). So \(\Phi\) is called eventually contractive. Then the Generalized Banach fixed-point theorem (see Theorem 2.1.4) tells us that there exists a unique fixed point \(\mu^*\) such that \(\Phi(\mu^*) = \mu^*\). And hence we have proven Theorem 2.4.9 (existence and uniqueness of a solution to (2.4.29)).
2.4. Collage method: SDE

The constant $c_{T,k}$ is also an indicator of the convergence rate. The closer this value to 0, the faster the values converge to the fixed point.

2.4.2 Inverse problem SDE case

The inverse problem consists now of finding an estimation of $g$, which is the functional

$$\int_{\mathbb{R}} g(X_t, y) d\mu_t(y) = \mathbb{E}[g]$$

from a sample of $X_t$. Now let $(X^1_t, X^2_t, ..., X^n_t), t \in [0, T]$ be an independent sample and $\mu_n$ the estimated law of the process. Then the following corollary of the collage theorem is presented.

**Theorem 2.4.12.** [Collage theorem for SDEs] Let $\mu_n \in M(C([0, T]))$ be the estimated law of the process. If $\mu$ is the law of process $X_t$, then there exists a constant $C$ such that the following estimate holds:

$$W_T(\mu, \mu_n) \leq CW_T(\Phi(\mu_n), \mu_n).$$  \hspace{1cm} (2.4.42)

**Proof.** Because $\mu$ is the law of the process, we know that $\Phi(\mu) = \mu$. We then have

$$W_T(\mu, \mu_n) \overset{\Delta}{=} W_T(\mu, \Phi(\mu_n)) + W_T(\Phi(\mu_n), \mu_n)$$

$$= W_T(\Phi(\mu), \Phi(\mu_n)) + W_T(\Phi(\mu_n), \mu_n)$$

$$\overset{\text{Lemma 4.10}}{\leq} c_T \int_0^T W_s(\mu, \mu_n) ds + W_T(\Phi(\mu_n), \mu_n)$$

$$\leq c_T \int_0^T W_T(\mu, \mu_n) ds + W_T(\Phi(\mu_n), \mu_n)$$

$$\leq c_T W_T(\mu, \mu_n) \cdot \int_0^T ds + W_T(\Phi(\mu_n), \mu_n)$$

$$= c_T \cdot T \cdot W_T(\mu, \mu_n) + W_T(\Phi(\mu_n), \mu_n)$$

By rearranging the terms we obtain the following inequality\(^2\)

$$W_T(\mu, \mu_n) \leq \frac{1}{1 - c_T \cdot T} W_T(\Phi(\mu_n), \mu_n) \hspace{1cm} (2.4.44)$$

The inverse problem is now reduced to the minimization of $W_T(\Phi(\mu_n), \mu_n)$ which is a function of the unknown coefficients of $g$. Due to the complexity of the Wasserstein metric, it is difficult to use this method in practice [7].

\(^2\)Note that we have to limit ourselves to case where $c_T \cdot T < 1$ in order to get a positive constant. If our end time $T$ is too large, then we have to split up the interval in smaller intervals in such a way that for each interval, the constraint on the constant is met.
Chapter 3

The passing function $\beta(t)$

Now we have seen the mathematical theory behind the collage method, we can apply this theory on our traffic flow problem. Before we can say anything about how fast cars should arrive at a queue, it is important to first look at how fast cars leave the queue.

3.1 Problem

The rate at which cars leave the queue at time $t$ is given by the function $\beta(t)$. However, it is yet unclear what this passing function looks like. If we disregard the fluctuations in acceleration and in the distance between cars, we expect that it will be an increasing function in time which converges to a constant value $\beta_{\text{max}}$. In order to find this passing function, we can use the mathematical theory, presented in Chapter 2.

For this end, we need data on the number of cars in the queue in front of a traffic light at the moment that the light is green. On these moments cars start to accelerate and leave the queue at certain times. Let us assume that we have an initial queue which is “long enough”. By this we mean that before the light switches to green, the number of cars in the queue is larger than the amount of cars that can leave in the next green phase. This means that we can disregard the last part of the queue and therefore the cars that arrive at the queue during the green phase. This means for our mathematical model that the arrival function can be considered zero.

$$\alpha(t) = 0, \text{ for all } t \in [0, T].$$  \hspace{1cm} (3.1.1)

Furthermore, if we only consider the queue length during the green phase, the ODE presented in Chapter 2 can be simplified to

$$\begin{cases}
\frac{dN}{dt}(t) = -\beta(t), \\
N(0) = N_0
\end{cases}$$  \hspace{1cm} (3.1.2)

By (3.1.2), it is obvious that once the queue length $N(t)$ is known in the form of data, the passing function $\beta(t)$ can simply be obtained by multiplying -1 with the derivative of $N(t)$ with respect to time. However, as we will see in Section 3.2, the data $N(t)$ will be in the form of piecewise constant functions. Therefore, the derivative of this function can be given in terms of the Dirac delta-function $\delta(t - t_k)$ which is in our case not very useful for defining the passing function $\beta(t)$. Therefore in Subsection 3.4, the passing function will be investigated via the collage method.
3.2 Traffic data

The data we obtained is shown in Table B.1 (Appendix B). This table contains the departure times (in seconds) of cars at the intersection of the Onze Lieve Vrouwestraat and the John F. Kennedy-laan (Eindhoven) during rush hour on May 21st and 22nd 2014. \( t = 0 \) is the moment the light switched to green. The data consists of 22 green phases. Note that the amount of cars that left the queue during a green phase is not the same for all samples. In sample 7 a total of 17 cars left the queue during one green phase, while in sample 8 only 11 cars were able to leave the queue. Note that in our case the green phase consists of the time the lights is green and orange. This is chosen because many cars still use the orange light to quickly pass the traffic light.

The passing times are easily measured with a split timer stopwatch. The timer starts at the moment the green phase starts (that is the time the light switches to green). Every time a car passes the traffic light the split timer button will be pressed and the passing time of that car is saved. The passing times of all cars can now be used to obtain data for the queue lengths. Because the queue was long enough in every sample, we can disregard the last part of the queue and only look at the first part. Because we disregard the last part of the queue, we do not precisely know the exact length of the initial queue. However, we can take any value for this \( N_0 \), since we are only interested in how the queue changes, instead of what values the queue length has.

Let us take \( N_0 = 30 \). This value is large enough, since in all samples there are no more then 17 cars leaving each phase. Then at the moment a car leaves (passing time), we know that the queue length drops by 1. So we can use the data of the passing times to obtain data for the queue lengths. In Figure 3.2 we show the queue lengths for all samples.

![Queue length (discrete)](image)

**Figure 3.1:** Discrete representation of sample 1.
3.2. Traffic data

Note that we can create a smoother path as follows. Each sample consists of data points \((t_k, N(t_k))\), where

\[
N(t_k) = N_0 - k, \tag{3.2.1}
\]

for \(k = 1\) to the total number of cars that left during that green phase. It is possible to connect these data points with straight lines in order to obtain a smoother path \(N(t)\). Furthermore these paths are now continuous. These paths are shown in Figure 3.3.

![Figure 3.2: Discrete representation of all samples.](image1)

![Figure 3.3: Linear connection between data points.](image2)
3.3 Approximation of $\beta(t)$ by differentiation

From the differential equation given in (3.1.2) it simply follows that the passing function is given by minus 1 times the derivative of the queue length. However, taking the derivative from a piecewise constant function $N(t)$ as shown in Figure 3.2 yields a function which is a sum of a number of dirac delta-functions $\delta(t - t_k)$. $\delta(t - t_k) = 0$ everywhere except at $t = t_k$, since here it has the value $+\infty$. This function is not useful to work with. Therefore it is better to look at ‘smooth’ representations of the data.

If we only demand the continuity of the queue lengths, it is enough to look at the paths shown in Figure 3.3. Between any two consecutive data points $k - 1$ and $k$, the queue lengths decreases linearly at the rate $\frac{N(t_k) - N(t_{k-1})}{t_k - t_{k-1}} = \frac{1}{t_k - t_{k-1}}$. This means that between two consecutive departure times, the derivative is constant. So this fact yields a piecewise constant passing function. We can do this for each sample and obtain the passing functions given in Figure 3.4.

The passing functions jump very rapidly due to the fluctuations in data and not much information about the passing function can be extracted. Furthermore, the behavior near $t = 0$ is not realistic. This can be explained by the fact that using the linear paths of $N_{\text{data}}(t)$, we throw away the fact that the passing rate should be 0 at time $t = 0$. We could resolve these issues by looking at a smooth interpolation of the data and then taking the derivative of this smooth function. The interpolation is done numerically. The important information which follows from the data is that near $t = 0$, the queue does decrease much (it is constant) due to the speed of the reaction of the drivers and the fact that cars have to accelerate. A polynomial fit will ignore this important information. Therefore we fit a function of the form

$$f(t) = \sum_{k=0}^{n} \frac{1}{\cosh \left( \frac{kT}{T} \right)}$$

(3.3.1)

to the data. This function is then used as the smooth interpolation. The resulting passing functions are shown in Figure 3.5. Also the average of these functions is plotted in 3.5.
3.4. Approximation of $\beta(t)$ by the collage method

As indicated in Chapter 2, the collage method tries to find an unknown function in a differential equation by minimizing the distance

$$d(\pi, T\pi).$$

(3.4.1) is the distance in some appropriate metric space, between a target function $\pi$ and the image of the target function under some operator $T$. In the case of ordinary differential equations, the operator $T$ is the Picard integral operator, in the sense of Definition (2.2.2).

Based on (3.1.2), we see that $S$ is given by

$$T(\beta) := N_0 - \int_0^t \beta(s)ds,$$

(3.4.2)

where the operator $T$ depends on the unknown passing function $\beta(t)$. Since we want that the difference between the queue length $N(t)$ and $N_{\text{data}}(t)$ is as small as possible for all $t \in [0, T]$, we take for $d$ the $L^2$-metric. The collage distance is then given by

$$\Delta := \left( \int_0^T \left( N_{\text{data}}(t) - N_0 + \int_0^t \beta(s)ds \right)^2 dt \right)^{\frac{1}{2}},$$

(3.4.3)

where $N_{\text{data}}(t)$ represents the number of cars in the queue which we derived from the data of passing times. Note that the collage method works if the operator $T$ is a contraction mapping. Since the operator $T$, given in (3.4.2), does not depend on this function $N(t)$, the operator is constant in $N(t)$ and therefore the right-hand side of the ODE is Lipschitz continuous in the function.

Figure 3.5: Passing functions obtained by taking the derivative from a smooth interpolation of the data.

3.5 gives us already a good view of what the passing function looks like in reality. In the next section the passing function is obtained via the collage method.

3.4 Approximation of $\beta(t)$ by the collage method
3.4. Approximation of $\beta(t)$ by the collage method

$N(t)$ with Lipschitz constant $c = 0$.

As mentioned, the collage distance depends on the unknown function $\beta(t)$. Since it is impossible to minimize this distance over all possible passing functions, we are going to look at two function classes: piecewise constant functions and linear splines. These will be explained in the next sections.

3.4.1 Piecewise constant functions

We assume $\beta(t)$ to be a piecewise constant function. Let $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$ be a partition of the interval $[0, T]$. Then the passing function is of the form

$$\sum_{k=1}^{n} \beta_k \cdot \chi_{[t_{k-1}, t_k)}(t),$$

(3.4.4)

where $\chi_{[t_{k-1}, t_k)}(t)$ is the characteristic function. The collage distance becomes then a function of the unknown coefficients $\beta_k$,

$$\Delta = \left( \int_0^T \left( N(t) - N_0 + \int_0^t \sum_{k=1}^{n} \beta_k \cdot \chi_{[t_{k-1}, t_k)}(s)ds \right)^2 dt \right)^{\frac{1}{2}}.$$  

(3.4.5)

Minimizing (3.4.5) gives the same solution as minimizing $\Delta^2$ and since it is more convenient to get rid of the square root we look at $\Delta^2$ from now on. Minimizing $\Delta^2$ will yield a linear system in the unknown coefficients $\beta_k$. In Appendix C.1, we show how this corresponding linear system is derived. This system is given by

$$\begin{pmatrix} \int_0^T J_1(t) \beta_1 dt \\ \int_0^T J_2(t) \beta_1 dt \\ \vdots \\ \int_0^T J_n(t) \beta_1 dt \\ \int_0^T J_1(t) \beta_2 dt \\ \int_0^T J_2(t) \beta_2 dt \\ \vdots \\ \int_0^T J_n(t) \beta_2 dt \\ \vdots \\ \vdots \\ \int_0^T J_1(t) \beta_n dt \\ \int_0^T J_2(t) \beta_n dt \\ \vdots \\ \int_0^T J_n(t) \beta_n dt \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} \int_0^T (N(t) - N_0) J_1(t) dt \\ \int_0^T (N(t) - N_0) J_2(t) dt \\ \vdots \\ \int_0^T (N(t) - N_0) J_n(t) dt \end{pmatrix}.$$  

(3.4.6)

Solving (3.4.6) for every sample of data yields the results shown in Figure 3.6. Also the average of these functions is plotted in 3.6.
3.4. Approximation of $\beta(t)$ by the collage method

![Figure 3.6: Piecewise constant passing functions obtained by using the linear paths from Figure 3.3.](image)

In Figure 3.6 we see that the passing rates jump rapidly due to the fluctuations in data. Using the smooth interpolation of the data yields the results in Figure 3.7.

![Figure 3.7: Piecewise constant passing functions obtained by using a smooth interpolation of the data.](image)

The resulting average passing function is very similar to the one obtained in Figure 3.5.
3.4. Approximation of \( \beta(t) \) by the collage method

3.4.2 Linear splines

We assume \( \beta(t) \) is a linear spline. Again, let \( 0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T \) be a partition of the interval \([0, T]\). Then the passing function is of the form

\[
\sum_{k=1}^{n} \beta_k(t) \cdot \chi_{(t_{k-1}, t_k)}(t),
\]

(3.4.7)

where \( \beta_k(t) \) is a linear function of the form

\[
\beta_k(t) = \beta_{k-1} + \frac{\beta_k - \beta_{k-1}}{t_k - t_{k-1}} (t - t_{k-1}) = \beta_{k-1} \left(1 - \frac{t - t_{k-1}}{\Delta t}\right) + \beta_k \left(\frac{t - t_{k-1}}{\Delta t}\right).
\]

(3.4.8)

Now we have \( n + 1 \) unknown coefficients \( \beta_k \) for \( k = 0, 1, \ldots, n \). The square of the collage distance becomes then a function of the unknown coefficients \( \beta_k \),

\[
\Delta = \int_0^T \left(N_{\text{data}}(t) - N_0 + \int_0^t \sum_{k=1}^n \beta_k(s) \cdot \chi_{(t_{k-1}, t_k)}(s) ds\right)^2 dt.
\]

(3.4.9)

Minimizing (3.4.9) will yield again a linear system in the unknown coefficients \( \beta_k \). In Appendix C.2, it is shown how this corresponding linear system is derived. Solving this linear system for every sample of data yields the results shown in Figure 3.8. Also the average of these functions is plotted.

Figure 3.8: Linear splines obtained by taking the derivative from the linear paths from Figure 3.3.

Note that the passing rates jump rapidly due to the fluctuations in data. Using the smooth interpolation of the data yields the results in Figure 3.9. The resulting average passing function is very similar to the one obtained in Figure 3.5.
3.5 Concluding remarks

We were able to investigate how the passing rates vary as a function of time via two methods. One method used simple derivation while the other method made use of the collage method. Both methods depend on the real life data of passing times. Due to the discontinuity of the queue lengths (which are integers) the resulting passing functions were not very smooth. However, using a smooth interpolation of the data, the methods resulted in a smooth passing function which is in line with our expectations.

From the figures in this section, we see that the passing functions start to diverge after time $t = 22$. This can be explained by the fact that after $t = 22$, the light switched to orange. Officially this would mean that cars have to stop, but in practice a lot of cars use this orange time as if it was green time and simply pass. However, from the data it appears that both scenarios occur. As a consequence, the results for $t$ larger than 22 are not considered, since this does not give us a right approximation of how cars leave the queue for longer times.

A passing function which agrees with our expectation and with the data is the function

$$\beta(t) = 0.6 \tanh(0.3t),$$

see Figure 3.10. From now on, this function will be used as our passing function $\beta(t)$. This function approximates our data well and is also convenient for future numerical computations. Furthermore, it has the property $\beta(0) = 0$ which corresponds with the reality.

