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Increasing contact line mobility by means of infrared laser illumination

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Increasing contact line mobility by means of infrared laser illumination

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Abstract

In a previous study, it was found that the mobility of a fluid-fluid-solid contact line can be improved by means of infrared illumination. The shape of a moving droplet is largely determined by the mobility of its contact line. I therefore studied how infrared laser illumination of a moving droplet changes its shape and the dynamics of motion, by conducting laboratory experiments using a combination of a turntable-like setup with infrared laser illumination. These experiments are complemented by numerical simulations of laser-induced temperature profiles and of the resulting droplet deformations.

In the experimental study, the substrate velocity, the laser power and the laser spot position have been varied. The results show that the mobility of the droplet contact line can be substantially increased by means of infrared laser illumination. The critical velocity of the droplet, i.e., the velocity at which the droplet leaves residual liquid behind on the substrate, increases with increasing laser power.

Numerical simulations of the temperature distribution of a 3-dimensional substrate, illuminated by a Gaussian intensity distribution, have been performed to obtain an estimate of the temperature distribution of the substrate during the conducted experiments. The temperature distribution depends on the substrate velocity and the total intensity of the illumination.

Numerical simulations of 2-dimensional sliding droplets on an inclined substrate show that the velocity of the droplet can be increased by applying a moving non-uniform temperature distribution. The receding contact angle increases with increasing velocity, due to the increased thermocapillary stresses.

A semi-analytical model of 2-dimensional sliding droplets has been numerically analyzed. The model describes the steady-state velocity of a droplet under the influence of gravity and a non-uniform temperature profile, as a function of the droplet shape. The model shows that a temperature profile provides an extra driving force. The steady-state velocity of the droplet is further increased if the temperature-dependency of the viscosity taken into account. This speed-up effect, i.e., the increase in the droplet mobility, is smaller for a narrower temperature profile.
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1 Introduction

1.1 Motivation

The movement of a liquid droplet in contact with a solid substrate involves the movement of a contact line. A contact line is the line along which the substrate, the liquid and the surrounding gas phases meet. The controlled movement of a contact line is necessary for applications in, e.g., coating, printing, or immersion lithography.

Immersion lithography is an improved method of photo-lithography, in which patterns are optically created on a silicon wafer to manufacture integrated circuits. The limiting resolution $W$ of the pattern is given by

$$W = k_1 \frac{\lambda}{n \sin \alpha_{ap}},$$

where $k_1$ is a process-dependent factor, $\lambda$ the light wavelength, $\alpha_{ap}$ the angular half aperture of the lens, and $n$ the refractive index of the medium between the lens and the substrate. The limiting resolution can be improved by placing a liquid with a refraction index $n>1$ between the lens and the wafer. This in analogy to the resolution enhancement in immersion microscopy. Due to the movement of the wafer underneath the lens, a residual thin liquid film can be left behind on the wafer. Defects can arise on the wafer due to this residual liquid [2], such as water marks, drying stains or micro bubbles. The break-up of the residual thin liquid film must therefore be controlled or the occurrence of a residual thin liquid film must be prevented. In this study, the prevention of a residual liquid film by means of infrared illumination has been investigated.

1.2 Literature overview

Podgorski et al. conducted controlled experiments of silicon oil droplets sliding down an inclined coated substrate [3]. The droplet velocity has been varied by changing the inclination angle. Different droplet shape regimes have been found at different velocities. At relatively low velocities, the droplet has a circular shape and obtains a pointed shape at higher velocities. At even higher velocities, the receding contact line obtains a $V$-shaped profile with a corner up to approximately 60°. At velocities above a critical value, the droplet loses residual liquid. Schwartz et al. numerically studied the droplet shape for droplets sliding down an incline [4]. The numerical model has been developed using the lubrication approximation, while the disjoining pressure model [5] has been used to simulate the dynamic contact angles of the droplet. The calculated shapes are consistent with the experimental results by Podgorski et al., i.e., all shape regimes could be numerically reproduced.

Snoeijer et al. conducted an experimental and analytical study on the droplet shape of droplets sliding down an incline [6]. Most importantly, they provided an analytical model for the opening angle of the receding contact line. Winkels et al. studied the dynamics of the droplet shape on a turntable-like setup and compared the results to the dynamics of droplets sliding down an incline [7]. In a turntable setup, a droplet is deposited on a substrate and attached to a needle. The substrate is subsequently moved with respect to the needle, causing the droplet to deform. The main difference between an inclined substrate setup and a turntable setup is the driving force. For sliding droplets, the droplet obtains a velocity due to gravitation, while the substrate is forced to move at a specific velocity in a turntable setup. The dynamics of the contact line shape of a sliding droplet and of a droplet in a turntable setup are qualitatively comparable.

Brochard provided an analytical description of droplet actuation by means of temperature gradients [8]. This thermocapillary actuation is compared with the actuation by means of chemical gradients on the surface. Ford et al. developed an analytical model for the thermocapillary motion of a droplet [9], based on the model by Brochard. In this analytical model, a moving droplet on a horizontal surface, due to a linear temperature profile has been considered. The migration velocity of the droplet has been determined by means of the lubrication theory, provided the liquid height profile of the droplet is specified. Brzoska et al. conducted an experimental study on the thermocapillary actuation of droplets on a substrate [10], while Smith conducted a numerically study [11]. Darhuber et al. demonstrated the thermocapillary actuation of small droplets on a substrate with a chemically prepared pattern [12], where the temperature profile has been induced by micro-heater arrays.
The deformation and break-up of thin liquid films by means of infrared laser illumination has been studied by Wedershoven [13]. It has been found that infrared illumination can influence the break-up process of a thin film. A thin liquid film was moved underneath the infrared laser beam with a certain velocity, creating a dry ‘track’ in the film. No residual liquid was left behind in the dry track, in contrast to the break-up induced by, e.g., a moving air-jet [14]. Wedershoven concluded that the dewetting velocity of the initial dry-spot is larger than the substrate velocity, i.e., the mobility of the contact line has been enhanced. A proof-of-concept experiment with an infrared illuminated droplet has been conducted by [15], in which the contact line mobility of the droplet has been enhanced.

1.3 Dynamics of droplets

In the present study, the effect of infrared laser illumination of moving droplets on the contact line shape has been investigated. Laboratory experiments have been conducted on a turntable-like setup. The dimensions of the considered droplets are in the order of the capillary length, capillary effects are therefore important. The capillary length is defined as [16],

$$l_{\text{cap}} = \sqrt{\frac{\gamma}{\rho g}},$$  \hspace{1cm} (1.2)

where \(\gamma\) represent the surface tension between the liquid and the surrounding gas, \(\rho\) the mass density of the liquid and \(g\) the gravitational acceleration. The surface tension is temperature-dependent, it decreases for increasing temperature. A gradient in the surface tension causes a shear stress in the liquid-gas interface, resulting in a flow within the liquid [17]. This thermocapillary flow is a type of Marangoni flow [18] and can be the driving force of a droplet. An important variable in the study of droplets and contact line movement is the dimensionless capillary number \(Ca\),

$$Ca \equiv \frac{\mu U}{\gamma},$$  \hspace{1cm} (1.3)

where \(U\) represents the velocity, scaled with the viscosity \(\mu\) of the liquid and the surface tension \(\gamma\) between the liquid and the surrounding gas. The capillary number indicates the ratio between viscous forces and the surface tension and is used as a scaled velocity.

As described in section 1.2, the receding contact line of a moving droplet has a round shape for velocities far below a critical value \(Ca \ll Ca_{cr}\). The contact line obtains a V-shaped profile at relatively high velocities \(Ca < Ca_{cr}\). The droplet leaves behind residual liquid at velocities above a critical value \(Ca > Ca_{cr}\). Figure 1 illustrates the droplet shape on a turntable-like setup for the different velocity regimes. The figure shows a droplet, deposited on a substrate and attached to a needle. The substrate is subsequently moved with a certain substrate velocity \(U_{sub}\) with respect to the needle. Top and side-views of the droplet are presented, at a relatively low substrate velocity (a), at a relatively high substrate velocity (b) and at a substrate velocity above the critical velocity (c).

![Schematic views of the droplet shape in a turntable-like experimental setup at different substrate velocities. The direction of the substrate velocity \(U_{sub}\) is indicated.](image)
The droplet shape is determined largely by the mobility of the contact line. It has therefore been studied how infrared laser illumination of a droplet changes the dynamics of the droplet shape. Due to the infrared laser illumination, the temperature of the substrate rises. The resulting temperature profile affects the surface tension and the viscosity of the liquid. Due to the temperature-dependency of the surface tension, a thermocapillary flow might be present. The viscosity decreases for increasing temperature. Therefore, the measured contact line mobility enhancement due to infrared laser illumination, is an effect of the presence of a thermocapillary flow and a variation in liquid properties.

Using the combination of a turntable-setup with infrared laser illumination, the influence of non-uniform temperature distributions on the shape of moving droplets has been measured. The conducted experiments are complemented by numerical simulations of laser-induced temperature profiles and of the resulting droplet deformations.

1.4 Outline

The experimental part of this study is presented in chapter 2. The turntable-setup and the infrared laser setup are discussed in respectively section 2.1 and 2.2. The experimental procedures are presented in section 2.3, which also focuses on the image processing and the calculation of the droplet shape parameters. The experimental setup has been calibrated, which is the focus of section 2.4. A parameter study, i.e., the variation of the laser power, the laser spot position and the substrate velocity, is presented in chapter 3.

To obtain an estimate of the substrate temperature distribution during the conducted experiments, a numerical model of a moving substrate, illuminated by a Gaussian intensity profile, has been considered in chapter 4.

A numerical model of a 2-dimensional droplet sliding down an inclined substrate has been considered in chapter 5. The inclination angle of the substrate has been varied, resulting in a steady-state droplet velocity. This velocity is subsequently increased by applying a moving non-uniform temperature distribution. The results are presented in section 5.6.

Chapter 6 presents a semi-analytical model of a 2-dimensional sliding droplet. The model describes the steady-state velocity of the droplet under the influence of gravity and a non-uniform temperature profile, as a function of the droplet shape. Section 6.1 focuses on the derivation of the model and the model is numerically analyzed in section 6.2.

The conclusions of this study are summarized in chapter 7. Furthermore, the developed Matlab-scripts for the image processing and the determination of the droplet shape parameters are presented in appendix A and the lubrication equation is derived in appendix B.
2 Experimental setup and procedures

In this chapter, the experimental setup and the used procedures are presented. The experimental turntable-setup is discussed in section 2.1. Subsequently, the used infrared laser source and the intensity profile of the laser beam are discussed in section 2.2. The experimental procedures are discussed in section 2.3 and the calibration of the experimental setup is discussed in section 2.4.

2.1 Turntable setup

Laboratory experiments have been conducted using a turntable setup. Figure 2 shows a schematic representation of the front-view (a), the side-view (b) and the top-view (c) of the experimental turntable setup. The main components of the setup are the rotating circular substrate (1), the concentric needle (5), the infrared laser beam (6) and the liquid droplet (7).

The circular transparent substrate (1) is placed on top of a circular glass plate (2) to obtain a stable and flat surface. The substrate consists of 480 µm thick Makrofol polycarbonate and the glass plate consists of 3.90 mm thick soda-lime glass. Both the substrate and the glass plate are 30 cm in diameter. The glass plate is connected at the center to a DC motor (4) (Maxon Motor, 136292) via a 636:1 ratio gearbox (Maxon Motor, 166182). The motor is connected to a computer via a controller (Maxon Motor, HEDL-5540A11) that allows to vary the rotation rate of the motor up to 4000 rpm in steps of 1 rpm. The gearbox transfers this rotation rate with a 636:1 ratio to the glass plate.

The concentric needle (5) consists of two hollow concentric cylinders with separate connections. The inner...
Figure 3: A microscopic bottom-view of the concentric needle. The inner and outer radius of the small cylinder are indicated by respectively \( r_1 \) and \( r_2 \). The inner and outer radius of the large cylinder are indicated by respectively \( r_3 \) and \( r_4 \).

cylinder constantly supplies liquid, while the outer cylinder constantly extracts a liquid-air mixture. The liquid is therefore constantly refreshed. The arrows on the needle in the figure indicate the direction of the flow. Figure 3 shows a microscope image of the needle as seen from the bottom. The different radii of the two cylinders are indicated. The small cylinder has an inner radius of \( r_1 = 0.41 \) mm and an outer radius of \( r_2 = 0.60 \) mm. The large cylinder has an inner radius of \( r_3 = 0.79 \) mm and an outer radius of \( r_4 = 1.28 \) mm. The dimensions are measured within 0.01 mm accuracy.

The concentric needle is positioned above the substrate, off-axis and close to the edge of the substrate. The distance from the center of the substrate to the needle is \( r_{\text{sub}} = (12.9 \pm 0.1) \) cm. This is much smaller than the outer radius of the needle, i.e., \( r_4/r_{\text{sub}} \approx 10^{-2} \). The circular motion of the substrate below the needle is therefore considered a rectilinear motion. Furthermore, the height of the needle with respect to the substrate is set using a linear translation stage with a precision of 5 \( \mu \)m.

The extracted mixture is pumped into a separator which separates the liquid and the air. A vacuum pump (KNF, N811KT.18) is connected to the air outlet of the separator, to provide the necessary pressure difference to extract the mixture. The ultimate vacuum of the pump is 290 mbara and the operating positive pressure is 2 barg. The pressure in the separator is controlled by a pressure regulator. The liquid for the supply is contained in a jar, positioned above the setup. The gravitational pressure provides the supply pressure, while the liquid supply rate is controlled using a valve.

Ethylene glycol has been chosen as the liquid to be investigated. Section 2.1.1 focuses on the liquid in more detail. A droplet (7) is obtained controlling the liquid supply rate and the mixture extraction rate. The settings for the liquid supply rate and the mixture extraction rate have been obtained by iteration, to obtain a stable droplet. The liquid supply rate is set to \( (7 \pm 1) \) mL/min. The pressure difference at the separator is set to \( (-8.0 \pm 0.5) \) kPas, resulting in an air flow of \( (1.5 \pm 0.5) \) L/min.

The infrared laser beam (6) has a wavelength of 1470 nm. Custom designed optics transform the shape of the laser intensity profile into an ellipse, at the position of the droplet. The optics are connected to the frame and can be positioned with linear translation stages. The angle of incidence \( \beta \) of the laser beam is set to 45° with respect to the substrate, so the needle does not obstruct the laser beam. The laser source, the optical system and the resulting laser intensity profile are discussed in more detail in section 2.2.

Two cameras image the droplet from the bottom (Allied Vision Technologies, Pike F-145B) and the side (Allied Vision Technologies, Guppy F-146B). The bottom of the droplet is imaged through the transparent glass plate and substrate, and is illuminated along the camera’s viewing direction. The viewing direction of the side-view camera is radially inwards with respect to the circular substrate. The illumination for the side-view is provided from behind the droplet. The edges of the droplet are clearly visible, because the droplet refracts the illumination light. Figure 14 shows typical images from both the bottom (a, b) and side-view (c, d), for a low (a, c) and a high (b, d) substrate velocity.

Both cameras are protected against the infrared radiation with a short-pass filter (Schott KG-3). This filter absorbs infrared radiation, but is transparent for visible light. The filter for the bottom-view camera is the most important, because the laser beam is directed directly into the lens. This filter limits the maximum usable power of the laser in this setup. The size of the pixels in the bottom and side-view images are respectively \( (4.91 \pm 0.02) \) \( \mu \)m/px and \( (9.22 \pm 0.04) \) \( \mu \)m/px. The frame rate of both cameras is set to 7.5 frames per second and the number of frames recorded per measurement is 100 frames per burst.
2.1.1 Liquid properties

Ethylene glycol has been chosen as the liquid to be investigated. Ethylene glycol is a non-volatile liquid, so evaporation effects are minimized. Furthermore, the critical velocity of ethylene glycol is in the range of the turntable setup. The viscosity \( \mu \), mass density \( \rho \) and surface tension \( \gamma \) can be expressed \([19, 20]\) as function of temperature \( T \) [°C]:

\[
\begin{align*}
\mu [\text{mPas}] &= \exp(-3.61359 + 986.519/(T + 127.861)), \\
\gamma [\text{mN/m}] &= 50.206 - 0.089 T, \\
\rho [\text{kg/m}^3] &= 1127.68 - 0.65816 T - 6.1765 \cdot 10^{-4} T^2.
\end{align*}
\]

These equations yield at a temperature of 20°C for the viscosity 21.29 mPas, the surface tension 48.43 mN/m and the mass density 1114 kg/m³. Figure 4 shows graphically the relative temperature dependence of the properties of ethylene glycol for temperatures between 20°C and 200°C. The viscosity (line), the surface tension (dotted line) and the mass density (dashed line) are normalized by their value at 20°C.

The viscosity of ethylene glycol shows the largest variation as function of temperature.

Ethylene glycol has been measured experimentally using a viscometer (Brookfield, lvdv-ii+pro) with a spindle (s18). The viscometer measures the torque at a fixed shear rate and calculates the viscosity. The measured viscosity at a temperature of 24°C is (17.6 ± 0.3) mPas, which is consistent with the calculated value of 17.86 mPas, using equation (2.1a).

Ethylene glycol has a thermal conductivity \( k \) of 0.3 W/(m K) and a heat capacity at constant pressure \( C_p \) of 2.5 kJ/(kg K) \([20]\). The thermal diffusivity \( \kappa \) of a material is defined in equation 2.2 \([17]\) and is for ethylene glycol equal to \( \kappa = 10^{-7} \text{m}^2/\text{s} \). The typical length scale \( L \) of the ethylene glycol droplets is 1 mm and the typical substrate velocity \( U \) during the experiments is 5 mm/s.

\[
\kappa \equiv \frac{k}{\rho C_p} \quad (2.2)
\]

The ratio between convective and diffusive heat transfer is represented by the thermal Peclet number \([17]\). A small thermal Peclet number indicates that diffusive heat transfer in the system is dominant. The thermal Peclet number is defined as,

\[
\text{Pe} \equiv \frac{LU}{\kappa}, \quad (2.3)
\]

where \( L \) represents the typical length and \( U \) the typical velocity. The thermal Peclet number for the droplet is Pe = 10. This indicates that the influence of the convective heat transfer is significant during the experiments.
The ratio between the heat transfer within the system and away from the system is represented by the Biot number \[ \text{Bi} = \frac{Lh_c}{k_t} \]. A small Biot number indicates that the temperature differences within the system are small. The Biot number is defined as,

\[ \text{Bi} \equiv \frac{Lh_c}{k_t} \]

(2.4)

where \( h_c \) represents the convective heat transfer coefficient. The Biot number for the droplet is \( \text{Bi} = 2 \times 10^{-2} \), calculated using the typical value for the convective heat transfer coefficient of \( h_c = 5 \text{W/(m}^2\text{K)} \). However, the extraction of the liquid-air mixture induces an air flow around the droplet, which results in a higher convective heat transfer coefficient and therefore a higher Biot number.

