MASTER

Hospital bed planning
a performance study

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Hospital bed planning

A performance study

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Abstract

For a hospital it is important that the number of beds is chosen such that on the one hand there are enough beds, such that almost no patients should be refused, and on the other hand there are not too many empty beds. In addition, an issue of concern for a hospital is the large variability in the number of occupied beds. The goal of this master thesis is to develop mathematical tools that may assist a hospital in dealing with the above issues. To accomplish this, we first analyze the arrival process of patients at a hospital, and the length-of-stay distribution of patients. Then we use a queueing model with time-dependent arrival rates depending on the type of week to approximate the mean number of occupied beds and its distribution. We compute the number of required beds for a given blocking probability. To reduce the variability of the number of occupied beds we first reduce the variability in the arrival process. Only when the arrival process is smooth, it would be beneficial to reduce the variance of the length-of-stay distribution.
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Chapter 1

Introduction

A patient arrives at a hospital when he or she needs medical care. He or she can only be admitted if there is a bed available. From the patient perspective it is important that at the time of arrival a bed is available with high probability. From the hospital perspective it is important to have a high bed occupancy rate. So the optimal trade-off between those two perspectives should be found. It is a challenge to determine the right number of beds, so that (almost) no patients should be refused and there are not too many empty beds.

At many departments the variability in the number of occupied beds is high. To reduce the variability in the number of occupied beds, the arrival process and length-of-stay of patients should be analyzed. There are two kinds of patient flows, elective and emergency, denoted by $el$ and $em$, and we distinguish two types of patients, clinical and daycare, denoted by $clin$ and $day$. An elective patient is planned and an emergency patient arrives unexpectedly. A daycare patient is a patient that leaves the hospital on the same day as he or she is admitted. A clinical patient could stay in the hospital during the night. The arrival process of emergency patients is exogenous and uncontrollable, the hospital has no influence on it. This is different for elective patients. The hospital has some influence on this arrival process. In most current hospital environments, there is a large variability in the number of arriving elective patients. Often this variability is even higher than for emergency patients. If it is possible to reduce the variability in the number of arriving elective patients by scheduling these patients in a better way, this should lead to a reduction of the variability in the number of occupied beds.

The goal of this study is to develop mathematical tools that may assist a hospital in dealing with the above issues.

1.1 Overview of the literature

Some interesting theses are [8] and [3]. The purpose of thesis [8] is to help and guide healthcare professionals making their organizations future-proof. The outcomes support the decision makers in realizing the best possible use of their resources. In [3] the main goal is to improve the patient flow in hospitals. This will happen by several operations and supply chain management concepts. So the patient flow in hospitals is an important issue. In both theses the authors want to make clear that it is important to look at the integral perspective in healthcare.

In [17] a survey of health care models that include multiple departments is given. That paper provides general overviews of the relationship that exists between major hospital departments and describes how these relationships are accounted for by researchers.
In [13] the non-stationary Poisson process with general service times and infinitely many servers is described. It is shown that the number of busy servers at time $t$ has a Poisson distribution. In this thesis we will use this result in Chapter 2.

In this thesis we use single station models for the patient flow. It is also possible to use network models, which are also described in [13]. In these models one will use routing between the different stations.

In [7] the mean number of occupied beds $m(t)$ at time $t$ for the $G_t/GI_t/\infty$ model is approximated by

$$m(t) = \int_0^\infty \lambda(t-s) \mathbb{P}(S(t-s) > s) ds,$$

with $S(u)$ being the service time of an arrival at time $u$ and $\lambda(u)$ the arrival rate at time $u$. In [7] the approximation for the variance of the number of occupied beds at time $t$ is

$$v(t) \approx z(t)m(t),$$

where

$$z(t) = 1 + c_2^2(t) - \frac{1}{E[S(t)]]} \int_0^\infty (\mathbb{P}(S(t) > x))^2 dx,$$

where $c_2^2(t) \approx \frac{\text{Var}[A(t) - A(t-\eta)]}{\int_{t-\eta}^t \lambda(u) du}$, for some $\eta > 0$ and $A(t)$ counts the number of arrivals before time $t$. For $c_2^2(t)$ one looks backward from $t$, since the number of occupied beds at time $t$ depends on the arrivals before $t$. The estimate should be robust in the interval length $\eta$ if $A(t) - A(t-\eta)$ is not too small and the variability is changing relatively slowly.

In [2] a time-varying queue with a two-time-scale service time is studied. This means that the service time of a patient has components in two time scales: the length-of-stay in days and the departure time in hours. The arrival process is periodic Poisson. In [2] it is mentioned that this queueing system has been used to study patient flows from the emergency department to hospital inpatient wards.

1.1.1 QED regime

The key tradeoff in running service operations is the one between service efficiency and quality. Queues that are both quality- and efficiency driven are called QED queues. In this regime the aim is to find the number of servers such that the balance between the service quality and the efficiency is good. So for example, in a hospital it involves finding the right number of beds such that a good balance is achieved between the occupancy rate (efficiency) and the probability of refusal (service quality). We make use of the square-root staffing rule: if the offered load is $\rho$ (arrival rate times average service time), then

$$c = \rho + \beta \sqrt{\rho}$$

is a staffing level that would balance quality and efficiency, where $\beta$ is a constant. This constant corresponds to the grade-of-service. QED approximations turn out to be accurate and robust. The square-root staffing rule was shown to be an asymptotically optimal staffing level. In [1] it is shown that the square-root rule rarely deviates from the actual optimal value by more than 1 server (for the $M/M/c$ queue). Paper [6] is motivated by [1], where the analytical assessment of the accuracy of the square-root staffing rule is discussed. This is very useful for approximations of relatively small systems.

In [5] the square-root staffing rule $\rho = c - \gamma \sqrt{c}$ for some fixed constant $\gamma$ is considered, which is asymptotically (as $c \to \infty$) equivalent with $c = \rho + \beta \sqrt{\rho}$ for some fixed constant $\beta$. Staffing via $\rho$ instead of $c$ gives mathematically more elegant derivations.

A common approach for accommodating time-inhomogeneity is to approximate a time-varying
arrival rate by a piecewise-constant function. An assumption then is that the arrival rate is slow-
varying with respect to the length of stay. Over this period where we assumed the arrival rate constant we apply steady-state results. Consider a time-varying queue, say $M_t/M/c_t + M$ with a time-varying arrival rate $\lambda(t)$. The goal is to find time-varying staffing levels that stabilize the delay probability at some level $\alpha$. Then this goal is achievable via the following time-varying square-root staffing rule

$$c(t) = \rho(t) + \beta(\alpha) \sqrt{\rho(t)}.$$  

In [11] it is shown that the time-varying square-root staffing rule is extremely robust in stabilizing the probability of delay. In this case it is assumed to change the staffing at any time and any level, which is rarely the case in practice.

### 1.1.2 Relation with call centers

In this subsection we mention a relation with call centers. The main problem in call centers is the following. A call center has $s_t$ permanent agents and $f_t$ flexible agents at time $t$ and has $N$ workplaces available. The objective of a call center is to meet a service level requirement by varying the number of flexible agents over the day. This service level requirement is defined as the fraction of customers that has waited less than the acceptable waiting time $\tau$ upon starting service. So, the problem of the call center and the hospital problem is the same: how many agents/beds do we need to reach a target performance measure.

There is many literature about call centers, (e.g. [16], [1]). This literature is not always directly applicable, because there are significant differences.

A main difference with call centers is the following: when at time $t$ a new customer arrives and there is no agent available, the customer joins an infinite buffer. In this buffer there are two opportunities: when there is a free agent the customer is in service or the customer abandons the system. So, a call center uses queues where a hospital blocks its patients.

Another difference is that the duration of a call relatively to the interarrival time of a customer is small in comparison with the length-of-stay of a patient relatively to the interarrival time of a patient. For example, a duration of a call is in minutes and the interarrival time of a customer is also in minutes. For a hospital the length-of-stay of a patient is in days and the intervals are in hours.

### 1.2 HOTflo Company

HOTflo Company was founded in 2010 with the aim to support the professionalization of capacity management and integrated management in hospitals, with high quality and intelligent software to offer in conjunction with a practical and systematical approach to change.

Using the apps for forecasting, capacity problems will not happen anymore, because you could see them coming in time. These different tools could be used for:

- Capacity management for outpatient clinic,
- Capacity management for the operating rooms,
- Capacity management for the wards,
- Optimization of the number of nurses,
- Discharge management,
- Operational control,
- Hospitalization planning.

HOTflo supports hospitals professionally in achieving change. A team of experienced consultants provides guidance at every step of the change process.

They also offer trainings, work classes and master classes. During these sessions it will be clear why the above described problems are important and how you could deal with them using their tools. A very good aspect of these sessions is the interaction with the participants. Also it is very useful that the participants have to do some assignment by using their tool(s). I was allowed to attend the work class beds- and nurse capacity. It made clear that the tools HOTflo develop and the consultancy are valuable for the hospitals. These tools show how well the hospital performed and how well the hospitals could do when some changes are made. It was a fantastic experience for me.

1.3 Overview of this thesis

This report is organized as follows. In Chapter 2 we describe how to compute the mean number of occupied beds and some other performance measures, like the blocking probability and the bed occupancy rate, for a known non-stationary Poisson process. Chapter 3 is devoted to model a realistic arrival process of patients and adapt the results of Chapter 2 to a stochastic Poisson parameter. The results of Chapters 2 and 3 are the basis for the analysis of the influence of changing parameters on the number of occupied beds in Chapter 4. Our main conclusion, and suggestions for further research are mentioned in Chapter 5.
Chapter 2

Offered-load approximation

In this chapter we derive an approximation for the distribution of the number of occupied beds at time $t$. First, we determine the distribution when there are infinitely many beds available. In this case all patients are admitted to the department. In the case of a finite number of beds we approximate the distribution by using the results of the infinite case. Based on this approximation we can compute the blocking probability, i.e. the probability that an arriving patient finds no free bed available, and the bed occupancy rate, i.e. the mean number of occupied beds relative to the number of available beds. The purpose of this chapter is to construct the general formulas which can be used by the arrival process models we will construct in Chapter 3. At the end we will be able to determine the required number of beds for a department with a target blocking probability $\varepsilon$.

We have two types of patient flows, elective and emergency. Also, we have two types of patient admissions, clinical and daycare. So, in total we distinguish four different types:

- elective, clinical patient,
- elective, daycare patient,
- emergency, clinical patient,
- emergency, daycare patient.

Some results we find in this chapter are that the number of occupied beds $M(t)$ with infinitely many beds follows a Poisson distribution with parameter

$$\rho(t) = \int_{-\infty}^{t} \left( \lambda_{\text{clin}}(u) \mathbb{P}(B_{\text{clin}}^{\text{el}} > t - u) + \lambda_{\text{clin}}(u) \mathbb{P}(B_{\text{clin}}^{\text{em}} > t - u) \right) du$$

$$+ \int_{-\infty}^{t} \left( \lambda_{\text{day}}(u) \mathbb{P}(B_{\text{day}}^{\text{el}} > t - u) + \lambda_{\text{day}}(u) \mathbb{P}(B_{\text{day}}^{\text{em}} > t - u) \right) du,$$

where $\lambda_{\text{clin}}(t)$ and $B_{\text{clin}}^{\text{el}}$, $\lambda_{\text{clin}}(t)$ and $B_{\text{clin}}^{\text{em}}$, $\lambda_{\text{day}}(t)$ and $B_{\text{day}}^{\text{el}}$, and $\lambda_{\text{day}}(t)$ and $B_{\text{day}}^{\text{em}}$ are the non-stationary arrival rate parameters and length-of-stay random variable for elective clinical, elective daycare, emergency clinical and emergency daycare patients respectively. Using this result we can derive formulas for the blocking probability

$$P_{\text{block}}(t) \approx \frac{\mathbb{P}(M(t) = c_t)}{\sum_{k=0}^{c_t} \mathbb{P}(M(t) = k)},$$

and bed occupancy rate with $c_t$ available beds

$$BOR(t) \approx \frac{\rho(t) \mathbb{P}(M(t) \leq c_t - 1)}{c_t \mathbb{P}(M(t) \leq c_t)}.$$
This chapter is organized as follows. In Section 2.1 we assume the number of beds is infinite, in Section 2.2 we use the results of Section 2.1 for a finite number of beds analysis. In Section 2.3 we will assume a sinusoidal arrival rate function and analyze the performance of our approximation results of Section 2.2. Section 2.4 is devoted to the computation on the number of required beds given a target blocking probability. In Section 2.5 we describe how to merge departments such that the previous formulas in this chapter could be used. In Section 2.6 we describe how to deal with a selection of specialism at a department. Section 2.7 is devoted to the discussion of the results.

2.1 Infinitely many beds

In this section we focus on the number of occupied beds in one, arbitrary, department of the hospital. We assume that the number of beds is infinite. In Section 2.2 the case of a finite number of beds is treated.

When patients arrive according to a nonhomogeneous Poisson process, in [4], the distribution of the number of occupied beds $M(t)$ at time $t$ is shown to be Poisson with parameter

$$
\rho(t) = \int_{-\infty}^{t} \left( \lambda^e(u) \mathbb{P}(B^e > t-u) + \lambda^m(u) \mathbb{P}(B^m > t-u) \right) \, du
$$

where $\lambda^e(t)$ and $B^e$, and $\lambda^m(t)$ and $B^m$ are the non-stationary arrival rate parameters and length-of-stay random variables for elective and emergency patients respectively. So,

$$
\mathbb{P}(M(t) = k) = e^{-\rho(t)} \frac{\rho(t)^k}{k!}, \quad k = 0, 1, ...
$$

We define the patient’s length-of-stay as the total time he or she stays at the department. The mean offered load $\rho_{(a,b]}$ to the department in the interval $(a,b]$ is defined by

$$
\rho_{(a,b]} := \frac{1}{b-a} \int_{a}^{b} \rho(t) \, dt.
$$

A hospital has two types of departments, clinical and daycare. In a clinical department the patient could stay during the night. In a daycare department there are no patients during the night. All patients should be discharged from the hospital or transferred to another department when the department closes at night. In the next two subsections we successively treat clinical departments and daycare departments. In subsection 2.1.3 we describe how to compute the mean number of occupied beds for clinical and daycare admissions to the two different types of departments.

2.1.1 Clinical department

In a hospital, patients leave the department only during the day. Departures during the night in principle do not occur. So, we have to rectify Formula (2.1) to get the right result for the number of occupied beds $M(t)$ at time $t$. In the following, three different approaches are mentioned for the computation of the parameter $\rho(t)$ as in Formula (2.1).

2.1.1.1 Approach 1: Patients could leave any moment

In this first approach the patients could leave the department at each moment of the day, including the night.
2.1.1.2 Approach 2: Remove all nights of length-of-stay

In this second approach all the nights are removed between time \( u \) and time \( t \). We define nights as the time that (almost) no patients leave the department.

So, we will have the following parameter

\[
\rho(t) = \int_{-\infty}^{t} (\lambda^c(u)P(B^c > t - u - N(u, t)) + \lambda^em(u)P(B^em > t - u - N(u, t)))du
\] (2.4)

where \( N(u, t) \) is defined as the number of nightly hours between time \( u \) and time \( t \).

