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Award date:
2015

Link to publication
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Eindhoven, March 2015
Abstract

Film-substrate systems are often characterized by poor adhesion between the film and the substrate. As a result, these systems tend to delaminate when loaded. The four-point bend test (4PB-test) is a conventionally used test setup for the determination of the interface toughness $G_c$. The presence of residual stresses is known to affect the outcome of this test. The effect of residual stresses is incorporated in the derivation of the energy release rate for three systems: film-substrate systems, film-substrate systems with an additional top layer, and film-substrate systems with an elasto-plastic substrate. The expressions for the energy release rates were validated using 2D numerical modeling. A rate-dependent interface model was adopted to model the delamination behaviour. Very good agreement was found between the numerical results and the analytical predictions. Two-layer systems are shown to be affected most by the presence of residual stresses. The addition of a third layer lowers the required bending moment for delamination significantly. In addition, the effect of residual stresses is reduced with both decreasing film thickness and increasing top layer thickness. The model for the elasto-plastic system was shown to be valid in the limit of the interface strength reaching zero. With increasing interface strength the effect of crack tip plasticity on the outcome of the 4PB-test was shown to be considerable. The size of the plastic zone was found to be affected by the film thickness for the thinnest film systems only.
1 Introduction

Film-substrate systems are often characterized by poor adhesion between the film and the substrate. As a result, these systems tend to delaminate when loaded. When loading conditions are mixed-mode, the four point bend test (4PB-test) is a conventionally used test setup for the determination of the interface toughness $G_c$, since the test provides fracture results for conditions of approximately equal normal and shear displacements on the interface crack [1]. Additionally, steady-state conditions prevail between the inner supports for which an analytical solution for the crack driving force, or energy release rate (ERR), can be derived. A typical 4PB-setup for a film-substrate system is depicted in Figure 1.1. The system consists of a top layer and a bottom layer, labeled 1 and 2 respectively, each layer with different elastic properties $E_i$ and $\nu_i$. Two external forces with magnitude $P/2$ are applied at distance $L$ from the supports, creating a constant bending moment region between the supports. A central notch exists through the thickness of the top layer and a symmetrical precrack is present along the interface.

Figure 1.1: Bilayer in a 4PB-setup, notched in the center with symmetrical interfacial cracks.

During the four-point bend test a displacement is forced on the loading lines as shown in Figure 1.1. The reaction forces in the supports are measured during the displacement increase from which the bending moment $M$ is computed. Subsequently a moment-displacement diagram can be obtained. A typical moment-displacement diagram is shown in Figure 1.2. Assuming no premature failure of the specimen due to excessive stresses, at a critical displacement the delamination process is initiated. After initiation, the reaction force and the bending moment remain constant throughout the delamination process. The critical moment is used to characterize the delamination resistance: through Griffith’s energy balance for brittle cracking the delamination toughness $G_c$ is related to the critical steady-state bending moment $M_{cr,ss}$.

If the applied initial imperfection, the notch, has not fully penetrated the film layer at the onset of loading, the initiation of the delamination process is characterized by a drop in reaction force caused by the propagation of the crack into the interface (Figure 1.2a). If the initial imperfection ends at the height of the interface, no such drop is observed. When the interface crack approaches a support, further advance is prevented by support reaction forces compressing the film-substrate system, see Figure 1.2c. As a result, the moment-displacement curve rises again. It is assumed the crack advances symmetrically along the film-substrate interface. The crack propagation is considered in a steady-state manner: the crack with length $a$ advances with a non-changing crack tip geometry and the ERR per unite advance is not dependent on crack length [2]. Essentially the stress field near the crack tip is unchanged if the observer translates with the moving crack tip during delamination. A steady-state propagation of the crack is reached quickly, i.e, when the crack tip length $a$ is several times larger than the film thickness [2], which is the smallest relevant dimension in film-substrate systems. Steady-state behaviour is observed in the moment-displacement curve of a 4PB-test (Figure 1.2) by the horizontal moment plateau at the critical value $M_{cr,ss}$. This is the critical moment at which steady state delamination occurs, and corresponds to the critical load as $M_{cr,ss} = P_{cr,ss}L/2b$.

The height of the moment plateau $M_{cr,ss}$ is, apart from the geometric and (elastic) material parameters,
affected by internal loading, residual stress $\sigma_{R}$, and plastic dissipation also. These effects are studied in this paper. Tensile residual stresses are known to lower the moment required for delamination [3, 4], whereas compressive residual stresses can result in higher values of $M_{cr,ss}$. The addition of a layer on top of the film increases the stored strain energy above the crack front, therefore lowering the critical load $P_{cr}$.

The origin of the residual stress in the film is a mismatch strain between the top layer and the bottom layer, caused by thermal expansion mismatch, phase transitions, grain growth, point defect annihilation, etc [5]. When during manufacturing a film with a specific mismatch strain is connected to an undeformed substrate, the substrate both experiences a compensating axial strain and a curvature where the corresponding stress pattern equilibrates the residual stress in the top layer. When the film-substrate system is subsequently used in a 4PB-test, the curvature of the system is further modified as a result of the application of an external couple $M$. The magnitude of the residual stress can be computed, in case of thermal mismatch strains, from the difference coefficient of thermal expansion (CTE) and the temperature change in the cooling down period after processing, or can be obtained by measurement of the radius of curvature. In the latter situation, the curvature is obtained and can be related to the residual (film) stress using Stoney’s equation for thin films, or from an extended version of Stoney’s equation [6] in case of arbitrary film thickness. More general types of residual loading, that is residual curvatures and offset strains in the individual film and/or substrate, are discussed in [7]. The residual stresses in this paper are restricted to a mismatch strain between the film and the substrate, which is representative for residual loadings due to CTE mismatch.

The addition of an additional top layer can be helpful when the critical loading exceeds the capacity of the bilayer to resist vertical cracking due to excessive substrate stresses. This is done extensively in the Integrated Circuit (IC) industry, for example, where low-k film stacks are sandwiched between two silicon substrates (e.g.[8, 9, 10]), which are subjected to bending moment loading in a four-point bend test to find the adhesion energy of the weakest interface. However, often the film properties as well as the presence of residual stress loading that is residual curvatures and offset strains in the individual film and/or substrate, are neglected for the sake of simplicity, so that the multi-layered system can be considered as a two-layer system. The addition of a third layer in the analytical model in combination with the presence of a residual film stress in this model can be a first step to elucidate the significance of the neglect of the film stack properties and residual stresses.

If layers have non-linear material properties, the film-substrate stiffness lowers towards the moment plateau. In case of plastic behaviour, these effects are irreversible and subsequently the energy considered with the plastic deformations is dissipated. The effect of plastic deformations on the macroscopic energy release rate is known to be substantial [11, 12]. In the IC-industry elasto-plastic material layers, e.g. copper layers, are often encountered in film stacks, sandwiched between substrates. The thickness of the yielding layers strongly affect the adhesion energy measured from the four-point bend test. Models as the embedded process zone model (EPZ, see [13]) and the plasticity-free strip model (SSV, see [14]) have been set up to properly describe the dissipative behaviour of the elasto-plastic materials and consequently match the data found in experiments. Elasto-plastic behaviour in film-substrate systems with residual stresses, however, have not been not fully described in a four-point bend
environment. If the film is elasto-plastic, the strain energy produced by the film is increasingly limited with growing plastic deformations. As a result, since the stored energy above the crack plane essentially drives the delamination, the energy release rate cannot reach a sufficient value for a successful four-point bend test; the stresses in the elastic substrate exceed the material strength long before delamination is possible. Thus, film-substrates with a elasto-plastic film are not relevant in four-point bending. The opposite case, however, may be subjected to four-point bending. When the substrate is elasto-plastic the film in four-point bending continues to store strain energy until the delamination process is initiated. The effect of plastic deformations in this case is studied. Here, crack tip plasticity and plasticity in the bulk material far from the crack are distinguished. The former is dependent on the film thickness [15], but only up to a certain point. The material-based length scale $R_0$ in relation to the film thickness plays an important role in the magnitude of plastic effects. When film thickness exceeds this material-based length scale a few times the plastic zone size shows to be unchanged for further increase of film thickness.