### 2.1.2 Temperature dependence quasi-static contact angle

The temperature-dependency of the quasi-static contact angles of ethylene glycol has been measured. The quasi-static contact angle is the contact angle while the contact line is moving very slowly. An experimental setup with a hotplate has been used for the measurements.

Figure 5 shows a schematic side-view of the hotplate-setup. A polycarbonate substrate (3) is placed on a hotplate (4) (IKA, C-MAG HS 7) to vary the temperature of the substrate. A droplet of ethylene glycol (2) has been deposited on the substrate using a needle (1). The needle has an inner diameter of 1.54 mm and an outer diameter of 1.84 mm and remains submerged in the liquid. The needle is connected to a syringe pump (KDS, KDS 88) to slowly apply and extract liquid from the droplet. The droplet is imaged from the side with a camera (6) (Thorlabs, DCC1645C) and illuminated (7) from the opposite side. An enclosure (5) is placed over the substrate and the droplet to minimize evaporation. The enclosure has a length, width and height of respectively (6.0 × 6.5 × 5) cm. It has two transparent sides for the imaging of the droplet and a half-open circular aperture in the top with a radius of 1 cm for the connection of the needle.

The quasi-static advancing contact angle is measured by slowly applying more liquid to the droplet. The volume of the droplet increases and so does the contact angle, while the contact line position remains constant. Once a certain contact angle is obtained, the contact line starts to move slowly forwards and the quasi-static advancing contact angle is measured. The same method is used to measure the quasi-static receding contact angle, by extracting liquid from the droplet.

Figure 6 shows the contact angle and the contact line position during a typical measurement as a function of time. The contact angle and the contact line position are measured at the right side of the droplet, as seen from the camera. The contact angle is presented by circles, corresponding to the left vertical axis. The contact line position is presented by a line, corresponding to the right vertical axis. The gray background indicates the time during which liquid is supplied to the droplet and the white background indicates the time during which liquid is extracted from the droplet. The dashed line is added as a guide to the eye for the contact angle data.

Liquid is initially supplied to the droplet. The figure shows that the contact angle increases and reaches a constant value at 20 s, the quasi-static advancing contact angle, while the contact line position increases.

**Figure 5:** Schematic side-view of the experimental hotplate setup. The needle (1), the droplet (2), the substrate (3), the hotplate (4), the enclosure (5), the camera (6), and the illumination (7) are indicated.
Liquid is extracted from the droplet again after 58 s. The contact angle starts to decrease, while the position of the contact line remains constant. Around 97 s the contact line starts to move backwards and around 102 s the contact angle has again a constant value, the quasi-static receding contact angle. The absolute velocity of the contact line has been (26 ± 4) µm/s during the measurements.

Figure 7 shows the measured quasi-static advancing contact angles (circles) and quasi-static receding contact angles (squares) at temperatures of 25°C, 30°C, 90°C and 120°C. A guide line connects the mean angle per temperature. The figure shows that for increasing temperature, the advancing contact angle slightly increases and the receding contact angle slightly decreases. However, more measurements are necessary to draw a conclusion due to the spreading of the data.

Measurements by Petke et al. state that the quasi-static advancing and receding contact angles depend linearly on the temperature [21]. A linear fit through the quasi-static advancing and receding contact angle provide a temperature-dependency of respectively (0.05 ± 0.03) deg/°C and (−0.09 ± 0.05) deg/°C. The literature values [21] state a temperature dependency of respectively -0.03 deg/°C and 0.12 deg/°C. These values have the opposite sign as the measured results. Also, the absolute values of the measured contact angles are inconsistent with the literature values. It could be that the contact line was moving too fast during the measurements or that the surface conditions of the polycarbonate substrates are not identical.
Figure 7: The quasi-static receding and advancing contact angle as a function of temperature.
2.2 Infrared laser setup

The laser source used in this study is a water cooled diode laser (Lumics, LU1470C020). It has a maximum optical power of 20 W at a wavelength of 1470 nm with a spectral width of 10 nm FWHM. The laser beam is transmitted from the source to the setup by means of an optical fiber with a core diameter of 400 μm and a numerical aperture of 0.22. A visible red pilot laser of 1 W is integrated for the initial alignment.

The diverging infrared laser beam from the optical fiber is focused in one direction by custom designed optics. A schematic side-view of the optical system is shown in figure 8. The path of the infrared laser beam is indicated by the straight lines. The beam from the optical fiber (1) is collimated by means of a collimator lens (2) (Thorlabs, f230sma-1550). The collimated beam is then focused in one direction by means of a cylindrical lens (3) at the focal point (4). The cylindrical lens consists of optical glass (Thorlabs, N-BK7) with a diameter of 22 mm and it has a numerical aperture of 0.138.

Because the laser beam is only focused in one direction by the cylindrical lens, the intensity profile at the focal point has an approximately elliptical shape. In the experimental turntable-setup, the long axis of the ellipse is oriented perpendicular to the substrate velocity. So, the receding side of the droplet is illuminated approximately uniformly in the direction perpendicular to the substrate velocity.

The position of the laser spot on the substrate has been determined using a thermocouple. The thermocouple (Omega, CHAL-0005) has a head diameter of 13 μm and has been glued to a polycarbonate substrate. The thermocouple is connected to a thermocouple-to-voltage converter (Omega, TAC80B-K) and the voltage has been measured with a multimeter (HP, 34401A). The thermocouple is positioned in the turntable setup at approximately the position of the receding side of the droplet. The highest temperature reading is achieved when the point of highest intensity of the laser spot coincides with the thermocouple head. The thermocouple is observed with the bottom-view camera and the position of the thermocouple is calculated afterwards. The position of the laser spot can be varied with respect to this position by attached stages with a precision of 5 μm.

2.2.1 Laser power

The total optical laser power can be set by adjusting the operating current of the laser source. Measurements of the laser power have been performed using a thermal head power sensor (Thorlabs, S314C) with corresponding console (PM100USB). Figure 9a shows the measured laser power as a function of the operating current. The figure shows that the laser power does not depend linearly on the operating current. A different method to set the laser power has therefore been used.

The operating signal of the laser source controller has been modulated with a duty cycle signal to vary the optical power. The operating current is set to a constant value. The duty cycle signal consists of a periodic signal which is a fraction \(a\) ‘on’ and a fraction \((1-a)\) ‘off’, where \(0 < a < 1\). This periodic signal is continuously sent as the operating signal to the controller at a rate of 50 Hz. The operating current has been set to a constant value of 3 A.

Figure 9b shows the optical laser power as a function of the duty cycle fraction \(a\). The figure shows that the laser power depends linearly on the duty cycle fraction. The data has been fitted with a linear function, forced through the origin, resulting in a function for the laser power of \(P \ [W] = (6.24 \pm 0.02)a\).
2.2.2 Scanning slit measurement

The intensity profile $I(x, y)$ of the infrared laser beam has been measured using a scanning slit measurement setup. The scanning slit method is discussed in this section and the measurement of the intensity profile is discussed in section 2.2.3. Figure 10 shows a schematic view of the setup, where the intensity profile is illustrated by the elliptic shape and the slit by the opening in the rectangle. The slit width $\delta$, the scan velocity $U$, the $x'$-axis in the scan direction, the $x$ and $y$-axis of the elliptical intensity profile and the angle $\alpha$ between the scan direction and the large axis of the elliptical intensity profile are indicated. The transmitted power through the slit is measured by a power sensor and both the power sensor and the slit are connected to a linear stage. The width of the slit is much smaller than the width of the laser beam in both the $x$ and $y$ direction, respectively $w_x$ and $w_y$.

With a scanning slit measurement, the width of the laser beam in the scan direction can easily be determined. It can be shown analytically \[22\] that the measured integrated intensity profile has the same width as the laser beam, when a Gaussian laser intensity profile with total power $P$ is assumed, \[ I(x, y) = \frac{2P}{\pi w_x w_y} \exp \left[ -2 \left( \frac{x^2}{w_x^2} + \frac{y^2}{w_y^2} \right) \right]. \] (2.5)

With this definition of the intensity profile, the beam widths $w_x$ and $w_y$ are defined as the distance between the point of highest intensity and $\exp(-2) \approx 14\%$ of the intensity profile. Assuming a scan direction in the $x$-direction, the integrated intensity $F(x')$ passing the slit is found by integrating the Gaussian intensity profile $I(x, y)$ over the width of the slit $\delta$ at position $x'$,

\[
F(x') = \frac{2P}{\pi w_x w_y} \int_{x'-\delta/2}^{x'+\delta/2} \int_{-\infty}^{\infty} \exp \left[ -2 \left( \frac{x^2}{w_x^2} + \frac{y^2}{w_y^2} \right) \right] dy \, dx
\]

\[
= \frac{P}{\pi w_x} \int_{x'-\delta/2}^{x'+\delta/2} \frac{\delta}{w_x} \exp \left( -2 \frac{x^2}{w_x^2} \right) dx
\]

\[
= \frac{P}{2} \left( \text{erf} \left( \frac{\sqrt{2}}{w_x} (x' + \delta/2) \right) - \text{erf} \left( \frac{\sqrt{2}}{w_x} (x' - \delta/2) \right) \right).
\]

Equation (2.6b) is obtained with use of the standard integral as stated in equation (2.7) and equation (2.6c) is obtained by rewriting the equation with use of the error function $\text{erf}(x)$ as stated in equation
When the slit width is much smaller than the beam width, $\delta \ll w_x$, the error functions in equation (2.6c) can be expanded with the relation [23],

$$\operatorname{erf}(x_0 + \epsilon) \approx \operatorname{erf}(x_0) + 2\sqrt{\frac{\pi}{\epsilon}} \exp\left(-x_0^2\right) \epsilon \left(1 - \epsilon x_0 + \frac{1}{3} \epsilon^2 (2x_0^2 - 1)\right) + \mathcal{O}(\epsilon^4).$$

The expansion of the error functions results into an equation for the integrated power through the slit,

$$F(x') = \sqrt{\frac{2}{\pi w_x^2}} P\delta \exp\left(-\frac{2x'^2}{w_x^2}\right) \left[1 + \frac{\delta^2}{6w_x^2} \left(\frac{4x'^2}{w_x^2} - 1\right)\right] + \mathcal{O}\left(\frac{\delta^5}{w_x^5}\right),$$

$$\approx \sqrt{\frac{2}{\pi w_x^2}} P\delta \exp\left(-\frac{2x'^2}{w_x^2}\right).$$

This equation states that the resulting integrated intensity profile has the same width $w_x$ as the original Gaussian intensity profile, as stated equation 2.5. The same derivation holds for the $y$-direction. So, a scanning slit measurement of a Gaussian intensity profile $I(x, y)$ results in an integrated intensity profile $F(x')$ with the same width in the scan direction as the original profile.

### 2.2.3 Characterization of the laser intensity profile

The scanning slit method has been used to measure the intensity profile of the infrared laser at the focal plane. The scanning slit method is discussed in section 2.2.2. A metal plate with a slit has been placed above a power sensor (Thorlabs, S314C) with corresponding console (PM100USB). The power sensor has been placed on a linear translation stage (Standa, 8MTF). A schematic side-view of the setup is shown in figure 11. The power sensor head is presented by the black rectangle. The slit width $\delta$, the scan velocity $U$, the $x'$-axis in the scan direction, and the angle of incidence $\beta$ of the laser are indicated. A schematic top view of the scanning slit method is shown in figure 10, where also the angle $\alpha$ between the scan direction and the large axis of the elliptical intensity profile is indicated. The measurements have been performed with a slit width of $(55.8 \pm 0.4) \mu m$ and a scan velocity of $20 \mu m/s$. 

---

**Figure 10:** A schematic representation of the scanning slit measurement setup. The intensity profile is illustrated by the elliptic shape and the slit by the opening in the rectangle. The slit width $\delta$, the scan velocity $U$, the $x'$-axis in the scan direction, the $x$ and $y$-axis of the elliptical intensity profile and the angle $\alpha$ between the scan direction and the large axis of the elliptical intensity profile are indicated.
Figure 11: A schematic side-view of the scanning slit measurement setup. The power sensor head is represented by the black rectangle. The slit width $\delta$, the scan velocity $U$, the $x'$ axis in the scan direction, and the angle of incidence $\beta$ of the laser are indicated.

The integrated intensity profile of three measurements are presented in figure 12. The laser beam is directed at normal incidence during the first two shown measurements, $\beta = 0^\circ$. The elliptical intensity profile has been scanned in the direction of the short and the long axis, respectively $\alpha = 90^\circ$ and $\alpha = 0^\circ$.

The results of these two measurements are presented by respectively the solid and dotted line. The results show that the maximum intensity of the scan in the direction of the small axis, $\alpha = 90^\circ$, is much higher and the profile much narrower than of the scan in the direction of the large axis, $\alpha = 0^\circ$.

The substrate in the turntable setup is illuminated at a 45° angle by the laser beam. Therefore, a third measurements has been performed with an angle of incidence of $\beta = 45^\circ$. The elliptical intensity profile has been scanned in the direction of the small axis again, $\alpha = 90^\circ$. Figure 12 shows the resulting profile by the dashed line. The results shows that the maximum and the width of the intensity profile are respectively lower and broader with respect to the measurement at normal incidence, ($\beta = 0^\circ$, $\alpha = 90^\circ$), due to the spreading of the beam over a larger area.

The measured intensity profiles are fitted with equation (2.6c) to calculate the beam width. The profiles can also be fitted with equation (2.11), which provides the same results for the beam width. Figure 12 shows the results by the thin lines for the three measurements. The double beam width $2w$ of the short axis of the elliptical intensity profile is $(1.188 \pm 0.002)$ mm, for the long axis $(6.44 \pm 0.04)$ mm. The double beam width at an angle of incident $\beta = 45^\circ$ is $(1.636 \pm 0.002)$ mm.

Figure 12: The measured intensity profile of the infrared laser beam. The angle of incidence $\beta$ of the laser beam has been varied, as is the angle between the large axis of the elliptical intensity profile and the scan direction $\alpha$.

The intensity profile has also been measured at different positions with respect to the focal plane. The laser beam has been positioned at normal incidence, $\beta = 0^\circ$. The measured profiles have been fitted with equation (2.6c) to find the width of the profile. Figure 13 shows the measured beam width as a function...
Figure 13: The beam width of the measured intensity profiles as a function of the displacement from the focal plane. A guide line connecting the mean value per displacement is added.

of the displacement from the focal plane and the uncertainty of the fitted values. The measurements have been repeated and a guide line connecting the mean value per displacement is added. It can be seen that the width of the beam increases as the displacement from the focal plane increases. However, the increase in beam width is small, up to 6% in this graph, due to the small numerical aperture of the cylindrical lens.
2.3 Experimental procedures

The experimental turntable setup has been prepared before every experiment. A clean substrate has been used for every series of experiments. The substrates are delivered with a protective foil, which is attached by a static charge. After removal of the protective foil on the bottom side of the substrate, the substrate is pressed against the glass plate using a paint roller to remove air between the plates. However, air is still trapped. This might cause fluctuations in the distance between the top of the substrate and the needle during the experiment. This will be discussed in section 2.4.1.

After the substrate is placed on the glass plate, the protective foil on the top side is removed. Both the substrate and the glass plate are discharged using a static control bar (Simco-Ion, MEB). The glass plate is subsequently placed on the air-bearings and connected to the motor. The needle height is set to the desired value and the laser beam is set to the desired position. The laser source is turned on at an operating current of 3A and the duty cycle fraction is set to obtain the desired optical laser power output.

The simultaneous recorded images from the bottom and side-view camera are analyzed and processed after the experiments. The following sections will deal with the shape of the droplet (2.3.1), the analysis of the shape of the droplet in the images (2.3.2) and the parameterization of the detected shapes (2.3.3).

2.3.1 Shape of the droplet

The variation of the droplet shape during the performed experiments has been quantified by key parameters of the droplet shape. Figure 14 shows typical bottom-view (a, b) and side-view (c, d) images of the droplet. The images have been taken from experiments with no infrared illumination and a substrate velocity that is relatively low (a, c) and relatively high (b, d), i.e., close to the critical velocity at which residual droplets are left behind on the substrate. A mirror image of the droplet is visible in the side-view images, due to reflections on the transparent glass plate and substrate. The horizontal line in the side-view images indicate the position of the substrate. The velocity of the substrate is indicated by $U$.

It can be seen in the bottom-view images (a, b) that the droplet tail has a round shape for a relatively low substrate velocity and a pointed shape for a relatively high substrate velocity. The considered droplet shape parameters in the bottom-view are the curvature radius $r$ and the opening angle $\alpha$ of the tip. These
properties are indicated in the figures on the right side of the droplet, to keep the droplet visible. The curvature radius is primarily a useful parameter for low substrate velocities, while the opening angle is primarily a useful parameter for high substrate velocities.

In the side-view images (c, d), the considered parameters are the advancing contact angle $\theta_a$ and the receding contact angle $\theta_r$. These contact angles are indicated in the figures. Because the receding side of the droplet is the area of importance, only the receding angle has been calculated. Also, the advancing contact angle is hard to determine with the current setup at relatively high substrate velocities.

The calculated parameters for a single image are not always representative for the experiment. Horizontal and vertical fluctuations of the droplet shape are present during the experiment and the detection of the shape of the droplet in an image is not always perfect. Therefore, multiple images are recorded for every experimental result. The images are processed separately and the resulting parameters are averaged and a standard deviation is calculated. To improve the accuracy of the results, the detection of the droplet and the calculated values are checked for errors per image and if necessary not taken into account during the averaging procedure.

### 2.3.2 Image processing

The analysis of the droplet shape is performed with different methods for the bottom and side-view images, due to the contrast differences. The contrast of the bottom-view images is low, as can be seen in the typical images shown in figure 14(a, b). The contrast in the side-view images is large, as can be seen in the typical images shown in figure 14(c, d). The droplet is very dark and the background is very bright. The image processing is performed using the commercial software package Matlab (The MathWorks, 2014a). The Matlab scripts to perform the image processing are listed in appendix A.

The processing of a bottom-view image is visualized in figure 15, where the processing of the typical bottom image shown in figure 14b is shown. The bottom-view image is first cropped, only the part of the image containing the droplet tail is processed further (a). Because of the low contrast, a built-in edge detection algorithm is used with a Canny filter [24] to find the edges in the image. The algorithm returns a black and white image with the found edges (b). It can be seen that the detection of the droplet shape is not perfect, edges within the droplet and in the background are also found. Furthermore, the edge around the droplet shape is not always continuous. So, additional steps are necessary to find the droplet shape. First a vertical line is added at the beginning of the droplet tail (c), to enclose the tail, and edges on the left side of this line are removed. To create a continuous edge around the droplet tail, white pixels a distance 4 pixels apart from each other are connected (d). The used technique rounds up sharp corners in the edges, as can be seen in the image around the edges close to the vertical line. To obtain only the perimeter of the droplet tail, the holes in the image are filled (e). This means that all pixels completely surrounded by an edge are made white. The droplet in the image is now represented by a white shape and only this shape is selected to be processed further (f). The perimeter of this shape is calculated (g) and finally the pixels representing the perimeter of the droplet shape are recalculated to $(x, y)$-pairs and plotted as an overlay on the original image (h). These data are used to calculate the curvature radius and the opening angle of the droplet shape and is discussed in section 2.3.3.