Note that this passing function fits to the data from one particular traffic light. It is possible that this passing function is different when considering traffic lights at other intersections. However, it is reasonable to assume that those passing functions will not differ much from $\beta(t) = 0.6 \tanh(0.3t)$.
3.5. Concluding remarks

Figure 3.10: Linear splines obtained by the collage method from a smooth interpolation of the data.

Note that the steep curve of the passing functions near $t = 0$ is due to the acceleration of the cars. For larger $t$, the cars that pass are driving at a constant speed (the maximum speed). Since the passing rate is in cars $\cdot s^{-1}$, we can express the constant rate of cars in terms of the maximum speed and the distance between cars

$$\beta = \frac{u_{\text{max}}}{\Delta L},$$

(3.5.1)

where $u_{\text{max}}$ is the maximum speed and $\Delta L$ the distance between the front of two consecutive cars.
Chapter 4

Arrival rate control

In Chapter 3 we solved an inverse problem for our model. For this we needed data of our queue lengths \( N_{\text{data}}(t) \), \( t > 0 \). From this data we retrieved information on how cars leave the queue. Since we now know more about how cars leave the queue, we can use this and derive some expressions for how cars should arrive at the queue.

As indicated in Chapter 1, if we know how the traffic lights are set and how cars drive away when there is green light, we can then control the number of cars in the queue by controlling the arrival rates. There are a few options which will be explained in the next sections.

4.1 Optimization options

4.1.1 Arrival rate threshold

Due to the finite length of the street, there is a certain maximum capacity. This means that there is a maximum amount of cars \( N_{\text{max}} \) allowed in the queue\(^1\). If the queue exceeds this maximum value, the end of the queue will block the previous intersection. If that happens, also other directions at this intersection will be hindered. This will only lead to more irritation, aggression and accidents. Since it is highly undesirable that cars block an intersection, this must be prevented.

One way to do this is to monitor the arrival rate \( \alpha \). It is useful to have an indication for the arrival rate when the queue is getting too large. If this happens, the arrival rate has to be lowered by traffic lights or by decreasing the velocities of arriving cars. If in the future cars would drive autonomously, an optimal set of velocities could be computed for all cars that are going to enter the queue.

How to find such an indicator \( \alpha \) can be explained as follows. Assume we know the parameters \( T_0, T_1, T, N_0 \) and the passing function \( \beta(t) \). Since there is a maximum number of cars \( N_{\text{max}} \) allowed in the queue, it is desirable to always have a queue for which the length is smaller than this maximum. So we want

\[
N(t, \alpha) < N_{\text{max}} \quad \text{for all } t \in [0, T].
\]  

(4.1.1)

When we are in the case of heavy traffic (e.g. rush hour), we want that the throughput is as large as possible without the queue exceeding the maximum value. So what we are interested in, is finding the \( \alpha \) such that the queue is as close as possible to \( N_{\text{max}} \). This is equivalent with the

---

\(^1\)This maximum amount should be less than the theoretical maximum value which is equal to the length of the street divided by the average length of one car. This is recommended for safety reasons, e.g. to deal with cars that are longer than the average car length.
4.1. Optimization options

inverse problem

\[
\min_{\alpha} |N_{\max} - \sup_{t \in [0,T]} N(t, \alpha(t))| = \min_{\alpha} \sup_{t \in [0,T]} |N_{\max} - N(t, \alpha(t))| \\
= \min_{\alpha} \|N_{\max} - N(\cdot, \alpha(\cdot))\|_{L^\infty([0,T])}.
\]

(4.1.2)

However, the collage method with the \(L^\infty([0,T])\)-metric is hard and analytically not possible. Therefore it is better to look at another option.

4.1.2 Target function

Instead of demanding that the queue length \(N(t)\) does not exceed \(N_{\max}\) for all \(t \in [0,T]\) we want to know which arrival rates make sure that our queue length is as close as possible to a desired function \(N_{\text{target}}(t)\) for all \(t \in [0,T]\). Similar to the case of finding the passing function, the metric we can use is the \(L^2([0,T])\)-metric. This means that the collage distance is given by

\[
\Delta = \left( \int_0^T (N_{\text{target}}(t) - N(t))^2 \, dt \right)^{\frac{1}{2}}.
\]

(4.1.3)

It is yet unclear what this desired target function \(N_{\text{target}}(t)\) looks like. There are several options for this target function. A target function \(N_{\text{target}}(t)\) indicates how many cars should be in the queue at each time. A list of some target functions that can be used is given below.

- \(N_{\text{target}}(t)\) is constant.
  The most simple target function is just the constant function
  \[
  N_{\text{target}}(t) = N_{\text{goal}},
  \]
  where \(N_{\text{goal}} \in \mathbb{R}\). However, keeping the queue length constant is extremely hard, especially when we consider the cases where \(N_0 \neq N_{\text{goal}}\). These are the cases where the initial queue length is not equal to the desired queue length. Looking for the arrival function \(\alpha\) which minimizes the distance between \(N(t, \alpha(t))\) and \(N_{\text{goal}}\) will give unrealistic arrival rates for \(t\) close to 0, since here the difference is relatively large (\(\sim \mathcal{O}(N_{\text{goal}} - N_0)\)). Note that this target function can be used in cases of heavy traffic. Because of the heavy traffic scenario, we want the throughput to be as large as possible. This means that in all cases we want the queue to be as close as possible to a given maximum number of cars. This is optimal since we assume that the traffic light settings as well as the passing function of the cars is given. So maximizing throughput is equivalent with keeping the queue as full as possible by letting as many cars as possible enter this queue.

- \(N_{\text{target}}(t)\) a function that increases from \(N_0\) to \(N_{\text{goal}}\).
  When \(N_0 \neq N_{\text{goal}}\), we can use a target function which is equal to \(N_0\) at time 0, and then slowly increases to a value \(N_{\text{goal}}\). How fast this function increases to the value \(N_{\text{goal}}\) can vary, but it is recommended to keep the increase below a certain \(\alpha_{\text{max}}\) which is the maximum arrival rate. Since cars have a maximum allowed velocity and the distance between cars is considered to be a certain safe distance, then the time between two consecutive cars is at a minimum, say \(dt_{\text{min}}\). The maximum arrival rate is then equal to \(\frac{1}{dt_{\text{min}}}\).

- \(N_{\text{target}}(t)\) that leads to strictly positive arrival rates.
  One of the drawbacks of the previous target functions is that whenever the light switches to red again, no cars are allowed to leave and the number of cars in the queue is already equal to the value of \(N_{\text{goal}}\). This means that there are also no cars allowed to enter the queue, since then the queue exceeds the value of \(N_{\text{goal}}\) while there are no cars that are allowed to leave anymore due to the red light. Therefore it is sometimes wise to let the target function
4.2. Finding optimal arrival rates

also increase at the interval after the light switched to red. This means that it is possible to create a target function which makes sure the arrival rate is strictly positive for all \( t \in [0, T] \). This has a positive effect to the mood of the drivers. Simply because it is often more desired to have a lower velocity for a longer time period then to actually stop and then move to a higher velocity.

- **Other type of functions** \( N_{\text{target}}(t) \).

There are many other functions that could act as a target function. Any function which is positive and bounded from above by \( N_{\text{max}} \) on \([0, T]\) can be chosen as target function.

In the next section, the approach of a target function will be used in order to derive some expressions for which the arrival rates can be found.

**4.2 Finding optimal arrival rates**

Assume that \( T_0, T_1, T \) are known parameters and the passing function \( \beta(t) \) is known. The differential equation is given by

\[
\begin{aligned}
\frac{dN(t)}{dt} &= \alpha(t) - \beta(t - T_0) \cdot \chi_{[T_0, T_1]}(t) \quad \text{for } t \in [0, T], \\
N(0) &= N_0,
\end{aligned}
\]

(4.2.1)

where \( \alpha(t) \) is an unknown function. Let \( N_{\text{target}}(t) \) be one of the target functions discussed in the previous section. We can use the collage method and find the arrival function such that the queue length (solution to (4.2.1)) is as close as possible to \( N_{\text{target}}(t) \).

In order to use the collage method, we define the Picard integral operator \( T \) as given by

\[
T(\alpha(\cdot)) := N_0 + \int_0^t \alpha(s)ds - \int_0^t \beta(s)ds \cdot \chi_{[T_0, T_1]}(t) - \int_0^t \beta(s)ds \cdot H_{T_1}(t), \quad \text{for } t \in [0, T]
\]

(4.2.2)

where the operator \( T \) is a function of the unknown arrival function \( \alpha(t) \). Furthermore we take for \( d \) the \( L^2 \)-metric. Since we want that the difference between the queue length \( N(t) \) and \( N_{\text{data}}(t) \) is as small as possible for all \( t \in [0, T] \), the collage distance is then given by

\[
\Delta = \left( \int_0^T (N_{\text{data}}(t) - N_0 - \int_0^t \alpha(s)ds + \int_0^t \beta(s)ds \cdot \chi_{[T_0, T_1]}(t) + \int_0^t \beta(s)ds \cdot H_{T_1}(t))^2 dt \right)^{\frac{1}{2}}.
\]

(4.2.3)

As in the case of the unknown passing function, it is impossible to minimize (4.2.3) over all possible functions. Therefore we are going to look at four function classes: polynomials, sums of cosines, piecewise constant functions and linear splines. These will be explained in the next sections.

For each class of functions, we illustrate the method to two numerical examples. In both case the values for the parameters are chosen:

\[
\begin{align*}
T_0 &= 20, \\
T_1 &= 50, \\
T &= 60.
\end{align*}
\]

(4.2.4)

Also, for the passing functions we take the functions as concluded in Section 3.5:

\[
\beta(t) = 0.6 \tanh (0.3t).
\]

(4.2.5)

However, the target functions are different.
4.2. Finding optimal arrival rates

- Case 1: Let \( N_0 = 10 \). We take \( n_{\text{target}} \equiv N_0 \). This means that initially, we are already in the desired position. We expect that the amount of cars that should arrive per second is identical to the amount of cars that leave per second. In other words, the arrival function such that the queue length fits best to the target function is exactly equal to the passing function;

- Case 2: Let \( N_0 = 4 \). Now we take for \( n_{\text{target}} \) a parabola which is equal to \( N_{\text{max}} = 10 \) at \( t = 20 \) and \( 0 \) at from \( t = T_1 \). This target function models the behavior of a queue growing until a certain moment from which the queue will decrease to 0. All cars should be able to leave during the green time. See Figure 4.1 for a plot of the target function.

![Figure 4.1: Target function in which all cars should be able to leave during the green phase.](image.png)

4.2.1 Polynomials

Assume \( \alpha(t) \) is of the form

\[
\sum_{k=0}^{n} \alpha_k t^k.
\]  

(4.2.6)

Then the square of the collage distance becomes a function of the unknown parameters \( \alpha_k \).

\[
\Delta = \left( \int_0^T (N_{\text{target}}(t) - N_0 - \int_0^t \sum_{k=0}^{n} \alpha_k s^k ds + \int_0^t \beta(s) ds \cdot \chi_{[T_0, T_1]}(t) + \int_0^t \beta(s) ds \cdot H_{T_1}(t) )^2 dt \right)^\frac{1}{2}.
\]  

(4.2.7)
4.2. Finding optimal arrival rates

The resulting linear system obtained by the collage method is derived in Appendix D.1 and given by

\[
\begin{pmatrix}
\frac{T^3}{3} & \frac{T^4}{8} & \cdots & \frac{T^{n+3}}{(n+1)(n+3)} \\
\frac{T^4}{8} & \frac{T^5}{20} & \cdots & \frac{T^{n+4}}{2(n+1)(n+4)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{T^{n+3}}{(n+1)(n+3)} & \frac{T^{n+4}}{2(n+1)(n+4)} & \cdots & \frac{T^{2n+3}}{(n+1)^2(n+3)}
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n
\end{pmatrix}
= \begin{pmatrix}
\int_0^T (N_{\text{target}}(t) - N_0 + J(t)) \cdot td t \\
\frac{1}{2} \int_0^T (N_{\text{target}}(t) - N_0 + J(t)) \cdot t^2 dt \\
\vdots \\
\frac{1}{n+1} \int_0^T (N_{\text{target}}(t) - N_0 + J(t)) \cdot t^{n+1} dt
\end{pmatrix},
\]

(4.2.8)

where \( J(t) \) is given by

\[
J(t) = \int_0^t \beta(s) ds \cdot \chi_{[T_0, T_1]}(t) + \int_0^t \beta(s) ds \cdot H_{T_1}(t)
\]

(4.2.9)

for convenience. This \( J(t) \) is assumed to be known since it contains the passing function and the traffic light parameters. Solving this linear system for different values of \( n \) (degree of the polynomial) for case 1 yields the arrival functions shown in Figure 4.2. Note that the functions approximate the passing function pretty good, but oscillate near the boundaries of the interval \([0, T]\). Higher order polynomials show a better approximation in the center of the interval, but show worse behavior near the boundaries. This is also clear from Table 4.1. In this table the collage errors, defined in (4.2.7), are shown for different values of \( n \).

![Figure 4.2: Polynomial arrival function for case 1.](image-url)

For case 2, the polynomial passing functions show even worse results. In Figure 4.3 we see that the polynomials oscillate heavily and obtain unrealistic (negative) values. The approximation near the boundaries is far worse than in case 1. Therefore, polynomial interpolation is not suitable for our problem.
4.2. Finding optimal arrival rates

Figure 4.3: Polynomial arrival function for case 2.

Table 4.1: Results polynomials case 1 and 2.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Error case 1</th>
<th>Error case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.592118</td>
<td>1.64920</td>
</tr>
<tr>
<td>8</td>
<td>0.281559</td>
<td>0.74197</td>
</tr>
<tr>
<td>16</td>
<td>0.156974</td>
<td>0.44482</td>
</tr>
<tr>
<td>32</td>
<td>0.085149</td>
<td>0.60817</td>
</tr>
<tr>
<td>64</td>
<td>0.310741</td>
<td>0.25848</td>
</tr>
</tbody>
</table>

4.2.2 Sums of cosines

Since the approximation of the arrival function by polynomials was not satisfying, we try the approximation by sums of cosines. Let $\alpha(t)$ be a function of the type

$$\sum_{k=0}^{n} \alpha_k \cos \left( \frac{k\pi t}{T} \right). \quad (4.2.10)$$

Now similar steps as in Section 4.2.1 can be taken in order to solve the inverse problem for $\alpha(t)$. See Appendix D.2 for a detailed description of the collage method. Solving the linear system for both numerical examples results in Figures 4.4 and 4.5. Both figures show a good approximation with no unrealistic arrival rates and also the error is small for $n$ large (Table 4.2).
4.2. Finding optimal arrival rates

Figure 4.4: Arrival function in terms of sums of cosines for case 1.

Figure 4.5: Arrival function in terms of sums of cosines for case 2.

The oscillatory behavior at $t = T_1$ in Figure 4.4 is called **Gibbs phenomenon** and occurs in Fourier approximations at functions that have a jump discontinuity. Since our passing term contains a jump discontinuity, so does the approximated arrival function. This Gibbs phenomenon means that there is an overshoot near the discontinuity and this overshoot does not decrease when $n \to \infty$. The Figures show a nice behavior. Furthermore, Table 4.2 shows that the collage error is very small, which implies a good approximation of the arrival function.
4.2. Finding optimal arrival rates

Table 4.2: Results sums of cosines case 1 and 2.

<table>
<thead>
<tr>
<th>n</th>
<th>Error case 1</th>
<th>Error case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.231212</td>
<td>0.069821</td>
</tr>
<tr>
<td>32</td>
<td>0.089649</td>
<td>0.014002</td>
</tr>
<tr>
<td>64</td>
<td>0.030772</td>
<td>0.002219</td>
</tr>
<tr>
<td>128</td>
<td>0.011261</td>
<td>0.000418</td>
</tr>
</tbody>
</table>

We can also use the same classes of functions we used for the passing function: piecewise constant functions and linear splines. These will be treated in the following two sections.