Since the specimen width of the analyzed film-substrate systems is much larger than the specimen thickness, the thickness direction can be assumed to be infinite, i.e. plane strain conditions prevail. To this end the elastic material stiffness $E_i$ is replaced with the plane strain modulus $\tilde{E}_i = E_i/(1 - \nu_i^2)$, where $\nu_i$ is the Poisson’s ratio. In the analytical procedures used in this paper, the process zone of the crack is assumed to be small, so the Small Scale Yielding (SSY) assumption is justified and Linear Elastic Fracture Mechanics (LEFM) holds. This assumption holds for brittle interface cracking. In the case of delamination with plastic effects the SSY solution is used as a reference case, its value relating to the ERR in the limit of the plastic effects near the cracktip vanishing. It is assumed the crack advances along the bi-material interface, and that crack-branching into the film or substrate does not occur. Though the mode mix, or phase angle, is in the order of $\pi/4$ for film-substrate systems of equal material properties, it can range from 35 to 60 degrees for specific material combinations [1].

In the study of the delamination failure mechanism, a rate-dependent interface model from [18] is applied which phenomenologically captures the characteristics of the fracture process through a traction-separation law, which describes the tractions across the interface as a function of the adjoining material face separations. For linear elastic systems with brittle interfaces only the interfacial toughness $G_c$, which is fact is the energy consumed by the process zone during development of the traction-separation curve, is of interest. This approach to fracture is the known brittle fracture criterion deduced from Griffith’s energy balance. In case the film-substrate interface is adjoined to an elasto-plastic material, in addition to the fracture toughness $G_c$ the interfacial strength $t^u$ is of main importance, and to a lesser extend also the shape of the traction-separation curve [19]. Toughness $G_c$ and interface strength $t^u$ are system properties, valid for a combination of materials bonded during a specific processing method.

The paper is organized as follows: Section 2 contains the derivations of the energy release rates for film-substrate systems, film-substrate systems with an additional top layer, and film-substrate systems with an elasto-plastic substrate. These systems are preloaded with residual stresses. Using an upstream-downstream approach, in which the energy states of the uncracked and cracked sections are considered, the steady-state energy release rate can be obtained for these systems in a four-point bend setup. For the elasto-plastic substrate system Von Mises plasticity with Ramberg-Osgood hardening is used. Crack tip plasticity is neglected in the analysis so that the derived solution is valid in the limit case of plastic deformations near the crack tip reaching zero. In Section 3 the interface damage model for the finite element simulations is introduced. The traction-separation law and the rate-dependent kinetic law for interface fracture are specified. Then a description is given of the finite element models. The boundary conditions are discussed, as well as the used element types and mesh density. The results are discussed in Section 4. The expression for the energy release rates from Section 2 are validated using numerical simulations. Trends in the energy release rates are discussed, as well as the significance of the presence of residual stresses in the systems. The significance is further illustrated by application of the extended energy release rate equation to experimental results from [20]. The effect of crack tip plasticity is discussed for the elasto-plastic film-substrate system. The interfacial strength is used as a control parameter to adjust the amount of plastic deformations near the crack tip. Subsequently, the effect of energy dissipation in the crack tip region due to the plastic deformations on the outcome of the 4PB-test is compared to the results obtained from the limit case of Section 2. In Section 5, the main analysis results are summarized.
2 Derivation of the energy release rate

2.1 Film-substrate system with residual stress

![Figure 2.1: Film-substrate system subjected to couple M and residual stress \(\sigma_R\) in the film.](image)

The steady-state energy release rate (ERR), \(G_{ss}\), for a delaminating crack in Figure 2.1 can be deduced analytically by consideration of the energy balance

\[
\Delta W = \Delta F - \Delta U \tag{2.1}
\]

where \(\Delta W\) is the interfacial energy consumed during delamination, \(\Delta U\) is the drop in strain energy as a result of delamination, and \(\Delta F\) is the external work applied to the film-substrate system. The steady-state energy release rate is given by the consumed interfacial energy per unit delamination \(a\),

\[
G_{ss} = \frac{\partial \Delta W}{\partial a}. \tag{2.2}
\]

The difference in strain energy is found by subtraction of the strain energy in the uncracked and cracked beam, recognizing the states in those two regions show the drop in strain energy. Those regions are from here on called upstream and downstream, respectively. Both the strain energy \(U\) and the conjugated work \(W\) are derived from the stress and strain profiles in the upstream and downstream sections. These are derived in the framework of ordinary beam theory, i.e., plane sections remain plane and perpendicular to the neutral line after deformation. The determination of the strain and stress profiles in the sections is performed by requiring force and moment equilibrium at those sections.

The strain energy in the upstream and downstream direction are found by integration of the strain energy density over the section height. The conjugated work produced by the external bending moment \(M\) during delamination can be computed as the product of the bending moment \(M\) and the energetically-conjugated rotation, \(\Delta F = M \Delta \phi\). The full derivation of \(G_{ss}\) is discussed in detail in Appendix A.

The expression of the ERR, \(G_{ss}\), is formulated in terms of the geometric parameters \((h_1, h_2)\), the plane strain elastic moduli \((\bar{E}_1, \bar{E}_2)\), with \(\bar{E}_i = E_i/(1 - v_i^2)\), and the loading parameters \((M\) and \(\sigma_R)\) as

\[
G_{ss} = 6M^2 \left( \frac{1}{\bar{E}_2h_2^2} - \frac{(\bar{E}_1h_1 + \bar{E}_2h_2)}{\alpha} \right) + \sigma_R^2 \bar{E}_1h_1h_2(\bar{E}_1h_1^3 + \bar{E}_2h_2^3) + 12\sigma_R M \bar{E}_1h_1 \bar{E}_2h_2(h_1 + h_2) \frac{2\bar{E}_1\alpha}{2(\bar{E}_1\alpha)} \tag{2.3}
\]

with \(\alpha = \bar{E}_1h_1^2 + 4\bar{E}_1h_1^2\bar{E}_2h_2 + 6\bar{E}_1h_1^2\bar{E}_2h_2^2 + 4\bar{E}_1h_1\bar{E}_2h_2^3 + \bar{E}_2^2h_2^4\).

It is observed that the expression for \(G_{ss}\) essentially is composed of three parts, namely a part due to the four point bend test loading \(M\), a part due to the residual stress \(\sigma_R\) in the top layer, and an interaction part. When the residual stress is absent, the result reduces to the known expression for the energy release rate for the case of pure bending, expression (10) in [1].

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2.2 Film-substrate system with residual stress and additional top layer

In Figure 2.2 a three-layer system is shown in a 4PB-test environment. The additional top layer is denoted by the subscript \( d \), whereas the film and the substrate are denoted by the subscripts 1 and 2 respectively. A symmetrical pre-crack is present throughout the thickness of the superlayer and the film. The film layer is loaded with residual stress \( \sigma_R \).

\[
G_{ss} = 6M^2 \left( \frac{1}{E_2h_2^3} - \frac{(E_1h_1 + E_2h_2 + E_dh_d)}{\gamma} \right) + \frac{\sigma_R^2h_1}{2E_1} \left( -3E_1h_1 \left( E_2h_2 - E_dh_d \right) + E_2h_2^2 - E_dh_d^2 \right) + \frac{\gamma (E_2h_2 + E_dh_d)}{\gamma} \left( \frac{E_1h_1^2 + E_dh_d^2}{\beta} - \frac{E_dh_d (E_1h_1 + E_2h_2 + E_dh_d)}{\beta} \right) + 6\sigma_RMh_1 \left( \frac{h_1 (E_2h_2 - E_dh_d) + E_2h_2^2 - E_dh_d^2}{\gamma} \right),
\]

with

\[
\beta = E_1^2h_1^4 + 4E_1h_1^3E_dh_d + 6E_1h_1^2E_dh_d^2 + 4E_1h_1E_dh_d^3 + E_d^3h_d^4 \]
\[
\gamma = E_1^2h_1^4 + 4E_1h_1^3 (E_2h_2 + E_dh_d) + h_1^2 \left( 6E_1h_1^2 + 12E_dh_2^2h_d \right) E_2 + 6E_1E_dh_d^3 \]
\[
+ h_1 \left( 4h_2 \left( 3h_d \left( h_2 + h_d \right) E_1 + E_1h_1^2 \right) E_2 + 4E_1E_dh_d^2 \right) + E_2h_2^4 + 4E_2h_2^2E_dh_d \left( h_2^2 + \frac{3}{2}h_2h_d + h_d^2 \right) + E_d^2h_d^4.
\]

The result for the ERR is similar to the case of the film-substrate system. Again, it consists of three components: a component due to the bending moment, a component due to the residual stresses, and an interaction...
part. Equation (2.4) reduces to equation (2.3) for the film-substrate if the top layer thickness \( h_d \) is set to zero. In another limit case, in the situation of the residual stress \( \sigma_R \) vanishing the equation for a three-layer system as presented by Hofinger et al. [21] is found.