The side-view images are processed with a different method. The processing of the typical side image shown in figure 14d is visualized in figure 16. The side-view image is first cropped to only contain the advancing and receding part of the droplet (a). The complement of the image is taken, to make the area of importance in the image white (b). The image is transformed into a black and white image, using a very low threshold (c). This is possible due to the large contrast in the image. In order to have the droplet shape presented as a white shape in the image, the holes in the image are filled (d). Only this shape is selected to remove any noise in the background. The perimeter of this shape is calculated (e) and the pixels representing the perimeter of the droplet shape are recalculated to $(x, y)$-pairs and plotted as an overlay on the original image (f). These data are used to calculate the receding contact angle of the droplet and is discussed in section 2.3.3.
Figure 15: Different steps in the processing of a bottom-view image to obtain the droplet shape perimeter. The cropping of the original image (a), the found edges with the built-in edge-detection algorithm (b), the adding of a vertical line (c), the closing of the edges (d), the filling of the holes (e), the selection of the droplet shape (f), the calculation of the droplet shape perimeter (g), and the overlay of the resulting perimeter on the image (h) are shown.
Figure 16: Different steps in the processing of a side-view image to obtain the perimeter of the droplet shape. The cropping of the original image (a), the complement of the image (b), the black-and-white conversion (c), the filling of the holes (d), the calculating of the droplet shape perimeter (e), and the overlay of the resulting perimeter on the image (f) are shown.
2.3.3 Calculation of shape parameters

The calculation of the droplet shape parameters in the bottom and side-view images is discussed in this section. The parameters are calculated with use of the extracted perimeter data of the droplet shape in the corresponding images.

First the calculation of the droplet shape parameters in the bottom-view images is discussed. The parameters are the curvature radius and the opening angle. The curvature radius is determined by a fit of a circle through the tip of the droplet shape. The opening angle is determined by a fit of two straight lines through the side of the droplet. Figure 17 shows the perimeter of the droplet tip and the resulting fits, for a round droplet shape (a, b) and a pointed droplet shape (c, d). Figures (a) and (c) show the pixel positions of the perimeter of the droplet shape. Figures (b) and (d) show the resulting circle and lines. The data are taken from the processing of the corresponding images shown in figure 14(a, b). However, the graph is rotated anticlockwise with respect to the original images.

The maximum of the perimeter in the y-direction is considered the tip of the droplet. If multiple data points have the maximum value, the rounded averaged value is used as the x-position of the tip. All points within 10 pixels of the tip, in the negative y-direction, are used to fit a circle. These points are indicated in figure 17(a, c) by blue squares. The radius of the fitted circle is considered the curvature radius of the tip.

The straight lines through the left and the right of the droplet shape are fitted independently. For the right side, all pixels within 10 pixels of the first point not used for the circle fit, in the positive x-direction, are used to fit the straight line. The same method is used for the left side, but in the negative x-direction. These points are indicated in figure 17(a, c) by red squares. The derivatives of the two lines is used to calculate the opening angle of the droplet shape. Due to the definition of the data points for the fit of the straight lines, the data always contains different y-values. Therefore the maximum opening angle for a perfect circular droplet shape does not increase to 180°.

The resulting circle and straight lines for the data in figure 17(a, c) is shown in respectively figure 17(b, d). The figure shows that for a circular droplet shape the curvature radius is large, as is the opening angle, in contrast to a pointed droplet shape.

The considered droplet shape parameter in the side-view images is the receding contact angle. The receding contact angle is determined by a fit of a straight line through the tip of the droplet. Figure 18 shows the perimeter of the receding side of the droplet shape and the resulting fits, for images from an experiment with a relatively low substrate velocity (a) and a relatively high substrate velocity (b). The data are taken from the processing of the corresponding images shown in figure 14(c, d).

The maximum of the perimeter in the x-direction is considered the tip of the droplet. If multiple data points have the maximum value, the rounded averaged value is used as the y-position of the tip. All points within 10 pixels of the tip along the interface, are used to fit a straight line. These points are indicated in figure 18 by red squares, where also the fitted line is shown. The derivative of this line is used to calculate the receding contact angle. With this definition, the contact angle is measured at a distance of approximately 92 µm from the contact line, measured along the interface.
Figure 17: The perimeter of the droplet tip from a typical bottom-view image with a circular droplet shape (a) and a pointed droplet shape (c). The fitted circle and straight lines through these data are respectively shown in figure (b) and (d).
Figure 18: The resulting perimeter data and the calculated properties of the shape of the droplet from typical images.
2.4 Calibration of the experimental setup

2.4.1 Fluctuations of the substrate surface

The parameters of the droplet shape depend on the needle height with respect to the substrate surface. The fluctuations of the vertical position of the substrate surface with respect to the needle have been measured using a confocal sensor (MICRO-EPSON, IFS 2405) placed at the position of the needle. The confocal sensor measures the absolute distance from the sensor to the top of the substrate. The measured distance while the substrate is rotating provides the vertical fluctuations of the substrate surface.

Figure 19 shows the measured fluctuations in the experimental setup for a single rotation of the substrate, for four different experiments. The fluctuations are measured for a substrate which is simply placed on top of the glass plate, experiment 1, resulting in fluctuations of $(69 \pm 3) \mu m$ peak to peak. Subsequently, this substrate is pressed against the glass plate using a paint roller to remove the air between the glass plate and the substrate in experiment 2. However, it is visible that air is still trapped. This results in fluctuations of $(48 \pm 3) \mu m$ peak to peak. It has been tried to remove the air between the glass plate and the substrate. The same method is repeated with another substrate, experiment 3 and 4, resulting in fluctuations of $(136 \pm 3) \mu m$ and $(25 \pm 3) \mu m$ peak to peak, for respectively simply placing the substrate on the glass plate and by pressing it with a paint roller.

The large peaks in the fluctuations are due to air trapped between the glass and the substrate, while the other peaks are relatively small. For the experiments performed in this study, the substrate has been pressed to the glass plate to remove the air between them. Fluctuations between the needle and the top of the substrate are still present, but are considered not to influence the results. The large peaks are approximately 10% of the distance between the needle and the substrate surface.

The fluctuations have also been examined with use of images from the side-view camera. The fluctuations of the substrate can be determined in the images by looking at the vertical position of the tip of the droplet at the position of the advancing and receding contact line. To measure the fluctuations, a droplet is formed on a substrate which is rotating at 2 rpm. The fluctuations are shown in figure 20 for two rotations of the substrate. The circles indicate the fluctuations at the position of the receding contact line and the triangles indicate the fluctuations at the position of the advancing contact line. Mainly due to one large peak, the peak to peak values are $(73 \pm 9) \mu m$ and $(82 \pm 9) \mu m$ for the fluctuations at respectively the position of the receding and advancing contact line.
Figure 20: The position of the substrate surface, as measured with the side-view camera. The position of the substrate is determined at the receding (circles) and the advancing (triangles) contact line of a droplet.

2.4.2 Needle height

An experiment has been performed to study the influence of the distance between the substrate surface and the needle, the needle height. In this experiment the receding contact angle has been measured for different needle heights. The substrate velocity has been set to a constant value of \((12.84 \pm 0.02) \text{ mm/s}\), while the needle height has been increased from 175 \(\mu\text{m}\) to 576 \(\mu\text{m}\).

The results from this measurement are shown in figure 21, showing that the needle height has a significant effect on the receding contact angle for low needle heights. This is probably due to the method to calculate the angle. For the lowest needle height, the droplet height in the images is only 14 pixels. A large part of the droplet will therefore be taken into account during the calculation of the receding contact angle. For higher needle heights, a relatively smaller part of the perimeter of the droplet is used to calculate the receding contact angle. Furthermore, the size of a pixel in the side-view images is \((9.22 \pm 0.04) \mu\text{m}\). Therefore, the needle height is set to \((0.65 \pm 0.03) \text{ mm}\) during the experiments.

2.4.3 Low substrate velocities

For relatively low substrate velocities, the droplet shape as seen from the bottom is circular. The contact line can therefore be considered locally straight. For a straight contact line, the dynamic advancing and receding contact angle \(\theta_{a,r}^3\) can be described as function of the capillary number \(Ca\) according to the so-called Cox-Voinov relation [25, 26],

\[
\theta_{a,r}^3 = \theta_0^3 \pm 9 Ca \ln \left( \frac{x}{l} \right),
\]

where \(\theta_0\) represents a static contact angle, \(x\) the position of the measurement of the contact angle as measured along the interface and \(l\) a microscopic distance. The positive sign corresponds to the advancing contact angle, the negative sign to the receding contact angle. At higher substrate velocities, the droplet obtains a pointed shape and the Cox-Voinov relation is no longer valid.

A measurement of the receding contact angle of an ethylene glycol droplet as function of substrate velocity has been conducted with use of the turntable setup. The substrate velocity has been varied up to \((6.42 \pm 0.07) \text{ mm/s}\), which corresponds to a capillary number of \((2.29 \pm 0.03) \cdot 10^{-3}\).

The measured receding contact angles as function of the substrate velocity are shown in figure 22. It can be seen that the receding contact angle initially decreases as the substrate velocity increases. The results have been fitted with a Cox-Voinov like function of the form,

\[
\theta_r = \sqrt[a]{bU},
\]

\[\text{(2.13)}\]
Needle height [µm]

Receding contact angle [deg]

Figure 21: The receding contact angle as function of the distance between the top of the substrate and the needle.

where $\theta_r$ represents the receding contact angle, $U$ the substrate velocity and $a, b$ the fit coefficients. Results for substrate velocities smaller than 3.5 mm/s have been taken into account in the fit procedure. The results are weighted with the inverse of their uncertainties. The resulting fit coefficients are $a = (7.8 \pm 0.7) \cdot 10^4$ and $b = (-1.7 \pm 0.3) \cdot 10^4$. This corresponds with a receding contact angle of $(41 \pm 1)^\circ$ for a droplet in rest.

Figure 23 shows the measured receding contact angles to the third power. These results are fitted with the function,

$$\theta_r^3 = a + bU,$$

with the same fitting procedure and gives the same result for the fitting coefficients. The uncertainty of the fitting coefficients is shown graphically as the dashed lines.

Figure 23 shows that the relation between the receding contact angle to the third $\theta^3_r$ and the substrate velocity is linear for substrate velocities up to approximately 4.3 mm/s, or a capillary number of $1.5 \cdot 10^{-3}$. The experimental results from this measurement are also shown in figure 29, as indicated by the circles. It can be seen from the graphs in this figure, that the tip radius has changed to a constant value at this substrate velocity. Also, the opening angle of the droplet shape is approximate 90° at this substrate velocity. This indicates that the droplet has obtained a pointed shape. Therefore, the Cox-Voinov relation is no longer valid.
Figure 22: The receding contact angle as function of the substrate velocity. The measurement results have been fitted with the Cox-Voinov like equation 2.13.

Figure 23: The receding contact angle to the third power as function of the substrate velocity. The measurement results have been fitted with the Cox-Voinov like equation 2.14.
3 Experimental results

3.1 Variation of laser power

The influence of the intensity of the laser beam on the droplet shape has been studied. A droplet of ethylene glycol has been deposited on the substrate, which is moving at a constant velocity of \((4.28 \pm 0.05)\) mm/s. This corresponds to a capillary number of \(1.6 \times 10^{-3}\). The distance \(d\) between the edge of the needle and the laser spot has been set to 0.50 mm and 0.62 mm. The position of the laser spot is defined at the point of maximum intensity of the laser intensity profile. The total optical power of the laser beam has been varied between 0 W and 4.37 W by varying the duty cycle fraction from 0 to 0.7.

Figure 24 shows the bottom–view of the droplet for laser powers of 0 W, 0.62 W, 1.25 W and 1.87 W during the experiment. The laser distance \(d\) and the scale of the images are indicated. The droplet shape is pointed when the laser is off, as shown in figure 24a, and obtains a more circular shape for higher laser powers, as shown in figure 24(b-d). The opening angle and the curvature radius therefore increase for higher laser powers.

Figure 25 shows the droplet shape parameters as a function of the total power of the laser. That is, the curvature radius \((a)\), the opening angle \((b)\) and the receding contact angle \((c)\). The results are shown for two different values of the distance between the needle and laser, that is, 0.50 mm (circles) and 0.62 mm (squares). A smooth line is added to the results of both distances as a guide line to the eye.

The figure shows that the curvature radius, the opening angle and the receding contact angle increase for increasing laser power, that is, the effect of the infrared radiation increases as the total power of the laser increases. The increase of the curvature radius and the opening angle corresponds to a more circular droplet shape. For powers higher than 2.5 W the curvature radius and the opening angle saturate to a constant value.

When the laser is off, the curvature radius is \((0.06 \pm 0.01)\) mm due to the substrate velocity. Due to the increasing laser power, the radius increases to \((1.0 \pm 0.2)\) mm. The opening angle simultaneously increases from \((87 \pm 8)^\circ\) to \((152 \pm 6)^\circ\).

The results of the experiment with a laser distance of 0.5 mm show that the receding contact angle increases over the total power range. However, the results from the experiment with a laser distance of 0.62 mm show a decrease of the receding contact angle at a power of 2 W–2.5 W and for powers higher than 3.5 W. Figure 22 showed that for low substrate velocities the receding contact angle is approximate 45°. Furthermore, the results in section 3.3 show that the receding contact angle indeed increases for higher laser powers. Therefore, it is assumed that the results from the experiment with a laser distance of 0.50 mm show correct results. The receding contact angle of the droplet is \((9 \pm 2)^\circ\) due to the substrate velocity and it increases to \((32 \pm 3)^\circ\) due to the increasing laser power.

![Figure 24](image24.png)

**Figure 24:** bottom–view images of the droplet for different optical laser powers \(P\). Taken from the experiment with a distance between the needle and the laser of 0.5 mm.
Figure 25: The influence of the total optical power of the infrared laser beam on the droplet shape parameters, that is, the curvature radius (a), the opening angle (b) and the receding contact angle (c). The measurements have been performed with a distance between the needle and the point of maximum intensity of the laser beam of 0.50 mm (squares) and 0.62 mm (circles), at a substrate velocity of $(4.28 \pm 0.05)$ mm/s.
3.2 Variation of the laser spot position

The influence of the position of the laser beam on the droplet shape has been studied. A droplet of the liquid ethylene glycol has been deposited on the substrate, which is moving with a constant velocity of $(4.28 \pm 0.05) \text{ mm/s}$. This corresponds to a capillary number of $1.6 \cdot 10^{-3}$. The total optical power of the laser beam has been set to $(1.873 \pm 0.006) \text{ W}$. The distance $d$ between the edge of the needle and the laser spot has been varied between 0.85 mm and 2.35 mm. The position of the laser spot is defined at the point of maximum intensity of the laser intensity profile.

The droplet shape is pointed when the laser is off. Figure 26 shows bottom-view images of the droplet when the laser is on, for laser distances $d$ of 0.85 mm, 1.05 mm, 1.15 mm and 1.65 mm. The laser distance $d$ and the scale of the images are indicated, as is the distance between the needle edge and the droplet apex, droplet length $l$. Figure 26d also shows the laser intensity profile at 14%, 43% and 72% of the maximum value. The images show that the droplet has a more circular shape for small laser distances and that the droplet length increases as the laser distance increases. This shows that the effect of the infrared laser intensity profile on the droplet shape decreases for increasing laser distances.

![Figure 26: Bottom–view images of the droplet for different laser distances $d$, while the laser power is set to 1.87 W and the substrate velocity to 4.3 mm/s.](image)

Figure 27 shows the relation between the droplet length and the laser spot distance. The figure shows that the droplet length $l$ increases for increasing laser distances $d$, although the rate of increase reduces for larger laser distances. The figure also indicates the value for the droplet length when the laser is off. This value indicates that the droplet length increases further to $(1.12 \pm 0.05) \text{ mm}$ for laser distances greater than 2.35 mm.

The increase of the droplet length weakens for laser distances larger than approximately 1.4 mm. The droplet length is approximately 0.85 mm at that point. The distance between laser and the droplet apex is then $d - l \approx 0.55 \text{ mm}$. The width of the laser intensity profile is 0.82 mm, as is discussed in section 2.2.3 and shown in figure 12. The laser intensity at a distance of 0.55 mm from the center, is at 0.20% of its maximum value. For larger laser spot distances, the intensity at the droplet position decreases significantly.

Figure 28 shows the droplet shape parameters as a function of the laser spot distance. That is, the curvature radius (a), the opening angle (b) and the receding contact angle (c). The figure also indicates the parameter values when the laser is off. The figure shows that for increasing laser spot distance, all parameters decrease to this value. The rate of decrease of the parameters also reduces for laser spot distances greater than approximately 1.4 mm. This indicates that for the infrared laser illumination to be effective, the laser spot must be positioned near the contact line.
Figure 27: The droplet length as a function of the laser spot distance. Also indicated is the droplet length when the laser is turned off.
Figure 28: The influence of the laser spot distance on the droplet shape parameters, that is, the curvature radius (a), the opening angle (b) and the receding contact angle (c). The experiments have been performed with a total optical laser power of (1.873 ± 0.006) W and a substrate velocity of (4.28 ± 0.05) mm/s.
3.3 Critical velocity

The influence of the substrate velocity on the droplet shape has been studied, as well as the influence of the infrared laser beam. A droplet of ethylene glycol has been deposited on the substrate. The substrate velocity has been varied between (0.21 ± 0.02) mm/s and (9.42 ± 0.09) mm/s, corresponding to capillary numbers of respectively (0.075 ± 0.007)·10^{-3} and (3.36 ± 0.03)·10^{-3}. This has been repeated for total optical laser powers of 0 W, 0.6 W, 1.9 W and 3.7 W. The distance \( d \) between the edge of the needle and the laser spot has been set to (0.50 ± 0.01) mm. The position of the laser spot is defined at the point of maximum intensity of the laser intensity profile. The substrate velocity has been increased until residual droplets are left behind on the substrate.

Figure 29 shows the droplet shape parameters as a function of the substrate velocity. That is, the curvature radius (a), the opening angle (b) and the receding contact angle (c), for a total optical laser power of 0 W (circles), 0.6 W (squares), 1.9 W (diamonds) and 3.7 W (triangles). Capillary numbers corresponding to the substrate velocities are also indicated. The capillary numbers are calculated with the value for the viscosity and the surface tension at a constant temperature of 24°C, as calculated with respectively equation (2.1a) and (2.1b). During the measurement of the data point with the highest substrate velocity for every series, residual droplets have been left behind on the substrate.

