MASTER

Control of systems with actuation and non-collocated discontinuous friction

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Award date:
2006

Link to publication
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DCT 2006.113
Summary

Friction occurs in almost all mechanical motion systems. It is a complex phenomenon and its presence may result in undesired dynamic behaviour and in an inferior performance of a motion system. Therefore, friction plays an important role in the control designs for this class of systems.

In this work, the focus is on systems with actuation and non-collocated discontinuous friction, i.e. systems with unactuated masses subject to friction. We consider systems that can be written as linear systems in the forward path with set-valued nonlinearities in the feedback loop. The goal is to design a state-feedback controller and an output-feedback controller for the latter class of systems that stabilizes the equilibrium of the closed-loop system.

First, we introduce an experimental rotor dynamic set-up that can be considered as a benchmark for the examined class of systems. This system consists of two inertias coupled by a flexibility. The first inertia is actuated and the second inertia is subject to friction. Due to the non-collocated nature of the friction and actuation, standard techniques can not be employed for friction compensation. The control goal is to stabilize the constant velocity equilibria of the system (that exist for constant inputs at the first inertia). It is important to note that the system configuration and control problem considered here is encountered in many industrial systems, such as drilling systems for oil exploration, printers etc.

A parameter identification is performed for the model which describes the rotor dynamic system. The parameter estimation is validated by experiments. The presented model with the estimated parameters is a predictive model for the the steady-state behaviour of the experimental set-up.

Furthermore, a model-based observer is presented, which is used for the output-feedback controller. The observer error converges exponential to zero if the observer error dynamics satisfies an expression of the circle criterion. The observer is applied to the rotor dynamic system and its performance is shown by means of both simulations and experiments.

Next, a control design is presented with a linear state-feedback control law. If the closed-loop system satisfies the circle criterion, then the equilibrium of the closed-loop system is globally asymptotically stable. However, for the examined rotor dynamic system, we show that the control design based on the circle criterion is not feasible. Therefore, we propose a second control design based on the Popov criterion that obtains global asymptotical stability of the equilibrium of the closed-loop system.

The proposed observer and a state-feedback controller are used to construct an output-feedback controller. For this purpose, the controller which is based on the circle criterion or the Popov criterion can be used. We show that these controllers in combination with the observer, introduced earlier, constitute output-feedback designs that render the equilibrium globally asymptotically stable.

Finally, the output-feedback controller based on the Popov criterion is implemented on the experimental rotor dynamic set-up. In experiments it is shown that the output-feedback controller is able to control the rotor dynamic set-up for large range of constant inputs to the desired setpoint and the region with stable equilibria is substantially extended (compared to the open-loop system).
Samenvatting

Wrijving komt voor in bijna alle mechanische bewegingssystemen. Het is een complex verschijnsel en de aanwezigheid van wrijving kan resulteren in ongewenst dynamisch gedrag en in een inferieure prestatie van het bewegingssysteem. Daarom speelt wrijving een belangrijke rol in het ontwerpen van een regelaar voor bewegingssystemen.

De focus in deze thesis is gericht op systemen die bestaan uit aangedreven massa’s en niet aangedreven massa’s onderhevig aan wrijving. We beschouwen systemen die geschreven kunnen worden als een lineair systeem met set-valued niet-lineairiteiten in de terugkoppelingsslus. Het doel is om een toestandsterugkoppeling-regelaar en om een uitgangsterugkoppeling-regelaar te ontwerpen voor de laatstgenoemde klasse van systemen om het evenwichtspunt van het closed-loop systeem te stabiliseren.

We introduceren eerst een experimentele rotor dynamische opstelling, die behoort tot de te beschouwen klasse van systemen. Het systeem bestaat uit twee massatraagheden die zijn gekoppeld door een flexibiliteit. De eerste massatraagheid is aangedreven en de tweede massatraagheid is onderhevig aan wrijving. Doordat wrijving uitgeoefend wordt op een massatraagheid die niet aangedreven is, kunnen we geen directe wrijvingscompensatiemethode toepassen. Het regeldoel is om de evenwichtspunten met constante snelheid te stabiliseren (deze evenwichtspunten bestaan voor een constant ingangskoppel bij de eerste massatraagheid). Het is belangrijk om te noemen dat de genoemde systeemconfiguratie en het bijbehorende regelprobleem vaker voorkomt in industriële systemen, zoals boorsystemen voor oliewinning, printers etc.

Een parameter identificatie is uitgevoerd voor het model dat het rotor dynamisch systeem beschrijft. De parameterschatting is gevalideerd met behulp van experimenten. Het gepresenteerde model met de geschatte parameters is een voorspellend model voor het steady-state gedrag van de experimentele opstelling.

Een waarnemer gebaseerd op het systeem model is gepresenteerd, die gebruikt zal worden voor een uitgangsterugkoppeling-regelaar. De waarnemerfout convergeert exponentieel naar nul als de waarnemerfout-dynamica voldoet aan het circle-criterium. De waarnemer is toegepast op het rotor dynamisch systeem en de prestatie van de waarnemer blijkt uit de simulaties en de experimenten.

Vervolgens wordt een regelaarontwerp gepresenteerd met een lineaire toestandsterugkoppeling-regelwet. Als het closed-loop systeem voldoet aan het circle-criterium, dan is het evenwichtspunt van het closed-loop systeem globaal asymptotisch stabil. We laten echter zien dat het regelaarontwerp gebaseerd op het circle-criterium voor het rotor dynamisch systeem niet toepasbaar is. Daarom presenteren we een tweede regelaarontwerp gebaseerd op het Popov-criterium dat globaal asymptotische stabiliteit van het evenwichtspunt van het closed-loop systeem garandeert.

De waarnemer en toestandsterugkoppeling-regelaar zijn gebruikt om een uitgangsterugkoppeling-regelaar te construeren. Voor dit doel kunnen we zowel de toestandsterugkoppeling-regelaar gebaseerd op het circle-criterium als de toestandsterugkoppeling-regelaar gebaseerd op het Popov-criterium gebruiken. We tonen dat deze regelaars in combinatie met de eerder geïntroduceerde waarnemer, beide een uitgangsterugkoppeling-regelaar vormen die het evenwichtspunt globaal asymptotisch stabel maakt.

Tenslotte is de uitgangsterugkoppeling-regelaar gebaseerd op het Popov-criterium toegepast op de experimentele rotor dynamische opstelling. De experimenten laten zien dat de uitgangsterugkoppeling-regelaar in staat is om de rotor dynamische opstelling te regelen voor een grote reeks van constante ingangsensitiviteiten en het gebied met stabiele evenwichtspunten substantieel uitgebreid ( vergeleken met het open-loop systeem).
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Chapter 1

Introduction

Friction occurs in almost all mechanical motion systems, for example in systems for domestic use, systems for transport, industrial systems etc. Friction is a complex phenomenon and its presence may result in undesired behaviour and in an inferior performance of the motion system. Therefore, friction plays an important part in the control designs for these motion systems. An important aim of the control engineer is to design a controller, which stabilizes the motion system and increases the performance of the motion system, despite the presence of friction.

1.1 Control of systems with friction

Friction appears at the physical interface between two surfaces in contact. There are different friction models to describe friction phenomena. An overview of friction models is given in e.g. [5] and [24]. Examples of static friction characteristics are given in Figure 1.1. One can think of Coulomb friction, see Figure 1.1(a), or viscous friction, see Figure 1.1(b). Also, it is possible to model the friction as a nonlinear function of the velocity. If the friction force is a function of the velocity and it is decreasing for the low velocities, then it is said that the Stribeck effect is present, see Figure 1.1(c).

The friction models will be incorporated in the model of the whole system. The model which describes the dynamic behaviour of the whole system, should be accurate in order to be useful for controller design purposes. However, model uncertainties will be present in the model in practice. Friction may have a large contribution to the model uncertainties, since it is a sensitive phenomenon. Namely, the friction phenomena can constantly or more gradually change due to sensitivity to changing conditions such as temperature, humidity, contaminations etc. A consequence is that the parameters of the modeled friction need to be estimated again and again. If the change is large, then even the friction model has to be changed. If a controller is designed to deal with only a certain friction situation, then the performance of the system may be bad when the friction situation changes. Therefore, it is desired that the controlled system has some robustness with respect to changes of the friction characteristic.

In this thesis, we design a controller, that deals with a motion system containing friction modeled by a set-valued (discontinuous) friction law. Note that the latter modeling choice is made to properly account for the sticking effect of friction.

A common approach to deal with friction is to compensate it by the control input of the motion system. Consequently, we can design more easily a controller for the motion system. Consider for example, the one-mass system in Figure 1.2. Suppose the mass is subject to a certain (highly nonlinear) friction
force \( F_f \). By choosing the control \( F_u = F_f + v \), we compensate the friction force \( F_f \) and we are able to apply a relative simple controller for the new input \( v \). Different types of compensation techniques, also in combination with adaptation mechanisms (for adapting of the parameters of the friction model), are discussed in the literature in e.g. [5] and [24]. It should be noted that non-exact compensation of the friction can lead to undesired phenomena. Friction compensation in a controlled one-link robot is discussed in [19] for exact friction compensation and non-exact friction compensation. A model-based compensation technique is applied to the one-link robot, where an observer is included to estimate the velocity, which is needed in compensating \( F_f \). It is shown that undercompensation leads to the existence of an equilibrium set and overcompensation leads to limit cycling.

However, there is a class of systems for which it is not possible to use a direct friction compensation technique. This class of systems contains systems with actuation and non-collocated friction, i.e. systems with unactuated masses subject to friction. The presence of an unactuated mass subject to friction implies that the friction acting on the unactuated mass can not be directly compensated. Consider, for example, the two-mass system in Figure 1.3, with one actuated mass subject to friction which is flexibly connected to another mass which is not actuated but which is also subject to friction. For this type of systems, we are not able to use a direct compensation technique for the friction acting on the unactuated mass, and so, another control approach is required. An approach with friction compensation for a two-body system with, possibly discontinuous, load friction and joint flexibility and damping is discussed in [28]. A model reference adaptive control scheme is presented using an adaptive friction compensation controller structure. The adaptation mechanism of the parameters operates if the velocity of the load is not zero. Boundedness of the tracking error is achieved. However, this method does not guarantee a zero tracking error.

An available technique for nonlinear systems is feedback linearization, see [16]. Input-output lineariza-
1.1. Control of systems with friction

Figure 1.3: Two-mass system with non-collocated friction and actuation.

tion renders the input-output mapping linear. And full-state linearization renders the entire state equations linear after a suitable feedback coordinate transformation. Linear control theory for designing a controller can be applied after obtaining a (partly) linear system. However, both linearization methods require smoothness of the system. Another method for the stabilization of systems with nonlinearities is backstepping [16]. For this method we need the derivatives of the nonlinear elements of the systems. Therefore, we can also not apply this method to systems with discontinuous elements.

An adapted backstepping method, which is called the multi-state backstepping approach, is presented in [17]. This approach is for systems with nondifferentiable, bounded, uncertain nonlinearities and it yields the design of a variable structure control in each step of the procedure. The variable structure control consists of a sliding control and reaching control. But the multi-state backstepping approach is only applicable to a restricted class of systems: single-input-single-output systems, where the friction forces are relatively small.

In this thesis, we will present a control design for motion systems with actuation and non-collocated discontinuous friction. The aim of the control design is to control these type of systems to a chosen setpoint.

An example of this type of systems is a rotary drilling system, see Figure 1.4, which is used for the exploration of oil and gas. The motor at the top drives a rotary table, which is a storage unit for kinetic energy. The rotary table is connected to the drill-string which consists of drill pipes, with a total length up to several kilometres. The drill-string is connected to the bottom-hole-assembly (including the drill-bit which creates the borehole) and is a low-stiffness connection between the rotary table and the bottom-hole-assembly. The drill-bit is subject to the contact forces (including friction) due to the contact between the drill-bit and the ground and rocks, which can lead to so-called stick-slip vibrations. The control goal is to rotate the drill-bit with a constant velocity for a high efficiency of the rotary drilling system and to prevent failure of the components of the system.

Another example is a printing system, which consists of a printhead, a belt, two wheels and a printhead guidance, see Figure 1.5. A motor is connected to one of the wheels. The belt connects the driven wheel with the undriven wheel. The printhead is attached to the belt and its motion is guided by the printhead guidance. The belt is a flexible connection between the driven wheel and the printhead. The goal is to ensure fast and accurate motion of the printhead for good printing results, while friction is present in the bearings of the wheels and in the contact between the printhead and the printhead guidance.

The last example is a driveline of a ship, see Figure 1.6. The screw of the ship is driven by an engine via a long shaft which is a low stiffness connection between the engine and the screw. The screw is subject to the friction in the bearings of the shaft and to the contact forces between the contact of the screw and the water. The friction can lead to undesired vibrations in the driveline. The control problem is to rotate the screw with a constant velocity for a constant propulsion of the ship and prevention of failure of drive parts.
1.2 Approach and objectives

The first step for control is to develop a dynamical model for the system to be controlled. This dynamical model can generally be written in state-space form. An important step for an accurate model is parameter identification. The next step is to design a state-feedback controller, which controls the system and is robust with respect to changes in the friction to a certain extent. To achieve such a controller design, the circle criterion and the Popov criterion are used, see e.g. [4] and [16].

If the entire state is measured, then we can use the full state in a state-feedback controller. However, in general not all state components can be measured, because it is not possible or it is too expensive. For such a situation, we need an estimation of the unmeasured state components. Therefore, we use an observer, that reconstructs the state of the system. An observer design suitable to deal with systems which contain discontinuous elements is used in this work, see [14].

The state-feedback controller and the observer are used to construct an output-feedback controller, where the controller uses state estimations from the observer. However, a separation principle for general nonlinear systems is absent. The combination of a state-feedback controller and an observer can lead to finite escape time. Therefore, we will provide a proof for the stability of the controller/observer combination for a system of the considered class.
An experimental set-up, which typically exhibits the structure as depicted in Figure 1.2, is available at the DCT-laboratory of the Technische Universiteit Eindhoven. It is a rotor dynamic system and consists of a driven upper disc which is connected via a string to a lower disc. Both discs are subject to friction. The friction of the lower disc can be described by a friction model that contains the Stribeck effect. The dynamic behaviour of the set-up shows interesting nonlinear phenomena for changing, though constant, input motor-voltages. For instance limit cycles, several bifurcations and coexistence of steady-state solutions. Extensive research of the rotor dynamic system is performed and discussed in [12, 21, 22, 23, 29].

Parameter identification is important for the rotor dynamic set-up, since the friction acting on the set-up is very sensitive to certain conditions. We will apply a different parameter estimation procedure than is discussed in [12, 21].

The output-feedback controller in this thesis will be applied to the experimental rotor dynamic set-up.

We focus on motion systems with actuation and non-collocated discontinuous friction. The objectives for this class of systems are the following:

- To present an observer, which reconstructs the states of the system.
- Design a state-feedback controller which is able to control the motion system to a properly chosen setpoint.
- Provide a stability proof of the combination of the observer and the state-feedback controller design.

The objectives for the experimental rotor dynamic set-up are:

- Perform a parameter identification for the rotor dynamic set-up.
- Design of an output-feedback controller for the rotor dynamic system and implementation of this output-feedback controller on the experimental set-up.

1.3 Outline of the thesis

Now, we continue with the outline of this thesis. In Chapter 2, we discuss elementary theory which supports the theory of the other chapters. These preliminaries involve discontinuous systems, passivity theory and the notion of absolute stability. The rotor dynamic system is presented in Chapter 3. First, the system is described, then the model of the rotor dynamic system is given and the performed parameter estimation is explained. We analyse the dynamic behaviour and we verify the analysis with experimental results. In Chapter 4, an observer design is introduced for Lur’e-type systems with discontinuities in the feedback loop. The observer design is applied to the rotor dynamic system and simulations and experiments are performed. We present two state-feedback controller designs in Chapter 5: one of the controller designs is based on the circle criterion and the another is based on the Popov criterion. The controller designs based on the Popov criterion is applied to the rotor dynamic system and simulations
are performed. The output-feedback control design is discussed in Chapter 6. Simulations and experiments for the rotor dynamic system illustrate the performance of the proposed control design. Finally, we finish this thesis with Chapter 7, where we give conclusions and recommendations.
Chapter 2

Preliminaries

Preliminary theory, needed for the discussion of the observer and the controller designs in this thesis, is presented in this chapter. In the first section, systems with a discontinuous right-hand side are discussed. An extensive discussion of this class of systems is given in [18]. It is possible to transform systems with a discontinuous right-hand side to systems modeled by differential inclusions, according to the approach of A.F. Filippov, [11].

Next, the passivity property is discussed for both functions and dynamical systems. Also, positive realness conditions for the transfer function of linear time-invariant systems are presented. These notions are used for the observer and the controller designs. We use the definitions of passivity from [16]. The notion of passivity is important in the scope of absolute stability theory which is discussed in the last section. Two criteria are presented to obtain absolute stability: the circle criterion and the Popov criterion. Here, we use the theory related to absolute stability presented in [16].

2.1 Discontinuous systems

Filippov’s solution concept is introduced in this section. We consider a system with a discontinuous right-hand side described by differential equations:

\[
\dot{x} = \begin{cases} 
\varphi_- (x), & x \leq x_e, \\
\varphi_+ (x), & x > x_e.
\end{cases}
\]

(2.1)

where the discontinuity occurs at \(x_e\) and \(x \in \mathbb{R}^n\).

The theory of A.F. Filippov extends a system described by differential equations with discontinuous right-hand side to a system described by differential inclusions [18]. This theory yields a convexification of the discontinuous right-hand side, which renders the right-hand side set-valued.

An set-valued extension is applied to the right-hand side of system (2.1), such that the system is described by differential inclusion:

\[
\dot{x} \in \begin{cases} 
\varphi_- (x), & x < x_e, \\
[\varphi_- (x_e), \varphi_+ (x_e)], & x = x_e, \\
\varphi_+ (x), & x > x_e.
\end{cases}
\]

(2.2)

Filippov’s solution concept guarantees the existence of a solution for a system with a discontinuous right-hand side (2.1) for almost all \(t \in \mathbb{R}\), if the convexified system (2.2) exhibits a right-hand side that
is upper semi-continuous, closed, convex and bounded for all \( x \in \mathbb{R}^n \). For more discussion about this result, see [11] and [18].

Figure 2.1(a) show an example of a discontinuous mapping. After convexification (according to (2.2)), the discontinuous mapping becomes a set-valued mapping, see Figure 2.1(b).

A model of a mechanical system with dry friction modeled by a discontinuous friction law is a typical example of a differential equation with a discontinuous right-hand side. Application of Filippov’s convexification concept has often a physical meaning for these mechanical systems due to the fact that for zero velocity it is sensible to allow the friction force to take values from a set in order to appropriately model the sticking effect.

Besides the use of the Filippov’s solution concept, it is also possible to describe the model of a system directly by differential inclusions. In this thesis, we consider systems that are described by differential inclusions (i.e. with a set-valued right-hand side).

### 2.2 Passivity

Passivity is a property of a system that can be expressed in terms of energy for a mechanical system. A mechanical system is called passive if the dissipated amount of energy over any period of time is greater or equal to the increase of the amount of the stored energy in the system.

A memoryless function \( w = \varphi(t, z) : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^p \) is passive if the following requirement holds, [16]:

\[
w^T z \geq 0, \quad \forall (t, z).
\]  

(2.3)

Let us introduce the sector condition. A memoryless function \( w = \varphi(t, z) : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^p \) belongs to the sector \([K_\alpha, K_\beta]\) with \( K_{\alpha, \beta} = K_\alpha - K_\beta = K_{\alpha, \beta}^T > 0 \) if

\[
[\varphi(t, z) - K_\alpha z]^T [\varphi(t, z) - K_\beta z] \leq 0, \quad \forall (t, z),
\]

(2.4)

with the matrices \( K_\alpha \) and \( K_\beta \) defined as:

\[
K_\alpha = \text{diag}(k_{\alpha,1}, \ldots, k_{\alpha,p}), \quad K_\beta = \text{diag}(k_{\beta,1}, \ldots, k_{\beta,p}).
\]

(2.5)

Consider the scalar functions in Figure 2.2. Then, the image of the scalar function \( w = \varphi(t, z) \) belongs to a sector whose boundaries are described by the lines \( w = k_\alpha z \) and \( w = k_\beta z \), where \( k_\beta > k_\alpha \). We say
that the scalar function belongs to the sector \([k_\alpha, k_\beta]\) or \(\varphi(t, z) \in [k_\alpha, k_\beta]\).

If the function \(w = \varphi(t, z)\) is passive, then it belongs to the sector \([0, \infty]\). Figure 2.2(a) shows an image of a scalar function which is not passive, while 2.2(b) shows an image of a scalar function which is passive.

Now, we will define passivity for dynamical systems [16]. Consider the dynamical system

$$\begin{align*}
\dot{x} &= f(x, w) \\
z &= h(x, w),
\end{align*}$$

(2.6)

with \(x \in \mathbb{R}^n\) the state of the system and \(w \in \mathbb{R}^p\) the input of the system, \(z \in \mathbb{R}^p\) the output of the system. The system has the same number of inputs and outputs. Assume that an equilibrium point of the system (2.6) for \(z = 0\) is \(x = 0\) (i.e. \(f(0, 0) = 0\) and \(h(0, 0) = 0\)).

Let the function \(V(x)\) be the storage function of system (2.6), which is a continuously differentiable positive semi-definite function. Then, the system (2.6) is said to be passive if

$$w^T z \geq \dot{V} = \frac{\partial V}{\partial x} f(x, w), \quad \forall (x, w) \in \mathbb{R}^n \times \mathbb{R}^p.$$  

(2.7)

Moreover, the system (2.6) is said to be strictly passive if

$$w^T z > \dot{V}, \quad \forall (x, w) \neq (0, 0).$$  

(2.8)

When a linear time-invariant system is (strictly) passive, then it has a (strictly) positive real transfer function. We will only present the definition of strictly positive realness for transfer functions of linear time-invariant systems, where the conditions are stated in the frequency domain. Consider the following linear system:

$$\begin{align*}
\dot{x} &= Ax + Gw \\
z &= Hx + Dw,
\end{align*}$$

(2.9)

where the equilibrium point of the system (2.9) for \(w = 0\) is \(x = 0\).

We define \(G(s) = H(sI - A)^{-1}G + D\) as the proper rational transfer function matrix from input \(w\) to output \(z\) of system (2.9). Then, \(G(s)\) is strictly positive real, see [16], if and only if

- \(G(s)\) is Hurwitz; that is, poles of all elements of \(G(s)\) have negative real parts,
\[ w(t) = \varphi(t, z(t)), \]

where \( x \in \mathbb{R}^n, w, z \in \mathbb{R}^p, (A, B) \) is controllable, \((A, C)\) is observable. The nonlinearity \( \varphi \) is required to satisfy a sector condition. For all nonlinearities satisfying the sector condition, the origin \( x = 0 \) is an equilibrium point of the system (2.10). Then, the system (2.10) is absolutely stable if the origin is globally uniformly asymptotically stable for any (set-valued) nonlinearity \( \varphi \) in the given sector [16].

A method to prove absolute stability for system (2.10) is the circle criterion. The circle criterion guarantees absolute stability if the transfer function of the linear part of system (2.10) is strictly positive real and the nonlinearity in the feedback loop is passive. This is a particular case for which the circle criterion can be used in order to prove absolute stability. The circle criterion is also used for the observer design in Chapter 4 and the control design in Chapter 5. More discussion about the circle criterion will be given in these chapters. The Popov criterion approach relaxes the condition of strict positive realness for the transfer function of the linear part of system (2.10) by a loop transformation with a dynamic multiplier. The Popov criterion is used for a second control design in Chapter 5, where more discussion is given about the Popov criterion.
Chapter 3

Rotor dynamic system

In this chapter, an experimental set-up of a rotor dynamic system is introduced. The rotor dynamic system consists of an upper disc driven by an electric motor, a steel string, a lower disc and a brake mechanism. The upper disc is connected to the lower disc by a steel string, see Figure 3.1, which is a low-stiffness connection between the discs. A brake disc is connected to the lower disc. A brake device is attached to the brake disc to exert a normal force to it. Oil is supplied to the brake disc to create an oil layer between the brake blocks and the brake disc. This combination of the brake device with the oil creates a friction characteristic with a Stribeck effect or with a so-called negative damping characteristic. It should be noted that extensive research has been performed with respect to the rotor dynamic set-up. The modeling, parameter estimation and analysis of the set-up are extensively discussed in [12, 21, 22, 23, 29]. The parts of the experimental rotor dynamic set-up is described in more detail Section 3.1. A model of the experimental rotor dynamic set-up is presented in Section 3.2. A set-valued force law is used to model the friction, because the friction present in the set-up shows a distinct sticking effect that is very important for the overall dynamic behaviour. Section 3.3 describes the identification of the parameters of the model. The friction (mainly the friction acting on the lower disc) acting on the rotor dynamic set-up is sensitive to conditions such as temperature, contaminations, amount of oil etc. Therefore, we have to estimate the parameters of the friction model for each experimental session. The dynamic analysis of the model of the experimental rotor dynamic set-up is discussed in Section 3.4 where a summary of the analysis of the steady-state behaviour to the rotor dynamic system, is given as is presented in [21]. Finally, the experimental results are discussed in Section 3.5.

3.1 Experimental set-up

The experimental rotor dynamic set-up is located in the DCT-lab (Dynamics and Control Technology laboratory) at the Technische Universiteit Eindhoven. A photo of the rotor dynamic set-up is shown in Figure 3.1. The experimental set-up consists of the following main parts:

- electronic equipment: computer, power amplifier;
- DC-motor;
- two rotational discs: an upper and a lower disc;
- a low stiffness string between the discs;
- additional brake device applied to the lower disc;
- measurement devices.
The outer dimensions of the experimental set-up are \((\text{length} \times \text{width} \times \text{depth})\): 
\(1.5 \times 1.0 \times 3.0\) m.

**Drive part with the upper disc/Upper part of the set-up**
The upper steel disc is suspended on a very stiff frame of steel beams, see Figure 3.2, and is driven by a DC-motor via a reduction (a gear box with a reduction ratio of \(3969/289\)). The input voltage for the motor is generated in a computer in a Matlab/Simulink environment and is applied to the motor via a DAC (Digital to Analogue Converter) and a power amplifier. The input voltage is limited to the range \([-5V,5V]\). The upper disc has only rotational freedom. A low stiffness steel string connects the upper disc to the lower brass disc. An encoder is attached to the top of the motor, which is connected to the shaft of the motor.

**Lower part of the set-up**
The lower part of the experimental rotor dynamic set-up is shown in Figure 3.3. A brake disc is connected to the lower disc via a stiff shaft. A bearing house is connected to the upper part of the shaft. Also a shaft is connected to the lower side of the lower disc. An encoder is attached to this shaft. Again a bearing housing is attached to lower part of the shaft. The lower part can rotate around its geometric center and it is also free to move in lateral directions. However, in this work we fix the lateral constraints.
of the lower disc of the set-up and we only consider rotational motions of both discs.

A brake device is fixed to the upper bearing housing of the lower disc, see Figure 3.4, and it contains two brake bronze blocks. We can apply with these brake blocks an adjustable normal force to the brake disc. The brake device is calibrated to obtain the relation between the exerted normal force of the brake device and the setting of the brake device, see Appendix B. To create friction that includes the Striebeck effect, or the so-called negative damping effect, oil is added to the brake disc. The used type of oil is ondina oil. For more information about the Striebeck effect, see e.g. [5] and [24]. The brake blocks have a wedge shape at the edges, this shape ensures that oil is supplied between the brake blocks and the brake disc when the lower disc rotates. An oil box with felt stripes is attached to the upper bearing housing. These felt stripes distribute the oil to the upper and lower side of the brake disc due to the capillary effect of the felt. It also makes it possible to add oil in a reproducible way.
Measurement devices
The angular positions of the upper and lower disc are measured using incremental encoders. An encoder is placed at the outer end of the motor, see Figure 3.2. The other encoder is placed at the end of the shaft, which is connected to the lower side of the lower disc, see Figure 3.3. The encoders measure the counts per revolution. The number of counts per revolution of the encoder of the motor for the upper disc is increased by the reduction ratio of the gearbox, since the encoder is connected to the motor shaft before the gearbox and we want to measure the counts per revolution for the upper disc. A quadratic decoder, an electronic circuit placed in the computer, increases the number of counts per revolution for both encoders. The final obtained counts per revolution for the upper disc is 54934.26, and for the lower disc 40000.
A force sensor is fixed at one side of the brake in order to measure the applied friction force of the brake device exerted to the brake disc.
For more technical details of the rotor dynamic set-up, see Appendix A.

3.2 Model of the rotor dynamic set-up

In this section, we present the model of the experimental rotor dynamic system.
Several assumptions are made for the modeling of the experimental rotor dynamic set-up:
- Damping in the string in torsional direction is negligible with respect to the damping in the bearings at the upper and lower disc;
- The lower disc does not move in lateral direction, because the constraints are fixed;
- The steel string is massless.

A schematic model of the experimental rotor dynamic set-up is drawn in Figure 3.5.
The power amplifier, the DC-motor and the gear box are considered as a single component for the modeling of the rotor dynamic set-up. We define $u$ as the input voltage to the power amplifier. The relation between the torque of the DC-motor and the input voltage $u$ is modeled by a linear relation where the torque of the DC-motor acting on the upper disc equals the multiplication of the input voltage $u$ by a motor constant $k_m$. 
The system has two degrees of freedom. The upper disc and lower disc both have rotational freedom. The equations of motion for the upper disc and the lower disc are given by

\[
\begin{align*}
J_u \ddot{\theta}_u + k \theta_u (\theta_u - \theta_l) + T_{fu}(\dot{\theta}_u) - k_m u &= 0 \\
J_l \ddot{\theta}_l - k \theta_u (\theta_u - \theta_l) + T_{fl}(\dot{\theta}_l) &= 0,
\end{align*}
\]  

(3.1)

with \( \theta_u \) the angular position of the upper disc and \( \theta_l \) the angular position of the lower disc. An overview of the used model variables and parameters is given in Table 3.1. With the equations of motion in (3.1), the state-space equations can formulated for the model of the experimental rotor dynamic set-up. The following definition of the state vector is adopted:

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \omega_u \\ \omega_l \end{bmatrix} = \begin{bmatrix} \theta_u - \theta_l \\ \dot{\theta}_u \\ \dot{\theta}_l \end{bmatrix}.
\]

(3.2)

Note that the model is reduced, so that only the difference \( \theta_u - \theta_l \) is included in the state vector. The state-space equations of the rotor dynamic system are then given by

\[
\begin{align*}
\dot{x}_1 &= x_2 - x_3 \\
\dot{x}_2 &= \frac{1}{J_u} \left[ -k g x_1 - T_{fu}(x_2) + k_m u \right] \\
\dot{x}_3 &= \frac{1}{J_l} \left[ k g x_1 - T_{fl}(x_3) \right].
\end{align*}
\]  

(3.3)

The set-valued functions \( T_{fu}(x_2) \) and \( T_{fl}(x_3) \) are the friction models of the friction at the upper and lower disc, respectively. These friction models are discontinuous in \( x_2 \) and \( x_3 \), respectively. Set-valued force laws are used to model the friction acting on the upper and lower disc to account for the sticking effect in both characteristics.
Table 3.1: Description of the variables and parameters of the model.

<table>
<thead>
<tr>
<th>symbol</th>
<th>meaning</th>
<th>unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta_l)</td>
<td>angular displacement of the lower disc</td>
<td>rad</td>
</tr>
<tr>
<td>(\theta_u)</td>
<td>angular displacement of the upper disc</td>
<td>rad</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>(\theta_u - \theta_l)</td>
<td>rad</td>
</tr>
<tr>
<td>(\omega_l = \dot{\theta}_l)</td>
<td>angular velocity of the lower disc</td>
<td>rad/s</td>
</tr>
<tr>
<td>(\omega_u = \dot{\theta}_u)</td>
<td>angular velocity of the upper disc</td>
<td>rad/s</td>
</tr>
<tr>
<td>(J_l)</td>
<td>moment of inertia of the lower disc</td>
<td>kg(\cdot)m(^2)/rad</td>
</tr>
<tr>
<td>(J_u)</td>
<td>moment of inertia of the upper disc</td>
<td>kg(\cdot)m(^2)/rad</td>
</tr>
<tr>
<td>(k_\theta)</td>
<td>torsional stiffness of the string</td>
<td>Nm/(\cdot)rad</td>
</tr>
<tr>
<td>(k_m)</td>
<td>motor constant</td>
<td>Nm/V</td>
</tr>
<tr>
<td>(u)</td>
<td>input voltage to the power amplifier</td>
<td>V</td>
</tr>
<tr>
<td>(T_{fu})</td>
<td>friction torque acting on the lower disc</td>
<td>Nm</td>
</tr>
<tr>
<td>(T_{fu})</td>
<td>friction torque acting on the upper disc</td>
<td>Nm</td>
</tr>
</tbody>
</table>

The friction \(T_{fu}\) acting on the upper disc, see Figure 3.6(a), is described by the algebraic inclusion

\[
T_{fu}(x_2) \in \begin{cases} 
T_{cu}(x_2) \text{sgn}(x_2) & \text{for } x_2 \neq 0 \\
[-T_{su} + \Delta T_{su}, T_{su} + \Delta T_{su}] & \text{for } x_2 = 0,
\end{cases}
\]

(3.4)

where the function \(T_{cu}(x_2)\) is given by

\[
T_{cu}(x_2) = T_{su} + \Delta T_{su} \text{sgn}(x_2) + b_u|x_2| + \Delta b_u x_2.
\]

(3.5)

The friction \(T_{fl}\) acting on the lower disc, see Figure 3.6(b), is described by the algebraic inclusion

\[
T_{fl}(x_3) \in \begin{cases} 
T_{cl}(x_3) \text{sgn}(x_3) & \text{for } x_3 \neq 0 \\
[-T_{sl}, T_{sl}] & \text{for } x_3 = 0,
\end{cases}
\]

(3.6)

where the continuous function \(T_{cl}(x_3)\) is given by

\[
T_{cl}(x_3) = T_{cl} + (T_{sl} - T_{cl})e^{-\frac{|x_3|}{\delta_{sl}}} + b_l |x_3|.
\]

(3.7)

Let us first discuss the equilibria of the system (3.3). See [21] for an extensive discussion of the equilibria. From the differential equation for \(x_1\) we can see that the equilibrium values for \(x_2\) and \(x_3\) are equal for a constant input \(u = u_c\). We denote the equilibria of the model of the rotor dynamic system (3.3) for constant input voltage \(u_c\) by \(x_{eq} := [x_{1eq} \ x_{2eq} \ x_{3eq}]^T = [\alpha_{eq} \ \omega_{eq} \ \omega_{eq}]^T\).