**Symmetric model**

It is not uncommon to create a three-layer sample by gluing together two film-substrate samples, the film faces oriented towards each other. Then the top layer has equal properties as the substrate, i.e. it has the same height \( (h_d = h_2) \) and elastic properties \( (E_d = E_2) \). The two film layers can be viewed as one new film layer, with the same elastic properties \( \bar{E}_1 \) and internal loading \( \sigma_R \), but with double thickness \( 2h_1 \). The new sample consists of three layers, hence the analytical three-layer model can be applied. Replacing the additional top layer properties for the substrate properties, and the film thickness to \( 2h_1 \), the ERR becomes

\[
G_{ss} = 6M^2 \left( \frac{1}{E_2h_2^2} - \frac{1}{8 E_1h_1^4 + 3h_1^2h_2^2 + h_2^4} \right) \bar{E}_2 + \frac{\sigma_R^2 h_1 h_2 \bar{E}_2}{E_1} \left( \frac{1}{E_1h_1 + E_2h_2} - \frac{8E_1h_1^4 + E_2h_2^4}{16E_1^2h_1^4 + 32E_1h_1^2E_2h_2^2 + 24E_1h_1E_2h_2^2 + 8E_1h_1E_2h_2^2 + E_2h_2^4} \right)
\]

The ERR is now computed from the original film-substrate parameters. It can be seen from (2.6) the interaction component \( G_{RM} \) is absent. The symmetry in the sample forces the interaction contributions of the superlayer and the substrate to cancel out.

**2.3 Film-substrate system with residual stress and elasto-plastic substrate**

Consider a film-substrate system as sketched in Figure 2.1, but with elasto-plastic material behaviour for the substrate, instead of the previously assumed elastic behaviour. In order to determine an expression for the ERR for such a system, again an energy balance is considered. However, with respect to the balance in equation (2.7), it is extended with a dissipation component, \( \Delta D \). It becomes

\[
\Delta W = \Delta F - \Delta U - \Delta D.
\]

In this extended energy balance it is seen the available energy for crack extension is lowered by the dissipation \( \Delta D \). This component includes all energy-consuming processes related to the plastification of substrate material. Two plasticity regions are distinguished, see Figure 2.3. First, global plasticity as the plasticity occurring in the uncracked and cracked section, far from the crack tip (see stress profiles of the downstream (b) and upstream (c) sections in Figure 2.3). And second, local plasticity at the crack tip (plastic strain zone near crack tip in Figure 2.3). As the crack progresses, it leaves behind a trail of plastically deformed but partially unloaded material in its wake [6].
During the cracking process, in the transition from the uncracked to the cracked section, the stresses and strains are increased as a result of loss of section height, from which it may be assumed unloading effects from the upstream to the downstream section are small. With this assumption, the stress and strain distribution in these two sections, far from the crack tip, can be considered from a nonlinear elastic analysis. Subsequently, the plastic dissipation $\Delta D$ in the transition between these sections is completely produced by local crack tip plasticity.

As a first approximation for the ERR for film-substrate systems with an elasto-plastic substrate, a global upstream-downstream approach similar to analysis of the previous film-substrate systems is applied. Since local crack tip effects are disregarded in a global energy balance approach, the outcome from this analysis is only valid in the case of small scale yielding (SSY) near the crack tip, or $\Delta D$ reaching zero. Since crack tip plasticity is primarily controlled by the ratio of interface strength and yield strength, denoted as $t = t_u/\sigma_y$ [15], the SSY solution provides information for systems where the interface strength is low. In the next section, the local crack tip effects are taken into account using numerical modeling.

### 2.3.1 Plasticity modeling

The plastic behaviour is modeled using J2-plasticity theory, and the Ramberg Osgood relation is taken as a hardening law. For uni-axial tension, this hardening law is

$$
\varepsilon = \frac{\sigma}{E} + \alpha \frac{\sigma_0}{E} \left( \frac{\sigma}{\sigma_0} \right)^n,
$$

where $\sigma$ and $\varepsilon$ are the uni-axial stress and strain, $E$ is the elastic modulus, $\alpha$ is the yield offset, and $n$ is the strain hardening exponent. Since the 4PB-test in plane strain conditions represents a case of multi-axial loading, the uni-axial stress-strain relation needs to be adjusted and expressed in terms of the multi-axial components. This is done using the equivalent stress ($\sigma_m$) and equivalent strain ($\varepsilon_m$) as defined in the J2-yield criterion. Since the equivalent stress and strain reduce to the uni-axial components in case of uni-axial tension, the Ramberg-Osgood hardening law from equation (2.8) is assumed to represent the relation between the equivalent stress and strain, $\sigma_m$ and $\varepsilon_m$, through which the multi-axial stress and strain components are related, see Appendix 2.

### 2.3.2 Analytical approach using deformation theory

Since it is assumed global unloading effects in the transition from the uncracked to the cracked section are negligible, the elasto-plastic material behaviour is identical to nonlinear elastic material behaviour, and deformation theory of plasticity can be applied. In this theoretical framework, the stresses and strains are interrelated using Hooke’s law, provided that the current values for elastic modulus $E$, or the secans modulus $E_{\sec}$, and Poisson’s ratio $\nu$ are used. Both depend on the current state of equivalent stress and equivalent strain, $\sigma_m$ and $\varepsilon_m$, through which the multi-axial stress and strain components are related, see Appendix 2.

For the elaboration of both force and moment equilibrium, a stress-strain relation is needed in the longitudinal direction (parallel to the neutral line). These are formulated in terms of the equivalent stress $\sigma_m$ and equivalent strain $\varepsilon_m$ as

$$
\sigma_x = \frac{\sigma_m}{\sqrt{\nu^2 - \nu + 1}}
$$

$$
\varepsilon_x = \frac{(\nu + 1)}{2} \left[ -\frac{\nu}{E_{\sec}} \sigma_x + \sqrt{\frac{4\varepsilon_m^2 - 3\varepsilon_m^2}{E_{\sec}^2 - \sigma_x^2}} \right],
$$

where the current Poisson’s ratio is $\nu = 1/2 - [1/2 - v_c]E_{\sec}/E$ and the current elastic modulus equals $E_{\sec} = \sigma_m/\varepsilon_m$. Using equation (2.9) and requiring zero net force and a net bending moment equal to $M$, the neutral line and subsequently the stress and strain profiles can be determined. These are used to determine both the strain energies and curvatures in the upstream and downstream direction. More details, along with the rest of the derivation of the ERR, are given in Appendix 2. The expression for the ERR is given by

$$
G_{ss} = (\kappa_d - \kappa_u) M - (U_d - U_u),
$$

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where $\kappa$ and $U$ are the section curvature and strain energy respectively. The subscripts $d$ and $u$ refer to the downstream and upstream section.
3 Numerical modeling

3.1 Review of the interface damage model

The numerical modeling of the interface between the film and substrate is done with interface elements. For the interface elements used in this paper a damage model as presented by Alfaro et al. in [18] is used. The interface elements are equipped with a mixed-mode damage model, where the mode-mixity is derived from an energy criterion. The model employs a rate-dependent damage kinetic law, which is used to avoid numerical convergence problems due to crack initiation. The model is reviewed below.

The interface tractions $t_i$ and the relative displacements $v_i$ consist of, in the case of 2D plane strain modeling, two components: $i \in \{1, 2\}$, with 1 denoting the normal direction, and 2 denoting the tangential direction at the interface, respectively.

A linear softening law is adopted for the traction-separation behaviour, which is depicted in Figure 3.1. The fracture toughness is equal to the area under the traction-separation curve, as seen in Figure 3.1. Damage is initiated at an effective relative displacement $v_0$, and damage completion is reached at the effective relative displacement $v_u$. Tractions and relative displacements are related by the following formulation

$$t_i = (1 - d)C_{ij}v_j - dC_{ij}\delta_{1j}(-v_1)$$

where $i, j \in \{1, 2\}, (3.1)$

where the damage parameter $d$ is bounded as $0 \leq d \leq 1$. The initial, undamaged, state corresponds to $d = 0$, and $d = 1$ refers to the case of complete damage, or zero resistance against any additional traction. The elastic stiffness tensor is $C_{ij} = K\delta_{ij}$, with $K$ the initial stiffness and $\delta_{ij}$ the Kronecker delta symbol. The Macauley brackets $\langle \cdot \rangle$ in the last term of (3.1) prevents penetration of two opposite crack faces, by describing these faces to interact elastically in the normal direction during contact.