Using this approach we also should remove the nights from the data, where the patient arrivals and departures are stored, to compute the length-of-stay, otherwise we will use two different measures.

2.1.1.3 Approach 3: Remove only last night of length-of-stay

In this third approach only the last night is removed between time \( u \) and time \( t \). To determine \( \rho_*(t) \) for \( t \) at an epoch during the night, this is the same as the last time patients could leave the department except for nightly arrivals.

We define \( T \) as the time at the beginning of the night and define \( S(T, t) \) as the number of arrivals between time \( T \) and \( t \) and \( \rho(t) \) as in Equation (2.1). Then, we will have the following parameter

\[
\rho_*(t) = \begin{cases} 
\rho(T) + S(T, t), & \text{if } t \text{ is during the night}, \\
\rho(t), & \text{if } t \text{ is during the day}.
\end{cases}
\] (2.5)
For example, if we assume that no patients leave the department from 21:00:00 until 10:00:00, then in Table 2.1.1 there are some examples where \( t - u \) is determined by the different approaches. Notice the significant difference between approaches 1 and 3 when \( t \) is shifted from 10:00:00 to 9:00:00. In Figure 2.1.1 the three approaches are shown for one week. Approach 1 almost completely misses the daily peaks that are present in the real data. Approach 2 catches those peaks, but in a rather inaccurate way. Approach 3 captures those peaks in a reasonably accurate way.

\[
\begin{array}{cccc}
\text{Time } u & \text{Time } t & \text{Approach 1} & \text{Approach 2} & \text{Approach 3} \\
2014-01-06 09:00:00 & 2014-01-08 10:00:00 & 49 & 22 & 49 \\
2014-01-06 09:00:00 & 2014-01-08 09:00:00 & 48 & 22 & 36 \\
\end{array}
\]

Table 2.1.1

Closing department
When the department is closed, e.g. every weekend or some day, the number of occupied beds \( \rho_*(t) \) should be zero. When the department will close at time \( M \), the mean number of occupied beds at time \( t \) will become

\[
\rho(t) = \int_{-\infty}^{t} (\lambda^l(u)P(t-u < B^l \leq M-u) + \lambda^m(u)P(t-u < B^m \leq M-u)) \, du.
\]

(2.6)

The random variables \( B^l \) and \( B^m \) are bounded from above by \( M-u \). This is done because the patient has a maximum length-of-stay for this department. At time \( M \) he or she has left the hospital or is transferred to another department.

2.1.2 Daycare department
At a daycare department, patients only arrive during the day and leave again the same day. At night there are no occupied beds. So the maximum length-of-stay of a daycare patient is one day. For a daycare department we investigate two approaches. The first approach is the same as approach 1 of subsection 2.1.1, except for the fact that at night there are no occupied beds. The second approach uses day-dependent length-of-stay distributions. For a daycare patient there is also a maximum length-of-stay. When the department closes at 22:00:00 and a patient arrives at 20:00:00, his maximum length-of-stay will be two hours at this department. Then the probability that this patient is still at the department at 21:00:00 is \( P(1 < B < 2) \) and not \( P(B > 1) \). This will lead to a zero probability that the patient is still at the department at 22:00:00, instead of \( P(B > 2) \). As above, \( M \) is the time the department will close.

2.1.2.1 Approach 1: Each day same length-of-stay distribution
In this first approach each day the same length-of-stay distribution with different parameters is used. For this approach we will have the following equation for \( \rho(t) \):

\[
\rho(t) = \begin{cases} 0, & \text{if } t \text{ is during the night} \\ \int_{T_1}^{t} (\lambda^l(u)P(t-u < B^l \leq M-u) + \lambda^m(u)P(t-u < B^m \leq M-u)) \, du, & \text{if } t \text{ is during the day} \end{cases}
\]

(2.7)

where \( T_1 \) is the starting time of the day. It should be clear that \( \rho(t) = 0 \) when \( t \) is at night, because the department is closed.
2.1.2.2 Approach 2: Day-dependent length-of-stay distribution

For this second approach we will have the following equation for $\rho(t)$:

$$\rho(t) = \begin{cases} 
0, & \text{if } t \text{ is during the night} \\
\int_{T_1}^{t} \left( \lambda^{cl}(u) \mathbb{P}(t-u < B_u^{cl} \leq M-u) + \lambda^{em}(u) \mathbb{P}(t-u < B_u^{em} \leq M-u) \right) \, du, & \text{if } t \text{ is during the day}.
\end{cases}$$

(2.8)

where $B_u$ is the length-of-stay random variable at time $u$, which is weekday-dependent, and $T_1$ is the starting time of the day.

![Figure 2.1.2: The mean number of occupied beds in two daycare departments where the two different approaches for the length-of-stay distribution are given.](image)

It seems that the daycare department in Figure 2.1.2a has a day-dependent length-of-stay distribution. The effect of a day-dependent length-of-stay distribution for the daycare department in Figure 2.1.2b is minimal.

2.1.3 Two types of admissions

There are two types of admissions: clinical and daycare. A clinical type of admission is a patient who could stay in the hospital during the night. A daycare type of admission is a patient who will leave the hospital the same day. For these two types of admissions in the two types of departments we compute the mean number of occupied beds differently, which is explained in the following two subsections.

2.1.3.1 Clinical type of admissions

In most cases a clinical type of admission can be found in a clinical department. In this case we approximate the mean number of occupied beds $\rho_{\text{cl}}^{\text{clin}}(t)$ as in Section 2.1.1.

In some cases a clinical type of admission can be found in a daycare department. In this case we approximate the mean number of occupied beds $\rho_{\text{cl}}^{\text{day}}(t)$ as in Section 2.1.2. When this patient could not leave the hospital, but the department is closing, he or she will be transferred to another department.
2.1.3.2 Daycare type of admissions

In most cases a daycare type of admission occurs in a daycare department, but sometimes it may also occur in a clinical department. In both cases we approximate the mean number of occupied beds $\rho_{\text{day}}^{\text{clin}}(t)$ and $\rho_{\text{day}}^{\text{day}}(t)$ as in Section 2.1.2.

Then the mean number of occupied beds $\rho(t)$ at time $t$, as in Equation (2.1), will be:

$$\rho(t) = \begin{cases} 
\rho_{\text{clin}}^{\text{clin}}(t) + \rho_{\text{clin}}^{\text{day}}(t), & \text{if department is open at night,} \\
\rho_{\text{day}}^{\text{clin}}(t) + \rho_{\text{day}}^{\text{day}}(t), & \text{if department is closed at night.}
\end{cases} \tag{2.9}$$

2.1.4 Different length-of-stay distributions

In order to use Formula (2.1) we have to know the length-of-stay distribution. The question that we discuss in this subsection is whether the choice of distribution has a significant effect on $\rho(t)$. In Figure 2.1.3 this is tested for one clinical department and in Figure 2.1.5 for one daycare department. We have used the following distributions: Exponential, Gamma, Weibull and the Log-Normal. For a clinical department the influence of the length-of-stay distribution is minimal. In Figure 2.1.4 the different length-of-stay distribution functions are shown for elective and emergency patients. These distribution functions are almost the same. So, using a different length-of-stay distribution has a minimal influence on Formula (2.5).

For a daycare department there is some difference between the different distributions. In Figure 2.1.5 it is shown that the Exponential distribution is the least accurate one, because the patient’s length-of-stay could not be fit well by an Exponential distribution. In Figure 2.1.6 it is shown how well the different length-of-stay distributions follow the Empirical distribution function. This makes clear why the Exponential length-of-stay distribution is the least accurate one.
Figure 2.1.3: The mean number of occupied beds for different length-of-stay distributions for one clinical department.

Figure 2.1.4: Length-of-stay distribution function for elective and emergency patients.
Figure 2.1.5: Daycare department where the length-of-stay distribution is day-dependent, in this case every day the same distribution with other parameters.
Figure 2.1.6: Daily length-of-stay distribution function for elective patients.
2.2 Finitely many beds

In this section we focus on two performance measures, the blocking probability and the bed occupancy rate, when there is only a finite number of beds. First, we approximate the distribution of the number of occupied beds when there are \( c \) beds available. This approximation is based on the results for the case when there are infinitely many beds available (Section 2.1).

Let \( M^c(t) \) be the number of occupied beds at time \( t \) when there are \( c \) beds available. We shall use results obtained in Massey and Whitt [14] for the so-called non-stationary Erlang loss model. That is a model with \( c \) servers at time \( t, t \geq 0 \), arrival rate \( \lambda(t) \) at time \( t \), and service time distribution \( B(\cdot) \). Massey and Whitt [14] suggest an approximation for the probability distribution of the number of occupied servers at time \( t \). Translated to our setting, we obtain the following approximation for the number of occupied beds at time \( t \)

\[
\mathbb{P}(M^c(t) = m) \approx \frac{\mathbb{P}(M(t) = m)}{\sum_{k=0}^c \mathbb{P}(M(t) = k)}, \quad \text{for } m = 0, 1, \ldots, c,
\]

(2.10)

where \( \mathbb{P}(M(t) = m) \) is given in Formula (2.2). In [14] it is mentioned that this approximation provides a good estimation for the peak time for a loss queue with a small blocking probability. This is because Formula (2.1) for the mean number of occupied beds provides a good estimation for the peak time for a loss queue with a small blocking probability.

In [14] the error of the blocking probability \( P_{\text{block}}(t) \) is bounded. It is shown that

\[
\sup_{0 \leq \tau \leq t} \left| \mathbb{P}(M^c(\tau) = c_\tau) - \frac{\mathbb{P}(M(\tau) = c_\tau)}{\mathbb{P}(M(\tau) \leq c_\tau)} \right| \leq 2 \int_0^t \frac{\mathbb{P}(M(\tau) = c_\tau)}{\mathbb{P}(M(\tau) \leq c_\tau)} \left| \frac{d\rho(\tau)}{d\tau} \right| d\tau.
\]

(2.12)

This bound implies that the approximation should perform better when the arrival rate \( \lambda(t) \) changes more slowly and when the blocking probability is low.

In [12] the blocking probability \( P^{[a,b]}_{\text{block}} \) for an arbitrary arriving patient in the interval \((a,b]\) is given by

\[
P^{[a,b]}_{\text{block}} = \begin{cases} \int_a^b (\lambda^a(t) + \lambda^m(t)) \mathbb{P}(M^c(t) = c_{[a,b]}) dt, & \text{if } \int_a^b (\lambda^a(t) + \lambda^m(t)) dt > 0, \\ 0, & \text{otherwise}, \end{cases}
\]

(2.13)

where \( c_{[a,b]} \) is the fixed number of beds in the interval \((a,b]\).

2.2.1.1 Normal approximation

When the mean number of occupied beds \( \rho(t) \) is quite large, we will use a Normal approximation for the Poisson distribution. In [15] it is shown that for one decimal accuracy in the probability
mass function, the minimal value of \( \rho(t) \) should be larger or equal than 3. For two decimals accuracy in the probability mass function we need \( \rho(t) \geq 20 \). For three decimals accuracy in the probability mass function we need \( \rho(t) \geq 188 \). In our case two decimals should be enough to approximate the mean number of occupied beds. We will have the following approximations:

\[
\mathbb{P}(M(t) = c) \approx \phi \left( \frac{c - \rho(t) + \frac{1}{2}}{\sqrt{\rho(t)}} \right), \quad (2.14)
\]

\[
\mathbb{P}(M(t) \leq c) \approx \Phi \left( \frac{c - \rho(t) + \frac{1}{2}}{\sqrt{\rho(t)}} \right), \quad (2.15)
\]

with \( \phi(x) \) the standard normal probability density function and \( \Phi(x) \) the standard normal cumulative distribution function. In Formula (2.13) the exact blocking probabilities are given. Define \( P^{[a,b]}_{\text{block}}(c_{(a,b)}) \) as the blocking probability when \( c_{(a,b)} \) is the number of beds in the interval \((a, b]\). We will have the following approximation for the blocking probability if \( \int_a^b (\lambda^{\text{el}}(t) + \lambda^{\text{em}}(t)) \rho(t) \, dt > 0 \):

\[
P^{[a,b]}_{\text{block}}(c_{(a,b)}) \approx \frac{\int_a^b (\lambda^{\text{el}}(t) + \lambda^{\text{em}}(t)) \rho(t) \, dt}{\int_a^b (\lambda^{\text{el}}(t) + \lambda^{\text{em}}(t)) \, dt}, \quad (2.16)
\]

and \( P^{[a,b]}_{\text{block}}(c_{(a,b)}) = 0 \) otherwise, where

\[
\hat{\rho}(t) = \mathbb{I}_{(\rho(t) < 20)} \frac{\mathbb{P}(M(t) = c_{(a,b)})}{\mathbb{P}(M(t) \leq c_{(a,b)})} + \mathbb{I}_{(\rho(t) \geq 20)} \frac{\phi \left( \frac{c_{(a,b)} - \rho(t) + \frac{1}{2}}{\sqrt{\rho(t)}} \right)}{\Phi \left( \frac{c_{(a,b)} - \rho(t) + \frac{1}{2}}{\sqrt{\rho(t)}} \right)}. \quad (2.17)
\]

### 2.2.2 The bed occupancy rate

In this subsection we define the bed occupancy rate. The bed occupancy rate is the expected ratio of the number of occupied beds and the total available number of beds. In Lemma 2.1 we derive an expression for the mean number of occupied beds at time \( t \).

**Lemma 2.1.** Given Approximation (2.10), the expected number of occupied beds \( \rho_{c_t}(t) := \mathbb{E}[M^{\text{el}}(t)] \) at time \( t \) is given by

\[
\rho_{c_t}(t) \approx \rho(t) \frac{\mathbb{P}(M(t) \leq c_t - 1)}{\mathbb{P}(M(t) \leq c_t)}. \quad (2.18)
\]

**Proof.** We can write

\[
\rho_{c_t}(t) = \sum_{m=0}^{c_t} m \mathbb{P}(M^{\text{el}}(t) = m)
\]

\[
\approx \sum_{m=1}^{c_t} m \frac{\mathbb{P}(M(t) = m)}{\mathbb{P}(M(t) \leq c_t)}
\]

\[
= \sum_{m=1}^{c_t} me^{-\rho(t)} \frac{\eta(t)^m}{m!} \frac{\mathbb{P}(M(t) \leq c_t)}{\mathbb{P}(M(t) \leq c_t)}
\]

\[
= \rho(t) \sum_{m=1}^{c_t} e^{-\rho(t)} \frac{(\rho(t))^{m-1}}{(m-1)!} \frac{\mathbb{P}(M(t) \leq c_t)}{\mathbb{P}(M(t) \leq c_t)}
\]

\[
= \rho(t) \frac{\mathbb{P}(M(t) \leq c_t - 1)}{\mathbb{P}(M(t) \leq c_t)},
\]

where the first step follows by the definition of expectation, the second from Formula (2.10), the third from Formula (2.2), the fourth by calculus and the fifth again from Formula (2.2).
Having the mean number of occupied beds at time $t$, we define the expected bed occupancy rate $BOR(t)$ at time $t$ by

$$BOR(t) = \frac{\rho_c(t)}{c_t},$$

the fraction of the mean number of occupied beds $\rho_c(t)$ and the available beds $c_t$.