4.2.3 Piecewise constant functions

As in Section 3.4.1, assume the unknown function is of the form

\[
\sum_{k=1}^{n} \alpha_k \cdot \chi_{[t_{k-1}, t_k)}(t), \tag{4.2.11}
\]

where \( \chi_{[t_{k-1}, t_k)}(t) \) is the characteristic function, for a partition \( 0 = t_0 < t_1 < ... < t_{n-1} < t_n = T \) of the interval \([0, T]\). The collage method yields a similar linear system as in (3.4.6). Except the minus sign is not present and the right-hand side vector contains an extra term in the integral. This right-hand side vector is now given by

\[
\begin{pmatrix}
\int_0^T (N_{\text{data}}(t) - N_0 + J(t)) \cdot I_1(t) \, dt \\
\int_0^T (N_{\text{data}}(t) - N_0 + J(t)) \cdot I_2(t) \, dt \\
\vdots \\
\int_0^T (N_{\text{data}}(t) - N_0 + J(t)) \cdot I_n(t) \, dt
\end{pmatrix},
\tag{4.2.12}
\]

where

\[
J(t) = \int_0^t \beta(s) \, ds \cdot \chi_{[T_0, T_1)}(t) + \int_0^t \beta(s) \, ds \cdot H_{T_1}(t),
\tag{4.2.13}
\]

\[
I_m(t) = (t - t_{m-1}) \cdot \chi_{[t_{m-1}, t_m)}(t) + \Delta t \cdot H_{t_m}(t).
\]

The results of solving this system for case 1 and 2 are shown in Figures 4.6 and 4.7.
4.2. Finding optimal arrival rates

The arrival functions look very similar to Figures 4.4 and 4.5 for the results of arrival functions which were sums of cosines. Together with Table 4.3 we conclude that these approximations are also pretty good.
4.2. Finding optimal arrival rates

Table 4.3: Results piecewise constant functions case 1 and 2.

<table>
<thead>
<tr>
<th>n</th>
<th>Error case 1</th>
<th>Error case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.442376</td>
<td>0.189936</td>
</tr>
<tr>
<td>32</td>
<td>0.145936</td>
<td>0.036522</td>
</tr>
<tr>
<td>64</td>
<td>0.050961</td>
<td>0.009375</td>
</tr>
<tr>
<td>128</td>
<td>0.01785</td>
<td>0.004245</td>
</tr>
</tbody>
</table>

4.2.4 Linear splines

Finally, assume $\alpha(t)$ is of the form

$$\sum_{k=1}^{n} \alpha_k(t) \cdot \chi_{[t_{k-1}, t_k)}(t),$$

(4.2.14)

on a partition $0 = t_0 < t_1 < ... < t_{n-1} < t_n = T$ of the interval $[0, T]$. $\alpha_k(t)$ is a linear function of the form

$$\alpha_k(t) = \alpha_{k-1} + \frac{\alpha_k - \alpha_{k-1}}{t_k - t_{k-1}} (t - t_{k-1}) = \alpha_{k-1} \left(1 - \frac{t - t_{k-1}}{\Delta t}\right) + \alpha_k \left(\frac{t - t_{k-1}}{\Delta t}\right).$$

(4.2.15)

This means that the arrival functions are now piecewise linear (linear splines). The results of the collage method for both cases are shown in Figure 4.8 and 4.9.

![Figure 4.8: Piecewise linear arrival functions for case 1.](image)
4.2. Finding optimal arrival rates

The collage error shown in Table 4.4 is large but decreases when \( n \) increases. The behavior in case 2 for \( n = 128 \) near \( t = 0 \) is strange and is due to numerical errors.

Table 4.4: Results piecewise linear functions case 1 and 2.

<table>
<thead>
<tr>
<th>( n )</th>
<th>Error case 1</th>
<th>Error case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>5.82002</td>
<td>3.55174</td>
</tr>
<tr>
<td>32</td>
<td>2.90179</td>
<td>1.87433</td>
</tr>
<tr>
<td>64</td>
<td>1.45324</td>
<td>0.96147</td>
</tr>
<tr>
<td>128</td>
<td>0.72594</td>
<td>0.49755</td>
</tr>
</tbody>
</table>
Chapter 5

Traffic light control

In urban traffic traffic, the arrival of cars at a queue often depends on the traffic light settings of the previous intersection. In this chapter we will investigate what influence this observation has on the arrival rates. We consider a model with two traffic lights 5.1 and with three traffic lights 5.2.

5.1 Two traffic light scenario

5.1.1 Traffic scenario

We start with a very simplified version of our problem. Let us consider the scenario shown in Figure 5.1.

Traffic lights operate in cycles. During the first period of the cycle only traffic lights 1 and 2a get green light (not for the entire period), while the other lights stay red. Then in the next period of the cycle, only traffic light 1 and 2b get green light for some time, while lights 2a and 2c stay red for the entire cycle. After that lights 1 and 2c get green light. Then the cycle ends and starts over with the period in which only traffic lights 1 and 2a get green light. By modeling this intersection in this way, for each cycle there is only one traffic light at intersection 2 (either 2a, 2b or 2c) which influences the queue at light 1.

Figure 5.1: Traffic flow scenario with 2 traffic lights ($n = 2$).
5.1. Two traffic light scenario

From now on we do not distinguish between the directions 2a, 2b and 2c but call it traffic light 2. We again assume a heavy traffic scenario for which we will also choose the fixed target function

\[ N_{\text{target}}(t) = N_{\text{max}}. \] (5.1.1)

Here \( N_{\text{max}} \) is not necessarily the maximum road capacity. Another assumption is that the cycle time of light 1 is equal to the cycle time of light 2, namely \( T \).

The question is now:

**How should traffic light 2 be set such that the queue at traffic light 1 is as close as possible to a given target queue?**

For the queue at traffic light 1 we have the same model as before (see (1.3.2))

\[
\begin{align*}
\frac{dN^{(1)}}{dt}(t) &= \alpha^{(1)}(t) - \beta^{(1)}(t - T_0^{(1)}) \cdot \chi_{[T_0^{(1)},T_1^{(1)}]}(t) \quad \text{for } t \in [0, T], \\
N^{(1)}(0) &= N_0^{(1)}. 
\end{align*}
\] (5.1.2)

Note the superscript in (5.1.2) at \( T^{(k)}, N^{(k)}, \alpha^{(k)}(t) \) and \( \beta^{(k)}(t) \) \((k = 1)\), which indicates that functions and parameter belong to traffic light 1.

The key observation is that the number of cars that arrive per second at queue 1 depends on the rate of cars that leave queue 2 and on the traffic light settings. So we can write the arrival function \( \alpha^{(1)}(t) \) in terms of:

- The settings of traffic light 2: \( T_0^{(2)}, T_1^{(2)} \).
- The passing function of light 2: \( \beta^{(2)}(t) \).

So this means that the arrival function of queue 1 is equal to

\[ \alpha^{(1)}(t) = \beta^{(2)}(t - T_0^{(2)}) \cdot \chi_{[T_0^{(2)},T_1^{(2)}]}(t). \] (5.1.3)

Since we are interested in the traffic light settings of light 2, the only unknown parameters are \( T_0^{(2)}, T_1^{(2)} \).

5.1.2 Collage method

We can use the collage method and find the values for the parameters \( T_0^{(2)}, T_1^{(2)} \) such that the queue length at light 1 at time \( t \) is as close as possible to the target queue \( N_{\text{target}}(t) \). Since the ODE is now given by

\[
\begin{align*}
\frac{dN^{(1)}}{dt}(t) &= \beta^{(2)}(t - T_0^{(2)}) \cdot \chi_{[T_0^{(2)},T_1^{(2)}]}(t) - \beta^{(1)}(t - T_0^{(1)}) \cdot \chi_{[T_0^{(1)},T_1^{(1)}]}(t) \quad \text{for } t \in [0, T], \\
N^{(1)}(0) &= N_0^{(1)}. 
\end{align*}
\] (5.1.4)

The collage error is then given by

\[
\Delta^2(T_0^{(2)}, T_1^{(2)}) = \int_0^T \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^t \beta^{(2)}(s - T_0^{(2)}) \cdot \chi_{[T_0^{(2)},T_1^{(2)}]}(s) ds + \int_0^t \beta^{(1)}(s - T_0^{(1)}) \cdot \chi_{[T_0^{(1)},T_1^{(1)}]}(s) ds \right)^2 dt. \] (5.1.5)
5.1. Two traffic light scenario

For convenience, we define the last term $J^{(k)}(t)$ as

$$J^{(1)}(t) = \int_0^t \beta^{(1)}(s - T^{(1)}_0) \cdot \chi_{[T^{(1)}_0, T^{(1)}_1]}(s) ds$$

$$= \int_0^{t-T^{(1)}_0} \beta^{(1)}(s) ds \cdot \chi_{[T^{(1)}_0, T^{(1)}_1]}(t) + \int_{T^{(1)}_0-T^{(1)}_0}^{T^{(1)}_1} \beta^{(1)}(s) ds \cdot H_{T^{(1)}_1}(t)$$

(5.1.6)

since this term contains only known parameters and functions. The integral term with the unknown parameters $T^2_0, T^2_1$ in (5.1.5) can be written as

$$\int_0^t \beta^{(2)}(s - T^{(2)}_0) \cdot \chi_{[T^{(2)}_0, T^{(2)}_1]}(s) ds = \int_0^{t-T^{(2)}_0} \beta^{(2)}(s) ds \cdot \chi_{[T^{(2)}_0, T^{(2)}_1]}(t) + \int_{T^{(2)}_0-T^{(2)}_0}^{T^{(2)}_1-T^{(2)}_0} \beta^{(2)}(s) ds \cdot H_{T^{(2)}_1}(t).$$

(5.1.7)

Since the characteristic function and the heaviside function in these terms also depend on the unknown parameters $T^{(2)}_0, T^{(2)}_1$, it is not wise to differentiate (5.1.5) with respect to $T^{(2)}_0, T^{(2)}_1$. This would lead to an expression containing the derivative of the heaviside function, which is the dirac-delta function. It is analytically not convenient to work with this function in this setting. Therefore we try a different approach.

Since the characteristic function and the heaviside function are either 0 or 1 at a certain interval, we can split up the square of the collage distance into three integral terms. Note that the square of the collage error $\Delta^2(T^{(2)}_0, T^{(2)}_1)$ is given by

$$\int_0^T \left( N_{\text{target}}(t) - N^{(1)}_0 - \int_0^{t-T^{(2)}_0} \beta^{(2)}(s) ds \cdot \chi_{[T^{(2)}_0, T^{(2)}_1]}(t) - \int_0^{T^{(2)}_1-T^{(2)}_0} \beta^{(2)}(s) ds \cdot H_{T^{(2)}_1}(t) + J^{(1)}(t) \right)^2 dt$$

$$= \int_0^{T^{(2)}_0} \left( N_{\text{target}}(t) - N^{(1)}_0 + J^{(1)}(t) \right)^2 dt =: \Delta^2_1$$

$$+ \int_{T^{(2)}_0}^{T^{(2)}_1} \left( N_{\text{target}}(t) - N^{(1)}_0 - \int_0^{t-T^{(2)}_0} \beta^{(2)}(s) ds + J^{(1)}(t) \right)^2 dt =: \Delta^2_2$$

$$+ \int_{T^{(2)}_1}^T \left( N_{\text{target}}(t) - N^{(1)}_0 - \int_0^{T^{(2)}_1-T^{(2)}_0} \beta^{(2)}(s) ds + J^{(1)}(t) \right)^2 dt =: \Delta^2_3.$$

(5.1.8)

Since we have that

$$\frac{\partial \Delta^2}{\partial T^{(2)}_k} = \frac{\partial \Delta^2_1}{\partial T^{(2)}_k} + \frac{\partial \Delta^2_2}{\partial T^{(2)}_k} + \frac{\partial \Delta^2_3}{\partial T^{(2)}_k},$$

(5.1.9)

for $k = 0, 1$ we can take the derivative of each term separately with respect to both parameters $T^{(2)}_0, T^{(2)}_1$. Because the boundaries of the integrals also depend on the parameters $T^{(2)}_0, T^{(2)}_1$, we need the Leibniz Integral Rule in order to take the derivatives.
5.1. Two traffic light scenario

Theorem 5.1.1. [Leibniz Integral Rule]

\[
\frac{\partial}{\partial z} \int_{a(z)}^{b(z)} f(x, z) \, dx = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z} \, dx + f(b(z), z) \frac{\partial b}{\partial z} - f(a(z), z) \frac{\partial a}{\partial z}
\] (5.1.10)

In what follows, we will derive the derivatives of the collage error with respect to the unknown parameters.

- Derivatives with respect to \( T_0^{(2)} \):

\[
- \frac{\partial \Delta_1^2}{\partial T_0^{(2)}} = \left( N_{\text{target}}(T_0^{(2)}) - N_0^{(1)} + J_0^{(1)}(T_0^{(2)}) \right)^2
\] (5.1.11)

\[
- \frac{\partial \Delta_2^2}{\partial T_0^{(2)}} =
\int_{T_0^{(2)}}^{T_0^{(1)}} 2 \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{t-T_0^{(2)}} \beta(t-s) \, ds + J_0^{(1)}(t) \right) \cdot \beta(t-T_0^{(2)}) \, dt
\]

\[
- \frac{\partial \Delta_3^2}{\partial T_0^{(2)}} =
\int_{T_0^{(2)}}^{T} 2 \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T-T_0^{(2)}} \beta(t-s) \, ds + J_0^{(1)}(t) \right) \cdot \beta(t-T_0^{(2)}) \, dt.
\] (5.1.12)

If we sum these three derivatives, we see that (5.1.11) cancels with the second term of (5.1.12). So we are left with two integral terms.

\[
\frac{\partial \Delta_1^2}{\partial T_0^{(2)}} = 2 \int_{T_0^{(2)}}^{T_0^{(1)}} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{t-T_0^{(2)}} \beta(t-s) \, ds + J_0^{(1)}(t) \right) \cdot \beta(t-T_0^{(2)}) \, dt
\]

\[
+ 2 \beta(T-T_0^{(2)}) \int_{T_0^{(2)}}^{T} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T-T_0^{(2)}} \beta(t-s) \, ds + J_0^{(1)}(t) \right) \, dt.
\] (5.1.14)

- Derivatives with respect to \( T_0^{(2)} \)

\[
- \frac{\partial \Delta_1^2}{\partial T_0^{(2)}} = 0
\] (5.1.15)

\[
- \frac{\partial \Delta_2^2}{\partial T_0^{(2)}} =
\left( N_{\text{target}}(T_0^{(2)}) - N_0^{(1)} - \int_0^{T_0^{(2)}} \beta(s) \, ds + J_0^{(1)}(T_0^{(2)}) \right)^2
\] (5.1.16)
5.1. Two traffic light scenario

\[- \frac{\partial \Delta^2}{\partial T_1^{(2)}} = \]
\[
\int_{T_1^{(2)}}^T 2 \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{1-T_0^{(2)}} \beta^{(2)}(s) ds + J^{(1)}(t) \right) \cdot \beta^{(2)}(T_1^{(2)} - T_0^{(2)}) \cdot (-1) dt
\]
\[
- \left( N_{\text{target}}(T_1^{(2)}) - N_0^{(1)} - \int_0^{T_1^{(2)}-T_0^{(2)}} \beta^{(2)}(s) ds + J^{(1)}(T_1^{(2)}) \right)^2.
\]
(5.1.17)

If we sum these three derivatives, we see that (5.1.16) cancels with the second term of (5.1.17). So we are left with one integral term.

\[
\frac{\partial \Delta^2}{\partial T_1^{(2)}} = -2\beta^{(2)}(T_1^{(2)} - T_0^{(2)}) \int_{T_1^{(2)}}^T \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)}-T_0^{(2)}} \beta^{(2)}(s) ds + J^{(1)}(t) \right) dt.
\]
(5.1.18)

Now we have the two derivatives, we want these derivatives to be equal to 0 in order to find the critical points. If we look closely to (5.1.14) and (5.1.18), we see that -1 times (5.1.18) equals the second term in (5.1.14). But solving an equation of the form

\[
\begin{pmatrix}
2a + 2b \\
-2b
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]
(5.1.19)
is equivalent with solving

\[
\begin{pmatrix}
a \\
b
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
(5.1.20)

So we end up with the system

\[
\begin{pmatrix}
\int_{T_1^{(2)}}^T \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{1-T_0^{(2)}} \beta^{(2)}(s) ds + J^{(1)}(t) \right) \cdot \beta^{(2)}(t - T_0^{(2)}) dt \\
\beta^{(2)}(T_1^{(2)} - T_0^{(2)}) \cdot \int_{T_1^{(2)}}^T \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)}-T_0^{(2)}} \beta^{(2)}(s) ds + J^{(1)}(t) \right) dt
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]
(5.1.21)

where we define

\[
\begin{pmatrix}
F_1(T_0^{(2)}, T_1^{(2)}) \\
F_2(T_0^{(2)}, T_1^{(2)})
\end{pmatrix} :=
\begin{pmatrix}
\int_{T_1^{(2)}}^T \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{1-T_0^{(2)}} \beta^{(2)}(s) ds + J^{(1)}(t) \right) \cdot \beta^{(2)}(t - T_0^{(2)}) dt \\
\beta^{(2)}(T_1^{(2)} - T_0^{(2)}) \cdot \int_{T_1^{(2)}}^T \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)}-T_0^{(2)}} \beta^{(2)}(s) ds + J^{(1)}(t) \right) dt
\end{pmatrix}.
\]
(5.1.22)

Unfortunately this system is nonlinear in the parameters $T_0^{(2)}, T_1^{(2)}$, so in order to solve this we will use Newton’s method.