A deformation history variable $\kappa$ is introduced from which both the equivalent traction $t$ and the damage $d$ can be computed. The onset of damage relates to $\kappa = v^0$ and damage completion is reflected by $\kappa = v^u$, corresponding to $d = 0$ and $d = 1$, respectively. For a given value of the damage history variable $\kappa$ the corresponding traction is equal to $Kv^0(\kappa - v^0)/(v^u - v^0)$ (see figure 3.1). The steepness of the line through the origin and the traction at effective relative displacement $\kappa$ represents the elastic stiffness of the interface element with damage $d = d(\kappa)$ and is equal to $(1 - d)K$. The damage parameter $d$ in terms of history damage parameter $\kappa$ is given by

$$d = \hat{d}(\kappa) = \frac{v^u(\kappa - v^0)}{\kappa(v^u - v^0)}.$$

(3.2)

Figure 3.1: Traction-separation law as in [18].
Since the employed model is rate-dependent, the process of damage evolution is not set by the deformation only, but also by its rate. The actual value of $\kappa$ consequently is obtained from an inverted form of (3.2), where the damage $d$ is updated for the rate effect $\dot{d}$. This rate is described by a specific kinetic law, which is formulated as

$$
\dot{d} = \begin{cases} 
\frac{\dot{F}(\lambda, \kappa)}{\eta} & \text{for } \lambda \geq \kappa \text{ and } v^0 < \kappa < v^u, \\
0 & \text{for } 0 \leq \lambda < \kappa \text{ or } \kappa = v^u,
\end{cases} \tag{3.3}
$$

where $\dot{F}(\lambda, \kappa)$ is the damage loading function and $\eta$ is a relaxation parameter. The upper expression in (3.3) describes the damage rate when the effective deformation $\lambda$ exceeds the threshold $\kappa$, whereas the lower expression sets the rate of damage equal to zero when the threshold value has not yet been reached, the interface element is in a state of unloading, or the damage process has completed. The deformation measure $\lambda$ is equal to the length of the vector of relative crack face displacements, $\lambda = |v| = \sqrt{v_I^2 + v_II^2}$. The loading function has the form

$$
\dot{F}(\lambda, \kappa) = \dot{f}(\lambda) - \dot{d}(\kappa) = \frac{v^u(\lambda - v^0)}{\lambda(v^u - v^0)} - \frac{v^u(\kappa - v^0)}{\kappa(v^u - v^0)}. \tag{3.4}
$$

It is seen in (3.4) that the form of $\dot{f}(\lambda)$ is similar as $\dot{d}(\kappa)$. Further, it is noted the kinetic law turns into the rate-independent loading condition when the relaxing parameter is going to zero, $\eta \rightarrow 0$.

The equivalent crack face displacements $v^0$ and $v^u$ are dependent on the relation between the normal and shear displacements, $v_1$ and $v_2$, at the interface. The mixed-mode behaviour is represented by the following parameter:

$$
\beta = \frac{v_2}{v_2 + (v_1)}. \tag{3.5}
$$

Following this definition, pure mode I loading is reflected by $v_2 = 0$, and thus by $\beta = 0$, whereas pure mode II loading is denoted by $v_1 = 0$, and thus $\beta = 1$. Adopting a mixed-mode failure criterion from linear elastic fracture mechanics, a relation can be obtained between the relative crack face displacements, $v_1$ and $v_2$, and the effective relative displacements, $v^0$ and $v^u$, through the parameter $\beta$. The mixed-mode criterion adopted has the form regularly used for the description of brittle interfacial fracture,

$$
\frac{G_I}{G_{I,c}} + \frac{G_{II}}{G_{II,c}} = 1, \tag{3.6}
$$

where $G_I$ and $G_{II}$ are the mode I and mode II energy release rates, and $G_{I,c}$ and $G_{II,c}$ the associated toughnesses for mode I and mode II loading. Rewriting (3.5) to express the normal displacement in terms of mixed-mode parameter $\beta$ and shear displacement, $v_1 = v_2(1 - \beta)/\beta$, (3.6) leads to

$$
v^u = \dot{v}^u(\beta) = \frac{2(1 + 2\beta^2 - 2\beta)}{Kv^0} \left[ \frac{(1 - \beta)^2}{G_{I,c}} + \frac{\beta^2}{G_{II,c}} \right]^{-1}. \tag{3.7}
$$

An explicit expression for $v^0$ can be found by inserting $G_{I,c}$ and $G_{II,c}$ in terms of their displacements $v^0_1$ and $v^0_2$ into (3.7) and replacing $v^u$, $v^0_1$ and $v^0_2$ by the corresponding values of $v^0$, $v^0_1$ and $v^0_2$. The equation then can be solved for $v^0$, leading to

$$
v^0 = \dot{v}^0(\beta) = v^0_1v^0_2 \sqrt{\frac{1 + 2\beta^2 - 2\beta}{(\beta v^0_1)^2 + ((1 - \beta) v^0_2)^2}}. \tag{3.8}
$$

Both (3.7) and (3.8) have mixed-mode parameter $\beta$ as the only variable. In the current study, the mode I and II toughnesses are assumed to be identical, as well as the traction-separation curves associated to mode I and II. Hence, with $G_{I,c} = G_{II,c}$ and $v^0_1 = v^0_2$, equations (3.7) and (3.8) are simplified to
\[
\begin{align*}
\nu^0 &= \nu_1^0, \\
\nu^a &= \frac{2G_c}{K_1^0}.
\end{align*}
\] (3.9)

As can be seen from (3.9), the effective relative displacements at damage nucleation and damage completion are identical to the individual mode I and II damage nucleation and completion displacements.

### 3.2 Numerical simulations

#### 3.2.1 Geometry and boundary conditions

The interface damage model as presented in the previous section is used to study the delamination behaviour of three types of residually stressed specimens in a four-point bend test setup: film-substrate systems, film-substrate systems with an additional top layer, and film-substrate systems with an elasto-plastic substrate. The considered systems are symmetrical, since the crack is assumed to propagate symmetrically along the bi-material interface. Hence, only half of the setup is modeled, see Figure 3.2. At the center of the span symmetry is warranted by subjecting the half-beam to a rolling support along the substrate height. The initial crack is simulated as having penetrated the top layer(s) and deflected into the interface over a distance \(a\), see Figure 3.2, which is achieved by modeling the layers above the interface only up to a set distance from the symmetry plane. In order to achieve steady state delamination, distance \(a\) needs to exceed local geometric parameters a few times [2]. The vertical support is allowed to move in the direction of the span, in order to prevent appearance of membrane forces.

The bending moment loading is generated displacement-controlled at a distance \(L\) from the inner vertical supports. The enforced displacement is denoted by \(u\) and produces a bending moment equal to the reaction force \(R\) in the vertical supports multiplied with the distance \(L\), or \(M = RL\). The residual film stresses are applied in an initial time-step. These are relaxed before the bending moment loading is applied in the next time-step. After relaxing the initial stresses, the displacement loading \(u\) is increased and a moment-displacement curve is set up. If the displacement load is sufficient and the modeled specimen long enough in span for the crack to develop, a moment plateau can be extracted from the figure. This bending moment is the critical steady state delamination moment \(M_{cr,ss}\) for the modeled specimen. Its value is used to validate the expressions for the energy release rates derived in section 2.

#### 3.2.2 Finite element discretisation

The numerical analyses are performed in a small-displacement, small-strain framework. Since brittle elastic two and three-layer systems usually have poor delamination resistance, deformations at the time of crack initiation are small and therefore the small-displacement, small-strain assumption is justified. For the film-substrate system with elasto-plastic substrate the deformations at the time of delamination are not necessarily small, yet the film material strength limits the deformation of the specimen. As a consequence, deformations are not allowed to be very large and the small-displacement, small-strain framework is applicable. Additionally, simulations of the considered systems with large-displacement formulations have shown only to differ marginally from the small-displacement simulations.