The expected bed occupancy rate $BOR_{[a,b]}$ in the interval $(a,b]$ is defined by

$$BOR_{[a,b]} = \frac{\int_a^b \rho_c(t)dt}{\int_a^b c_t dt}, \quad (2.19)$$

where $\int_a^b \rho_c(t)dt$ is the integrated mean number of occupied beds in the interval $(a,b]$ and $\int_a^b c_t dt$ the integrated mean number of available beds.

### 2.3 Sinusoidal arrival rates

In this section we test the accuracy of the approximation for the mean number of occupied beds $\rho_c(t)$ (Formula (2.18)). We assume an Exponential length-of-stay distribution with parameter $\mu$. We will take a smooth arrival rate function, in this case a sinusoidal.

We consider the sinusoidal arrival rate function

$$\lambda(t) = \alpha + \beta \sin(\gamma t), \quad (2.20)$$

for positive constants $\alpha, \beta$ and $\gamma$. When the length-of-stay distribution is Exponential we can compute the exact solution. We assume the number of beds is $c$. Then we will have the following equations to solve:

$$P_0'(t) = \mu P_1(t) - \lambda(t) P_0(t),$$

$$P_i'(t) = \lambda(t) P_{i-1}(t) + (i+1) \mu P_{i+1}(t) - (\lambda(t) + i \mu) P_i(t), \quad \text{for } i = 1, \ldots, c-1$$

$$P_c'(t) = \lambda(t) P_{c-1}(t) - c \mu P_c(t),$$

$$P_0(t) + P_1(t) + \ldots + P_c(t) = 1,$$

$$P_0(0) = 1, \quad P_k(0) = 0, \quad \text{for } k = 1, \ldots, c.$$ \quad (2.21)

This system will lead to the following mean number of occupied beds $m(t)$ at time $t$:

$$m(t) = \sum_{i=0}^c i P_i(t). \quad (2.22)$$

In the following figures we will investigate how well the approximation of the mean number of occupied beds $\rho_c(t)$ performs for different arrival rate functions $\lambda(t)$ or mean length-of-stay $\frac{1}{\mu}$. 

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(a) Arrival rate function $\lambda(t) = 5 + \beta \sin(2\pi t)$ and $\mu = \frac{2}{\pi}$.

(b) Arrival rate function $\lambda(t) = 15 + \beta \sin(2\pi t)$ and $\mu = 2$.

Figure 2.3.1

(a) Arrival rate function $\lambda(t) = 5 + 5 \sin(\gamma t)$ and $\mu = \frac{2}{\pi}$.

(b) Arrival rate function $\lambda(t) = 15 + 5 \sin(\gamma t)$ and $\mu = 2$.

Figure 2.3.2
In the previous figures the mean number of occupied beds \( \rho_c(t) \), with \( c = 15 \) available beds, for \( \lambda(t) = \alpha + \beta \sin(\gamma t) \) is computed. In these examples we change \( \alpha, \beta, \gamma \) and \( \mu \) and see the impact on the mean number of occupied beds. In Figure 2.3.1 we vary \( \beta \) and keep the fraction \( \frac{\alpha}{\mu} \) and \( \gamma \) fixed. In Figure 2.3.2 we vary \( \gamma \) and keep the fraction \( \frac{\alpha}{\mu} \) and \( \beta \) fixed. In Figure 2.3.3 we vary \( \alpha \) and keep \( \beta, \gamma \) and \( \mu \) fixed. In Figure 2.3.4 we vary \( \mu \) and keep \( \alpha, \beta \) and \( \gamma \) fixed.

When the number of occupied beds \( \rho_c(t) \) is close to \( c \), the approximation becomes less accurate. This is the same as the smaller the blocking probability, the more accurate the approximation, which is also observed in [14].
2.4 Determining the number of required beds

In this section we focus on determining the number of required beds such that the target blocking probability is \( \varepsilon \). Since most hospitals work with specified shifts, we assume here that the number of required beds changes only at specific times. So, the total period has to be divided into smaller intervals. In these intervals we compute the number of required beds. In Subsection 2.4.1 an algorithm is described to determine the number of required beds.

When the target blocking probability is \( \varepsilon \) we will determine the required number of beds over the whole period \((t_1, t_2]\).

Divide the period \((t_1, t_2]\) in smaller intervals \((a_1, a_2], (a_2, a_3], \ldots, (a_{n-1}, a_n]\) for which we determine the number of beds \(c_{(a_1, a_2]}, c_{(a_2, a_3]}, \ldots, c_{(a_{n-1}, a_n]}\). During the interval \((a_i, a_{i+1}]\) the number of beds will not change.

For each of the intervals we compute the number of required beds \(c_{(a_i, a_{i+1}]}, \) for \(i = 1, \ldots, n - 1\), by finding the highest value for which \( P_{\text{block}}^{(a_i, a_{i+1}]}(c_{(a_i, a_{i+1}]}) \leq \varepsilon \). When \( P_{\text{block}}^{(a_i, a_{i+1}]}(c_{(a_i, a_{i+1}]}) = 0\), we take the number of required beds by rounding \( P_{\text{block}}^{(a_i, a_{i+1}]}\).

This procedure will give a blocking probability over the period \((t_1, t_2]\) smaller than \(\varepsilon\). Define \(c_{(t_1, t_2]} := (c_{(a_1, a_2]}, \ldots, c_{(a_{n-1}, a_n]}\) as the vector with the number of required beds during the period \((t_1, t_2]\). Algorithm 1 will give the number of required beds \(c_{(a_1, a_{i+1}]}\) such that the blocking probability over \((t_1, t_2]\) is almost equal to \(\varepsilon\). This algorithm uses only the intervals where the blocking probability is positive.

Define \(c_{(t_1, t_2]}^1\) as the smallest number of required beds where the blocking probability is smaller than or equal to \(\varepsilon\) and \(p_{(t_1, t_2]}^1\) as the corresponding blocking probabilities.

\[
\begin{align*}
c_{(t_1, t_2]}^1 &= \{c_{(a_i, a_{i+1}]} | 0 < P_{\text{block}}^{(a_i, a_{i+1}]}(c_{(a_i, a_{i+1}]})) \leq \varepsilon, P_{\text{block}}^{(a_i, a_{i+1}]}(c_{(a_i, a_{i+1}]} - 1) > \varepsilon \} \\
p_{(t_1, t_2]}^1 &= \{P_{\text{block}}^{(a_i, a_{i+1}]}(c_{(a_i, a_{i+1}]})) | 0 < P_{\text{block}}^{(a_i, a_{i+1}]}(c_{(a_i, a_{i+1}]})) \leq \varepsilon, P_{\text{block}}^{(a_i, a_{i+1}]}(c_{(a_i, a_{i+1}]} - 1) > \varepsilon \} 
\end{align*}
\] (2.23)

Let \(c_{(t_1, t_2]}^2 = (c_{(t_1, t_2]}^1 - 1)\) be the largest number of required beds such that the blocking probability is larger than \(\varepsilon\) and let \(p_{(t_1, t_2]}^2\) be the corresponding blocking probabilities.

2.4.1 Algorithm

This algorithm will return a vector of required beds \(c_{(t_1, t_2]}\) with blocking probability \(\varepsilon\) for an arbitrary customer.

Define \(c_{(t_1, t_2]}^\text{alg}\) as the vector with the number of required beds in the algorithm and \(p_{(t_1, t_2]}^\text{alg}\) as the vector of the corresponding blocking probabilities. Assume these vectors have \(L\) elements.

The first step is to determine the initial vector. We will take for each element \(1, \ldots, L\) the number of beds such that the related blocking probability is the closest to the target blocking probability \(\varepsilon\). Define \(x_{(t_1, t_2]}\) as the \(i\)-th element of \(x_{(t_1, t_2]}\). We will do for \(i = 1, \ldots, L\):

\[
\begin{align*}
c_{(t_1, t_2]}^\text{alg} &= \begin{cases} 
c_{(t_1, t_2],i}^1 & \text{if } |p_{(t_1, t_2],i}^1 - \varepsilon| \leq |p_{(t_1, t_2],i}^2 - \varepsilon|, \\
c_{(t_1, t_2],i}^2 & \text{if } |p_{(t_1, t_2],i}^1 - \varepsilon| > |p_{(t_1, t_2],i}^2 - \varepsilon|. 
\end{cases} \\
p_{(t_1, t_2]}^\text{alg} &= \begin{cases} 
p_{(t_1, t_2],i}^1 & \text{if } |x_{(t_1, t_2],i}^1 - \varepsilon| \leq |x_{(t_1, t_2],i}^2 - \varepsilon|, \\
p_{(t_1, t_2],i}^2 & \text{if } |x_{(t_1, t_2],i}^1 - \varepsilon| > |x_{(t_1, t_2],i}^2 - \varepsilon|.
\end{cases} 
\end{align*}
\] (2.25)

The idea of the algorithm is to change the entry in the vector and see the influence of the swap on the blocking probability. So, in the first step we first change \(c_{(t_1, t_2],i}^\text{alg}\) into \(c_{(t_1, t_2],i}^1\) if
\( c_{(t_1,t_2),1}^{\text{alg}} = c_{(t_1,t_2),1}^{2} \) and otherwise into \( c_{(t_1,t_2),1}^{2} \). All other coordinates stay the same. Secondly, we only change the second coordinate and we repeat this until all \( L \) coordinates changed once.

When the mean of the initial blocking probability vector \( \bar{p}_{(t_1,t_2)}^{\text{alg}} \) is smaller than \( \varepsilon \) we investigate which change in the blocking probability is the closest to \( \varepsilon \) and stays below this target value. When the mean of the initial blocking probability vector \( \bar{p}_{(t_1,t_2)}^{\text{alg}} \) is larger than \( \varepsilon \) we investigate which change in the blocking probability is the closest to \( \varepsilon \) and stays above this target value. This procedure is repeated until the number of required beds does not change any more.

Input for this algorithm are the vectors \( c_{(t_1,t_2),1}^{1}, c_{(t_1,t_2),2}^{2}, p_{(t_1,t_2)}^{1} \) and \( p_{(t_1,t_2)}^{2} \). Algorithm 1 concerns the case that the mean of the initial blocking probability vector \( \bar{p}_{(t_1,t_2)}^{\text{alg}} \) is smaller than \( \varepsilon \).

**Example 2.1.** In this example we will make clear how the algorithm works. The target blocking probability \( \varepsilon = 0.05 \).

Assume

\[
\begin{align*}
c_{(t_1,t_2),1}^{1} &= (5, 8, 4, 7), \\
p_{(t_1,t_2)}^{1} &= (0.03, 0.04, 0.035, 0.028), \\
c_{(t_1,t_2),1}^{2} &= (4, 7, 3, 6), \\
p_{(t_1,t_2)}^{2} &= (0.06, 0.07, 0.08, 0.075).
\end{align*}
\]

Then the initial vectors are:

\[
\begin{align*}
c_{(t_1,t_2),1}^{\text{alg}} &= (4, 8, 4, 7), \\
p_{(t_1,t_2)}^{\text{alg}} &= (0.06, 0.04, 0.035, 0.028).
\end{align*}
\]

The mean of the vector \( p_{(t_1,t_2)}^{\text{alg}} \) is defined by \( \bar{p}_{(t_1,t_2)}^{\text{alg}} \). In this case \( \bar{p}_{(t_1,t_2)}^{\text{alg}} = 0.04075 \).

The first step is:

- swap element 1: \( c_{(t_1,t_2),1}^{\text{alg}} = (5, 8, 4, 7), p_{(t_1,t_2)}^{\text{alg}} = (0.03, 0.04, 0.035, 0.028), \bar{p}_{(t_1,t_2)}^{\text{alg}} = 0.03325, \)
- swap element 2: \( c_{(t_1,t_2),1}^{\text{alg}} = (4, 7, 4, 7), p_{(t_1,t_2)}^{\text{alg}} = (0.06, 0.07, 0.035, 0.028), \bar{p}_{(t_1,t_2)}^{\text{alg}} = 0.04825, \)
- swap element 3: \( c_{(t_1,t_2),1}^{\text{alg}} = (4, 8, 3, 7), p_{(t_1,t_2)}^{\text{alg}} = (0.06, 0.04, 0.08, 0.028), \bar{p}_{(t_1,t_2)}^{\text{alg}} = 0.052, \)
- swap element 4: \( c_{(t_1,t_2),1}^{\text{alg}} = (4, 8, 4, 6), p_{(t_1,t_2)}^{\text{alg}} = (0.06, 0.04, 0.035, 0.075), \bar{p}_{(t_1,t_2)}^{\text{alg}} = 0.0525. \)

Swapping element 2 will be the outcome of step 1, because \( \bar{p}_{(t_1,t_2)}^{\text{alg}} = 0.04825 \) is the closest of the two \( \bar{p}_{(t_1,t_2)}^{\text{alg}} \) values lower than \( \varepsilon = 0.05 \). So, we will have:

\[
\begin{align*}
c_{(t_1,t_2),1}^{\text{alg}} &= (4, 7, 4, 7), \\
p_{(t_1,t_2)}^{\text{alg}} &= (0.06, 0.07, 0.035, 0.028).
\end{align*}
\]

The second step is:

- swap element 1: \( c_{(t_1,t_2),1}^{\text{alg}} = (5, 7, 4, 7), p_{(t_1,t_2)}^{\text{alg}} = (0.03, 0.07, 0.035, 0.028), \bar{p}_{(t_1,t_2)}^{\text{alg}} = 0.04075, \)
- swap element 2: \( c_{(t_1,t_2),1}^{\text{alg}} = (4, 8, 4, 7), p_{(t_1,t_2)}^{\text{alg}} = (0.06, 0.04, 0.035, 0.028), \bar{p}_{(t_1,t_2)}^{\text{alg}} = 0.04075, \)
- swap element 3: \( c_{(t_1,t_2),1}^{\text{alg}} = (4, 7, 3, 7), p_{(t_1,t_2)}^{\text{alg}} = (0.06, 0.07, 0.08, 0.028), \bar{p}_{(t_1,t_2)}^{\text{alg}} = 0.0595, \)
- swap element 4: \( c_{(t_1,t_2),1}^{\text{alg}} = (4, 7, 4, 6), p_{(t_1,t_2)}^{\text{alg}} = (0.06, 0.07, 0.035, 0.075), \bar{p}_{(t_1,t_2)}^{\text{alg}} = 0.06. \)

Swapping some element will not lead to a closer blocking probability, so the algorithm stops and the output of the number of required beds is \( c_{(t_1,t_2),1}^{\text{alg}} = (4, 7, 4, 7) \), which will give a blocking probability of 0.04825.
Algorithm 1

\[ L = |c_{alg}^{t_1,t_2}| \]
\[ B = p_{alg}^{t_1,t_2} \]
\[ \bar{B} = \sum_{i=1}^{L} b_i \]
\[ V = (0,0,\ldots,0) \]

for \( i = 1, \ldots, L \) do
  if \( c_{alg}^{t_1,t_2,i} = c_{alg}^{t_1,t_2,i} \) then
    \( V_i = 1 \)
  else
    \( V_i = 2 \)
  end if
end for

\( I = 1 \)