5.1.3 Newton’s method in $\mathbb{R}^n$

Let $F = (F_1, F_2, \ldots, F_n)$ be a function from $\mathbb{R}^n \to \mathbb{R}^n$, from which we would like to find the zeros. To find the zeros, we first look at the linearization of $F$ around a point $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$. Then for $x = (x_1, x_2, \ldots, x_n)$ near $\hat{x}$, the linearization is given by

\[
F(x) \approx F(\hat{x}) + \frac{\partial F}{\partial x_1}(x_1 - \hat{x}_1) + \ldots + \frac{\partial F}{\partial x_n}(x_n - \hat{x}_n).
\]
(5.1.23)
5.1. Two traffic light scenario

In matrix notation we have

\[ F(x) \approx F(\hat{x}) + J(\hat{x})(x - \hat{x}), \]  

(5.1.24)

where \( J \) is the Jacobian matrix, defined by

\[
J = \begin{pmatrix}
J_{11} & J_{12} & \ldots & J_{1n} \\
J_{21} & J_{22} & \ldots & J_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
J_{n1} & J_{n2} & \ldots & J_{nn}
\end{pmatrix} := \begin{pmatrix}
\frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \ldots & \frac{\partial F_1}{\partial x_n} \\
\frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \ldots & \frac{\partial F_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \ldots & \frac{\partial F_n}{\partial x_n}
\end{pmatrix}.
\]  

(5.1.25)

We can start a sequence by choosing an initial guess for our zeros, \( x^0 \) and defining the next vector in our sequence as

\[ x^{n+1} = x^n - J^{-1}(x^n) \cdot F(x^n), \]  

(5.1.26)

where \( J^{-1} \) is the inverse of the Jacobian. The sequence defined in (5.1.26) will almost always converge to the root of the function \( F \). Note that the sequence is dependent on the initial guess. If an initial guess is not close enough to the root, the sequence will not converge. Furthermore, if the function \( F \) has multiple roots, different initial guesses could lead to different roots. It is therefore recommended to investigate the results of Newton’s method more thoroughly.

For numerical computations, we also need a stopping criterion. We want the method to stop if \( F(x^n) \) is close enough to 0. By this we mean that the method stops when

\[ \| F(x^n) \|_2 < \text{error}, \]  

(5.1.27)

where we take error = 10^{-10}.

Before we can use Newton’s method, we first have to find the Jacobian matrix. Note that Newton’s method typically fails if the function is not differentiable. Our function contains the term \( J^{(1)}(t) \) which contains the characteristic and heaviside function. But since the function \( J^{(1)}(t) \) does not depend on our variable parameters \( T_0^{(2)}, T_1^{(2)} \), these functions will not disturb the differentiability of our function \( F \).

Since our vector function \( F \) is a function from \( \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), our Jacobian matrix will be a 2 \times 2-matrix.

- \( J_{11} \):
  \[
  \frac{\partial F_1}{\partial T_0^{(2)}} = \int_{T_0^{(2)}}^{T_1^{(2)}} \left[ (\beta^{(2)}(t - T_0^{(2)}))^2 - (\beta^{(2)}(t - T_0^{(2)}))' \right] \cdot \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{t-T_0^{(2)}} \beta^{(2)}(s) ds + J^{(1)}(t) \right) \, dt,
  \]  

(5.1.28)

where \( (\beta^{(2)}(t - T_0^{(2)}))' \) is the derivative of \( \beta(\cdot) \) at \( t - T_0^{(2)} \).

- \( J_{12} \):
  \[
  \frac{\partial F_1}{\partial T_1^{(2)}} = \beta^{(2)}(T_1^{(2)} - T_0^{(2)}) \cdot \left( N_{\text{target}}(T_1^{(2)}) - N_0^{(1)} - \int_{T_0^{(2)}}^{T_1^{(2)}} \beta^{(2)}(s) ds + J^{(1)}(T_1^{(2)}) \right).
  \]  

(5.1.29)
5.1. Two traffic light scenario

- \( J_{21} \):

\[
\frac{\partial F_2}{\partial T_0^{(2)}} = (\beta^{(2)}(T_1^{(2)} - T_0^{(2)}))^2(T - T_1^{(2)})
\]

\[
-(\beta^{(2)}(T_1^{(2)} - T_0^{(2)}))' \cdot \int_{T_1^{(2)}}^{T} \left( N_{\text{target}}(t) - N_{0}^{(1)} - \int_{0}^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s)ds + J^{(1)}(t) \right) dt
\]

\[
(5.1.30)
\]

- \( J_{22} \):

\[
\frac{\partial F_2}{\partial T_1^{(2)}} = \beta^{(2)}(T_1^{(2)} - T_0^{(2)}) \left[ \int_{T_1^{(2)}}^{T} -\beta^{(2)}(T_1^{(2)} - T_0^{(2)}) dt \right.
\]

\[
- \left( N_{\text{target}}(T_1^{(2)}) - N_{0}^{(1)} - \int_{0}^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s)ds + J^{(1)}(T_1^{(2)}) \right]
\]

\[
+ \left( \beta^{(2)}(T_1^{(2)} - T_0^{(2)}))' \cdot \int_{T_1^{(2)}}^{T} \left( N_{\text{target}}(t) - N_{0}^{(1)} - \int_{0}^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s)ds + J^{(1)}(t) \right) dt
\]

\[
(5.1.31)
\]

5.1.4 Numerical results

For traffic light 1, we first take the same settings as before:

\[
T_0^{(1)} = 20,
T_1^{(1)} = 50,
T = 60.
\]

(5.1.32)

For traffic light 2, the cycle time will be the same as for cycle 1. As passing function for the cars at traffic light 1 and 2 we take

\[
\beta^{(1)}(t) = 0.6 \tanh(0.3t),
\]

(5.1.33)

similar as before. The target function we keep fixed and equal to

\[
N_{\text{target}}(t) \equiv 10.
\]

(5.1.34)

In the next sections we will display the results if we vary the initial queue length \( N_0^{(1)} \) (5.1.4) and if we vary the traffic light settings for light 1 (5.1.4). It could also be plausible that the passing function of light 2 differs from the passing function of light 1. This can be caused by the fact that one street could be positioned under a slope or that the street is curved right before the intersection. This effect will be investigated in section 5.1.4. But first, one numerical example will be examined in detail.

A numerical example in detail

As mentioned we take \( \beta^{(1)}(t) = \beta^{(2)}(t) \) and as initial queue we choose \( N_0^{(1)} = 6 \). Newton’s method is implemented in Mathematica. For several initial guesses of \( T_0^{(2)}, T_1^{(2)} \) we show the solution of Newton’s method in Table 5.1. The solutions of Newton’s method are the roots of the function \( F_2 \) given in (5.1.22). In other words, these are the critical points of the collage distance.

We are interested in the values of \( T_0^{(2)}, T_1^{(2)} \) for which the collage distance is minimal. So in Table 5.1 also the collage distance is shown for each solution of Newton’s method.
Table 5.1: Results of Newton’s method for several initial guesses

<table>
<thead>
<tr>
<th>Initial guess for ((T_0^{(2)}, T_1^{(2)}))</th>
<th>Solution ((T_0^{(2)}, T_1^{(2)}))</th>
<th>Number of iterations</th>
<th>Collage error</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>((0, 0))</td>
<td>1</td>
<td>97.20</td>
</tr>
<tr>
<td>((0, 60))</td>
<td>((14.03, 60))</td>
<td>4</td>
<td>19.82</td>
</tr>
<tr>
<td>((0, 50))</td>
<td>((14.03, 60))</td>
<td>8</td>
<td>19.82</td>
</tr>
<tr>
<td>((20, 60))</td>
<td>((14.03, 60))</td>
<td>3</td>
<td>19.82</td>
</tr>
<tr>
<td>((20, 50))</td>
<td>((12.45, 49.08))</td>
<td>4</td>
<td>16.48</td>
</tr>
<tr>
<td>((50, 60))</td>
<td>((60, 60))</td>
<td>18</td>
<td>97.20</td>
</tr>
</tbody>
</table>

Note that we are explicitly interested in the case where \(T_1^{(2)} > T_0^{(2)}\). The other way around would make no sense for our model, since we explicitly model \(T_0^{(2)}\) as the time the green phase starts and \(T_1^{(2)}\) the green phase ends. One could argue that it is possible to start with a green phase, then a red phase and end with a green phase again. However, this case will not be considered in this thesis.

If the initial guess is of the form \((T_0^{(2)}, T_1^{(2)})\), where \(T_0^{(2)} = T_1^{(2)}\) then the method stops after 1 iteration and gives these values as critical points. Indeed, all solutions of the form \(T_0^{(2)} = T_1^{(2)}\) are critical points. However, these critical points are global maxima, see also Figure 5.2.

![3D plot of the collage error](image)

Figure 5.2: 3D plot of the collage error.

There are 2 critical points which satisfy \(T_1^{(2)} > T_0^{(2)}\). \((14.03, 60)\) is a critical point for the collage distance, but is no global minima. Furthermore, the collage error is relative large \(> 1\). This can be explained by the fact that there is a large difference between \(N_0^{(1)}\) and \(N_{\text{tar}}\). This means that it takes some time before the approximate queue is equal to the target queue. Because this difference is integrated we obtain the large collage error. But since we are interested in the parameters which minimize the distance, we do not care about the absolute size of the error. Furthermore,
5.1. Two traffic light scenario

in this case Newton’s method leads to no problems. It converges to a critical point for any initial
guess \((T_0^{(2)}, T_1^{(2)}) \in [0, T] \times [0, T]\). Of course, the solutions found by Newton’s method need to
be compared to each other by looking at the collage error for these solutions. Then the set of
solutions that give the smallest collage error leads to the global minimum of the collage error. In
Figure 5.2 we see a 3D plot of the collage error. The collage error is symmetrical around the line
\(T_0^{(2)} = T_1^{(2)}\). Here we also see that all points of the form \((T_0^{(2)}, T_1^{(2)})\) where \(T_0^{(2)} = T_1^{(2)}\) are global
maxima, which means that there are infinitely many critical points. Figure 5.3 shows a contour-
plot of the collage error. The blue color shows the region where the collage error is relatively
small. It is clear that the minimum lies around \((13, 50)\). Note that there is also another minimum
around \((50, 13)\), but this is a minimum we are not interested in since \(T_0^{(2)} > T_1^{(2)}\) in this case.

We can zoom in at the minimum around \((13, 50)\), see Figure 5.4 and conclude that \((14.03, 60)\) are
indeed the values for the parameters \(T_0^{(2)}, T_1^{(2)}\) which minimize the collage distance. Since we
obtained the minimal collage error where the initial guess was equal to the traffic light settings
of intersection 1, we will use this initial guess from now on. It is reasonable to assume that the
settings for traffic light 2 are close to traffic light 1 if the initial queue is not far from \(N_{\text{target}}\).

Varying initial queue length

We will investigate what happens if we vary the initial queue length. We keep all other parame-
ters fixed and equal to the example in the previous section. This means that we will investigate
for several initial queue lengths what the traffic light settings of light 2 will be, when the traffic
light settings of intersection 1 are given. We expect that the green phase of light 2 will increase
as the initial queue length of light 1 will decrease. Since then, there is a larger difference between
the initial queue length \(N_0^{(1)}\) and the target queue length \(N_{\text{max}}\). This leads to the results in Table
5.2.
5.1. Two traffic light scenario

Table 5.2: Results for different initial queue lengths

<table>
<thead>
<tr>
<th>Initial queue $N_0$</th>
<th>Solution $(T_0^2, T_1^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>(23.40, 50.07)</td>
</tr>
<tr>
<td>11</td>
<td>(21.71, 50.05)</td>
</tr>
<tr>
<td>10</td>
<td>(20, 50)</td>
</tr>
<tr>
<td>9</td>
<td>(18.24, 49.91)</td>
</tr>
<tr>
<td>8</td>
<td>(16.41, 49.73)</td>
</tr>
<tr>
<td>7</td>
<td>(14.48, 49.46)</td>
</tr>
<tr>
<td>6</td>
<td>(12.45, 49.08)</td>
</tr>
<tr>
<td>5</td>
<td>(10.31, 48.55)</td>
</tr>
<tr>
<td>4</td>
<td>(8.04, 47.85)</td>
</tr>
<tr>
<td>3</td>
<td>(5.64, 46.95)</td>
</tr>
<tr>
<td>2</td>
<td>(3.07, 45.78)</td>
</tr>
<tr>
<td>1</td>
<td>(0.29, 44.23)</td>
</tr>
</tbody>
</table>

If $N_0^{(1)} = 10$, then $N_0^{(1)} = N_{\text{target}}$ and therefore the optimal settings of light 2 are equal to the settings of light 1. Furthermore, if the initial queue is smaller than the target queue, it is better to shift the green phase of intersection 2 to an earlier moment than the green phase of intersection 1. Furthermore, the duration of the green phase increases as the initial queue length decreases as expected.

Varying settings of traffic light 1

Now we will investigate what happens for the traffic light settings of light 2, when we vary the settings for light 1. We will keep $N_0^{(1)} = 6$ and $N_{\text{target}} = 10$ fixed. Of course, if we reduce the duration of the green phase of light 1, we expect that the green phase of light 2 will also decrease. The results for several traffic light settings are shown in Table 5.3. The first column denotes the settings for traffic light 1.

Table 5.3: Results for different settings of light 1

<table>
<thead>
<tr>
<th>Settings $(T_0^{(1)}, T_1^{(1)})$</th>
<th>$\Delta^{(1)} := T_1^{(1)} - T_0^{(1)}$</th>
<th>Solution $(T_0^{(2)}, T_1^{(2)})$</th>
<th>$\Delta^{(2)} := T_1^{(2)} - T_0^{(2)}$</th>
<th>$\Delta^{(2)} - \Delta^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(20,55)</td>
<td>35</td>
<td>(12.59, 54.20)</td>
<td>41.61</td>
<td>6.61</td>
</tr>
<tr>
<td>(20,50)</td>
<td>30</td>
<td>(12.45, 49.08)</td>
<td>36.63</td>
<td>6.63</td>
</tr>
<tr>
<td>(20,45)</td>
<td>25</td>
<td>(12.24, 43.87)</td>
<td>31.63</td>
<td>6.63</td>
</tr>
<tr>
<td>(20,40)</td>
<td>20</td>
<td>(11.91, 38.53)</td>
<td>26.62</td>
<td>6.62</td>
</tr>
<tr>
<td>(20,35)</td>
<td>15</td>
<td>(11.25, 32.83)</td>
<td>21.58</td>
<td>6.58</td>
</tr>
<tr>
<td>(20,30)</td>
<td>10</td>
<td>(8.44, 24.73)</td>
<td>16.29</td>
<td>6.29</td>
</tr>
</tbody>
</table>

In the last column of 5.3 are the differences in green phases shown. Note that the differences in length of the green phases is approximately the same.
Varying passing function

Finally, we will investigate the effect of the passing function. Until now, we took the same passing function for every intersection. However, it is reasonable to assume that different intersections have different passing functions due to curves, slopes, etc. We will vary the passing function of light 2 and keep the passing function of light 1 fixed to $\beta^{(1)}(t) = 0.6 \tanh(0.3t)$. This leads to the results in Figure 5.5. We assume the passing function of light 2 is of the form $\beta^{(2)}(t) = a \tanh(bt)$ and vary the coefficients $a, b$. The coefficient $a$ defines the maximum passing rate while the coefficient $b$ is responsible for the speed at which this maximum rate is reached. The larger $b$, the faster the passing function is at the constant maximum arrival rate $a$.

Figure 5.5: Carpet plot. Different values for $a, b$ in the passing function lead to different optimal traffic light settings $T_0^{(2)}$ (x-axis) and $T_1^{(2)}$ (y-axis).

Note that the differences in the solutions $(T_0^{(2)}, T_1^{(2)})$ do not vary much if the coefficient $a$ is varied. However, the solutions do show a significant difference if the maximum passing rate, defined by coefficient $b$, is varied. Therefore it is important that the (averaged) passing function is known in order to implement these traffic light settings at an intersection.
5.2 Three traffic light scenario

5.2.1 Traffic scenario

Let us consider the scenario shown in Figure 5.6. This is an extension of the scenario treated in 5.1.