The film and substrate, as well as the additional top layer, are modeled using plane strain biquadratic (8-node) quadrilateral elements CPE8 in ABAQUS v.6.12.1. The intermaterial interface is modeled using 4-node cohesive elements COH2D4 with the cohesive law as presented in Section 3.1. The interface elements are modeled with zero thickness. Its bottom nodes are connected to the substrate using tie-constraints, while the top nodes are tied to the film. A contact formulation is used to prevent the film from penetrating the substrate when these layers are compressed. This layer compression may result from the oscillatory character of the stress field in front of the crack tip, as well as from the enforced displacements near the supports.

As is illustrated in Figure 3.2, the finite element mesh is relatively fine near material interface. It becomes coarser near the bottom edge of the specimen in order to limit mesh size and computational time. The approximately squared elements near the interface have a size \(\Delta\), see Figure 3.2. It is important \(\Delta\) is small enough
to properly capture the fracture process. An important length scale is the length of the cohesive zone, \( u^w \) in Figure 3.1. Alfaro et al. [18] chose the size of their elements near the delamination tip such that \( \Delta/u^w \) was equal to 2.5. Accordingly, in this study the meshsize near the interface is chosen to be 2. It is noted mesh refinement studies for several specimens in this paper have shown mesh convergence for approximately \( \Delta/u^w < 4 \).

For the film-substrate system with an elasto-plastic substrate, in contrary to the derivation of the ERR in Section 2.3, crack tip plasticity and unloading effects are accounted for in the numerical simulations. The plastic material behaviour is modeled using a rate-independent flow formulation with isotropic hardening. As the hardening law, the Ramberg-Osgood constitutive relation as indicated in Section 2.3 is used. The effects of plastic deformations near the advancing crack are compared to the semi-analytical results from Section 2.3 in the next section. The presence of plasticity gives rise to another important length scale, namely the height of the plastic zone near the crack tip. In this context a much used reference length is \( R_0 \) [16],

\[
R_0 = \frac{1}{3\pi} \frac{E G_c}{\sigma_y^2}, \tag{3.10}
\]

where \( E \) denotes the substrate plane strain modulus, \( G_c \) the interface toughness, and \( \sigma_y \) is the yield stress of the substrate material. Equation (3.10) gives the height of the plastic zone in case of a semi-infinite substrate in mode I loading, assuming that the plastic dissipation is small. In general it should be regarded as a material-based length quantity. The magnitude of \( R_0 \) is, along with the cohesive zone length \( v^w \), important for the mesh size near the delamination front. The plastic zone size determines the amount of energy dissipation near the advancing crack tip, and consequently also the height of the critical steady-state delamination moment \( M_{ct,ss} \) found in the moment-displacement curve. It is therefore important that the plastic zone near the crack tip is properly meshed. In the system of Figure 3.2 the value of \( R_0 \) is much greater than the cohesive zone length and
therefore it is not decisive for the mesh size. Nonetheless is $R_0$ small with respect to other geometric parameters (e.g. $h_2/R_0 = 27.5$ in Fig. 3.2), so that it needs to be considered in determining the mesh size distribution.
4 Results and discussion

4.1 Film-substrate system

4.1.1 Components of the ERR

In section 2.1 expression (2.3) was derived for the energy release rate of film-substrate systems loaded with residual film stresses in a four-point bend test setup. An alternative representation of (2.3) may be given in terms of dimensionless parameters. The expression for the steady state ERR is then presented in terms of relative height $\lambda = h_2/h_1$ and relative elastic modulus $R = E_2/E_1$. The result is again composed of three parts: $G_M$ due to the applied bending moment $M$; $G_R$ due to the residual stress $\sigma_R$ in the film; and $G_{RM}$ due to the interaction of the bending moment and the residual stress. The total ERR and its individual components are:

$$G_{ss} = G_M + G_R + G_{RM}$$  \hspace{1cm} (4.1)

$$G_M = \frac{6M^2}{E_2h_2^3} \left( \frac{3R\lambda(1+\lambda)^2 + (1 + R\lambda)}{3R\lambda(1+\lambda)^2 + (1 + R\lambda)(1 + R\lambda^3)} \right)$$  \hspace{1cm} (4.2)

$$G_R = \frac{\sigma_R^2 h_1}{2E_1} \left( \frac{R\lambda(1 + R\lambda^3)}{3R\lambda(1+\lambda)^2 + (1 + R\lambda)(1 + R\lambda^3)} \right)$$  \hspace{1cm} (4.3)

$$G_{RM} = \frac{6M\sigma_R}{E_1h_2} \left( \frac{R\lambda^2(1+\lambda)}{3R\lambda(1+\lambda)^2 + (1 + R\lambda)(1 + R\lambda^3)} \right)$$  \hspace{1cm} (4.4)

Trends in the dimensionless ERR of the individual components are shown in Figure 4.1. The dimensionless presentation of these components essentially represent the terms between brackets in equations (4.2), (4.3), and (4.4). On the horizontal axis the thickness ratio $h_1/h_2$ is shown, whereas the vertical axis shows the dimensionless components of the ERR. The results are plotted for different values of $R$. A low value of $R$ corresponds to a relative compliant substrate, whereas a high value indicates a relative stiff substrate. With $R$ ranging from 0.1 to 10.0, most material combinations are covered. Thickness distributions range from the relative thickness 0.01 to 1.0.

From Figure 4.1a it can be observed that bilayers with stiff films and substrate thickness $h_2$ reach high values of $G_M$ already at low film thickness ($h_1/h_2 \approx 0.3$), while systems with compliant films are not very sensitive to thickness changes when films are thin. The differences diminish with increasing top layer thickness, and asymptote towards approximately 6.

The effect of the residual stresses can be extracted from Figure 4.1b. For thin films, compliance differences in the film-substrate system are barely affecting the residual stress component of ERR, all converging to $GE_1/(\sigma_R^2 h_1) = 0.5$ in the limit of $h_1/h_2 \rightarrow 0$. With increasing thickness ratio $h_1/h_2$, or for decreasing $h_2$ and constant $h_1$, film-substrate systems with stiff films are affected most: they show a quick drop in energy release rate. Film-substrates with stiff substrates show smaller sensibility towards increasing thickness ratio. It is noted both components $G_M$ and $G_R$ are always positive, since the loading parameters $M$ and $\sigma_R$ both appear squared in both expressions (4.2) and (4.3).

The interaction component, $G_{RM}$, however, can also obtain negative values for its contribution to the total ERR $G_{ss}$. This occurs when $M$ and $\sigma_R$ bear different signs. This is only the case when $\sigma_R$ is negative, since $M$ is always positive in a 4PB test. It is interesting to see the interaction component shows a maximum value for a specific thickness ratio, which depends on $R$. 

Effect of residual stress on the delamination of film-substrate systems under bending 17
Figure 4.1: Dimensionless components the ERR of a film-substrate system with residual film stress under pure bending: a) component due to $M$, b) component due to $\sigma_R$, c) interaction component of $M$ and $\sigma_R$. 

Effect of residual stress on the delamination of film-substrate systems under bending


4.1.2 Total ERR

The relative magnitude of the three ERR components inherently depends on the relative magnitude of the bending moment and residual stress loading. In order to inspect trends in the total ERR, equations (4.1) are non-dimensionalized using a relative loading parameter $\tau$. The dimensionless loading $\tau$ is formulated as

$$\tau = \frac{\sigma_R}{\sigma_M},$$

(4.5)

where the residual stress is related to the maximum tensile bending stress per unit depth in cracked section of the substrate, $\sigma_M = 6M/h^2$. Using (4.5), equations (4.2) to (4.4) are reworked into

$$\frac{G_M E_2 h^3}{M^2} = 6 \left( \frac{3RL(1+\lambda)^2 + (1 + R\lambda)}{3RL(1+\lambda)^2 + (1 + R\lambda)(1 + R\lambda^3)} \right),$$

(4.6)

$$\frac{G_R E_2 h^3}{M^2} = 18\tau^2 \left( \frac{R^2(1 + R\lambda^3)}{3RL(1+\lambda)^2 + (1 + R\lambda)(1 + R\lambda^3)} \right),$$

(4.7)

$$\frac{G_{RM} E_2 h^3}{M^2} = 36\tau \left( \frac{R^2(1 + \lambda)}{3RL(1+\lambda)^2 + (1 + R\lambda)(1 + R\lambda^3)} \right).$$

(4.8)