Figure 29a shows the curvature radius of the droplet shape as a function of the substrate velocity. The curvature radii have been normalized for this graph, because the initial radius depends on the volume of the droplet. For all four series, the measured radii for the first four velocities have been averaged to obtain a normalization constant. The used normalization constants for the series with a laser power of 0 W, 0.6 W, 1.9 W and 3.7 W are respectively 1.14 mm, 0.97 mm, 0.84 mm and 1.02 mm. The figure shows that the curvature radius is constant for small substrate velocities and decreases for higher substrate velocities. A smaller curvature radius corresponds with a more pointed droplet shape. The substrate velocity at which the radius decreases, depends on the total laser power. This velocity is higher for higher laser powers.

Figure 29b shows the same effect for the opening angle. The opening angle is constant for small substrate velocities and decreases for higher substrate velocities. A smaller opening angle corresponds to a more pointed droplet shape. The substrate velocity at which the opening angle decreases also depends on the total laser power. The figure also shows that the minimal opening angle is (68 ± 3)°. Residual droplet are left behind on the substrate if the opening angle is smaller than this angle.

Figure 29c shows the receding contact angle as a function of the substrate velocity. The figure shows that the receding contact angle continuously decreases for an increasing substrate velocity, to a minimum of (5 ± 1)°.

The droplet shape parameters as a function of substrate velocity show that the droplet obtains a more pointed shape for increasing substrate velocities. The critical values of the parameters, corresponding to the values at the critical velocity, are not dependent on the laser power. However, the substrate velocity at which this minimum value is obtained, does depend on the total laser power.

Because residual droplets have been left behind during the measurement of the data point with the highest substrate velocity, the substrate velocity has been larger than the critical velocity of the droplet. The mean substrate velocity of the two data points with the highest velocity is considered the critical velocity of the droplet, for every series. Figure 30 shows the critical velocity as a function of the total laser power, as well as the critical capillary number. It shows that the critical velocity increases for increasing total laser power.
Figure 29: The influence of the substrate velocity on the droplet shape parameters, that is, the curvature radius (a), the opening angle (b) and the receding contact angle (c). The experiments have been performed with different total optical laser powers.
Figure 30: The critical substrate velocity as function of the total laser power.
4 Substrate temperature distribution

To obtain an estimate of the substrate temperature distribution during the conducted experiments, a numerical model of a moving substrate, illuminated by a Gaussian intensity profile, has been considered. The dimensions of the Gaussian intensity profile correspond to the dimensions of the experimental laser intensity profile, as measured in section 2.2.3. The motion of the substrate is considered rectilinear and steady. The presence of the polycarbonate substrate and the glass plate in the experiments is taken into account. However, the presence of the liquid is disregarded. The simulations have been performed using a time-dependent finite-element solver, using the commercial software package Comsol (COMSOL Ltd, 3.5a).

4.1 Numerical model

The problem has a symmetry-plane parallel to the direction of the substrate velocity. Only one side of the problem has been considered in the model, to reduce the complexity of the model. Figure 31 shows the considered geometry for the numerical model. The geometry consists of two stacked domains, upper domain A represents the polycarbonate substrate and bottom domain B represents the supporting glass plate. The two domains have approximately the same thickness as the corresponding values in the experimental setup. That is, a height of 0.5 mm for domain A and 3.9 mm for domain B. The x and y-dimensions of the domains have been chosen larger than the dimensions of the infrared laser intensity profile, to satisfy the chosen boundary conditions. The substrate velocity $U_{\text{sub}}$ is directed in the positive x-direction.

The upper surface of domain A is positioned at the $z = 0$ plane and the domain extends in the negative $z$-direction up to $z_1 = -0.5$ mm. Domain B extends in the negative $z$-direction from $z_1 = -0.5$ mm up to $z_2 = -4.4$ mm. The symmetry-plane is positioned at the $y = 0$ plane and the domains extend to $y_2 = 8$ mm. Both domains are positioned in the $x$-direction from $x_1 = -1.8$ mm to $x_2 = 18.2$ mm. The maximum mesh element size is 500 $\mu$m at the area of importance. Simulations with a finer grid provide the same results.

![Figure 31: The geometry of the numerical model for the temperature simulations of a 3-dimensional moving substrate consisting of two stacked domains. A Gaussian heat source is indicated by the elliptical shape. The center of the elliptical shape is positioned at the origin of the domain.](image)

The evolution of the temperature distribution of a moving substrate, illuminated by an intensity profile, can be described by the heat conduction and convection equation [27],

$$\frac{DT}{Dt} - \kappa \nabla^2 T = \frac{Q(t, x, y, z)}{\rho C_p},$$

where $\rho$ represents the mass density, $C_p$ the specific heat at constant pressure and $\kappa$ the diffusivity. The diffusivity is defined as $\kappa \equiv \frac{k}{\rho C_p}$, where $k$ represents the thermal conductivity of the substrate. A source term is represented by $Q(t, x, y, z)$. For a substrate moving in the $x$-direction and a time-independent source term, the equation can be written as,

$$\rho C_p \frac{\partial T}{\partial t} + \rho C_p U_{\text{sub}} \frac{\partial T}{\partial x} - \kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = Q(x, y, z).$$

The heat conduction and convection equation is numerically solved on the geometrical model until a steady-state is reached. Different properties for the top and bottom domain are taken into account. The mass density, the specific heat and the thermal conductivity of domain A, representing the polycarbonate
substrate, are set to respectively $\rho_A = 1200 \text{kg/m}^3$, $C_p,A = 1240 \text{J/(kg K)}$ and $k_A = 0.22 \text{W/(m K)}$. The same parameters in domain $B$, representing the glass plate, are set to respectively $\rho_B = 2500 \text{kg/m}^3$, $C_p,B = 750 \text{J/(kg K)}$ and $k_B = 1 \text{W/(m K)}$.

The initial temperature is set to $T_i = 20^\circ\text{C}$. The boundary condition at the planes $y = 0$ and $y = y_2$ is that of thermal insulation, $\vec{n} \cdot \nabla T = 0$, where $\vec{n}$ is the normal vector of the computational domain. The temperature at the boundary plane $x = -x_1$ is set to a constant temperature, $T = T_i$. A convective flux boundary condition is applied to the plane $x = x_2$, allowing heat to be transported out of the computational domain. A heat flux $q$ is applied as the boundary condition for the top surface of domain $A$ and the bottom surface of domain $B$. This flux consists of convective and radiation losses and is described as [27],

$$q = h_c(T_i - T) + \sigma_{SB} (T_i^4 - T^4),$$

where $h_c$ is the convective heat transfer coefficient and $\sigma_{SB}$ the Stefan–Boltzmann-coefficient. The first term represents convective losses and the second term represents radiation losses. The heat transfer coefficient is set at the top surface of domain $A$ to $h_{ht,A} = 5 \text{W/(m}^2\text{K})$, and at the bottom surface of domain $B$ to $h_{ht,B} = 7 \text{W/(m}^2\text{K})$.

A Gaussian laser intensity profile has been considered as the heat source $Q(x,y,z)$. The width $w$ of the intensity profile in the $x$ and $y$-direction are approximately those from the experimental study. Namely, $w_x = 0.6 \text{mm}$ and $w_y = 3.2 \text{mm}$. The expression for the source term of the conduction and convection equation is described as,

$$Q(x,y,z) = \frac{2A(1-R)P}{\pi w_x w_y} \exp \left(-\frac{2x^2}{w_x^2}\right) \exp \left(-\frac{2y^2}{w_y^2}\right) \exp \left(-A |z|\right),$$

where $A$ represents the absorption coefficient, $R$ the reflection coefficient and $P$ the total power. The absorption coefficient and the reflection coefficient are set to respectively $A = 100 \text{m}^{-1}$ and $R = 0.04$. The total power has been varied between 0.6 W and 3.7 W. In this expression for the heat source, the in-plane profile of the intensity profile is considered independent of the $z$-direction. The angle of incidence is set to 0° and the spreading of the beam width is neglected.
4.2 Steady-state temperature distribution

The substrate velocity has been set to $U_{sub} = 4.3 \text{ mm/s}$ and the laser power to $P = 1.9 \text{ W}$. These values are typical for the performed experiments. The steady-state temperature increase $\Delta T(x, y, z) = T(x, y, z) - T_i$ of the substrate is illustrated in figure 32. The point of highest intensity of the intensity profile is located at the origin, which is indicated in the figure by a cross.

Figure 32a shows the temperature increase $\Delta T(x, y, z = 0)$ of the top substrate surface as a function of the $x$ and $y$-position. The maximum temperature increase is $6.7^\circ \text{C}$. The figure shows that the point of maximum temperature is not located at the origin but displaced in the direction of the substrate velocity, i.e., at $x = 0.75 \text{ mm}$. Furthermore, the temperature profile is narrow in the negative $x$-direction, but broad in the $y$-direction at the position of the origin.

Figure 32b shows the temperature increase $\Delta T(x, y = 0, z)$ of the symmetry plane as a function of the $x$ and $z$-position. The figure shows that the point of maximum temperature is located at the substrate surface, but displaced in the direction of the substrate velocity. The temperature decreases for decreasing values of $z$, that is, farther into the substrate.

The results in figure 32 show that for typical values for the substrate velocity and the laser power, the point of maximum temperature is located at the top substrate surface, but moved in the direction of the substrate velocity with respect to the position of the maximum intensity.

![Figure 32](image)

(a) The steady-state temperature increase $\Delta T(x, y, z = 0)$ of the substrate at the top substrate surface.

![Figure 32b](image)

(b) The steady-state temperature increase $\Delta T(x, y = 0, z)$ of the substrate in the symmetry plane.

**Figure 32:** The steady-state temperature increase $\Delta T(x, y, z)$ at the top substrate surface (a) and in the symmetry plane (b).
4.3 Steady-state temperature profiles

The laser power $P$ has been varied between the values 0.6 W, 1.9 W and 3.7 W, while the substrate velocity has been set to $U_{sub} = 4.3$ mm/s. These values have been used in the experimental study. Figure 33 shows the steady-state temperature increase of the top substrate surface along the symmetry plane $\Delta T(x, y = 0, z = 0)$, for the different laser powers. The point of maximum temperature is indicated by a dot. The figure shows that the maximum temperature increase rises for increasing laser power approximately linear. The temperature increase is sufficiently small for the non-linear radiative losses to be negligible. However, the position of the point of maximum temperature remains constant at $x = 0.74$ mm. At a position of $x \approx -0.8$ mm the temperature increase is small for all laser powers.

Also the substrate velocity $U_{sub}$ has been varied, between 2 mm/s and 10 mm/s, while the laser power $P$ has been set to 1.9 W. These values have also been used in the experimental study. Figure 34 shows the steady-state temperature increase of the substrate surface along the symmetry plane $\Delta T(x, y = 0, z = 0)$, for the different substrate velocities. The point of maximum temperature is indicated by a dot. The figure shows that for increasing substrate velocity the maximum temperature rise decreases from 14°C to 3°C, while the position of the maximum temperature increases from $x = 0.55$ mm to $x = 0.87$ mm. The temperature increase at position $x \approx -0.8$ mm is again small.

Figure 35 shows the maximum temperature increase as a function of the substrate velocity. The corresponding thermal Peclet number $Pe$ of the substrate is indicated in the figure. The Peclet number is calculated with equation (2.3), using the width of the intensity profile as the typical length scale $L = 0.6$ mm, the diffusivity of polycarbonate $\kappa = 1.5 \cdot 10^{-7}$, and the substrate velocity as the typical velocity. The relation between the maximum temperature increase and the velocity in this regime is proportional to $Pe^{-1}$ [13, 28].

The results in figure 33 and figure 34 indicate that the maximum temperature increase for the considered values is between approximately 14.1°C and 3°C, for respectively the lowest and highest substrate velocity. The temperature increase of the substrate decreases for increasing substrate velocity or decreasing laser power. However, the position of the maximum temperature increase is not at the origin, but displaced in the direction of the substrate velocity. This offset increases for increasing substrate velocity, but is independent of the laser power.

In the experimental setup, the needle is positioned at the negative x-side. The temperature profile on this side of the origin is narrow. This indicates that the contact line should be relatively close to the point of maximum intensity of the infrared laser intensity profile.

![Figure 33: The steady-state temperature increase of the top substrate surface in the symmetry plane $\Delta T(x, y = 0, z = 0)$, for different laser powers. The point of maximum temperature increase is indicated by a dot.](image)
Figure 34: The steady-state temperature increase of the top substrate surface in the symmetry plane \( \Delta T(x, y = 0, z = 0) \), for different substrate velocities. The point of maximum temperature increase is indicated by a dot.

Figure 35: The maximum steady-state temperature increase as a function of the substrate velocity. The thermal Peclet number is indicated.
4.4 Comparison

The results from the simulations suggest that the temperature increase of the substrate at a distance \(0.8\) mm in the negative \(x\)-direction from the origin is very small. However, the experimental study suggests that the influence of the infrared laser beam extends to a larger distance, as discussed in section 3.2.

The presence of the liquid is disregarded in the simulations. The absorption coefficient of ethylene glycol is unknown, but considered comparable with the absorption coefficient of tri-ethylene glycol, which is in the order of \(A = 700\,\text{m}^{-1}\) [13]. This is larger than the absorption coefficient of the substrate and the glass plate. The receding side of the droplet is illuminated by the laser beam and could therefore have an influence on the temperature distribution of the substrate and the droplet.

Furthermore, the angle of incidence \(\beta\) is set to 0° and the spreading of the beam width is neglected in the performed simulations. The source term has been adjusted to incorporate both effects. The angle of incidence is set to 45° and the spreading of the beam width is calculated with use of the numerical aperture of the cylindrical lens, \(\text{NA} = 0.138\). Figure 36 shows the steady-state temperature profile \(\Delta T(x, y, z) = T(x, y, z) - T_i\) at the top substrate surface \(\Delta T(x, y, z = 0)\) (a) and in the symmetry plane \(\Delta T(x, y = 0, z)\) (b) as simulated with the adjusted source term. The origin is indicated by a cross. The figure shows a maximum temperature increase of 7.5°C, which is higher than with the initial source term. The position of the maximum temperature increase is slightly increased to \(x = 0.77\) mm. Most importantly, the temperature increase profile in the negative \(x\)-direction is broader than with the initial source term.

Figure 37 shows the temperature increase of the top substrate surface at the symmetry plane \(\Delta T(x, y = 0, z = 0)\) for the initial source term (\(\beta = 0°\)) and for the adjusted source term (\(\beta = 45°\)). The figure shows that the steady-state temperature increase is higher and the temperature increase of the top substrate surface extends more in the negative \(x\)-direction.

(a) The steady-state temperature increase \(\Delta T(x, y = 0, z)\) of the top substrate surface.

(b) The steady-state temperature increase \(\Delta T(x, y = 0, z)\) of the substrate in the symmetry plane.

Figure 36: The steady-state temperature increase \(\Delta T(x, y, z)\) at the top substrate surface (a) and in the symmetry plane (b), as simulated with the adjusted source term.
Figure 37: The steady-state temperature increase $\Delta T(x, y = 0, z = 0)$ of the top substrate surface as calculated with the initial source term ($\beta = 0^\circ$) and with the adjusted source term ($\beta = 45^\circ$).

### 4.5 Conclusion

The numerical simulations of the steady-state temperature distribution of the substrate during the conducted experiments show that the temperature increase depends on the substrate velocity and the total laser power. The position of the maximum temperature increase is displaced with respect to the position of maximum intensity, in the direction of the substrate velocity. The maximum temperature increase of the substrate has been found to be between 3°C and 14°C. The temperature increase at the position of the droplet is significantly lower. However, the presence of the liquid is disregarded in the numerical model.
5 Numerical models and procedures

The influence of a non-uniform temperature distribution on the shape of a droplet has been studied numerically. Simulations of 3-dimensional droplets attached to a needle are numerically complex. Therefore, a numerical model of 2-dimensional droplets sliding down an incline has been considered. Experimental results in literature indicate that the dynamics of the contact line shape of droplets on an inclined substrate and of droplets in a turntable-setup are qualitatively comparable [7].

The numerical model has been used to calculate the steady-state velocity of a sliding droplet as a function of the inclination angle. Subsequently, the steady-state velocity has been increased by imposing a moving non-uniform temperature profile on the substrate. The simulations are performed using a time-dependent finite-element solver from the commercial software package Comsol (COMSOL Ltd, 3.5a). The results are processed using the commercial software package Matlab (The MathWorks, R2010a).

Figure 38a shows a schematic representation of a 2-dimensional droplet on an inclined substrate with inclination angle $\alpha$ and temperature $T_0$. The droplet obtains a steady-state velocity $U_{grav}$ due to gravity, referred to as the gravitational steady-state velocity, which depends on the inclination angle. The steady-state droplet shape, with corresponding advancing contact angle $\theta_a$ and receding contact angle $\theta_r$, is obtained as a function of the gravitational steady-state velocity. The droplet velocity is increased by imposing a moving non-uniform temperature profile $T(x)$ on the substrate. Figure 38b shows a schematic representation of a 2-dimensional sliding droplet on an inclined substrate with inclination angle $\alpha$ and a moving non-uniform temperature profile $T(x)$. The temperature profile is moved across the substrate with velocity $U_T$, higher than the gravitational steady-state velocity of the droplet $U_{grav}$. If the temperature profile is ‘strong’ enough, the droplet obtains a new steady-state velocity $U_{temp}$, equal to the velocity of the temperature profile $U_T$.

The imposed temperature profile $T(x)$ has been calculated using a separate numerical model. In this model the steady-state temperature profile of a moving substrate, illuminated by a Gaussian intensity profile, has been calculated. The temperature distribution within the droplet is considered to be independent of height. Therefore, the temperature profile at the liquid-air interface is equal to the temperature profile at the liquid-solid interface, i.e., the substrate. This corresponds to a system with a small thermal Peclet number and a small Biot number, which are defined in respectively equation (2.3) and equation (2.4).