\( M \) is a \( L \) by \( L \) matrix

Define \( M_{k,} \) as the \( k \)-th row of the matrix

while \( I \neq -1 \) do
  for \( k = 1, \ldots, L \) do
    \( M_{k,} = B \)
    for \( j = 1, \ldots, L \) do
      \( M_{k,j} = B_{j} \)
    end for
    if \( V_k = 1 \) then
      \( M_{k,k} = p_{alg}^{t_1,t_2,k} \)
    else
      \( M_{k,k} = p_{alg}^{t_1,t_2,k} \)
    end if
  end for
  \( I = -1 \)
  for \( k = 1, \ldots, L \) do
    \( MLO = \bar{B} \)
    if \( |\varepsilon - MLO| > |\varepsilon - M_k,| \) AND \( M_k, \leq \varepsilon \) then
      \( B = M_k, \)
      \( I = k \)
      \( \bar{B} = M_k, \)
    end if
  end for
  if \( I > 0 \) then
    if \( V_I = 1 \) then
      \( V_I = 2 \)
    else
      \( V_I = 1 \)
    end if
  end if
end while

for \( i = 1, \ldots, L \) do
  if \( V_i = 1 \) then
    \( b_i = c_{alg}^{t_1,t_2,i} \)
  else
    \( b_i = c_{alg}^{t_1,t_2,i} \)
  end if
end for

Output: \( b \) (vector with the number of beds)
2.5 Merging departments

In this section we describe how we could merge \( m \) departments to determine the number of required beds. Set \( D \) as the set of the selected \( m \) departments. Define \( \rho_u(t) \) as the mean number of occupied beds \( \rho(t) \) at time \( t \) for department \( u \) (2.27) and

\[
\rho^\text{total}(t) = \sum_{u \in D} \rho_u(t) \tag{2.28}
\]

as the total mean number of occupied beds at time \( t \) when merging \( m \) departments. Then the distribution of the number of occupied beds is a Poisson distribution with parameter \( \rho^\text{total}(t) \). We know this because if \( X_1, \ldots, X_m \) are independent Poisson random variables where \( X_i \) has parameter \( \lambda_i \), then

\[
X_1 + \ldots + X_m \text{ has a Poisson distribution with parameter } \lambda_1 + \ldots + \lambda_m. \tag{2.29}
\]

The impact of merging departments on the number of required beds could be beneficial. Suppose department 1 has number of required beds \( b_1 \) and department 2 has number of required beds \( b_2 \). When merging departments 1 and 2 the number of required beds becomes \( b_{1,2} \). Then \( b_{1,2} \leq b_1 + b_2 \).

In the Table 2.5.1 we find the result of number of required beds \( b_{1,2} \) at time \( t \) with target blocking probability \( \varepsilon = 0.05 \). In Figures 2.5.1, 2.5.2, 2.5.3 and 2.5.4 the corresponding number of required beds \( b_1 + b_2 \) is shown, for all combinations \( \rho_1(t) \) and \( \rho_2(t) \) such that \( \rho^\text{total}(t) = \rho_1(t) + \rho_2(t) \).

In these figures we see that \( b_{1,2} \leq b_1 + b_2 \). This result shows that the number of required beds could reduce when we merge two or more departments. Theoretically this is a nice result, however merging departments will not always be practical.

<table>
<thead>
<tr>
<th>( \rho^\text{total}(t) )</th>
<th>( b_{1,2} )</th>
<th>Figure</th>
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<tr>
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<td>105</td>
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</table>

Table 2.5.1

When we merge all departments of the hospital, then we could compute the number of required beds for the whole hospital. This means that when a patient arrives at the hospital, he or she is blocked at this hospital and has to be moved to another hospital. The probability that a patient is blocked at the hospital should be very low.
2.6 Selection of specialisms and departments

In this section we pose the following question: does the following two selections influence the arrival process? The two selections are the following. When we select not all specialisms at a department and selecting one specialism for all departments.

At a department patients of different specialisms could be present. So, it could be interesting to compute the mean number of occupied beds not for all specialisms on the department, but only for a few specialisms. For example, we will do this to see how many beds are occupied by patients at the wrong department. We will describe what will change for the computation of $\rho(t)$, when we have infinitely many beds available.
When we select at one department not all specialisms, the computation of the mean number of occupied beds \( \rho(t) \) does not change. The only difference is that we will compute \( \lambda(t) \) not for all arriving patients at the department, but only for the selected patients. When we are only interested in a selection of arriving patients this is still a non-stationary Poisson process. Here we use the following property of Poisson processes.

Consider a Poisson process \( \{N(t), t \geq 0\} \) having rate \( \lambda \), and suppose that each time an event occurs it is classified as either a type I or a type II event. Suppose further that each event is classified as a type I event with probability \( p \) or a type II event with probability \( 1 - p \), independently of all other events. Let \( N_1(t) \) and \( N_2(t) \) denote respectively the number of type I and type II events occurring in \([0, t]\). Then \( \{N_1(t), t \geq 0\} \) and \( \{N_2(t), t \geq 0\} \) are both Poisson processes having respective rates \( \lambda p \) and \( \lambda (1 - p) \).

So the arriving patients arrive according to a non-stationary Poisson process and this concludes that the number of occupied beds is still represented by a Poisson distribution.

When we select one specialism for all departments, for each department separately we compute the mean number of occupied beds \( \rho_* (t) \) at time \( t \) during the night by

\[
\rho_* (t) = \rho(T) + S(T, t),
\]

where \( T \) is the time when the night starts and \( S(T, t) \) as the number of arrivals between time \( T \) and \( t \). When the arriving patients follow a non-stationary Poisson process then the distribution of the number of occupied beds is Poisson. This lead to the following approximation for the blocking probability for an arbitrary arriving patient in the interval \( (a, b] \):

\[
P^{(a, b]}_{\text{block}} = \begin{cases} 
\frac{\int_{a}^{b} (\lambda^i(t) + \lambda^m(t)) d\lambda^i(t) + \lambda^m(t)) d\lambda^m(t)}{\int_{a}^{b} (\lambda^i(t) + \lambda^m(t)) d\lambda^i(t) + \lambda^m(t)) d\lambda^m(t)} , & \text{if } \int_{a}^{b} (\lambda^i(t) + \lambda^m(t)) d\lambda^i(t) + \lambda^m(t)) d\lambda^m(t) > 0, \\
0, & \text{otherwise.} 
\end{cases}
\]

Having a target blocking probability we are able to compute the number of required beds by using the algorithm in Section 2.4.1. In Section 2.5 we presented how to merge departments and see the impact on the number of required beds. In Section 2.6 we showed that the arrival process, when one takes only a selection of specialisms on the department(s), is still represented by a non-stationary Poisson process.

### 2.7 Discussion of the results

In this chapter the goal was to compute the number of required beds given a target blocking probability \( \varepsilon \). First, we compute the mean number of occupied beds \( \rho(t) \) at time \( t \) in the case that there are infinitely many beds available by Equation (2.1). During the night, almost no patient leave the department. We have solved this problem by computing the mean number of occupied beds \( \rho_*(t) \) at time \( t \) during the night by

\[
\rho_*(t) = \rho(T) + S(T, t),
\]

and let \( \rho_*^2 = (\rho_*^1 - 1) \) be the largest number of required beds such that the bed occupancy rate is larger than \( \delta \) and let \( \rho_*^2 \) be the corresponding bed occupancy rate. Then we could still use the algorithm
in Subsection 2.4.1 to get the number of required beds $c_{(t_1, t_2)}$. So, we could compute the number of required beds when we have a target blocking probability $\varepsilon$ or a target bed occupancy rate $\delta$. 
Chapter 3

Models for patient arrival processes

In this chapter we describe two models for the arrival process of patients. The purpose of this chapter is to present a model that computes the non-stationary arrival rate $\lambda(t)$. We need this arrival rate $\lambda(t)$ to use the equations in Chapter 2.

In [9], for each department we have tested whether the elective and emergency patients arrive according to a Poisson process. We have performed this in the following way. For each elective/emergency patient we first decide at which interval of hours of the weekday he or she arrives at the department. For each of these 168 intervals we test whether the Poisson distribution fits the number of arrivals by using the Chi-Squared goodness-of-fit test. The main conclusion of [9] is that each hourly interval the number of arriving patients follows the Poisson distribution with at most three different parameters. So, at interval $k = 1, \ldots, 168$ the distribution of the number of arriving patients is Poisson with parameter

$$\lambda^k = \begin{cases} 
\lambda_1^k, & \text{with probability } p_1^k, \\
\lambda_2^k, & \text{with probability } p_2^k, \\
\lambda_3^k, & \text{with probability } 1 - p_1^k - p_2^k.
\end{cases}$$

This chapter is organized as follows. In Section 3.1 the non-stationary Poisson process is shown with cyclic parameter $\lambda(t)$ during each week. In Section 3.2 we describe a non-stationary Poisson process where the parameter $\lambda(t)$ depends on the type of week.

The following formulas will be defined for elective patients. However, similar formulas apply for emergency patients, unless mentioned otherwise.

3.1 Non-stationary Poisson process with a cyclic intensity function

In this section we assume that the patients arrive according to a non-stationary Poisson process. In every hour of the week we determine the arrival rate from the data. We will define $N_w$ as the set of week numbers. Define $X_{ij}^{el}$ as the number of elective arrivals during the $j$-th hour of the $i$-th week, for $i \in N_w$ and $j = 1, \ldots, 168$. The total number of elective arrivals during the $i$-th week is defined by

$$X_i^{el} := \sum_{j=1}^{168} X_{ij}^{el}. \quad (3.1)$$

30
Then

\[ Y_{ij}^{el} := \frac{X_{ij}^{el}}{X_i^{el}}, \quad (3.2) \]

is the fraction of elective arrivals during the \( j \)-th hour of the \( i \)-th week. The fraction of elective arrivals during the \( j \)-th hour of a random week is defined by

\[ Z_j^{el} := \frac{1}{|N_w|} \sum_{i \in N_w} Y_{ij}^{el}. \quad (3.3) \]

Then \( Z_1^{el}, \ldots, Z_{168}^{el} \) is the standard week profile for elective patients. The expected number of elective arrivals in one week is given by

\[ X^{el} = \frac{1}{|N_w|} \sum_{i \in N_w} X_i^{el}. \quad (3.4) \]

In this first model we generate the number of elective arrivals during the \( j \)-th hour of a week by a Poisson random variable with parameter \( X^{el} \cdot Z_j^{el} \).

To determine the number of occupied beds \( M(t) \) at time \( t \) we use Formula (2.1), with

\[ \lambda^{el}(u) = X^{el} \cdot Z_j^{el} \cdot \mathbb{I}_{[j-1,j)}(u \mod 168) \quad (3.5) \]

and

\[ \lambda^{em}(u) = X^{em} \cdot Z_j^{em} \cdot \mathbb{I}_{[j-1,j)}(u \mod 168). \quad (3.6) \]

### 3.2 Non-stationary Poisson process with a stochastic intensity function

In this section we will extend the arrival process in Section 3.1 and drop the assumption of a cyclic intensity function for each week.

For every hourly interval we will have at most three parameters \( \lambda_1, \lambda_2 \) or \( \lambda_3 \). First compute \( Y_{ij}^{el} \) (Formula (3.2)) and \( Z_j^{el} \) (Formula (3.3)) for \( i \in N_w \) and \( j = 1, \ldots, 168 \).

Then define the standard deviation of the \( Y_{ij} \) as

\[ \sigma_j^{el} := \sqrt{\frac{1}{|N_w| - 1} \sum_{i \in N_w} (Y_{ij}^{el} - Z_j^{el})^2}. \quad (3.7) \]

For \( j = 1, \ldots, 168 \) define

\[ \Lambda_j^{el,1} = \{ i : Z_j^{el} - \sigma_j^{el} \geq Y_{ij}^{el} \}, \]

\[ \Lambda_j^{el,2} = \{ i : Z_j^{el} - \sigma_j^{el} \leq Y_{ij}^{el} \leq Z_j^{el} + \sigma_j^{el} \}, \]

\[ \Lambda_j^{el,3} = \{ i : Z_j^{el} + \sigma_j^{el} < Y_{ij}^{el} \}. \quad (3.8) \]

Denote \(|\Lambda_j^{el,1}\)| as the number of elements in \( \Lambda_j^{el,1} \). Then we define

\[ Z_j^{el,u} := \frac{1}{|\Lambda_j^{el,u}|} \sum_{k \in \Lambda_j^{el,u}} Y_{kj}^{el,u}, \text{ for } u = 1, 2, 3. \quad (3.9) \]

We generate the number of elective arrivals during the \( j \)-th hour of a week by a Poisson random variable with parameter

\[ \lambda_j^{el} = \begin{cases} X^{el} Z_j^{el,1}, & \text{with probability } \frac{|\Lambda_j^{el,1}|}{|N_w|}, \\ X^{el} Z_j^{el,2}, & \text{with probability } \frac{|\Lambda_j^{el,2}|}{|N_w|}, \\ X^{el} Z_j^{el,3}, & \text{with probability } \frac{|\Lambda_j^{el,3}|}{|N_w|}. \end{cases} \quad (3.10) \]
At this moment the parameters are not dependent on the week. In practice, one could think of a period where more/less patients arrive. So, we will take into account the type of week.

**Type of week**

We shall distinguish three types of weeks, a calm, normal or busy week. For week \( i \) we can define the total number of elective arrivals by \( X_i^e \) (Formula (3.1)). The mean number of weekly arrivals is defined by \( X_i^e \) (Formula (3.4)) and the standard deviation by \( \sigma_{X_i^e} \).

Then \( \lambda_{j,k}^e \) is the parameter of the \( j \)-th hour from the \( k \)-th type week, where \( k = 1 \) means a calm week, \( k = 2 \) means a normal week and \( k = 3 \) means a busy week. For \( i \in N_u \) we call week \( i \):

- a calm week when \( X_i^e < X_i^e - \sigma_{X_i^e} \),
- a normal week when \( X_i^e - \sigma_{X_i^e} \leq X_i^e \leq X_i^e + \sigma_{X_i^e} \),
- a busy week when \( X_i^e > X_i^e + \sigma_{X_i^e} \).

Define \( |\lambda_{j,k}^{e,u}| \) as the total number of parameters with type \( k \) of hour \( u \) for \( u = 1, 2, 3 \). Then

\[
\lambda_{j,k}^e = \begin{cases} 
X_i^{e,1}_{j,1}, & \text{with probability } P_{j,k}^{e,1} = \frac{|\lambda_{j,k}^{e,1}|}{|\lambda_{j,k}^{e,1}| + |\lambda_{j,k}^{e,2}| + |\lambda_{j,k}^{e,3}|}, \\
X_i^{e,2}_{j,2}, & \text{with probability } P_{j,k}^{e,2} = \frac{|\lambda_{j,k}^{e,2}|}{|\lambda_{j,k}^{e,1}| + |\lambda_{j,k}^{e,2}| + |\lambda_{j,k}^{e,3}|}, \\
X_i^{e,3}_{j,3}, & \text{with probability } P_{j,k}^{e,3} = \frac{|\lambda_{j,k}^{e,3}|}{|\lambda_{j,k}^{e,1}| + |\lambda_{j,k}^{e,2}| + |\lambda_{j,k}^{e,3}|}. 
\end{cases}
\]  

(3.11)

In words, for every one-hour interval we could have at most three parameter values. For each week-type \( k \) we compute which parameter value(s) and how many times the parameter value(s) happen for this one-hour interval.