Trends in the total value of $G_{ss}$ are illustrated for different values of $\tau$ in Figure 4.2a. Here, the non-dimensional ERR is plotted for different values of $h_1/h_2$. The film is three times as stiff as the substrate, or $R = 1/3$, which is representative for a lot of film-substrate systems [bron]. Negative values for $\tau$ represent compressive residual stresses in the film, while positive values denote tensile residual stresses. Using $\tau$, the effect of the residual stress $\sigma_R$ on the total ERR is examined. It is seen tensile residual stresses ($\tau = 0.5$ and $\tau = 1$) increase the ERR, while compressive residual stresses ($\tau = -0.5$ and $\tau = -1$) decrease the ERR with respect to the initial stress-free case ($\tau = 0$). The latter is not obvious however. When the compressive residual stresses become substantial, the ERR is increased instead of decreased with respect to the reference case without residual stresses. This effect is clarified by observing that the relative loading $\tau$ appears squared in the residual stress component $G_R$, equation (4.7), and subsequently eventually outvalues the $\tau$ term in the interaction component $G_{RM}$, equation (4.8), which is the case when

$$\tau < -\frac{2\lambda^2 (1 + \lambda)}{1 + R\lambda^3}.$$  

(4.9)

An example film-substrate system is shown in Figure 4.2b. Assuming brittle interface fracture, Griffith’s energy criterion states that delamination occurs at a critical value of the ERR $G_c$. The system shown has an interfacial toughness of $G_c = 2N/m$ and its film is three times stiffer than its substrate, or $R = 1/3$. The critical bending moments needed for steady state delamination, $M_{cr,ss}$, are presented in a dimensionless fashion using a reference moment $M_{cr,ref}$, which is defined as the critical bending moment needed for steady-state delamination when $h_1/h_2 = 1$ and the bilayer is initially stress-free. The bending moments are plotted for varying relative thicknesses $h_1/h_2$, for three levels of residual film stress: $-50$MPa, $0$MPa, and $50$MPa. The solid lines denote the analytical results using an inverse form of the expression for the ERR, equation (2.3), while the dots represent the plateau values found in the moment-displacement curves from the numerical simulations as described in Section 3.2. It is seen the numerical simulations are approximated very closely by the analytical predictions. Figure 4.2b shows the effect of the presence of residual stresses on the outcome of a four-point bend test, which is subsequently used to determine an interface toughness. Analogous to the lines in Figure 4.2 a, tensile residual film stresses show the drop in needed critical bending moment for delamination. Around $\lambda = 0.90$ the curve intersects the horizontal axis, indicating bilayer systems with thicker films cannot be delaminated in a 4PB-test: they delaminate spontaneously due to the residual stresses only. This phenomenon is discussed below. Conversely, negative residual film stresses increase the resistance against delamination in a 4PB-test. However, only up to thickness ratios of $h_1/h_2 \approx 0.9$, where the $-50$MPa line and the $0$MPa line intercept. From this point on the positive residual stress component $G_R$ outvalues the negative interaction component $G_{RM}$.

4.1.3 Analytical results from literature

The expression for the ERR, equation (2.3), was compared to analytical results found in literature. Discrepancies were found in comparison with the residual stress component $G_R$ in [3], which differs by a factor...
\[
\frac{G_{R,Char}}{G_R} = \frac{3\lambda(1 + \lambda)}{(1 + R\lambda^3)} + \frac{(1 + R\lambda)}{R(1 + \lambda)} - \frac{3R\lambda(1 + \lambda)^2}{(1 + R\lambda^3)(1 + R\lambda)}.
\] (4.10)

More recently, Delette et al. gave expressions for the ERR in terms of \(\lambda\) and \(R\) for film-substrate systems loaded with residual film stress in a four-point bend test. The interaction component \(G_{RM}\) from equation (4.4) differs from the interaction component from [4], equation (37), by a factor \(G_{RM,Del}/G_{RM} = \lambda\).

Figure 4.2: a) Dimensionless presentation of the total ERR for different values of loading \(\tau\) and thickness ratios \(h_1/h_2\), b) Critical bending moments of an example system with different thickness ratios for three magnitudes of residual stress.

4.1.4 Critical residual film stress

Since residual stresses in a film-substrate system contribute to the ERR, the situation may occur in which the residual stresses alone are sufficient to initiate and drive delamination, see also [22]. This phenomenon is seen Figure 4.2b with the tensile residually stressed specimen, near the relative thickness \(h_1/h_2 \approx 0.96\). Here, the
residual stress alone was sufficient to initiate delamination. Subsequently, a numerical four-point bend test for $h_1/h_2 = 1$ was not possible.

For a given specimen with layer thicknesses $h_1$ and $h_2$, and plane strain moduli $E_1$ and $E_2$, a critical residual stress $\sigma_{R, crit}$ exists. The magnitude of $\sigma_{R, crit}$ is obtained from (2.3), by setting the external bending moment $M$ to zero, equating the ERR to the fracture toughness $G_c$, and solving for $\sigma_R$. The critical residual stress $\sigma_{R, crit}$ then is

\[
\sigma_{R, crit} = \pm \sqrt{\frac{\alpha}{E_2h_2(E_1h_1^2 + E_2h_2^2)\frac{2E_1G_c}{h_1}}} = \pm \sqrt{\frac{3RA(1 + \lambda)^2 + (1 + R\lambda)(1 + R\lambda^3)2E_2G_c}{R^2(1 + R\lambda^3)\frac{h_2}{h_1}}}
\]

where $\alpha$ is given by (2.3) and $G_c$ is the delamination toughness of the interface. The critical stress is computed from (4.11) provided brittle fracture occurs, or Griffith’s fracture criterion is adopted. The lower line in (4.11) represents the critical stress in terms of relative parameters $R$ and $\lambda$.

### 4.1.5 Residual stresses in sprayed NiAl system

The significance of residual stresses is further demonstrated by application of equation (2.3) to experimental results of [20], where Nickel-Aluminum (NiAl) is sprayed on a carbon steel substrate using shape deposition manufacturing (SDM) by means of plasma-spray deposition. In SDM with plasma-spray deposition molten metal (NiAl) droplets are sprayed onto the substrate. Since this process is executed at high temperatures, residual stresses are likely to be present in the film-substrate system after cool-down. The specific test specimen to be compared has a NiAl film of $h_1 = 1.9$mm and a carbon steel substrate of $h_2 = 1.5$mm. The elastic moduli are $E_1 = 68$GPa and $E_2 = 190$GPa, with Poisson’s ratios of $\nu_1 = 0.19$ and $\nu_2 = 0.33$. The NiAl sprayed specimen was subjected to four-point bending and the critical bending moment at which steady-state delamination was observed was $M_{cr, ss} = 1.66$Nmm. Disregarding any residual stresses in the computation of the critical ERR, $G_c$ is equal to 53.10N/m. However, assuming the presence of residual stresses as a result of the mismatch in coefficient of thermal expansion (CTE) in the cooling down process, a first approximation may be given by assuming a constant mismatch strain between the deposited film and the substrate. Consequently, given the differences in CTE and the cool-down temperature, a mismatch stress may be assumed of 50MPa. Using equation (2.3), the ERR components $G_M$, $G_R$, and $G_{RM}$ have the magnitudes 53.10N/m, 6.37N/m, and 26.56N/m respectively. The total ERR at the critical load $M_{cr, ss}$ then is $G_c = 86.03$N/m, which is nearly twice as high as in the an initial stress-free system. It shows neglect the of residual stresses can cause the determined interfacial toughness to be significantly underestimated.

### 4.2 Film-substrate system with residual stress and additional top layer

In this section the effect of the additional top layer on a film-substrate system with residual film stresses on the outcome of the 4PB-test is examined. This is done by comparing two different three-layer systems with residual film stresses. Analogous to the example film-substrate system of the previous section, a brittle system is examined: the interfacial toughness is $G_c = 2N/m$. The relative material stiffness of the film and substrate is $R = 1/3$. Figure 4.3a shows a three-layer system with relative film thickness $h_2/h_1 = 10$, whereas in Figure 4.3b a system with $h_2/h_1 = 100$ is depicted. For both systems the critical bending moment required for steady-state delamination is plotted on the vertical axis, using an inverse form of the ERR equation (2.4). This moment is made dimensionless with a reference moment $M_{ref}$, which is the critical bending moment needed for steady-state delamination when $h_2/h_1 = 1$. The critical bending moment $M_{cr, ss}$ is plotted for different relative top layer thicknesses $h_2/h_1$. In the range 0.01 < $h_2/h_1 < 1$ three values of residual stress (−50MPa, 0MPa, and 50MPa) are plotted. For $h_2/h_1 = 10$, the results reduce to those found in the film-substrate system, whereas the line with residual stress $\sigma_R = 0$ represents the result found in [21]. The dots in Figure 4.3a represent the plateau values from the moment-displacement plot from the numerical simulations. These can be seen to closely
resemble the analytical results from section 2.2. Since the results in Figure 4.3a show stronger characteristics and are in close approximation of the analytical results, the analytical results from Figure 4.3b are not validated using numerical models to prevent ambiguity.