Note that traffic lights 2 and 3 influence the queue at light 1. This problem looks similar to the one in the previous section, however, there is a difference. Previously, we assumed we could split the cycle into different periods. In each period, only two traffic lights could become green, e.g. first light 2a and 1, then 2b and 1, etc. Now, we assume that there is only one period, in which light 1, 2 and 3 become green for a certain amount of time. We need an extra constraint on the traffic lights in order to prevent accidents: lights 2 and 3 cannot both be green at the same time.

We will not regard the other traffic light at the intersection. This direction will automatically get green light when the other two directions are both red and have had their green phase in the cycle. We again assume a heavy traffic scenario for which we will also choose the fixed target function

\[ N_{\text{target}}(t) = N_{\text{max}}. \]  

Here \( N_{\text{max}} \) is not necessarily the maximum road capacity. It is better to let \( N_{\text{max}} \) be a little smaller than the road capacity due to the fact that not all cars are equally long and the distance between the cars is not the same between all cars. Another assumption is that the cycle times of lights 2 and 3 are equal to the cycle time of light 1, namely \( T \).

The question is now:

*How should traffic light 2 and 3 be set such that the queue at traffic light 1 is as close as possible to a given target queue?*

For the queue at traffic light 1 we have the same model as previously

\[
\begin{cases}
\frac{dN^{(1)}}{dt}(t) = \alpha^{(1)}(t) - \beta^{(1)}(t) \cdot (t - T_0^{(1)}) \cdot \chi_{[T_0^{(1)}, T_1^{(1)}]}(t), & \text{for } t \in [0, T] \\
N^{(1)}(0) = N_0^{(1)}
\end{cases}
\]  

(5.2.2)
5.2. Three traffic light scenario

Note the superscript which indicates that it concerns traffic light 1.

The key observation is that the arrival function $\alpha^{(1)}(t)$ depends on:

- The settings of traffic light 2: $T_0^{(2)}, T_1^{(2)}$.
- The passing function of light 2: $\beta^{(2)}(t)$.
- The settings of traffic light 3: $T_0^{(3)}, T_1^{(3)}$.
- The passing function of light 3: $\beta^{(3)}(t)$.

Due to safety reasons, the intervals $[T_0^{(2)}, T_1^{(2)}]$ and $[T_0^{(3)}, T_1^{(3)}]$ should satisfy

$$[T_0^{(2)}, T_1^{(2)}] \cap [T_0^{(3)}, T_1^{(3)}] = \emptyset. \quad (5.2.3)$$

In order to stay away from constraint optimization, we assume that there is a safety parameter $\tau$ which equals the time between light 2 switching to red and light 3 switching to green. This means that $T_0^{(3)} \geq T_1^{(2)} + \tau$. For simplification of our model, we choose $T_0^{(3)} = T_1^{(2)} + \tau$. So this means that the arrival function of queue 1 is equal to

$$\alpha^{(1)}(t) = \beta^{(2)}(t - T_0^{(2)}) \cdot \chi_{[T_0^{(2)}, T_1^{(2)}]}(t) + \beta^{(3)}(t - T_1^{(2)} - \tau) \cdot \chi_{[T_1^{(2)} + \tau, T_0^{(3)}]}(t). \quad (5.2.4)$$

Since we are interested in the traffic light settings of light 2 and 3, the only unknown parameters are $T_0^{(2)}, T_1^{(2)}, T_0^{(3)}$. We are interested in the values of these parameters such that the queue at light 1 is as close as possible to $N_{\text{target}}(t)$. This can be solved similar to Section 5.1.

5.2.2 Collage method

The collage error is then given by

$$\Delta^2(T_0^{(2)}, T_1^{(2)}, T_0^{(3)}) = \int_0^T \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^t \beta^{(2)}(s - T_0^{(2)}) \cdot \chi_{[T_0^{(2)}, T_1^{(2)}]}(s) ds - \int_0^t \beta^{(3)}(s - T_1^{(2)} - \tau) \cdot \chi_{[T_1^{(2)} + \tau, T_0^{(3)}]}(s) ds \right)^2 dt, \quad (5.2.5)$$

where for convenience, we define the integral terms as

$$J^{(1)}(t) = \int_0^{t-T_0^{(1)}} \beta^{(1)}(s) ds \cdot \chi_{[T_0^{(1)}, T_1^{(1)}]}(t) + \int_0^{t-T_0^{(1)}} \beta^{(1)}(s) ds \cdot H_{T_1^{(1)}}(t),$$

$$J^{(2)}(t, T_0^{(2)}, T_1^{(2)}, T_0^{(3)}) = \int_0^{t-T_0^{(2)}} \beta^{(2)}(s) ds \cdot \chi_{[T_0^{(2)}, T_1^{(2)}]}(t) + \int_0^{t-T_0^{(2)}} \beta^{(2)}(s) ds \cdot H_{T_1^{(2)}}(t),$$

$$J^{(3)}(t, T_0^{(2)}, T_1^{(2)}, T_0^{(3)}) = \int_0^{t-T_0^{(3)} - \tau} \beta^{(3)}(s) ds \cdot \chi_{[T_1^{(2)} + \tau, T_0^{(3)}]}(t) + \int_0^{t-T_0^{(3)} - \tau} \beta^{(3)}(s) ds \cdot H_{T_1^{(3)}}(t). \quad (5.2.6)$$

We can write the collage error $\Delta^2(T_0^{(2)}, T_1^{(2)}, T_0^{(3)})$ as

$$\int_0^T \left( N_{\text{target}}(t) - N_0^{(1)} - J^{(2)}(t, T_0^{(2)}, T_1^{(2)}, T_0^{(3)}) - J^{(3)}(t, T_0^{(2)}, T_1^{(2)}, T_0^{(3)}) + J^{(1)}(t) \right)^2 dt. \quad (5.2.7)$$

Similarly to Section 5.1, we first differentiate (5.2.7) with respect to the three variable parameters $T_0^{(2)}, T_1^{(2)}, T_0^{(3)}$. We can divide a cycle in 5 phases:
5.2. Three traffic light scenario

- \([0, T_0^{(2)})\): both light 2 and 3 are red (possibly the remaining direction has green light, but we do not consider this);
- \([T_0^{(2)}, T_1^{(2)})\): light 2 is green, light 3 is red;
- \([T_1^{(2)}, T_1^{(2)} + \tau)\): light 2 and 3 are both red for safety reasons;
- \([T_1^{(2)} + \tau, T_1^{(3)})\): light 2 is red, light 3 is green;
- \([T_1^{(3)}, T]\): both light 2 and 3 are red (possibly the remaining direction has green light, but we do not consider this).

Due to the characteristic and heaviside functions in (5.2.7), we can split the collage error in 5 separate integrals. This is done in order to avoid the dirac-delta functions which we would obtain if we differentiate (5.2.7) directly. In Appendix E.1 the derivatives are presented.

Since we want these derivatives to be equal to 0, we obtain a system of the form

\[
\begin{pmatrix}
F_1(T_0^{(2)}, T_1^{(2)}, T_1^{(3)}) \\
F_2(T_0^{(2)}, T_1^{(2)}, T_1^{(3)}) \\
F_3(T_0^{(2)}, T_1^{(2)}, T_1^{(3)})
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\tag{5.2.8}
\]

This nonlinear system we will solve by using Newton’s method. For Newton’s method we need the Jacobian matrix. Because we have \(E = (F_1, F_2, F_3)\) and three variable parameters \(T_0^{(2)}, T_1^{(2)}, T_1^{(3)}\), the Jacobian will be a \(3 \times 3\)-matrix. This matrix is presented in Appendix E.2.

Instead of computing numerical examples for this scenario, we will apply this method to a real traffic situation.
5.3 Application to a real traffic situation: Beek

In Beek (Limburg, The Netherlands) lie two intersections on the Maastrichterlaan, see Figure 5.7. The Maastrichterlaan is the main road through Beek. One intersection intersects the Maastrichterlaan with the Raadhuisstraat and the other intersects the Maastrichterlaan with the Stationsstraat. These two intersections are sketched in Figure 5.8.

Figure 5.7: Two traffic intersections in Beek (Limburg, The Netherlands).

Figure 5.8: Sketch of the traffic situation in Beek. Traffic lights 2 and 3 affect the queue at light 1.
5.3. Application to a real traffic situation: Beek

The situation is symmetrical. We see that the two intersections lie close to each other. Clearly, the queue at light 1 (the blue rectangle in Figure 5.8) should not become too large, since then it would block the road and the intersection previous to light 1. The length of this queue can be controlled by the traffic lights 2 and 3. Note that the arrival function of the queue at light 1 is not simply the sum of the passing functions of traffic lights 2 and 3, since it is possible that cars leave light 2/3 and, instead of joining queue 1, head straight forward. We could model this by adding extra coefficients $\lambda_2, \lambda_3$ to the passing functions of light 2 and 3 which model the fraction of cars that do join the queue of light 1. Then $0 \leq \lambda_2, \lambda_3 \leq 1$. Besides this, we could immediately apply the framework we derived in the previous section and calculate the optimal settings for light 2 and 3, given the settings of light 1.

Given the length of the street and the average length of a car plus distance between cars, we pose that the maximum capacity of the road before light 1 is equal to 6. So

$$N_{\text{max}} = 6. \quad (5.3.1)$$

We assume a heavy traffic scenario, so we want as many cars as possible to enter the queue, so for the target function we take

$$N_{\text{target}}(t) \equiv N_{\text{max}}. \quad (5.3.2)$$

For the settings of traffic light 1 we take the values

$$T_0^{(1)} = 20, \quad T_1^{(1)} = 50, \quad T = 60, \quad T = 5. \quad (5.3.3)$$

Since we do not know the exact fraction of cars at light 2 and 3 which actually join queue 1, we compute the optimal settings for light 2 and 3 for different ratios $\lambda_2, \lambda_3$. We also vary the initial queue length at light 1, $N_0^{(1)}$. The results are shown in Table 5.4.

<table>
<thead>
<tr>
<th>$N_0^{(1)}$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>Solution $(T_0^{(2)}, T_1^{(2)}, T_1^{(3)})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>(15, 15, 50) or (20, 50, 55)</td>
</tr>
<tr>
<td>6</td>
<td>0.8</td>
<td>1</td>
<td>(15, 15, 50)</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0.8</td>
<td>(20, 50, 55)</td>
</tr>
<tr>
<td>6</td>
<td>0.8</td>
<td>0.8</td>
<td>(16.93, 58.21, 60) or (12.01, 12.01, 60.)</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>(7.26, 14.51, 49.52)</td>
</tr>
<tr>
<td>3</td>
<td>0.8</td>
<td>1</td>
<td>(5.89, 14.98, 49.53)</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.8</td>
<td>(3.52, 10.22, 52.83)</td>
</tr>
<tr>
<td>3</td>
<td>0.8</td>
<td>0.8</td>
<td>(2.37, 10.72, 52.83)</td>
</tr>
</tbody>
</table>

If the initial queue already equals the maximum/desired queue length and $\lambda_2 = \lambda_3 = 1$, there are two optimal settings. The first one is $(T_0^{(2)}, T_1^{(2)}, T_1^{(3)}) = (15, 15, 50)$. For this setting, light 2 will not get green, since $T_1^{(2)} - T_0^{(2)} = 15 - 15 = 0$. While light 2 will get green light for $T_1^{(2)} - (T_2^{(2)} + \tau) = 50 - (15 + 5) = 30$ at exactly the moment light 1 has the same amount of green time. In the other optimal solution, $(T_0^{(2)}, T_1^{(2)}, T_1^{(3)}) = (20, 50, 55)$, it is the other way around. There we have that light 2 is green from $t = T_0^{(2)} = 20$ to $t = T_1^{(2)} = 50$ and light 3 does not get green light, since $T_1^{(3)} - (T_2^{(2)} + \tau) = 55 - (50 + 5) = 0$. A way to resolve this is to use two cycles. The first cycle light 1 and 2 form a pair, while in the next cycle light 1 and 3 form a pair. This is
5.3. Application to a real traffic situation: Beek

similar to the case in Section 5.1, where we also made a distinction between different cycles. The
behavior is similar for any other case where $\lambda_2 = \lambda_3$.

In fact, all the numerical results show that if the initial queue $N_0^{(1)}$ already satisfies the desired
queue length, then it is optimal to let only one light get green while the other light should remain
red for the entire cycle. So in this case it would be recommended to make two separate cycles,
one for light 2 and one for light 3, because else there would be a direction which would not get
any green light.

In the case where the initial queue is less then the desired queue we also see an interesting be-
behavior. Because of the ordering (first light 2 and then light 3) we see that light 2 makes sure the
queue is complemented to the desired amount of cars, while light 3 makes sure the amount of
cars that leave during the green light of light 1 is complemented again.

A nice way to resolve is to fix the parameter $T_1^{(2)}$ and set it equal to e.g. 30. By doing this
we demand that both light 2 and 3 get a more or less equal amount of green light. By this we
mean that the balance between the green phase of light 2 and light 3 is more in equilibrium. If
we do this, and make a contourplot of the collage distance where the other two parameters are
$T_0^{(2)}, T_1^{(3)}$, we obtain Figures 5.9 to 5.12.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig5_9.png}
\includegraphics[width=0.4\textwidth]{fig5_10.png}
\caption{Figure 5.9: Contourplot of the collage error.}
\caption{Figure 5.10: Contourplot of the collage error (zoomed).}
\end{figure}
5.3. Application to a real traffic situation: Beek

Since we are only interested in the values where \( T_{1}^{(3)} \geq T_{0}^{(2)} \), we see that the minimal collage error is obtained for the values \((T_{0}^{(2)}, T_{1}^{(3)}) \approx (10.45, 52.7)\). This gives us another way of finding the traffic light settings which keep the queue length of light 1 as close as possible to a desired length.
Chapter 6

Conclusions and future research

In this thesis we studied a few basic inverse problems in urban traffic flow. They are all meant to prevent gridlocks at intersections. We present here a summary of the results we obtained. Furthermore, we provide a few suggestions for eventual future research.

6.1 Summary of results

We introduced a simple model for the evolution of the length of a queue of cars at traffic lights. We designed a way to control the length of the queues by controlling the way cars arrive at the traffic light. This was done via the introduction of inverse problems. Since our model consisted of a simple differential equation, we were able to apply the collage method. As far as we know, this is the first application of the collage method to urban traffic flow. After an extensive literature study of the collage method, we introduced several inverse problems in urban traffic flow.

Before one can draw any conclusions on how cars should enter an intersection, one needs to know how cars leave that intersection, the first topic we studied was the leaving of cars at an intersection (passing). We obtained data of the passing times of cars at a traffic light. These passing times were used to derive data of queue lengths. We showed that we could use the framework of the collage method and solve the inverse problem for the passing function. Here we used the data of the queue lengths in order to solve an inverse problem. As in other applications of the collage method [21], one usually makes use of data in order to find the parameters for which the solution fits best to the data. However, in the next chapter we did not have any data for the optimal queue lengths. Hence, we introduced the notion of a target queue $N_{\text{target}}$ which would be our desired queue length. We were looking for the arrival function for which the queue was equal (as close as possible) to the target queue. This throws another look on urban traffic flow optimization. Since our model contains discontinuous functions like the characteristic function and the heaviside function, we tried several classes of functions in order to apply the collage method. It turned out that the piecewise constant functions are the most appropriate candidates to represent the arrival functions.

Because urban traffic is often a large network of intersections, the arrival of cars at an intersection is connected to the passing of cars at a previous intersection. In fact, the traffic light settings of the previous intersection define the arrival of cars at the next intersection. This observation was used in the last chapter. We adapted our model slightly and applied again the collage method. Because of the dependence of the characteristic function on the unknown traffic light parameters, the collage method resulted in a nonlinear system. We were able to solve this system by a multidimensional Newton’s method. For a list of numerical examples we obtained optimal traffic light settings such that the queue length was as close as possible to a desired queue length.
6.2 Discussion

Furthermore we were able to apply the framework to a real traffic situation in Beek (Limburg, The Netherlands). By solving the inverse problem for this situation we came to several recommendations for the traffic light settings in the case of heavy traffic.

6.2 Discussion

Although we did an attempt to formulate a model which can be used to prevent gridlocks, there is always room for more research. A few possible research directions are given below.