The equilibria of system (3.3) satisfy the following equilibrium equations:

\[
\begin{align*}
\omega_{eq} - \omega_{eq} &= 0 \\
-k_\theta \alpha_{eq} + k_m u_c - T_{fu}(\omega_{eq}) &= 0 \\
-T_{fl}(\omega_{eq}) + k_\theta \alpha_{eq} &= 0.
\end{align*}
\]

(3.8)

From the equilibrium equations (3.8) we can obtain the following expression:

\[
k_m u_c = T_{fl}(\omega_{eq}) + T_{fu}(\omega_{eq}).
\]

(3.9)
We divide the equilibria in two situations:

1. equilibria for which \( \omega_{eq} \neq 0 \),
2. equilibria for which \( \omega_{eq} = 0 \).

If \( \omega_{eq} > 0 \), then the input voltage must be higher than a minimum input voltage to overcome the static friction in positive direction. The velocity \( \omega_{eq} \) will be higher than zero if

\[
 u_c > u_{\varepsilon p} := \frac{T_{su} + \Delta T_{su} + T_{sl}}{k_m}. \tag{3.10}
\]

If the desired equilibrium velocity is smaller than zero, then the corresponding input voltage has to be lower than a certain value to overcome the static friction in negative direction. The equilibrium velocity \( \omega_{eq} \) is smaller than zero if

\[
 u_c < u_{\varepsilon n} := \frac{-T_{su} + \Delta T_{su} - T_{sl}}{k_m}. \tag{3.11}
\]

In general, equation (3.9) can have more than one solution for \( \omega_{eq} \), see Figure 3.7(a). However, we assume that the sum of the two functions, \( T_{fl}(\omega_{eq}) + T_{fu}(\omega_{eq}) \), results in a nondecreasing function, see Figure 3.7(b). This means that the rotor dynamic system has only one unique isolated equilibrium point for a given \( u_c > u_{\varepsilon p} \).

We now consider the equilibria for \( \omega_{eq} = 0 \). Such equilibria only exist when the constant input voltage satisfies the condition

\[
 u_{\varepsilon n} \leq u_c \leq u_{\varepsilon p}. \tag{3.12}
\]

The equilibrium equations (3.8) imply that these equilibria constitute an equilibrium set \( E \), with

\[
 E = \{ x \in \mathbb{R}^3 | \dot{\alpha} = \omega_{eq} = 0, \alpha \in [\alpha_{min}, \alpha_{max}] \}, \tag{3.13}
\]

with

\[
 \alpha_{\min} = \max \left( \frac{k_m u_c - T_{su} - \Delta T_{su}}{k_\theta}, -\frac{T_{sl}}{k_\theta} \right), \quad \alpha_{\max} = \min \left( \frac{k_m u_c + T_{su} - \Delta T_{su}}{k_\theta}, \frac{T_{sl}}{k_\theta} \right). \tag{3.14}
\]
An accurate parameter identification of the rotor dynamic model is necessary in order to have a model that describes the experimental set-up with high accuracy. The parameters are also important for control of the rotor dynamic system. Since we aim to stabilize (by means of control) the equilibrium point of the uncontrolled system, it is necessary that the controller is provided with the correct setpoint; these correct equilibrium values need to be calculated with the help of the model. We use different methods to obtain parameter values for the rotor dynamic model. A nonlinear least-squares technique for estimation of the parameters of the rotor dynamic model is presented in \[12, 21\]. The nonlinear least-squares optimization technique is used to minimize the difference between the measured responses of the experimental rotor dynamic set-up to an appropriate input signal and the simulated responses of the rotor dynamic model to the same input signal over a certain time period. This input signal has the property of persistence of excitation.

The parameters for the upper part \(k_m, J_u\) and the friction model \(T_{fu}\) are taken from the estimated parameters presented in \[21\]. A validation experiment shows that these parameter values are reliable for the experimental set-up. Validation measurements of the steady-state responses are performed for the rotor dynamic system. In order to obtain stable equilibria for a range of constant input voltages, the brake device is disconnected from the set-up (it will be explained in the next section, that the friction acting on the lower disc, caused by the brake device, may lead to stable periodic solutions). The value of the friction acting on the upper disc \(T_{fu}\) for a constant input voltage can be calculated from the corresponding equations of motion and is used to construct the diagram in Figure 3.8. The velocity, denoted by the circles in Figure 3.8, is based on the observer \(4.31\), which is applied on the experimental set-up.

The friction at the lower disc is very sensitive to conditions such as temperature, humidity and the amount of oil in the contact area of the brake. Therefore, the parameters of the friction model of the lower disc have to be estimated again for each new experimental session.

The nonlinear least-squares technique provides us with parameters for the rotor dynamic model which describes transient behaviour well. The method presented in \[21\] gives us performance criteria for the fit between the responses, related to the persistently excited input signal, of the experiments and the simulations. However, the focus in this thesis is on steady-state behaviour. Namely, we want to control the system for a certain setpoint and in order to provide the controller with the proper setpoint, the equilibria (steady-state behaviour) of the uncontrolled system has to be modeled accurately.
For a check whether the estimated parameters yield a predictive model for the steady-state behaviour, we use a diagram to depict the steady-state solutions as a function of the constant input voltage: a so-called bifurcation diagram.

A nonlinear dynamical system can have different types of steady-state solutions, such as an equilibrium point or equilibrium set, a periodic solution or a quasi-periodic solution depending on the initial conditions of the system. Small changes of certain parameter of the system can cause a qualitative change of the steady-state behaviour. A bifurcation diagram is a way to show graphically the changes in the steady-state behaviour with respect to a certain parameter. This parameter is called the bifurcation parameter. For the bifurcation diagram of the rotor dynamic system we use the constant input voltage $u_c$ as the bifurcation parameter. The bifurcation diagram can show different branches which represent equilibrium points or periodic solutions. Bifurcation diagrams can be constructed as a function of $u_c$ in terms of the equilibrium velocity $\omega_{eq}$ and the difference between the positions of upper and lower disc $\alpha_{eq}$ for an equilibrium. We can compare the diagrams based on experimental measurements (using the aforementioned observer) to the obtained bifurcation diagrams from the simulations and check whether the estimated parameters constitute a predictive model for the steady-state behaviour of the rotor dynamic system.

Besides the check with the bifurcation diagrams for $\omega_{eq}$ and $\alpha_{eq}$, we can also study the responses of the rotor dynamic system especially the periodic solutions, and check whether the period time of the period solutions is the same for the measurements and the simulations.

The nonlinear least-squares technique needs good initial guesses for the parameters to be estimated, because the small number of sets with initial parameter values, which converge to parameters with values for which the difference between measurements and simulations is substantially minimized. Since our focus on steady-state behaviour of the rotor dynamic system, we will perform extra checks on the quality of the parameter fit (in terms of the match on the level of the bifurcation diagram). Since we will apply extra checks and the nonlinear least-squares technique is a delicate technique to apply, we choose to estimate the parameters in a different way. We start with an initial guess based on the parameter set found in [21]. Then we construct a bifurcation diagram from the performed simulations with the rotor dynamic system for the initial parameter set. The bifurcation diagrams are compared with the experimental results. If necessary, the parameter set is adapted, based on the insight in the friction model of the lower disc. And again, bifurcation diagrams are constructed from simulations with the rotor dynamic system with the current parameter set. This iterative procedure ends when the simulated bifurcation diagrams match the experimental results.
Now, we introduce a particular friction characteristic for the rotor dynamic set-up. We consider this situation as the nominal case for the rotor dynamic system. Besides the nominal case, we consider another friction characteristic for the rotor dynamic set-up. This situation is used to prove that the observer and the output-feedback controller work for the rotor dynamic system with different friction characteristics. By changing the temperature and the normal force of the brake device, a different friction characteristic can be obtained. We continue this chapter with the parameter identification in this section, the analysis of the dynamic behaviour in Section 3.4 and the experimental results are given in Section 3.5 for the nominal case (called friction characteristic I). The same approach can be applied to the rotor dynamic system with friction characteristic II. The main information and results for friction characteristic II are shown in Appendix I.

Table 3.2 shows the obtained estimated parameters for the rotor dynamic system for the nominal case. The normal force of the brake is approximately 17.4 N and the temperature is between 24-30 degrees Celsius around the area of the brake disc.

<table>
<thead>
<tr>
<th>parameter</th>
<th>estimated value</th>
<th>unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_m$</td>
<td>4.3228</td>
<td>[Nm/V]</td>
</tr>
<tr>
<td>$J_u$</td>
<td>0.4765</td>
<td>[kg m$^2$]</td>
</tr>
<tr>
<td>$T_{su}$</td>
<td>0.37975</td>
<td>[Nm]</td>
</tr>
<tr>
<td>$\Delta T_{su}$</td>
<td>-0.00575</td>
<td>[Nm]</td>
</tr>
<tr>
<td>$b_u$</td>
<td>2.4245</td>
<td>[kg m$^2$/rad s]</td>
</tr>
<tr>
<td>$\Delta b_u$</td>
<td>-0.0084</td>
<td>[kg m$^2$/rad s]</td>
</tr>
<tr>
<td>$k_\theta$</td>
<td>0.075</td>
<td>[Nm/rad]</td>
</tr>
<tr>
<td>$J_i$</td>
<td>0.035</td>
<td>[kg m$^2$]</td>
</tr>
<tr>
<td>$T_{sl}$</td>
<td>0.26</td>
<td>[Nm]</td>
</tr>
<tr>
<td>$T_{cl}$</td>
<td>0.05</td>
<td>[Nm]</td>
</tr>
<tr>
<td>$\omega_{sl}$</td>
<td>2.2</td>
<td>[rad/s]</td>
</tr>
<tr>
<td>$\delta_{sl}$</td>
<td>1.5</td>
<td>[-]</td>
</tr>
<tr>
<td>$b_l$</td>
<td>0.009</td>
<td>[kg m$^2$/rad s]</td>
</tr>
</tbody>
</table>

The friction models, corresponding to the estimated model parameters, are shown in Figure 3.9 and Figure 3.10, for the upper disc and for the lower disc, respectively. The friction model for the friction acting on the lower disc is called a humped friction model. It is clearly seen in Figure 3.10(b) that the friction first increases for low velocities and then decreases.

Afterwards, in Section 3.5, the bifurcation diagrams and the time responses for both experiments and simulations are placed and discussed. In that section, the experimental results will be discussed in general. The dynamic behaviour of the rotor dynamic system will be analyzed in Section 3.4 to before discussing the experimental results in Section 3.5.

### 3.4 Analysis of the dynamic behaviour

Extensive research on the analysis of the dynamic behaviour of the rotor dynamic system is made in [21]. A short overview of the results for the analysis of stability of the equilibria for $u_c \geq 0$ will be given here.

Local stability properties of the equilibria can be investigated by means of linearization at least away
from the points of discontinuity. This can only be done for the isolated equilibrium points \( \omega_{eq} > 0 \), because it is not possible to linearize the discontinuities at \( \omega_{eq} = 0 \). The equilibrium system for \( u_c > u_{ep} \) is locally stable if

\[
\left. \frac{dT_{cl}}{d\omega} \right|_{\omega_l=\omega_{eq}} > d_{min},
\]

(3.15)

where \( d_{min} \) is a certain threshold depending on the system parameters (see page 60 in [21]). Equation (3.15) reflects that the local asymptotic stability of the equilibria (for \( \omega_{eq} > 0 \)) is guaranteed if the frictional damping acting on the lower disc exceeds the threshold \( d_{min} \).

In [21], Lyapunov’s stability theorem is used to investigate the non-local stability properties of the equilibrium sets and equilibrium points. An overview of the results of this investigation is given in Figure 3.11 where the stability properties are related to the constant input \( u_c \) voltage and the equilibrium velocity \( \omega_{eq} \). The overview applies for the two friction characteristics of the rotor dynamic system: the nominal case and friction characteristic II. Region I, with the equilibrium set where \( \omega_{eq} = 0 \) and with the equilibrium points for small equilibrium velocities, is locally asymptotically stable. For higher input voltages, there is region II for which the equilibrium points of the rotor dynamic system are unstable. The frictional damping acting on the lower disc is below the threshold in this region. In region III,
equilibrium points are locally asymptotically stable. The equilibrium points for region IV are globally asymptotically stable. For more explanation about the stability properties, see [21].

We continue with the bifurcation diagrams for the rotor dynamic system with the constant input voltage $u_c$ as the bifurcation parameter. The bifurcation diagrams give a graphical presentation of the different steady-state solutions of the rotor dynamic system for a constant input voltage $u_c$ depending on the initial conditions. The bifurcation diagrams are constructed based on simulations for the estimated parameters in Table 3.2.

For the simulations, we need numerical solutions of the rotor dynamic system. Many computational problems are encountered when trying to obtain a numerical solution of the system described by differential inclusions. A method to overcome this problem is to replace the discontinuous vector field by a continuous vector field, for example replacing a set-valued sign-function by an $\arctan$-function. However this results in stiff differential equations and these are numerically expensive to solve, especially close to the original discontinuity. To overcome this difficulty we use the switch model [18] as a numerical technique for the determination of solutions of the differential inclusion describing the rotor dynamic system. The switch model does not have the problem of the expensive computation near to the discontinuity. For the applied switch model of the rotor dynamic system (3.3), see Appendix C.

The constructed bifurcation diagrams based on simulations will be validated by experiments in Section 3.5. The bifurcation diagram for the velocity $\omega_l$ of the lower disc is depicted in Figure 3.12 and the bifurcation diagram for the difference between the positions of the discs, $\alpha$, is depicted in Figure 3.13. The results of the simulations and of the experiments are plotted in these bifurcation diagrams. When a stable periodic solution occurs, the maximum and minimum values of the velocity $\omega_l$, and the difference between the position of the discs, $\alpha$, are plotted in the bifurcation diagrams. Now, we only consider the simulation results with the stable equilibria branch (thick solid line), the unstable equilibria branch (dotted line) and the stable periodic solution branch (solid line).

It appears that a stable periodic solution occurs for a range of input voltages. These stable periodic solutions are called limit cycles. A limit cycle response for constant input voltage $u_c = 1.8$ V is depicted in Figure 3.15, where we only consider the simulation (dashed line). This limit cycling implies that the

![Figure 3.11: Stability properties related to the input voltage $u_c$.](image-url)
velocity at the lower disc decreases and comes to a standstill. Since the upper disc continues rotating, a torque is built up in the steel string due to the torsion. The torque of the steel string acts on the lower disc and at some time instant, it overcomes the static friction at the lower disc. Consequently, the lower disc starts rotating again and the velocity increases. At some time instant torsion is present in the steel string in opposite direction and now the torque of the steel string acting on the lower disc slows down the lower disc. The lower disc comes to a standstill again and the cycle repeats again. The upper disc rotates at approximately the same velocity for the whole cycle. It is only slightly influenced by the torque exerted by the torsion of the steel string.

The bifurcation diagrams in Figure 3.12 and Figure 3.13 show an equilibrium branch $e_1$, which is the equilibrium set (3.13). The branch $e_1$ with the equilibrium set is connected to the unstable branch $e_2$ and to the branch $p_1$. The branch $p_1$ represents stable periodic solutions with stick-slip, which are the limit cycles for which the period time is displayed in Figure 3.14. One would expect a bifurcation point where branch $e_1$ is connected to $e_2$ and $p_1$, but in fact there are two bifurcation points, which means that there are also more branches than displayed in the bifurcation diagram. This is extensively discussed in [21]. The focus in this thesis is on control of a large range of voltages and not especially on the dynamics at low voltages. Therefore, these bifurcation points and branches with solutions at low voltages are not discussed in detail here.

The branch $e_2$ represents the locally unstable equilibria for which condition (3.15) is not satisfied. There also exists a stable periodic solution for the corresponding input voltages. A subcritical Hopf bifurcation occurs where branch $e_2$ is connected to the stable branch $e_3$ and an unstable periodic branch which is not depicted. This unstable periodic branch consists of unstable periodic solutions without stick-slip, see for more information [21]. The branch $e_3$ consists of isolated locally asymptotically stable equilibria. The other end of the aforementioned unstable branch, which is not depicted, is connected to the stable branch $p_1$. The connection point represents a fold bifurcation point. The bifurcation point is a non-smooth transition from periodic solutions with stick-slip to periodic solutions without stick-slip. Therefore it is a discontinuous fold bifurcation.

Figure 3.12: Bifurcation diagram with the velocity of the lower disc for positive input voltages.
The validity of the model with the estimated parameters is verified with experiments. The conditions for the friction acting on the lower disc are created by lubricating the brake disc with ondina oil 68 and applying a normal force with the brake device on the brake disc of 17.4 N. The quality of the model with the estimated parameters is investigated by several steady-state experiments. The steady-state experiments are performed for a constant input voltage and lasted long enough to fade out transient effects. The state $\alpha$, which represents the difference between the positions of the upper and lower disc, is measured indirectly by the two angular encoders. The other states, $\omega_u$ and $\omega_l$, velocities of the upper and lower disc, respectively, are estimated by the observer design presented in Chapter 4. Steady-state responses for several constant input voltages are measured. A limit cycle response is shown in Figure 3.15 for $u_c = 2.8$ V, where the solid line represents the experimental result and the dashed line represents the simulated result. Two steady-state equilibrium responses are depicted in Figure 3.16 and Figure 3.17. These figures indicate the good match between the experimental and model results. However, the equilibrium values for $\alpha$ and $\omega_l$ in Figure 3.16 and Figure 3.17 for $u_c = 2.8$ V and $u_c = 3.5$ V, respectively, are not really constant in experiments. The responses are fluctuating around a certain value. This phenomenon is probably due to some unmodeled dynamics which are present in the experimental set-up. In spite of these perturbations, we regard this solution as an equilibrium. The figures of other steady-state responses are shown in Appendix G: Figures G.1–G.5.

The steady-state experiments for different constant input voltages are used to experimentally validate bifurcation diagrams with $u_c$ as a bifurcation diagram (in terms of $\alpha$ and $\omega_l$), constructed from the simulations, see Figure 3.12 and Figure 3.13. An equilibrium point is marked with "x". The mean value of the response which is regarded as an equilibrium is taken when some perturbations are present, see for example Figure 3.16 and Figure 3.17. The maximum and minimum value of the state $\alpha$ and $\omega_l$ are plotted in the bifurcation diagrams with the mark "o" for a periodic solution. The results of the experiments are plotted together with the simulation results, where the stable equilibria branch is depicted with a thick solid line, the unstable equilibria branch is depicted with a dotted line and the
stable periodic solutions is depicted with a solid line. The bifurcation diagrams are shown in Figures 3.12 and 3.13.

No measurements are performed for the lower voltages, since the focus in this thesis is on control of a large range of voltages and not especially on the dynamics at low voltages.

For the region with constant input voltages up to approx. 2.7 V, we see only stable limit cycles. In the region with the input voltages from approx. 2.7 V up to approx. 4.5 V two stable solutions coexist. In this region, the steady-state solution can be an equilibrium point or a periodic solution depending on the initial conditions. For constant input voltages higher than 4.5 V, only an equilibrium point occurs.

The periodic time of the periodic solutions from the experiments are measured and plotted in Figure 3.14. The period times obtained from simulations are plotted as solid line in Figure 3.14.

If we consider the bifurcation diagrams, Figures 3.12 and 3.13, and the diagram with the period times of the periodic solutions, Figure 3.14, we can conclude that there is a good match between the experiments and the simulations. Some unmodeled dynamics are present in the experimental set-up, especially influencing the equilibria for the lower input voltages. Nevertheless, these unmodeled dynamics are not dominating the modeled dynamics. Therefore, the model with the estimated parameters is a predictive

Figure 3.14: Diagram with the period times of the periodic solutions for positive input voltages.

Figure 3.15: Measured and simulated limit cycle responses of the rotor dynamic system for \( u_c = 1.8 \) V.
We have already mentioned that there are perturbations present in the experimental responses, especially for equilibria at lower constant input voltages. The bifurcation diagram with $u_c$ as a bifurcation parameter with respect to $\omega_l$ is plotted once more in Figure 3.18. Instead of taking the mean value for the states when a solution is regarded as an equilibrium (as in Figure 3.12), the minimum and maximum value of the responses are taken and plotted in the bifurcation diagrams, marked with "x" in case of equilibria. If we consider the line with equilibria starting at 2.7 V, the bifurcation diagrams show that the perturbations decrease when the constant input voltage increases.

The time responses of the equilibria are investigated by performing frequency-domain analysis on the response $\omega_l(t)$. This is done for the responses to the constant input voltages 2.8 and 3.5 V, see Figure 3.19, which corresponds to the responses depicted in Figure 3.16 and Figure 3.17. The rotational frequency is plotted with a vertical dashed line in the diagrams. From the power spectral density diagrams in these figures we see that there is a dominant spectral component at 0.24 Hz for both $u_c = 2.8$ V and $u_c = 3.5$ V. This frequency is due to the mechanical resonance frequency, which is independent of the rotational frequency. The mechanical resonance frequency is also observed in [21]. For all the constant
input voltages $u_c$, the rotational frequency represents another dominant spectral component. This indicates that probably unmodeled position dependent friction is present in the experimental set-up. Twice the rotational frequency is also a dominant spectral component for all the input voltages.

The bifurcation diagrams in Figure 3.12 and 3.13 clearly show the two bifurcations for voltages higher than 0.5 V in experiments, which are discussed in Section 3.4 for the model. Responses for experiments in which $u_c$ is changed such that it passes a bifurcation value are shown in Appendix G.

### 3.6 Discussion

The experimental rotor dynamic set-up has been described in this chapter. The set-up exhibits nonlinear phenomena, such as periodic solutions with stick-slip, coexistence of steady-state solutions, bifurcations and regions with different stability properties of the steady-state solutions. All these phenomena occur
for both friction characteristics: the nominal case and friction characteristic II.
A cause for the limit cycles is the negative damping present in the friction acting on the lower disc. The presented model together with the estimated parameters is accurate enough to predict the steady-state behaviour of the experimental set-up, although unmodeled dynamics are present. It seems that the friction model of the lower disc does not entirely captures the real friction. From frequency-domain analysis of the experimental responses, we notice that in practice, there are phenomena indicating a small position-dependent friction effect.
The bifurcation diagrams show a large region with stable limit cycles and an unstable equilibrium and a large region with coexistence of stable equilibria and stable limit cycles. For higher constant input voltages there is a region with stable equilibria. This region is limited for the experimental set-up, since the input voltage to the DC-motor is limited.
The last mentioned region with only stable equilibria is important for practical use from a mechanical engineering point of view. If we consider the example from Chapter 1, the rotary drilling system (see Figure 1.4), then it is desired that the drill-bit of the rotary drilling system rotates with a constant velocity. This is more efficient than if the drill-bit of the rotary drilling system would exhibit limit cycles. Moreover, limit cycles could lead to premature failure of parts of the rotary drilling system. So the regions with only limit cycles and with coexistence of solutions are not desired for the rotor dynamic system if we link this system to corresponding mechanical systems in practice. This makes it interesting to find a controller for the rotor dynamic system which extends the region with stable equilibria and thereby reduces the other regions.
Observer design

In this chapter, a model-based observer design is presented for systems in Lur’e-type form, i.e. linear systems with (set-valued) decoupled nonlinearities in the feedback loop. In the scope of the observer design, the requirement for the set-valued nonlinearities is monotonicity. A circle criterion design is used to construct an observer which renders the observer error dynamics absolutely stable. The observer error dynamics can also be written as a Lur’e-type system. For absolute stability of the observer error dynamics, we demand strict positive realness of the transfer function of the linear part of the observer error system. The requirement for the strict positive realness of the transfer function of the linear part of the observer error system is represented by a matrix inequality. The presented observer design can be applied if the set-valued decoupled nonlinearities of the system to be observed are monotone and the matrix inequality, concerning the observer error system, is feasible.

The observer design was first presented by Arcak and Kokotović in e.g. [1] and [3] for systems with continuous nonlinearities. Different extensions are proposed for this observer, for example an extension for systems with multivariable monotone nonlinearities in [10]. Another useful extension to systems with non-smooth and discontinuous mappings is presented in [14] and [15]. We use the latter extension for application to systems with discontinuous elements.

The observer design in [14] and [15] is applied to the experimental rotor dynamic system. In these papers, the non-smooth nonlinear friction of the upper disc is modeled as viscous friction. In this thesis, the non-smooth nonlinear friction at the upper disc is also taken into account in the design of the observer. The observer is implemented online and the results of simulations and experiments are shown in Sections 4.4 and 4.5.

4.1 Class of systems

We consider a class of Lur’e-type systems that can be represented as a linear dynamic system with set-valued decoupled nonlinearities in the feedback loop, see Figure 4.1. The set-valued nonlinearity can be defined by an algebraic inclusion and contains the discontinuous elements of the dynamical system. The approach can also be applied to systems with a continuous (though non-smooth) nonlinearity in the feedback loop.
Consider systems of the form (see Figure 4.1):

\[
\begin{align*}
\dot{x} &= Ax + Gw + \psi(y, u) \\
z &= Hx \\
w_i &\in -\varphi_i(z_i) \quad i = 1, \ldots, p \\
y &= Cx,
\end{align*}
\]

where \( w = [w_1, \ldots, w_p]^T \) and \( \varphi = [\varphi_1, \ldots, \varphi_p]^T \), and with the system state \( x \in \mathbb{R}^n \), measured output \( y \in \mathbb{R}^l \), \( z, w \in \mathbb{R}^p \), control input \( u \in \mathbb{R}^m \) and \( \varphi_i : \mathbb{R} \rightarrow \mathbb{R}, i = 1, \ldots, p \), are memoryless set-valued functions. The matrix \( G \in \mathbb{R}^{n \times p} \) is full rank and \( \psi(y, u) : \mathbb{R}^l \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a continuous nonlinearity.

We assume that each set-valued mapping \( \varphi_i \) is an upper semi-continuous mapping, which is convex and does have a closed image. This assumption guarantees the existence of solutions of the system (4.1), and indirectly the existence of solutions of the observer (proposed in the next section).

We now introduce a condition for the set-valued nonlinearity in the feedback loop. This condition is necessary to prove the asymptotic stability of the origin of the observer error dynamics. Each set-valued function \( \varphi_i(\cdot) \) has to be monotone or nondecreasing. The set-valued function \( \varphi_i(\cdot) \) is monotone or nondecreasing if

\[
(a - b)[\varphi_i(a) - \varphi_i(b)] \geq 0 \quad \forall \quad a, b \in \mathbb{R}.
\]

### 4.2 Observer design based on the circle criterion

The following observer is proposed for system (4.1) (see Chapter 2 in [14]).

\[
\begin{align*}
\dot{\hat{x}} &= A\hat{x} + L(y - C\hat{x}) + G\hat{w} + \psi(y, u) \\
\dot{\hat{z}} &= H\hat{x} + N(y - C\hat{x}) \\
\hat{w}_i &\in -\varphi_i(\hat{z}_i) \quad i = 1, \ldots, p,
\end{align*}
\]

where \( \hat{w} = [\hat{w}_1, \ldots, \hat{w}_p]^T \), and with \( \hat{x} \in \mathbb{R}^n \) the estimation of the state \( x \), \( \hat{z} \in \mathbb{R}^p \) the estimation of \( z \), \( \hat{w} \in \mathbb{R}^p \), \( H\hat{x}(0) \in \text{dom } \varphi \) and the matrices \( N \in \mathbb{R}^{p \times l}, L \in \mathbb{R}^{n \times l} \) are to be designed.

The observer can be considered as a Filippov system with two different inputs \( u \) and \( y \). As in Section 2.1, it follows that solutions for such systems exist (given the condition on the set-valued mapping \( \varphi_i \) stated in the previous section).

For a certain class of systems, uniqueness of solutions of the presented observer (4.3) is proven by [14]. We do not discuss the uniqueness of solutions of the observer, since we will prove that the equilibrium
point of the observer error system is globally asymptotically stable. So even if the solution of the observer would be non-unique, all solutions still converge to the solution of the system to be observed.

In order to show that the observer asymptotically reconstructs the state of system (4.1), we consider the observer error dynamics. The observer error $e$ is defined as $e = x - \hat{x}$. Then the observer error dynamics is given by:

$$\dot{e} = (A - LC)e + Gk$$

where $k, q \in \mathbb{R}^p$. The set-valued nonlinearity $\phi$ is derived from the set-valued nonlinearity $\varphi$ through two transformation steps. These transformations do not alter the monotonicity property of the transformed set-valued nonlinearity $\varphi$. Thus, we can state the monotonicity property (4.2) for $\varphi_i$, for $i = 1, \ldots, p$. This monotonicity property can be derived for each individual scalar function $\phi_i$ in $\phi = [\phi_1 \ldots \phi_p]^T$:

$$(a - b)[\phi_i(t, a) - \phi_i(t, b)] \geq 0 \quad \forall \quad a, b \in \mathbb{R} \text{ and any fixed } t.$$
Then, the definitions given by (4.5) and (4.6) can be applied to the expression (4.2) and this results in

$$
\phi_i(t, q_i)q_i \geq 0 \quad \forall q_i, \text{ and any fixed } t,
$$  

(4.9)

which means that the graph of the set-valued function $\phi_i(t, q_i)$ can be drawn in the first and third quadrant for any fixed $t$.

The observer design goal is finding gain matrices $N$ and $L$ for the observer (4.3) that will guarantee that the equilibrium point $e = 0$ of the observer error dynamics (4.7) is globally asymptotically stable.

The structure of the error dynamics is a linear system with a set-valued function in the feedback path. The circle criterion design is available for these type of system with a continuous Lipschitz nonlinearity [16]. This circle criterion design can be extended to systems with upper semi-continuous set-valued mappings [14]. The circle criterion design renders the linear part of the error system strictly passive. If the linear part of the error system is strictly passive (in combination with the already assumed sector condition of the set-valued nonlinearity in the feedback loop) then the equilibrium point of the error dynamics is globally asymptotically stable. Therefore, we desire strict passivity of the observer error dynamics or strict positive realness of the transfer function of the linear part of the observer error dynamics.

In order to assess the stability properties of $e = 0$ of (4.7), we seek a Lyapunov candidate function

$$
V = e^T P e,
$$  

(4.10)

with $P = P^T > 0$. The time-derivative of $V$ satisfies

$$
\dot{V} = e^T P e + e^T P \dot{e} = e^T [(A - L C)^T P + P(A - L C)] e + k^T G^T P e + e^T P G k
$$  

(4.11)

$$
= \begin{bmatrix}
e \\
k
\end{bmatrix}^T \begin{bmatrix}
(A - L C)^T P + P(A - L C) & PG \\
G^T P & 0
\end{bmatrix} \begin{bmatrix}
e \\
k
\end{bmatrix}.
$$  

(4.12)

Assume that the transfer function of the linear part of the error dynamics is strictly positive real, i.e. the following matrix inequality is feasible

$$
F(P) = \begin{bmatrix}
(A - L C)^T P + P(A - L C) + \varepsilon I & PG - (H - N C)^T \\
G^T P & (H - N C)
\end{bmatrix} \leq 0,
$$  

(4.13)

for $P = P^T > 0$ and a scalar $\varepsilon > 0$.

Note that the frequency domain SPR conditions stated in Chapter 2 are equivalent to the matrix inequality condition (4.13). Then the expression for $\dot{V}$ can be written as

$$
\dot{V} = \begin{bmatrix}
e \\
k
\end{bmatrix}^T \begin{bmatrix}
(A - L C)^T P + P(A - L C) & PG \\
G^T P & 0
\end{bmatrix} \begin{bmatrix}
e \\
k
\end{bmatrix} \leq \begin{bmatrix}
e \\
k
\end{bmatrix}^T \begin{bmatrix}
-\varepsilon I & (H - N C)^T \\
(H - N C) & 0
\end{bmatrix} \begin{bmatrix}
e \\
k
\end{bmatrix},
$$  

(4.14)

which is equal to

$$
\dot{V} \leq -\varepsilon e^T e + k^T (H - N C)e + e^T (H - N C)^T k.
$$  

(4.15)
With the expression for $q$ in (4.7) and condition (4.9) (which implies that $k^T q \geq 0$), we can write (4.15) as:

$$
\dot{V} \leq -\varepsilon e^T e + 2k^T q < 0. \quad (4.16)
$$

The fact that the derivative of the Lyapunov function is smaller than zero for all $e \neq 0$, implies that $e = 0$ is an asymptotically stable equilibrium point of the observer error dynamics. Due to radial unboundedness of $V$, we can conclude global asymptotic stability of the equilibrium.