### 3.2.1 Jump probabilities

In this subsection we will compute the probability that week \( i \) is type \( k \). Define \( W_i \) as the type of week \( i \). Define \( p_{j,k} \) as the probability that week \( i + 1 \) is of type \( j \) given that week \( i \) is of type \( j \), where \( j,k = 1, 2, 3 \). Define

\[
U_{j,k} = |\{i|W_i = j,W_{i+1} = k\}|, 
\]  

(3.12)

as the total number of week-type \( j \) followed by week-type \( k \). So

\[
p_{j,k} = P(W_{i+1} = k|W_i = j) = \frac{U_{j,k}}{\sum_{m=1}^{3} U_{j,m}}. \]  

(3.13)

### 3.2.2 Approximation of the mean number of occupied beds

In this subsection we will describe how to approximate the mean number of occupied beds \( \rho(t) \). We approximate Formula (2.1) by

\[
\rho(t) \approx \int_{t-T}^{t} (\lambda^{e}(u)P(B^{e} > t-u) + \lambda^{em}(u)P(B^{em} > t-u))du, 
\]  

(3.14)

where \( T \) is defined as the maximum length-of-stay, rounded up in hours, a patient stayed at the department. We will divide the interval \([t-T, t]\) into the calendar weeks. We assume there are \( n \) weeks in this interval. Define \( N \) as the total number of weeks we use to determine the type of week and the parameters \( \lambda^{e}(t) \) and \( \lambda^{em}(t) \) by

\[
N = \max\{n, 4\}. 
\]  

(3.15)
This means that when $n < 4$ we will use $N = 4$ weeks for our computations.

To compute Formula (3.14) we have to determine the arrival rates $\lambda_{el}(u)$ and $\lambda_{em}(u)$. These arrival rates depend on the type of week. Assuming $u \in [j-1, j)$, for hour $j$ in week-type $k$ we will have

$$
\lambda_{el}(u) = E(\lambda_{el}^{i,k})
= X_{el}^{i}(Z_{j}^{el,1} P_{j,k}^{el,1} + Z_{j}^{el,2} P_{j,k}^{el,2} + Z_{j}^{el,3} P_{j,k}^{el,3}).

$$

(3.16)

Define the probability that week 1 is of type $i_1$ as $P(W_1 = i_1)$. We can use the jump probabilities $p_{i_1,i_2}$ to compute the probability that week 2 is of type $i_2$. We continue this process until we reach week $n$. The probability that $W_2 = i_2, W_3 = i_3, ..., W_n = i_n$ given that $W_1 = i_1$ is:

$$
P(W_2 = i_2, W_3 = i_3, ..., W_n = i_n|W_1 = i_1) = P(W_2 = i_2|W_1 = i_1) P(W_3 = i_3|W_2 = i_2) ... P(W_n = i_n|W_{n-1} = i_{n-1})

\approx \prod_{k=i_1}^{i_{n-1}} p_{i_k,i_{k+1}} = \prod_{k=i_1}^{i_n} p_{i_k,i_{k+1}},

(3.17)

where $i_k + 1 = i_{k+1}$.

Formula (3.14) approximates the mean number of occupied beds. We have the following approximation of the mean number of occupied elective beds:

$$
\rho_{el}(t) = \int_{-\infty}^{t} \lambda_{el}(u) P(B_{el} > t-u) du
\approx \sum_{i_1 = 1}^{3} P(W_1 = i_1) \sum_{i_2 = 1}^{3} \sum_{i_3 = 1}^{3} \left( \prod_{i=1}^{n} p_{i,i+1} \right) \int_{t-T}^{t} \lambda_{el}(u) P(B_{el} > t-u) du,

(3.18)

with $\lambda_{el}(u)$ as in Formula (3.16).

Formula (3.18) uses Formula (3.16) and Formula (3.17) to approximate the mean number of elective occupied beds. We compute for every possible sequence the mean number of occupied beds

$$
\int_{t-T}^{t} \lambda_{el}(u) P(B_{el} > t-u) du

(3.19)

and multiply this with the probability that this route happens. A sequence means a vector of length $n$ with possible week-types. There are $3^n$ possible sequences. One possible sequence for $n = 3$ is: 1(calm), 2(normal), 2(normal).

**Example 3.1.** We want to know the mean number of occupied beds at time $t=2014-06-01 12:00:00$, this is week 22. Suppose the probability that a patient stays more than 3 weeks (504 hours) at the department is $P(B > 504) \approx 0$, then the important weeks are weeks 19, 20, 21 and 22. There are $3^4 = 81$ possible sequences. These 81 sequences have probability $p_i$, $i = 1, ..., 81$, that this will happen.

<table>
<thead>
<tr>
<th>Route</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>81</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 19</td>
<td>calm</td>
<td>calm</td>
<td>...</td>
<td>busy</td>
</tr>
<tr>
<td>Week 20</td>
<td>calm</td>
<td>calm</td>
<td>...</td>
<td>busy</td>
</tr>
<tr>
<td>Week 21</td>
<td>calm</td>
<td>calm</td>
<td>...</td>
<td>busy</td>
</tr>
<tr>
<td>Week 22</td>
<td>calm</td>
<td>normal</td>
<td>...</td>
<td>busy</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mean number of occupied beds at time $t$ following route $i$</th>
<th>$\rho_1(t)$</th>
<th>$\rho_2(t)$</th>
<th>...</th>
<th>$\rho_{81}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>$p_1$</td>
<td>$p_2$</td>
<td>...</td>
<td>$p_{81}$</td>
</tr>
</tbody>
</table>

So the mean number of occupied beds at time $t$ is $\rho(t) = p_1 \rho_1(t) + ... + p_{81} \rho_{81}(t)$.
3.2.3 Holidays

In this subsection we will describe how to deal with the number of arriving patients during holidays. We will use the data of 2013 for computing the expected number of occupied beds in 2014. First we will describe how we will do this. We assume that a week starts at Sunday and ends at Saturday.

In our setting we define week 1 as the first week of the new year which includes 1 January. Then we define week 2 as the week which starts the first Sunday after 1 January. Week 53 always includes 31 December. In the next table the date of the data which is used for the total arriving number of patients is shown.

<table>
<thead>
<tr>
<th>Week</th>
<th>Sunday</th>
<th>Monday</th>
<th>Tuesday</th>
<th>Wednesday</th>
<th>Thursday</th>
<th>Friday</th>
<th>Saturday</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
<td></td>
<td></td>
<td>(2013-01-04)</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>(2013-01-02)</td>
<td></td>
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<td></td>
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<td></td>
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</tr>
</tbody>
</table>

In this model we should also take the holidays into account. On most of these days almost no elective patients arrive at the hospital. There are holidays where only the weekday changes (e.g. New Year’s Day) and holidays where the date changes (e.g. Easter, summer holiday).

When the date in 2013 was a holiday, but in 2014 it is not, we should predict the expected number of arrivals. We could not just use the arrivals of this date on 2013, because it was a special day and in 2014 it is just a normal day. For example, the department is closed at Easter Monday. In 2013 this was 2013-04-01, week 14, and in 2014 this was 2014-04-21, week 17. In 2014 the Monday at week 14 will be a normal day. To compute the above mentioned expectation, we will use the previous two weeks and the next two weeks where the Monday is a normal day. In Section 3.2.3.1 we describe how to do this.

We will describe what could happen to the new holiday date given that this day last year was a normal day. Then there are a few possibilities what could happen for the new holiday date:

- the department is closed.
- there are no/less (elective) arrivals.
- it is a normal day.

When the department is closed, the number of occupied beds is zero. When there are no (elective) arrivals, we will use the same number of transfers as during this day last year. When the day is treated like a normal day, nothing will happen.

3.2.3.1 Predict the expected number of arrivals

In this subsection we will present a way to compute the expected number of arrivals at holidays when these are treated like normal days. We will compute the expected number of arrivals at this day by Lagrange polynomial interpolation.

Lagrange polynomial interpolation

Given a set of \( k + 1 \) data points \( (x_0, y_0), ..., (x_k, y_k) \), where no two \( x_j \) are the same. The Lagrange
interpolating polynomial is the polynomial \( L(x) \) of degree smaller than or equal to \( k \) that passes through the \( k + 1 \) points \((x_0, y_0), \ldots, (x_k, y_k)\) and is given by

\[
L(x) = \sum_{j=0}^{k} y_j l_j(x),
\]

where

\[
l_j(x) = \prod_{0 \leq m \leq k \atop m \neq j} \frac{x - x_m}{x_j - x_m}, \quad \text{for } 0 \leq j \leq k.
\]

In our case we have the following data points: \((x_1, y_1), (x_2, y_2), (x_4, y_4), (x_5, y_5)\). We are interested in the value of \( L(x_3) \), the number of arrivals at \( x_3 \). It could happen that the value of \( L(x_3) \) is negative, then in our case \( y_3 = \max\{0, L(x_3)\} \), because the number of arrivals could not be negative.

**Example 3.2.** We want to compute the expected number of arrivals during 2014-03-31 (10:00:00-11:00:00]. For the number of arrivals at 2014-03-31 we use the arrivals of 2013-04-01, but this was Easter Monday and the department was closed. So, we have to predict the number of arrivals at 2014-03-31 by Lagrange polynomial interpolation as described above. Define

\[
y_1 := \text{the number of arrivals during 2014-03-17 (10:00:00-11:00:00]} = 3,
\]
\[
y_2 := \text{the number of arrivals during 2014-03-24 (10:00:00-11:00:00]} = 3,
\]
\[
y_4 := \text{the number of arrivals during 2014-04-07 (10:00:00-11:00:00]} = 0,
\]
\[
y_5 := \text{the number of arrivals during 2014-04-14 (10:00:00-11:00:00]} = 1,
\]

and \( x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5 \). Then we use this Lagrange polynomial interpolation to find \( y_3 \), the number of arrivals during 2014-03-31 (10:00:00-11:00:00]. Then

\[
l_1(x) = \left(\frac{x-x_2}{x_1-x_2}\right) \left(\frac{x-x_4}{x_1-x_4}\right) \left(\frac{x-x_5}{x_1-x_5}\right) = \frac{10}{3} - \frac{19}{6} x + \frac{11}{12} x^2 - \frac{1}{12} x^3,
\]
\[
l_2(x) = \left(\frac{x-x_1}{x_2-x_1}\right) \left(\frac{x-x_4}{x_2-x_4}\right) \left(\frac{x-x_5}{x_2-x_5}\right) = -\frac{10}{3} + \frac{29}{6} x - \frac{5}{3} x^2 + \frac{1}{6} x^3,
\]
\[
l_4(x) = \left(\frac{x-x_1}{x_4-x_1}\right) \left(\frac{x-x_2}{x_4-x_2}\right) \left(\frac{x-x_5}{x_4-x_5}\right) = \frac{5}{3} - \frac{17}{6} x + \frac{4}{3} x^2 - \frac{1}{6} x^3,
\]
\[
l_5(x) = \left(\frac{x-x_1}{x_5-x_1}\right) \left(\frac{x-x_2}{x_5-x_2}\right) \left(\frac{x-x_4}{x_5-x_4}\right) = \frac{2}{3} + \frac{7}{6} x - \frac{7}{12} x^2 + \frac{1}{12} x^3.
\]

This will give us

\[
L(x) = \frac{2}{3} + \frac{37}{6} x - \frac{17}{6} x^2 + \frac{1}{3} x^3,
\]

which is shown in Figure 3.2.1. So, \( y_3 = L(x_3) = L(3) = \frac{4}{3} \).
3.2.4 Example

In this subsection an example is shown where we will compute the mean number of occupied beds $\rho(t)$ by the model described in this section with the holidays. We will look at a daycare department. The department is closed at the weekend, so then the number of required beds is zero. In Figure 3.2.2 we determine the required number of beds per day such that the blocking probability is 0.05. This will give a bed occupancy rate of 33.68\%.

In Figure 3.2.3 we determine the required number of beds per hour such that the blocking probability is 0.05. This will give a bed occupancy rate of 71.50\%.

In both of Figures 3.2.2 and 3.2.3 it is shown that the department is closed on Easter Monday (2014-04-21). We conclude from the difference between the two bed occupancy rates that the number of occupied beds is not equally spread over the day, which is what we should expect.

In practice it should be useful to determine the number of required beds during the working hours, for example intervals of 4 or 8 hours. This means that the number of required beds does not change during these intervals. This will be the intervals $[a_i, a_{i+1}]$ in Section 2.4.
The number of required beds

Figure 3.2.2: The number of required beds per day for a department in the period 2014/03/31 00:00:00 - 2014/05/02 23:00:00 with $\varepsilon = 0.05$
3.2.5 The variability of the number of occupied beds

In this subsection we measure the variability of the number of occupied beds, when infinitely many beds are available. For the variability in the number of occupied beds we will use the index of dispersion as performance measure. The index of dispersion $I(t)$ for the number of occupied beds at time $t$ is defined by

$$I(t) = \frac{\text{Var}(M(t))}{\mathbb{E}[M(t)]}. \quad (3.22)$$

To compute the index of dispersion $I([a, b])$ during an interval $[a, b]$ we break up the interval $[a, b]$ into some number $n$ of subintervals $[t_i, t_{i+1})$ for $i = 1, ..., n$, i.e. $a = t_1, t_2, ..., t_n, t_{n+1} = b$.

Define

$$\eta = \frac{1}{n+1} \sum_{i=1}^{n+1} M(t_i), \quad (3.23)$$

$$\nu = \frac{1}{n} \sum_{i=1}^{n} (M(t_i) - \eta)^2. \quad (3.24)$$
The index of dispersion $I([a, b])$ will be approximated by

$$I([a, b]) \approx \frac{\nu}{\eta}. \quad (3.25)$$

In Lemma 3.1 we compute the variance of the number of occupied beds in the interval $[t_1, t_n]$. The mean number of occupied beds in the interval $[t_1, t_n]$ is

$$\frac{1}{n} \sum_{i=1}^{n} \rho(t_i).$$

**Lemma 3.1.** Let $X_1, \ldots, X_n$ be independent Poisson random variables with parameters $\lambda_1, \ldots, \lambda_n$. Define $X = \frac{1}{n-1} \sum_{i=1}^{n} \left( X_i - \frac{1}{n} \sum_{j=1}^{n} X_j \right)^2$. Then the mean of $X$ is

$$E[X] = \frac{1}{n} \sum_{i=1}^{n} (\lambda_i + \lambda_i^2) - \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \lambda_i \lambda_j. \quad (3.26)$$

The proof can be found in the Appendix in Section 6.1.