- This thesis provides a model in which for several problems, the optimal arrival functions are given. However, before these methods are ready for real life application, some more work is needed to be done. The arrival rates which were obtained, can be translated to recommended velocities of vehicles. It is not yet possible to let the vehicles drive at these recommended velocities. For this, some global network is needed to communicate between cars. Furthermore, it is better if cars drive autonomously. Because then the cars would really drive at the recommended velocity. If cars would still be controlled by human drivers, it is impossible to let all cars drive according to the recommended velocities, since there are always drivers which do not follow these recommendations. A good topic for further research could be to think about how the methods and results obtained in this thesis can be implemented in real traffic situations.

- The model we presented could be extended with randomness or stochasticity to make the equations fit more to the real fluctuations in traffic flow. Furthermore, the traffic lights could be equipped with sensors in order to fit the current traffic situation.

- Furthermore, our model fits to one intersection. However, in urban traffic flow, usually a network of intersections is considered. Our model could be extended to multiple intersections connecting each other. For this network the collage method could still be used, except that the dimension of the system would increase. In our model we had only one target function. A network of $m$ intersections would need $m$ target functions, yielding $m$ arrival functions. In other words a vector of target functions is needed

$$
\begin{pmatrix}
N_{\text{target}}^{(1)} \\
N_{\text{target}}^{(2)} \\
\vdots \\
N_{\text{target}}^{(m)}
\end{pmatrix}.
$$

- The model could be extended in order to detect gridlocks. For this, the arrival and passing rates should be expressed in terms of Dirac measures instead of continuous functions. The metric which one needs to minimize with respect to the unknown arrival rates will be the Wasserstein metric. In this framework, one can identify the arrival rates which would lead to a gridlock.

Despite of the fact that the implementation of the results and methods obtained in this thesis would require more time and research, we believe that this thesis adds a promising new direction in the prevention of gridlocks in urban traffic flow.
Bibliography


Appendix A

Derivation of model equation

Let the discrete equations of the queue length be given by

\[
N(t) = \begin{cases} 
N(0) + \int_0^t \alpha(s) \, ds & \text{for } 0 \leq t < T_0 \\
N(T_0) + \int_{T_0}^t \alpha(s) \, ds - \int_{T_0}^t \beta(s - T_0) \, ds & \text{for } T_0 \leq t < T_1 \\
N(T_1) + \int_{T_1}^t \alpha(s) \, ds & \text{for } T_1 \leq t < T.
\end{cases}
\] (A.0.1)

Then there are three cases:

- **0 ≤ t < T₀:** For \( \Delta t \) small, we have that

  \[
  N(t) = N(0) + \int_0^t \alpha(s) \, ds,
  \]

  \[
  N(t + \Delta t) = N(0) + \int_0^{t+\Delta t} \alpha(s) \, ds.
  \] (A.0.2)

  Subtracting the first one from the second one yields

  \[
  N(t + \Delta t) - N(t) = \int_0^{t+\Delta t} \alpha(s) \, ds - \int_0^t \alpha(s) \, ds.
  \] (A.0.3)

  The two integrals in (A.0.3) we can write as one integral \( \int_t^{t+\Delta t} \alpha(s) \, ds \). Dividing by small \( \Delta t \) and taking the limit \( \lim_{\Delta t \to 0} \) yields the differential equation

  \[
  \frac{dN}{dt}(t) = \alpha(t), \quad \text{for } t \in [0, T_0].
  \] (A.0.4)

- **T₀ ≤ t < T₁:** For \( \Delta t \) small, we have that

  \[
  N(t) = N(T_0) + \int_{T_0}^t \alpha(s) \, ds - \int_{T_0}^t \beta(s - T_0) \, ds,
  \]

  \[
  N(t + \Delta t) = N(T_0) + \int_{T_0}^{t+\Delta t} \alpha(s) \, ds - \int_{T_0}^{t+\Delta t} \beta(s - T_0) \, ds.
  \] (A.0.5)
As before, subtracting, dividing by $\Delta t$ and taking the limit yields

$$
\frac{dN}{dt}(t) = \alpha(t) - \beta(t - T_0), \quad \text{for } t \in [T_0, T_1].
$$

(A.0.6)

- $T_1 \leq t < T$: For $\Delta t$ small, we have that

$$
N(t) = N(T_1) + \int_{T_1}^{t} \alpha(s)ds,
$$

$$
N(t + \Delta t) = N(T_1) + \int_{T_1}^{t+\Delta t} \alpha(s)ds.
$$

(A.0.7)

Following the same steps as before we obtain

$$
\frac{dN}{dt}(t) = \alpha(t), \quad \text{for } t \in [T_1, T].
$$

(A.0.8)

Combining equation (A.0.4), (A.0.6) and (A.0.8) together with the definition of the characteristic function

$$
\chi_{[T_0, T_1]}(t) = \begin{cases} 
1 & \text{for } t \in [T_0, T_1), \\
0 & \text{for } t \notin [T_0, T_1).
\end{cases}
$$

(A.0.9)

yields the differential equation

$$
\begin{cases}
\frac{dN}{dt}(t) = \alpha(t) - \beta(t - T_0) \cdot \chi_{[T_0, T_1]}(t), & \text{for } t \in [0, T] \\
N(0) = N_0.
\end{cases}
$$

(A.0.10)
Appendix B

Traffic data: passing times

The table with data of the passing times is shown on the next page.
Table B.1: Passing times

<table>
<thead>
<tr>
<th>Sample</th>
<th>Car 1</th>
<th>Car 2</th>
<th>Car 3</th>
<th>Car 4</th>
<th>Car 5</th>
<th>Car 6</th>
<th>Car 7</th>
<th>Car 8</th>
<th>Car 9</th>
<th>Car 10</th>
<th>Car 11</th>
<th>Car 12</th>
<th>Car 13</th>
<th>Car 14</th>
<th>Car 15</th>
<th>Car 16</th>
<th>Car 17</th>
</tr>
</thead>
</table>

This table contains the departure times (in seconds) of cars at the intersection of the Onze Lieve Vrouwestraat and the John F. Kennedylaan (Eindhoven) during rush hour on May 21st and 22nd 2014. \( t = 0 \) is the moment the light switched to green. The data consists of 22 green phases. Note that the amount of cars that left the queue during a green phase is not the same for all samples. In sample 7 a total of 17 cars left the queue during one green phase, while in sample 8 only 11 cars were able to leave the queue.
Appendix C

Passing function: linear system

C.1 Case of piecewise constant functions

Let \( \beta(t) \) be an unknown, piecewise constant function and \( 0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T \) a partition. Then the square of the collage distance is given by

\[
\Delta = \int_0^T \left( N_{\text{data}}(t) - N_0 + \sum_{k=1}^{n} \beta_k \cdot \chi_{[t_{k-1}, t_k)}(s) \right)^2 dt. \tag{C.1.1}
\]

We can rewrite (C.1.1) as

\[
\int_0^T \left( N_{\text{data}}(t) - N_0 + \sum_{k=1}^{n} \beta_k \int_0^t \chi_{[t_{k-1}, t_k)}(s) \right)^2 dt. \tag{C.1.2}
\]

Let us assume that the partition of the interval is equidistant and the distance between \( t_{k-1} \) and \( t_k \) is equal to \( \Delta t \). Let us define

\[
J_k(t) := \int_0^t \chi_{[t_{k-1}, t_k)}(s) \tag{C.1.3}
\]

Since the characteristic function \( \chi_{[a,b]}(\cdot) \) is equal to 1 in the interval \([a, b]\) and 0 elsewhere, we have that

\[
J_k(t) = (t - t_{k-1}) \cdot \chi_{(t_{k-1}, t_k)}(t) + \Delta t \cdot H_{t_k}(t). \tag{C.1.4}
\]

We can find the minimizers of (C.1.1) by taking the derivative with respect to the unknown coefficients \( \beta_k \). This will give

\[
\frac{\partial \Delta^2}{\partial \beta_m} = \int_0^T 2 \left( N_{\text{data}}(t) - N_0 + \sum_{k=1}^{n} \beta_k J(t) \right) J_m(t) dt = 0, \tag{C.1.5}
\]

for \( m = 1, 2, \ldots, n \). (C.1.5) has to be equal to 0 since we want to find the minimum. Rewriting (C.1.5) yields

\[
- \sum_{k=1}^{n} \beta_k \int_0^T J_k(t)J_m(t) dt = \int_0^T (N_{\text{data}}(t) - N_0) J_m(t) dt \tag{C.1.6}
\]

Now let us define

\[
A_{m,k} := \int_0^T J_k(t)J_m(t) dt, \tag{C.1.7}
\]

\[
b_m := \int_0^T (N_{\text{data}}(t) - N_0) J_m(t) dt. \tag{C.1.7}
\]
Then we end up with a linear system in the unknown coefficients $\beta_k$

$$
\begin{pmatrix}
\int_0^T J_1(t) J_1(t) dt & \int_0^T J_2(t) J_1(t) dt & \cdots & \int_0^T J_n(t) J_1(t) dt \\
\int_0^T J_1(t) J_2(t) dt & \int_0^T J_2(t) J_2(t) dt & \cdots & \int_0^T J_n(t) J_2(t) dt \\
\vdots & \vdots & \ddots & \vdots \\
\int_0^T J_1(t) J_n(t) dt & \int_0^T J_2(t) J_n(t) dt & \cdots & \int_0^T J_n(t) J_n(t) dt
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix}
= 
\begin{pmatrix}
\int_0^T (N_{\text{data}}(t) - N_0) J_1(t) dt \\
\int_0^T (N_{\text{data}}(t) - N_0) J_2(t) dt \\
\vdots \\
\int_0^T (N_{\text{data}}(t) - N_0) J_n(t) dt
\end{pmatrix}
$$

Numerically solving this linear system has a high computational cost, since there are $O(n^2 + n)$ integrals that have to be computed. In order to speed up the computations, the integrals in the matrix $A$ can be computed analytically. Note that

$$
A_{k,m} = \int_0^T (t - t_{k-1})(t - t_{m-1}) \cdot \chi_{[t_{k-1}, t_k)}(t) \cdot \chi_{[t_{m-1}, t_m)}(t) dt
+ \int_0^T (t - t_{k-1}) \Delta t \cdot \chi_{[t_{k-1}, t_k)}(t) \cdot H_{t_{m}}(t) dt
+ \int_0^T (t - t_{m-1}) \Delta t \cdot \chi_{[t_{m-1}, t_m)}(t) \cdot H_{t_{k}}(t) dt
+ \int_0^T (\Delta t)^2 \cdot H_{t_{k}}(t) \cdot H_{t_{m}}(t) dt.
$$

Because of the characteristic functions and heaviside functions, several terms are 0. We can distinguish three cases:

- $k = m$:

$$
A_{k,m} = \int_{t_{k-1}}^{t_k} (t - t_{k-1})^2 dt + \int_{t_k}^{T} (\Delta t)^2 dt = \frac{1}{3} (\Delta t)^3 + (\Delta t)^2 (T - t_k).
$$

- $k < m$:

$$
A_{k,m} = \int_{t_{m-1}}^{t_m} (t - t_{k-1}) \Delta t dt + \int_{t_m}^{T} (\Delta t)^2 dt = \frac{1}{2} (\Delta t)^3 + (\Delta t)^2 (T - t_m).
$$

- $k > m$: This case is equal to $k < m$, except that the indices are swapped. So

$$
A_{k,m} = \frac{1}{2} (\Delta t)^3 + (\Delta t)^2 (T - t_k).
$$

### C.2 Case of linear splines

Let $\beta(t)$ be an unknown, linear spline and $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$ a partition. Then the square of the collage distance is given by

$$
\Delta = \int_0^T \left(N_{\text{data}}(t) - N_0 + \sum_{k=1}^{n} \beta_k(s) \cdot \chi_{[t_{k-1}, t_k)}(s) ds\right)^2 dt.
$$
C.2. Case of linear splines

First we look at the second integral term.

\[
\int_0^T \beta(s) ds = \int_0^t \sum_{k=1}^n \beta_k(s) \cdot \chi_{[t_{k-1}, t_k]}(s) ds
\]

\[
= \int_0^t \sum_{k=1}^n \beta_{k-1} \left( 1 - \frac{s-t_{k-1}}{\Delta t} \right) + \beta_k \left( \frac{s-t_{k-1}}{\Delta t} \right) \cdot \chi_{[t_{k-1}, t_k]}(s) ds
\]

\[
= \sum_{k=1}^n \int_0^t \beta_{k-1} \left( 1 - \frac{s-t_{k-1}}{\Delta t} \right) \chi_{[t_{k-1}, t_k]}(s) ds + \int_0^t \beta_k \left( \frac{s-t_{k-1}}{\Delta t} \right) \chi_{[t_{k-1}, t_k]}(s) ds
\]

\[
= \sum_{k=1}^n \beta_{k-1} I_{1,k}(t) + \sum_{k=1}^n \beta_k I_{2,k}(t)
\]

Then the two terms \( I_{1,k}(t), I_{2,k}(t) \) can be computed analytically.

\[
I_{1,k}(t) = \int_0^t \left( 1 - \frac{s-t_{k-1}}{\Delta t} \right) \chi_{[t_{k-1}, t_k]}(s) ds
\]

\[
= \int_{t_{k-1}}^t \left( 1 - \frac{s-t_{k-1}}{\Delta t} \right) ds \cdot \chi_{[t_{k-1}, t_k]}(t) + \int_{t_{k-1}}^t \left( 1 - \frac{s-t_{k-1}}{\Delta t} \right) ds \cdot H_{t_k}(t)
\]

\[
= \left( t - t_{k-1} - \frac{t^2 - t_{k-1}^2 - 2t \cdot t_k - 1}{2\Delta t} \right) \chi_{[t_{k-1}, t_k]}(t) + \left( \Delta t - \frac{t_k^2 - t_{k-1}^2 - 2t_k \cdot t_{k-1}}{2\Delta t} \right) H_{t_k}(t)
\]

\[
= \left( t - t_{k-1} - \frac{(t-t_{k-1})^2}{2\Delta t} \right) \chi_{[t_{k-1}, t_k]}(t) + \left( \frac{\Delta t}{2} \right) H_{t_k}(t).
\]

(C.2.3)

Similar for \( I_{2,k}(t) \) we obtain

\[
I_{2,k}(t) = \left( \frac{(t-t_{k-1})^2}{2\Delta t} \right) \chi_{[t_{k-1}, t_k]}(t) + \left( \frac{\Delta t}{2} \right) H_{t_k}(t).
\]

(C.2.4)

Rearranging the sums in C.2.2 yields

\[
\sum_{k=1}^n \beta_{k-1} I_{1,k}(t) + \sum_{k=1}^n \beta_k I_{2,k}(t) = \sum_{k=0}^{n-1} \beta_k I_{1,k+1}(t) + \sum_{k=1}^n \beta_k I_{2,k}(t)
\]

\[
= \beta_0 I_{1,1}(t) + \sum_{k=1}^{n-1} \beta_k \left( I_{1,k+1}(t) + I_{2,k}(t) \right) + \beta_n I_{2,n}(t).
\]

(C.2.5)

We can find the minimizers of (C.2.1) by taking the derivative with respect to the unknown coefficients \( \beta_j, j = 0, 1, ..., n \). We can distinguish three cases:
C.2. Case of linear splines

- $j = 0$: Taking the derivative of (C.2.1) with respect to $\beta_0$ and rearranging the terms yields
  \[
  -\beta_0 \int_0^t I_{1,1}(t)I_{1,1}(t)dt - \sum_{k=1}^{n-1} \beta_k \int_0^T (I_{1,k+1}(t) + I_{2,k}(t)) I_{1,1}(t) dt - \beta_n \int_0^t I_{2,n}(t)I_{1,1}(t)dt
  \]
  \[
  = \int_0^T (N_{\text{data}}(t) - N_0) I_{1,1}(t)dt.
  \]
  (C.2.6)

- $j = 1, 2, .., n - 1$: Analogously we find
  \[
  -\beta_0 \int_0^t I_{1,1}(t) (I_{1,j+1}(t) + I_{2,j}(t))dt - \sum_{k=1}^{n-1} \beta_k \int_0^T (I_{1,k+1}(t) + I_{2,k}(t))(I_{1,j+1}(t) + I_{2,j}(t))dt
  \]
  \[
  = -\beta_n \int_0^t I_{2,n}(t)(I_{1,j+1}(t) + I_{2,j}(t))dt \int_0^T (N_{\text{data}}(t) - N_0) (I_{1,j+1}(t) + I_{2,j}(t))dt.
  \]
  (C.2.7)

- $j = n$: Analogously we find
  \[
  -\beta_0 \int_0^t I_{1,1}(t)I_{2,n}(t)dt - \sum_{k=1}^{n-1} \beta_k \int_0^T (I_{1,k+1}(t) + I_{2,k}(t))I_{2,n}(t) dt - \beta_n \int_0^t I_{2,n}(t)I_{2,n}(t)dt
  \]
  \[
  = \int_0^T (N_{\text{data}}(t) - N_0) I_{2,n}(t)dt.
  \]
  (C.2.8)