Note that the output-dependent nonlinearity $\psi(y, u)$ can even be allowed to depend on the observed state $\hat{x}$. Such a nonlinearity $\psi(\hat{x}, y, u)$ is canceled in the error system and, thus, can not destabilize the observer error system.

We assume that the set-valued nonlinearity $\phi$ belongs to the sector $[0, \infty]$. For the circle criterion this means that the condition for the linear system becomes strict passivity (i.e. strict positive realness for the transfer function). With the circle criterion it is possible to relax the condition for the linear system if the set-valued nonlinearity $\phi$ belongs to the sector $[0, k_u]$ for $k_u > 0$. This is not possible for the actual design since the nonlinearity $\phi$ contains discontinuities. We want the observer to reconstruct the system state $x(t)$ for all $t \in \mathbb{R}$. The set-valued nonlinearity $\phi$ constantly changes over time. It can happen that the discontinuity in $\phi$ occurs at the origin, i.e. $\phi$ belongs to $[0, \infty]$. Therefore we assume that $\phi \in [0, \infty]$ and the condition for the linear part of the error system is strict passivity.

In Appendix D.2, we show that the following bound holds:

$$
||e||^2 \leq \frac{V_0}{\lambda_{\min}(P)} e^{-\beta t}, \quad (4.17)
$$

where $V_0 = e_0^T P e_0$ and $\beta = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$. Note that if $Q = \varepsilon I$, then $\lambda_{\min}(Q) = \varepsilon$. Since (4.17) holds, we conclude that the origin of the observer error dynamics (4.7) is globally exponentially stable.

There is a comment to be made on the presented observer (4.3) about the absolute stability. Absolute stability can be obtained for the observer error system, i.e. the origin of the observer error dynamics is globally exponentially stable for any set-valued nonlinearity $\phi$ in the sector $[0, \infty]$. However, this does not imply global exponential stability for the origin of the observer error dynamics for any set-valued nonlinearity $\phi$ in the sector $[0, \infty]$, only for the modeled set-valued nonlinearity $\varphi$.

### 4.3 Application to the rotor dynamic system

The rotor dynamic system is discussed in Chapter 3. We apply the design of the observer to the rotor dynamic system for the nominal case (friction characteristic I). The parameters for the nominal case of the rotor dynamic system can be found in Table 3.2. The results of the simulations and the results of the experiments for the nominal case are shown and discussed in Section 4.4 and Section 4.5, respectively. Besides the application of the observer design to the rotor dynamic system for the nominal case, the observer design will also be applied to the rotor dynamic system for the friction characteristic II. These results are discussed in Appendix J.

The observer design, presented in the previous section, will be applied to the rotor dynamic system. The model of the rotor dynamic system must be in the considered Lur’e form: a linear system with a monotone nonlinearity in the feedback loop. The necessary transformation, needed to write the system in such a form, is performed for the rotor dynamic system (3.3) and is described in Appendix F.1. The angular positions $\theta_u$ and $\theta_l$ of the upper and lower disc are measured by encoders.
We consider the model of the rotor dynamic system as in (F.7). The state \( \alpha = \theta_u - \theta_l \) is measured and this output will be added to the model:

\[
\begin{align*}
\dot{\xi} &= A\xi + Bv + Gw \\
z &= H\xi \\
w &\in -\varphi(z) \\
y &= C\xi,
\end{align*}
\]

where state \( \xi \in \mathbb{R}^3 \), \( w, z \in \mathbb{R}^2 \), control input \( v \in \mathbb{R} \), measured output \( y \in \mathbb{R} \), and \( \varphi_i : \mathbb{R} \rightarrow \mathbb{R} \) for \( i = 1, 2 \). The matrices \( A, B, G, H, C \) and \( \varphi(z) \), are given by

\[
\begin{align*}
A &= \begin{bmatrix} 0 & 1 & -1 \\ -\frac{k_u}{J_u} & 0 & 0 \\ \frac{k_l}{J_l} & 0 & 0 \end{bmatrix}, & B &= \begin{bmatrix} 0 \\ \frac{k_m}{J_u} \\ 0 \end{bmatrix}, & G &= \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{J_u} & 0 & 0 \\ 0 & \frac{1}{J_l} & 0 \end{bmatrix},
\end{align*}
\]

\[
H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \varphi(z) = \begin{bmatrix} T_{fu, tr2}(z_1) \\ T_{fl, tr2}(z_2) \end{bmatrix}.
\]

The transformed friction torques acting on the upper and lower disc are shown in Figure 4.3 (see Appendix F.1 for the definition of the transformed friction torques) for a constant input voltage \( u_c = 1.0 \) V. From Figure 4.3(a) it can be seen that the upper friction \( T_{fu, tr2} \) satisfies the monotonicity property (4.2). Clearly, the lower friction \( T_{fl, tr2} \), see Figure 4.3(b), is not monotone. Therefore the observer design cannot be applied to the current form of the model of the rotor dynamic system.

In order to overcome this problem, we transform the system via loop transformations, such that the lower friction \( T_{fl, tr3} \) derived from this transformation is monotone. The loop transformation yields an output feedback for the linear system and an input feedforward for the set-valued nonlinearity. The rotor dynamic system (4.18) with the loop transformation matrix \( M \) is depicted in Figure 4.4(a). The loop transformation matrix \( M \) is given by

\[
M = \begin{bmatrix} -b & 0 \\ 0 & m \end{bmatrix},
\]

where \( b = b_u + \Delta b_u = 2.4161 \) Nms/rad and \( m = 0.1 \) Nms/rad. Further information for the use of this loop transformation matrix \( M \) is given below. The minimum linear damping needed to render
4.3. Application to the rotor dynamic system

the transformed friction $T_{fl, tr3}$ monotone (i.e. to ensure that the derivative of $T_{fl, tr3}$ with respect to $\xi_3$ is larger or equal to zero $\forall \xi_3 \neq 0$), is less than the chosen value for $m$. In that way, a certain level of robustness with respect to rendering a changed friction $T_{fl, tr3}^*$ monotone is obtained, where $T_{fl, tr3}^*$ represents the actual friction acting on the lower disc which differs with the modeled friction model $T_{fl, tr3}$ (due to unavoidable modeling errors).

By including the loop transformation matrix (4.21) in the structure of system (4.18), we obtain the following system:

\[
\begin{align*}
\dot{\xi} &= A_{tr} \xi + B v + G \tilde{w} \\
\dot{z} &= H \xi \\
\dot{\tilde{w}} &= -\varphi_{tr}(z) \\
y &= C \xi, \\
\end{align*}
\]

where state $\xi \in \mathbb{R}^3$, $\tilde{w}, z \in \mathbb{R}^2$, input $v \in \mathbb{R}$, measured output $y \in \mathbb{R}$, and $\varphi_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$. This system is depicted in Figure 4.4(b). The form of the following matrices given by

\[
A_{tr} = A + GMH = \begin{bmatrix} 0 & -\frac{k_m}{J_u} & -\frac{b}{J_u} \\
\frac{b}{J_u} & 0 & 0 \\
0 & \frac{k_m}{J_u} & 0 \end{bmatrix}, \\
B = \begin{bmatrix} 0 \\
\frac{k_m}{J_u} \\
0 \end{bmatrix}, \\
G = \begin{bmatrix} 0 & 0 \\
0 & 0 \\
0 & \frac{1}{J_u} \end{bmatrix}, \\
\]

\[
H = \begin{bmatrix} 0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix}, \\
\varphi_{tr}(z) = \begin{bmatrix} \varphi_{tr,1}(z_1) \\
\varphi_{tr,2}(z_2) \end{bmatrix} = \begin{bmatrix} T_{fu, tr3}(z_1) \\
T_{fl, tr3}(z_2) \end{bmatrix}. 
\]

The expressions for $T_{fu, tr3}$ and $T_{fl, tr3}$ are given by

\[
T_{fu, tr3}(z_1) = T_{fu, tr2}(z_1) - b z_1, \\
T_{fl, tr3}(z_2) = T_{fl, tr2}(z_2) + m z_2. 
\]

The reasoning for adding negative viscous damping in $T_{fu, tr3}$ is explained at the end of the following paragraph. The friction torques $T_{fu, tr3}$ and $T_{fl, tr3}$ are plotted in Figure 4.5 (friction transformed with respect to input voltage $u_c = 1.0 \text{ V}$).

The loop transformation adds viscous friction to the friction model $T_{fl, tr2}$ to create a new friction model, $T_{fl, tr3}$, which is monotone and thus satisfies (4.2). The loop transformation also takes viscous friction out of the friction model $T_{fu, tr2}$ and creates a new friction model $T_{fu, tr3}$ which is still monotone. The changes in the friction models are compensated by changes in the linear system. Negative viscous damping is added with respect to $\xi_3$ and viscous damping with respect to $\xi_2$. Adding viscous damping with respect to $\xi_2$ makes it easier to render the transformed linear system strictly passive.

The proposed observer for system (4.22), according to (4.3), is

\[
\dot{\hat{\xi}} = A_{tr} \hat{\xi} + L(y - C \xi) + G \tilde{w} + B v \\
\dot{\hat{w}} = -\varphi_{tr}(\hat{z}) \\
\dot{\hat{z}} = H \xi + N(y - C \xi) \\
y = C \xi, 
\]

where $\hat{\xi} \in \mathbb{R}^3$, $\hat{z}, \hat{w} \in \mathbb{R}^2$, $H \hat{x}(0) \in \text{dom } \varphi$ and the observer matrices $N \in \mathbb{R}^{2 \times 1}$, $L \in \mathbb{R}^{3 \times 1}$ are to be designed.
Application of the observer (4.27) results in following observer error dynamics:

\[
\begin{align*}
\dot{e} &= (A_{tr} - LC)e + Gk \\
k &\in -\phi(t, q) \\
q &= (H - NC)e,
\end{align*}
\]

(4.28)

with \( e \in \mathbb{R}^3, k, q \in \mathbb{R}^2 \) and \( \phi_i : \mathbb{R} \to \mathbb{R} \) for \( i = 1, 2 \).

Two conditions have to be satisfied to make \( e = 0 \) a globally asymptotically stable equilibrium point of the observer error dynamics. The first condition demands monotonicity of the set-valued nonlinearity \( \varphi \) in the feedback loop. This condition is satisfied for the transformed set-valued friction models \( T_{f_{u,tr}} \) and \( T_{f_{l,tr}} \). The set-valued nonlinearity \( \varphi \) in the feedback of the observer error system belongs to the sector \([0, \infty)\) since the condition for monotonicity for \( \varphi \) is satisfied. This is necessary for the application of the circle criterion.

The second condition (4.13) demands strictly positive realness of the transfer function of the linear part of the observer error dynamics. Appropriate matrices \( N \) and \( L \) have to be found to meet this condition. A numerical LMI solving tool within the Matlab program [9, 8] is used to solve the matrix inequality (4.13) for the matrices \( N \) and \( L \). The used algorithm by the LMI solving tool is \textit{sdph} [25]. Two solutions
for (4.13) are shown on the next pages. The two solutions will be used for comparison. The observer gains $L_1$ and $N_1$, the matrix $P_1$ and the scalar $\varepsilon_1$ for the first solution are given by

$$L_1 = \begin{bmatrix} 13.8 \\ -4.37 \\ -165 \end{bmatrix}, \quad N_1 = \begin{bmatrix} -0.572 \\ -7.07 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 3.1398 & 0.2726 & 0.2474 \\ 0.2726 & 0.4765 & 0.0000 \\ 0.2474 & 0.0000 & 0.0350 \end{bmatrix}, \quad \varepsilon_1 = 0.154. \quad (4.29)$$

The eigenvalues of $F(P_1)$ are: $-1.4899$, $-4.1335$, $-70.141$, $0$, $0$. The observer gains $L_2$ and $N_2$, the matrix $P_2$ and the scalar $\varepsilon_2$ for the second solution are given by

$$L_2 = \begin{bmatrix} 195 \\ -312 \\ -9080 \end{bmatrix}, \quad N_2 = \begin{bmatrix} -2.22 \\ -37.8 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 63.3492 & 1.0578 & 1.3230 \\ 1.0578 & 0.4765 & 0.0000 \\ 1.3230 & 0.0000 & 0.0350 \end{bmatrix}, \quad \varepsilon_2 = 2.20. \quad (4.30)$$

The eigenvalues of $F(P_2)$ are: $-12.915$, $-0.6783$, $-0.0691$, $0$, $0$. When $\varepsilon$ increases, then the corresponding observer gains also increase. A higher value for the variable $\varepsilon$ moves the eigenvalues of the linear system to the left in the complex plane. The derivative of the Lyapunov function, as in (4.16), will become more negative semi-definite. This implies a higher convergence velocity of the observer error $e$ to its equilibrium point $e = 0$.

### 4.4 Simulation results

Simulations are performed for the obtained gain matrices $L$ and $N$ of the observer for the two solutions discussed in Section 4.3. The model of the system and the observer are solved with a fixed-step solver with the help of a numerical technique, the so-called switch model [18]; this numerical technique allows to numerically integrate the differential inclusions with sliding modes. The fixed-step solver algorithm is ode1 which is available within the Matlab program.

Figures 4.6 show the responses and the observer errors of the rotor dynamic system and the observers with the gains $L_1$ and $N_1$, $L_2$ and $N_2$, respectively, for the input voltage $u = 2.0$ V for a time interval of 0.7 seconds. The initial condition for the rotor dynamic system is $\xi = [0 \quad 0 \quad 0]^T$, the initial condition for the observers is $\hat{\xi} = [4 \quad 4 \quad 4]^T$. The solution of the rotor dynamic system is a periodic solution with stick-slip (the stick phase cannot be seen in Figure 4.6, since we only show 0.7 seconds). Both observer states converge to the rotor dynamic state. The observer error of the observer with the gains $L_2$ and $N_2$ becomes larger initially than the observer with the gains $L_1$ and $N_1$, but the solutions $\dot{\omega}_l$ and $\hat{\omega}_l$ finally converge faster to the state components $\omega_u$ and $\omega_l$ of the rotor dynamic system.

The quadratic vector norm of the observer error $\|e\|_2$ of the two observers, and the upper bound of the observer error, (4.17), are plotted in Figure 4.7 for the input voltage $u = 2.0$ V during five seconds. The simulations depicted in Figure 4.7 show that the upper bound of the observer error holds, but it is very conservative.

In Figure 4.8, the response $\dot{\omega}_l$ of the observer with the gains $L_1$ and $N_1$, the rotor dynamic system limit cycle response $\omega_l$ and the observer error $|\omega_l - \dot{\omega}_l|$ are depicted for the input voltage $u = 3.5$ V. The initial condition for the rotor dynamic system is $\xi = [3.5 \quad 6 \quad 0]^T$ and for the observer $\hat{\xi} = [4 \quad 4 \quad 4]^T$. The observer is able to estimate the rotor dynamic state for the slip phase and the stick phase.

Next, we show the response $\dot{\omega}_l$ of the observer with the gains $L_1$ and $N_1$ and the response of the rotor dynamic system where the input voltage is decreased from $u = 2.7$ V with a step to $u = 2.2$ V at $t = 0.25$ s. The initial condition for the rotor dynamic system is $\xi = [\alpha_{eq} \quad \omega_{eq} \quad \omega_{eq}]^T$, i.e. the equilibrium values corresponding to the input voltage $u = 2.7$ V. The initial condition for the observer
Figure 4.6: Responses of the rotor dynamic system and the two observers with different gains for the input voltage $u_c = 2.0$ V.

Figure 4.7: Quadratic vector norm of the observer error of the two observers with different gains for the input voltage $u_c = 2.0$ V with the error bounds.

is $\hat{\xi} = [0 \ 3 \ 3]^T$. The response of the rotor dynamic system and the response of the observer with the gains $L_1$ and $N_1$ can be seen in Figure 4.9. Although the input voltage is changed, the observer error still converges to zero, i.e. $\hat{\xi}$ converges to $\xi$.

The simulations for the observer with the gains $L_1$ and $N_1$, $L_2$ and $N_2$ show that the observer state components $\hat{\alpha}$, $\hat{\omega}_u$ and $\hat{\omega}_l$ converge to the rotor dynamic system state components $\alpha$, $\omega_u$ and $\omega_l$. The observer with the gains $L_2$ and $N_2$ converges faster than the observer with the gains $L_1$ and $N_1$. This is in accordance with the calculated values for $\varepsilon$, (4.29), (4.30), since the matrix inequality that computes
4.4. Simulation results

![Graph](image)

(a) Rotor dynamic system state component \( \omega_l \) and observer state component \( \hat{\omega}_l \).

(b) Observer error \(|\omega_l - \hat{\omega}_l|\).

Figure 4.8: Response and error of the rotor dynamic system and the observer with the gains \( L_1 \) and \( N_1 \) for the input voltage \( u_c = 3.5 \) V.

![Graph](image)

(a) Rotor dynamic system state component \( \alpha \) (solid line) and observer state component \( \hat{\alpha} \) (dashed line).

(b) Rotor dynamic system state component \( \omega_u \) (solid line) and observer state component \( \hat{\omega}_u \) (dashed line).

(c) Rotor dynamic system state component \( \omega_l \) (solid line) and observer state component \( \hat{\omega}_l \) (dashed line).

Figure 4.9: Responses of the rotor dynamic system and the observer with the gains \( L_1 \) and \( N_1 \), the input voltage decreases with a step on \( t = 0.25 \) s from \( u_c = 2.7 \) V to \( u_c = 2.2 \) V.

The gains \( L_2 \) and \( N_2 \) has the largest value for \( \varepsilon \). This results in the largest negative eigenvalues of the linear part of the system and the most negative derivative of the Lyapunov function for the same system. These facts give an indication for the largest convergence velocity, which seems to be confirmed by the simulations.

The observer design is also applied to the rotor dynamic system for the friction characteristic II, see Appendix J. Since the linear system is the same as for the nominal case, we can use the same gains: \( L_1, N_1 \) and \( L_2, N_2 \). Of course, the parameters of the observer are different; the parameters for the friction characteristic II are displayed in Table I.1. The performed simulations for the observer applied to the rotor dynamic system for friction characteristic II are discussed in Appendix J.1. The conclusions do not differ from the conclusion of the simulations with the observer applied to the rotor dynamic system for the nominal case. The observer states converge to the rotor dynamic system states and the same conclusion about the convergence velocity of the two observers can be drawn.
4.5 Experimental results

The next step is to implement the observer (4.27) on the experimental drill-string set-up. Several experiments are performed for the different pairs of gains: $L_1$ and $N_1$, $L_2$ and $N_2$. The considered friction characteristic is the nominal case (friction characteristic I).

The state component $\hat{\alpha}$ is measured in the experimental drill-string set-up, so the observer error component $(\alpha - \hat{\alpha})$ can be computed. Unfortunately, the state components $\omega_u$ and $\omega_l$ are not measured and thus we cannot check the observer error with respect to these state components. Therefore, we compare the results of the current observer design (4.27) with the results from an alternative observer. The measured positions $\theta_u$ and $\theta_l$ are filtered by the alternative observer to obtain the velocities $\dot{\theta}_u$ and $\dot{\theta}_l$, respectively. The transfer function of this alternative observer is:

$$H_{obs} = \frac{200s}{s + 200}, \quad s \in \mathbb{C}, \quad (4.31)$$

with $\theta_u$ as input and $\dot{\theta}_u$ as output or with $\theta_l$ as input and $\dot{\theta}_l$ as output. The observer (4.31) differentiates the input signal and applies a low-pass filter.

The first experiments are performed for an input voltage $u_c = 2.0 \, \text{V}$ for the two pairs of gains: $L_1$ and $N_1$, $L_2$ and $N_2$, see Figures 4.10-4.12. The initial condition for the experimental drill-string set-up is $\xi = [0 \ 0 \ 0]^T$ and the initial condition for the observers is $\hat{\xi} = [4 \ 4 \ 4]^T$. The solution of the experimental drill-string set-up is a periodic solution with stick-slip.

The state component $\hat{\alpha}$ of the observer with the gains $L_1$ and $N_1$, the state component $\alpha$ of the experimental set-up and the observer error $(\alpha - \hat{\alpha})$ can be seen in Figure 4.10. The observer state component $\hat{\alpha}$ converges to the measured state component $\alpha$ within a second. The steady-state observer error $(\alpha - \hat{\alpha})$ is $\approx 10^{-2}$, and its average value is negative. In Figure 4.11, the observer with the gains $L_1$ and $N_1$ is compared to the alternative observer (4.31). The initial conditions for the state components $\dot{\theta}_u$ and $\dot{\theta}_l$ of the alternative observers are zero. Figure 4.11(b) and 4.11(d) show a small difference between the observed state components of the two different observers. The state component $\dot{\theta}_u$ of the alternative observer contains high frequencies, see Figure 4.11(b).

The state component $\hat{\alpha}$ of the observer with the gains $L_2$ and $N_2$ converges faster to the state component $\alpha$ of the experimental set-up than the state component $\hat{\alpha}$ of the observer with the gains $L_1$ and $N_1$, see Figures 4.12(a) and 4.12(b). The steady-state observer error $(\alpha - \hat{\alpha})$ is $\approx 10^{-4}$, see Figure 4.12(c), and the average value is negative, as we also noticed in the previous experiment. It means that $\hat{\alpha}$ is larger than $\alpha$. A possible cause can be that the modeled friction acting on the lower disc is higher than the actual friction. When the actual friction is increased by touching the lower disc by hand, the observer error $(\alpha - \hat{\alpha})$ becomes positive, see Figure 4.13. At approximately $t = 12 \, \text{s}$, the friction acting on the lower disc is not increased anymore and the observer error becomes negative again.

We notice a good resemblance between the experiments and the simulations with both observers for $u_c = 2.0 \, \text{V}$, when we compare the experimental responses $\hat{\alpha}$ and errors $(\alpha - \hat{\alpha})$ of the observers in Figures 4.10(a), 4.10(b), 4.12(a), 4.12(b) with the simulated responses and errors in Figure 4.6(a) and Figure 4.6(b).

4.6 Discussion

Simulations and experiments with the observer design (4.27) are performed with two pairs of gains: $L_1$ and $N_1$ (4.29), $L_2$ and $N_2$ (4.30). The simulations show convergence of the observed state to the
4.6. Discussion

Figure 4.10: Comparison of the measured state component \( \alpha \) with the state component \( \hat{\alpha} \) of the observer with the gains \( L_1 \) and \( N_1 \) for \( u_c = 2.0 \) V.

Figure 4.11: Comparison of the responses \( \hat{\omega}_u \) and \( \hat{\omega}_l \) of the observer with the gains \( L_1 \) and \( N_1 \) (black dashed line) and the responses \( \hat{\omega}_u \) and \( \hat{\omega}_l \) of the observer \( \frac{200 s}{s+200} \) (grey solid line) for \( u_c = 2.0 \) V.

rotor dynamic system state, see Section 4.4. The convergence velocity is higher for a higher observer gain. For the experiments we can only check the observer error with respect to the state component \( \alpha \), since we only measure the state component \( \alpha \). The conclusion is that the observer state component \( \alpha \) converges to the state component \( \alpha \). The convergence velocity of the observer with the gains \( L_2 \) and \( N_2 \) is higher than the observer with the gains \( L_1 \) and \( N_1 \). Unlike the simulations, the observer error \( (\alpha - \hat{\alpha}) \) is not the same for the two observers; the higher the gain, the smaller the observer error \( (\alpha - \hat{\alpha}) \) in steady-state.

For all the experiments, the average value of the observer error \( (\alpha - \hat{\alpha}) \) is negative which can be caused by model uncertainties. The experimental results match the simulation results if we compare \( \hat{\alpha} \). We prefer the observer with the gains \( L_2 \) and \( N_2 \), since the experimental results show fast convergence and
the smallest steady-state observer error \((\alpha - \hat{\alpha})\) in experiments compared to the other observer. The observer error with respect to \(\omega_u\) and \(\omega_l\) for the experiments can not be calculated, since the state components \(\omega_u\) and \(\omega_l\) are not measured. The obtained state components \(\hat{\omega}_u\) and \(\hat{\omega}_l\) of the two observers with the different pairs of gains are compared with observer state components of an alternative observer (4.31) and the differences between the observed state components \(\hat{\omega}_u\) and \(\hat{\omega}_l\) are small. We can conclude that the comparison of the two observer designs gives the indication that the observed state components \(\hat{\omega}_u\) and \(\hat{\omega}_l\) of the considered observer converge to the state components \(\omega_u\) and \(\omega_l\).

The observer (4.3) is also applied to the rotor dynamic system for the friction condition II, see Appendix J. We again apply two observers with different gains for this system. The simulations show convergence of the observer state to the rotor dynamic system state, see Appendix J.1, with a very small steady-state observer error. The experiments for the observer with the gains \(L_2\) and \(N_2\) are performed, see Appendix J.2. We only show the experiments for one observer, since we already performed a comparison between two observers for the rotor dynamic system for the nominal case. From these experiments, we can conclude that the observed state component \(\hat{\alpha}\) converges to the measured state component \(\alpha\).

We can not draw a conclusion for the observer error with respect to \(\omega_u\) and \(\omega_l\), because these state components are not measured in the experimental set-up. But the results of the applied output-feedback control design in Chapter 6 will show that the output-feedback controller is able to control the experimental drill-string set-up provided with the observed state components \(\hat{\alpha}, \hat{\omega}_u\) and \(\hat{\omega}_l\).
In this chapter, we will propose a linear state-feedback control design for a class of Lur’e-type systems: a linear part in the forward path with decoupled, and in our case set-valued, nonlinearities in the feedback loop. The design aim is to find a control gain for the linear state-feedback control law, which stabilizes the closed-loop system.

Two approaches towards such a control aim are presented in this chapter: a circle criterion design and a Popov criterion design. Both control designs aim at rendering the closed-loop system absolutely stable, i.e. the equilibrium of the closed-loop system is globally asymptotically stable as long as the set-valued nonlinearity belongs to the sector $[0, \infty)$.

We consider a particular case for which the circle criterion guarantees absolute stability for a system of the considered class; when the transfer function of the linear part of the system is strictly positive real (SPR) and the set-valued nonlinearity belongs to $[0, \infty)$, i.e. the set-valued nonlinearity in the feedback loop is passive. This approach is discussed in general in e.g. [16] and for control designs in [4]. The feasibility of the circle criterion design for systems with one nonlinearity in the feedback loop is discussed in [2] and [4]. Loop transformations can be applied to the system to ensure that the transformed set-valued nonlinearity belongs to the sector $[0, \infty)$; see [16].

The requirement of strict positive realness for the transfer function of the linear part of the system can be relaxed by a transformation with a dynamic multiplier. This results in a transformed linear system and a transformed set-valued nonlinearity. This is the approach of the Popov criterion. The closed-loop system is absolutely stable if the transformed linear system is SPR and the transformed set-valued nonlinearity is passive. The Popov criterion is discussed for system with continuous nonlinearities in e.g. [4], [13] and [16]. In this thesis, we present a proof of the application of the Popov criterion to systems with set-valued nonlinearities in the feedback loop. Note that the essential difficulty herein is the fact that Popov-inspired Lyapunov function becomes non-smooth.

Finally, we design a controller for the rotor dynamic system with two different friction characteristics (I and II). A proof is given for the infeasibility of the design of a controller using the circle criterion. Therefore, a controller is designed using the Popov criterion. Only simulations for the closed-loop system are performed, since we do not measure the entire state in the experimental set-up. The controller will be combined with the observer, presented in Chapter 4, in the next chapter to construct a dynamic output-feedback controller.
Consider the system to be controlled in the following Lur'e form, see Figure 5.1(a):

\[
\begin{align*}
\dot{x} &= Ax + Gw + Bu \\
z &= Hx \\
w_i &\in -\varphi_i(z_i), \quad i = 1, \ldots, p,
\end{align*}
\]  

(5.1)

where the system state \(x\in\mathbb{R}^n\), the input \(u\in\mathbb{R}^m\), \(w, z\in\mathbb{R}^p\) and \(\varphi_i : \mathbb{R} \to \mathbb{R}, i = 1 \ldots p\), are decoupled set-valued nonlinearities represented by upper semi-continuous, convex set-valued mappings with a closed image. The origin \(x = 0\) is the equilibrium point of system (5.1), this also implies that \(0 \in \varphi_i(0), i = 1, \ldots, p\).

In case of the presence of non-collocated friction, there exists at least one \(i (i = 1, \ldots, n)\) such that \(G_{ij} \neq 0\), for some \(j = 1, \ldots, p\), and \(B_{ij} = 0\) for all \(j = 1, \ldots, m\).

We propose the linear state-feedback control law

\[
u = Kx,
\]

(5.2)

where \(K \in \mathbb{R}^{m \times n}\) is the feedback gain matrix. Consequently, the resulting closed-loop system is described by the differential inclusion

\[
\begin{align*}
\dot{x} &= (A + BK)x + Gw \\
z &= Hx, \\
w_i &\in -\varphi_i(z_i), \quad i = 1, \ldots, p,
\end{align*}
\]  

(5.3a)

(5.3b)

where \(x \in \mathbb{R}^n\). We assume that \(((A + BK), G)\) is controllable and \(((A + BK), H)\) is observable. Moreover, the origin \(x = 0\) is the equilibrium point of system (5.3). The closed-loop model is depicted in Figure 5.1(b). The transfer function \(G_{cl}(s)\) of the linear part (5.3a) of the closed-loop system is:

\[
G_{cl}(s) = H(sI - (A + BK))^{-1}G.
\]

(5.4)

The control goal is to induce absolute stability for the closed-loop system (5.3) by designing the feedback matrix \(K\), i.e. the origin of the closed-loop system is globally asymptotically stable for any set-valued nonlinearity in the feedback loop belonging to the \([0, \infty]\) sector. Two design methods are presented in the next two sections to achieve absolute stability for the closed-loop system.
5.2 Control design based on the circle criterion

A particular case of the circle criterion states that the closed-loop system (5.3) is absolute stable if the transfer function of the linear system is strictly positive real and the set-valued nonlinearities are passive. The aim is to render the closed-loop system absolutely stable by finding a control gain for which the closed-loop system satisfies the circle criterion.

The requirement for passivity of the set-valued function $w \in -\varphi(z)$ is

$$w^T z \leq 0, \quad \text{for } w \in -\varphi(z). \quad (5.5)$$

This means that the set-valued nonlinearity $\varphi_i$, for $i = 1, \ldots, p$, is passive as long as it belongs to the sector $[0, \infty]$. If the set-valued nonlinearity is not passive, then it may be possible to render the set-valued nonlinearity passive by loop transformations. Also, it is possible to remove an "excess" of passivity by performing loop transformations. Loop transformations for rendering set-valued nonlinearities passive, are discussed in Appendix E.1. As a consequence, due to loop transformations, we obtain a transformed system with a transformed linear part in the forward path. The transformed linear time-invariant part may contain a transition matrix. The circle criterion approach for the design of a controller for the latter class of systems is discussed in Appendix E.2.

Next, we adopt the following quadratic Lyapunov candidate function to investigate the passivity properties of the linear system (5.3a) and the stability properties of the total closed-loop system (5.3):

$$V(x) = \frac{1}{2} x^T P x, \quad (5.6)$$

with $P = P^T > 0$. If we consider the Lyapunov function $V(x)$ as the storage function for the linear system (5.3a), then the following inequality must hold for strict passivity of the linear system:

$$\dot{V} < w^T z. \quad (5.7)$$

If, we fill in the derivative of the Lyapunov candidate function (5.6) and use the equation $z = Hx$ of (5.3a), we can write expression (5.7) as follows:

$$\frac{1}{2}(x^T [(A + BK)^T P + P(A + BK)] x + w^T G^T P x + x^T PGw) < \frac{1}{2}(w^T Hx + x^T H^T w). \quad (5.8)$$

The inequality (5.8) can be written in a matrix form as shown below:

$$\frac{1}{2} \begin{bmatrix} x & w \end{bmatrix}^T \begin{bmatrix} (A + BK)^T P + P(A + BK) & PG - H^T \\ G^T P - H & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0. \quad (5.9)$$

From (5.9) we can derive the condition for strict positive realness of the transfer function of the linear system (5.3a). More specifically, if the following matrix inequality $F_1(P, K)$ is satisfied

$$F_1(P, K) := \begin{bmatrix} (A + BK)^T P + P(A + BK) & PG - H^T \\ G^T P - H & 0 \end{bmatrix} < 0, \quad (5.10)$$

then the linear dynamic system (5.3a) is SPR.

Thus, if the conditions (5.5) and (5.10) are satisfied, then we can write (5.7) as $\dot{V} - w^T z < 0$ and the origin of the closed-loop system (5.3) is globally asymptotically stable since $\dot{V} < 0$. Moreover, the origin of the closed-loop system (5.3) is absolutely stable.
Robustness with respect to the set-valued nonlinearity can be obtained by rendering the closed-loop system absolutely stable. This is desired when the set-valued nonlinearity changes in time (in a quasi stationary fashion), while it still satisfies the sector $[0, \infty]$. Another reason for aiming at absolute stability is to control the closed-loop system for a range of set-points with the same control gain $K$. An example is the rotor dynamic system (3.3), where we give a constant input to obtain a desired equilibrium velocity of the upper disc and the lower disc. Besides this constant input, we may need an additional input to control the rotor dynamic system such that the discs rotate with the equilibrium velocity. The reason we need an input to this system is that the presence of friction induces unstable periodic solutions and coexistence of solutions (see the bifurcation diagrams of the rotor dynamic system (3.3) in Figure 3.12 and Figure 3.13). A single controller is desired, which is able to control the rotor dynamic system for a range of constants inputs.