5.1 Governing equations

5.1.1 Lubrication equation

The flows in this numerical study are considered to be thin liquid film flows. A thin liquid film is characterized by a typical film height $H$ which is much smaller than the typical longitudinal length $L$, $H/L = \varepsilon \ll 1$. Figure 39 shows a schematic representation of a thin film, where the film height $h(x)$ and the velocities $u_x$ and $u_z$ are indicated. The liquid-solid interface is positioned at $z = 0$ and the liquid-air interface is positioned at $z = h(x)$. A thin film flow can be described with the lubrication equation, which is derived from the equations for momentum conservation and mass conservation of the liquid. The origin of the lubrication equation is discussed in this section, the full derivation is presented in appendix B.
The liquid in this numerical study is considered Newtonian and incompressible. Momentum conservation is then described by the Navier-Stokes equation,

$$\rho \frac{D \vec{u}}{Dt} = -\nabla P + \mu \nabla^2 \vec{u},$$

(5.1)

where $\rho$ represents the mass density, $\vec{u} = (u_x, u_y, u_z)$ the velocity, $P$ the dynamic pressure and $\mu$ the dynamic viscosity. The dynamic pressure is defined as,

$$P \equiv p + \rho \vec{g} \cdot \vec{r} = p + p_{grav},$$

(5.2)

where $p$ represents the pressure, $\vec{g}$ the gravitational acceleration and $\vec{r}$ the position vector. The equation for the conservation of mass is given by the continuity equation,

$$\nabla \cdot \vec{u} = 0.$$  

(5.3)

Furthermore, the Reynolds number based on the typical film height is,

$$Re = \frac{\rho U H}{\mu}.$$  

(5.4)

The important terms of the Navier-Stokes equation (5.1) and the continuity equation (5.3) are found by a scaling analysis. The scaled continuity equation provides a relation for the characteristic longitudinal velocity scale $U$ and the characteristic transverse velocity scale $V$,

$$\frac{V}{U} = \varepsilon.$$  

Under the assumption that $\varepsilon^2 \ll 1$ and $\varepsilon Re H \ll 1$, the scaled Navier-Stokes equation reduces to the equations which are commonly referred to as the lubrication approximation. The equations are in dimensional form,

$$\mu \frac{\partial^2 u_x}{\partial z^2} = \frac{\partial P}{\partial x},$$

(5.5a)

$$\mu \frac{\partial^2 u_y}{\partial z^2} = \frac{\partial P}{\partial y},$$

(5.5b)

$$\frac{\partial P}{\partial z} = 0.$$  

(5.5c)

With use of the appropriate boundary conditions, these equations are used to obtain the lubrication equation. For the boundary at the liquid-solid interface at $z = 0$, the no-slip and no-penetration boundary conditions are chosen,

$$\vec{u}(x, y, z = 0) = (0, 0, 0).$$  

(5.6)

The stress balance is chosen as boundary condition for the liquid-air interface at $z = h$. The stress balance can be expressed in a normal and tangential component [17], which are respectively,

$$p_1 - p_2 + \vec{n} \cdot \left(\vec{T}_2 - \vec{T}_1\right) \cdot \vec{n} + 2\kappa \gamma = 0,$$

(5.7)

$$\vec{t} \cdot \left(\vec{T}_2 - \vec{T}_1\right) \cdot \vec{n} + \vec{t} \cdot \nabla_s \gamma = 0,$$

(5.8)

where $\vec{n}$ represents the normal vector on the interface, $\vec{t}$ a tangential vector, $\vec{T}$ the stress tensor, $2\kappa$ the curvature of the liquid air interface, and $\gamma$ the surface tension. The normal component of the stress balance provides a boundary condition for the pressure,

$$p(h) = p_{amb} - \gamma \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right).$$  

(5.9)
The disjoining pressure as a function of the liquid height. The disjoining pressure at liquid height $h_s$ is indicated.

The tangential component of the stress balance provides two boundary conditions for the velocity in respectively the $x$ and $y$-direction,

$$
\mu \frac{\partial u_x}{\partial z} = \frac{\partial \gamma}{\partial x} \quad \text{and} \quad \mu \frac{\partial u_y}{\partial z} = \frac{\partial \gamma}{\partial y}.
$$

These boundary equations are used to integrate the equations of the lubrication approximation, as stated in equation (5.5), resulting in a relation for the evolution of the liquid height. This relation is commonly referred to as the lubrication equation,

$$
\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{3\mu} \frac{\partial P}{\partial x} h^3 - \frac{1}{2\mu} \frac{\partial \gamma}{\partial x} h^2 \right) + \frac{\partial}{\partial y} \left( \frac{1}{3\mu} \frac{\partial P}{\partial y} h^3 - \frac{1}{2\mu} \frac{\partial \gamma}{\partial y} h^2 \right),
$$

with a dynamic pressure of,

$$
P = p_{amb} - \gamma \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) + p_{grav}.
$$

### 5.1.2 Disjoining pressure

The lubrication equation describes the evolution of the thin liquid film in the system. However, the contact line of the droplet must be modeled separately. In this numerical study, the disjoining pressure model has been used to model the contact lines. The disjoining pressure represents the effect of intermolecular interactions between the liquid and the adjacent solid and gas on the liquid pressure. A phenomenological model for the disjoining pressure as function of liquid height $\Pi(h)$ has been used in this study to model the contact line,

$$
\Pi(h) = \gamma \left( \frac{1 - \cos \theta_0}{h_s} \right) \left( \frac{m - 1}{n - m} \right) \left[ \left( \frac{h_s}{h} \right)^n - \left( \frac{h_s}{h} \right)^m \right],
$$

where $\gamma$ represents the surface tension and $\theta_0$ a steady-state contact angle. The constants $m$ and $n$ describe the strength of the disjoining pressure and $m < n$. The constant $h_s$ represents a liquid height for which the disjoining pressure is 0. In order to use the disjoining pressure to model the contact line of the droplet, the liquid height on the area outside the droplet is set to $h_s$. In this study, the used values for the parameters $(n, m, h_s)$ are $(3, 2, 10 \text{nm})$.

Figure 40 shows the disjoining pressure as a function of the liquid height. For this graph, the surface tension of ethylene glycol at 293 K of 48.43 mN/m has been used for $\gamma$ and an angle of 5° has been used for the contact angle $\theta_0$. The figure shows that the disjoining pressure is positive for liquid heights lower than the constant $h_s$ and increases to infinity for $h \to 0$. It is negative for liquid heights higher than the constant $h_s$. For increasing liquid heights, the disjoining pressure has an absolute minimum and increases asymptotically to zero.
5.1.3 Heat conduction and convection equation

The steady-state temperature profile of a moving substrate, illuminated by a Gaussian intensity profile has been considered as the moving temperature profile in the sliding droplet simulations. The evolution of the temperature distribution $T = T(x, y, z, t)$ is described by the heat conduction and convection equation \[ DT \frac{Dt}{Dt} - \kappa \nabla^2 T = \frac{Q(t, x, y, z)}{\rho C_p}, \] (5.14) where $\rho$ represents the mass density, $C_p$ the specific heat at constant pressure and $\kappa$ the diffusivity. The diffusivity is defined as $\kappa \equiv \frac{k}{\rho C_p}$, where $k$ represents the thermal conductivity of the substrate. A source term is represented by $Q(t, x, y, z)$. In the numerical models, the direction of the substrate velocity $U_{sub}$ is considered in the positive $x$-direction and the intensity profile $I(x, y, z)$ time-independent. The heat conduction and convection equation can therefore be written as, \[ \rho C_p \frac{\partial T}{\partial t} + \rho C_p U_{sub} \frac{\partial T}{\partial x} - k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = I(x, y, z). \] (5.15)

5.2 Numerical models

5.2.1 Sliding droplet

Figure 41 shows the considered geometry for the 2-dimensional sliding droplets simulations. A 1-dimensional substrate extends along the $x$-axis from position $x_1$ at -2.5 mm to position $x_2$ at 2.5 mm. The maximum mesh size for the finite-element solver is initially 1 $\mu$m around the contact line. When steady-state is obtained the maximum mesh size is decreased to 0.1 $\mu$m and the simulation is restarted with the steady-state as the initial condition. This provides the same solution as if the problem were solved with the fine mesh directly, but considerably faster. A further mesh refinement provides the same solution. The maximum mesh size for the rest of the domain is set to 100 $\mu$m.

The simulations are performed in the rest frame of the steady-state droplet. This is achieved by moving the substrate with velocity $U_{sub}$ in opposite direction as the sliding droplet. This velocity is not known in advance and is therefore adjusted by iteration. This speeds up the convergence of the simulation. The droplet slides in the negative $x$-direction, so the substrate moves in the positive $x$-direction. Due to the substrate movement, a liquid flux $\Phi$ is present at the boundaries of the geometry. An inward flux at position $x_1$ of $\Phi_1 = h_s U_{sub}$ and an outward flux at position $x_2$ is $\Phi_2 = h U_{sub}$. The pressure at positions $x_1$ and $x_2$ is set to the gravitational pressure.

The moving substrate is incorporated in the lubrication equation. The general lubrication equation is stated in equation (5.11) and is adjusted for the moving substrate by adding the term $\frac{\partial}{\partial z} (h U_{sub})$, \[ \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{3\mu} \frac{\partial P}{\partial x} h^3 - \frac{1}{2\mu} \frac{\partial}{\partial x} h^2 + U_{sub} h \right), \] (5.16a) \[ P = \rho g (h \cos \alpha + x \sin \alpha) - \gamma \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) - \Pi. \] (5.16b)

The equation for the dynamic pressure $P$ consists of the gravitational term for an incline with inclination angle $\alpha$, the capillary pressure term and the disjoining pressure $\Pi$, which is stated in equation (5.13). The surface tension $\gamma$ is temperature-dependent, as are the viscosity $\mu$ and the mass density $\rho$. The thermocapillary stress term $\frac{\partial \gamma}{\partial x}$ can be rewritten as a function of the temperature gradient, \[ \frac{\partial \gamma}{\partial x} = \frac{\partial \gamma}{\partial T} \frac{\partial T}{\partial x}. \] (5.17)

Figure 41: The geometry for the numerical model of a 2-dimensional sliding droplet.
The temperature-dependent viscosity, surface tension and mass density of ethylene glycol have been used for the corresponding liquid properties. These properties are described by the equations stated in equation (2.1) and graphically shown in figure 4, both given in section 2.1.1. The initial liquid height on the domain around the droplet has been set to the constant \( h_s \), from the equation for the disjoining pressure \( \Pi \). A symmetric parabola has been considered for the initial liquid height profile of the droplet,

\[
h_{\text{init}}(x) = a_2 x^2 + a_0,
\]

where the coefficients \( a_2 = -\frac{1}{\Delta x} \tan \theta_0 \) and \( a_0 = h_s + \frac{1}{2} \Delta x \tan \theta_0 \) are obtained by determining an initial contact angle \( \theta_0 \) and an initial width \( \Delta x \) for the droplet. The contact angle \( \theta_0 \) must be small enough for the lubrication equation to be valid, i.e., \( H/L = \varepsilon \ll 1 \).

An initial width of \( \Delta x = 1.77 \text{ mm} \) and a contact angle of \( \theta_0 = 5^\circ \) are used in this study. This corresponds to the coefficients \( a_2 = -49.43 \text{ m}^{-1} \) and \( a_0 = 3.872 \times 10^{-5} \text{ m} \). The initial maximum value of the droplet height profile is then \( H_{\text{max}} = 3.872 \times 10^{-5} \text{ m} \), which results in \( H/L = H_{\text{max}}/\Delta x = 2.10^{-2} \ll 1 \). The steady-state height profile of a droplet on a horizontal substrate is discussed in section 5.3, as is the method to determine the advancing contact angle and the receding contact angle of the droplet.

### 5.2.2 Temperature profile of a moving substrate

The temperature profile of a moving substrate, illuminated by a Gaussian intensity profile has been considered. Figure 42 shows a schematic representation of the system. The system is considered in the rest frame of the intensity profile. The substrate is moving in the positive \( x \)-direction with velocity \( U_{\text{sub}} \). A Gaussian intensity profile \( I(x) \) with beam width \( w_x \) is indicated in the figure.

**Figure 42:** Schematic representation of a moving substrate, illuminated by a Gaussian intensity profile \( I(x) \) with width \( w \). The substrate velocity \( U_{\text{sub}} \) is indicated.

The considered geometry for the simulations of the temperature profile of a moving substrate, is identical to the geometry for the 2-dimensional sliding droplet simulations, as is shown in figure 41. A 1-dimensional substrate extends along the \( x \)-axis from position \( x_1 \) at -2.5 mm to position \( x_2 \) at 2.5 mm. The maximum mesh size for the finite-element solver is 1 \( \mu \text{m} \). A finer mesh provides identical solutions. The substrate is moving with velocity \( U_{\text{sub}} \) in the positive \( x \)-direction. The boundary condition at \( x_1 \) is that of thermal insulation, \( \frac{\partial T}{\partial x} = 0 \). The boundary condition at \( x_2 \) is that of convective flux, which allows heat to be transported out of the computational domain. The heat conduction and convection equation, as stated in equation (5.15), is solved until a steady-state is reached. For this model, the heat conduction and convection equation can be written as,

\[
\rho C_p \frac{\partial T}{\partial t} + \rho C_p U_{\text{sub}} \frac{\partial T}{\partial x} - k \frac{\partial^2 T}{\partial x^2} = Q(x),
\]

The properties of polycarbonate are used for the parameters. The mass density \( \rho \) is set to 1200 kg/m\(^3\) and the specific heat at constant pressure \( C_p \) to 1240 J/kgK. However, a small thermal conductivity \( k \) of \( 2.2 \times 10^{-4} \text{ W/Km} \) is used, due to the small considered substrate velocities. The source term \( Q(x) \) consists of a Gaussian intensity profile \( I(x) \),

\[
Q(x) = A(1 - R)I(x) = \frac{2A(1-R)}{\sqrt{2\pi w_x c_y}} \exp \left( -\frac{2x^2}{w_x^2} \right),
\]

where \( A \) represents the absorption coefficient, \( R \) the reflection coefficient and \( P/c_y \) the total optical power per unit length. The absorption coefficient \( A \) is set to 30 m\(^{-1}\), the reflection coefficient \( R \) to 0.04, the laser width \( w_x \) to 600 \( \mu\text{m} \), the power \( P \) to 1 mW, and \( c_y \) to 600 \( \mu\text{m} \). The considered substrate velocities and the resulting temperature profiles are discussed in section 5.5.
5.3 Numerical methods

The 2-dimensional droplet has been simulated on a substrate with no inclination to obtain the steady-state liquid height profile of a droplet in rest. Figure 43 shows the initial parabola and the steady-state height profile for a droplet in rest with respectively a line and a dashed line. The parabola and the steady-state height profile are symmetric in the $x$-direction and only the right sides are shown in the figure. The steady-state height profile of the droplet in rest has been used as the initial condition for subsequent simulations. For the parameter studies, the solution for a specific parameter has been used as the initial condition for the subsequent simulations. This gives the same steady-state solution as when the droplet in rest, or the initial parabola, is used as the initial condition for the height profile.

The steady-state height profile of a droplet is continuous, as is its derivative. Therefore, a definition for the position to calculate the contact angle must be made. The angle $\theta(x)$ between the tangent of the height profile and the horizontal axis as a function of the $x$-position is,

$$\theta(x) = \arctan \left( \frac{\partial h}{\partial x} \right). \quad (5.21)$$

Figure 44 shows the angle $\theta(x)$ as a function of position for the initial parabola and steady-state height profile of a droplet in rest. Figure (a) shows the angle over the complete domain. The angle shows absolute extrema, close to the location of the contact lines, for both the initial parabola and the height profile of the droplet in rest. A close up of the absolute minimum is shown in figure (b). It shows that the angle of the height profile of the droplet in rest is continuous. Both extrema have been used to calculate the contact angles on the left and right side of the droplet. Their value is considered the contact angle and their position the contact line position. The droplet slides in the negative $x$-direction, so the advancing contact angle $\theta_a$ is located at the left side of the droplet and the receding contact angle $\theta_r$ at the right side of the droplet. The contact angles are therefore defined as,

$$\theta_a = \max \left[ \arctan \left( \frac{\partial h}{\partial x} \right) \right], \quad (5.22)$$

$$\theta_r = \min \left[ \arctan \left( \frac{\partial h}{\partial x} \right) \right]. \quad (5.23)$$

The minimum value of the angle is negative, so the absolute value is taken to obtain a positive angle. The advancing and receding contact angle are calculated directly within Comsol.

To study the influence of the inclination angle $\alpha$, simulations of a droplet on an incline with inclination angles between 0° and 40° have been performed. The temperature is set to a constant value of $T = 293$ K. The steady-state liquid height profile and the steady-state velocity of the droplet have been obtained. The results of these simulations are presented and discussed in section 5.4. Subsequently, the inclination angle has been set to a constant value of $\alpha = 2°$. The previous solution of the simulation with an inclination angle of 2° has been used as the initial condition. A temperature profile $T(x)$ has been applied over the domain, moving with a velocity $U_T$ with respect to the substrate. The calculation of the imposed moving non-uniform temperature profiles is discussed in section 5.5. The velocity of the temperature profile has been varied between 1 and 30 times the gravitational steady-state velocity $U_{grav}$ of the droplet. The same method has been used with a constant inclination angle of $\alpha = 10°$. The results of these simulations are presented and discussed in section 5.6.
Figure 43: The initial parabola and the steady-state height profile of the droplet in rest. Both profiles are symmetric in the $x$-direction and only the right sides are shown.

Figure 44: The angle of the initial parabola and the steady-state height profile of the droplet in rest (a) and a close-up of the absolute minimum (b).
5.4 Variation of inclination angle

The numerical model as described in section 5.2.1 has been used to perform simulations of a droplet sliding down an incline with inclination angle $\alpha$. Figure 38a shows a schematic representation of the system. The temperature of the substrate has been set to a constant value $T_0$ of 293 K. The inclination angle has been varied between 0° and 40° and the gravitational steady-state velocity and the liquid height profile of the droplet have been obtained as a function of the inclination angle.

Figure 45 shows the steady-state liquid height profile for droplets sliding down an incline with an inclination angle $\alpha$ of 8°, 16°, 24°, 32° and 40°. The figure shows that for increasing inclination angle, the length of the droplet increases and the position of the maximum height decreases. Figure 46 shows the angle $\theta(x)$ of the height profiles, as calculated with equation (5.21). It shows that for increasing inclination angle, both the maximum and the minimum increase. This corresponds to an increasing advancing contact angle and a decreasing receding contact angle.

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**Figure 45:** The steady-state liquid height profile of droplets sliding down an incline with inclination angle $\alpha$, at a constant temperature of $T_0 = 293$ K.