Then the linear system $-A\beta = b$ is shown on the next page.
\[
\begin{pmatrix}
\int_a^T (I_{1,1}(t))^2 \, dt & \int_a^T (I_{1,2}(t) + I_{2,1}(t))I_{1,1}(t) \, dt & \cdots & \int_a^T (I_{1,n}(t) + I_{2,n-1}(t))I_{1,1}(t) \, dt & \int_a^T I_{2,n}(t)I_{1,1}(t) \, dt \\
\int_a^T I_{1,1}(t) (I_{1,2}(t) + I_{2,1}(t)) \, dt & \int_a^T (I_{1,2}(t) + I_{2,1}(t))^2 \, dt & \cdots & \int_a^T (I_{1,n}(t) + I_{2,n-1}(t))(I_{1,2}(t) + I_{2,1}(t)) \, dt & \int_a^T I_{2,n}(t)(I_{1,2}(t) + I_{2,1}(t)) \, dt \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\int_a^T I_{1,1}(t) (I_{1,n}(t) + I_{2,n-1}(t)) \, dt & \int_a^T (I_{1,n}(t) + I_{2,n-1}(t))(I_{1,2}(t) + I_{2,1}(t)) \, dt & \cdots & \int_a^T (I_{1,n}(t) + I_{2,n-1}(t))^2 \, dt & \int_a^T I_{2,n}(t)(I_{1,n}(t) + I_{2,n-1}(t)) \, dt \\
\int_a^T I_{1,1}(t)I_{2,n}(t) \, dt & \int_a^T (I_{1,2}(t) + I_{2,1}(t))I_{2,n}(t) \, dt & \cdots & \int_a^T (I_{1,n}(t) + I_{2,n-1}(t))I_{2,n}(t) \, dt & \int_a^T (I_{2,n}(t))^2 \, dt \\
\end{pmatrix} \cdot \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n \\
\end{pmatrix} = \begin{pmatrix}
\int_a^T (N_{\text{data}}(t) - N_{0})I_{1,1}(t) \, dt \\
\int_a^T (N_{\text{data}}(t) - N_{0})(I_{1,2}(t) + I_{2,1}(t)) \, dt \\
\vdots \\
\int_a^T (N_{\text{data}}(t) - N_{0})I_{2,n}(t) \, dt \\
\end{pmatrix}
\]
Appendix D

Arrival rates: linear system

D.1 Case of polynomials

Let \( \alpha(t) \) be an unknown polynomial. Then the square of the collage distance is given by

\[
\Delta = \left( \int_0^T (N_{\text{target}}(t) - N_0 - \sum_{k=0}^n \alpha_k s^k ds + \int_0^t \beta(s)ds \cdot \chi(T_0, T_1)(t) + \int_0^t \beta(s)ds \cdot H_{T_1}(t))^2 dt \right)^{\frac{1}{2}}.
\]

We can rewrite (D.1.1) as

\[
\int_0^T \left( N_{\text{target}}(t) - N_0 - \sum_{k=0}^n \alpha_k s^k ds + \int_0^t \beta(s)ds \cdot \chi(T_0, T_1)(t) + \int_0^t \beta(s)ds \cdot H_{T_1}(t) \right)^2 dt.
\]

Let us define

\[
I_k(t) := \int_0^t s^k ds = \frac{1}{k+1} t^{k+1}.
\]

We can find the minimizers of (D.1.1) by taking the derivative with respect to the unknown coefficients \( \alpha_k \). This will give

\[
\frac{\partial \Delta^2}{\partial \alpha_m} = \int_0^T 2 \left( N_{\text{target}}(t) - N_0 - \sum_{k=0}^n \alpha_k I_k(t) + \int_0^t \beta(s)ds \cdot \chi(T_0, T_1)(t) + \int_0^t \beta(s)ds \cdot H_{T_1}(t) \right) I_m(t) dt \neq 0,
\]

for \( m = 0, 1, ..., n \). (D.1.4) has to be equal to 0 since we want to find the minimum. Rewriting (D.1.4) yields

\[
\sum_{k=0}^n \alpha_k \int_0^T I_k(t) I_m(t) dt = \int_0^T \left( N_{\text{target}}(t) - N_0 + \int_0^t \beta(s)ds \cdot \chi(T_0, T_1)(t) + \int_0^t \beta(s)ds \cdot H_{T_1}(t) \right) I_m(t) dt
\]

Now let us define

\[
A_{m,k} := \int_0^T I_k(t) I_m(t) dt,
\]

\[
b_m := \int_0^T \left( N_{\text{target}}(t) - N_0 + \int_0^t \beta(s)ds \cdot \chi(T_0, T_1)(t) + \int_0^t \beta(s)ds \cdot H_{T_1}(t) \right) I_m(t) dt.
\]
D.2. Case of sums of cosines

Then we end up with a linear system in the unknown coefficients $\alpha_k$. Note that we can calculate the integrals in the matrix $A$ analytically

$$A_{m,k} = \int_0^T I_k(t) I_m(t) dt$$

$$= \frac{1}{(k+1)(m+1)} \int_0^T t^{k+m+2} dt$$

$$= \frac{1}{(k+1)(m+1)(k+m+3)} T^{k+m+3},$$

for $k, m = 0, 1, \ldots, n$. This results in the following linear system

$$\begin{pmatrix}
\frac{T^3}{3} & \frac{T^4}{8} & \cdots & \frac{T^{n+3}}{(n+1)(n+3)} \\
\frac{T^4}{8} & \frac{T^5}{20} & \cdots & \frac{T^{n+4}}{2(n+1)(n+4)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{T^{n+3}}{(n+1)(n+3)} & \frac{T^{n+4}}{2(n+1)(n+4)} & \cdots & \frac{T^{2n+3}}{(n+1)^2(n+3)}
\end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \int_0^T (N_{\text{target}}(t) - N_0 + J(t)) \cdot t dt \\ \frac{1}{2} \int_0^T (N_{\text{target}}(t) - N_0 + J(t)) \cdot t^2 dt \\ \vdots \\ \frac{1}{n+1} \int_0^T (N_{\text{target}}(t) - N_0 + J(t)) \cdot t^n dt \end{pmatrix}.$$

The condition number for $n = 32$ of the matrix is of $\mathcal{O}(10^5)$.

D.2 Case of sums of cosines

Let $\alpha(t)$ be an unknown function of the form

$$\sum_{k=0}^n \alpha_k \cos \left( \frac{k \pi t}{T} \right).$$

Then the square of the collage distance is given by

$$\Delta = \left( \int_0^T (N_{\text{target}}(t) - N_0 - \int_0^t \sum_{k=0}^n \alpha_k \cos \left( \frac{k \pi s}{T} \right) ds + \int_0^t \beta(s) ds \cdot \chi(T_0, T_1)(t) + \int_0^t \beta(s) ds \cdot H_{T_1}(t))^2 dt \right)^{\frac{1}{2}}.$$  \hfill (D.2.2)

We can rewrite (D.2.2) as

$$\int_0^T \left( N_{\text{target}}(t) - N_0 - \sum_{k=0}^n \alpha_k \int_0^t \cos \left( \frac{k \pi s}{T} \right) ds + \int_0^t \beta(s) ds \cdot \chi(T_0, T_1)(t) + \int_0^t \beta(s) ds \cdot H_{T_1}(t) \right)^2 dt.$$  \hfill (D.2.3)

Let us define

$$I_k(t) := \int_0^t \cos \left( \frac{k \pi s}{T} \right) ds = \begin{cases} \frac{T}{k \pi} \sin \left( \frac{k \pi t}{T} \right) & \text{if } k \geq 1 \\
t & \text{if } k = 0 \end{cases}.$$  \hfill (D.2.4)

We can find the minimizers of (D.2.2) by taking the derivative with respect to the unknown coefficients $\alpha_k$. Like before $\frac{\partial \Delta^2}{\partial \alpha_m} = 0$ yields an expression of the form

$$\sum_{k=0}^n \alpha_k \int_0^T I_k(t) I_m(t) dt = \int_0^T (N_{\text{target}}(t) - N_0 + J(t)) I_m(t) dt,$$  \hfill (D.2.5)
D.2. Case of sums of cosines

where for convenience we write

\[ J(t) = \int_0^t \beta(s) ds \cdot \chi_{[T_0,T_1)}(t) + \int_0^t \beta(s) ds \cdot H_{T_1}(t). \]

(D.2.6)

Now define

\[ A_{m,k} := \int_0^T I_k(t) I_m(t) dt. \]

(D.2.7)

We can distinguish five cases:

- **\( m = k = 0 \):** From (D.2.4) we know that \( I_0(t) = t \), so

\[ A_{m,k} = \int_0^T t^2 dt = \frac{1}{3} T^3. \]

(D.2.8)

- **\( m = 0, k \neq 0 \):** By using (D.2.4) and partial integration we obtain

\[ A_{m,k} = \int_0^T \frac{T}{k\pi} \sin \left( \frac{k\pi t}{T} \right) \cdot t dt \]

\[ = -\frac{T^3}{k^2\pi^2} \cos (k\pi) + \frac{T^3}{k^3\pi^3} \sin (k\pi) \]

\[ = -\frac{T^3}{k^2\pi^2} \cos (k\pi) \]

\[ = \frac{(-1)^{k+1}T^3}{k^2\pi^2}. \]

(D.2.9)

The last equality in (D.2.9) follows from the fact that \( k \) is an integer and hence \( \sin (k\pi) \) for all \( k \).

- **\( m \neq 0, k = 0 \):** Similar to the previous case, we obtain

\[ A_{m,k} = \frac{(-1)^{m+1}T^3}{m^2\pi^2}. \]

(D.2.10)

- **\( m = k \):** We have that

\[ A_{m,k} = \int_0^T \left( \frac{T}{k\pi} \right)^2 \sin^2 \left( \frac{k\pi t}{T} \right) dt \]

\[ = \frac{T^3}{2k^2\pi^2} - \frac{T^3}{4k^3\pi^3} \sin (k\pi) \]

\[ = \frac{T^3}{2k^2\pi^2}. \]

(D.2.11)

- **\( m \neq k \neq 0 \):** Note that we then have

\[ A_{m,k} = \int_0^T \frac{T}{k\pi} \frac{T}{m\pi} \sin \left( \frac{k\pi t}{T} \right) \sin \left( \frac{m\pi t}{T} \right) dt \]

\[ = \frac{T^1}{2km\pi^2} \int_0^T \cos \left( \frac{(k - m)2\pi t}{T} \right) - \cos \left( \frac{(k + m)2\pi t}{T} \right) dt \]

\[ = 0. \]

(D.2.12)
D.2. Case of sums of cosines

This results in a linear system in the \( n \) unknown coefficients \( \alpha_k \).

\[
\begin{pmatrix}
\frac{T^3}{3} & \frac{T^3}{\pi^2} & \cdots & \frac{(-1)^n T^3}{n^2 \pi^2} \\
\frac{T^3}{\pi^2} & \frac{T^3}{2 \pi^2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{(-1)^n T^3}{n^2 \pi^2} & 0 & \cdots & \frac{T^3}{2 n^2 \pi^2}
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n
\end{pmatrix}
=
\begin{pmatrix}
\int_0^T (N_{\text{target}}(t) - N_0 + J(t)) I_1(t) dt \\
\int_0^T (N_{\text{target}}(t) - N_0 + J(t)) I_2(t) dt \\
\vdots \\
\int_0^T (N_{\text{target}}(t) - N_0 + J(t)) I_n(t) dt
\end{pmatrix}.
\]  

Note that the matrix \( A \) is only nonzero at the first row, first column and the diagonal.
Appendix E

Collage method for a traffic flow scenario with three traffic lights

E.1 Nonlinear system

The collage error is given by

$$\Delta^2(T_0^{(2)}, T_1^{(2)}, T_1^{(3)}) = \int_0^T \left( N_{\text{target}}(t) - N_0^{(1)} - J^{(2)}(t, T_0^{(2)}, T_1^{(2)}, T_1^{(3)}) - J^{(3)}(t, T_0^{(2)}, T_1^{(2)}, T_1^{(3)}) + J^{(1)}(t) \right)^2 dt,$$

(E.1.1)

where for simplicity

$$J^{(1)}(t) = \int_0^{t-T_0^{(1)}} \beta^{(1)}(s) ds \cdot \chi_{[T_0^{(1)}, T_1^{(1)}]}(t) + \int_0^{T_1^{(1)}-T_0^{(1)}} \beta^{(1)}(s) ds \cdot H_{T_0^{(1)}}(t),$$

$$J^{(2)}(t, T_0^{(2)}, T_1^{(2)}, T_1^{(3)}) = \int_0^{t-T_0^{(2)}} \beta^{(2)}(s) ds \cdot \chi_{[T_0^{(2)}, T_1^{(2)}]}(t) + \int_0^{T_1^{(2)}-T_0^{(2)}} \beta^{(2)}(s) ds \cdot H_{T_0^{(2)}}(t),$$

$$J^{(3)}(t, T_0^{(2)}, T_1^{(2)}, T_1^{(3)}) = \int_0^{t-T_1^{(2)}-\tau} \beta^{(3)}(s) ds \cdot \chi_{[T_1^{(2)}+\tau, T_1^{(3)}]}(t) + \int_0^{T_1^{(3)}-T_1^{(2)}-\tau} \beta^{(3)}(s) ds \cdot H_{T_0^{(2)}}(t).$$

(E.1.2)

Note that in $J^{(1)}(t)$ the parameters $T_0^{(1)}, T_1^{(1)}$ are known. A cycle consists of 5 phases:

- $[0, T_0^{(2)}]$: both light 2 and 3 are red (possibly the remaining direction has green light, but we do not consider this);
- $[T_0^{(2)}, T_1^{(2)}]$: light 2 is green, light 3 is red;
- $[T_1^{(2)}, T_1^{(2)}+\tau]$: light 2 and 3 are both red for safety reasons;
- $[T_1^{(2)}+\tau, T_1^{(3)}]$: light 2 is red, light 3 is green;
- $[T_1^{(3)}, T]$: both light 2 and 3 are red (possibly the remaining direction has green light, but we do not consider this).
E.1. Nonlinear system

Because the collage error given in (E.1.1) consists of characteristic and heaviside functions which are equal to 0 or 1, we can split up the collage error into 5 integrals:

\[
\Delta^2(T^{(2)}_0, T^{(2)}_1, T^{(3)}_1) = \int_0^{T^{(2)}_0} \left( N_{\text{target}}(t) - N_0^{(1)} + J^{(1)}(t) \right)^2 dt
\]

\[
= \Delta^2_1
\]

\[
+ \int_{T^{(2)}_0}^{T^{(2)}_1} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_{0}^{t-T^{(2)}_0} \beta^{(2)}(s) ds + J^{(1)}(t) \right)^2 dt
\]

\[
= \Delta^2_2
\]

\[
+ \int_{T^{(2)}_1}^{T^{(3)}_1+\tau} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_{0}^{T^{(2)}_1-T^{(2)}_0} \beta^{(2)}(s) ds + J^{(1)}(t) \right)^2 dt
\]

\[
= \Delta^2_3
\]

\[
+ \int_{T^{(3)}_1}^{T} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_{0}^{T^{(2)}_1-T^{(2)}_0} \beta^{(2)}(s) ds - \int_{0}^{T^{(3)}_1-T^{(2)}_1} \beta^{(3)}(s) ds + J^{(1)}(t) \right)^2 dt
\]

\[
= \Delta^2_4
\]

\[
\text{And}
\]

\[
\Delta^2(T^{(2)}_0, T^{(2)}_1, T^{(3)}_1) = \int_0^{T^{(2)}_0} \left( N_{\text{target}}(t) - N_0^{(1)} + J^{(1)}(t) \right)^2 dt
\]

\[
= \Delta^2_5
\]

Since we have that

\[
\frac{\partial \Delta^2}{\partial t} = \frac{\partial \Delta^2_1}{\partial t} + \frac{\partial \Delta^2_2}{\partial t} + \frac{\partial \Delta^2_3}{\partial t} + \frac{\partial \Delta^2_4}{\partial t} + \frac{\partial \Delta^2_5}{\partial t},
\]

(E.1.4)

for \( t = T^{(2)}_0, T^{(2)}_1, T^{(3)}_1 \), we can take the derivative of each of the integrals in (E.1.3) separately and then sum them. Without explicitly deriving the derivatives here, we will merely present the final result.