It is possible to represent the rotor dynamic system for a constant input as a constant linear system (linear system does not change if the constant input changes) in the forward path and a set-valued nonlinearity in the feedback loop (changing with changing desired setpoint). Namely, the set-valued nonlinearity changes if the constant input changes. If we want to design one controller for all possible equilibria of the rotor dynamic system, then we have another requirement for the set-valued nonlinearity. This approach with the specific requirement is discussed in Appendix E.3.

### 5.3 Control design based on the Popov criterion

The circle criterion, presented in the previous section, requires strict passivity (or strict positive realness of the transfer function) for the linear system (5.3a), while the set-valued nonlinearity in the feedback path (5.3b) has to be passive. However the condition of strict positive realness for the transfer function of the linear part (5.3a) of the system can be relaxed by applying a loop transformation with a dynamic multiplier characterized by a linear transfer function $M(s)$. The linear subsystem can be post-multiplied by a dynamic multiplier, see Figure 5.3(a). The post-multiplication can be nullified by pre-multiplication of the set-valued nonlinearity in the feedback loop with the inverse of the dynamic multiplier. The idea of the loop transformation is to transform the original system into a feedback connection of two passive dynamical systems. This is the approach of the Popov criterion [16].

We consider once more the system to be controlled given by (5.3). The equilibrium point is the origin. The transfer function $G_{cl}(s)$ is given by (5.4). We assume that the discontinuity for $\varphi_i$ for $i = 1, \ldots, p$, occurs at $z_i = z_{i,e}$, see Figure 5.2 for an example. The work presented here can readily be employed in the case $\varphi(z)$ exhibits multiple points of discontinuity, where we assume that these points are separated and finite in number.
The transfer function of the dynamic multiplier $M(s)$ is given by
\[
M(s) = I + \Gamma s,
\] (5.11)
where $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_p) \in \mathbb{R}^{p \times p}$, with $\gamma_i > 0$ for $i = 1, \ldots, p$.

A requirement is that the inverse of $M(s)$ must be (strictly) passive, because the multiplication of the set-valued nonlinearity (5.3b) in the feedback loop with the inverse of the dynamic multiplier must yield a passive system.

The linear part of the system is post-multiplied with the dynamic multiplier which results in the output $\tilde{z}$ for the transformed linear system:
\[
\tilde{z} = z + \Gamma \dot{z} \\
= Hx + \Gamma \dot{x} \\
= Hx + \Gamma H((A + BK)x + Gw) \\
= (H + \Gamma H(A + BK))x + \Gamma HGw,
\] (5.12)
with $\tilde{z} \in \mathbb{R}^p$. The transformed closed-loop system can be represented as feedback interconnection of two dynamical systems: $H_1$ and $H_2$, see Figure 5.3(b):
\[
H_1 = \begin{cases} 
\dot{x} = (A + BK)x + Gw \\
\dot{z} = Hx + Dw,
\end{cases}
\] (5.13a)
\[
H_2 = \begin{cases} 
\dot{z}_i = -\frac{1}{\gamma_i} z_i + \frac{1}{\gamma_i} \hat{z}_i \\
- w_i \in \varphi_i(z_i) & i = 1, \ldots, p
\end{cases}
\] (5.13b)

with the definitions for the matrices $\hat{H}$ and $\hat{D}$ given by
\[
\hat{H} = H + \Gamma H(A + BK), \quad \hat{D} = \Gamma HG.
\] (5.14)

The aim is to find a gain $\Gamma$ for the dynamic multiplier and a control gain $K$ which results in a feedback interconnection of two passive dynamical systems. We will show that the dynamical systems, $H_1$ (5.13a) and $H_2$ (5.13b), are strictly passive and passive, respectively, if two conditions are satisfied. Then, we will investigate the stability of the origin of the closed-loop system (5.13).
Let us first consider the transformed linear system $H_1$ (5.13a) in the forward path of Figure 5.3(a). We adopt the following Lyapunov candidate function for the system $H_1$, which is also used as a storage function to derive a condition for strict passivity of $H_1$:

$$V_1(x) = \frac{1}{2}x^TPx,$$

with $P = P^T > 0$. We can derive the condition for strict passivity of the system $H_1$ in the same way as is done in Section 5.2:

$$\begin{bmatrix} x \\ w \end{bmatrix}^TF_2(P, K)\begin{bmatrix} x \\ w \end{bmatrix} = \dot{V}_1 - \ddot{z}^Tw < 0.$$  \hspace{1cm} (5.16)

Expression $F_2(P, K)$ can be written in a matrix inequality form. This matrix inequality has to be satisfied for strict passivity (or strict positive realness of the transfer function) of the transformed linear system $H_1$:

$$F_2(P, K) = \begin{bmatrix} (A + BK)^TP + P(A + BK) & PG - \tilde{H}^T \\ G^TP - \tilde{H} & -\tilde{D} - \tilde{D}^T \end{bmatrix} < 0.$$  \hspace{1cm} (5.17)

If the matrix inequality (5.17) holds, then the strict passivity requirement $\dot{V}_1 < \ddot{z}^Tw$ is satisfied, since (5.16) holds.

Let us now consider system $H_2$ (5.13b) in the feedback path of Figure 5.3(a). We adopt the following Lyapunov candidate function $V_2(z)$ for the system $H_2$:

$$V_2(z) = \sum_{i=1}^{p} V_{2,i}(z_i),$$

with

$$V_{2,i}(z_i) = \gamma_i \int_{0}^{z_i} \varphi_i(s)ds \quad \text{for} \quad i = 1, \ldots, p.$$  \hspace{1cm} (5.19)

This Lyapunov function will also be used as a storage function to prove passivity for the system $H_2$.

The functions $V_{2,i}(z_i)$ are continuous in $z_i$ but not differentiable for all $z_i$ due to the discontinuous nature of the mappings $\varphi_i(z_i)$ at $z_i = z_{i,e}$. Therefore, in order to calculate the derivative of $V_{2,i}(z_i)$ with respect to time for $z_i = z_{i,e}$ for some $i = 1, \ldots, p$, we use the concept of a subderivative or generalized directional derivative, which was introduced in [26].

Consider a continuous function $f(x)$. Then, the function

$$df(x)(v) = \lim_{h \rightarrow 0} \frac{f(x + hv) - f(x)}{h}$$

(5.20)

defines the subderivative of $f$ at $x$ in the direction of $v$.

As it was mentioned before, $V_{2,i}$, for $i = 1, \ldots, p$, is a continuous function in $z_i$ and differentiable for all $z_i$ but $z_i = z_{i,e}$. In order to overcome this difficulty, we use the subderivative of $V_2$ as it is defined in (5.20). Note that the subderivative of $V_2$ for all $z_i \neq z_{i,e}$ can be expressed as the normal Lie derivative of $V_2$ in the direction of $\dot{z}_i$, since $V_2(z_i)$ is differentiable with respect to $z_i$ as long as $z_i \neq z_{i,e}$. Therefore, for all $z_i \neq z_{i,e}$, the subderivative of $V_{2,i}$ can be expressed as:

$$dV_{2,i}(z_i)(\dot{z}_i) = \dot{V}_{2,i}(z_i) = \partial V_{2,i}(z_i) \dot{z}_i = \gamma_i \varphi_i(z_i) \dot{z}_i = \gamma_i \varphi_i(z_i)(-\frac{1}{\gamma_i}z_i + \frac{1}{\gamma_i}z_{\dot{i}}) = w_i z_i - w_i \tilde{z}_i \quad \forall z_i \neq z_{i,e}.$$  \hspace{1cm} (5.21)
The subderivative of $V_2_i$ in the direction of $\dot{z}_i$ for $z_i = z_{i,e}$ is evaluated:

$$dV_2_i(z_{i,e})(\dot{z}_i) = \lim_{h \to 0} \frac{V_2_i(z_{i,e} + \dot{z}_i h) - V_2_i(z_{i,e})}{h}$$

$$= \lim_{h \to 0} \frac{\gamma_i \int_0^{z_{i,e} + \dot{z}_i h} \phi_i(s)ds - \gamma_i \int_0^{z_{i,e}} \phi_i(s)ds}{h}$$

$$= \lim_{h \to 0} \frac{\gamma_i \int_0^{\dot{z}_i h} \phi_i(s + z_{i,e})ds}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\gamma_i}{h} \int_0^{\dot{z}_i h} \phi_i(s + z_{i,e})ds}{h}$$

$$= \lim_{h \to 0} \frac{\partial}{\partial h} \left( \gamma_i \int_0^{\dot{z}_i h} \phi_i(s + z_{i,e})ds \right)$$

with $s^* = s - z_{i,e}$ and $ds^* = ds$. (5.22)

At this point we are going to examine the solution of $dV_2_i$ for $z_i = z_{i,e}$ more closely. $dV_2_i(z_{i,e})(\dot{z}_i)$ depends on $\phi_i(z_{i,e}) \in [\phi_i(z_{i,e}^-), \phi_i(z_{i,e}^+)]$ and $\dot{z}_i$. We will investigate these variables by investigating the solution $x(t)$ of (5.13) at $t = t^*$, where $t^*$ is the time instant that the system resides at the discontinuity, i.e. at $z_i = z_{i,e}$.

From a theoretical point of view, the vector field of system (5.13) can behave in six ways on the discontinuity $z_i = z_{i,e}$. These are schematically depicted in Figure 5.4. In this figure, $f_i^+$ and $f_i^-$ are the right-hand sides of the vector field of (5.13) for $z_i \downarrow z_{i,e}$ and $z_i \uparrow z_{i,e}$, respectively. For more details the reader is referred to [18].

![Figure 5.4: The vector field of the system (5.1) on the discontinuity.](image-url)

Let us examine first the case where $z_i(t^*) = z_{i,e}$ and $\dot{z}_i(t^*) > 0$ (this implies $\dot{z}(t^*) > z_{i,e}$). In [7], it was shown that systems in the form of (5.1) with maximal monotone mappings cannot exhibit repulsive sliding modes. For the considered case this means that the situation $A$ in Figure 5.4 cannot occur. Furthermore, the situations $B, C, D$ and $F$ cannot occur by definition since $\dot{z}_i(t^*) > 0$ i.e. the solution will leave the surface $z_i = z_{i,e}$ immediately after $t = t^*$. Consequently, when $z_i(t^*) = z_{i,e}$ and $\dot{z}_i(t^*) > 0$, then the solution is on a transversal intersection (situation $E$ in Figure 5.4). This implies that the specific value of $w_i \in [-\phi_i(z_{i,e}^-), \phi_i(z_{i,e}^+)]$ in (5.13b) does not affect the solution $x(t)$ since the solution resides on the discontinuity only for a singleton. Hence, the value of $w_i(t^*)$ does not affect $dV_2_i(z_{i,e})(\dot{z}_i)$ in this case. Using similar reasoning for the case that $z_i(t^*) = z_{i,e}$ and $\dot{z}_i(t^*) < 0$ (this implies $\dot{z}_i(t^*) < z_{i,e}$) we derive similar results (where a transversal intersection in situation $F$ occurs). Let examine finally the case where $z_i(t^*) = z_{i,e}$ and $\dot{z}_i(t^*) = 0$. In this case, it is obvious that $dV_2_i(z_{i,e})(\dot{z}_i) = 0$, see (5.22). The aforementioned analysis learns that for $z_i = z_{i,e}$ the specific value of $w_i$, with $w_i \in -\phi_i(z_{i,e}^+)$, is of no concern and can be chosen arbitrarily for all $\dot{z}_i$. 
The latter fact can be used to rewrite (5.22) as follows:

\[ dV_{2,i}(z_i)(\dot{z}_i) = \begin{cases} 
-\gamma_1 w_i \dot{z}_i & \text{if } \dot{z}_i > z_{i,e} \\
-\gamma_1 w_i \dot{z}_i = 0 & \text{if } \dot{z}_i = z_{i,e} \\
-\gamma_1 w_i \dot{z}_i & \text{if } \dot{z}_i < z_{i,e}.
\end{cases} \]

\[ = \begin{cases} 
-w_i (\dot{z}_i - z_{i,e}) & \text{if } \dot{z}_i > z_{i,e} \\
-w_i (\dot{z}_i - z_{i,e}) & \text{if } \dot{z}_i = z_{i,e} \\
-w_i (\dot{z}_i - z_{i,e}) & \text{if } \dot{z}_i < z_{i,e}.
\end{cases} \]

\[ = w_i z_{i,e} - w_i \dot{z}_i \quad \text{for } w_i \in -\varphi_1(z_{i,e}). \tag{5.23} \]

Note that the expression for \( dV_2(z_i)(\dot{z}_i) \) for \( z_i \neq z_{i,e} \), in (5.21), and for \( z_i = z_{i,e} \), in (5.23), conform. Therefore, we can write

\[ dV_2(z) = \sum_{i=1}^{p} dV_{2,i}(z_i)(\dot{z}_i) = \sum_{i=1}^{p} w_i z_i - w_i \dot{z}_i = w^T z - w^T \dot{z}, \quad \forall z. \tag{5.24} \]

If system \( H_2 \) is passive then \( dV(z)(\dot{z}) \leq -w^T \dot{z} \) must hold. Using (5.24), this inequality can be written as:

\[ dV_2(z)(\dot{z}) = w^T z - w^T \dot{z} \leq -w^T \dot{z}. \tag{5.25} \]

For passivity of system \( H_2 \), the following condition must hold to satisfy requirement (5.25):

\[ w^T z \leq 0. \tag{5.26} \]

This condition implies passivity of \( \varphi_i \), for \( i = 1, \ldots, p \). If \( \varphi_i \) does not belong to the sector \([0, \infty]\) or there is an "excess" of passivity, loop transformations can be performed to the set-valued nonlinearities and to the linear system. Such loop transformations are described in Appendix E.1.

Now, we investigate the stability of the closed-loop system (5.13). We adopt the following Lyapunov candidate function, which is a combination of the Lyapunov functions for the two systems \( H_1 \) and \( H_2 \):

\[ V(x, z) = V_1(x) + \sum_{i=1}^{p} V_{2,i}(z_i). \tag{5.27} \]

The time-derivative of \( V(x, z) \) is given by

\[ \dot{V} = \dot{V}_1 + \sum_{i=1}^{p} dV_{2,i}(z_i)(\dot{z}_i) = \begin{bmatrix} x \\ w \end{bmatrix}^T F_2(P, K) \begin{bmatrix} x \\ w \end{bmatrix} + \dot{z}^T w + dV_2(z)(\dot{z}) \]

\[ = \begin{bmatrix} x \\ w \end{bmatrix}^T F_2(P, K) \begin{bmatrix} x \\ w \end{bmatrix} + \dot{z}^T w + z^T w - \dot{z}^T w \]

\[ = \begin{bmatrix} x \\ w \end{bmatrix}^T F_2(P, K) \begin{bmatrix} x \\ w \end{bmatrix} + z^T w. \tag{5.28} \]

The condition for \( F_2(P, K) \) is given by (5.17), and the condition for \( z^T w \) is given by (5.26). If these two conditions hold, then \( \dot{V} < 0 \) and the origin of the closed-loop system (5.3) is globally asymptotically stable for any set-valued nonlinearity belonging to the sector \([0, \infty]\). Moreover, the closed-loop system (5.3) is said to be absolutely stable.

If we want to control system (5.3) with one linear state-feedback control law for a range of equilibria, then we can apply the approach that is presented in Appendix E.3. Monotonicity is required for the set-valued nonlinearity and \( 0 \in \varphi_i(0) \) for \( i = 1, \ldots, p \). If the set-valued nonlinearity is not monotone, a loop transformation may be possible to render the transformed set-valued nonlinearity monotone.
5.4 Control design based on the circle criterion for the rotor dynamic system

We want to design a controller based on the circle criterion for the rotor dynamic system (3.3) for the nominal case (friction characteristic I). For the application of a controller based on the circle criterion, the rotor dynamic system (3.3) needs to be written as a Lur’e-type system with the origin as an equilibrium. The transformation of the rotor dynamic system (3.3) to the rotor dynamic system (F.7) in Lur’e-type form is described in Appendix F.1. The system (F.7) has two set-valued nonlinearities in the feedback loop. We want to design a controller based on the circle criterion for this system. However, we are not able to obtain a solution for the control gain with the available numerical tool (more information about the numerical tool will be given in the next section).

Feasibility checks are available for the application of a controller based on the circle criterion to Lur’e-type systems with one linearity in the feedback loop, see [2] and [4]. A transformation of the rotor dynamic system (F.7), with two set-valued nonlinearities in the feedback loop, is needed to be able to check the feasibility conditions for the application of a circle criterion based controller. Since the input is applied to the upper disc, we will cancel partially the friction acting at the upper disc with an appropriate friction compensation design to obtain a transformed model with just one set-valued friction model. This transformation is described in Appendix F.2. The rotor dynamic system with one set-valued nonlinearity in the feedback loop is given by (F.14).

The aim is to find a controller for which the closed-loop system satisfies the circle criterion. It is required for the approach of the circle criterion that the set-valued nonlinearity must be monotone and globally asymptotically stable for all equilibria of the closed-loop rotor dynamic system. This means that the set-valued nonlinearity must be monotone and 0 ∈ φ(0) as we demanded in Appendix E.3.

The current set-valued nonlinearity T_{fl, tr2} (F.11) of rotor dynamic system (F.14) is not monotone, which can be seen in Figure F.7(b). The requirement 0 ∈ φ(0) holds for the set-valued nonlinearity of the rotor dynamic system (F.14). To render the set-valued nonlinearity of the rotor dynamic system (F.14) monotone, we apply a loop transformation. This loop transformation can be compared to the loop transformation applied to the rotor dynamic system in Section 4.3. The current loop transformation is described in Appendix F.3. The transformed rotor dynamic system is given by:

\[
\begin{align*}
\dot{\xi} &= A_{\text{tr}2}\xi + Bv_{\text{control}} + G_{\text{tr}}\tilde{w} \\
z &= H_{\text{tr}}\xi \\
\tilde{w} &\in -\varphi_{\text{tr}}(z)
\end{align*}
\] (5.29)

with the new input \(v_{\text{control}} \in \mathbb{R}, \tilde{w}, z \in \mathbb{R}, \varphi_{\text{tr}} : \mathbb{R} \rightarrow \mathbb{R}\), and where

\[
\varphi_{\text{tr}}(z) = T_{fl, tr3}(z).
\] (5.30)

The set-valued nonlinearity \(T_{fl, tr3}\) is given by

\[
T_{fl, tr3}(z) = T_{fl, tr2}(z) + mz,
\] (5.31)

where \(m = 0.1\) Nms/rad and \(T_{fl, tr2}\) described by (F.11). The minimum linear damping needed to render the transformed friction map \(T_{fl, tr3}\) monotone (i.e. the derivative of \(T_{fl, tr3}\) with respect to \(\xi_3\) is larger or equal to zero), is less than the chosen value for \(m\). If the friction map \(T_{fl, tr2}\) changes slightly due to some conditions (temperature, humidity etc.), then the value for \(m\) would still be able to render the transformed friction map \(T_{fl, tr3}\) monotone. The set-valued nonlinearity \(T_{fl, tr3}\), which is monotone, is plotted in Figure F.10.
The system matrices in (5.29) are given by

\[
A_{tr2} = \begin{bmatrix}
0 & -\frac{\Delta b_u}{J_u} & 1 & -1 \\
-\frac{1}{J_u} & 0 & \frac{1}{J_u} (b_u - \Delta b_u) & 0 \\
0 & 0 & \frac{m}{J_l} & \frac{k_2}{J_l} \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
\frac{k_2}{J_u} \\
0 \\
\end{bmatrix}, \quad G_{tr} = \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix}, \quad H_{tr}^T = \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix}.
\] (5.32)

We consider the nominal case (friction characteristic I) in this section. The corresponding parameters for the rotor dynamic system (5.29) for the nominal case can be found in Table 3.2. The transformed system (5.29) (after application of the input that compensation partially the friction acting at the upper disc) is different from the rotor dynamic system (3.3), since we consider a different input. Nevertheless, the steady-state behaviour of system (5.29) does not change qualitatively with respect to the original system, but only quantitatively. The bifurcation diagrams for the transformed system (5.29), with the constant input together with the compensation input as the bifurcation parameter, are shown in Appendix F.3.

We propose for the rotor dynamic system (5.29) the following linear state-feedback law:

\[
v_{\text{control}} = K\xi,
\] (5.33)

with the feedback gain \(K = [k_1 \quad k_2 \quad k_3]\).

Now, we can apply the feasibility conditions, discussed in [2] and [4], to the rotor dynamic system (5.29), which is discussed in Appendix H.1. The design of a controller based on the circle criterion is not feasible for the rotor dynamic system (5.29). The absence of a damping element in the link between the upper disc and the lower disc is a cause for the infeasibility.

### 5.5 Control design based on the Popov criterion for the rotor dynamic system

In the previous section, we showed that the application of a controller based on the circle criterion to the rotor dynamic system (5.29) is infeasible and we did not obtain a numerical solution for the application of a controller based on the circle criterion to rotor dynamic system (F.7). Therefore, we will continue with the design of the linear state-feedback controller (5.33) using the Popov criterion. We are not able to obtain a controller based on the Popov criterion for the rotor dynamic system given by (F.7) (a system with two set-valued nonlinearities in the feedback loop) with the available numerical tool. Therefore, we continue with the rotor dynamic system (5.29), where the friction acting at the upper disc is partly compensated.

If we apply the control law (5.33), then the closed-loop system becomes

\[
\begin{align*}
\dot{\xi} &= (A_{tr2} + BK)\xi + G_{tr}\tilde{w} \\
z &= H_{tr}\xi \\
\tilde{w} &\in -\varphi_{tr}(z).
\end{align*}
\] (5.34)

The transfer function \(G_{cl}(s)\) of the linear part of the closed-loop system is

\[
G_{cl}(s) = H_{tr}(sI - (A_{tr2} + BK))^{-1}G_{tr}
\] (5.35)

The transformation, introduced in Section 5.3, is depicted in Figure 5.5(a).
5.5. Control design based on the Popov criterion for the rotor dynamic system

(a) Popov transformation of the rotor dynamic system.

(b) Transformed rotor dynamic system after transformation with dynamic multiplier $M(s)$.

Figure 5.5: Transformation of the rotor dynamic system.

The transfer function of the dynamic multiplier is chosen as:

$$M(s) = 1 + \gamma s.$$  \hspace{2cm} (5.36)

Consequently, the transformed system becomes, see Figure 5.5(b)

$$H_1 = \left\{ \begin{array}{l}
\dot{\xi} = (A_{tr2} + BK)\xi + G_{tr}\tilde{w} \\
\tilde{z} = \tilde{H}\xi + \tilde{D}\tilde{w},
\end{array} \right. \hspace{2cm} \text{and} \hspace{2cm} (5.37a)$$

$$H_2 = \left\{ \begin{array}{l}
\dot{z} = -\frac{1}{\gamma}z + \frac{1}{\gamma}\tilde{z} \\
\tilde{w} \in -\varphi_{tr}(z),
\end{array} \right. \hspace{2cm} \text{where the matrices } \tilde{H} \text{ and } \tilde{D} \text{ are given by}$$

$$\tilde{H} = H_{tr} + \gamma H_{tr} (A_{tr2} + BK) = \left[ \begin{array}{cc} \gamma \frac{k_a}{J} & 0 \\
1 + \gamma \frac{m}{J} & 1
\end{array} \right], \hspace{2cm} \tilde{D} = \gamma H_{tr} G_{tr} = \left[ \begin{array}{c} \frac{\gamma}{J} \\
\gamma s
\end{array} \right]. \hspace{2cm} (5.38)$$

The transfer function from $\tilde{w}$ to $\tilde{z}$ of the linear system $H_1$ described by (5.37a) is given by:

$$H_1(s) = G_{cl}(s)M(s) = \tilde{H}(sI - (A_{tr2} + BK))^{-1}G_{tr} + \tilde{D}. \hspace{2cm} (5.39)$$

The system (5.37) is schematically depicted in Figure 5.5(b). To prove stability using the Popov criterion approach, two conditions have to be satisfied; as it is stated in Section 5.3. For the transformed rotor dynamic system (5.37), these conditions are:

$$F_2(P, K, \gamma) = \begin{bmatrix}
(A_{tr2} + BK)^T P + P(A_{tr2} + BK) & PG_{tr} - \tilde{H}^T \\
G_{tr}^T P - \tilde{H} & -\tilde{D} - \tilde{D}^T
\end{bmatrix} < 0 \hspace{2cm} (5.40)$$

for $P = P^T > 0$ and

$$\tilde{w}^T z = -\varphi_{tr}^T z \leq 0, \hspace{2cm} \forall z. \hspace{2cm} (5.41)$$

The second condition (5.41), which implies passivity of the set-valued nonlinearity $\varphi_{tr}$, is satisfied. The first condition (5.40) is a matrix inequality which has to be solved for $K$, $\gamma$ and $P$. As mentioned, $K$ is the control gain of (5.33), $\gamma$ is the constant of the dynamic multiplier in (5.36) and $P$ is the square matrix of the Lyapunov function $V_1(\xi) = \frac{1}{2}\xi^T P \xi$ (analagous to (5.15)).
A numerical solver for LMIs is used (see [9, 8]), in order to solve the matrix inequality (5.40) for a given K and γ. The used algorithm by the LMI solving tool is *sdpic* [25]. Note that this LMI-toolbox can only solve semi-definite LMIs. Therefore, we have to adapt matrix inequality (5.40) as follows:

\[
F_3(P, Q, K, \gamma) = \begin{bmatrix}
(\text{tr} + BK)^T P + P(\text{tr} + BK) + Q & \text{tr} + BK \\
\text{tr} + BK & -D - \tilde{D}^T
\end{bmatrix} \leq 0,
\]

with \( P = P^T > 0 \) and \( Q = Q^T > 0 \).

We solve matrix inequality (5.42) for a chosen K and γ with the LMI-solver for two situations with the system parameters of the nominal case, see Table 3.2. The performance of the two controllers are compared by performing simulations for the closed-loop systems.

A first solution that satisfies condition (5.42) is:

\[
K_1 = \begin{bmatrix}
15.9 & 1.57 & 27.6
\end{bmatrix}, \quad \gamma_1 = 10,
\]

\[
P_1 = \begin{bmatrix}
3.6243 & 0.4311 & 6.3725 \\
0.4311 & 0.0702 & 0.7414 \\
6.3725 & 0.7414 & 11.6411
\end{bmatrix}, \quad Q_1 = \begin{bmatrix}
16.6997 & 2.2968 & 29.1096 \\
2.2968 & 0.3522 & 3.9707 \\
29.1096 & 3.9707 & 50.8145
\end{bmatrix}.
\]

The eigenvalues of \( P_1 \) are 15.2, 0.108, 0.0175; the eigenvalues of \( Q_1 \) are: 67.8, 0.0508, 0.0076.

A second solution of the matrix inequality (5.42) is given below:

\[
K_2 = \begin{bmatrix}
23.5 & 2.07 & 46.3
\end{bmatrix}, \quad \gamma_2 = 10,
\]

\[
P_2 = \begin{bmatrix}
3.2436 & 0.3196 & 6.4400 \\
0.3196 & 0.0442 & 0.6234 \\
6.4400 & 0.6234 & 13.3737
\end{bmatrix}, \quad Q_2 = \begin{bmatrix}
19.5373 & 2.2389 & 38.5324 \\
2.2389 & 0.2881 & 4.3878 \\
38.5324 & 4.3878 & 76.0788
\end{bmatrix}.
\]

The eigenvalues of \( P_2 \) are 16.5, 0.118, 0.0122; the eigenvalues of \( Q_2 \) are: 95.9, 0.0423, 0.0089.

The SPR conditions for both solutions are checked in Appendix H.2.

Simulations of the closed-loop systems with the control gains \( K_1 \) and \( K_2 \), respectively, are performed and discussed in Appendix H.3. These simulations confirm that the equilibria of the rotor dynamic system with both controllers are globally asymptotically stable.

### 5.6 Discussion

The rotor dynamic system is transformed into a linear system with one set-valued nonlinearity in the feedback loop by applying a control law that cancels the friction acting on the upper disc. We proved that it is not feasible to design a linear state-feedback control law using the circle criterion. Therefore, we designed two linear state-feedback controllers using the Popov criterion approach. We use the controllers for the rotor dynamic system with friction characteristics I and II. Several simulations are performed for the closed-loop systems with the two control gains, \( K_1 \) and \( K_2 \) in Appendix H.3. These simulations cannot be directly verified by experiments, because the entire state is not measured in the experiments. The closed-loop systems for both control gains converge to the equilibrium values.
Output-feedback control design

In the previous chapters, we presented an observer design and two control designs. Now, we will combine the observer design together with the control designs to construct output-feedback control designs. With these observer/controller designs, we would be able to control a system for which not the entire state is measured. Due to the fact that the separation principle does not hold for general nonlinear systems, we have to prove that the interconnected system that consists of the observer/controller combination and the considered Lur’e-type system, is stable. The fact that the observer error converges globally exponentially to zero does not guarantee that the interconnected system state converges to its equilibrium. Namely, the output-feedback implementation may lead to finite escape time before the observer estimates converge. The observer error can be considered as an external input to the controlled system which will vanish after some time. In [27], results are given which states that if the external input goes to zero, then the boundedness of the state of the controlled system (together with its global asymptotic stability for zero observer error) is enough for convergence of the state to its equilibrium. We will use this theory to prove that the equilibrium point of the interconnected system is globally asymptotically stable. First, we discuss the output-feedback controller with the controller based on the circle criterion. Next, we prove the stability of the output-feedback controller with the controller based on the Popov criterion. For the stability of the latter design, we require that the nonlinearity in the feedback loop of the system to be controlled (in the Lur’e type form) can be bounded by a linear function.

The output-feedback design is applied to the rotor dynamic system. We show simulations for the output-feedback control scheme to illustrate its effectiveness. Then, the output-feedback control is implemented on the experimental set-up for both the case of friction characteristic I and friction characteristic II. The results of the experiments are discussed and we show that the stabilization of the desired equilibria can be achieved experimentally using the proposed output-feedback controller.

6.1 Output-feedback control design for Lur’e-type systems with set-valued nonlinearities

Consider the Lur’e-type system given by (5.1) with $p = 1$, such that $\varphi : \mathbb{R} \to \mathbb{R}$ is a scalar set-valued nonlinearity. The presented work in this section can be extended to Lur’e type systems with multiple set-valued nonlinearities in the feedback loop.
We propose the following observer-based output-feedback control law:

\[ u = K \hat{x}, \tag{6.1} \]

with \( K^T \in \mathbb{R}^n \) and the observer (which is presented in Chapter 4) given by

\[
\begin{align*}
\dot{x} &= Ax + L(y - C\hat{x}) + G\hat{w} + Bu \\
\dot{\hat{x}} &= H\hat{x} + N(y - C\hat{x}) \\
\hat{w} &\in -\varphi(\hat{z}),
\end{align*}
\tag{6.2}
\]

where \( \hat{x} \in \mathbb{R}^n \) is the observed state and \( L \in \mathbb{R}^{n \times l}, N \in \mathbb{R}^{1 \times l} \) are observer gain matrices.

The observer error is defined as \( e = x - \hat{x} \). In Chapter 4, we conclude that the origin of the observer error dynamics is globally asymptotically stable if there exist matrices \( P_o = P_o^T > 0 \), \( L, N \) and a positive constant \( \varepsilon > 0 \), such that

\[
F(P_o) = \begin{bmatrix} (A - LC)^T P_o + P_o(A - LC) + \varepsilon I & P_o G - (H - NC)^T P_o G \\ G^T P_o - (H - NC)^T & 0 \end{bmatrix} \leq 0,
\tag{6.3}
\]

and the set-valued nonlinearity \( \varphi(z) \) is monotone, i.e.

\[
(a - b)[\varphi(a) - \varphi(b)] \geq 0, \quad \forall \quad a, b \in \mathbb{R}.
\tag{6.4}
\]

The closed-loop system, after the application of control law (6.1), is given by

\[
\begin{align*}
\dot{x} &= (A + BK)x + Gw - BKe \\
z &= Hx, \\
w &\in -\varphi(z),
\end{align*}
\tag{6.5a}
\]

\[
\begin{align*}
\dot{\hat{e}} &= (A - LC)e + Gk \\
q &= (H - NC)e \\
k &\in -\phi(t, q),
\end{align*}
\tag{6.5b}
\]

with \( k, q \in \mathbb{R} \) and where we consider the observer error \( e \) as an input to system (6.5). The closed-loop system (6.5) is depicted in Figure 6.1. The set-valued nonlinearity \( \phi \) is defined as:

\[
\phi(t, q) := \varphi(z) - \varphi(\hat{z}).
\tag{6.6}
\]

In the next sections, we will prove that the equilibrium point \((x, e) = (0, 0)\) for system (6.5) is globally asymptotically stable under specific conditions and for a suitable control gain \( K \). In the proposed approach, we will utilize the theory presented in [27].
The adopted reasoning is as follows: the equilibrium point \( (x, e) = (0, 0) \) of system (6.5) is globally asymptotically stable if

1. the origin \( e = 0 \) of the observer error dynamics is exponentially stable,
2. the origin \( x = 0 \) of system (6.5) is globally asymptotically stable for the observer error input \( e = 0 \),
3. and the state \( x(t) \) of system (6.5) is bounded for any bounded observer error input \( e(t) \).

In Section 6.2, a controller is presented based on the circle criterion. The origin \( x = 0 \) of system (6.5) is globally asymptotically stable for observer error input \( e = 0 \) if the linear part (6.5a) of system (6.5) is strictly passive and the set-valued nonlinearity in the feedback (6.5b) belongs to the sector \([0, \infty]\). Moreover, it is proven that the state \( x(t) \) of system (6.5) is bounded for any bounded observer error input \( e(t) \).