![Graph](image)

Figure 4.3: Bilayer with superlayer system subjected to couple $M$ and residual stress $\sigma_R$ in the top layer of the bilayer.

It is observed that the required bending moment $M_{cr,ss}$ is lowered significantly with increasing top layer thickness. Systems with thinner films react more strongly to the increase of top layer thickness than systems with thicker films. For example, in Figure 4.3 it is seen that in the case of a negative residual stress equal to $-50$ MPa, the addition of a superlayer with thickness $h_d = h_2$, the required bending moment $M_{cr,ss}$ reduces by a factor 3.28. However, there is one exception seen in Figure 4.3a. The system with $\sigma_R = 50$ MPa is not easier delaminated with the addition of an additional top layer. It seems that the diminishing influence of the residual stress $\sigma_R$ is neutralized by the influence of the increasing top layer thickness. In the study of negative residual film stresses in the previous section it was seen that negative residual stresses increase the resistance against delamination only up to a certain layer thickness ratio. This phenomenon for three-layer systems is reflected in Figure 4.3a, where the line $\sigma_R = -50$ MPa decreases the critical bending moment needed for delamination.
for \( h_2/h_2 > 0.4 \). Also, the effect of the residual stress diminishes with increasing top layer height, as can be observed by the tendency of the curves to come together, especially when films are thin. This is due to the increasing symmetry of the system and consequently the decrease of the effective moment arm of the residual film stresses. A mismatch is seen between the residual stress curves and the initial stress-free curve, which is caused by the compensating axial specimen strain caused by the residual film stress. Thus, the need to measure residual stresses and to incorporate those in the formulation computing the ERR diminishes when films are thin and the system becomes more symmetric. In Figure 4.3a, the curves for \( \sigma_R = -50 \) MPa and \( \sigma_R = 50 \) MPa coalesce at \( h_d = h_2 \), since the interaction component in the ERR vanishes here and the component solely due to residual stress becomes independent of its sign. This is reflected the absence of an interaction term between the moment \( M \) and the residual stress \( \sigma_R \) in equation (2.6), where the superlayer thickness equals the substrate thickness. For the thin film variant in Figure 4.3b, the existence of negative residual stresses is beneficial for the specimen delamination resistance, in contrast to the variant with \( h_2/h_1 = 10 \) in Figure 4.3a. The net contribution of the negative residual stresses is determined by the relative magnitude of the interaction term and the residual stress term in equation (2.4).

### 4.2.1 Critical residual film stress

Similarly as for the film-substrate system, the extended three-layer system has a critical residual film stress, which alone is sufficient to initiate and drive delamination. The magnitude of this critical stress, \( \sigma_{R,crit} \) is computed from the residual stress component of (2.4). Brittle fracture is assumed so that delamination occurs at a critical level of the ERR, \( G_c \). Reworking the equation gives

\[
\sigma_{R,crit} = \pm \sqrt{\left( -3E_1h_1 (E_2h_2 - E_2h_d) + E_2h_2^2 - E_2h_d^2 \right)^2 + \gamma \left( E_2h_2 + E_2h_d \right) - \frac{E_2h_d (E_1h_1^2 + E_2h_d^2)}{\beta}} \frac{2E_1G_c}{h_1}.
\]

### 4.3 Film-substrate system with residual stress and elasto-plastic substrate

In Section 2.3 a method was presented for the determination of the ERR of film-substrate systems with an elasto-plastic substrate, preloaded with residual film stresses, in a four-point bend test environment. The presence of plasticity was noted in two main regions: the upstream and downstream sections far from the crack tip, and near the crack tip. Since only the former are considered in an upstream-downstream approach, plastic deformations in the vicinity of the crack tip are disregarded. The derived results from the global energy approach are therefore valid only if crack tip plasticity is negligible. Using numerical models as described in Section 3.2, the range of validity of the limit case solution from Section 2.3 is studied. Wei et al. [15] indicated that the interface strength \( t^u \) plays a key role in the size of the crack tip plasticity. Specifically, the relative magnitude of interface strength \( t^u \) with respect to the yielding stress \( \sigma_y \) is important. A relative strength is introduced as

\[
t = \frac{t^u}{\sigma_y}.
\]

For higher values of \( t \) the plastic zone size is increased, hence the significance of plastic deformations on the outcome of a four-point bend test is increased. In addition to the relative strength \( t \), the hardening power to a lesser extend plays a role in the size of the plastic zone [6]. With higher hardening rate, hence with lower values of hardening power \( n \), the plastic zone size will be smaller. This effect, however, will not be studied in the present paper, as the crack tip plasticity is primarily controlled by parameter \( t \).

Two sets of simulations are presented in Figure 4.4. In the studied systems the elasto-plastic substrate initially is three times as compliant as the film. The interfacial toughness is set to \( G_c = 32 \) N/m. The substrate height for 4.4a is 0.5mm, while it is 5mm for 4.4b. The (initial) elastic modulus is \( E = 70 \) GPa with \( v = 0.3 \). The
Ranberg-Osgood yield offset is $\alpha = 3/7$ and the hardening power is set to $n = 5$. The yield stress is equal to $\sigma_y = 120MPa$. The critical bending moment needed for steady-state delamination $M_{cr,ss}$ relative to the reference moment $M_{ref}$ is assigned to the vertical axis, where the reference moment $M_{ref}$ is the critical bending moment required for steady-state delamination in the limit case of the interface strength reaching zero, for thickness ratio $h_1/h_2 = 1$.

In both figures 4.4a and 4.4b the dashed lines represent the elastic solution and the elasto-plastic solution from section 2.1 and 2.3. It is seen the elastic solution deviates most from the elasto-plastic line in Figure 4.4a when films are thin. In Figure 4.4b, these lines nearly coincide. Great deviation of the two lines in 4.4a points out high upstream and downstream plastic deformations, while in b the substrate upstream and downstream plasticity is negligible. It is interesting to see the elasto-plastic line lies below the elastic solution: the moment needed for delamination is lower when the substrate is elasto-plastic, provided crack tip plasticity is neglected.

An explanation is found using Figure 2.3. Here, the stress and strain profiles for the downstream (a and b) and the upstream (c and d), subjected to bending moment $M$, are given. The dotted lines present the stresses and the strains in case the entire system is elastic. In case the substrate is elasto-plastic with the elastic modulus from the elastic system as the initial modulus, the stress and strain profiles are depicted by the full lines. It can be observed from the figure that the upstream direction stress and strain profile of the elastic and elasto-plastic case are nearly the same. In the downstream section, however, are the profiles significantly different. Here, as less section height is available, higher stresses are needed to equilibrate the bending moment $M$. Since the magnitude of stress is limited by degradation of the material stiffness with increasing loading, higher strains are required to equilibrate the moment $M$. It is seen from Figure 2.3a, the strains in the elasto-plastic case are nearly twice as high as in the elastic case. Hence, the section curvature of the downstream section in the elasto-plastic case is significantly higher than in the elastic case. As the external bending moment $M$ produces more work when the difference in curvature of the upstream and downstream section is higher the ERR is also increased, see equation (2.10). The strain energy component of the ERR is affected less severely. The effect of elasto-plastic substrate behaviour is that more energy is produced by the external bending moment $M$ than in the case the substrate is elastic. Hence, the critical bending moment needed for steady-state delamination at interface toughness $G_c$ is lower for an elasto-plastic substrate system than for an elastic system.

The critical bending moments are plotted for four values of the relative interface strength $t$. It is seen from both figure 4.4a and b that the system with $t = 0.05$ lies on top of the elasto-plastic limit solution computed from Section 2.3. Systems where the interface strength is $t = 0.5$ show no significant deviations from the elasto-plastic limit case. When the interface strength is equal to the yield stress deviations up to 14 per cent ($h_1/h_2 \approx 0.05$ in Fig. 4.4a) in the critical bending moment $M_{cr,ss}$ are found. When the interface strength starts to outvalue the yield stress, crack tip plasticity starts to influence the outcome of the four-point bend test significantly. For $t = 2$, the moment $M_{cr,ss}$ can differ from the limit case up to 32 per cent ($h_1/h_2 \approx 0.025$ in Fig. 4.4b).