We know that the number of occupied beds $M(t)$ at time $t$ is Poisson distributed with parameter $\rho(t)$. Then the index of dispersion $I_{[t_1, t_n]}$ is given by

$$I_{[t_1, t_n]} = \frac{1}{n} \sum_{i=1}^{n} (\rho(t_i) + \rho(t_i)^2) - \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \rho(t_i) \rho(t_j). \quad (3.27)$$

**3.3 Discussion of the results**

In this chapter we have presented two models for the arrival process. In the first model (Section 3.1) we use a non-stationary Poisson process with a cyclic intensity function, where the period is one week. The effect on the mean number of occupied beds $\rho(t)$ at time $t$ of this arrival process is that the mean number of occupied beds is also cyclic with a period of one week. The second model (Section 3.2) is an extension of the first model. We distinguish three types of weeks, a calm, normal or busy week. This type of week would be determined by the number of arriving patients. For each type of week we computed the arriving parameter and the probability that this week is of type calm, normal or busy. We also take into account the holidays, because the number of elective arrivals could be different on these days. This would give the arrival rate $\lambda(t)$ at time $t$.

One could use this model to estimate the arrival parameter $\lambda(t)$ at time $t$. When this parameter is known, we compute the mean number of occupied beds $\rho(t)$ at time $t$. Then the number of occupied beds $M(t)$ at time $t$ is Poisson distributed with parameter $\rho(t)$. One is now able to compute the number of required beds as described in Section 2.4.
Chapter 4

Impact of changing parameters on the number of occupied beds

In this chapter we investigate the influence of changing parameters on the number of occupied beds. The purpose of doing this is to show that planning elective patients differently could lead to a smaller variability of the number of occupied beds. Further we will show that changing the parameters of the length-of-stay distribution is only beneficial when the variability in the arrival process is small.

Throughout this chapter we will use the same department in the examples. This department has mean length-of-stay $E[B^{el}] = 31.9$ hours and variance $Var(B^{el}) = 1077.2$ hours for elective patients and mean length-of-stay $E[B^{em}] = 90.0$ hours and variance $Var(B^{em}) = 10014.8$ hours for emergency patients. Define $t_s = 2014-01-01 00:00:00$, $t_e = 2014-12-31 23:00:00$ and $M([t_s, t_e])$ the mean number of occupied beds during $[t_s, t_e]$. The index of dispersion of the number of occupied beds $I([t_s, t_e]) = 1.71$. In 2013 there arrived 3886 elective patients and 284 emergency patients.

In Section 4.1 we look at the distribution of the number of occupied beds when the arrival process is not Poisson. We look at the impact of the length of stay in Section 4.2, the total production (more/less patients to treat) in Section 4.3 and the variability of the number of occupied beds in Section 4.4.

4.1 General arrival process

In this section we describe how we can approximate the number of occupied beds when the arrival process is not Poisson. In Section 4.4 we will change the number of elective arrivals such that the index of dispersion of the number of occupied beds reduce to a target $I_{\text{target}}$. This means that the elective arrival process will change into some deterministic arrival process during each hour.

4.1.1 The blocking probability

In this subsection we will approximate the blocking probability when the arrival process is not Poisson. In [10] it is shown that the number of occupied beds $M$ in the $G/G/\infty$ model becomes approximately normally distributed

$$M \approx N(\alpha, z\alpha),$$

and is asymptotically exact for $\alpha$, where $\alpha$ is the mean number of occupied beds and $z$ is called the heavy-traffic peakedness, which is

$$z = 1 + \frac{\alpha^2 - 1}{E[B]} \int_0^\infty P(B > x)^2 \, dx,$$
where the random variable $B$ is the length-of-stay. We also have that $0 \leq \int_0^\infty \frac{P(B > x)^2}{E[B]} \leq 1$ because
\[ \int_0^\infty P(B > x)^2 dx \leq \int_0^\infty P(B > x)^2 dx = E[B]. \]

For a renewal arrival process, the variability parameter $c^2_\alpha$ is the squared coefficient of variation of an interarrival time. In general
\[ c^2_\alpha = \lim_{t \to \infty} \frac{\text{Var}(Q(t))}{E[Q(t)]}, \quad (4.3) \]
where $Q(t)$ counts the number of arrivals before time $t$, which is defined as Equation (10) in [19]. When we have a Poisson arrival process, then $c^2_\alpha = 1$ and also $z = 1$. Then the number of occupied beds $M$ is approximately Normally distributed with mean $\alpha$ and variance $\alpha$. In Chapter 2 it is shown that the number of occupied beds is Poisson distributed. In this chapter is also shown that when $\alpha$ is large, then the Poisson distribution with parameter $\alpha$ is asymptotically exact to a Normal distribution with mean $\alpha$ and variance $\alpha$. This is the connection between the distribution of the number of occupied beds in Chapter 2 and this chapter.

In [10] the following blocking probability $P_{\text{block}}$ is used for the $G/G/c/c$ model:
\[ P_{\text{block}} = 1 - \frac{E[M_c]}{\alpha} \approx \sqrt{\frac{z}{\alpha \Phi((c-\alpha)/\sqrt{\alpha z})}}, \quad (4.4) \]
where $M_c$ is the number of occupied beds in the $G/G/c/c$ model and
\[ P(M_c = k) \approx \frac{P(M = k)}{P(M \leq c)} \text{ for } 0 \leq k \leq c. \quad (4.5) \]

### 4.1.1.1 Time-varying arrivals

In this subsection we will present the blocking probability when we deal with time-varying arrivals. The approximation of the number of occupied beds $M_c(t)$ at time $t$ is asymptotically exact when $\alpha(t)$ is large. The probability that $P(M_c(t) = k)$ is approximated by
\[ P(M_c(t) = k) \approx \frac{P(M(t) = k)}{P(M(t) \leq c)} \text{ for } 0 \leq k \leq c. \quad (4.6) \]

The blocking probability $P_{\text{block}}(t)$ at time $t$ for the $G_t/G/c_t/c_t$ model is approximated by
\[ P_{\text{block}}(t) = 1 - \frac{E[M_c(t)]}{\alpha(t)} \approx \sqrt{\frac{z(t)}{\alpha(t)} \Phi((c_t - \alpha(t))/\sqrt{\alpha(t)z(t)})}, \quad (4.7) \]
where $\alpha(t)$ defines the mean number of occupied beds at time $t$ and
\[ z(t) = 1 + c^2_\alpha(t) - \frac{1}{E[B]} \int_0^\infty P(B > x)^2 dx, \quad (4.8) \]
is called the heavy-traffic peakedness as in [7], where $c^2_\alpha(t) \approx \frac{\text{Var}(Q(t)-Q(t-\eta))}{\int_{t-\eta}^t \lambda(u) du}$ is the time-dependent generalization of the asymptotic variability parameter in Equation (4.3).

Define $z_{el}(t)$ and $z_{em}(t)$ as the peakedness for elective and emergency patients respectively. Then the total number of occupied beds $M(t)$ at time $t$ is approximated by a Normal distribution with mean
\[ \rho_{el}(t) + \rho_{em}(t) \quad (4.9) \]
and variance
\[ z_{el}(t)\rho_{el}(t) + z_{em}(t)\rho_{em}(t), \quad (4.10) \]
where $\rho_e(t)$ and $\rho_{\text{em}}(t)$ denote the mean number of occupied beds at time $t$ for elective and emergency patients respectively. Then the blocking probability $P_{\text{block}}^{(a,b)}$ for an arbitrary arriving patient in the interval $(a,b]$ is approximated by

$$
P_{\text{block}}^{(a,b)} \approx \begin{cases} 
\int_a^b (\lambda^e(t) + \lambda^\text{em}(t)) P_{\text{block}}(t) \, dt, & \text{if } \int_a^b (\lambda^e(t) + \lambda^\text{em}(t)) \, dt > 0, \\
0, & \text{otherwise.}
\end{cases}
$$

(4.11)

The number of required beds

For combining the blocking probability as in Equation (4.11) with the algorithm in Section 2.4 we are able to compute the number of required beds given a target blocking probability $\varepsilon$ or a target bed occupancy rate $\delta$.

### 4.2 The length-of-stay distribution

In this section we examine the influence of the length-of-stay distribution. We will look at three different variations

i. the mean length-of-stay reduces and the variance is the same,

ii. the mean length-of-stay is the same and the variance reduces,

iii. the mean length-of-stay reduces and the variance reduces, such that the index of dispersion of the length-of-stay is the same.

In Subsection 2.1.4 we have shown that the influence of the length-of-stay distribution on the mean number of occupied beds is minimal. So, we assume the length-of-stay distribution is Log-Normal with parameters $\mu$ and $\sigma$. Then the mean length-of-stay is

$$
e^{\mu + \frac{\sigma^2}{2}}
$$

and the variance

$$(e^{\sigma^2} - 1)e^{2\mu + \sigma^2}.
$$

(4.13)

Define $\theta$ as the target mean and $\nu$ as the target variance. Then we have to solve the following two equations

$$
\theta = e^{\mu + \frac{\sigma^2}{2}}, \quad \nu = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}.
$$

(4.14)

We can solve this for $\mu$ and $\sigma$ and have the parameters for the length-of-stay distribution. The solution of the system of equations (4.14) is

$$
\mu = \ln(\theta) - \frac{1}{2} \ln \left( \frac{\nu}{\theta^2} + 1 \right), \\
\sigma^2 = \ln \left( \frac{\nu}{\theta^2} + 1 \right).
$$

(4.15)

In the case of a day-dependent (time-dependent) length-of-stay distribution there are some variations. The mean length-of-stay could be changed each weekday or only a few weekdays. The procedure is still the same: get the new parameters $\mu$ and $\sigma$ of that weekday by using equations (4.15).
**Example 4.1.** In this example we will test the impact of changing the mean and variance of the length-of-stay distribution of elective patients on the variability of the number of occupied beds. The results are displayed in Table 4.2.1. In the first column of this table denotes the case which is given in the beginning of this section. We get these results by using the model in Section 3.2 and changing the parameters of the length-of-stay distribution as described above.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\mathbb{E}[B]$</th>
<th>$\text{Var}(B)$</th>
<th>$\mathbb{E}[M([t_a,t_c]])$</th>
<th>$\text{Var}(M([t_a,t_c]))$</th>
<th>$I([t_a,t_c])$</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>31.9</td>
<td>1077.2</td>
<td>18.96</td>
<td>32.42</td>
<td>1.71</td>
</tr>
<tr>
<td>ii</td>
<td>28</td>
<td>1077.2</td>
<td>17.21</td>
<td>29.08</td>
<td>1.69</td>
</tr>
<tr>
<td>ii</td>
<td>31.9</td>
<td>538.6</td>
<td>18.98</td>
<td>34.35</td>
<td>1.81</td>
</tr>
<tr>
<td>iii</td>
<td>28</td>
<td>945.5</td>
<td>17.22</td>
<td>29.27</td>
<td>1.70</td>
</tr>
</tbody>
</table>

Table 4.2.1: The impact of changing the parameters of the length-of-stay distribution on the number of occupied beds.

### 4.2.1 Discussion of the results

In Figures 4.2.1, 4.2.3, 4.2.5 and 4.2.7 the different parameters, as in Table 4.2.1, are compared to the original parameters of the length-of-stay distribution. In Figures 4.2.2, 4.2.4, 4.2.6 and 4.2.8 the differences between the mean number of occupied beds when changing the parameters and the original mean number of occupied beds are shown.

The figures and tables suggest the following. Reducing the mean length-of-stay lowers the mean number of occupied beds. This is clear, because the mean length-of-stay of a patient reduces. In Table 4.2.1 and in Figure 4.2.2 this is shown.

Another conclusion is that reducing the variance of the length-of-stay distribution lead to a higher variability of the number of occupied beds when the variability of the arrival process is larger than one. This result is also shown in Table 4.2.1. We explain this by Equation (4.8). When $c_2^2(t) > 1$ the deterministic length-of-stay $D$ gives the highest peakedness $z(t) = c_2^2(t)$, because

$$\int_0^{\infty} P(B > x)^2 dx = \int_0^D 1 dx = D,$$

and $E[B] = D$. Thus, we have $z(t) = c_2^2(t)$. So, reducing the variance of the length-of-stay distribution will lead to a higher variability of the number of occupied beds when $c_2^2(t) > 1$.

We can conclude that the mean and variance of the length-of-stay distribution did not reduce the variability of the number of occupied beds when the variability of the arrival process is large ($c_2^2(t) > 1$), which is also concluded in [18]. But when the variability of the arrival process is low ($c_2^2(t) < 1$), reducing the variability of the length-of-stay distribution is reducing the variability of the number of occupied beds.
Figure 4.2.1: The mean number of occupied beds when $\mathbb{E}[B^3] = 28$ hours.

Figure 4.2.2: The difference between the mean number of occupied beds when $\mathbb{E}[B^3] = 28$ hours and $\mathbb{E}[B^3] = 31.9$ hours.

Figure 4.2.3: The mean number of occupied beds when $\text{Var}(B^3) = 538.6$ hours.

Figure 4.2.4: The difference between the mean number of occupied beds when $\text{Var}(B^3) = 538.6$ hours and $\text{Var}(B^3) = 1077.2$ hours.
4.3 The production

In this section we will add $S$ elective patients during the period $P = [t_1, t_2]$. Define $\kappa$ as the total number of elective patients that arrived in the interval $P$. Then the fraction of extra elective patients is $\nu = \frac{S}{\kappa}$.
We use three different methods. In method 1 we add every time a fraction of the extra elective patients. In methods 2 and 3 we find the moment where the average total number of occupied beds is the lowest and add the extra elective patients there. In method 3 we use a more realistic way of adding the patients by using the seasonality. Each of the three methods will be discussed below and the results will be discussed in Subsection 4.3.4.

4.3.1 Method 1

In this first method we multiply each number of elective arrivals with \( v \). So, each time-interval where elective patients are planned, the total number of arriving elective patients becomes larger.

In Figure 4.3.1 the mean number of occupied beds is shown and the mean number of occupied beds when adding 5% elective patients. In Figure 4.3.2 the difference of the mean number of occupied beds when adding 5% elective patients and the original mean number of occupied beds as in Section 3.2 is shown.

Figure 4.3.1: The mean number of occupied beds for a department in the period 2014/01/01 00:00:00 - 2014/12/31 23:00:00 when adding 5% elective patients according to method 1.
Define \( \tau_j \) as the interval \((j - 1, j]\) of one hour. Define
\[
\tau = \{\tau_j\mid (\text{elective}) \text{ patient can arrive in interval } (j - 1, j]\},
\]
as the set of intervals in which (elective) patients can arrive. Then we compute for every \( \tau_j \) the mean number of occupied beds \( \xi_{\tau_j} \) during the interval \([\tau_j, \tau_j + T]\), where \( T \) is the mean length-of-stay of elective patients. So,
\[
\xi_{\tau_j} = \frac{\sum_{i=t \in \tau_j} \rho(i) T}{T},
\]
where \( \rho(i) \) is as in Section 3.2.2. Then we add one patient in the interval
\[
\tau_j^* = \arg \min_j \xi_{\tau_j},
\]
where the mean number of occupied beds is minimal. We repeat this process until all \( S \) patients are added.

In this method we have to compute \( \rho(t) \) for all \( t \) where this extra patient has influence on the number of occupied beds. This could be very expensive when many patients should be added.

We can approximate the number of elective occupied beds at time \( t \) by adding the elective arrivals during the interval \([t - \mathbb{E}(B^{el}), t]\) and the number of emergency occupied beds at time \( t \) by adding the emergency arrivals during the interval \([t - \mathbb{E}(B^{em}), t]\).