In Section 6.3, it is shown that the requirement of strict positive realness for the transfer function of the linear system (6.5a) of system (6.5) can be relaxed by a loop transformation with a dynamic multiplier (i.e. with a Popov criterion approach). The origin \( x = 0 \) of system (6.5) is globally asymptotically stable for observer error input \( e = 0 \) if the transformed linear part is strictly passive and the dynamical system in the feedback loop is passive. Furthermore, we prove that the state \( x(t) \) of system (6.5) is bounded for any bounded observer error input \( e(t) \) if the set-valued nonlinearity (6.5b) in the feedback loop of system (6.5) can be bounded by a linear function.

### 6.2 Control design based on the circle criterion

First, we want to use the circle criterion to prove absolute stability for system (6.5), with the input \( e = 0 \). To investigate the stability properties of the origin of the system (6.5) we adopt the following quadratic Lyapunov candidate function

\[
V(x) = \frac{1}{2} x^T P_c x,
\]

with \( P_c = P_c^T > 0 \). The derivative of the Lyapunov candidate function (6.7) is:

\[
\dot{V} = \frac{1}{2} x^T P_c x + \frac{1}{2} \dot{x}^T P_c \dot{x} = \frac{1}{2} x^T \left[ (A + BK)^T P_c + P_c (A + BK) \right] x + w^T P_c x + x^T P_c G w - e^T K^T B^T P_c x - x^T P_c B K e].
\]

If we desire strict passivity for the linear system (6.5a) with the input \( e = 0 \), then the following inequality must hold for strict passivity of linear system (6.5a):

\[
\dot{V} < wz.
\]

If we use the equation \( z = H x \) of (6.5a), we can write inequality (6.9) as:

\[
\dot{V} < \frac{1}{2} wz + \frac{1}{2} \dot{x}^T H w = \frac{1}{2} (w H x + x^T H^T w).
\]

If we combine (6.8) and (6.10), then we derive the following inequality

\[
\frac{1}{2} x^T \left[ (A + BK)^T P_c + P_c (A + BK) \right] x + w^T P_c x + x^T P_c G w - w H x - x^T H^T w - e^T K^T B^T P_c x - x^T P_c B K e] < 0.
\]
Inequality (6.11) can be written as:

\[
\frac{1}{2} \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} (A + BK)^T P_c + P_c(A + BK) & P_c G - H^T \\ G^T P_c - H \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} - \frac{1}{2} e^T K^T B^T P_c x - \frac{1}{2} x^T P_c B K e < 0.
\]

We can derive a matrix inequality for the strict positive realness of the transfer function of the linear system with input \( e = 0 \) from the above inequality:

\[
F_1(P_c, K) := \begin{bmatrix} (A + BK)^T P_c + P_c(A + BK) & P_c G - H^T \\ G^T P_c - H \end{bmatrix} < 0,
\]

with \( P_c = P_c^T > 0 \). From (6.9) and (6.13), we can conclude that if

\[
w z \leq 0,
\]

then the derivative of the Lyapunov function (6.7) \( \dot{V} < 0 \) and the equilibrium point \( x = 0 \) of system (6.5) is globally asymptotically stable for \( e = 0 \).

Now, we will show that the state \( x(t) \) of system (6.5) is bounded for bounded observer error \( e(t) \). Hereto, we rewrite \( \dot{V} \), as in (6.8) as:

\[
\dot{V} = \frac{1}{2} \begin{bmatrix} x \\ w \end{bmatrix}^T F_1(P_c, K) \begin{bmatrix} x \\ w \end{bmatrix} - \frac{1}{2} e^T K^T B^T P_c x + \frac{1}{2} x^T P_c B K e + z w.
\]

Using inequalities (6.13), (6.15) we obtain

\[
\dot{V} < -\frac{1}{2} e^T K^T B^T P_c x - \frac{1}{2} x^T P_c B K e.
\]

Let us now show that the right-hand side of (6.16), which is due to the non-zero observer error \( e \), satisfies

\[
-\frac{1}{2} e^T K^T B^T P_c x - \frac{1}{2} x^T P_c B K e \leq |e| V,
\]

for \( |x| \geq R \), for some \( R > 0 \). Note hereto, that the right-hand side of (6.16) satisfies

\[
-\frac{1}{2} e^T K^T B^T P_c x - \frac{1}{2} x^T P_c B K e = -x^T P_c B K e \leq | - x^T P_c B K ||e| \leq |e| V.
\]

Let us now show that

\[
| - x^T P_c B K ||e| \leq |e| V,
\]

for \( x \geq R \) for some \( R > 0 \). The left-hand side of (6.19) can be rewritten as

\[
| - x^T P_c B K ||e| \leq |x^T P_c B K ||e| \leq |x||P_c B K||e| = \alpha |x||e|,
\]

with

\[
\alpha = |P_c B K|.
\]
In order to satisfy (6.19) we need to show that

$$\alpha|x| \leq x^TP_c x \quad \text{for} \quad |x| \geq R \quad \text{and some} \quad R > 0. \quad (6.22)$$

Consider the norm $|x|_{P_c} = \sqrt{x^TP_c x}$. Based on this norm, the right hand side of (6.22) takes the form:

$$x^TPx = |x|_{P_c}^2 \geq \beta|x|^2 \quad \text{for} \quad \beta = \lambda_{\min}(P_c) > 0. \quad (6.23)$$

Therefore, the requirement (6.22) can be rewritten as the more strict requirement

$$\alpha|x| \leq \beta|x|^2 \quad \text{for} \quad |x| \geq R \quad \text{and some} \quad R > 0, \quad (6.24)$$

where $\beta = \lambda_{\min}(P_c)$. It is easy to show that there always exists an $R > 0$ such that (6.24) is satisfied for $|x| \geq R$. For example, if we solve analytically the equality derived from (6.24) we can provide an exact value for $R$:

$$-\beta|x|^2 + \alpha|x| = 0$$

$$\Rightarrow |x| = \frac{\alpha}{\beta} \quad \text{or} \quad |x| = 0. \quad (6.25)$$

Therefore, we can choose $R = \frac{\alpha}{\beta}$.

If inequalities (6.13) and (6.14) hold, then inequality (6.17) holds and we can conclude that

$$\dot{V} < |e|V \quad \text{for} \quad |x| \geq R, \text{and some} \quad R > 0. \quad (6.26)$$

At this point, we will show that $V(x(t))$ is bounded, hence $x(t)$ is bounded and that closed-loop system (6.5) is stable. We let $T < t_f$, where $[0, t_f)$ is the maximal interval of existence of the closed-loop system. Moreover, based on (6.3) it holds that there exist $\kappa, \delta > 0$ such that $|e(t)| \leq \kappa|e(0)|e^{-\delta t}$ for all $t \in [0, T]$. From (6.26), if $|x| \geq R$, then $\dot{V} < |e|V$, that is

$$V(x(t)) \leq V(x(0)) \exp \left( \int_0^T |e(s)| ds \right)$$

$$\leq V(x(0)) \exp \left( \int_0^T |e(s)| ds \right)$$

$$= V(x(0)) \exp \left( \int_0^T k|e(0)| \exp (-\delta s) ds \right)$$

$$= V(x(0)) \exp \left( \frac{k|e(0)|}{\delta} (\exp (-\delta T) - 1) \right)$$

$$= V(x(0)) \exp \left( \frac{k|e(0)|}{\delta} (1 - \exp (-\delta T)) \right). \quad (6.27)$$

Consequently, $V(x(t))$ is bounded for any bounded $e(0)$; hence, the solutions $x(t)$ of the closed-loop system are bounded since $V(x)$ is a positive definite function. Clearly, the maximal interval of existence is $[0, \infty)$. Therefore, it holds that

$$V(x(t)) \leq \lim_{T \to \infty} V(x(0)) \exp \left( \frac{k|e(0)|}{\delta} (1 - \exp (-\delta T)) \right)$$

$$= V(x(0)) \exp \left( \frac{k|e(0)|}{\delta} \right), \quad \forall \quad t \in [0, \infty). \quad (6.28)$$

This implies stability of the origin of the closed-loop system, since $\forall \epsilon > 0$ there exist a $\delta_0 > 0$, such that if $V(x(0)) < \delta_0$ then $V(x(t)) < \epsilon \forall \ t \in [0, \infty)$. Namely, choose $\delta_0 = \frac{\epsilon}{\exp \left( \frac{k|e(0)|}{\delta} \right)}$.

Let us finally show that $x(t) \to 0$ as $t \to \infty$. Since firstly, $x = 0$ for the system (6.5) and for $e = 0$ is a globally asymptotically stable equilibrium point, secondly, the origin of the observer error dynamics is globally exponentially stable and, thirdly, boundedness of any $x(t)$ of (6.5) for any bounded input $e(t)$ is shown, we can conclude that $x(t) \to 0$ as $t \to \infty$ (herein, we used the converging-input-to-converging-state property as in [27]). Therefore, we can conclude that $(x, e) = (0, 0)$ is globally asymptotically stable.
6.3 Control design based on the Popov criterion

In the previous section, the requirement for the linear part of the closed-loop system (6.5) with the observer error input zero is strict passivity. However, it may be infeasible to achieve strict passivity of the linear part of the closed-loop system (we showed in Chapter 5 that such circle criterion-based design is not feasible for the rotor dynamic system). The Popov criterion can also be used to prove global asymptotic stability of the origin $x = 0$ of system (6.5) when the input $e = 0$. The requirement of strict passivity for the linear part (6.5a) of system (6.5), stated in the previous section, is relaxed by a loop transformation with a dynamic multiplier.

The transfer function of the linear part (6.5a) between $w$ and $z$ is $G_{cl}(s), s \in \mathbb{C}$. A cascade that represents the loop transformation of system (6.5) is shown in Figure 6.2. In this figure, we post-multiply the linear part (6.5a) of system (6.5) with a dynamic multiplier $M(s) = 1 + \gamma s$ and pre-multiply the nonlinear (feedback) part (6.5b) of system (6.5) with the inverse of $M(s)$. Using the dynamic multiplier $M(s)$ we try to transform the original system into a feedback connection of two passive systems. Now, $H_1$ represents a new linear system in the forward path and $H_2$ represents a new system in the feedback path.

$$H_1 = \begin{cases} \dot{x} = (A + BK)x + Gw - BK e \\ z = \tilde{H}x + \tilde{D}w + Ze \end{cases}, \quad (6.29a)$$

$$H_2 = \begin{cases} \dot{z} = -\frac{1}{\gamma}z + \frac{1}{\gamma} \tilde{z} \\ w \in -\varphi(z) \end{cases}, \quad (6.29b)$$

with the observer error dynamics given by (6.5c).

Herein, the matrices $\tilde{H}, \tilde{D}$ and $Z$ can be derived from:

$$\tilde{z} = z + \gamma \dot{z}$$

$$= Hx + \gamma H \dot{x}$$

$$= Hx + \gamma H[(A + BK)x + Gw - BK e]$$

$$= [H + \gamma H(A + BK)]x + \gamma HGw - \gamma HBK e$$

$$= \tilde{H}x + \tilde{D}w + Ze,$$

with

$$\tilde{H} = H + \gamma H(A + BK), \quad \tilde{D} = \gamma HG, \quad Z = -\gamma HBK. \quad (6.31)$$
To investigate the stability properties of the origin \((x, e) = (0, 0)\) of the system (6.29), we adopt the following Lyapunov candidate function \(V(x, z)\):

\[
V(x, z) = V_1(x) + V_2(z)
\]

with

\[
V_1(x) = \frac{1}{2} x^T P_c x, \quad P_c = P_c^T > 0,
\]

and

\[
V_2(z) = \gamma \int_0^z \varphi(s) ds.
\]

The Lyapunov function \(V_2(z)\) is not differentiable for all \(z\) due to the discontinuous nature of \(\varphi(z)\). In order to overcome this difficulty, we will use the subderivative of \(V_2\). In Chapter 5, the subderivative of the Lyapunov function of a similar system as in (6.29b) is derived. Therefore, we will adopt the result from Chapter 5. The subderivative \(dV_2(z)(\dot{z})\) of \(V_2(z)\) is given by

\[
dV_2(z)(\dot{z}) = wz - w\dot{z}.
\]

The time-derivative of \(V_1(x)\) as in (6.33) given by:

\[
\dot{V}_1 = \frac{1}{2} x^T P_c x + \frac{1}{2} x^T \dot{P}_c x
\]

\[
= \frac{1}{2} [x^T [(A + BK)^T P_c + P_c (A + BK)] x
\]

\[
+ w G^T P_c x + x^T P_c G w - e^T K^T B^T P_c x - x^T P_c B K e].
\]

If we desire strict passivity for the linear system (6.29a) with the input \(e = 0\), then the following inequality must hold for strict passivity of linear system (6.29a):

\[
\dot{V}_1 < wz.
\]

If we use the equation \(\dot{z} = Hx + \dot{D}w + Ze\) in (6.29a), we can write expression (6.37) to:

\[
\dot{V}_1 < \frac{1}{2} wz + \frac{1}{2} \dot{z} w = \frac{1}{2} (wHx + w\dot{D}w + wZe + x^T H^T w + w\dot{D}w + e^T Z^T w).
\]

If we combine (6.36) and (6.38), then we derive the following inequality

\[
\frac{1}{2} [x^T [(A + BK)^T P_c + P_c (A + BK)] x + w^T G^T P_c x + x^T P_c G w - wHx - x^T H^T w
\]

\[
- w\dot{D}w - w\dot{D}w - e^T K^T B^T P_c x - x^T P_c B K e - wZe - e^T Z^T w] < 0.
\]

Inequality (6.39) can be written as:

\[
\frac{1}{2} \left[ \begin{array}{c} x \\ w \end{array} \right]^T \left[ \begin{array}{cc} (A + BK)^T P_c + P_c (A + BK) & P_c G - \dot{H}^T \\ G^T P_c - H & -\dot{D} - \dot{D} \end{array} \right] \left[ \begin{array}{c} x \\ w \end{array} \right]
\]

\[
- \frac{1}{2} (e^T K^T B^T P_c x - x^T P_c B K e - wZe - e^T Z^T w) < 0.
\]

We can derive a matrix equality for strict positive realness of the transfer function of the linear system (6.29a) with input \(e = 0\) from the above inequality (6.40):
\[ F_2(P_c, K) := \begin{bmatrix} (A + BK)^TP_c + P_c(A + BK) \\ G^TP_c - \bar{H} \\ P_cG - \bar{H}^T \\ -\bar{D} - \bar{D}^T \end{bmatrix} < 0, \quad (6.41) \]

with \( P_c = P_c^T > 0 \). Now, we combine the derivative (6.36) of \( V_1(x) \) (where also used (6.41)) and the subderivative (6.35) of \( V_2(z) \).

\[
\dot{V} = \frac{1}{2} \begin{bmatrix} x \\ w \end{bmatrix}^T F_2(P_c, K) \begin{bmatrix} x \\ w \end{bmatrix} + wz - \frac{1}{2} e^TK^TB^TPx - \frac{1}{2} x^TPBK e \\
-\frac{1}{2} e^TZ^Tw - \frac{1}{2} wz + wz - w\dot{z}. \quad (6.42)
\]

\[
\dot{V} = \frac{1}{2} \begin{bmatrix} x \\ w \end{bmatrix}^T F_2(P_c, K) \begin{bmatrix} x \\ w \end{bmatrix} + wz - \frac{1}{2} e^TK^TB^TPx - \frac{1}{2} x^TPBK e - \frac{1}{2} e^T Z^Tw - \frac{1}{2} wz. \]

If we consider the observer error input \( e \) to be zero, then the equilibrium point \( x = 0 \) of system (6.29) is globally asymptotically stable if (6.41) holds and

\[
wz \leq 0, \quad (6.43)
\]

since for \( e = 0 \) the time-derivative of the Lyapunov function \( V(x, z) \) is given by

\[
\dot{V} = \frac{1}{2} \begin{bmatrix} x \\ w \end{bmatrix}^T F(P_c, K) \begin{bmatrix} x \\ w \end{bmatrix} + wz < 0. \quad (6.44)
\]

Now, we will show that the state \( x \) of system (6.29) is bounded for bounded \( e(t) \). Equation (6.42) is rewritten and using inequalities (6.41) and (6.43), we obtain the following inequality:

\[
\dot{V} < -\frac{1}{2} e^TK^TB^TP_c x - \frac{1}{2} x^TP_cBK e - \frac{1}{2} e^T Z^Tw - \frac{1}{2} wz. \quad (6.45)
\]

Let us now show that the right-hand side of (6.45), which is due to the non-zero observer error \( e \), satisfies

\[
-\frac{1}{2} e^TK^TB^TP_c x - \frac{1}{2} x^TP_cBK e - \frac{1}{2} e^T Z^Tw - \frac{1}{2} wz E \leq |e|V, \quad (6.46)
\]

for \( |x| \geq R \), for some \( R > 0 \). Note here to that the right-hand side of (6.45) satisfies

\[
-\frac{1}{2} e^TK^TB^TP_c x - \frac{1}{2} x^TP_cBK e - \frac{1}{2} e^T Z^Tw - \frac{1}{2} wz E = -wz e - x^TP_cBK e \leq |wz - x^TP_cBK| |e|. \quad (6.47)
\]

Let us now show that

\[
|wz - x^TP_cBK||e| \leq |e|V, \quad (6.48)
\]

for \( x \geq R \) for some \( R > 0 \). Given the fact that \( z\phi(z) \geq 0 \) (due to the fact that \( wz \geq 0 \) and \( w = -\phi(z) \)), the integral component of \( V \gamma \int_0^s \phi(s) ds \) is always positive or zero. Therefore, the requirement (6.48) can be replaced by the more strict requirement

\[
|wz - x^TP_cBK||e| \leq |e|V^* \leq |e|V \quad \text{for} \quad |x| \geq R \quad \text{and} \quad R > 0, \quad (6.49)
\]

with \( V^* = x^TPx \). Then, we assume that the set-valued nonlinearity \( \phi(z) \) is linearly bounded (with is typically the case for the friction-induced nonlinearities we consider in this thesis):

\[
|w| \leq \eta_1 + \eta_2 |z| \quad \forall \ w \in -\phi(z), \quad (6.50)
\]
Using condition (6.50), the left-hand side of (6.49), can be rewritten as

\[ | - w^T Z - x^T P_c B K | \leq |w^T Z| + |x^T P_c B K|^2 \]

\[ \leq |w||Z||e| + |x||P_c B K||e| \]

\[ \leq (\eta_1 + \eta_2 |x|)|Z||e| + |x||P_c B K||e| \]

\[ = \eta_1 |Z||e| + \eta_2 |x| |Z||e| + |x||P B K||e| \]

\[ = \eta_1 |Z||e| + (\eta_2 |Z| + |P B K||e||x| \]

\[ = \alpha_1 |e| + \alpha_2 |e||x| \]

\[ = (\alpha_2 |x| + \alpha_1)|e|, \]

with

\[ \alpha_1 = \eta_1 |Z| \]

\[ \alpha_2 = \eta_2 |Z| + |P_c B K|. \]  

(6.52)

In order to satisfy (6.49), we need to show that

\[ \alpha_2 |x| + \alpha_1 \leq x^T P_c x \quad \text{for} \quad |x| \geq R \quad \text{and some} \quad R > 0. \]  

(6.53)

Consider the norm \( |x|_{P_c} = \sqrt{x^T P_c x}. \) Based on this norm, the right-hand side of (6.53) takes the form:

\[ x^T P x = |x|^2 \geq \beta |x|^2 \quad \text{for} \quad \beta = \lambda_{\min}(P) > 0. \]  

(6.54)

Therefore, (6.53) can be rewritten as

\[ \alpha_2 |x| + \alpha_1 \leq \beta |x|^2 \quad \text{for} \quad |x| \geq R \quad \text{and some} \quad R > 0, \]  

(6.55)

where \( \beta = \lambda_{\min}(P_c). \)

It is easy to show that there always exists an \( R > 0 \) such that (6.55) is satisfied for \( |x| \geq R. \) For example, if we solve analytically the equality derived from (6.55) we can provide an exact value for \( R: \)

\[ -\beta |x|^2 + \alpha_2 |x| + \alpha_1 = 0 \]

\[ \Rightarrow |x| = -\alpha_2 + \sqrt{\frac{\alpha_2^2 + 4\alpha_1}{2\beta}} \]

\[ \Rightarrow |x| = \frac{\alpha_2}{2\beta} + \frac{\sqrt{\alpha_2^2 + 4\alpha_1}}{2\beta}. \]  

(6.56)

Therefore, we can choose \( R = \frac{\alpha_2}{2\beta} + \frac{\sqrt{\alpha_2^2 + 4\alpha_1}}{2\beta}, \) since \( \alpha_1, \alpha_2, \beta > 0. \)

Since equations (6.44) and (6.17) hold, we can conclude that

\[ \dot{V} < |e|V \quad \text{for} \quad |x| \geq R, \quad R > 0. \]  

(6.57)

For the same reasons as we discussed in Section 6.2, we conclude that the state \( x \) is bounded and that \( x(t) \to 0 \) as \( t \to \infty, \) i.e. equilibrium \( (x, e) = (0, 0) \) of system (6.5) is globally asymptotically stable, since firstly, \( x = 0 \) for the system (6.29) for \( e = 0 \) is globally asymptotically stable, secondly, the origin of the observer error dynamics is globally exponentially stable and, thirdly, boundedness of any \( x(t) \) of (6.29) for any bounded input \( e(t) \) is shown.
6.4 Control design based on the Popov criterion for the rotor dynamic system

The output-feedback controller which is based on the Popov criterion will be applied to the rotor dynamic system for the nominal case (friction characteristic I). In Chapter 5, we were not able to obtain solutions for the controllers based on the circle criterion and the Popov criterion, applied to the rotor dynamic system (F.7) (a system with two set-valued nonlinearities in the feedback loop). It also is shown that the state-feedback controller based on the circle criterion is not feasible for the rotor dynamic system (a system with one set-valued nonlinearity in the feedback loop). The output-feedback controller designs use the same principles as the state-feedback variants. Therefore we will apply the output-feedback controller based on the Popov criterion to the rotor dynamic system given by (5.29).

The observer with the gains \(L_2\) and \(N_2\) will be used for the output-feedback controller. The observer error is defined as \(e = \xi - \hat{\xi}\). In Section 4.3, it is proven that the origin of the corresponding observer error dynamics (4.28) is globally exponentially stable. It should be noted that the observer is based on the rotor dynamic system without the information of the compensation input that partially compensated the friction acting at the upper disc. However, the observer takes into account the input to the rotor dynamic system, which also includes the compensation input.

We consider the rotor dynamic system with one set-valued friction map in the feedback loop given by

\[
\begin{align*}
\dot{\xi} &= A_{tr2}\xi + Bv_{control} + G_{tr}\tilde{w} \\
z &= H_{tr}\xi \\
\tilde{w} &\in -\varphi_{tr}(z),
\end{align*}
\]

with

\[
A_{tr2} = \begin{bmatrix}
0 & 1 & -1 \\
-\frac{k_u}{J_u} & \frac{1}{J_u}(b_u - \Delta b_u) & 0 \\
0 & 0 & \frac{m}{J_l}
\end{bmatrix}, \quad B = \begin{bmatrix}
k_u \\
0 \\
0
\end{bmatrix}, \quad G_{tr} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]

\[
H_{tr} = \begin{bmatrix}
0 & 0 & 1
\end{bmatrix}, \quad \varphi_{tr}(z) = T_{fft3}(z).
\]

The set-valued nonlinearity \(\varphi_{tr}(z)\) is monotone. The transformation of the rotor dynamic system (3.3) to rotor dynamic system (6.58) is described in Appendix F.1 and Appendix F.

We employ the following control law for the rotor dynamic system (6.58):

\[v_{control} = K\hat{\xi},\]

with the feedback gain \(K = [k_1 \ k_2 \ k_3]\). The closed-loop rotor dynamic system is given by

\[
\begin{align*}
\dot{\hat{\xi}} &= (A_{tr2} + BK)\xi + G_{tr}\tilde{w} - BK\xi \\
z &= H_{tr}\hat{\xi}, \\
\tilde{w} &\in -\varphi_{tr}(z).
\end{align*}
\]

The choice for the control gain in the control law (6.61) is \(K_1\) as in (5.43). In Section 5.4, it is shown that for the state-feedback control situation (this is comparable to the rotor dynamic system with the output-feedback controller when the observer error \(e = 0\)) the origin \(\xi = 0\) of the closed-loop system is globally asymptotically stable. Consequently, the origin \(\xi = 0\) of system (6.62) for \(e = 0\) is globally asymptotically stable.
6.5 Simulation results

Simulations are performed for the output-feedback controller for the rotor dynamic system for the nominal case (friction characteristic I). We will show the responses of the rotor dynamic system for the state of the model of the form (3.3) with the equilibrium $\mathbf{x}_{eq} = [\alpha_{eq} \ \omega_{eq} \ \omega_{eq}]^T$. The equilibrium point $\mathbf{x}_{eq}$ of the model (3.3) corresponds to the origin $\mathbf{ξ} = 0$ of the model in Lur’e-type form (5.29). We employ the control law $u = u_c + u_{\text{comp}} + BK(\mathbf{x} - \mathbf{x}_{eq})$, which is equivalent to the control law $v_{\text{control}} = K\mathbf{ξ}$ applied to model (5.29). Note that we use the state of model (3.3), because it is favourable from the perspective of physical interpretation.

The simulations for the rotor dynamic system show a large control action for the state-feedback control (see Appendix H.3), when the controller is switched on. But the experimental set-up cannot execute this large control action, since its voltage range is limited. Therefore, we will also performed simulations where the input voltage $u$ of system (3.3) is limited to the range $[-5V, 5V]$.

We chose the control gain $K_1$ as in (5.43), because the closed-loop system with control gain $K_1$ converges faster to the setpoint than the closed-loop system with control gain $K_2$ when the input voltage is saturated (the conclusion about the convergence is the opposite for the rotor dynamic system without input voltage saturation, see Appendix H.3). Observer gains $L_2$ and $N_2$ as in (4.30) are chosen, because the observer error converges faster to zero for these gains than for the observer with the gains $L_1$ and $N_1$. The application of the output-feedback controller to the rotor dynamic system with friction characteristic II on simulation level is discussed in Appendix K.4.
The first simulation is performed for the input voltage \( u_c = 1.8 \) V. The initial condition for the system is \( \xi = [0 \ 0 \ 0]^T \). The solution for the open-loop rotor dynamic system (6.58) with friction characteristic I is a limit cycle (see also the bifurcation diagrams in Figure F.11(a) and Figure F.11(b)). We show the response of the rotor dynamic system with the unsaturated input voltage and the rotor dynamic system with the input voltage saturated in Figure 6.4. The control error is defined as \( x - x_{eq} \). The control errors for the two different systems are depicted in Figure 6.5. The states of the rotor dynamic system with the unsaturated input and the rotor dynamic system with the saturated input both converge to the setpoint. The state of the rotor dynamic system with the saturated input converges slower to the setpoint.

The response of the rotor dynamic system with unsaturated input and the response of the rotor dynamic system with saturated input for the input voltage \( u_c = 4.0 \) V are shown in Figure 6.6. The initial condition for the system is \( \xi = [0 \ 0 \ 0]^T \). Coexistence of solutions exists for the open-loop rotor dynamic system with friction characteristic I: a stable equilibrium and a stable limit cycle. The solution of the open-loop rotor dynamic system for the initial condition \( \xi = [0 \ 0 \ 0]^T \) is a limit cycle. We see that the state of the rotor dynamic system with the saturated input converges slower to the setpoint than the state of the rotor dynamic system with unsaturated input.

The bifurcation diagrams for the closed-loop rotor dynamic system with the output-feedback controller do not differ from the bifurcation diagrams of the closed-loop rotor dynamic system with the state-feedback controller, since the equilibria are also globally asymptotically stable. Therefore, see Figure H.8(a) and Figure H.8(b) for the bifurcation diagrams.

The saturation of the actuator may induce coexistence of steady-state solutions. This phenomenon is also discussed in Appendix H.3 for the state-feedback controller. In the next section, we will discuss the experimental results.

![Graphs showing responses of the closed-loop system](image_url)

Figure 6.4: Responses of the closed-loop system with controller with gain \( K_1 = [15.9 \ 1.57 \ 27.6] \) switched on at \( t = 0 \) s, for input voltage unsaturated (solid line) and for input voltage saturated (dashed line); observer with gains \( L_2 \) and \( N_2 \); the constant input voltage is \( u_c = 1.8 \) V; the equilibrium values are \( \alpha_{eq} = 1.55 \) rad and \( \omega_{eq} = 3.15 \) rad/s.
6.5. Simulation results

Figure 6.5: Control error for closed-loop system with controller with gain $K_1 = [15.9 \ 1.57 \ 27.6]$ switched on at $t = 0$ s, for input voltage unsaturated (solid line) and for input voltage saturated (dotted line); observer with gains $L_2$ and $N_2$; the constant input voltage is $u_c = 1.8$ V and the equilibrium values are $\alpha_{eq} = 1.55$ rad and $\omega_{eq} = 3.15$ rad/s.

Figure 6.6: Responses of the closed-loop system with controller with gain $K_1 = [15.9 \ 1.57 \ 27.6]$ switched on at $t = 0$ s, for input voltage unsaturated (solid line) and for input voltage saturated (dashed line); observer with gains $L_2$ and $N_2$; the constant input voltage is $u_c = 4.0$ V and the equilibrium values are $\alpha_{eq} = 1.52$ rad and $\omega_{eq} = 7.06$ rad/s.
6.6 Experimental results

The output-feedback controller is implemented on the experimental set-up. This is done for the rotor dynamic set-up with friction characteristic I. The output-feedback controller is also applied to the rotor dynamic set-up with friction characteristic II. This is discussed in Appendix K.5. We choose the same gains as we used for the simulations in the previous section. The control gain is $K_1$ as in (5.43) and the observer gains are $L_2$ and $N_2$ as in (4.30).

We choose to switch on the observer first and then to switch on the controller. However if we switch on the controller with the control gain $K_1$, then this leads to several undesired phenomena. The DC-motor reacts with a high frequency response which is highly undesirable for the DC-motor. Sometimes the output-feedback control is able to bring the system to an equilibrium for a while, but then chatter occurs and the system becomes unstable again. It seems that the observed lower velocity signal contains too high frequencies. The feedback of the velocity of the lower disc by the controller may cause the bad performance. To overcome this problem, we implement a low-pass filter which removes the high frequencies from the lower velocity estimate. The transfer function $H_{\omega_l}(s)$ of the low-pass filter is:

$$H_{\omega_l}(s) = \frac{1}{s + 1}, \quad s \in \mathbb{C}. \quad (6.63)$$

A frequency-domain analysis is performed for the equilibrium response and the limit cycle response of the rotor dynamic system for the input voltage of $u_c = 4.0$ V. The spectral analysis for the observed velocity of the lower disc $\hat{\omega}_l$ for the stable equilibrium solution is depicted in Figure 6.7(a). After application of the low-pass filter (6.63), we obtain the filtered observed velocity of the lower disc $\hat{\omega}_{l,\text{filtered}}$. The power spectral density of $\hat{\omega}_{l,\text{filtered}}$ for the stable equilibrium solution is depicted in Figure 6.7(b).

If we compare the latter power spectral density in Figure 6.7(b) with the power spectral density of $\hat{\omega}_l$ in Figure 6.7(a), then we see a substantial reduction of the power spectral density for frequencies above approximately $2\pi$ Hz. The power spectral density for the observed velocity of the lower disc $\hat{\omega}_l$ and the filtered observed velocity of the lower disc $\hat{\omega}_{l,\text{filtered}}$ are shown in Figure 6.7(c) and Figure 6.7(d), respectively, for the case of a stick-slip limit cycling solution. Again, we see a substantial reduction of the power spectral density for frequencies above approximately $2\pi$ Hz if we compare Figure 6.7(c) with Figure 6.7(d).

After the application of the low-pass filter (6.63), the output-feedback control is able to stabilize the experimental set-up from the constant input voltage of 1.4 V up to 5.0 V. A transient response for the input voltage $u_c = 4.0$ V is depicted in Figure 6.8. Solutions coexists for the open-loop rotor dynamic system (6.58): a stable equilibrium and a stable limit cycle (see also the bifurcation diagrams in Figure F.11(a) and Figure F.11(b)). The solution of the open-loop rotor dynamic system is a limit cycle for the first five seconds in Figure 6.8. The input voltage saturates several times after the controller is switched on. The closed-loop system converges to the equilibrium values, in spite of the saturation of the input voltage.

Next, we show a transient response for the constant input voltage $u_c = 2.5$ V in Figure 6.9. The solution for the open-loop rotor dynamic system (6.58) is a limit cycle. The closed-loop system converges to the equilibrium state within approx 12 seconds. We notice that the input voltage is saturated for approximately 0.5 seconds when the controller is switched on. The control error $x - x_{eq}$ is depicted in Figure 6.10. The control errors for the state components converge to non-zero but small values for the steady-state equilibrium solution.

The equilibrium response for $u_c = 1.4$ V is shown in Figure 6.11 (the solution for the open-loop rotor dynamic system (6.58) is a limit cycle). We observe that there are fluctuations present in the observed
state components in Figure 6.11. The power spectral density of the observed velocity of the lower disc in Figure 6.12 shows that the rotational frequency (depicted with a vertical dashed line) is dominant. The dominance of the rotational frequency indicates a position-dependent friction acting on the lower disc. We notice for a small region, from the constant input voltage $u_c = 1.4$ V to 1.5 V, that two steady-state solutions exist: a stable equilibrium and a stable limit cycle. It depends on the initial conditions to which stable steady-state solution the closed-loop system converges.