**Figure 46:** The angle of the steady-state liquid height profile of droplets sliding down an incline with inclination angle $\alpha$, at a constant temperature of $T_0 = 293$ K.
Figure 47 shows the steady-state velocity of the droplet as a function of the inclination angle. The corresponding capillary number is indicated on the right vertical axis in units of $10^{-3}$. The definition of the capillary number is stated in equation (1.3). The figure shows that the steady-state velocity of the droplet increases, for increasing inclination angle. However, the rate of increase reduces for higher inclination angles. The lateral component of the gravitational force is proportional to $\sin \alpha$. The steady-state velocity $U(\alpha)$ has therefore been fitted with a sine function of the form,

$$U(\alpha) = A_f \sin \omega_f \alpha,$$

(5.24)

with amplitude $A_f$ and angular frequency $\omega_f$. The resulting fit is graphically presented in figure 47 by the straight line. The fit gives the values,

$$A_f = (25.12 \pm 0.06) \mu m/s,$$

(5.25)

$$\omega_f = (1.568 \pm 0.004) \text{deg}^{-1}.$$

(5.26)

The advancing contact angle $\theta_a$ and the receding contact angle $\theta_r$ have been calculated from the obtained steady-state height profiles, using respectively equation (5.22) and (5.23). Figure 48 shows the contact angles as a function of the steady-state velocity of the droplet. The corresponding capillary number is indicated on the top horizontal axis in units of $10^{-3}$. The figure shows that the advancing contact angle increases for increasing velocity and the receding contact angle decreases with increasing velocity. The advancing and receding contact angles $\theta_{a,r}$ and the steady state velocity $U_{\text{grav}}$ have been fitted with a Cox-Voinov like function of the form,

$$\theta_{a,r} = \sqrt[3]{a_{a,r}^3 + b_{a,r} U},$$

(5.27)

with fit parameters $a_{a,r}$ and $b_{a,r}$. The advancing contact angles and the receding contact angles have been fitted separately and the resulting fits respectively provides the coefficients,

$$a_a = (4.869 \pm 0.001) \text{deg},$$

(5.28)

$$b_a = (3.092 \pm 0.005) \text{deg}/(\mu m s^{-1}),$$

(5.29)

$$a_r = (4.850 \pm 0.009) \text{deg},$$

(5.30)

$$b_r = (-2.66 \pm 0.04) \text{deg}/(\mu m s^{-1}).$$

(5.31)

The fits are graphically presented in figure 48, by the straight lines through the corresponding data. There is no distinction between an advancing and receding side for a droplet in rest, so the fit coefficients $a_a$ and $a_r$ should be equal. The values are indeed consistent within 1%. The rate of change for the advancing contact angle is higher than for the receding contact angle.
Figure 47: The steady-state droplet velocity as function of the inclination angle.

Figure 48: The advancing contact angle and receding contact angle as a function of the gravitational steady-state velocity of the droplet.
5.5 Temperature profiles

The numerical model as described in section 5.2.2 has been used to obtain the non-uniform temperature profile of a moving substrate, illuminated by a Gaussian intensity profile. As is discussed in section 5.3, two different temperature profiles have been calculated. The steady-state velocity of the sliding droplet is increased at a constant inclination angle of 2° and 10°. The temperature profile for the 2°-case has been calculated with a substrate velocity of \( U_{\text{sub}}^{(2°)} \). For the 10°-case, the temperature profile has been calculated with a substrate velocity of \( U_{\text{sub}}^{(10°)} \). For these velocities, the average velocity of the gravitational steady-state velocity of the sliding droplets has been used.

It has been found in section 5.4 that the gravitational steady-state velocity of a droplet at an incline with inclination angle 2° is \( U_2 = 1.4 \mu m/s \), whereas the gravitational steady-state velocity of the droplet at an incline with inclination angle 10° is \( U_{10°} = 6.8 \mu m/s \). The gravitational steady-state velocity at the highest considered inclination angle of 40° is \( U_{40°} = 22.3 \mu m/s \). The considered velocities for the moving substrate are,

\[
U_{\text{sub}}^{(2°)} = \frac{1}{2} U_{40°} + \frac{1}{2} U_2 = 11.8 \mu m/s,
\]

\[
U_{\text{sub}}^{(10°)} = \frac{1}{2} U_{40°} + \frac{1}{2} U_{10°} = 14.5 \mu m/s.
\]

The Gaussian intensity profile is given by equation 5.20. The point of maximum intensity is set to the position of 2 mm. This source term is used to calculate the temperature profile of the substrate, in the rest frame of the intensity profile.

Figure 49a shows the resulting temperature profiles. The left side of the domain, \( x<0 \), has a constant temperature of \( T_0 = 293 \) K, so only the right side of the domain, \( x>0 \), is shown. The temperature profiles have a maximum temperature of respectively 320 K and 315 K, corresponding with a temperature increase of respectively 27 K and 22 K. Figure 49b shows the gradient of the temperature profiles. It shows that the maximum temperature gradient is at the position of the point of maximum intensity of the source term, i.e., at \( x = 2 \) mm. The maximum of the temperature gradient is respectively 36 K/mm and 29 K/mm. The results in figure 49 show that the maximum temperature and maximum temperature gradient decreases for increasing substrate velocity. For relatively large substrate velocities, the temperature increase and the temperature gradient will diminish.

![Figure 49: The temperature profile (a) and its gradient (b) of the moving substrate, for substrate velocities \( U_{\text{sub}}^{(2°)} \) and \( U_{\text{sub}}^{(10°)} \).](image-url)
5.6 Variation of laser velocity

The numerical model as described in section 5.2.1 has been used to perform simulations of a droplet sliding down an incline with inclination angle $\alpha$, while a moving non-uniform temperature profile is imposed on the substrate to increase the steady-state droplet velocity. Figure 38b shows a schematic representation of the system.

The inclination angle $\alpha$ of the inclined substrate is set to 2°. A moving non-uniform temperature profile, induced by a Gaussian intensity profile, is imposed on the substrate and is moving with a velocity $U_T$ with respect to the substrate. The droplet slides down the incline due to gravity and the moving temperature profile. The resulting steady-state droplet velocity $U_{temp}$ is equal to the velocity of the temperature profile $U_T$. The simulation is performed in the rest frame of the laser. The same simulations have been performed with a constant inclination angle of 10°. The imposed temperature profiles are discussed in section 5.5.

Figure 50 shows the steady-state liquid height profile of droplets sliding down an incline with an inclination angle $\alpha$ of 2° with velocities between 8.2 $\mu$m/s and 24.6 $\mu$m/s, due gravitation and the moving temperature profile. The lowest velocity is higher than the gravitational steady-state velocity of the droplet at a 2° inclination. The imposed temperature profile is also shown in the figure, corresponding to the right vertical axis. The figure shows that for increasing velocity, the droplet length decreases and the position of the maximum increases. The opposite effect occurs when gravitation is the only driving force, as is shown in section 5.4. The position of the receding contact line also increases with increasing velocity. This results in a higher temperature at the receding contact line and in a higher thermocapillary driving force. However, the temperature increase is small with respect to the maximum temperature increase.

Figure 51 shows the angle of the profiles, as calculated with equation (5.21). For increasing velocities, the maximum angle increases and the minimum decreases. This corresponds with an increasing advancing contact angle and a decreasing receding contact angle. When gravitation is the only driving force, the receding contact angle decreases for increasing velocities, as is shown in section 5.4.

Figure 52a shows the advancing contact angles (circles) and the receding contact angles (squares) as a function of the droplet velocity ($T(x)$), as calculated with respectively equation (5.22) and equation (5.23). For comparison, the figure also shows the contact angles when only the lateral gravitational force is used to increase the steady-state velocity ($T_0$). These are the same data as shown in figure 48. The considered substrate velocities in the calculation of the temperature profiles are also indicated in the figure by the two vertical lines, for the 2° case ($U_{sub}^{(2°)}$) and the 10° case ($U_{sub}^{(10°)}$).

The figure shows that the advancing contact angles (circles) are identical for droplets driven by only gravitation and for droplets driven by both gravitation and the temperature profile. However, this is not the case for the receding contact angles (squares). Due to the non-uniform temperature profile, the receding contact angle increases with increasing velocity, in contrast to the receding contact angle of droplets driven by only gravity.

Figure 52b also shows the advancing contact angles (circles) and the receding contact angles (squares) as a function of the steady-state droplet velocity, but for a larger velocity range. The figure shows that both the advancing and receding contact angles keep increasing for increasing velocities. However, the rate of increase reduces for high velocities. Due to the relatively low obtained steady-state velocity with only gravitation as the driving force, the advancing contact angles cannot be compared.

Figure 52 also shows the advancing and receding contact angle for droplets with a temperature-independent viscosity on an inclination of 2° ($T(x), \alpha 2°, \mu_0$). The viscosity has been set to its value at $T = 293$ K. However, the results are consistent with the results from simulation with a temperature-dependent viscosity. This might be due to the small temperature increases at the receding contact line.

Because the receding contact angle increases for increasing velocities, a critical velocity cannot be defined. However, the simulations are performed 1-dimensional and the non-uniform temperature profile is kept constant with velocity. These results are therefore not consistent with potential experiments.
Figure 50: The steady-state liquid height profile of droplets sliding down an incline with an inclination angle $\theta$ of 2°, with different velocities, due to the lateral gravitational force and the temperature profile.

Figure 51: The angle of the steady-state liquid height profile of droplets sliding down an incline with an inclination angle $\theta$ of 2°, with different velocities, due to the lateral gravitational force and the temperature profile.
Figure 52: The advancing contact angle (circles) and the receding contact angle (squares) as a function of the steady-state velocity of the droplet, due to the imposed temperature profile \( T(x) \) and gravity \( (\alpha=2^\circ \text{ and } \alpha=10^\circ) \). For comparison, the figure also shows the contact angles when only the lateral gravitational force is used to increase the steady-state velocity \( T_0 \). This is the same data as shown in figure 48. The simulations have also been performed with a temperature-independent viscosity \( \mu_0 \). Figure (b) shows the results for a larger velocity range.
6 Semi-analytical model

In order to gain insight into the potential and the dominant mechanism of laser induced contact line mobility enhancement, the analytical model for the motion of a 2-dimensional droplet by Ford et al. has been adopted [9]. The original model determines the steady-state velocity of a 2-dimensional droplet due to thermocapillary stresses in the lubrication approximation. The lubrication theory is discussed in section 5.1.1 and the derivation is given in appendix B. Ford et al. only considered thermocapillary stresses as the driving mechanism, but in this study the gravitational force due to an incline is also incorporated in the model, following the theory by Dussan et al. [29]. Furthermore, the temperature-dependency of the liquid properties is taken into account, that is, the viscosity, the surface tension and the mass density. The analytical model is derived in the following section and is discussed and numerically analyzed in section 6.2.

6.1 Derivation

Figure 53 shows a schematic representation of the considered system. The droplet is positioned on an inclined plane with inclination angle $\alpha$ between $-R<x<R$, in the coordinate system in which the droplet is in rest. The droplet velocity $U$, the advancing contact angle $\theta_a$, the receding contact angle $\theta_r$, the inclination angle $\alpha$, and the temperature profile $T(x)$ are indicated in the figure.

Figure 53: A schematic representation of the considered system, consisting of a 2-dimensional droplet on an inclined plane. The droplet velocity $U$, the advancing contact angle $\theta_a$, the receding contact angle $\theta_r$, the inclination angle $\alpha$, and the temperature profile $T(x)$ are indicated.

The model considers the liquid height profile of the droplet $h(x)$ as an input parameter and provides a steady-state velocity of the droplet due to thermocapillary stresses and the gravitational force. The temperature distribution within the system is considered to be independent of height. Therefore, the temperature profile at the liquid-air interface is equal to the temperature profile at the liquid-solid interface. This corresponds to a system with a small thermal Peclet number and a small Biot number. A small Peclet number indicates that the diffusive heat transfer in the system is dominant and a small Biot number indicates that the temperature differences within the system are small. The Peclet number and the Biot number are defined in respectively equation (2.3) and equation (2.4).

The steady state droplet velocity $U$ is determined by means of a lateral force balance per unit length in the $z$-direction. The lateral gravitation force, the wall shear stress and the contact line forces are taken into account, but liquid-air friction is neglected. The solid–gas surface tension $\gamma_{sg}$, the solid–liquid surface tension $\gamma_{sl}$ and the liquid–gas surface tension $\gamma_{lg}$ define the contact line forces $F_l$ at the advancing $a$ and receding $r$ side,

$$F_l = (\gamma_{sg,a} - \gamma_{sl,a}) - (\gamma_{sg,r} - \gamma_{sl,r}),$$

(6.1)

$$= \gamma_{lg,a} \cos \theta_a - \gamma_{lg,r} \cos \theta_r,$$

(6.2)

by means of Young’s law $\gamma_{sg} - \gamma_{sl} = \gamma_{lg} \cos \theta$. For convenience, the liquid-gas surface tension $\gamma_{lg}$ is written as $\gamma$. The lateral force balance is then given by,

$$-\int_{-R}^{R} \mu \frac{\partial u_x}{\partial y} \bigg|_{y=0} \, dx + g \sin \alpha \int_{-R}^{R} \rho h \, dx + \gamma_a \cos \theta_a - \gamma_r \cos \theta_r = 0,$$

(6.3)

where $\mu$ represents the viscosity, $u_x$ the lateral liquid velocity, $g$ the gravitational constant, and $\rho$ the mass density. The first term represents the wall stresses, the second term the gravitational forces and the third
and fourth term the contact line forces. The lateral liquid velocity \( u_x \) is obtained from the lubrication approximation,

\[
    u_x = -\frac{1}{2\mu} \frac{\partial P}{\partial x} y^2 + c_1 y + c_0,
\]

where \( c_0 \) and \( c_1 \) are integration constants and \( P \) represents the dynamic pressure. The latter is defined as,

\[
    P \equiv p_0 - \frac{\partial^2 h}{\partial x^2} + \rho g (h \cos \alpha + x \sin \alpha),
\]

where \( p_0 \) represents the ambient pressure, the second term the capillary pressure and the last term the gravitational pressure.

The integration constant \( c_1 \) is determined by the thermocapillary shear stress boundary condition at the liquid-air interface \( y = h \). The thermocapillary shear stress is defined as \( \tau = \frac{\partial \gamma}{\partial T} \frac{\partial T}{\partial x} \) and is therefore related to the temperature profile \( T(x) \). The boundary condition relates the shear stress to the flow profile.

\[
    \tau = \mu \frac{\partial u_x}{\partial y} \bigg|_{y=h(x)}
\]

\[
    = \mu \left( -\frac{2}{2\mu} \frac{\partial P}{\partial x} h + c_1 \right) \Rightarrow \quad c_1 = \frac{1}{\mu} \left( \tau + \frac{\partial P}{\partial x} h \right)
\]

The integration constant \( c_0 \) is determined by the Navier slip condition with slip length \( b \) at the solid–liquid interface \( y = 0 \), in the rest frame of the droplet. The Navier slip boundary condition relates the slip velocity at the solid-liquid interface to the wall shear stress.

\[
    u_x(y = 0) = c_0 = -U + b \frac{\partial u_x}{\partial y} \bigg|_{y=0}
\]

\[
    = -U + b c_1
\]

\[
    = -U + b \left( \frac{1}{\mu} \left( \tau + \frac{\partial P}{\partial x} h \right) \right)
\]

With these integration constants, the lateral velocity profile can be written as,

\[
    u_x = -\frac{1}{2\mu} \frac{\partial P}{\partial x} y^2 + \frac{y + b}{\mu} \left( \tau + \frac{\partial P}{\partial x} h \right) - U.
\]

With use of this velocity, a condition for the pressure gradient \( \frac{\partial P}{\partial x} \) is found. In the rest frame of the droplet, the height-averaged velocity must be zero.

\[
    0 = \int_0^{h(x)} u_x dy
\]

\[
    = \left[ -\frac{1}{2\mu} \frac{\partial}{\partial x} \frac{y^3}{3} + \frac{y^2}{2\mu} \left( \tau + \frac{\partial P}{\partial x} h \right) + \frac{by}{\mu} \left( \tau + \frac{\partial P}{\partial x} h \right) - U y \right]_0^h
\]

\[
    = \frac{h^3}{3\mu} \frac{\partial P}{\partial x} + \frac{h^2}{2\mu} \tau + \frac{bh}{\mu} \left( \tau + \frac{\partial P}{\partial x} h \right) - U h
\]

\[
    = \frac{\partial P}{\partial x} \left( \frac{h^3}{3\mu} + \frac{bh^2}{\mu} \right) + \frac{h^2}{2\mu} \tau + \frac{bh \tau}{\mu} - U h
\]

Which yields an expression for the pressure gradient,

\[
    \frac{\partial P}{\partial x} = \frac{-\frac{3}{2} h \tau - 3b \tau + 3U \mu}{h^2 + 3bh}
\]
This expression is used in the force balance to calculate the velocity of the droplet.

\[
0 = -\int_{-R}^{R} \mu \frac{\partial u_x}{\partial y} \bigg|_{y=0} \, dx + g \sin \alpha \int_{-R}^{R} \rho h \, dx + \gamma_a \cos \theta_a - \gamma_r \cos \theta_r
\]  
(6.18)

\[
= \int_{-R}^{R} \left( \tau + \frac{\partial P}{\partial x} \right) \, dx + g \sin \alpha \int_{-R}^{R} \rho h \, dx + \gamma_a \cos \theta_a - \gamma_r \cos \theta_r
\]  
(6.19)

\[
= \int_{-R}^{R} \left( \tau + \frac{\frac{3}{2} h \tau - 3 b \tau + 3 U \mu}{h^2 + 3b h} \right) \, dx + g \sin \alpha \int_{-R}^{R} \rho h \, dx + \gamma_a \cos \theta_a - \gamma_r \cos \theta_r
\]  
(6.20)

\[
= \int_{-R}^{R} \left( \tau + \frac{\frac{3}{2} h \tau + 3 b \tau - 3 U \mu}{h + 3b} + g \sin \alpha \rho h \right) \, dx + \gamma_a \cos \theta_a - \gamma_r \cos \theta_r
\]  
(6.21)

Which can be solved for the steady-state velocity of the droplet \( U \),

\[
3U \int_{-R}^{R} \frac{\mu}{h + 3b} \, dx = \int_{-R}^{R} \left( -\tau \left[ 1 + \frac{\frac{3}{2} h - 3b}{h + 3b} \right] + \rho g h \sin \alpha \right) \, dx + \gamma_a \cos \theta_a - \gamma_r \cos \theta_r \Rightarrow
\]  
(6.22)

\[
U = \frac{1}{3J} \left[ \int_{-R}^{R} \left( 1 + \frac{\frac{1}{2} h(x) \tau(x)}{h(x) + 3b} + \rho g h(x) \sin \alpha \right) \, dx + \gamma_a \cos \theta_a - \gamma_r \cos \theta_r \right],
\]  
(6.23)

\[
\text{where } J \equiv \int_{-R}^{R} \frac{\mu(x)}{h(x) + 3b} \, dx
\]  
(6.24)
6.2 Results and discussion

The equation for the steady state droplet velocity, equation (6.23), has four different terms. The first term is the driving force of the droplet due to a temperature profile. The thermocapillary stress is defined as \( \tau = \frac{\partial \gamma}{\partial x} = \frac{\partial \gamma}{\partial T} \frac{\partial T}{\partial x} \). It therefore depends on the gradient of the temperature profile. The second term is the driving force of the droplet due to an incline. The third and fourth term depend on the advancing and receding contact angle, and the corresponding value of the surface tension at the contact lines. An analysis of equation (6.23) has been performed by integrating the equation numerically for different cases, using the commercial software package Matlab (The MathWorks, 2014a).