Because the derivatives have to be equal to 0 in order to find the critical points and since we have three unknown parameters, we obtain the nonlinear system of the form

\[
\begin{pmatrix}
F_1(T^{(2)}_0, T^{(2)}_1, T^{(3)}_1) \\
F_2(T^{(2)}_0, T^{(2)}_1, T^{(3)}_1) \\
F_3(T^{(2)}_0, T^{(2)}_1, T^{(3)}_1)
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
\]

(E.1.5)

where
E.2. Jacobian matrix

- \( F_1(T_0^{(2)}, T_1^{(2)}, T_1^{(3)}) = \frac{\partial \Delta^2}{\partial T_0^{(2)}} = \)
  \[
  \int_{T_0^{(2)}}^{T_1^{(2)}} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{t-T_0^{(2)}} \beta^{(2)}(s)ds + J^{(1)}(t) \right) \cdot \beta^{(2)}(t - T_0^{(2)}) dt \\
  + \int_{T_1^{(2)}}^{T_1^{(2)} + \tau} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s)ds + J^{(1)}(t) \right) dt \cdot \beta^{(2)}(T_1^{(2)} - T_0^{(2)}) \\
  + \int_{T_1^{(2)}}^{T_1^{(3)}} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s)ds \int_0^{t-T_1^{(2)} - \tau} \beta^{(3)}(s)ds + J^{(1)}(t) \right) dt \cdot \beta^{(2)}(T_1^{(2)} - T_0^{(2)}) \\
  + \int_{T_1^{(3)}}^{T} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s)ds \int_0^{T_1^{(3)} - T_1^{(2)} - \tau} \beta^{(3)}(s)ds + J^{(1)}(t) \right) dt \cdot \beta^{(2)}(T_1^{(2)} - T_0^{(2)}). \\
  \tag{E.1.6}
  \]

- \( F_2(T_0^{(2)}, T_1^{(2)}, T_1^{(3)}) = \frac{\partial \Delta^2}{\partial T_1^{(2)}} = \)
  \[
  \int_{T_1^{(2)}}^{T_1^{(2)} + \tau} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s)ds + J^{(1)}(t) \right) dt \cdot \beta^{(2)}(T_1^{(2)} - T_0^{(2)}) \\
  + \int_{T_1^{(2)}}^{T_1^{(2)} + \tau} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s)ds - \int_0^{t-T_1^{(2)} - \tau} \beta^{(3)}(s)ds + J^{(1)}(t) \right) dt \cdot \beta^{(2)}(T_1^{(2)} - T_0^{(2)}) \\
  - \int_{T_1^{(2)}}^{T_1^{(3)}} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s)ds - \int_0^{t-T_1^{(2)} - \tau} \beta^{(3)}(s)ds + J^{(1)}(t) \right) \beta^{(3)}(t - T_1^{(2)} - \tau) dt \\
  + \int_{T_1^{(3)}}^{T} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s)ds - \int_0^{T_1^{(3)} - T_1^{(2)} - \tau} \beta^{(3)}(s)ds + J^{(1)}(t) \right) dt \cdot \beta^{(2)}(T_1^{(2)} - T_0^{(2)}). \\
  \tag{E.1.7}
  \]

- \( F_3(T_0^{(2)}, T_1^{(2)}, T_1^{(3)}) = \frac{\partial \Delta^2}{\partial T_1^{(3)}} = \)
  \[
  \int_{T_1^{(3)}}^{T} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s)ds - \int_0^{T_1^{(3)} - T_1^{(2)} - \tau} \beta^{(3)}(s)ds + J^{(1)}(t) \right) dt \cdot \beta^{(3)}(T_1^{(3)} - T_1^{(2)} - \tau). \\
  \tag{E.1.8}
  \]

E.2 Jacobian matrix

Our system is of the form \( F = 0 \) where \( F = (F_1, F_2, F_3) \). The Jacobian matrix \( J \) which is needed in order to solve this nonlinear system, is a \( 3 \times 3 \) matrix:

\[
J = \begin{pmatrix}
  J_{11} & J_{12} & J_{13} \\
  J_{21} & J_{22} & J_{23} \\
  J_{31} & J_{32} & J_{33}
\end{pmatrix} := \begin{pmatrix}
  \frac{\partial F_1}{\partial T_0^{(2)}} & \frac{\partial F_1}{\partial T_1^{(2)}} & \cdots & \frac{\partial F_1}{\partial T_1^{(3)}} \\
  \frac{\partial F_2}{\partial T_0^{(2)}} & \frac{\partial F_2}{\partial T_1^{(2)}} & \cdots & \frac{\partial F_2}{\partial T_1^{(3)}} \\
  \frac{\partial F_3}{\partial T_0^{(2)}} & \frac{\partial F_3}{\partial T_1^{(2)}} & \cdots & \frac{\partial F_3}{\partial T_1^{(3)}}
\end{pmatrix}. \tag{E.2.1}
\]

The derivation of the 9 derivatives shown in this matrix are expensive to compute and write down, therefore only the end results are given:
E.2. Jacobian matrix

- \( J_{11} = \frac{\partial F_1}{\partial T_0^{(2)}} = \)

\[
\int_{T_0^{(2)}}^{T_1^{(2)}} (N_{\text{target}}(t) - N_0^{(1)} - \int_0^{t-T_0^{(2)}} \beta^{(2)}(s) ds + J^{(1)}(t)) (-\beta^{(2)}(t - T_0^{(2)})) dt + (\beta^{(2)}(t - T_0^{(2)}))^2 dt
\]

\[
- \int_{T_1^{(2)}}^{T_1^{(2)} + \tau} (N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s) ds + J^{(1)}(t)) dt \cdot (\beta^{(2)}(T_1^{(2)} - T_0^{(2)}))^2 \cdot \tau
\]

\[
- \int_{T_1^{(2)} + \tau}^{T_1^{(2)}} (N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s) ds - \int_0^{t-T_1^{(2)} - \tau} \beta^{(3)}(s) ds + J^{(1)}(t)) dt \cdot (\beta^{(2)}(T_1^{(2)} - T_0^{(2)}))^2
\]

\[
+ (\beta^{(2)}(T_1^{(2)} - T_0^{(2)}))^2 \cdot (T_1^{(3)} - T_1^{(2)} - \tau)
\]

\[
- \int_{T_1^{(2)} + \tau}^{T_1^{(2)}} \beta^{(3)}(t - T_1^{(2)} - \tau) dt \cdot \beta^{(2)}(T_1^{(2)} - T_0^{(2)}))
\]

\[
- \int_{T_1^{(2)}}^{T_0^{(2)}} (N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s) ds - \int_0^{t-T_1^{(2)} - \tau} \beta^{(3)}(s) ds + J^{(1)}(t)) dt \cdot (\beta^{(2)}(T_1^{(2)} - T_0^{(2)}))^2
\]

\[
+ (\beta^{(2)}(T_1^{(2)} - T_0^{(2)}))^2)
\]

- \( J_{21} = \frac{\partial F_2}{\partial T_0^{(2)}} = \)

\[
- \int_{T_1^{(2)}}^{T_1^{(2)} + \tau} (N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s) ds + J^{(1)}(t)) dt \cdot (\beta^{(2)}(T_1^{(2)} - T_0^{(2)}))^2 \cdot \tau
\]

\[
- \int_{T_1^{(2)} + \tau}^{T_1^{(2)}} (N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s) ds - \int_0^{t-T_1^{(2)} - \tau} \beta^{(3)}(s) ds + J^{(1)}(t)) dt \cdot (\beta^{(2)}(T_1^{(2)} - T_0^{(2)}))^2
\]

\[
+ (\beta^{(2)}(T_1^{(2)} - T_0^{(2)}))^2 \cdot (T_1^{(3)} - T_1^{(2)} - \tau)
\]

\[
- \int_{T_1^{(2)} + \tau}^{T_1^{(2)}} \beta^{(3)}(t - T_1^{(2)} - \tau) dt \cdot \beta^{(2)}(T_1^{(2)} - T_0^{(2)}))
\]

\[
- \int_{T_1^{(2)}}^{T_0^{(2)}} (N_{\text{target}}(t) - N_0^{(1)} - \int_0^{T_1^{(2)} - T_0^{(2)}} \beta^{(2)}(s) ds - \int_0^{t-T_1^{(2)} - \tau} \beta^{(3)}(s) ds + J^{(1)}(t)) dt \cdot (\beta^{(2)}(T_1^{(2)} - T_0^{(2)}))^2
\]

\[
+ (\beta^{(2)}(T_1^{(2)} - T_0^{(2)}))^2)
\]

- \( J_{31} = \frac{\partial F_3}{\partial T_0^{(2)}} = \)

\[
\beta^{(3)}(T_1^{(3)} - T_1^{(2)} - \tau) \cdot \beta^{(2)}(T_1^{(2)} - T_0^{(2)}), (T - T_1^{(3)})
\]

(E.2.2)

(E.2.3)

(E.2.4)
E.2. Jacobian matrix

\[ J_{12} = \frac{\partial F_1}{\partial T_1^{(2)}} = \]
\[
\left(\beta(2)(T_1^{(2)} - T_0^{(2)})\right)' \int_{T_1^{(2)}}^{T_1^{(2)}+\tau} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_{0}^{T_1^{(2)}-T_0^{(2)}} \beta(2)(s) ds + J^{(1)}(t) \right) dt \\
- (\beta(2)(T_1^{(2)} - T_0^{(2)})^2 \cdot \tau \\
+ (\beta(2)(T_1^{(2)} - T_0^{(2)})' \int_{T_1^{(2)}}^{T_1^{(2)}+\tau} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_{0}^{T_1^{(2)}-T_0^{(2)}} \beta(2)(s) ds - \int_{0}^{t-T_1^{(2)}-\tau} \beta(3)(s) ds + J^{(1)}(t) \right) dt \\
+ (\beta(2)(T_1^{(2)} - T_0^{(2)})' \int_{T_1^{(2)}}^{T_1^{(2)}+\tau} \beta(2)(T_1^{(2)} - T_0^{(2)}) + \beta(3)(t - T_1^{(2)} - \tau) dt \\
+ (\beta(2)(T_1^{(2)} - T_0^{(2)})' \int_{T_1^{(2)}}^{T_1^{(2)}+\tau} \left( -\beta(2)(T_1^{(2)} - T_0^{(2)}) + \beta(3)(T_1^{(2)} - T_1^{(2)} - \tau) \right) \cdot (T - T_1^{(2)}) \right). \\
\] (E.2.5)

\[ J_{22} = \frac{\partial F_2}{\partial T_1^{(2)}} = \]
\[
\left(\beta(2)(T_1^{(2)} - T_0^{(2)})\right)' \int_{T_1^{(2)}}^{T_1^{(2)}+\tau} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_{0}^{T_1^{(2)}-T_0^{(2)}} \beta(2)(s) ds + J^{(1)}(t) \right) dt \\
- (\beta(2)(T_1^{(2)} - T_0^{(2)})^2 \cdot \tau \\
- \beta(2)(T_1^{(2)} - T_0^{(2)}) \cdot \left( N_{\text{target}}(T_1^{(2)}) - N_0^{(1)} - \int_{0}^{T_1^{(2)}-T_0^{(2)}} \beta(2)(s) ds + J^{(1)}(T_1^{(2)}) \right) \\
\left(\beta(2)(T_1^{(2)} - T_0^{(2)})\right)' \int_{T_1^{(2)}}^{T_1^{(2)}+\tau} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_{0}^{T_1^{(2)}-T_0^{(2)}} \beta(2)(s) ds - \int_{0}^{t-T_1^{(2)}-\tau} \beta(3)(s) ds + J^{(1)}(t) \right) dt \\
\beta(2)(T_1^{(2)} - T_0^{(2)}) \int_{T_1^{(2)}}^{T_1^{(2)}+\tau} \beta(2)(T_1^{(2)} - T_0^{(2)}) + \beta(3)(t - T_1^{(2)} - \tau) dt \\
- \int_{T_1^{(2)}}^{T_1^{(2)}+\tau} \beta(3)(t - T_1^{(2)} - \tau) \cdot \left( -\beta(2)(T_1^{(2)} - T_0^{(2)}) + \beta(3)(t - T_1^{(2)} - \tau) \right) dt \\
+ \int_{T_1^{(2)}}^{T_1^{(2)}+\tau} \left(\beta(3)(t - T_1^{(2)} - \tau)\right)' \left( N_{\text{target}}(t) - N_0^{(1)} - \int_{0}^{T_1^{(2)}-T_0^{(2)}} \beta(2)(s) ds - \int_{0}^{t-T_1^{(2)}-\tau} \beta(3)(s) ds + J^{(1)}(t) \right) dt \\
+ (\beta(2)(T_1^{(2)} - T_0^{(2)})' \int_{T_1^{(2)}}^{T_1^{(2)}+\tau} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_{0}^{T_1^{(2)}-T_0^{(2)}} \beta(2)(s) ds - \int_{0}^{t-T_1^{(2)}-\tau} \beta(3)(s) ds + J^{(1)}(t) \right) dt \\
+ \beta(2)(T_1^{(2)} - T_0^{(2)}) \cdot \left[ -\beta(2)(T_1^{(2)} - T_0^{(2)}) + \beta(3)(T_1^{(2)} - T_1^{(2)} - \tau) \right] \cdot (T - T_1^{(2)}) \right). \\
\] (E.2.6)
E.2. Jacobian matrix

\[ J_{32} = \frac{\partial F_3}{\partial T_{1}^{(2)}} = \]
\[
(-\beta^3 (T_1^{(3)} - T_1^{(2)} - \tau))' \cdot \int_{T_{1}^{(3)}}^{T_{1}^{(2)}} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_{0}^{T_{1}^{(2)} - T_0^{(2)}} \beta^2(s) ds - \int_{0}^{T_{1}^{(2)} - \tau} - \beta^3(s) ds + J^{(1)}(t) \right) dt
\]
\[ + \beta^3 (T_1^{(3)} - T_1^{(2)} - \tau) \cdot \left( -\beta^2 (T_1^{(2)} - T_0^{(2)}) + \beta^3 (T_1^{(3)} - T_1^{(2)} - \tau) \right) \cdot (T - T_1^{(3)}).
\]
(E.2.7)

\[ J_{13} = \frac{\partial F_1}{\partial T_{1}^{(3)}} = \]
\[ -\beta^2 (T_1^{(2)} - T_0^{(2)}) \cdot \beta^3 (T_1^{(3)} - T_1^{(2)} - \tau) \cdot (T - T_1^{(3)}).
\]
(E.2.8)

\[ J_{23} = \frac{\partial F_2}{\partial T_{1}^{(3)}} = \]
\[ -\beta^2 (T_1^{(2)} - T_0^{(2)}) \cdot \beta^3 (T_1^{(3)} - T_1^{(2)} - \tau) \cdot (T - T_1^{(3)})
\]
\[ -\beta^3 (T_1^{(3)} - T_1^{(2)} - \tau) \cdot \left( N_{\text{target}}(T_1^{(3)}) - N_0^{(1)} - \int_{0}^{T_{1}^{(2)} - T_0^{(2)}} \beta^2(s) ds - \int_{0}^{T_{1}^{(3)} - T_1^{(2)} - \tau} \beta^3(s) ds + J^{(1)}(T_1^{(3)}) \right).
\]
(E.2.9)

\[ J_{33} = \frac{\partial F_3}{\partial T_{1}^{(3)}} = \]
\[
(\beta^3 (T_1^{(3)} - T_1^{(2)} - \tau))' \cdot \int_{T_{1}^{(3)}}^{T_{1}^{(2)}} \left( N_{\text{target}}(t) - N_0^{(1)} - \int_{0}^{T_{1}^{(2)} - T_0^{(2)}} \beta^2(s) ds - \int_{0}^{T_{1}^{(3)} - T_1^{(2)} - \tau} \beta^3(s) ds + J^{(1)}(t) \right) dt
\]
\[ -\beta^3 (T_1^{(3)} - T_1^{(2)} - \tau) \cdot \beta^3 (T_1^{(3)} - T_1^{(2)} - \tau) \cdot (T - T_1^{(3)})
\]
\[ -\beta^3 (T_1^{(3)} - T_1^{(2)} - \tau) \cdot \left( N_{\text{target}}(T_1^{(3)}) - N_0^{(1)} - \int_{0}^{T_{1}^{(2)} - T_0^{(2)}} \beta^2(s) ds - \int_{0}^{T_{1}^{(3)} - T_1^{(2)} - \tau} \beta^3(s) ds + J^{(1)}(T_1^{(3)}) \right).
\]
(E.2.10)
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