The output-feedback controller is not able to control the experimental rotor dynamic set-up for all equilibria corresponding to the constant input voltages. A stable equilibrium can not be obtained with the chosen output-feedback controller for the constant input voltages from 0.2 V up to 1.3 V. A limit cycle response is depicted in Figure 6.13 for the input voltage $u_c = 0.9$ V. It should be noted that the limit cycle responses for the closed-loop system differ from the limit cycle responses for the open-loop system. The aforementioned position-dependent friction could also be a probable cause of the limit cycle responses for the low constant input voltages. The position-dependent friction changes the friction in such a way that the output-feedback controller is not able to control the closed-loop system anymore. Moreover, the Popov criterion guarantees no absolute stability for systems with time-dependent friction.

The measured responses from the experiments are used to construct bifurcation diagrams with the constant input voltage $u_c$ as the bifurcation parameter. We compare the results from the simulations with the experimental results in these bifurcation diagrams, see Figure 6.14(a) and Figure 6.14(b).

We notice a good match between the simulations and the experiments for constant input voltages above $u_c = 1.4$ V. The control aim is achieved for these range of voltages, i.e. the controller is able to control the closed-loop experimental rotor dynamic set-up to its setpoint. For the voltages from $u_c = 0.2$ V up to 1.4 V, the simulations do not match with the experiments. The simulations shows stable equilibria, while the results of the experiments are stable limit cycles.

We have noticed that there are still fluctuations present in the equilibrium responses of the experimental rotor dynamic set-up for lower constant input voltages. The maximum and minimum values of these responses with fluctuations are depicted in the bifurcation diagram in Figure 6.15. The fluctuations increase when the constant input voltage decreases.

We compare the experimental bifurcation diagrams of the open-loop system with the experimental
Figure 6.8: Closed-loop system response with control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ switched on at $t = 5$ s; the constant input voltage $u_c = 4.0$ V, the equilibrium values are $\alpha_{eq} = 1.52$ rad and $\omega_{eq} = 7.06$ rad/s; observer gains are $L_2$ and $N_2$.

Figure 6.9: Closed-loop system response with control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ switched on at $t = 5$ s; the constant input voltage is $u_c = 2.5$ V, the equilibrium values are $\alpha_{eq} = 1.36$ rad and $\omega_{eq} = 4.40$ rad/s; observer gains are $L_2$ and $N_2$.

results of the closed-loop system in Figure 6.16(a) and Figure 6.16(b). These diagrams show that a large extension of the region with only stable equilibria is achieved by the output-feedback controller. The region with only stable equilibria for the open-loop system covers the constant input voltages $u_c = 4.5$ V up to 5.0 V. The closed-loop system extends the region with only stable equilibria to the constant input voltages from 1.5 V up to 5.0 V.
6.7. Discussion

We can apply the presented output-feedback controller to the rotor dynamic system for friction characteristic I and friction characteristic II. The simulations in Section 6.5 show that the rotor dynamic state converges to the equilibrium state when the input voltage is unsaturated. When the input voltage in the simulations is limited to the range $[-5V, 5V]$, then for almost every situation the rotor dynamic state converges to the equilibrium state. The saturation of the actuator sometimes induces coexistence of steady-state solutions: a stable equilibrium and a stable limit cycle. Next, the output feedback controller is implemented on the experimental rotor dynamic set-up for friction characteristic I and friction characteristic II. A low-pass filter is applied to the observed velocity of the lower disc to avoid undesired phenomena (e.g. chatter) when the observed velocity of the lower disc is fed back to the controller. After the application of the low-pass filter, the output-feedback controller is able to control the rotor dynamic set-up for large range of constant input voltages to its setpoint. The controller can not control the rotor dynamic set-up to the chosen setpoint for only a small range of low voltages. The solution for these range of voltages is either a stable limit cycle or a different stable equilibrium. A cause for
Figure 6.12: Power spectral density analysis of the equilibrium response $\dot{\omega}_l$ of the rotor dynamic system for the input voltage $u_c = 1.4$ V.

Figure 6.13: Closed-loop system response with control gain $K_1 = [15.9, 1.57, 27.6]$; the constant input voltage is $u_c = 0.9$ V, the equilibrium values are $\alpha_{eq} = 2.42$ rad and $\omega_{eq} = 1.52$ rad/s; observer gains are $L_2$ and $N_2$.

Figure 6.14: Bifurcation diagrams of the closed-loop system with control gain $K_1 = [15.9, 1.57, 27.6]$ for positive input voltages; observer gains are $L_2$ and $N_2$.

this lack of performance at these low input voltages may be the position-dependent friction acting on the lower disc. In spite of the undesired solutions for the lower voltages, the region with only stable equilibria is substantially extended (compared to the open-loop system) for friction characteristic I and friction characteristic II. We have shown that the rotor dynamic set-up is absolutely stable, since the same output-feedback controller can be used for a range of constant input voltages. When the constant voltage changes, then this implies that the linear part of the system remains the same and the set-valued nonlinearity changes within the sector $[0, \infty]$. The absolute stability can also be shown by the fact that the same output-feedback controller can be used for the experimental rotor dynamic set-up with friction characteristic I and for the experimental rotor dynamic set-up with friction characteristic II.
Figure 6.15: Bifurcation diagram of the closed-loop system with control gain $K_1 = [15.9 \ 1.57 \ 27.6]$; with the velocity of the lower disc for positive input voltages; with maximum and minimum values for measured $\hat{\omega}_{eq}$; observer gains are $L_2$ and $N_2$.

(a) Bifurcation diagram with the velocity of the lower disc.  
(b) Bifurcation diagram with the position difference between the upper and the lower disc.

Figure 6.16: Bifurcation diagrams of the closed-loop system with control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ for positive input voltages; observer gains are $L_2$ and $N_2$; comparison with the experimental bifurcation diagram of the open-loop system.
Conclusions and recommendations

In this chapter, we finish this thesis with conclusions and recommendations. We have explained that we focus on systems with actuation and non-collocated discontinuous friction. For this type of systems, we obtained the following contributions:

- Two state-feedback control designs which are able to control a motion system to a properly chosen setpoint.
- An output-feedback control design which is able to control a motion system to a properly chosen setpoint.

An experimental rotor dynamic set-up, which is a system with actuation and non-collocated discontinuous friction, is available at the DCT-laboratory of the Technische Universiteit Eindhoven. We applied the presented theory in this work to that experimental set-up. The main contributions for the rotor dynamic system are:

- Identification of the parameters of a model that describes the rotor dynamic system.
- Experimental implementation of an observer for the rotor dynamic system.
- Application of the presented output-feedback controller to the experimental rotor dynamic set-up.
- Validation of the output-feedback controller based on simulations and experiments.

7.1 Conclusions

A model-based observer is presented. The observer error converges exponentially to zero if the observer error dynamics satisfies the circle criterion for the chosen observer gains.

Next, two state-feedback controllers are presented with a linear state-feedback control law. Both designs are based on absolute stability theory. The first controller achieves absolute stability for the closed-loop system when the closed-loop system with a chosen control gain satisfies the circle criterion. However, it may be infeasible to satisfy the circle criterion for a given system. Therefore, a second controller is proposed. The imposed conditions on the closed-loop system for the circle criterion are relaxed by a loop transformation with a dynamic multiplier. The second control design aims at rendering the closed-loop system absolutely stable using a Popov criterion-type approach.

We use the observer and a state-feedback controller to construct an output-feedback controller. Herein, we can use the controller based on the circle criterion as well the controller based on the Popov criterion. Global asymptotic stability of the equilibrium of the closed-loop system is obtained for both output-feedback designs.
An experimental rotor dynamic set-up is available in the DCT-laboratory. This system consists of two inertias coupled by a flexibility where the first inertia is actuated and the second inertia is subject to friction. The control goal is to stabilize the constant velocity equilibria of the system (that exist for constant inputs at the first inertia). However, for such constant inputs the set-up exhibits a wide variety of nonlinear phenomena, such as limit cycling, coexistence of steady-state solutions and bifurcations. These phenomena are due to the friction acting on the system. The rotor dynamic system can be considered as a benchmark for systems with actuation and non-collocated friction. Moreover, it is important to note that the system configuration and control problem considered here is encountered in many industrial systems, such as drilling systems and printers.

An accurate model with model parameters of the rotor dynamic set-up is required for the design of a controller and an observer. The estimation of the model parameters (mainly friction parameters) is important, since the set-up is sensitive to conditions such as temperature, humidity, contaminations etc. The model parameter estimation is validated with experimental results. The obtained model with model parameters is predictive for the steady-state behaviour of the experimental rotor dynamic set-up.

The aforementioned observer is applied to the rotor dynamic system. One state component is measured in the experiments; for that state component we can conclude that the observer-based state estimation converges to the measured value with a small observer error in steady-state.

The state-feedback controller based on the circle criterion is not feasible for the rotor dynamic system. Therefore we applied the controller based on the Popov criterion to the rotor dynamic system. Since the state is not entirely measured in the experimental set-up, we can only perform simulations for the closed-loop rotor dynamic system. The simulations show that the closed-loop system converges to the desired setpoint.

Finally, the output-feedback controller based on the Popov criterion is implemented on the experimental rotor dynamic set-up. The output-feedback controller stabilizes the desired setpoint of the rotor dynamic set-up for large range of constant inputs, i.e. the setpoint is globally asymptotically stable. The controller can not control the rotor dynamic set-up to the desired setpoint for a small range of low constant inputs. A cause for this fact may be the position-dependent friction which is acting on the lower inertia. In spite of the undesired solutions for the lower constant inputs, the region with only stable equilibria is substantially extended (compared to the open-loop system).

### 7.2 Recommendations

For further research, we formulate the following recommendations:

- Extend the output-feedback designs to systems with several set-valued nonlinearities in the feedback loop.

- Extend the control design based on the Popov criterion to systems with nonlinearities in the feedback loop where each nonlinearity can contain several discontinuities.

- The presented control designs aimed at controlling a system at a setpoint. An interesting extension could be trajectory control for systems with actuation and non-collocated friction.

- In this thesis, we focus on controlling motion systems at a setpoint. This setpoint can include velocities or positions of the motion system masses. In order to control such system at a position setpoint, we use a model (higher order model) which is more complex with respect to the model we use in order to control the same system at a velocity setpoint. A higher-order model could make the application of the presented control designs more difficult or even infeasible. This problem is also encountered in [20]. Such a problem should be investigated.
• The applied observer for the rotor dynamic system uses as output injection the difference between the positions of the both inertias. The application of an observer that uses as output injection only the position of the actuated inertia is more difficult. The difficulties one faces when dealing with such an observer should be further investigated.

• The current control design based on the Popov approach is for Lur’e systems with time-independent nonlinearities. It could be useful to extend the Popov criterion to Lur’e systems with time-dependent nonlinearities, see e.g. [6].

• The search for gains for the controllers is not a constructive approach, since we have to choose some variables to obtain a linear matrix inequality. Writing the matrix inequalities to linear matrix inequalities with new variables is not trivial. A study to make the approach more constructive should make the design of the state-feedback controller or the output-feedback controller easier in practice.

• We proposed a specific structure of the dynamic multiplier in Section 5.3 and Section 6.3 for the control based on the Popov criterion. It may be possible that a different structure of the dynamic multiplier would extend the class of systems for which a controller based on the Popov criterion can be applied.

• For the current design of the output-feedback controller, we use a model-based observer which is not robust for unmodeled changes in the set-valued nonlinearity. Therefore, we propose to investigate the design of such a robust observer. Next, an output-feedback controller, based on this new observer, should be constructed which is robust for unmodeled changes in the set-valued nonlinearity.

• A linear static control law is used for the presented controllers. It should be investigated if dynamic control laws could be used within the presented control approaches. The application of the dynamic control laws may extend the class of system which can be rendered absolute stable.

• The magnitude of the control action is not taken into account in the design of the controllers. However, the control action may exceed the physical possibilities of the experimental set-up (as is noticed for the application of the controller to the experimental rotor dynamic set-up). It should be studied if conditions could be added in the design of a controller to limit the control action.

Recommendations for the rotor dynamic system are:

• Unmodeled dynamics are observed during the experiments with the rotor dynamic set-up. The experimental set-up should be aligned to reduce the presence of unmodeled dynamics, especially the position-dependent friction. The alignment should ensure that the discs and the steel string of the experimental set-up (when the constraints prevent lateral freedom of the lower disc) have only rotational freedom.

• An output-feedback controller based on the Popov criterion is designed for the rotor dynamic system with partial compensation of the non-smooth friction acting at the upper disc. The next step would be to design an output-feedback controller without any friction compensation, since overcompensation and undercompensation can lead to undesired dynamic behaviour of the rotor dynamic system.

• An observer should be designed which can reconstruct the state of the rotor dynamic system with only the information of the position of the upper disc. An example of a system for which it is difficult to obtain information about the position of the unactuated mass with non-collocated friction is a rotary drilling system, since down-hole measurements are hardly feasible. Therefore, it would be useful to show for the experimental rotor dynamic set-up that it would be possible to reconstruct the states with only information from the actuated mass.

• Design of an output-feedback controller for rotor dynamic system with lateral degrees of freedom for the lower disc.
Bibliography


Data of the experimental rotor dynamic set-up

Important data about the rotor dynamic system is given in the following table:

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Calibration of the brake device

The brake device, attached to the brake disc in the experimental set-up of the rotor dynamic system, exerts a normal force to the brake disc. This normal force can be adjusted by a screw, see Figure 3.4. The brake device is calibrated in order to determine the relation between the revolutions of the adjustment screw and the exerted normal force. This calibration is performed with the help of a force sensor. The result is shown in Figure B.1, where the marks “o” represented the measurements and the solid line represents the linear approximation.

![Graph](image)

Figure B.1: The normal force exerted by the brake device corresponding to the number of revolutions of the adjustment screw.
Switch model for the rotor dynamic system

The switch model is for the rotor dynamic system (3.3)-(3.7) defined as:

if \(|x_2| > \eta\)
\[
f_2 = \frac{1}{J_f} [-k_\theta x_1 + k_m u - T_{fu}(x_2)]
\]
\[
T_{fu} = \text{sgn}(x_2)(T_{su} + \Delta T_{su} \text{sgn}(x_2) + b_u|x_2| + \Delta b_u x_2)
\]
else
if \(|k_m u - k_\theta x_1| > |\text{sgn}(x_2)(T_{su} + \Delta T_{su} \text{sgn}(x_2))|\)
\[
f_2 = \frac{1}{J_f} [-k_\theta x_1 + k_m u - T_{fu}^{ad}(x_2)]
\]
else
\[
f_2 = -c_2 x_2
\]

if \(|x_3| > \eta\)
\[
f_3 = \frac{1}{J_f} [k_\theta x_1 - T_{fl}(x_3)]
\]
\[
T_{fl} = \text{sgn}(x_3)(T_{cl} + (T_{sl} - T_{cl})e^{-\frac{x_3}{\omega s}} + b_l|x_3|)
\]
else
if \(|k_\theta x_1| > T_{sl}\)
\[
f_3 = \frac{1}{J_f} [k_\theta x_1 - T_{fl}^{ad}(x_3)]
\]
else
\[
f_3 = -c_3 x_3
\]

\[
x_1 = x_2 - x_3
\]
\[
x_2 = f_2
\]
\[
x_3 = f_3
\]

with \(c_2 > 0\) and \(c_3 > 0\).
Additional theory for observer design

In this appendix, we first discuss the nonlinearity in the feedback loop of the observer error dynamics. Next, we show the exponential convergence of the observer error.

D.1 Nonlinearity in the feedback loop of the observer error dynamics

We consider the nonlinearity $\phi$ given by (4.5). The function $\phi = [\phi_1 \ldots \phi_p]^\top$ is ‘generated’ each time instant $t_n$. We desire a new function $\phi_i$, for $i = 1 \ldots p$, with just one variable. Therefore, the difference of $\phi_i(z_i(t))$ with a variable input $\hat{z}_i(t)$, with respect to a $\phi_i(z_i(t))$ with a ‘fixed’ $z_i(t)$, is calculated at a time instant $t_n$, see Figure D.1(a).

The function $\varphi_i$ can be transformed to obtain the new function $\phi_i$, for $i = 1 \ldots p$. Since we want to know the difference of $\phi_i(z_i(t))$ with respect to $\phi_i(z_i(t))$, we transform the functions $\varphi_i$, for $i = 1 \ldots p$, such that the points $(z_i(t_n), \varphi_i(z_i(t_n)))$ of the functions are shifted to the origin of the function. This creates the functions $\varphi_i(\hat{z}_i - z_i(t_n)) - \varphi_i(z_i(t_n))$ with the arguments $\hat{z}_i - z_i(t_n)$, for $i = 1 \ldots p$, where we consider $\varphi_i(z_i(t_n))$ and $z_i(t_n)$ as constants, generated at each time instant $t_n$. The transformed function does have the difference between $z_i$ and $\hat{z}_i$ as an argument, see Figure D.1(b).

To obtain the new function $\phi$, two transformation steps are performed. Take the negative function value, $\varphi_i(z_i(t_n)) - \varphi_i(\hat{z}_i) = \phi_i(z_i(t_n)) - \hat{z}_i = \phi_i(q_i)$, and take the negative input, $z_i(t_n) - \hat{z}_i = q_i$, this results in the function $\phi(t, q) = [\phi_1(t, q_1) \ldots \phi_p(t, q_p)]^\top$, see Figure D.1(c).

D.2 Exponential convergence of the observer error

If we use the comparison principle (Lemma 3.4, [16]), it is possible to show the exponential convergence of the observer error of the observer error dynamics (4.7). We again use the Lyapunov function in (4.10) with its time-derivative given in (4.16), which can be written as

$$\dot{V} \leq -e^\top Q e + 2k^\top q,$$

with $Q = Q^\top > 0$. Herein, the matrix $\varepsilon I$ is replaced by the matrix $Q$. The latter is done for practical reasons. Often, it is more easy to obtain a solution numerically for a certain positive definite matrix.
\[ \dot{V} \leq -e^T Q e. \]  

The right-hand side is made less negative because the term \( 2k^T q \leq 0 \) is omitted.

We can apply the following inequality, see Lemma 4.3, [16], to bound \( e^T Q e \):

\[ \lambda_{\text{min}}(Q)||e||_2^2 \leq e^T Q e \leq \lambda_{\text{max}}(Q)||e||_2^2, \]  

such that

\[ \dot{V} \leq -e^T Q e \leq -\lambda_{\text{min}}(Q)||e||_2^2. \]  

Similarly, bounds on \( e^T P e \) are given by

\[ \lambda_{\text{min}}(P)||e||_2^2 \leq e^T P e \leq \lambda_{\text{max}}(P)||e||_2^2. \]
Since $Q = Q^T > 0$, $\lambda_{\text{min}} > 0$, which means that the expression $-\lambda_{\text{min}}(Q)||e||^2 \leq 0$. We want to replace $||e||^2$ with an expression derived from (D.5). If inequality (D.4) still has to be satisfied, $||e||^2$ has to be replaced by an expression which is smaller or equal to $||e||^2$. Here, we derive from (D.5) that

$$
\frac{1}{\lambda_{\text{max}}(P)} e^T P e \leq ||e||^2.
$$

(D.6)

Consequently, (D.3) can be written as

$$
\dot{V} \leq -e^T Q e \leq -\lambda_{\text{min}}(Q)||e||^2 \leq -\frac{\lambda_{\text{min}}(Q)}{\lambda_{\text{max}}(P)} e^T P e.
$$

(D.7)

The right part of this inequality can be written as a function of the Lyapunov function $V$:

$$
\dot{V} \leq -\frac{\lambda_{\text{min}}(Q)}{\lambda_{\text{max}}(P)} V = -\beta V, \quad \beta := \frac{\lambda_{\text{min}}(Q)}{\lambda_{\text{max}}(P)} > 0.
$$

(D.8)

Now, let us exploit the comparison principle. If we write (D.8) as an equality, a differential equation in $V$ can be recognized:

$$
\dot{V} = -\beta V.
$$

(D.9)

The solution for this differential equation is

$$
V(t) = V_0 e^{-\beta t}, \quad V_0 = e_0^T P e_0
$$

(D.10)

Consequently, it holds that

$$
V = e^T P e \leq V_0 e^{-\beta t},
$$

(D.11)

and expression (D.5) is used to obtain

$$
\lambda_{\text{min}}(P)||e||^2 \leq V_0 e^{-\beta t}.
$$

(D.12)

Since (D.12) holds, we conclude that the origin of the observer error dynamics (4.7) is globally exponentially stable.

This inequality can be rewritten to an upper bound for the error of the observer state in time:

$$
||e||^2 \leq \frac{V_0}{\lambda_{\text{min}}(P)} e^{-\beta t},
$$

(D.13)

where $V_0 = e_0^T P e_0$ and $\beta$ as in (D.8). Note that if $Q = \varepsilon I$, then $\lambda_{\text{min}}(Q) = \varepsilon$. 

Additional theory for state-feedback control

In the first section, we present loop transformations which can be applied to render the set-valued nonlinearities in the feedback loop of Lur'e-type system to the sector \([0, \infty]\). The application of the control design based on the circle criterion to systems transformed by loop transformed is discussed in Section E.2. In the last section, we discuss the design of single controller for a Lur'e-type system for a range of equilibria corresponding to constant inputs.

### E.1 Loop transformations

Consider the set-valued nonlinearity \(\varphi_i \in [k_{\alpha,i}, k_{\beta,i}]\), for \(i = 1, \ldots, p\) and \(k_{\beta,i} - k_{\alpha,i} > 0\), see Figure E.1. If \(k_{\alpha,i} \neq 0\) or \(k_{\beta,i} \neq \infty\), then we can apply loop transformations which create a transformed linear system and a transformed nonlinearity \(\tilde{\varphi}_i \in [0, \infty]\), see [16]. If for example \(k_{\alpha,i} > 0\) and \(k_{\beta,i} < \infty\), then loop transformations can remove the "excess" of passivity. Also the opposite, we can remove a "shortage" of passivity by loop transformations.

We define the matrix \(K_\alpha\) as:

\[
K_\alpha = \text{diag}(k_{\alpha,1}, \ldots, k_{\alpha,p}), \quad \text{for} \quad i = 1, \ldots, p. \tag{E.1}
\]

Likewise, we define \(K_\beta\) as follows:

\[
K_\beta = \text{diag}(k_{\beta,1}, \ldots, k_{\beta,p}), \quad \text{for} \quad i = 1, \ldots, p. \tag{E.2}
\]

The loop transformation is performed as is depicted in Figure E.2. It consists of input feedforward, an input multiplication followed by output feedback for the set-valued nonlinearity and an output feedback, multiplication of the output and input feedforward for the linear system.

It should be noted that if robustness with respect to the set-valued nonlinearity is desired, then the sector \([k_{\alpha,i}, k_{\beta,i}]\) should be taken larger than the smallest possible sector which encloses the set-valued nonlinearity.

Since the set-valued nonlinearity \(\varphi\) belongs to the sector \([K_\alpha, K_\beta]\), we can state the following inequality (see [16]):

\[
(-w) - K_\alpha z^T ((-w) - K_\beta z) \leq 0, \quad \forall(t, z). \tag{E.3}
\]
The transformation applied on the closed-loop system (5.3) results in the transformed linear system:

\[
\begin{align*}
\dot{x} &= \tilde{A}_{cl}x + G\tilde{w} \\
\tilde{z} &= Hx + I\tilde{w}, \\
\tilde{w}_i &\in -\tilde{\varphi}_i(\tilde{z}_i), \quad i = 1, \ldots, p,
\end{align*}
\]  

(E.4a)  

(E.4b)  

with the matrices

\[
\begin{align*}
\tilde{A}_{cl} &= A + BK - GK_\alpha H, \quad \text{and} \quad \tilde{H} = (K_\beta - K_\alpha)H,
\end{align*}
\]  

(E.5)
and where \( I \) is the unity matrix.

The new input of the transformed nonlinearity \( \varphi \) is given by

\[
\tilde{z} = -(-w) + K_\beta z.
\]  
(E.6)

The new output of the transformed nonlinearity \( \varphi \) is given by

\[
(-\tilde{w}) = (-w) - K_\alpha z.
\]  
(E.7)

The multiplication of the output with the input of the transformed nonlinearity \( \varphi \) is given by

\[
-w^T \tilde{z} = [(-w) - K_\alpha z]^T [(-w) + K_\beta z].
\]  
(E.8)

We recognize (E.3) with opposite sign in (E.8). Therefore, the transformed set-valued nonlinearity \( \tilde{\varphi} \) (E.4b) belongs to the sector \([0, \infty)\), i.e.

\[
\tilde{w}^T \tilde{z} \leq 0.
\]  
(E.9)

This implies that the transformed set-valued nonlinearity \( \tilde{\varphi} \) is passive.

### E.2 Circle criterion control design after loop transformations

We consider the transformed closed-loop system (E.4). We assume that the transformed set-valued nonlinearity is passive after the application of the loop transformations in Appendix E.1.

Next, we investigate the passivity properties of the transformed linear system (E.4a) and the stability properties of the total closed-loop system (E.4). To investigate the stability properties of the origin of the transformed system (E.4) we adopt the quadratic Lyapunov candidate function (5.6).

The time-derivative of the Lyapunov candidate function (5.6) becomes:

\[
\dot{V} = \frac{1}{2} \dot{x}^T P x + \frac{1}{2} x^T P \dot{x} \\
= \frac{1}{2} (x^T (\tilde{A}_{cl}^T P + P \tilde{A}_{cl}) x + \tilde{w}^T G^T P x + x^T PG \tilde{w}).
\]  
(E.10)

If we consider the Lyapunov function \( V(x) \) as the storage function for the linear system, then the following inequality must hold for strict passivity of the linear system:

\[
\dot{V} < \tilde{w}^T \tilde{z}.
\]  
(E.11)

If we use the equation \( \tilde{z} = \tilde{H} x + I \tilde{w} \) of (E.4a), we can write expression (E.11) to:

\[
\dot{V} < \frac{1}{2} \tilde{w}^T \tilde{z} + \frac{1}{2} \tilde{z}^T \tilde{w} \\
= \frac{1}{2} (\tilde{w}^T \tilde{H} x + \tilde{w}^T I \tilde{w} + x^T \tilde{H}^T \tilde{w} + \tilde{w}^T I^T \tilde{w}).
\]  
(E.12)
If we combine (E.10) and (E.12), then we write the following expression

\[
\frac{1}{2} \begin{bmatrix} x^T (\tilde{A}_{cl}^T P + P \tilde{A}_{cl}) x + \tilde{w}^T G^T P x + x^T PG \tilde{w} \\
-\tilde{w}^T \tilde{H} x - x^T \tilde{H}^T \tilde{w} - \tilde{w}^T I \tilde{w} - \tilde{w}^T I^T \tilde{w} \end{bmatrix} < 0. \tag{E.13}
\]

We can write inequality (E.13) in a matrix form:

\[
\frac{1}{2} \begin{bmatrix} x \\ \tilde{w} \end{bmatrix}^T \begin{bmatrix} \tilde{A}_{cl}^T P + P \tilde{A}_{cl} & PG - \tilde{H}^T \\
G^T P - \tilde{H} & -2I \end{bmatrix} \begin{bmatrix} x \\ \tilde{w} \end{bmatrix} < 0. \tag{E.14}
\]

From (E.14) we can derive the condition for strict positive realness of the transformed linear system (E.4a); i.e. the following matrix inequality has to be satisfied:

\[
\begin{bmatrix} \tilde{A}_{cl}^T P + P \tilde{A}_{cl} & PG - \tilde{H}^T \\
G^T P - \tilde{H} & -2I \end{bmatrix} < 0, \tag{E.15}
\]

for \( P = P^T > 0 \).

If the conditions (E.9) and (E.15) are satisfied, then we can prove that the origin of the closed-loop system (E.4) is globally asymptotically stable. For this purpose we write (E.11) as:

\[
\dot{V} - \tilde{w}^T \tilde{z} < 0. \tag{E.16}
\]

From (E.9) we know that \( \tilde{w}^T \tilde{z} \leq 0 \), this implies that \( \dot{V} < 0 \), and thus the origin of the transformed closed-loop system (E.4) or the origin of the closed-loop system in the form of (5.3) is globally asymptotically stable.

If the conditions (E.9) and (E.15) hold, then the origin of the closed-loop system (5.3) is absolutely stable, since the origin is globally asymptotically stable for any (transformed) set-valued nonlinearity belonging to the sector \([0, \infty] \).

### E.3 Control for a range of equilibria

In Section 5.2, it is shown that the closed-loop system (5.3) is absolutely stable if transfer function of the linear part of the system is strictly positive real (SPR) and the set-valued nonlinearity in the feedback loop is passive. This implies that the origin of the closed-loop system (5.3) is globally asymptotically stable. In this appendix, we show that a range of equilibria of the closed-loop system (after application of a control law consisting of a constant input and a linear state-feedback, to the open-loop system (5.1)) are globally asymptotically stable for some conditions. A different equilibrium point can be obtained by applying a constant input to system (5.1). And a different equilibrium point means a different corresponding equilibrium input and output for the set-valued nonlinearities. If the equilibrium of the system is changed, then the origin is not the equilibrium point of the system, where this is required for the approach of the circle criterion. A coordinate transformation and a transformation of the set-valued nonlinearity are necessary to obtain a transformed system for which the origin is the equilibrium point.
Consider the open-loop system \( \dot{x} = Ax + Gw + Bu_c + Bv \) with input \( u = u_c + v \), with the constant input \( u_c \) and the new control input \( v \). Consequently, the system becomes

\[
\dot{x} = Ax + Gw + Bu_c + Bv \\
z = Hx \\
w_i \in -\varphi_i(z_i), \quad i = 1, \ldots, p,
\]

with \( u_c, v \in \mathbb{R}^m \). If \( u_c = 0 \), then the origin \( x = 0 \) is the equilibrium point of system (E.17), i.e. \( 0 \in \varphi_i(0), i = 1, \ldots, p \). If \( u_c \neq 0 \), then the equilibrium point is \( x = x_{eq} \). The equilibrium equations are

\[
A x_{eq} + Gw + Bu_c = 0 \\
z_{eq} = Hx_{eq} \\
w_i \in -\varphi_i(z_{eq,i}), \quad i = 1, \ldots, p.
\]

We apply a coordinate transformation \( \xi = x - x_{eq} \) to obtain a system with the origin as an equilibrium point. This results in the following system:

\[
\dot{\xi} = A\xi + A x_{eq} + Gw + Bu_c + Bv \\
\dot{z} = H\xi \\
w_i \in -\varphi_{tr,i}(z_i), \quad i = 1, \ldots, p,
\]

where \( \tilde{z} = z - H\xi_{eq} \) and \( \varphi_{tr,i}(\tilde{z}_i) = \varphi_i(\tilde{z}_i + z_{eq,i}) \), for \( i = 1, \ldots, p \), with \( z_{eq,i} = [H_{i1}, \ldots, H_{in}]x_{eq} \). The transformed set-valued nonlinearity \( \varphi_{tr,i} \), for the example in Figure E.3, is shown in Figure E.4(a).

We transform the set-valued nonlinearity \( \varphi_{tr,i}(\tilde{z}_i) \), for \( i = 1, \ldots, p \), into another set-valued nonlinearity \( \varphi_{tr2,i}(\tilde{z}_i) \) by adding a term. This change will be canceled in the linear part of the system. The changed system is given by

\[
\dot{\xi} = A\xi + A x_{eq} + G\tilde{w} + Bu_c + Bv - G\varphi(Hx_{eq}) \\
\dot{z} = H\xi \\
\tilde{w}_i \in -\varphi_{tr2,i}(z_i), \quad i = 1, \ldots, p,
\]
where $\varphi_{tr,i}(\tilde{z}_i) = \varphi_{tr,i}(\tilde{z}_i) - \varphi_i([H_{i1}, \ldots, H_{in}]x_{eq})$, for $i = 1, \ldots, p$. The transformed set-valued nonlinearity $\varphi_{tr2,i}$, for the example in Figure E.3 and Figure E.4(a), is shown in Figure E.4(b).

According to (E.18), the term $Ax_{eq} + Bu_c - G\varphi(Hx_{eq}) = 0$. Therefore, we can write system (E.20), after application of the proposed control law $v = K\xi$, as

$$
\dot{\xi} = (A + BK)\xi + G\tilde{w}
$$

$$
\tilde{z} = H\xi
$$

$$
\tilde{w}_i \in -\varphi_{tr2,i}(\tilde{z}_i), \quad i = 1, \ldots, p,
$$

which is a closed-loop system with the origin $\xi = 0$ as an equilibrium point corresponding to $v = 0$ (i.e. $u = u_c$). If the original set-valued nonlinearity $\varphi_i$, for $i = 1, \ldots, p$, satisfies the sector condition $[0, \infty)$, which implies that the mapping crosses the origin of the mapping, then the mapping of the transformed set-valued nonlinearity $\varphi_{tr2,i}$, for $i = 1, \ldots, p$, will also crosses the origin of its mapping. This means that the value of the transformed set-valued mapping corresponding to the equilibrium $\xi = 0$ is zero, which, of course, is necessary.