A 2-dimensional droplet of size \( 2R = 2 \text{ mm} \) and with a contact angle of \( \theta_0 = 5^\circ \) is considered. The liquid properties of ethylene glycol are used for the surface tension \( \gamma \), the viscosity \( \mu \) and the mass density \( \rho \). These properties are described by the equations stated in equation (2.1) and graphically shown in figure 4, which are both discussed in section 2.1.1. Unless stated otherwise, the slip length \( b \) has been set to 10 nm, the gravitational constant to \( g = 9.81 \text{ m/s}^2 \), the temperature to \( T = 293 \text{ K} = T_0 \), the inclination angle to \( \alpha = 0^\circ \), and both contact angles to \( \theta_r = \theta_a = 5^\circ = \theta_0 \). A symmetric parabola has been used as the liquid height profile \( h(x) \),

\[
h(x) = \begin{cases} 
0 & \text{if } x < -R \\
a_1 x^2 + a_2 & \text{if } -R \leq x \leq R \\
0 & \text{if } x > R,
\end{cases}
\]

with \( a_1 = -\frac{1}{2R^2} \tan \theta_0 = -43.74 \text{ m}^{-1} \) and \( a_2 = h_a + \frac{1}{2} R \tan \theta_0 = 43.74 \text{ µm} \). Figure 54 shows the liquid height profile as a function of position. The maximum value of the droplet height profile is then \( H_{\text{max}} = 43.74 \text{ µm} \), which results in \( H_{\text{max}}/2R = 2 \cdot 10^{-2} < 1 \). The lubrication approximation is therefore valid.

![Figure 54: The considered liquid height profile for the droplet as a function of position, as used in the calculations of the droplet velocity.](image)

The inclination angle \( \alpha \) has been varied between \( 0^\circ \) and \( 90^\circ \), while a constant temperature \( T = 293 \text{ K} \) is applied over the domain. Figure 55 shows the steady state velocity of the droplet as a function of the inclination angle. The gravitational force in the lateral direction scales with \( F_g \propto \sin \alpha \), which explains the sinusoidal relation between the velocity and the inclination angle. The calculated data has therefore been fitted with a sine function, resulting in an amplitude of \( 50.37 \text{ µm/s} \) and a period of \( 360^\circ \). The maximum steady state velocity at an inclination angle of \( 90^\circ \) is therefore \( 50.37 \text{ µm/s} \).

Next, temperature profiles have been applied to the domain. Three different temperature profiles have been considered. The first is a linearly decreasing temperature profile, \( T(x) = T_0 - \frac{\Delta T}{2R} (x - R) \),

where \( \Delta T \) is the temperature difference between the advancing and receding contact line and \( \frac{dT}{dx} = -\frac{\Delta T}{2R} \) the temperature gradient of the profile. The temperature difference \( \Delta T \) has been varied. The temperature
Figure 55: The droplet velocity as a function of the inclination angle, as calculated with equation (6.23).

at the advancing contact line, at position $x = R$, has been kept constant at a temperature of $T_0 = 293$ K, while the temperature at the receding side of the droplet is increased.

The second considered temperature profile is a smooth step function,

$$T(x) = T_0 + \frac{2\Delta T}{1 + \exp\left(10^4(x + R)\right)}.$$  \hspace{1cm} (6.27)

The temperature difference $\Delta T$ between the advancing and receding contact line has been varied, while the temperature at the advancing contact angle has been kept constant at a temperature of $T_0 = 293$ K. The third considered temperature profile is the same smooth step function as stated in equation (6.27), but narrower,

$$T(x) = T_0 + \frac{2\Delta T}{1 + \exp\left(10^5(x + R)\right)}.$$ \hspace{1cm} (6.28)

Figure 56 shows the three temperature profiles as a function of position for a temperature difference of $\Delta T = 50^\circ C$.

Figure 56: The considered temperature profiles for a temperature difference between the advancing and receding contact line of the droplet of $50^\circ C$. That is, the linearly decreasing temperature profile (linear) as stated in equation (6.26), the smooth step function (step 1) as stated in equation (6.27) and the smooth step function (step 2) as stated in equation (6.28).

First the influence of a linear decreasing temperature profile has been analyzed. The temperature profile is described by equation (6.26) and the temperature difference $\Delta T$ has been varied between $0^\circ$C and $50^\circ$C. Figure 57 shows the steady-state velocity as a function of the temperature difference $\Delta T$, for different inclination angles $\alpha$. Initially, the viscosity of the liquid has been considered temperature-dependent
\( \mu = \mu(T) \). The figure shows that the droplet obtains a velocity due to the temperature profile, even in the absence of an inclination. This is due to the thermocapillary stresses. The steady-state velocity increases for an increasing temperature difference.

Figure 57 also shows that the inclination angle provides an additional increase of the velocity. However, this additional increase depends on the temperature difference. This is due to the decreasing viscosity of the liquid for increasing temperature. Therefore, the same calculation of the steady-state velocity has been performed, but with a temperature-independent viscosity. The viscosity has been kept constant at \( \mu = \mu(T_0) = \mu_0 \) along the domain. The resulting velocity is also shown in figure 57 as a function of the temperature difference, for different inclination angles. It shows that the temperature-dependency of the viscosity has large influence for high temperature differences. A high temperature difference means a high absolute temperature at the receding side of the droplet and the viscosity decreases exponentially with increasing temperature. The viscosity has therefore a strong effect on the velocity of the droplet. Moreover, the additional increase of the velocity due to the incline is temperature-independent for the results with a temperature-independent viscosity.

The same calculation has been performed with a broad smooth step function as the temperature profile, as stated in equation (6.27). The temperature difference \( \Delta T \) has been varied between 0°C and 50°C, for different inclination angles \( \alpha \). Figure 58 shows the steady-state velocity as a function of the temperature difference. The viscosity has been considered either temperature-dependent (\( \mu = \mu(T) \)) and temperature-independent (\( \mu = \mu_0 \)). The velocity increases with increasing temperature difference. The effect of the inclination is again a velocity offset, which is temperature-independent for a temperature-independent viscosity. The resulting droplet velocity is smaller compared to the linear decreasing temperature profile, because the thermocapillary driving force is reduced.

The calculation has also been performed with a narrow smooth step function as the temperature profile, as stated in equation (6.28). Figure 59 shows the steady-state velocity as a function of the temperature difference \( \Delta T \). The resulting velocity is smaller than the velocity due to the linear temperature profile and due to the broad smooth step function as the temperature profile. The gradient of the narrow smooth step function is higher than the broad smooth step function, as stated in equation (6.27), but the effect on the steady-state velocity is smaller. The velocity increase due to the temperature-dependency of the viscosity is also smaller.

### 6.3 Conclusion

The analysis of the the semi-analytical model shows that a temperature profile provides an extra driving force due to thermocapillary stresses. The steady-state velocity of the droplet is further increased when the temperature-dependency of the viscosity is taken into account. This speed-up effect, i.e., the increase in the droplet mobility, is smaller for a narrower temperature profile. The thermocapillary forces depend on the temperature gradient, while the velocity increase due the temperature-dependency of the viscosity depends on the absolute temperature increase.
Figure 57: The steady-state droplet velocity as a function of the temperature difference of the linearly decreasing temperature profile, as stated in equation (6.26), for various inclination angles $\alpha$. The viscosity has been considered either temperature-dependent ($\mu = \mu(T)$) or temperature-independent ($\mu = \mu_0$).

Figure 58: The steady state droplet velocity as a function of the temperature difference with the smooth step function as temperature profile, as stated in equation (6.27), for various inclination angles $\alpha$. The viscosity has been considered either temperature-dependent ($\mu = \mu(T)$) or temperature-independent ($\mu = \mu_0$).

Figure 59: The steady state droplet velocity as a function of the temperature difference with the smooth step function as temperature profile, as stated in equation (6.28), for various inclination angles $\alpha$. The viscosity has been considered either temperature-dependent ($\mu = \mu(T)$) or temperature-independent ($\mu = \mu_0$).
7 Summary and conclusions

The shape of a moving droplet is largely determined by the mobility of its contact line. Therefore, it has been studied how infrared laser illumination of a moving droplet affects its shape and the dynamics of motion. Laboratory experiments have been conducted using a combination of a turntable-like setup with infrared laser illumination. These experiments are complemented by numerical simulations of laser-induced temperature profiles and of the resulting droplet deformations.

The main components of the turntable-setup are the rotating circular substrate, the concentric needle, the infrared laser beam, and the liquid droplet. The substrate consists of polycarbonate. The concentric needle consists of two hollow concentric cylinders. The inner cylinder constantly supplies liquid, while the outer cylinder constantly extracts a liquid-air mixture. The infrared laser beam is focused in one direction by an optical system, resulting in an intensity profile with an approximately elliptical shape. The long axis of the ellipse is directed perpendicular to the substrate velocity, so the receding side of the droplet is illuminated practically uniformly in this direction. Ethylene glycol has been chosen as the liquid to investigate, because it is a non-volatile liquid and the critical velocity is within the range of the current experimental setup.

The receding contact line of the moving droplet has a round shape for velocities far below the critical capillary number \( Ca \ll Ca_{cr} \). The contact line obtains a \( V \)-shaped profile at relatively high velocities \( Ca < Ca_{cr} \), while the droplet leaves behind residual liquid at velocities above a critical value \( Ca > Ca_{cr} \). The droplet shape has been parametrized by the curvature radius of the receding side, the opening angle of the droplet apex and the receding contact angle.

The laser power has been varied, while the laser spot position and the substrate velocity are kept constant. The results show that the receding side of the droplet initially has a pointed shape due to the substrate velocity. Due to infrared illumination, the droplet obtains a more rounded shape with increasing laser power. The same effect is observed when the laser spot distance to the droplet is decreased. This indicates that the critical velocity of the droplet has increased. Therefore, the combined variation of laser power and substrate velocity has been investigated. The results show that the velocity at which the droplet obtains a pointed shape increases with increasing laser power. Also the velocity at which the droplet leaves behind residual liquid on the substrate has been increased, i.e., the critical velocity is increased, by approximately a factor 2.

Numerical simulations of the temperature distribution of a 3-dimensional moving substrate, illuminated by a Gaussian intensity distribution, have been performed to obtain an estimate of the temperature distribution of the substrate during the conducted experiments. The temperature distribution depends on the substrate velocity and the total intensity of the illumination. The results show that the position of the maximum temperature is displaced with respect to the point of highest intensity. The temperature increase at positions larger than the beam width is small with respect to the maximum temperature increase. This is consistent with the experimental results, which show that the influence of the laser illumination increases for laser spot distances smaller than the laser beam width. However, the numerical results show a very small temperature increase at the position of the droplet. The presence of the liquid is disregarded in the numerical model, but could be responsible for a large temperature increase. There are indications that the absorption coefficient of ethylene glycol is large with respect to the absorption coefficient of the polycarbonate substrate and the glass plate.

Numerical simulations of 2-dimensional sliding droplets on an inclined substrate have been performed. The results show that the velocity of the droplet can be increased by applying a moving non-uniform temperature distribution. The receding contact angle increases with increasing velocity, due to the increased thermocapillary stresses. Therefore, a critical velocity cannot be defined.

Also an analytical model of 2-dimensional sliding droplets has been considered. The model describes the steady-state velocity of a droplet under the influence of gravity and a non-uniform temperature profile, as a function of the droplet shape. A numerical analysis of the model shows that a temperature profile provides an extra driving force due to thermocapillary stresses. The steady-state velocity of the droplet is further increased when the temperature-dependency of the viscosity is taken into account. This speed-up effect, i.e., the increase in the droplet mobility, is smaller for a narrower temperature profile.
7.1 Further research

For further research, the temperature distribution of the substrate could be studied in more detail. For example, the presence of the liquid can be taken into account during numerical simulations and experimental measurements of the temperature distribution could be performed. The temperature increase of the substrate might be important for technological applications. Also different material systems could be investigated, i.e., different liquids and different substrate materials. For example, water is a volatile liquid and has a high absorption coefficient for infrared light and a substrate of silicon is transparent for infrared light.

Furthermore, numerical simulations could be performed with a 3-dimensional sliding droplet. Literature suggests that all experimental shape regimes can be numerically reproduced [4]. A proof-of-concept simulation has been performed, using the same numerical model as discussed in chapter 5, but 3-dimensional. The lubrication equation has been solved, in combination with the disjoining pressure model. Figure 60 shows a 3-dimensional droplet sliding down an incline, as seen from above, at a velocity above the critical velocity. The figure shows the droplet at different time-steps. The direction of the substrate velocity $U_{sub}$ and the scale are indicated. The figure shows that the droplet loses residual liquid on the substrate, due to the high velocity. It could be studied how a non-uniform temperature distribution, induced by a Gaussian intensity profile, influences the droplet shape and if it is possible to prevent the losing of residual liquid on the substrate.

The feasibility to simulate a sliding droplet attached to a needle could also be investigated. The lubrication approximation is then no longer valid and multiple contact lines are introduced. The mesh size must be small along the contact lines, which makes the problem numerically complex.

![U_{sub}](image)

**Figure 60:** A 3-dimensional droplet sliding down an incline, as seen from above, at a velocity above the critical velocity. The figure shows the droplet at different time-steps. The direction of the substrate velocity $U_{sub}$ and the scale are indicated.


8 References


A Matlab functions and scripts

The analysis of the droplet shape and the calculation of the droplet shape parameters from the bottom and side view images are performed with the software Matlab (The Mathworks, R2014a 8.3.0.532). Some functions from the Image Processing Toolbox (v9.0) and the Curve Fitting Toolbox (v3.4.1) have been used. In this section, the four scripts to process the images from various experiments are presented.

The analysis of the droplet shape in the bottom view images is discussed in section 2.3.2, where the results of different steps in the analysis are shown in figure 15. These steps are performed with the Matlab script stated in box 1, where the different steps are commented. The considered bottom view image is cropped to only contain the droplet tail and is subsequently loaded into this function, together with a position $p$ somewhere on the droplet shape.

The analysis of the droplet shape in the side view images is also discussed in section 2.3.2 and the results of different steps in the analysis are shown in figure 16. Box 2 shows the commented Matlab script used for the analysis. The considered side view image is cropped to only contain the receding and advancing contact line and is subsequently loaded into this function, together with a position $p$ somewhere on the droplet shape.

The calculation of the droplet shape parameters, with use of the found droplet shape perimeters, is discussed in section 2.3.3. That is, the curvature radius, the opening angle and the receding contact angle of the droplet shape. The data points used in the calculation are shown in figure 17(a, c) for the bottom view images and in figure 18 for the side view images. The resulting droplet shape parameters in the bottom and side view images are shown in respectively figure 17(b, d) and figure 18. The script in box 3 is used to calculate the curvature radius and the opening angle from the bottom view images and the script in box 4 is used to calculate the receding contact angle from the side view images.
Box 1: Matlab function to detect droplet shape in bottom view images.

```matlab
function [x, y] = f_edgebottom(img, p)
% Function to detect droplet at x-position 'p' in image 'img'. Output is
% perimeter (x,y)-pair.

% Edge detection with 'Canny' filter, sigma 5
bw = edge(img, 'canny', [], 5);
% Add vertical line at point p
bw(:, p) = 1;
% Clear left side of point p
bw(:, 1:p-1) = 0;
% Close the image with the morphological 'close' function
disk = strel('disk', 2);
bw = imclose(bw, disk);
% Fill the holes in the image
bw = imfill(bw, 'holes');
% Only take the object in the middle of the screen
bw = bwselect(bw, p, round(size(bw,1)/2));
% Take the perimeter of the object
bw = bwperim(bw);
% Find the right perimeter of the droplet
y = [];
for k = 1:size(bw,1);
    temp = find(bw(k,:) == 1, 1, 'last');
    if ~isempty(temp)
        if temp > p
            y(end + 1) = temp;
            x(end + 1) = k;
        end
    end
end
end
```

Box 2: Matlab function to calculate droplet shape parameters in bottom view images.

```matlab
function [x, y, perim] = f_edgeside(img, p)
% Function to detect droplet at x-position 'p' in image 'img'
% Output is perimeter (x,y)-pair

% Take complement to make area of importance white
img = imcomplement(img);
% Create black-and-white image with low threshold (10)
bw = im2bw(img, 10/255);
% Close the top and bottom of the droplet
bw([1 end], :) = 1;
% Fill the droplet
bw = imfill(bw, 'holes');
% Open the top and bottom of the droplet
bw([1 end], :) = 0;
% Select only the droplet
bw = bwselect(bw, p(1), p(2));
% Take the perimeter of the droplet
perim = bwperim(bw);
% Calculate the x and y positions of the edge (perimeter)
[y, x] = find(perim == 1);
```
function [cr, cx, cy, gamma, tipx, tipy] = processbottomedge(x, y)

% This function calculates the curvature radius and opening angle from the edge in 'x' and 'y'. Output is respectively: circle radius, circle x-offset, circle y-offset, opening angle, x position tip, y position tip.

%%% DROPLET TIP
% Find the y value
maxy = max(y);
% Find the x value
midpx = find(y == maxy);
midpx = midpx(1) + round((midpx(end) - midpx(1))/2);

% FIT CIRCLE
% Determine points to fit (5 px from tip in negative y-direction)
yn = maxy - 5;
fitcx = x(y >= yn)';
fitcy = y(y >= yn)';
% Fit circle
circ = [fitcx fitcy ones(size(fitcx))]
    \(-\left(\text{fitcx}^2+\text{fitcy}^2\right)\);
cx = -.5*circ(1);
cy = -.5*circ(2);
cr = sqrt((circ(1)^2+circ(2)^2)/4 - circ(3));
ce = norm([fitcx fitcy ones(size(fitcx))] * circ - ((fitcx.^2+fitcy.^2))/2;

% FIT LINES
% Determine points to fit (10 px horizontally from data without 'circle fit data')
% Split first, otherwise midpx not correct
fitlxl = x(1:midpx);
fitlyl = y(1:midpx);
fitlxr = x(midpx:end);
fitlyr = y(midpx:end);
% Remove the points used to fit the circle data
listl = fitlyl < yn;
listr = fitlyr < yn;
fitlxl = fitlxl(listl);
fitlyl = fitlyl(listl);
fitlxr = fitlxr(listr);
fitlyr = fitlyr(listr);
% Only use the data 10 px horizontally
fitlxl = fitlxl(end-10:end);
fitlyl = fitlyl(end-10:end);
fitlxr = fitlxr(1:10);
fitlyr = fitlyr(1:10);
% Fit the line through the datapoints
[fitl gof] = fit(fitlxl', fitlyl', 'poly1');
[fitr gof] = fit(fitlxr', fitlyr', 'poly1');
% Calculate the opening angle
gamma = atand(1/fitl.p1) + atand(-1/fitr.p1);
Box 4: Matlab function to calculate droplet shape parameters in side view images.

```matlab
function [angle] = f_processside(x, y)
% This function calculates the receding contact angle from the edge in 'x' and 'y'.
% Output is receding contact angle

%%% DROPLET TIP
maxxp1 = find(xx == max(xx), 1, 'first');
maxxp2 = find(xx == max(xx), 1, 'last');
maxxp = maxxp1 + floor((maxxp2 - maxxp1)/2);
maxx = xx(maxxp);
maxy = yy(maxxp);

%%% FIT LINE
% Determine points to fit (10px from tip in positive y-direction)
list = (xx-maxx).^2 + (yy - maxy).^2 <= 10^2 & yy >= maxy;
fitx = xx(list);
fity = yy(list);
% Fit line through the datapoints
fitr = fit(fitx, fity, 'poly1');
% Calculate receding contact angle
angle = atand(abs(fitr.p1));
```
B  Full derivation of the lubrication equation

The flows in this numerical study are considered thin liquid film flows on a substrate, surrounded by air. A thin liquid film is characterized by a characteristic film height $H$ much smaller than the characteristic length scale $L$, $H/L = \varepsilon \ll 1$.