In Figure 4.3.3 the mean number of occupied beds is shown and the mean number of occupied beds when adding 5% elective patients. In Figure 4.3.4 the difference of the mean number of occupied beds when adding 5% elective patients and the original mean number of occupied beds as in Section 3.2 is shown.
4.3.3 Method 3

There could be seasonality in the number of elective arrivals. In method 2 this is not used, so a more realistic way of adding elective patients is the following. Determine the extra number of elective patients for each month, during the period $P$. Define $M = \{i | \text{month } i \text{ in interval } P\}$ as the set of months in interval $P$. For each month $i \in M$ we compute $A_i$, the number of arrivals in $P$ during month $i$. Define $A$ as the total number of elective arrivals during $P$. Then the fraction
of arrivals during month $i$ will be

$$F_i = \frac{A_i}{\bar{A}}.$$  \hfill (4.19)

Define $S_i$ as the extra elective patients for month $i \in M$. Then

$$S_i = F_i S.$$ \hfill (4.20)

For each month $i \in M$ we want to add the patients where the workload is minimal, such that the busy moments are avoided as much as possible. We will do this by applying method 2 each month.

One problem that still must be solved in this method is: How to get integer values of $S_i$ such that $\sum_{i \in M} S_i = S$?

Define $\gamma_i = \lfloor S_i \rfloor$, where $\lfloor S_i \rfloor$ is the nearest integer to $S_i$. If $\sum_{i \in M} \gamma_i = S$ we are done. The second possibility is $\sum_{i \in M} \gamma_i < S$. Then we do the following:

**Algorithm 2**

1. Define $M_k$ as the $k$-th element of the set $M$. In algorithm 2 we start with $M_1$. If $\gamma_{M_1}$ is a rational number we round it to the nearest integer above its current value and compute $\sum_{i \in M} \gamma_i$. If still $\sum_{i \in M} \gamma_i < S$, then we will do this for $M_2$. We repeat this until $\sum_{i \in M} \gamma_i = S$.

The third possibility is $\sum_{i \in M} \gamma_i > S$. Then we do the same as when $\sum_{i \in M} \gamma_i < S$, but instead of rounding it to the nearest integer above its current value we round it to the nearest integer below its current value.

In Figure 4.3.5 the mean number of occupied beds is shown and the mean number of occupied beds when adding 5% elective patients. In Figure 4.3.6 the difference of the mean number of occupied beds when adding 5% elective patients and the original mean number of occupied beds as in Section 3.2 is shown.
Figure 4.3.5: The mean number of occupied beds for a department in the period 2014/01/01
00:00:00 - 2014/12/31 23:00:00 when adding 5% elective patients according to method 3.

Figure 4.3.6: The difference of the mean number of occupied beds for a department in the period
2014/01/01 00:00:00 - 2014/12/31 23:00:00 when adding 5% elective patients according to method
3.

4.3.4 Discussion of the results

In this subsection we discuss the results of the three methods. In Table 4.3.1 we find the results
of the impact of adding elective patients by the different methods on the number of required beds
computed by the algorithm in Section 2.4. Define $\bar{\rho}_{[t_s, t_c]}$ as the mean number of occupied
beds in the interval $[t_s, t_c]$, $\bar{c}_{[t_s, t_c]}$ as the mean number of required beds $c_{[t_s, t_c]}$ in the
interval $[t_s, t_c]$ and $v_{[t_s, t_c]}$ as the variance of the number of required beds $c_{[t_s, t_c]}$ in the
interval $[t_s, t_c]$. 

50
We conclude that the mean number of occupied beds and the mean number of required beds after adding patients are almost the same for the three different methods. In method 1 the mean number of occupied beds is larger during the whole period in comparison to the original mean number of occupied beds. In methods 2 and 3 the mean number of occupied beds could be lower than the original mean number of occupied beds. The conclusion we draw is that when adding the elective patients in a smart way, the maximum number of required beds become not automatically larger and we could reduce the variability in the number of required beds.

The most realistic way of adding elective patients is method 3. In practice it could be the case for most of the departments that the number of elective patients depends on the season. Method 3 uses this, so when the summer is a calm period not all extra patients could be planned in the summer. It is possible that there are no more elective patients in the summer.

### 4.4 The variability in the number of occupied beds

In Section 4.2 and Section 4.3 we change the input parameters to see the impact on the output. In this section we do it the other way around. So, we pose the question: what should be your input to get the target output?

We do this because we know that reducing the variability of the arrival process will reduce the variability of the number of occupied beds. Our goal is to answer the question: if one reduces the index of dispersion of the number of occupied beds to $I_{\text{target}}$, how do you plan your elective patients? One restriction is that the total number of elective patients should stay the same over the period. In the following algorithm we will change the number of elective arrivals such that the index of dispersion of the number of occupied reduces. In this algorithm we will also look to the mean number of emergency beds. For example, if the emergency number of occupied beds is low we will schedule more elective patients then when the emergency number of occupied beds is large. So, the variability in the emergency number of occupied beds is significant on scheduling the elective patients such that the index of dispersion of the total number of occupied beds is $I_{\text{target}}$.

#### 4.4.1 Algorithm

In this subsection we will describe the algorithm we will use to reduce the variability of the number of occupied beds. We simulate the number of occupied beds as starting position of the algorithm. For each time-interval where patients arrive we take for each elective and clinical type of admission patient a Log-Normal length-of-stay random variable $B_{\text{el}}^{\text{clin}}$, for each elective and daycare type of admission patient a Log-Normal length-of-stay random variable $B_{\text{el}}^{\text{day}}$, for each emergency and clinical type of admission patient a Log-Normal length-of-stay random variable $B_{\text{em}}^{\text{clin}}$ and for each emergency and daycare type of admission patient a Log-Normal length-of-stay random variable $B_{\text{em}}^{\text{day}}$ for weekday $d$. Only when the departure time is at night, the patient stays until the next morning. This will give an approximation of the number of occupied beds.

The target index of dispersion of the number of occupied beds is $I_{\text{target}}$. Define $\Omega$ as the set of times where elective patients could arrive. Define $G_{s,\text{clin}}$ as the mean number of occupied beds

<table>
<thead>
<tr>
<th>Method</th>
<th>$\bar{c}(t_s, t_e)$</th>
<th>$c(t_s, t_e)$</th>
<th>$\min c(t_s, t_e)$</th>
<th>$\max c(t_s, t_e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>original</td>
<td>18.96</td>
<td>24.3</td>
<td>14.44</td>
<td>36</td>
</tr>
<tr>
<td>1</td>
<td>19.76</td>
<td>25.2</td>
<td>15.6</td>
<td>38</td>
</tr>
<tr>
<td>2</td>
<td>19.77</td>
<td>25.2</td>
<td>13.74</td>
<td>35</td>
</tr>
<tr>
<td>3</td>
<td>19.77</td>
<td>25.2</td>
<td>13.76</td>
<td>36</td>
</tr>
</tbody>
</table>

Table 4.3.1
for clinical types of admissions during the interval \([s, s + E(B_{\text{clin}}^d)]\), \(\forall s \in \Omega\) and \(G_{s,\text{day}}\) as the mean number of occupied beds for daycare types of admissions during the interval \([s, T_{\text{end}}]\), \(\forall s \in \Omega\), where

\[
T_{\text{end}} = \min\{s + E[B_{\text{clin}}^d], C_d\},
\]

where \(C_d\) the closing time of the department on weekday \(d\). Then define

\[
H_{\text{max,clin}} = (t_1, ..., t_n) \text{ such that } G_{t_1,\text{clin}} \geq G_{t_2,\text{clin}} \geq ... \geq G_{t_n,\text{clin}}, \forall t_i \in \Omega,
\]

\[
H_{\text{max,day}} = (t_1, ..., t_n) \text{ such that } G_{t_1,\text{day}} \geq G_{t_2,\text{day}} \geq ... \geq G_{t_n,\text{day}}, \forall t_i \in \Omega,
\]

\[
H_{\text{min,clin}} = (t_1, ..., t_n) \text{ such that } G_{t_1,\text{clin}} \leq G_{t_2,\text{clin}} \leq ... \leq G_{t_n,\text{clin}}, \forall t_i \in \Omega,
\]

\[
H_{\text{min,day}} = (t_1, ..., t_n) \text{ such that } G_{t_1,\text{day}} \leq G_{t_2,\text{day}} \leq ... \leq G_{t_n,\text{day}}, \forall t_i \in \Omega.
\]

\(H_{\text{max,type}}\) denotes the time intervals such that the mean number of occupied beds are ordered, where the maximum is the first element. \(H_{\text{min,type}}\) is the opposite of \(H_{\text{max,type}}\). Define \(H_{\text{max,clin},i_c}\) as the \(i_c\)-th element of the vector \(H_{\text{max,clin}}\), \(H_{\text{max,day},i_d}\) as the \(i_d\)-th element of the vector \(H_{\text{max,day}}\), \(H_{\text{min,clin},i}\) as the \(i\)-th element of the vector \(H_{\text{min,clin}}\) and \(H_{\text{min,day},i}\) as the \(i\)-th element of the vector \(H_{\text{min,day}}\).

We start with \(i_c = 1\) and \(i_d = 1\). Then define

\[
t_{\text{max}} = \begin{cases} H_{\text{max,clin},i_c}, & \text{if } G_{H_{\text{max,clin},i_c,\text{clin}}} > G_{H_{\text{max,day},i_d,\text{clin}}}, \\ H_{\text{max,day},i_d}, & \text{otherwise}, \end{cases}
\]

as the time-interval where the mean number of occupied beds is maximum. If \(t_{\text{max}} = H_{\text{max,clin},i_c}\), then \(t_{\text{min}} = H_{\text{min,clin},1}\) is the time-interval where the mean number of occupied beds is minimum. If \(t_{\text{max}} = H_{\text{max,day},i_d}\), then \(t_{\text{min}} = H_{\text{min,day},1}\) is the time-interval where the mean number of occupied beds is minimum.

Then at \(t_{\text{max}}\) we remove one elective patient and add this patient at time-interval \(t_{\text{min}}\). When at \(t_{\text{max}}\) no elective patients arrive we set \(i_c = i_c + 1\), if \(t_{\text{max}} = H_{\text{max,clin},i_c}\), or \(i_d = i_d + 1\), \(t_{\text{max}} = H_{\text{max,day},i_d}\), and compute \(t_{\text{max}}\) again. Then we simulate the number of occupied beds again and approximate the index of dispersion \(I\). When \(|I_{\text{target}} - I| > \varepsilon\) we repeat this process, where \(\varepsilon\) is a small number which denotes the maximum absolute difference to \(I_{\text{target}}\).

When we deal with a daycare department the algorithm is a bit different then we described above. Now, will describe these differences.

For the computation of the index of dispersion we only need to use the number of occupied beds when the department is open. Otherwise the number of occupied beds is zero. Again we define \(\Omega\) as the set of time-intervals where the elective patients could arrive and \(G_{s,\text{clin}}\) as the mean number of occupied beds for clinical types of admissions during the interval \([s, T_{\text{clin}}]\), \(\forall s \in \Omega\) and \(G_{s,\text{day}}\) as the mean number of occupied beds for daycare types of admissions during the interval \([s, T_{\text{end}}]\), \(\forall s \in \Omega\), where

\[
T_{\text{clin}} = \min\{s + E[B_{\text{clin}}^d], C_d\},
\]

\[
T_{\text{end}} = \min\{s + E[B_{\text{day}}^d], C_d\}.
\]

The rest of the algorithm will be the same.

### 4.4.2 The number of required beds

We want to see the impact of reducing the index of dispersion of the number of occupied beds on the number of required beds.
Define $A^{el}(t)$ as the number of elective arrivals in $(t - 1, t]$. Then $\lambda^{el}(t) = A^{el}(t)$ is the elective arrival rate in the interval $(t - 1, t]$. We compute $\rho^{el}(t)$ by

$$\rho^{el}(t) = \int_{-\infty}^{t} A^{el}(u)\mathbb{P}(B^{el} > t - u)du.$$ \hfill (4.24)

The number of emergency arrivals at time $t$ is represented by a Poisson process and $\rho^{em}(t)$ is computed in the same way as in Section 3.2. We compute the number of required beds as described in Section 4.1 with $\lambda^{el}(t) = A^{el}(t)$ and $\rho^{el}(t)$ as in Equation (4.24).

In Figure 4.4.1 the mean number of occupied beds is shown for an index of dispersion of 1.7, and also 1, of the number of occupied beds. In Figure 4.4.3 the corresponding number of required beds with blocking probability 0.05 is shown. When the index of dispersion of the number of occupied beds is 1.7 the mean number of required beds is 24.3. When reducing the index of dispersion of the number of occupied beds to 1 the mean number of required beds is 21.2. In this example, we can reduce the mean number of required beds with three when reducing the variability of the number of occupied beds.

![The mean number of occupied beds](image)

**Figure 4.4.1:** The mean number of occupied beds for the two cases (i) $I([t_s, t_e]) = 1.7$ and (ii) $I([t_s, t_e]) = 1$. 

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Figure 4.4.2: The difference between the mean number of occupied beds for the two cases (i) \( I([t_s, t_e]) = 1.7 \) and (ii) \( I([t_s, t_e]) = 1 \).

Figure 4.4.3: The number of required beds for the two cases (i) \( I([t_s, t_e]) = 1.7 \) and (ii) \( I([t_s, t_e]) = 1 \).
Difference between the number of required beds

Figure 4.4.4: The difference between the number of required beds for the two cases (i) \( I([t_s, t_c]) = 1.7 \) and (ii) \( I([t_s, t_c]) = 1 \).

### 4.4.3 Merging departments

In this subsection we will describe what will change to the algorithm discussed in Subsection 4.4.1 when we merge \( m \) departments. Let \( D \) be the set of the selected \( m \) departments.