Depending on the constant input $u_c$, we obtain a new system. But the linear part of the system (E.21) will be the same for all $u_c$. The set-valued nonlinearity $\varphi_{tr2,i}$, for $i = 1, \ldots, p$, will change due to the constant input $u_c$. The set-valued nonlinearity contains discontinuities. When the discontinuity for the set-valued nonlinearity $\varphi_i$ occurs at the origin of the mapping, then the set-valued nonlinearity $\varphi_i$ belongs to the sector $[k_1, \infty]$ for $k_1 \in \mathbb{R}$. Since we want to control the system for all equilibria, the linear state-feedback control law is to be designed for a set-valued nonlinearity that belongs to the sector $[k_1, \infty]$ for $k_1 \in \mathbb{R}$.

Consider a set-valued nonlinearity $\varphi_i$ in Figure E.5; it belongs to the sector $[k_1, \infty]$ for some $k_1 > 0$. $k_1 > 0$ is required for global asymptotic stability of one equilibrium of closed-loop system 5.3 with the controller based on the circle criterion, see Section 5.2. If the constant input $u_c$ changes, then the transformed set-valued nonlinearity $\varphi_{tr2,i}$ may be as depicted in Figure E.6(a), it also belongs to the sector $[k_1, \infty]$ for some $k_1 > 0$. For another constant input voltage $u_c$ the transformed set-valued nonlinearity $\varphi_{tr2,i}$ may be as depicted in Figure E.6(b). The transformed set-valued nonlinearity $\varphi_{tr2,i}$ in Figure E.6(b) does not belong to the sector $[k_1, \infty]$ for any $k_1 > 0$. The approach with the loop transformations as depicted in Figure E.2 can not be applied to the design of a linear state-feedback controller.
control law for the system as presented before in this section, since we want to control a system for all equilibria with one linear state-feedback control law. To ensure that the set-valued nonlinearity in the feedback loop belongs to the sector \([0, \infty]\), we demand monotonicity for the set-valued nonlinearity \(\varphi\):

\[
[\varphi_i(a) - \varphi_i(b)](a - b) > 0 \quad \text{for} \quad a, b \in \mathbb{R}, \quad \text{for} \quad i = 1, \ldots, p,
\]  

(E.22)

together with the requirement that \(0 \in \varphi_i(0)\) for \(i = 1, \ldots, p\).

We propose another loop transformation when a system is to be controlled by one linear state-feedback control law for all equilibria. First, we state the following inequality for the derivative of \(\varphi_i\) with respect to \(z_i\):

\[
\left. \frac{d\varphi_i}{d z_i} \right|_{z_i \neq z_{e,i}} \geq l_i \quad i = 1, \ldots, p,
\]  

(E.23)

where the discontinuity of the set-valued nonlinearity \(\varphi_i\) is located at \(z_{e,i}\).
The proposed loop transformation is an output-feedback for the linear part of the system and an input-feedforward for the set-valued nonlinearity as depicted in Figure E.7. The loop transformation matrix $L$ is defined as:

$$ L = \text{diag}(l_1, \ldots, l_p). \quad (E.24) $$

The loop transformation adds a linear term to the set-valued nonlinearity $\varphi_i$ to make it monotone if necessary. The loop transformation also removes a linear term from the set-valued nonlinearity $\varphi_i$ if possible.

Consequently after the loop transformation, the closed-loop system (E.21) becomes

$$
\begin{align*}
\dot{\xi} &= \tilde{A}_{cl} \xi + G \tilde{w} \\
\tilde{z} &= H \xi \\
\tilde{w}_i &= -\tilde{\varphi}_i(z_i), \quad i = 1, \ldots, p,
\end{align*}
$$

(E.25)

where $\tilde{A}_{cl} = A + BK - GLH$ and $\tilde{\varphi}_i(z_i) = \varphi_{tr2,i}(z_i) - l_i z_i$ for $i = 1, \ldots, p$.

The set-valued nonlinearity $\tilde{\varphi}_i$ belongs to the sector $[0, \infty]$ and thus $\tilde{w}^T \tilde{z} \leq 0$. The linear part of system (E.25) must be SPR, the matrix inequality which ensures SPR of the linear part of the system (E.25) can be stated as

$$
\begin{bmatrix}
\tilde{A}_{cl}^T P + P \tilde{A}_{cl} & PG - H^T \\
G^T P - H & 0
\end{bmatrix} < 0,
$$

(E.26)

with $P = P^T > 0$.

If the matrix inequality (E.26) holds for some $P = P^T > 0$ and $\tilde{w}^T \tilde{z} \leq 0$ due to the loop transformation, then system (E.25) is rendered absolutely stable for all equilibria by one linear state-feedback control law, since the set-valued nonlinearities corresponding to all these equilibria belong to the sector $[0, \infty]$. 

Figure E.7: Loop transformation of the closed-loop system for obtaining monotonicity for the set-valued nonlinearity $\varphi_{tr}$. 

Transformations of the rotor dynamic system

The rotor dynamic system is described by a model (3.3) with the state $x$, the control input $u$ and set-valued friction maps in Chapter 3. The rotor dynamic model is written as a Lur’e type system for the approach of the observer design and the control design, which is described in Section F.1. The two set-valued friction maps are in the feedback loop of the model.

In the second section, a transformation which transforms the rotor dynamic system (F.7) to a single-input-single-output (SISO) system with respect to set-valued nonlinearities in the feedback loop, is discussed. One of the set-valued friction maps in the feedback loop is partly compensated by the control input and the remaining friction is linear and will be included in the linear part of the system. In the last section, a loop transformation is applied to the SISO rotor dynamic system to render the set-valued nonlinearity in the feedback loop monotone. The monotonicity is required for the application of the controller based on the Popov criterion to the rotor dynamic system.

F.1 Coordinate transformation

We consider the rotor dynamic system (3.3). For a constant input voltage $u_c$, such that $x_{eq}$ is a unique isolated equilibrium point of (3.3), we employ a coordinate transformation. This coordinate transformation ensures that the origin is the unique equilibrium point of the transformed system. The coordinate transformation is convenient for the observer and controller designs in Chapter 4 and 5.

The new states are:

$$
\begin{align*}
\xi_1 &= x_1 - x_{1eq} = \alpha - \alpha_{eq} \\
\xi_2 &= x_2 - x_{2eq} = \omega_u - \omega_{eq} \\
\xi_3 &= x_3 - x_{3eq} = \omega_l - \omega_{eq}.
\end{align*}
$$

(F.1)

The control input is $u = u_c + v = \frac{1}{J_m}[k_\theta \alpha_{eq} + T_{fu}(\omega_{eq})] + v$, where $v$ is the input for the transformed model. The input voltage $v$ will be zero for the uncontrolled system. The state-space equations of the transformed system are given by

$$
\begin{align*}
\dot{\xi}_1 &= \xi_2 - \xi_3 \\
\dot{\xi}_2 &= \frac{1}{J_t}[-k_\theta \xi_1 - k_\theta \alpha_{eq} + k_m u_c - T_{fu, tr1}(\xi_2) + k_m v] \\
\dot{\xi}_3 &= \frac{1}{J_t}[k_\theta \xi_1 + k_\theta \alpha_{eq} - T_{fl, tr1}(\xi_3)].
\end{align*}
$$

(F.2)
with the transformed friction model for the upper disc $T_{fu, tr1}(\xi_2)$, see Figure F.1, as follows:

$$T_{fu, tr1}(\xi_2) \in \begin{cases} T_{cu, tr1}(\xi_2) \, \text{sgn}(\xi_2 + \omega_{eq}) & \text{for } \xi_2 \neq -\omega_{eq} \\ [-T_{su} + \Delta T_{su}, T_{su} + \Delta T_{su}] & \text{for } \xi_2 = -\omega_{eq}, \end{cases} \quad (F.3)$$

where the function $T_{cu, tr1}(\xi_2)$ is given by

$$T_{cu, tr1}(\xi_2) = T_{su} + \Delta T_{su} \, \text{sgn}(\xi_2 + \omega_{eq}) + b_u |\xi_2 + \omega_{eq}| + \Delta b_u (\xi_2 + \omega_{eq}); \quad (F.4)$$

and with the transformed friction model for the lower disc $T_{fl, tr1}(\xi_3)$, see Figure F.2:

$$T_{fl, tr1}(\xi_3) \in \begin{cases} T_{cl, tr1}(\xi_3) \, \text{sgn}(\xi_3 + \omega_{eq}) & \text{for } \xi_3 \neq -\omega_{eq} \\ [-T_{sl}, T_{sl}] & \text{for } \xi_3 = -\omega_{eq}, \end{cases} \quad (F.5)$$

where the continuous function $T_{cl, tr1}(\xi_3)$ is given by

$$T_{cl, tr1}(\xi_3) = T_{cl} + (T_{sl} - T_{cl}) e^{-\frac{\xi_3 + \omega_{eq}}{\delta_{sl}}} + b_l |\xi_3 + \omega_{eq}|. \quad (F.6)$$

![Figure F.1: Transformed upper friction model $T_{fu, tr1}(\xi_2)$.](image)

The transformed rotor dynamic model can be written into a form with a linear time-invariant system in the forward path and an upper semi-continuous nonlinearity in the feedback part. This form of the model is convenient for application of the circle criterion and the Popov criterion. These criterions will be discussed in the context of the observer and controller design in the Chapters 4 and 5. The constants $-k_{\theta} \alpha_{eq} + k_m u_c$ and $k_{\theta} \alpha_{eq}$ in the right-hand side of the differential equations for $\xi_2$ and $\xi_3$, (F.2), will be added to the friction models $T_{fu, tr1}(\xi_2)$ and $T_{fl, tr1}(\xi_3)$, see Figures F.1 and F.2 to create new friction models $T_{fu, tr2}(\xi_2)$ and $T_{fl, tr2}(\xi_3)$, see Figures F.4 and F.5. This results in a linear system for which zero input yields zero output.
Then the model of the rotor dynamic system can be written as follows, see also Figure F.3 for the structure of the model:

\[ \dot{x} = Ax + Bv + Gw \]
\[ z = Hx \]
\[ w \in -\varphi(z), \]

with \( x \in \mathbb{R}^3, z \in \mathbb{R}^2, \varphi : \mathbb{R}^2 \to \mathbb{R}^2, w \in \mathbb{R}^2 \) and with the matrices given by

\[
A = \begin{bmatrix} 0 & 1 & -1 \\ -\frac{k_u}{J_u} & 0 & 0 \\ \frac{k_v}{J_l} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{k_m}{J_u} \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},
\]
\[
H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \varphi(x) = \begin{bmatrix} \varphi_1(z_1) \\ \varphi_2(z_2) \end{bmatrix} = \begin{bmatrix} T_{fu,tr1}(z_1) \\ T_{fl,tr1}(z_2) \end{bmatrix}.
\]

The set-valued nonlinearities are given by

\[
T_{fu,tr2}(z_1) = T_{fu,tr1}(z_1) - k_m u_e + k_\theta \alpha_{eq}
\]
\[ = T_{fu,tr1}(z_1) - T_{fu}(\omega_{eq}), \]

\[
T_{fl,tr1}(0) = T_{fl}(\omega_{eq})
\]

Figure F.2: Transformed lower friction model \( T_{fl,tr1}(\xi_3) \).

Figure F.3: System with feedback connection.
We consider the rotor dynamic system (F.7) in Lur’e form. The Coulomb friction and the asymmetric part of the viscous friction acting on the upper disc (included in the friction map $T_{fu,tr2}$ of system (F.7)) are compensated by applying the following control law $v$ to the rotor dynamic system (F.7):

$$T_{fu,tr2}(z_1) = T_{fu,tr1}(z_1) - k_b \phi_{eq} = T_{fu,tr1}(z_1) - T_{fu}(\omega_{eq}). \quad (F.11)$$

### F.2 Friction compensation for the friction at the upper disc

We consider the rotor dynamic system (F.7) in Lur’e form. The Coulomb friction and the asymmetric part of the viscous friction acting on the upper disc (included in the friction map $T_{fu,tr2}$ of system (F.7)) are compensated by applying the following control law $v$ to the rotor dynamic system (F.7):
The feedback loop given by \( k_m v^*_{\text{comp}} \) with the original rotor dynamic system in Figure F.6(a). The reason for not compensating all the friction acting at the upper disc is that with the remaining viscous damping \( -\Delta b_u \xi_2 \) the dynamic behaviour of the transformed rotor dynamic system will not differ much with the original rotor dynamic system (F.7). The compensation control law as the new control input and where the compensation control law \( v_{\text{comp}} \) is given by

\[
v_{\text{comp}}(\xi_2) = \begin{cases} 
\frac{1}{k_m} (T_{su} + \Delta T_{su} \text{sgn}(\xi_2 + \omega_{eq}) + \Delta b_u [\xi_2 + \omega_{eq}] - T_{fu}(\omega_{eq})) & \text{for } \xi_2 \neq -\omega_{eq} \\
\frac{1}{k_m} (-T_{su} + \Delta T_{su} - T_{fu}(\omega_{eq}), T_{su} + \Delta T_{su} - T_{fu}(\omega_{eq})) & \text{for } \xi_2 = -\omega_{eq}.
\end{cases}
\]  

The set-valued nonlinearity \( T_{fu} \) is described by (3.4) and the equilibrium velocity \( \omega_{eq} \) corresponds to (3.8). The compensation control law \( v_{\text{comp}} \) partly compensate the friction acting on the upper disc. The remaining viscous damping \( -\frac{1}{J_u} (b_u - \Delta b_u) \xi_2 \) will be included in the linear part of the system. The reason for not compensating all the friction acting at the upper disc is that with the remaining viscous damping the dynamic behaviour of the transformed rotor dynamic system will not differ much with the original rotor dynamic system (F.7). The compensation \( k_m v_{\text{comp}} \) for rotor dynamic system (F.7) is depicted in Figure F.6(b). We will show for clarity the compensation \( k_m v^*_{\text{comp}} \) for rotor dynamic system (3.3) with \( v^*_{\text{comp}} \) as the compensation control law is depicted in Figure F.6(a). The compensation \( k_m v^*_{\text{comp}} \) represents a torque for system (3.3). The resulting viscous friction acting at the upper disc is shown in Figure F.7(a).

After the application of the control law for \( v \) in (F.12), the model of the rotor dynamic system (F.7) transforms to:

\[
\begin{align*}
\dot{\xi} &= A_{tr} \xi + B v_{\text{control}} + G_{tr} w \\
z &= H_{tr} \xi \\
w &\in -\varphi(z)
\end{align*}
\]  

with \( \xi \in \mathbb{R}^3, v_{\text{control}}, w, z, \varphi : \mathbb{R} \rightarrow \mathbb{R} \), and with the matrices and the set-valued nonlinearity \( \varphi \) in the feedback loop given by

\[
A_{tr} = \begin{bmatrix} 0 & 1 & -1 \\ -\frac{k_u}{J_u} & -\frac{1}{J_u} (b_u - \Delta b_u) & 0 \\ \frac{k_u}{J_l} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{k_l}{J_l} \end{bmatrix}, \quad G_{tr} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]  

(F.15)
The set-valued nonlinearity $T_{f_{l,tr}}$ is given by (F.11).

### F.3 Loop transformation of the rotor dynamic system

For the application of the controller based on the Popov criterion to the rotor dynamic system (F.14), the set-valued nonlinearity $\varphi$ in the feedback loop has to be monotone. But the set-valued nonlinearity is not monotone, see Figure F.7(b). In order to overcome this problem, the proposed loop transformation, presented in Appendix E.3 is applied to the rotor dynamic system (F.14). The loop transformation yields an output feedback for the linear system in the forward path and an input feedforward for the set-valued nonlinearity in the feedback loop, see Figure F.8. The value $m = 0.1 \text{ Nms/rad}$ is chosen for the loop transformation, which transforms the friction map $T_{f_{l,tr}}$ into the monotone friction map $T_{f_{l,tr}'}$.

The minimum linear damping needed to render the transformed friction $T_{f_{l,tr}'}$ monotone (i.e. the derivative of $T_{f_{l,tr}'}$ with respect to $\xi_3$ is larger or equal to zero), is less than the chosen value for $m$. In that way, robustness with respect to rendering a changed friction $T'_{f_{l,tr}'}$ monotone is obtained, where $T'_{f_{l,tr}'}$ represents the actual friction acting on the lower disc which is different from the modeled friction $T_{f_{l,tr}'}$.

The transformed system can be described by

$$\dot{\xi} = A_{tr2} \xi + B_{v\text{control}} \nu + G_{tr} \tilde{w}$$

$$z = H_{tr} \xi$$

$$\tilde{w} \in -\varphi_{tr}(z),$$

with $\varphi_{tr} : \mathbb{R} \rightarrow \mathbb{R}$, and where

$$\varphi_{tr}(z) = T_{f_{l,tr}'}(z).$$

(F.17)
The set-valued nonlinearity \( T_{fl,tr}^3 \) is given by
\[
T_{fl,tr}^3(z) = T_{fl,tr}^2(z) + mz. \tag{F.19}
\]

The transformed system (F.17) is depicted in Figure F.9 and the set-valued nonlinearity \( T_{fl,tr}^3 \) is plotted in Figure F.10.

The matrix \( A_{tr}^2 \) is given by
\[
A_{tr}^2 = \begin{bmatrix}
0 & 1 & -1 \\
\frac{k_u}{J_u} & -\frac{1}{J_u}(b_u - \Delta b_u) & 0 \\
\frac{m}{J_l} & 0 & \frac{m}{J_l}
\end{bmatrix}. \tag{F.20}
\]

The same analysis approach as presented in Chapter 3 can be applied to the new system. Therefore the analysis of the new system is not presented here. Note that the partial canceling of the friction at the upper disc shows that the qualitative dynamic behaviour is largely determined by the specific nonlinear friction at the lower disc. We show the bifurcation diagrams of system (F.17) in Figures F.11(a) and F.11(b). The transformed rotor dynamic system (F.17) is different from the original rotor dynamic system (3.3) (since we consider a different control input). Nevertheless, the steady-state behaviour of system (F.17) does not change qualitatively with respect to the original system (3.3), but quantitatively.

This can be seen in the bifurcation diagrams in the Figures F.11(a) and F.11(b) for system (F.17) and the bifurcation diagrams in the Figures 3.12 and 3.13 for system (3.3) (the bifurcation parameter is different for system (F.17) compared to system (3.3)).

Note that the equilibria of the new system (F.17) differ from the original system (3.3).
Figure F.10: Model of the friction $T_{fl,tr3}$ acting on the lower disc.

Figure F.11: Bifurcation diagrams of the open-loop system (F.17) with compensated friction at the upper disc for positive input voltages.
Dynamic behaviour of the rotor dynamic system for friction characteristic I

Steady-state and transient responses of the rotor dynamic system for the nominal case (friction characteristic I) are presented in this Appendix. The corresponding parameters can be found in Table 3.2.

G.1 Steady-state responses

The steady-state responses for several constant input voltages are shown in Figures G.1-G.5, where the solid lines represent the experimental result and the dashed lines represent the simulated results. Only one stable solution exists for the input voltage $u_c = 1.3$ V, a periodic solution, see Figure G.1. Coexistence of stable solutions appear for the input voltages $u_c = 3.0$ V and $u_c = 3.5$ V, see Figure G.2-G.4, where the equilibrium response to the constant input voltage $u_c = 3.5$ V is already shown in Figure 3.17. The experimental responses for the equilibria show some perturbations. When the constant input voltage is further increased then for the input voltage of $u_c = 4.5$ V there exists only an stable equilibrium point, Figure G.5.

The simulation responses and experimental responses, presented in Figures G.1-G.5, show that there is a good match between the experiments and simulations for steady-state solutions.

![Figure G.1](image1.png)

Figure G.1: Measured and simulated limit cycle responses of the rotor dynamic system for $u_c = 1.3$ V.
Figure G.2: Measured and simulated limit cycle responses of the rotor dynamic system for $u_c = 3.0$ V.

Figure G.3: Measured and simulated equilibrium responses of the rotor dynamic system for $u_c = 3.0$ V.

Figure G.4: Measured and simulated limit cycle responses of the rotor dynamic system for $u_c = 3.5$ V.
Figure G.6 shows the Hopf bifurcation, a transition from an equilibrium point to a periodic solution while the input voltage is decreased with a small step, see the bifurcation diagrams in Figure 3.12 and Figure 3.13. The discontinuous fold bifurcation is shown in Figure G.7. The solution goes from a periodic solution to an equilibrium point after some time. It should be noted that the input voltage is increased before \( t = 0 \) s in Figure G.7.
Figure G.7: Measured responses of the discontinuous fold bifurcation.
Appendix H

Application of the control design to the rotor dynamic system with friction characteristic I

In the first section, the feasibility conditions of the application of a circle-criterion-inspired state-feedback controller are applied to the rotor dynamic system. The result is that the application of a circle-criterion-inspired state-feedback controller is not feasible. Therefore, two Popov-criterion-inspired state-feedback controllers are designed for the rotor dynamic system in Section 5.5. The Popov-criterion-inspired state-feedback controllers render the transfer function of the transformed linear part of the rotor dynamic system strictly positive real which is shown in the second section. In the last section, simulations with the two state-feedback controller are performed and discussed.

H.1 Feasibility conditions for the controller based on the circle criterion

The feasibility conditions, discussed in [2] and [4], will be applied to the rotor dynamic system (5.29). Hereto, we write the system (5.29) into a normal form by a state and feedback transformation to check the feasibility conditions:

\[\begin{align*}
\dot{y}_1 &= y_2 + g_1 \tilde{w} \\
\dot{y}_2 &= y_3 + g_2 \tilde{w} \\
\dot{y}_3 &= \tilde{v} + g_3 \tilde{w},
\end{align*}\]  

where \( \tilde{v} \) is the new input and \( g_1 = \frac{j}{J}, g_2 = \frac{m}{J} \) and \( g_3 = \frac{j}{J} \left( \frac{m^2}{J} - \frac{k\theta}{J} \right) \).

The coordinate transformation is applied according to the following definitions:

\[\begin{align*}
y_1 &= \xi_3 = z \\
y_2 &= \frac{k_{2}}{J} \xi_1 + \frac{m}{J} \xi_3 \\
y_3 &= \frac{k_{2m}}{J} \xi_1 + \frac{k_{2}}{J} \xi_2 + \left( \frac{m^2}{J} - \frac{k\theta}{J} \right) \xi_3.
\end{align*}\]  

The expression for the new control input \( \tilde{v} \) is:

\[\tilde{v} = \frac{k_{q} k_{m}}{J l_{u}} v_{control} + \frac{k_q}{J l} \left( \frac{m^2}{J^2_l} - \frac{k_q}{J_l} - \frac{k_q}{J_u} \right) \xi_1 + \frac{k_q}{J_l} \left( \frac{\Delta b_u}{J_u} - \frac{b_u}{J_u} + \frac{m^2}{J_l} \right) \xi_2 + \frac{m}{J_l} \left( \frac{m}{J_l} - 2k\theta \right) \xi_3.\]
If the system (5.29) with the linear state-feedback control law (5.33) is feasible using the circle criterion, then the following conditions must be satisfied: \( g_1 > 0 \) and \( g_2 < 0 \), see [2] and [4]. The circle criterion design for the system (5.29) with a linear state-feedback control law (5.33) is not feasible because \( g_2 > 0 \). This is caused by the negative damping \( m\xi_3 \) term in system (5.29) and the absence of a damping element in the link between the upper disc and the lower disc.

### H.2 Strict positive realness for the nominal case

Here, we check the strict positive realness of the linear part of the rotor dynamic system for the solutions found in Section 5.4. First, we consider the solution with \( K_1 \), \( P_1 \) and \( Q_1 \).

The transfer function \( G_{cl,K_1} \) (given by (5.35) with \( K = K_1 \)) of the closed-loop system without Popov transformation is:

\[
G_{cl,K_1} = \frac{28.57s^2 + 552.8s + 4126}{s^3 + 16.49s^2 + 91.26s + 165.4}.
\] (H.4)

And the corresponding transfer function \( H_{1,K_1}(s) \) (given by (5.39) with \( K = K_1 \)) of the transformed system is

\[
H_{1,K_1}(s) = \frac{285.7s^3 + 5557s^2 + 4.181 \cdot 10^4s + 4126}{s^3 + 16.49s^2 + 91.26s + 165.4}.
\] (H.5)

We check the frequency-domain conditions of strict positive realness, given in Section 2.2, for the closed-loop system with the gain \( K_1 \). The poles of \( H_{1,K_1}(s) \) have negative real parts: \(-6.2288 + 1.4867i\), \(-6.2288 - 1.4867i\) and \(-4.0340\). The Nyquist plot of the frequency response function \( H_{1,K_1}(j\omega) \) is shown in Figure H.1(b). The Nyquist plot of the frequency response function \( G_{cl,K_1}(j\omega) \) is shown in Figure H.1(a) for comparison. The Nyquist plot of the frequency response function \( H_{1,K_1}(j\omega) \) shows that the condition \( H_{1,K_1}(j\omega) + H_{1,K_1}^T(j\omega) \) is positive definite for all \( \omega \in \mathbb{R} \), while the frequency response function \( G_{cl,K_1}(j\omega) \) does not satisfy the condition.
H.2. Strict positive realness for the nominal case

The last condition is that $H_{1,K_1}(\infty) + H_{1,K_1}^T(\infty)$ is positive definite, which can be proven by the following expression of $H_{1,K_1}(\infty)$:

$$
\lim_{j\omega \to \infty} H_{1,K_1}(j\omega) = \lim_{j\omega \to \infty} \frac{285.7(j\omega)^3 + 5557(j\omega)^2 + 4.181 \cdot 10^4 j\omega + 4126}{(j\omega)^3 + 16.49(j\omega)^2 + 91.26 j\omega + 165.4} = 285.7.
$$

Then, we consider the second solution with $K_2$, $P_2$ and $Q_2$.

The transfer function $G_{cl,K_2}$ (given by (5.35) with $K = K_2$) of the closed-loop system without Popov transformation and the transfer function $H_{1,K_2}(s)$ (given by (5.39) with $K = K_2$) of the transformed system are, respectively,

$$
G_{cl,K_2} = \frac{28.57s^2 + 682.4s + 6096}{s^3 + 21.03s^2 + 147.2s + 341.7},
$$

$$
H_{1,K_2}(s) = \frac{285.7s^3 + 6853s^2 + 6.164 \cdot 10^4 s + 6096}{s^3 + 21.03s^2 + 147.2s + 341.7}.
$$

We also check the frequency-domain conditions of strict positive realness, given in Section 2.2, for the closed-loop system with the gain $K_1$. The poles of $H_{1,K_2}(s)$ are: $-7.6261 + 1.0022i$, $-7.6261 - 1.0022i$ and $-5.7754$. The Nyquist plots of the frequency response functions $G_{cl,K_1}(j\omega)$ and $H_{1,K_1}(j\omega)$, respectively, are shown in Figure H.2. These figures show that $H_{1,K_1}(j\omega) + H_{1,K_1}^T(j\omega)$ is positive definite for all $\omega \in \mathbb{R}$, while the frequency response function $G_{cl,K_1}(j\omega)$ does not satisfy the condition.

$H_{1,K_2}(\infty) + H_{1,K_2}^T(\infty)$ is positive definite, which can be proven by the following expression of $H_{1,K_2}(\infty)$:

$$
\lim_{j\omega \to \infty} H_{1,K_2}(j\omega) = \lim_{j\omega \to \infty} \frac{285.7(j\omega)^3 + 6853(j\omega)^2 + 6.164 \cdot 10^4 j\omega + 6096}{(j\omega)^3 + 21.03(j\omega)^2 + 147.2j\omega + 341.7} = 285.7.
$$

Figure H.2: Nyquist plots for control gain matrix $K_2$. 

(a) Nyquist plot of the system $G_{cl,K_2}(j\omega)$ with control gain $K_2$. 

(b) Nyquist plot of the transformed system $H_{1,K_2}(j\omega)M$ with control gain $K_2$. 


H.3 Simulations for state-feedback control of rotor dynamic system

In this section, we present simulations for the rotor dynamic system with the two controllers, with the gains $K_1$ and $K_2$, respectively. The simulations are performed for the closed-loop system with the parameters of the nominal case, see Table 3.2. First, we simulate the closed-loop system for the input voltage $u_c = 1.8$ V and $u_c = 4.0$ V. Next, we perform simulations for the same input voltages, but the actual input to the rotor dynamic system is saturated for voltages outside the range $[-5V, 5V]$. This is done, because the input voltage for the experimental set-up is limited to that range. Also simulations are performed for the rotor dynamic system for friction characteristic II. These results are discussed in Appendix K. The performed simulations can not be verified by experiments, since not all state components are measured. Therefore, we discuss only simulations in this chapter. We will show the responses of the rotor dynamic system for the state of the model of the form (3.3) with the equilibrium variables $x_{eq} = [\alpha_{eq}, \omega_{eq}, \omega_{eq}]^T$. The equilibrium point $x_{eq}$ of the model (3.3) corresponds to the origin $\xi = 0$ of the model in Lur’e-type form (5.29). We employ the control law $u = u_c + u_{comp} + BK(x - x_{eq})$, which is equivalent to the control law $v_{control} = K\xi$ applied to model (5.29). Note that we use the state of model (3.3), because it is favourable from the perspective of physical interpretation.

The response of the rotor dynamic system is shown in Figure H.3 for the input voltage $u_c = 1.8$ V, where the controller with the gain $K_1$ is switched on at $t = 10$ s. The solution for the open-loop rotor dynamic system (5.29) is a limit cycle (see also the bifurcation diagrams in Figure F.11(a) and Figure F.11(b)). The system with the controller switched on converges to the equilibrium values $\alpha_{eq} = 1.55$ rad and $\omega_{eq} = 3.15$ rad/s within 2 seconds. The closed-loop system with gain $K_1$ is compared with the closed-loop system with gain $K_2$, see Figure H.4 for the responses of the closed-loop systems and Figure H.5 for the control signals. When the controller is switched on, the control voltage $v$ and the responses $\alpha$ and $\omega$ initially increase more for the closed-loop system with gain $K_2$ than the closed-loop system with gain $K_1$. The closed-loop system with gain $K_2$ also converges faster to the equilibrium values and reaches faster an error which is numerically zero, this can be seen in Figure H.5. The controller is able to control the solution of the closed-loop rotor dynamic system (5.37) to an equilibrium (instead of the limit cycle solution of the open-loop rotor dynamic system (5.29)).

Now, we show the response of the closed-loop system (5.37) with gain $K_1$ for the input voltage $u_c = 4.0$ V in Figure H.6. Solutions coexists for the open-loop rotor dynamic system (5.29): a stable equilibrium and a stable limit cycle. The solution of the open-loop rotor dynamic system is a limit cycle for the first ten seconds. Here, the control voltage decreases first when the controller is switched on at $t = 10$ s. In this case, the controller is switched on at a different position in the limit cycle, compared to the previous simulation for the input voltage $u_c = 1.8$ V. The responses of the closed-loop system (5.37) with gain $K_1$ are compared to the closed-loop system with gain $K_2$ in the Figures H.7. Again, the closed-loop system with gain $K_2$ converges faster to the equilibrium values $\alpha_{eq} = 1.52$ rad and $\omega_{eq} = 7.06$ rad/s. The controller is able to control the closed-loop system (5.37) to a stable equilibrium. Only one solution exists for the closed-loop rotor dynamic system: a stable equilibrium (instead of the coexistence of solutions of the open-loop system (5.29)).

We construct bifurcation diagrams for the closed-loop system with respect to $\omega_l$ and $\alpha$ with $u_c$ as bifurcation parameter. The bifurcation diagrams are given in Figure H.8(a) and Figure H.8(b). The bifurcation diagrams are the same for the two controllers, since both controllers achieve absolute stability.

Each constant input $u_c$ of the bifurcation diagrams represents an unique Lur’e-type system. The linear system is the same for every constant input $u_c$, but the set-valued nonlinearity in the feedback loop is different and unique for every constant input $u_c$ (see also Appendix F.1 and Appendix E.3 for more information). Thus, We can conclude from the bifurcation diagrams that the simulations confirm the absolute stability of the closed-loop rotor dynamic system (5.37), since we achieved global asymptotic stability of the equilibrium of the rotor dynamic system (5.37) for several set-valued nonlinearities in the sector $[0, \infty]$. 
Figure H.3: Closed-loop system response for control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ switched on at $t = 10$ s; the constant input voltage is $u_c = 1.8$ V, the equilibrium values are $\alpha_{eq} = 1.55$ rad and $\omega_{eq} = 3.15$ rad/s.

Figure H.4: Comparison between the closed-loop system responses for the control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ (solid line) with those for the control gain $K_2 = [23.5 \ 2.07 \ 46.3]$ (dashed line); controllers are switched on at $t = 1$ s; the constant input voltage is $u_c = 1.8$ V, the equilibrium values are $\alpha_{eq} = 1.55$ rad and $\omega_{eq} = 3.15$ rad/s.
Figure H.5: Comparison of the error signals between the closed-loop system with control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ (black solid line) with the closed-loop system with control gain $K_2 = [23.5 \ 2.07 \ 46.3]$ (grey dashed line); controllers are switched on at $t = 1 \text{ s}$; the constant input voltage is $u_c = 1.8 \text{ V}$, the equilibrium values are $\alpha_{eq} = 1.55 \text{ rad}$ and $\omega_{eq} = 3.15 \text{ rad/s}$.

Figure H.6: Closed-loop system response for control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ switched on at $t = 10 \text{ s}$; the constant input voltage is $u_c = 4.0 \text{ V}$, the equilibrium values are $\alpha_{eq} = 1.52 \text{ rad}$ and $\omega_{eq} = 7.06 \text{ rad/s}$. 
Figure H.7: Comparison between the closed-loop system response for control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ (solid line) with those for the control gain $K_2 = [23.5 \ 2.07 \ 46.3]$ (dashed line); controllers are switched on at $t = 1$ s; the constant input voltage is $u_c = 4.0$ V, the equilibrium values are $\alpha_{eq} = 1.52$ rad and $\omega_{eq} = 7.06$ rad/s.