A schematic representation of a thin film is presented in figure 39. The film height $h$, the $z$ axis and the $x$ axis are indicated. The solid-liquid interface is positioned at $z = 0$, and the solid-air interface is positioned at $z = h$.

A thin film flow can be described with the lubrication equation. This equation can be derived from the equations for momentum and mass conservation. The liquids in this study are considered Newtonian and incompressible. The momentum conservation is then described by the Navier-Stokes equation,

$$\rho \frac{D \vec{u}}{Dt} = -\nabla P + \mu \nabla^2 \vec{u}, \quad (B.1)$$

where $\rho$ represents the mass density, $\vec{u}$ the velocity, $\mu$ the dynamic viscosity, and $P$ the dynamic pressure of the fluid. The dynamic pressure is defined as,

$$P \equiv p + \rho \vec{g} \cdot \vec{r}, \quad (B.2)$$

where $p$ represents the pressure, $\vec{g}$ the gravitational acceleration and $\vec{r}$ the position vector. The equation for the conservation of mass for an incompressible liquid is given by the continuity equation,

$$\nabla \cdot \vec{u} = 0 \quad (B.3)$$

The lubrication approximation

To find the important terms of the equations, the Navier-Stokes and continuity equations are scaled with the following scalings,

$$\begin{align*}
(\bar{x}, \bar{y}, \bar{z}) &= \left(\frac{x}{L}, \frac{y}{L}, \frac{z}{H}\right), \\
(\bar{u}_x, \bar{u}_y, \bar{u}_z) &= \left(\frac{u_x}{U}, \frac{u_y}{U}, \frac{u_z}{V}\right), \\
\bar{t} &= \frac{t}{L/U} \quad \text{and} \quad \bar{P} = \frac{P}{P_s},
\end{align*} \quad (B.4)$$

where $U$ and $V$ are respectively a characteristic longitudinal and transverse velocity of the thin film flow. The dynamic pressure is scaled by the characteristic pressure scale $P_s$. Furthermore, the Reynolds number $Re_H$ based on the film height is defined as,

$$Re_H \equiv \frac{\rho U H}{\mu} \quad (B.7)$$

Information about the scale for the characteristic pressure $P_s$ can be obtained by considering a unidirectional, stationary flow, i.e. $\vec{u} = (u(z), 0, 0)$. If this flow is inserted in the Navier-Stokes equation and scaled variables are substituted, the characteristic pressure follows as,

$$P_s = \frac{L}{\mu H^2 U} = \frac{1}{\varepsilon^2} \frac{\mu U}{L}. \quad (B.8)$$

The scaled continuity equation obtains the form,

$$\frac{U}{L} \frac{\partial \bar{u}_x}{\partial \bar{x}} + \frac{U}{L} \frac{\partial \bar{u}_y}{\partial \bar{y}} + \frac{V}{H} \frac{\partial \bar{u}_z}{\partial \bar{z}} = 0. \quad (B.9)$$

If this equation is scaled properly, the terms must be of comparable magnitude. This implies for the transverse velocity scale $V = \varepsilon U$.

With this information, the Navier-Stokes equation can be simplified. The $x$-direction of the Navier-Stokes equation has the form,

$$\rho \left( \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right) = \frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right). \quad (B.10)$$
The $y$ and $z$-directions of the Navier-Stokes equation have a similar form. These equations can be non-dimensionalized using the obtained scalings. The following equations are then obtained for respectively the $x$, $y$ and $z$-direction of the non-dimensionalized Navier-Stokes equation.

\[
\varepsilon \text{Re} \left( \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right) = \frac{\partial P}{\partial x} + \frac{\partial^2 u_x}{\partial x^2} + \varepsilon^2 \left( \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) \tag{B.11a}
\]

\[
\varepsilon \text{Re} \left( \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} \right) = \frac{\partial P}{\partial y} + \frac{\partial^2 u_y}{\partial y^2} + \varepsilon^2 \left( \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2} \right) \tag{B.11b}
\]

\[
\varepsilon^3 \text{Re} \left( \frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \varepsilon^2 \frac{\partial^2 u_z}{\partial z^2} + \varepsilon^4 \left( \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} \right) \tag{B.11c}
\]

With the assumption that $\varepsilon^2 \ll 1$ and $\varepsilon \text{Re} \ll 1$ these equations reduce to the equations which are commonly referred to as the lubrication approximation. In dimensional form,

\[
\mu \frac{\partial^2 u_x}{\partial z^2} = \frac{\partial P}{\partial x}, \tag{B.12a}
\]

\[
\mu \frac{\partial^2 u_y}{\partial z^2} = \frac{\partial P}{\partial y}, \tag{B.12b}
\]

\[
\frac{\partial P}{\partial z} = 0. \tag{B.12c}
\]

**Boundary conditions**

Before the lubrication approximation can be used to obtain the lubrication equation, the applicable boundary conditions must be found. For the boundary at the solid-solid interface at $z = 0$, the no-slip and no-penetration boundary conditions are chosen,

\[
\vec{u}(x, y, z = 0) = (0, 0, 0). \tag{B.13}
\]

The stress balance applies to the liquid-air interface at $z = h$. The stress balance can be expressed in a normal and tangential component [17], which are respectively,

\[
p_1 - p_2 + \vec{n} \cdot \left( \frac{\partial}{\partial t} \vec{T}_2 + \vec{T}_1 \right) \cdot \vec{n} + 2\kappa \gamma = 0, \tag{B.14}
\]

\[
\vec{t} \cdot \left( \frac{\partial \vec{T}_2}{\partial t} - \frac{\partial \vec{T}_1}{\partial t} \right) \cdot \vec{n} + \vec{t} \cdot \nabla \gamma = 0, \tag{B.15}
\]

where $\vec{n}$ represents the normal vector on the interface, $\vec{t}$ a tangential vector and $\vec{T}$ the viscous stress tensor. The components of the viscous stress tensor for an Newtonian fluid can be described as,

\[
T_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \tag{B.16}
\]

where $\delta_{ij}$ represents the Kronecker delta function. The viscosity of the liquid is much larger than the viscosity of the air and a flow driven by gas convection is not considered, so the stress contributions of the air will be neglected. Also, due to the small density of the air compared to the density of the liquid, the hydrostatic pressure contribution in the gas is neglected and the pressure of the gas is set to the (constant) ambient pressure, i.e. $p_2 = p_{\text{amb}}$.

The partial derivatives of the height will be written with the short notation $h_i = \frac{\partial h}{\partial x_i}$, as will the second derivatives $h_{ij} = \frac{\partial^2 h}{\partial x_i \partial x_j}$. The derivatives $h_x$ and $h_y$ scale with $\varepsilon \ll 1$. Terms scaling with $\varepsilon^2$ are neglected.

The normal vector $\vec{n}$ and the tangential vectors $\vec{t}$ on the liquid-air surface, are defined respectively as,

\[
\vec{n} = \frac{1}{\sqrt{1 + h_x^2 + h_y^2}} (-h_x, -h_y, 1), \tag{B.17}
\]

\[
\vec{t}_1 = \frac{1}{\sqrt{1 + h_x^2}} (1, 0, h_x), \tag{B.18}
\]

\[
\vec{t}_2 = \frac{1}{\sqrt{h_x^2 h_y^2 + (h_x^2 + 1)^2 + h_y^2}} (-h_x h_y, h_x^2 + 1, h_y). \tag{B.19}
\]
The gradient along the liquid-air interface is defined as,
\[ \nabla_s = \frac{1}{1 + h_x^2 + h_y^2} \left( \begin{array}{c} \frac{h_x}{\sqrt{1 + h_x^2}} \nabla_x T - h_x h_y \frac{\partial}{\partial y} \\
- h_x h_y \frac{\partial}{\partial x} + \frac{h_x^2}{\sqrt{1 + h_x^2}} \frac{\partial}{\partial y} 
\end{array} \right) \approx \left( \begin{array}{c} \frac{\partial}{\partial x} \\
0 \end{array} \right). \] \tag{B.20}

With these definitions, the third term in equation B.14 can be calculated and scaled in the lubrication approximation as,
\[ \vec{n} \cdot \vec{T} \cdot \vec{n} = \frac{1}{1 + h_x^2 + h_y^2} (-h_x, -h_y, 1) \cdot \left( \begin{array}{ccc} T_{xx} & T_{xy} & T_{xz} \\
T_{xy} & T_{yy} & T_{yz} \\
T_{xz} & T_{yz} & T_{zz} \end{array} \right) \cdot \left( \begin{array}{c} -h_x \\
-h_y \\
1 \end{array} \right) \tag{B.21} \]
\[ \varepsilon^2 \leq -2h_x T_{xx} - 2h_y T_{yz} + T_{zz} \tag{B.22} \]
\[ = -2h_x \mu \left( \frac{\partial u_x}{\partial x} + h_y \frac{\partial u_y}{\partial x} \right) - 2h_y \mu \left( \frac{\partial u_y}{\partial y} + h_x \frac{\partial u_x}{\partial y} \right) + 2 \mu \left( \frac{\partial u_z}{\partial z} \right) \tag{B.23} \]
\[ \varepsilon^2 \leq -2h_x \mu \left( \frac{\partial u_x}{\partial z} \right) - 2h_y \mu \left( \frac{\partial u_y}{\partial z} \right) + 2 \mu \left( \frac{\partial u_z}{\partial z} \right) \tag{B.24} \]
The last term in equation B.14 is calculated and scaled as,
\[ 2\kappa \gamma = -\gamma \left( \nabla_s \cdot \vec{n} \right) = \gamma \frac{h_x x [1 + h_y^2] + h_y [1 + h_x^2]}{1 + h_x^2 + h_y^2} - 2h_x h_y h_{xy} \tag{B.25} \]
\[ \varepsilon^2 \leq \gamma (h_{xx} + h_{yy}) \tag{B.26} \]
The normal component of the stress balance, equation B.14, obtains with these solutions the form,
\[ p_1 - p_{amb} - 2h_x \mu \left( \frac{\partial u_x}{\partial x} \right) - 2h_y \mu \left( \frac{\partial u_y}{\partial y} \right) + 2 \mu \left( \frac{\partial u_z}{\partial z} \right) + \gamma (h_{xx} + h_{yy}) = 0. \tag{B.27} \]
The first two terms scale with \( \mu U^2 \), the third, fourth and fifth term scale with \( \mu U^2 \) and the last surface tension term scale with \( \gamma \). The surface tension effects are balanced by the viscous effects in a thin film flow. The scale for the pressure should therefore be comparable with the scales for the surface tension effects, i.e. \( \frac{\mu U^2}{\gamma} \sim \frac{\gamma}{\epsilon^2} \). The third, fourth and fifth term scale with a factor \( \varepsilon^2 \) smaller than the pressure and surface tension terms and can therefore be neglected. The final form of the normal component of the stress balance provides the boundary condition:
\[ p(h) = p_{amb} - \gamma \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right). \tag{B.28} \]
The first term of the tangential component of the stress balance is calculated and scaled as,
\[ \vec{T} \cdot \nabla_s \gamma = \frac{1}{1 + h_x^2 + h_y^2} \frac{1}{\sqrt{1 + h_x^2}} (1, 0, h_x) \cdot \left( \begin{array}{ccc} T_{xx} & T_{xy} & T_{xz} \\
T_{xy} & T_{yy} & T_{yz} \\
T_{xz} & T_{yz} & T_{zz} \end{array} \right) \cdot \left( \begin{array}{c} -h_x \\
-h_y \\
1 \end{array} \right) \tag{B.29} \]
\[ = \frac{1}{1 + h_x^2 + h_y^2} \frac{1}{\sqrt{1 + h_x^2}} (-h_x T_{xx} - h_y T_{xy} + T_{zz} - h_x^2 T_{xx} - h_y T_{yz} + h_x T_{xz}) \tag{B.30} \]
\[ \varepsilon^2 \leq -h_x T_{xx} - h_y T_{xy} + T_{zz} + h_x T_{xz} \tag{B.31} \]
\[ \varepsilon^2 \leq \mu \frac{\partial u_x}{\partial z}, \tag{B.32} \]
because all stress tensor terms scale with \( \varepsilon^2 \), except the \( \mu \frac{\partial u_x}{\partial z} \) term in \( T_{xz} \). This term scales with \( \mu U^2 \) and is therefore the only term of importance.

The last term of the tangential component of the stress balance can be calculated and scaled as,
\[ \vec{T} \cdot \nabla_s \gamma = \frac{1}{1 + h_x^2 + h_y^2} \frac{1}{\sqrt{1 + h_x^2}} (1, 0, h_x) \cdot \left( \begin{array}{c} [1 + h_y^2] \frac{\partial}{\partial x} - h_x h_y \frac{\partial}{\partial y} \\
-h_x h_y \frac{\partial}{\partial x} + [1 + h_x^2] \frac{\partial}{\partial y} \end{array} \right) \gamma \tag{B.33} \]
\[ \varepsilon^2 \leq \frac{\partial \gamma}{\partial x}. \tag{B.34} \]
The same procedure is applicable for the other tangential vector $\vec{t}_2$. The tangential component of the stress balance provides the boundary conditions at $z = h$,

\begin{align*}
\mu \frac{\partial u_x}{\partial z} &= \frac{\partial \gamma}{\partial x}, \\
\mu \frac{\partial u_y}{\partial z} &= \frac{\partial \gamma}{\partial y}.
\end{align*}

(B.35)

(B.36)

**Lubrication equation**

The last equation of the lubrication approximation, equation (B.12c), indicates that the dynamic pressure is independent of the $z$-coordinate. The remaining two equations of the lubrication approximation, equations (B.12a) and (B.12b), can therefore be integrated over $z$ to obtain the equations for the velocities $u_x$ and $u_y$,

\begin{align*}
u_x &= \frac{1}{2\mu} \frac{\partial P}{\partial x} z^2 + z f_x(x, y) + g_x(x, y), \\
&= \frac{1}{2\mu} \frac{\partial P}{\partial x} z^2 + z f_x(x, y) + g_x(x, y), \\
\end{align*}

(B.37)

(B.38)

where $f_x$, $f_y$, $g_x$ and $g_y$ are functions, independent of $z$, which can be determined using the no-slip boundary condition and the boundary condition obtained from the tangential component of the stress balance. The equations for the velocity then become,

\begin{align*}
u_x &= \frac{1}{2\mu} \frac{\partial P}{\partial x} (z^2 - 2hz) + \frac{1}{\mu} \frac{\partial \gamma}{\partial x} z, \\
&= \frac{1}{2\mu} \frac{\partial P}{\partial x} (z^2 - 2hz) + \frac{1}{\mu} \frac{\partial \gamma}{\partial x} z, \\
\end{align*}

(B.39)

(B.40)

These equations can be integrated over $z$ again from $z = 0$ to $z = h$ to obtain the volume fluxes in the $x$ and $y$-direction,

\begin{align*}
Q_x &= \int_0^h u_x(z) dz = -\frac{1}{3\mu} \frac{\partial P}{\partial x} h^3 + \frac{1}{2\mu} \tau_x h^2, \\
Q_y &= \int_0^h u_y(z) dz = -\frac{1}{3\mu} \frac{\partial P}{\partial y} h^3 + \frac{1}{2\mu} \tau_y h^2, \\
\end{align*}

(B.41a)

(B.41b)

where $\tau_x = \frac{\partial \gamma}{\partial x}$ and $\tau_y = \frac{\partial \gamma}{\partial y}$. Also the continuity equation can be integrated over the local film thickness to obtain an expression for the volume fluxes.

\[ \int_0^h \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) dz = 0 \]

(B.42)

This integral can be rewritten using the integration rule of Leibniz,

\[ \frac{\partial}{\partial x} \int_A^{B(x)} F(x, y, z) dz = \int_A^B \frac{\partial F}{\partial x} dz + \frac{dA}{dx} F(x, y, B(x)) - \frac{dB}{dx} F(x, y, A(x)), \]

(B.43)

So, the integrals can be written as,

\begin{align*}
\int_0^h \frac{\partial u_x}{\partial x} dz &= \frac{\partial}{\partial x} \int_0^h u_x(z) dz - \frac{\partial h}{\partial x} u_x(h) = \frac{\partial Q_x}{\partial x} - \frac{\partial h}{\partial x} u_x(h), \\
\int_0^h \frac{\partial u_y}{\partial y} dz &= \frac{\partial}{\partial y} \int_0^h u_y(z) dz - \frac{\partial h}{\partial y} u_y(h) = \frac{\partial Q_y}{\partial y} - \frac{\partial h}{\partial y} u_y(h).
\end{align*}

(B.44)

(B.45)

Resulting in,

\[ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \frac{\partial h}{\partial x} u_x(h) + \frac{\partial h}{\partial y} u_y(h) - u_z(h) - u_z(0) = 0 \]

(B.46)
The term $u_z(0) = 0$ due to the no-slip condition. The term $u_z(h)$ can be rewritten using the equation for the material derivative of the function $f = h(x, y, z, t) - z = 0$, at the position of the liquid-gas interface,

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + u_x \frac{\partial f}{\partial x} + u_y \frac{\partial f}{\partial y} + u_z \frac{\partial f}{\partial z}, \quad (B.47)$$

$$= \frac{\partial h}{\partial t} + u_x \frac{\partial h}{\partial x} + u_y \frac{\partial h}{\partial y} - u_z(h) = 0. \quad (B.48)$$

This equation is solved for $h_z(h)$ and substituted in equation B.46. The obtained equation relates the time-derivative of the liquid height with the volume fluxes,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial h}{\partial t} = 0 \quad (B.49)$$

The obtained equations for the volume flux, as stated in equation B.41, are substituted in this equation. The resulting equation is the lubrication equation,

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{3\mu} \frac{\partial P}{\partial x} h^3 - \frac{1}{2\mu} \tau_x h^2 \right) + \frac{\partial}{\partial y} \left( \frac{1}{3\mu} \frac{\partial P}{\partial y} h^3 - \frac{1}{2\mu} \tau_y h^2 \right). \quad (B.50)$$

According to equation (B.12c) the dynamic pressure $P$ is independent of $z$ and thus considered equal to its value at the gas-liquid interface. Using the definition of the dynamic pressure in equation B.2, the dynamic pressure can be written as,

$$P = p_{amb} - \gamma \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) + \rho gh. \quad (B.51)$$