#### Changes in algorithm

Define \( E[B^u_{\text{clin},u}] \) as the mean length-of-stay for an elective and clinical patient at department \( u \) and \( E[B^u_{\text{day},u}] \) as the mean length-of-stay for an elective and daycare patient at department \( u \) and weekday \( d \). For each department \( u \in D \) we define \( \Omega_u \) as the set of times where elective patients could arrive at department \( u \). For each department \( u \in D \) we define \( G^u_{\text{clin}} \) as the total mean number of occupied beds for clinical types of admissions of all \( m \) departments during the interval \( [s, s + E[B^u_{\text{clin},u}]] \), \( \forall s \in \Omega_u \) and \( G^u_{\text{day}} \) as the total mean number of occupied beds for daycare types of admissions of all \( m \) departments during the interval \( [s, s + E[B^u_{\text{day},u}]] \), \( \forall s \in \Omega_u \). Then define

\[
\begin{align*}
H^u_{\text{max,clin}} &= (t_1, ..., t_n) \text{ such that } G^u_{t_1,\text{clin}} \geq G^u_{t_2,\text{clin}} \geq ... \geq G^u_{t_n,\text{clin}}, \forall t_i \in \Omega_u, \\
H^u_{\text{max,day}} &= (t_1, ..., t_n) \text{ such that } G^u_{t_1,\text{day}} \geq G^u_{t_2,\text{day}} \geq ... \geq G^u_{t_n,\text{day}}, \forall t_i \in \Omega_u, \\
H^u_{\text{min,clin}} &= (t_1, ..., t_n) \text{ such that } G^u_{t_1,\text{clin}} \leq G^u_{t_2,\text{clin}} \leq ... \leq G^u_{t_n,\text{clin}}, \forall t_i \in \Omega_u, \\
H^u_{\text{min,day}} &= (t_1, ..., t_n) \text{ such that } G^u_{t_1,\text{day}} \leq G^u_{t_2,\text{day}} \leq ... \leq G^u_{t_n,\text{day}}, \forall t_i \in \Omega_u,
\end{align*}
\]

(4.25)

for each department \( u \in D \). Assume \( H^u_{\text{max,clin},i^u} \) as the \( i^u \)-th element of the vector \( H^u_{\text{max,clin}} \) at department \( u \) and \( H^u_{\text{max,day},i^u} \) as the \( i^u \)-th element of the vector \( H^u_{\text{max,day}} \) at department \( u \). Then

\[
\begin{align*}
t_{\text{max}} &= \text{arg max}_{u \in D} \{ G^u_{H^u_{\text{max,clin},i^u},\text{clin}}, G^u_{H^u_{\text{max,day},i^u},\text{day}} \} \\
t_{\text{min}} &= \text{min} H^u_{\text{min,clin}}
\end{align*}
\]

(4.26)

(4.27)
is the time-interval where the mean number of occupied beds is minimum at department $u$.
If $t_{\text{max}} = H_{\text{max,day},i}^u$, then
\[
    t_{\text{min}} = \min H_{\text{min,day}}^u
\]
is the time-interval where the mean number of occupied beds is minimum at department $u$.
At $t_{\text{max}}$ we remove one elective patient of the corresponding department $u$ and add this elective patient at $t_{\text{min}}$.

**Distribution of the number of occupied beds**

In Subsection 4.1.1.1 the total number of occupied beds $M(t)$ at time $t$ is approximated by a Normal distribution. Assume $X_1, ..., X_m$ to be independent Normal random variables where $X_i$ has parameters $\mu_i$ and $\sigma_i^2$, then
\[
    X_1 + ... + X_m \text{ has a Normal distribution with parameters } \mu_1 + ... + \mu_m \text{ and } \sigma_1^2 + ... + \sigma_m^2.
\]
Define
\[
    \rho_{\text{el,clin}}^u(t) \text{ as the elective and clinical mean number of occupied beds } \rho_{\text{el,clin}}(t) \text{ at time } t \text{ for department } u,
\]
\[
    \rho_{\text{el,day}}^u(t) \text{ as the elective and daycare mean number of occupied beds } \rho_{\text{el,day}}(t) \text{ at time } t \text{ for department } u,
\]
\[
    \rho_{\text{em,clin}}^u(t) \text{ as the emergency and clinical mean number of occupied beds } \rho_{\text{em,clin}}(t) \text{ at time } t \text{ for department } u,
\]
\[
    \rho_{\text{em,day}}^u(t) \text{ as the emergency and daycare mean number of occupied beds } \rho_{\text{em,day}}(t) \text{ at time } t \text{ for department } u,
\]
\[
    z_{\text{el,clin}}^u(t) \text{ as the peakedness for elective and clinical patients at time } t \text{ for department } u,
\]
\[
    z_{\text{el,day}}^u(t) \text{ as the peakedness for elective and daycare patients at time } t \text{ for department } u,
\]
\[
    z_{\text{em,clin}}^u(t) \text{ as the peakedness for emergency and clinical patients at time } t \text{ for department } u,
\]
\[
    z_{\text{em,day}}^u(t) \text{ as the peakedness for emergency and daycare patients at time } t \text{ for department } u.
\]
When merging $m$ departments, the number of occupied beds is Normally distributed with
\[
    \rho_{\text{total}}^u(t) = \sum_{u \in D} \left( \rho_{\text{el,clin}}^u(t) + \rho_{\text{em,clin}}^u(t) + \rho_{\text{el,day}}^u(t) + \rho_{\text{em,day}}^u(t) \right)
\]
as the mean number of occupied beds at time $t$ and
\[
    \upsilon_{\text{total}}^u(t) = \sum_{u \in D} \left( z_{\text{el,clin}}^u(t) \rho_{\text{el,clin}}^u(t) + z_{\text{el,day}}^u(t) \rho_{\text{el,day}}^u(t) + z_{\text{em,clin}}^u(t) \rho_{\text{em,clin}}^u(t) + z_{\text{em,day}}^u(t) \rho_{\text{em,day}}^u(t) \right)
\]
as the variance of the number of occupied beds at time $t$.

**4.4.4 Discussion of the results**

The essence of the algorithm is to raise the downs and reduce the peaks of the number of occupied beds. This makes it possible to reduce the variability of the number of occupied beds. This reduction makes the number of occupied beds and required beds more stable. This has the effect of less variability in the amount of work and this means less variability in the number of nurses. Reducing the index of dispersion of the number of occupied beds could reduce the number of required beds.

The algorithm which determines the planning of the elective patients would not take into account the seasonality. The algorithm in Section 4.4.1 reduces the peaks and increases the downs of the mean number of occupied beds. When the seasonality of the elective arrivals is significant, divide the total period (e.g. one year) into smaller intervals (e.g. months) to get a more realistic way of what variability of the number of occupied beds is possible. Then choose the target index of dispersion for each interval separately.
Another possibility of implementing seasonality could be to determine a maximum number of elective arrivals during each interval (e.g. month). This prevents that you let more patients arriving during a period than there are available.
Chapter 5

Conclusions and suggestions for further research

In this chapter we present the main conclusions of this thesis (Section 5.1), indicate how one may use the results of this thesis (Section 5.2) and give some suggestions for further research (Section 5.3).

5.1 Conclusions

In Chapter 2 we assume that the arrival rate function $\lambda(t)$ is known. In that case we can compute the mean number of occupied beds $\rho(t)$ and some performance measures as the blocking probability and bed occupancy rate. When there are infinitely many beds available we know that the distribution of the number of occupied beds $M(t)$ is Poisson with parameter $\rho(t)$. However, when there are finitely many beds $c$ available we obtain the following approximation for the number of occupied beds $M_c(t)$ at time $t$

$$P(M_c(t) = m) \approx \sum_{k=0}^{c} P(M(t) = k), \quad \text{for } m = 0, 1, ..., c,$$

(5.1)

the blocking probability $P_{\text{block}}(t)$ at time $t$ by

$$P_{\text{block}}(t) := P(M_c(t) = c) \approx \frac{P(M(t) = c)}{\sum_{k=0}^{c} P(M(t) = k)},$$

(5.2)

and the bed occupancy rate $\rho_c(t)$ at time $t$ by

$$\rho_c(t) \approx \rho(t) \frac{P(M(t) \leq c-1)}{P(M(t) \leq c)}.$$

(5.3)

Chapter 3 is devoted to the computation of the arrival rate function $\lambda(t)$ at time $t$. Using that result, we are be able to use the results of Chapter 2. In this chapter we use the result of [9]. The conclusion of this paper is that patients arrive according to a non-stationary Poisson process with a stochastic parameter. We distinguish three types of weeks, a calm, normal or busy week. This type of week is determined by the number of arriving patients. For each type of week we compute the arriving parameter and the probability that this week is of type calm, normal or busy. This leads to the arrival rate $\lambda(t)$ at time $t$.

In Chapter 4 we study the impact of changing the parameters on the number of occupied beds and on the number of required beds. Based on this study, we conclude the following.

First, we changed the mean and variance of the length-of-stay distribution. We conclude that the
mean and variance of the length-of-stay distribution are not reducing the variability of the number of occupied beds when the variability of the arrival process is large. However, when the variability of the arrival process is low, then reducing the variability of the length-of-stay distribution reduce the variability of the number of occupied beds.

Secondly, we add some extra elective patients $S$. For each month we determine how many extra elective patients arrive. Then we add these patients, in this month, where the mean number of occupied beds is lowest. So, the extra patients are added in the period where the workload is the lowest and we also use seasonality in the arrivals. This is a realistic and smart way of adding these patients, because the maximum number of required beds become not automatically larger.

Lastly, when the variability of the number of occupied beds reduces, we want to see how to plan the elective patients. Raise the downs and reduce the peaks of the number of occupied beds reduce the variability. Reducing the variability of the number of occupied beds could reduce the number of required beds.

5.2 How to use the results of this thesis

In this report we developed a mathematical model to compute the number of required beds for a hospital. One could use the model and results of this report in the following way.

Choose input for the following items:

- select the department(s),
- select the specialism(s),
- select the type of admission: clinical and/or daycare,
- select the patient flow: elective and/or emergency,
- select the starting date $t_1$ and ending date $t_2$,
- select the time-intervals that number of beds stay the same,
- select target blocking probability $\varepsilon$ or target bed occupancy rate $\delta$.

There could be more input parameters, but in these thesis we only looked to the above described items. When these are chosen we can compute the mean number of occupied beds $\rho(t)$, for the selection, as in Section 3.2. Using data of the history (e.g. previous year), we will forecast the corresponding number of required beds as computed in Section 2.4. Then one can obtain the following performance measures:

- the blocking probability for an arbitrary arriving patient in the interval $[t_1, t_2]$ by Equation (2.13),
- the mean bed occupancy rate in the interval $[t_1, t_2]$ is given by Equation (2.19),
- the mean number of occupied beds in the interval $[t_1, t_2]$ is given by $\frac{1}{t_2-t_1} \int_{t_1}^{t_2} \rho(u) du$,
- the index of dispersion of the number of occupied beds in the interval $[t_1, t_2]$ is given by Equation (3.27).

Another interesting option resulting from our research is that one may study the impact of changing the parameters on the number of occupied beds and on the number of required beds. In Section 4.2 we described how one could change the length-of-stay distribution and when reducing the variability in the length-of-stay distribution is beneficial for the variability in the number of occupied beds. We also presented, in Section 4.3, an algorithm which determines the time interval where one should add elective patients. This algorithm finds the time interval where the mean
number of occupied beds is lowest and adds one extra elective patient. One is able to add elective patients in a smart way. In Section 4.4 we presented an algorithm which computes how the schedule of the elective arrivals should be to reduce the variability in the number of occupied beds.

Using the results of this thesis one could build different scenarios and show the hospitals that changes in planning elective patients or reducing the length-of-stay lead to a better hospital environment. This thesis gives the tool to investigate where and how one could reduce the number of required beds, and thus save costs.

5.3 Suggestions for further research

In this section we present some suggestions for further research.

A result of Chapter 4 that reducing the variability in the number of occupied beds is reducing the variability in the arrival process. Only when this arrival process is already smooth, reducing the variability in the length-of-stay will reduce the variability in the number of occupied beds. So, the elective patient arrival process is important. The schedules of the operating rooms play a big role in the elective arrival process. Looking the other way around, when we know how our elective arrival process should behave to reach a target variability, it could be interesting to determine how our operating room schedule has to be. So, the connection between the number of beds and the operating room schedule are important.

In this thesis we used a stochastic intensity function for the non-stationary Poisson process. We could also think of more a refined mathematical model for the arrival process: $\lambda(t)$ is a random variable, e.g. Gamma. Britt Mathijsen and Johan van Leeuwaarden do research on representing the arrival rate parameter by a Gamma random variable. A data analysis will estimate the parameters for the random variable. It would be interesting to implement such an arrival process for the patients.

Systems with two-time-scale service times are studied in [2]. The length-of-stay of a patient is affected by different factors: the patient’s medical condition determines how many days the patient needs to spend in the hospital to recover, and when the patient is to be discharged on a day, the time-of-day when he or she leaves the hospital is driven by operational factors other than her medical conditions. The length-of-stay $B$ is defined by

$$B = \text{LOS} + h_{\text{dis}} - h_{\text{adm}},$$

where LOS denotes the number of days that the customer occupies a bed, and $h_{\text{adm}} \in (0, 1)$ and $h_{\text{dis}} \in (0, 1)$ represent the time-of-day when the patient is admitted and leaves the system, respectively. Using this length-of-stay random variable one could investigate the impact of shifting the discharging time on the number of occupied beds. What could be the impact on the number of occupied beds by not discharging them in the morning, but in the afternoon?

In this thesis we presented a model which computes the number of required beds. However, as described in this section there are further research opportunities.
Chapter 6

Appendix

6.1 Proof of Lemma 3.1

Proof. We will prove Equation (3.26). We have

\[ \mathbb{E}[X] = \mathbb{E} \left[ \frac{1}{n-1} \sum_{i=1}^{n} \left( X_i - \frac{1}{n} \sum_{j=1}^{n} X_j \right)^2 \right]. \] (6.1)

We can write

\[ \sum_{i=1}^{n} \left( X_i - \frac{1}{n} \sum_{j=1}^{n} X_j \right)^2 = \sum_{i=1}^{n} \left( X_i^2 - 2X_i \frac{1}{n} \sum_{j=1}^{n} X_j + \frac{1}{n^2} \left( \sum_{j=1}^{n} X_j \right)^2 \right). \] (6.2)

We can split the sum in Equation (6.2) in three terms, so we will split the proof in two parts.

Part I:

\[ \mathbb{E} \left[ \frac{1}{n-1} \sum_{i=1}^{n} X_i \right] = \frac{1}{n-1} \mathbb{E} \left[ \sum_{i=1}^{n} X_i \right] = \frac{1}{n-1} \sum_{i=1}^{n} \mathbb{E}[X_i] = \frac{1}{n-1} \sum_{i=1}^{n} (\lambda_i^2 + \lambda_i). \] (6.3)
Part II:

\[
\begin{align*}
\mathbb{E} \left[ -\frac{2}{n(n-1)} \sum_{i=1}^{n} X_i \sum_{j=1}^{n} X_j + \frac{1}{n(n-1)} \left( \sum_{j=1}^{n} X_j \right)^2 \right] \\
= - \mathbb{E} \left[ \frac{1}{n(n-1)} \left( \sum_{j=1}^{n} X_j \right)^2 \right] \\
= - \frac{1}{n(n-1)} \mathbb{E} \left[ \sum_{i=1}^{n} X_i^2 + 2 \sum_{i=1}^{n} \sum_{j \neq i} X_i X_j \right] \\
= - \frac{1}{n(n-1)} \sum_{i=1}^{n} (\lambda_i^2 + \lambda_i) - \frac{2}{n(n-1)} \mathbb{E} \left[ \sum_{i=1}^{n} \sum_{j \neq i} X_i X_j \right] \\
= - \frac{1}{n(n-1)} \sum_{i=1}^{n} (\lambda_i^2 + \lambda_i) - \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \mathbb{E}[X_i] \mathbb{E}[X_j] \\
= - \frac{1}{n(n-1)} \sum_{i=1}^{n} (\lambda_i^2 + \lambda_i) - \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \lambda_i \lambda_j.
\end{align*}
\]

So, in total we have

\[
\mathbb{E}[X] = \frac{1}{n-1} \sum_{i=1}^{n} (\lambda_i^2 + \lambda_i) - \frac{1}{n(n-1)} \sum_{i=1}^{n} (\lambda_i^2 + \lambda_i) - \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \lambda_i \lambda_j
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} (\lambda_i^2 + \lambda_i) - \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \lambda_i \lambda_j.
\]
Bibliography