Figure H.8: Bifurcation diagrams of the closed-loop system for the two controllers for positive input voltages.
The simulations for the rotor dynamic system show a large control action, when the controller is switched on. But the experimental set-up cannot execute this large control action, since its voltage range is limited. The following simulations are performed where the input voltage $u$ of system (3.3) is limited to the range $[-5\,\text{V}, 5\,\text{V}]$.

The responses of the closed-loop system (5.37) with the control gains $K_1$ and $K_2$, respectively, are compared in Figure H.9. This figure clearly shows that the control voltage is saturated. However, the responses still converge to desired equilibrium. The closed-loop system with gain $K_1$ converges faster to the equilibrium values $\alpha = 1.55\,\text{rad}$ and $\omega_{eq} = 3.15\,\text{rad/s}$ than the closed-loop system with gain $K_2$. This is the opposite result compared to the simulations where the input voltage is not limited. The actuator saturation clearly influences the convergence of the closed-loop rotor dynamic system (5.37).

The response of the closed-loop system with gain $K_2$ is shown in Figure H.10. The controller is switched on at a different position of the limit cycle and the solution of the closed-loop system is again a periodic solution, a limit cycle. Here, the actuator saturation induces another limit cycle and the controller is not able to control the closed-loop rotor dynamic system to the equilibrium. This simulations shows that, with actuator saturation, the global asymptotic stability is indeed not guaranteed.

Next, we compare the responses of the closed-loop systems with the gains, $K_1$ and $K_2$, respectively, for the input voltage $u_c = 4.0\,\text{V}$ in Figure H.11. Again, the closed-loop system (5.37) with gain $K_1$ converges faster to the equilibrium values $\alpha = 1.52\,\text{rad}$ and $\omega_{eq} = 7.06\,\text{rad/s}$, than the closed-loop system with gain $K_2$. This is the opposite of what the simulations without saturation of the control voltage showed. The controllers are able to control the closed-loop rotor dynamic system (5.37) to a stable equilibrium, despite the actuator saturation.
Figure H.10: Closed-loop system response for the control gain $K_2 = [23.5 \ 2.07 \ 46.3]$; controller is switched on at $t = 5$ s; the constant input voltage is $u_c = 1.8$ V, the equilibrium values are $\alpha_{eq} = 1.55$ rad and $\omega_{eq} = 3.15$ rad/s (input voltage saturated).

Figure H.11: Comparison between the closed-loop system response for the control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ (solid line) with those for the control gain $K_2 = [23.5 \ 2.07 \ 46.3]$ (dashed line); controllers are switched on at $t = 5$ s; the constant input voltage is $u_c = 4.0$ V, the equilibrium values are $\alpha_{eq} = 1.52$ rad and $\omega_{eq} = 7.06$ rad/s (input voltage saturated).
We can conclude from the simulations without input saturation that the closed-loop systems with the gains $K_1$ and $K_2$, respectively, both converge to the equilibrium values. The closed-loop system with gain $K_2$ converges faster to the equilibrium values than the closed-loop system with gain $K_1$. If the input voltage is saturated, then the closed-loop system with gain $K_1$ converges faster to the equilibrium values than the closed-loop system with gain $K_2$. Therefore, in this case we have the opposite results related to the gains $K_1$ and $K_2$ with respect to the case of the simulations without input saturation.

The entries of the gain $K_2$ are larger than the entries of the gain $K_1$. Note that we feed back the state multiplied by the feedback gain. This implies that for a system state which differs from the equilibrium, the control voltage is larger for the closed-loop system with gain $K_2$ than for the closed-loop system with gain $K_1$. When the input voltage is limited, the control voltage is more often saturated for the closed-loop system with gain $K_2$ than for the closed-loop system with gain $K_1$. If the input voltage is saturated, then we have in fact a different system, for which we did not prove stability. We observe that the convergence is slower when the input voltage is saturated in the simulations. Moreover, the closed-loop system does not always converge to the equilibrium values. Whether or not convergence to the equilibrium is attained for the saturated case also depend on when the controller is switched on with respect to the initial condition of the closed-loop system. Since the closed-loop system with the gain $K_1$ converges faster to the equilibrium state in case of input saturation (considering the situation where the state actually converges to the equilibrium), we prefer the latter linear state-feedback control law.

The simulations for friction characteristic II in Appendix K show qualitative similar results for unsaturated input voltages and saturated input voltages. Here, the same controllers as for the nominal case are used, since the linear part of the closed-loop system is equal to the linear part of the closed-loop system for the nominal case. For this situation, we use the absolute stability property and apply the feedback gains $K_1$ and $K_2$. 
Rotordynamische Modellierung und Validierung für Friction Characteristic II


Tabelle I.1: Werte der geschätzten Parameter für Friction Characteristic II.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Schätzwert</th>
<th>Einheit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_m$</td>
<td>4.3228</td>
<td>[Nm/V]</td>
</tr>
<tr>
<td>$J_u$</td>
<td>0.4765</td>
<td>[kg m$^2$]</td>
</tr>
<tr>
<td>$T_{su}$</td>
<td>0.37975</td>
<td>[Nm]</td>
</tr>
<tr>
<td>$\Delta T_{su}$</td>
<td>-0.00575</td>
<td>[Nm]</td>
</tr>
<tr>
<td>$b_u$</td>
<td>2.4245</td>
<td>[kg m$^2$/rad s]</td>
</tr>
<tr>
<td>$\Delta b_u$</td>
<td>-0.0084</td>
<td>[kg m$^2$/rad s]</td>
</tr>
<tr>
<td>$k_\theta$</td>
<td>0.075</td>
<td>[Nm/rad]</td>
</tr>
<tr>
<td>$J_l$</td>
<td>0.035</td>
<td>[kg m$^2$]</td>
</tr>
<tr>
<td>$T_{sl}$</td>
<td>0.24</td>
<td>[Nm]</td>
</tr>
<tr>
<td>$T_{cl}$</td>
<td>0.02</td>
<td>[Nm]</td>
</tr>
<tr>
<td>$\omega_{sl}$</td>
<td>2.0</td>
<td>[rad/s]</td>
</tr>
<tr>
<td>$\delta_{sl}$</td>
<td>2.2</td>
<td>[-]</td>
</tr>
<tr>
<td>$\alpha_l$</td>
<td>0.01</td>
<td>[kg m$^2$/rad s]</td>
</tr>
</tbody>
</table>

Die Parameter des Friction Model für die obere Scheibe sind dieselben wie für den nominalen Fall, siehe Abbildung 3.9. Das Friction Model der unteren Scheibe ist in Abbildung I.1 dargestellt. Es ist möglich, die gleiche Analyse wie in Abschnitt 3.4 durchzuführen. Deshalb verweisen wir auf Abbildung 3.11 für die Stabilitäts Eigenschaften des Rotordynamischen Systems für Friction Characteristic II.


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Figure I.1: Friction torque of the lower disc for the estimated parameters (friction characteristic I).

Figure I.2: Bifurcation diagram with velocity of the lower disc for positive input voltages.
Figure I.3: Bifurcation diagram with the difference between the positions of the discs for positive input voltages.

Figure I.4: Diagram with the period times of the periodic solutions.
We continue with a discussion of the experimental results, which are compared with results from the simulations. We already see in the bifurcation diagrams, depicted in Figure I.2 and Figure I.3, a good similarity between the simulations and the experiments. The responses from the simulations are depicted with dotted line and the experimental results with a solid line in Figure I.5 to Figure I.11. Figure I.5 and Figure I.6 show limit cycles responses, for $\alpha$ and $\omega_l$, for the input voltages $u_c = 1.2$ V and $u_c = 2.0$ V, respectively. These input voltages belongs to the region with only stable limit cycles, see Figures I.2-I.4. Equilibrium responses and limit cycle responses from the region with coexistence of solutions are shown in the Figures I.7-I.10 for the input voltages of $u_c = 2.8$ V and $u_c = 3.5$ V. Figure I.11 shows the equilibrium responses $\alpha$ and $\omega_l$ for the input voltage $u_c = 4.2$ V, from the region with only stable equilibria.

The bifurcation diagrams depicted in Figure I.2 and Figure I.3 and the time responses depicted in Figures I.5-I.11, show that there is a good match between the simulations and the experiments for steady-state solutions.

![Figure I.5](image1.png)  
*Figure I.5: Measured and simulated limit cycle responses of the rotor dynamic system for $u_c = 1.2$ V.*

![Figure I.6](image2.png)  
*Figure I.6: Measured and simulated limit cycle responses of the rotor dynamic system for $u_c = 2.0$ V.*
Figure I.7: Measured and simulated limit cycle responses of the rotor dynamic system for $u_c = 2.8$ V.

Figure I.8: Measured and simulated equilibrium responses of the rotor dynamic system for $u_c = 2.8$ V.

Figure I.9: Measured and simulated limit cycle responses of the rotor dynamic system for $u_c = 3.5$ V.
Figure I.10: Measured and simulated equilibrium responses of the rotor dynamic system for $u_c = 3.5$ V.

Figure I.11: Measured and simulated equilibrium responses of the rotor dynamic system for $u_c = 4.2$ V.
Appendix J

Application of the observer design to the rotor dynamic system for friction characteristic II

We apply the observer design of Chapter 4 to the rotor dynamic system with the friction characteristic II. The model parameters can be found in Table I.1. These model parameters imply a rotor dynamic system with the same linear system as for the rotor dynamic system for the nominal case, see Section 4.3. The lower friction is not monotone, so a loop transformation is needed to render the set-valued nonlinearity monotone. We apply the same loop transformation with the loop transformation matrix $M$ (4.21), as for the rotor dynamic system for the nominal case, see Figure 4.4(a). We also use the same two pairs of gains for the observer: $N_1$ and $L_1$, $N_2$ and $L_2$, see (4.29) and (4.30). In the following section, the performance of the observers are compared in the simulations.

J.1 Simulation results of the observer design

We start with Figures J.1 and J.2, where the observer responses for the constant input voltage $u_c = 1.5$ V are displayed. The solution of the rotor dynamic system is a periodic solution with stick-slip. Initial condition for the rotor dynamic system is $\xi = [0 \ 0 \ 0]^T$. The initial condition for the two observers $N_1$ and $L_1$, $N_2$ and $L_2$, respectively, is $\hat{\xi} = [4 \ 4 \ 4]^T$. We compare the convergence of the solution of the two observers in Figure J.1 and Figure J.2. The higher the gains, the faster $\hat{\alpha}$ converges to $\alpha$. The higher the gain, the larger the error for $\hat{\omega}_u$ and $\hat{\omega}_l$ during the first time instants, but the faster the ultimate convergence to $\omega_u$ and $\omega_l$, respectively.

A transient is shown in Figures J.3 and J.4 for the observer with gains $N_1$ and $L_1$. The input voltage is $u_c = 2.3$ V for the first ten seconds, on $t = 2$ s, the input voltage is decreased with a step to $u_c = 1.8$ V. The solution of the rotor dynamic system changes from an equilibrium point to a periodic solution with stick-slip. The initial condition of the rotor dynamic system is $\xi = [\alpha_{eq} \ \omega_{eq} \ \omega_{eq}]^T$, the equilibrium values are corresponding to the input voltage $u_c = 2.3$ V. The initial condition for observer is $\hat{\xi} = [0 \ 0 \ 0]^T$. When the input voltage is changed, the observer error still converges to zero, see Figure J.4.
Figure J.1: Responses of the rotor dynamic system and the two observers with different gains for the input voltage $u_c = 1.5$ V.

Figure J.2: Observer error of the two observers with different gains for the input voltage $u_c = 1.5$ V.
Figure J.3: Responses of the rotor dynamic system and the observer with the gains $L_1$ and $N_1$, the input voltage decreases with a step on $t = 2$ s from $u_c = 2.3$ V to $u_c = 1.8$ V.

Figure J.4: Observer error of observer with the gains $L_1$ and $N_1$, the input voltage decreases with a step on $t = 2$ s from $u_c = 2.3$ V to $u_c = 1.8$ V.
J.2 Experimental results of the observer design

We implement the observer with the gains \( L_2 \) and \( N_2 \) on the experimental set-up. We prefer the observer with the gains \( L_2 \) and \( N_2 \), since it converges the fastest of all three observers, which are compared in the previous section.

The first observer experiment is performed for an input voltage \( u_c = 2.0 \, \text{V} \) for the observer. The solution of the rotor dynamic system is a limit cycle. Figure J.5 shows the state component \( \dot{\alpha} \) and the observer error component \( \alpha - \dot{\alpha} \) for the observer with the gains \( L_2 \) and \( N_2 \). The initial condition for the experimental rotor dynamic set-up is \( \xi = [0 \ 0 \ 0]^T \) and the initial condition for the observer is \( \dot{\xi} = [4 \ 4 \ 4]^T \).

The next observer experiment is performed for an input voltage \( u_c = 3.0 \, \text{V} \). The solution of the rotor dynamic system is a limit cycle. The state component \( \dot{\alpha} \) and the observer error component \( \alpha - \dot{\alpha} \) for the observer with the gains \( L_2 \) and \( N_2 \) are shown for a steady-state limit cycle response in Figure J.6.

The last observer experiment is performed for an input voltage \( u_c = 4.0 \, \text{V} \). The solution of the rotor dynamic system is an equilibrium point. The state component \( \dot{\alpha} \) and the observer error component \( \alpha - \dot{\alpha} \) for the observer with the gains \( L_2 \) and \( N_2 \) are shown for the steady-state equilibrium response in Figure J.7. The state components \( \hat{\omega}_u \) and \( \hat{\omega}_l \) the observer with the gains \( L_2 \) and \( N_2 \) are compared with the state components of the alternative observer (4.31) in Figure J.8.

The conclusion is that the observer state component \( \dot{\alpha} \) converge to the rotor dynamic state \( \alpha \). For all the experiments, the average value of the observer error \( \alpha - \dot{\alpha} \) is negative. This means that \( \dot{\alpha} \) is larger than \( \alpha \). A possible cause can be that the modeled friction acting on the lower disc is higher than the actual friction, as we also mentioned in Section 4.5.

![Graph](image1)

(a) Measured and observed state components \( \alpha \) (solid line) and \( \dot{\alpha} \) (dashed line), respectively.

![Graph](image2)

(b) The observer error \( |\alpha - \dot{\alpha}| \).

![Graph](image3)

(c) The observer error \( \alpha - \dot{\alpha} \).

Figure J.5: Comparison of the measured state component \( \alpha \) with the state component \( \dot{\alpha} \) of the observer with the gains \( L_2 \) and \( N_2 \) for \( u_c = 2.0 \, \text{V} \).
Figure J.6: Estimation $\hat{\alpha}$ and the observer error $\alpha - \hat{\alpha}$ of the observer with the gains $L_2$ and $N_2$ for $u_c = 3.0$ V.

Figure J.7: Estimation $\hat{\alpha}$ and the observer error $\alpha - \hat{\alpha}$ of the observer with the gains $L_2$ and $N_2$ for $u_c = 4.0$ V.

Figure J.8: Comparison of the responses $\hat{\omega}_u$ and $\hat{\omega}_l$ of the observer with the gains $L_2$ and $N_2$ (black line) and the responses $\hat{\omega}_u$ and $\hat{\omega}_l$ of the observer $\frac{200s}{s+200}$ (grey line) for $u_c = 4.0$ V.
Appendix J. Application of the observer design to the rotor dynamic system for friction characteristic II
Appendix K

Application of the control design to the rotor dynamic system with friction characteristic II

A control law will be designed for the rotor dynamic system (3.3) for friction characteristic II. The difference with the nominal case is that only the parameters of the lower friction model are different. Therefore we can apply the same transformation to the rotor dynamic system for friction characteristic II as described in Appendix F, to write the system to a Lur’e type system (F.14). The upper set-valued friction map is partly compensated such that the resulting friction acting at the upper disc is linear. The compensation of the friction acting at the upper disc is described in Appendix F.2. The transformed rotor dynamic system is given by (F.14). The lower friction model is depicted in Figure K.1(a). Also it is possible to apply the same loop transformation to the set-valued nonlinearity to render it monotone as described in Appendix F.3 (which is required for the application of the control design). The loop transformation is performed with $m = 0.1 \text{ Nms/rad}$; the transformed monotone friction model is shown in Figure K.1(b). The transformed rotor dynamic system after the loop transformation is given by (F.17) with friction characteristic II. The parameters of friction characteristic II can be found in Table I.1.

The corresponding bifurcation diagrams for the transformed system (F.17) for friction characteristic II are shown in Figure K.2(a) and Figure K.2(b).

![Bifurcation Diagrams](image)

(a) Model of the friction $T_{f1,TR2}$ acting on the lower disc.  
(b) Model of the friction $T_{f1,TR3}$ acting on the lower disc.

**Figure K.1:** Friction model and transformed for friction characteristic II.
As we did in the nominal case, we can draw the conclusion for friction characteristic II, that the design of a control law using the circle criterion for the examined rotor dynamic system is not feasible. The same proof as is discussed in Section 5.4 can be given. Then, we continue with the design of a control law using the Popov criterion. Since the linear part of the transformed rotor dynamic system (F.17) is the same as for the nominal case (we even use the same loop transformation parameter \(m\)), we use the absolute stability property and apply the same control gains \(K_1\) and \(K_2\) as computed in Section 5.5.

The closed-loop rotor dynamic system is given by

\[
\dot{\xi} = (A_{tr2} + BK)\xi + G_{tr}\tilde{w} \\
z = H_{tr}\xi \\
\tilde{w} \in -\varphi_{tr}(z),
\]

(K.1)

with the matrices \(A_{tr2}, B, G_{tr}, H_{tr}\) and \(\varphi_{tr}(z)\) given by (5.30) and (5.32).

In the next section, we perform simulations for the rotor dynamic system for friction characteristic II.

### K.1 State-feedback control design based on Popov criterion

Simulations are performed for the input voltage of \(u_c = 1.8\) V, where the open-loop system exhibits an asymptotically stable stick-slip limit cycle. The response of the rotor dynamic system is shown in Figure K.3 for the input voltage \(u_c = 1.8\) V, where the controller with the gain \(K_1\) is switched on at \(t = 10\) s. The system with the controller switched on converges to the equilibrium values \(\alpha_{eq} = 1.21\) rad and \(\omega_{eq} = 3.17\) rad/s within 2 seconds. The closed-loop system with gain \(K_1\) is compared with the closed-loop system with gain \(K_2\), see Figure K.4 for the responses of the closed-loop systems. When the controller is switched on, the control voltage \(v\) and responses \(\alpha\) and \(\omega_u\) increase more for the closed-loop system with gain \(K_2\) than the closed-loop system with gain \(K_1\). The closed-loop system with gain
$K_2$ also converges faster to the equilibrium values. The controller is able to control the closed-loop rotor dynamic system (K.1) to a stable equilibrium.

We show the response of the closed-loop system with gain $K_1$ for the input voltage $u_{c} = 4.0 \text{ V}$ in Figure K.5. Coexistence of solutions exists for the open-loop rotor dynamic system (F.17): a stable equilibrium and a stable limit cycle. The solution of the open-loop rotor dynamic system is a limit cycle for the first ten seconds. The responses of the closed-loop system with gain $K_1$ are compared to the closed-loop system with gain $K_2$ in the Figures K.6. Again, the closed-loop system with gain $K_2$ converges faster to the equilibrium values $\alpha_{eq} = 0.88 \text{ rad}$ and $\omega_{eq} = 7.07 \text{ rad/s}$. Only one solution exists for the closed-loop rotor dynamic system (K.1): a stable equilibrium (in stead of the coexistence of solutions of the open-loop system (F.17)).

The bifurcation diagram of the closed-loop system (K.1) for the two controllers is depicted in Figure K.7(a) and Figure K.7(b). Obviously, the bifurcation diagrams are the same for the two controllers. We can conclude from the bifurcation diagrams that the simulations confirm the absolute stability of the closed-loop rotor dynamic system (K.1).

Figure K.3: Closed-loop system responses with control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ switched on at $t = 10 \text{ s}$; the constant input voltage is $u_{c} = 1.8 \text{ V}$, the equilibrium values are $\alpha_{eq} = 1.21 \text{ rad}$ and $\omega_{eq} = 3.17 \text{ rad/s}$. 
Figure K.4: Comparison between the closed-loop system responses for the control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ (solid line) with those for the control gain $K_2 = [23.5 \ 2.07 \ 46.3]$ (dashed line); controllers are switched on at $t = 1$ s; the constant input voltage is $u_c = 1.8$ V, the equilibrium values are $\alpha_{eq} = 1.21$ rad and $\omega_{eq} = 3.17$ rad/s.

Figure K.5: Closed-loop system responses with control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ switched on at $t = 10$ s; the constant input voltage is $u_c = 4.0$ V, the equilibrium values are $\alpha_{eq} = 0.88$ rad and $\omega_{eq} = 7.07$ rad/s.
Figure K.6: Comparison between the closed-loop system responses for the control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ (solid line) with those for the control gain $K_2 = [23.5 \ 2.07 \ 46.3]$ (dashed line); controllers are switched on at $t = 1$ s; the constant input voltage is $u_c = 4.0$ V, the equilibrium values are $\alpha_{eq} = 0.88$ rad and $\omega_{eq} = 7.07$ rad/s.

Figure K.7: Bifurcation diagrams of the closed-loop system for the two controllers for positive input voltages (friction characteristic II for friction at the lower disc).
The simulations for the rotor dynamic system show a large control action, when the controller is switched on. But the experimental set-up cannot execute this large control action, since its voltage range is limited. The following simulations are performed where the input voltage \( u_c \) of the rotor dynamic system is limited to the range \([-5V, 5V]\).

The responses of the closed-loop system (K.1) with the gains, \( K_1 \) and \( K_2 \), respectively, for \( u_c = 1.8 \) V are compared in Figure K.8. The closed-loop system with gain \( K_1 \) converges faster to the equilibrium values \( \alpha = 1.21 \) rad and \( \omega_{eq} = 3.17 \) rad/s than the closed-loop system with gain \( K_2 \). This is the opposite result compared to the simulations where the input voltage is not limited. The controllers are able to control the closed-loop rotor dynamic system (K.1) to a stable equilibrium, despite the actuator saturation.

The response of the closed-loop system with gain \( K_2 \) is shown in Figure K.9. The controller is switched on at a different position of the limit cycle and the solution of the closed-loop system is again a periodic solution, a limit cycle. This means that the actuator saturation can lead to coexistence of solutions.

We compare the responses of the closed-loop systems with the gains, \( K_1 \) and \( K_2 \), respectively, for the input voltage \( u_c = 4.0 \) V in Figure K.10. Again, the closed-loop system with gain \( K_1 \) converges faster to the equilibrium values \( \alpha = 0.88 \) rad and \( \omega_{eq} = 7.07 \) rad/s, than the closed-loop system with gain \( K_2 \). This is the opposite of what the simulations without saturation of the control voltage showed. Here, the controllers are able to control the closed-loop system (K.1) to a stable equilibrium, despite the actuator saturation.
Figure K.9: Closed-loop system responses with control gain $K_2 = [23.5 \ 2.07 \ 46.3]$; controller is switched on at $t = 5$ s; the constant input voltage is $u_c = 1.8$ V, the equilibrium values are $\alpha_{eq} = 1.21$ rad and $\omega_{eq} = 3.17$ rad/s (input voltage saturated).

Figure K.10: Comparison between the closed-loop system responses for the control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ (solid line) with those for the control gain $K_2 = [23.5 \ 2.07 \ 46.3]$ (dashed line); controllers are switched on at $t = 5$ s; the constant input voltage is $u_c = 4.0$ V, the equilibrium values are $\alpha_{eq} = 0.88$ rad and $\omega_{eq} = 7.07$ rad/s (input voltage saturated).
K.3 Output-feedback control design based on Popov criterion

In this Appendix, the output-feedback design from Chapter 6 is applied to the rotor dynamic system with friction characteristic II. The linear part of the rotor dynamic system (F.17) is the same for friction characteristic I and friction characteristic II. Only the set-valued nonlinearity $\varphi_{tr}(z)$ differs between the systems with the different friction characteristics. Therefore, we use the same observer gains as in $L_2$ and $N_2$ (4.30). In Chapter 4, we showed that the origin of the observer error dynamics is globally exponentially stable. Also the same control gain $K_1$ as in (5.43) is used. The closed-loop rotor dynamic system is given by (6.62). In Chapter 6, we showed that the origin $\xi = 0$ of system (6.62) with friction characteristic II for $e = 0$ is globally asymptotically stable. It can easily be shown that the bound (6.50) holds for the set-valued nonlinearity $\tilde{w} \in -\varphi(z)$.

Then, we can conclude that $\xi(t) \to 0$ as $t \to \infty$, i.e. equilibrium $(\xi, e) = (0, 0)$ of system (6.62) with friction characteristic II is globally asymptotically stable, since firstly, $\xi = 0$ for the system (6.62) with friction characteristic II for $e = 0$ is globally asymptotically stable, secondly, the origin of the observer error dynamics is globally exponentially stable and, thirdly, boundedness of any $\xi(t)$ of (6.62) for any bounded input $e(t)$ is shown.

K.4 Simulations output-feedback design

Simulations are performed for the output-feedback controller for the rotor dynamic system. We choose the control gain as in $K_1$ (5.43). The observer gains as in $L_2$ and $N_2$ (4.30) are used.

The first simulation is performed for the input voltage $u_c = 1.8$ V. The initial condition for the system is $\xi = [0 \ 0 \ 0]^T$. The solution for the open-loop rotor dynamic system (F.17) is a limit cycle (see also the bifurcation diagrams in Figure K.2(a) and Figure K.2(b)). We show the response of both the rotor dynamic system with the unsaturated input voltage and the rotor dynamic system with the input voltage saturated in Figure K.11. The states of the rotor dynamic system with the unsaturated input and the rotor dynamic system with the saturated input both converge to the setpoint. However, the state of the rotor dynamic system with the saturated input converges slower to the setpoint.

The response of the rotor dynamic system with unsaturated input and the response of the rotor dynamic system with saturated input for the input voltage $u_c = 4.0$ V are shown in Figure K.12. Coexistence of solutions exists for the open-loop rotor dynamic system (F.17): a stable equilibrium and a stable limit cycle. The solution of the open-loop rotor dynamic system for the initial condition $\xi = [0 \ 0 \ 0]^T$ is a limit cycle. We see that the state of the rotor dynamic system with the saturated input converges slower to the setpoint than the state of the rotor dynamic system with unsaturated input.

It is also possible that the actuator saturation induces a limit cycle solution, depending on the constant input voltage and the initial conditions.
Figure K.11: Closed-loop system responses with control gain $K = [15.9 \ 1.57 \ 27.6]$ switched on at $t = 0$ s, for input voltage unsaturated (solid line) and for input voltage saturated (dashed line); the constant input voltage is $u_c = 1.8$ V, the equilibrium values are $\alpha_{eq} = 1.21$ rad and $\omega_{eq} = 3.17$ rad/s; observer gains are $L_2$ and $N_2$.

Figure K.12: Closed-loop system responses with control gain $K = [15.9 \ 1.57 \ 27.6]$ switched on at $t = 0$ s, for input voltage unsaturated (solid line) and for input voltage saturated (dashed line); the constant input voltage is $u_c = 4.0$ V, the equilibrium values are $\alpha_{eq} = 0.88$ rad and $\omega_{eq} = 7.07$ rad/s; observer gains are $L_2$ and $N_2$. 
K.5 Experimental results output feedback design

The output-feedback controller is implemented on the experimental set-up. We choose the same gains as we used for the simulations in the previous section; the control is $K_1$ as in (5.43) and the observer gains are $L_2$ and $N_2$ as in (4.30). We apply the same low-pass filter (6.63) as for friction characteristic I, since the same undesired phenomena occur. From the constant input voltage of 1.6 V up to 5.0 V, the controller is able to obtain a stable equilibrium for the closed-loop system. The transient response for the input voltage $u_c = 3.8$ V is depicted in Figure K.13. Coexistence of solutions exists for the open-loop rotor dynamic system (F.17): a stable equilibrium and a stable limit cycle (see also the bifurcation diagrams in Figure K.2(a) and Figure K.2(b)). The solution of the open-loop rotor dynamic system is a limit cycle for the first ten seconds in 6.8. The input voltage saturates two times after the controller is switched on. The closed-loop system converges to the equilibrium values within approximately 10 seconds.

The response for the input voltage $u_c = 3.2$ V is shown in Figure K.14 (coexistence of solutions exists for the open-loop rotor dynamic system (F.17): a stable equilibrium and a stable limit cycle). The control error $\dot{\omega}_l - \omega_{eq}$ fluctuates for the steady-state solution, with the rotational frequency, see Figure K.14(c).

Next, we show a transient response for the constant input voltage $u_c = 1.8$ V in Figure K.15. The solution for the open-loop rotor dynamic system (F.17) is a limit cycle. The closed-loop system converges to the equilibrium state within approximately 20 seconds. We notice that the input voltage saturates for a short time when the controller is switched on.

The equilibrium response for $u_c = 1.4$ V is shown in Figure K.16 (the solution for the open-loop rotor dynamic system (F.17) is a limit cycle). We observe that there are fluctuations present in the observed state components in Figure K.16. These fluctuations may be caused by position-dependent friction, as we also observed for friction characteristic I (see Section 6.6 for a discussion).

In Figure K.17, the resulting limit-cycling response of the experimental rotor dynamic set-up is depicted for the constant input voltage $u_c = 1.3$ V. The limit-cycling response differs from the limit-cycling response of the open-loop rotor dynamic system (F.17). Finally, we show the response for the constant input voltage $u_c = 0.5$ V in Figure K.18. When the controller is switched, the system goes to a stable equilibrium. But this equilibrium is different from the setpoint.

The measured responses from the experiments are used to construct bifurcation diagrams with the constant input voltage $u_c$ as the bifurcation parameter. We compare the results from the simulations with the experimental results in these bifurcation diagrams, see Figure K.19(a) and Figure K.19(b). The experiments match the simulations well above the constant input voltage $u_c = 1.5$ V. The control aim is achieved for these range of voltages i.e. the controller is able to control the closed-loop experimental rotor dynamic set-up to its setpoint.

For the voltages from $u_c = 0.2$ V up to 1.5 V, the simulations do not match with the experiments. The simulations shows stable equilibria, while the results of the experiments are different stable equilibria and stable limit cycles. We notice a good match between the simulations and the experiments for constant input voltages above $u_c = 1.6$ V. For these input voltages the controller ensures the asymptotic stabilization of the setpoint and thereby the avoidance of stick-slip limit cycling.

We compare the experimental bifurcation diagrams of the open-loop system with the experimental results of the closed-loop system in Figure K.20(a) and Figure K.20(b). These diagrams show a large extension of the region with only stable equilibria. The region with only stable equilibria for the open-loop system covers the constant input voltages $u_c = 4.3$ V up to 5.0 V. The closed-loop system extends the region with only stable equilibria to the constant input voltages $u_c = 1.7$ V up to 5.0 V.
Figure K.13: Closed-loop system responses with control gain $K = [15.9 \quad 1.57 \quad 27.6]$ switched on at $t = 10$ s; the constant input voltage $u_c = 3.8$ V, the equilibrium values are $\alpha_{eq} = 1.16$ rad and $\omega_{eq} = 6.72$ rad/s; observer gains are $L_2$ and $N_2$.

Figure K.14: Closed-loop system responses with control gain $K = [15.9 \quad 1.57 \quad 27.6]$; the constant input voltage is $u_c = 3.2$ V, the equilibrium values are $\alpha_{eq} = 1.02$ rad and $\omega_{eq} = 5.65$ rad/s; observer gains are $L_2$ and $N_2$. 
Figure K.15: Closed-loop system responses with control gain $K = [15.9, 1.57, 27.6]$ switched on at $t = 2 \text{s}$; the constant input voltage is $u_c = 1.8 \text{ V}$, the equilibrium values are $\alpha_{eq} = 0.88 \text{ rad}$ and $\omega_{eq} = 3.17 \text{ rad/s}$; observer gains are $L_2$ and $N_2$.

Figure K.16: Closed-loop system responses with control gain $K = [15.9, 1.57, 27.6]$; the constant input voltage is $u_c = 1.6 \text{ V}$, the equilibrium values are $\alpha_{eq} = 0.99 \text{ rad}$ and $\omega_{eq} = 2.81 \text{ rad/s}$; observer gains are $L_2$ and $N_2$.

Figure K.17: Closed-loop system responses with control gain $K = [15.9, 1.57, 27.6]$; the constant input voltage is $u_c = 1.3 \text{ V}$, the equilibrium values are $\alpha_{eq} = 1.35 \text{ rad}$ and $\omega_{eq} = 2.27 \text{ rad/s}$; observer gains are $L_2$ and $N_2$. 
Figure K.18: Closed-loop system responses with control gain $K = [15.9 \ 1.57 \ 27.6]$ switched on at $t = 9$ s; the constant input voltage is $u_c = 0.5$ V, the equilibrium values are $\alpha_{eq} = 2.94$ rad and $\omega_{eq} = 0.80$ rad/s; observer gains are $L_2$ and $N_2$.

Figure K.19: Bifurcation diagrams of the closed-loop system with control gain $K_1 = [15.9 \ 1.57 \ 27.6]$ for positive input voltages (friction characteristic II for friction at the lower disc); observer gains are $L_2$ and $N_2$. 
Figure K.20: Bifurcation diagrams of the closed-loop system with control gain $K_1 = [15.9, 1.57, 27.6]$ for positive input voltages (friction characteristic II for friction at the lower disc); comparison with the simulated bifurcation diagram of the open-loop system.