What is interesting to see is that the influence of plastic deformations near the crack tip on the 4PB-test is relatively invariant of the thickness ratio $h_1/h_2$ for both the systems in figure 4.4, that is the difference between the line corresponding to $t = 2$ and the elasto-plastic limit case differs not greatly with changing thickness ratio. Wei et al. [15] considered a film-substrate system with an elasto-plastic semi-infinite substrate, loaded with residual film stresses. They showed that from the geometric parameters primarily the local length scale $h_1/R_0$ is of interest in the magnitude of the plastic zone. As $h_2$ is larger than $h_1$ and $h_1/h_2$ does not affect the outcome of the 4PB-test greatly, the absolute value of $h_2$ seems to be irrelevant to the local crack tip zone height in the example systems of Figure 4.4. Wei et al. [15] showed that the crack tip plasticity only varies with $h_1$ up to $h_1/R_0 = 6$ for a film-substrate system with a semi-infinite substrate, after which the amount of plastic deformation stays constant. However, it was seen that no significant changes occur after $h_1/R_0 = 2$. It is seen from figure 4.4a only thin film systems with $h_1/R_0 < 1$ do experience less plasticity effects as the differences in critical moments $M_{cr,ss}$ become smaller, though this only seems to hold for systems with $t < 1$. The same goes for the lines in figure 4.4b for the thinnest film systems. Except for the thinnest film systems, the plastic zone size is unaffected by both $h_1$ and $h_2$.

With increasing bending moment the stresses in the upstream and downstream section rise. Since the substrate stiffness deteriorates with increasing plastic deformations, the film is increasingly carrying more load. Consequently, the material strength of the film material may be reached before high bending moments arise in the two-layer system. Large plastic deformations, in addition to high levels of bending moment $M$, therefore
should warrant caution with respect to the magnitude of the film stresses.

In Figure 4.4 the significance of plastic deformations of the substrate sections is indicated by the difference of the dashed elastic solution line and the dashed elasto-plastic solution line. For example, the elasto-plastic solution for the thinnest systems modeled in Figure 4.4a deviate significantly from the elastic solution, and hence the film stresses are really high here. Initiating delamination of high-strength interface specimens with very thin films therefore can be difficult to accomplish.

Figure 4.4: Two example film-substrate systems with an elasto-plastic substrate, with different values for the interface strength.
4.3.1 Systems with residual stresses

As was done for the elastic two and three-layer systems, the effect of residual stresses is studied in an example system. The critical bending moments required for steady-state delamination with the presence of residual film stress of magnitude $-50\text{MPa}$, $0\text{MPa}$, and $50\text{MPa}$ are sketched in Figure 4.5. This is done for the systems studied in Figure 4.4a and b. The effect of the residual stress for two relative interface strengths $t = 2$ and $t = 0$ are shown in Figure 4.5. The lower lines in Figure 4.5a and b are the results computed from the energy balance in Section ??, while the top lines are the results from the numerical simulations with relative interface strength $t = 2$. It is seen in both figures Figure 4.5a and b that the systems with negative residual film stress are harder to delaminate. However this tendency is less apparent in the systems with strong interfaces $t = 2$. In the thin samples of Figure 4.5a, the line corresponding to $\sigma_R = -50\text{MPa}$ for most values of $h_1/h_2$ seems to coincide with the initial stress-free systems. From b it is observed the line corresponding to $t = 2$ and $\sigma_R = -50\text{MPa}$ intersects with the initial stress-free line and ceases to be of positive influence with respect to delamination resistance. A similar trend was seen in both the elastic systems of Figure 4.3b and Figure 4.4a. The systems with strong interfaces, that is interfaces with $t = 2$, are less affected by the presence of residual stresses than the weak interface samples. This is due to the fact that the strong interface samples already experience higher film stresses at the higher loads $M_{cr,ss}$, so that a difference of $50\text{MPa}$ has not as great an effect as in the less stressed films in the weak interface samples.
Figure 4.5: Two example film-substrate systems with an elasto-plastic substrate subjected to residual film stress $\sigma_R$, for a weak and a strong interface.
5 Summary of results and concluding remarks

Equations for energy release rates (ERR) for several film-substrate systems loaded with residual stresses in a four-point bend (4PB) test were derived. The first system was the entirely elastic film-substrate system with a residual film stress. The presence of this stress was shown to either ease the delamination process, in terms of required bending moment loading for steady-state delamination. Compressive residual stresses were shown to delay the crack propagation, though only up to a certain thickness ratio, the value of which is dependent of the film-substrate material properties. The effect of the presence of residual stresses is illustrated exemplary at a) by the moment-displacement curves of Figure 5.1, where the top line represents the film-substrate system with compressive residual film stresses, the middle line shows the initial stress-free system, and the bottom line denotes the tensile residually stressed film-substrate system. The significance of the presence of residual stresses is clear from the example in the figure; the plateau value of the moment-displacement curve is affected about 30 per cent with respect to the initial stress-free system. The importance of the inclusion of residual stresses was shown for 4PB-test experiments also. The delamination toughness of the considered NiAl sprayed system was found to be almost twice as high as in the absence of the residual stress.

![Figure 5.1: Bilayer system subjected to couple \( M \) and residual stress \( \sigma_R \) in the top layer. a) represents the original system, b) is the original system with an additional top layer, c) shows the results for the case the substrate is elasto-plastic, d) represents the elasto-plastic substrate system with a strong interface.](image)

In the case delamination of a film-substrate is hard to achieve, because the stresses in the film or substrate exceed the material strength of one of the subsequent layers, due to either inadequate film thickness or insufficient film stiffness, the specimen performance in the 4PB-test can be greatly improved by addition of an additional top layer. In Figure 5.1, this is illustrated by the lines belonging to b). The addition of half a substrate on top of the film shows a decrease in critical bending moment \( M_{cr,ss} \) of almost 50 per cent with respect to the original system. This drop in required loading can make a 4PB-test setup for the determination of an interface toughness feasible for weak systems. The presence of residual stresses in this three-layer system alters the outcome of the 4PB-test less severely than in the two-layer case. In fact, it diminishes greatly for thin films, especially when the top layer has similar elastic properties as well as a similar layer height as the...
substrate. An expression for the ERR was derived to include the effect of the residual film stresses.

As a third residually stressed system, a film-substrate system with an elasto-plastic substrate was studied. For this case also an ERR was derived in the form of a simple numerical procedure. The procedure was shown to yield the ERR in the limit case of the interfacial strength going to zero. In this limit, it was seen delamination could occur at lower loading levels than in the elastic case, see lines c) in Figure 5.1. Analysis showed that this was due a larger difference in section curvatures of the uncracked and cracked sections in the case of elasto-plastic material behaviour, which resulted in greater values of conjugated work.

The plateau values in Figure 5.1 for lines a), b), and c) were all derived by the combination of the analytical expressions for the ERR in combination with Griffith’s criterion, which is the energy criterion for brittle cracking. These results were validated using numerical models. Plateau values for the elasto-plastic system with a strong interface were derived from numerical simulations, see lines d). Steady-state values in lines c) are the limit case of the interface strength reaching zero for system d). The significance of plastic deformation and subsequently energy dissipation are illustrated by the comparison of lines d) and c) in Figure 5.1. In the shown example, the needed moment to initiate delamination for the initial stress-free case is 44 per cent higher in case the interface strength outvalues the yield stress by a factor two. It shows that the interfacial strength, along with the interface toughness, is a parameter of primary importance in the study of delamination resistance. The effect of residual film stresses is diminished with increasing interface strength. The reason for this is the decreasing magnitude of these stresses with respect to the stresses present in the specimen due to the 4PB-test loading. Higher stresses prior to delamination also reduce the possibility of a successful 4PB-test, as material strengths may be exceeded. To this end, caution is to be taken when the plastic deformations grow large. An estimate for this situation is The plastic zone size was shown to be largely invariant of the relative layer thickness for the specimens shown. Only for very thin film systems the effects of crack tip plasticity diminished.

Although the effects of crack tip plasticity on the outcome of the four-point bend test were demonstrated, the determination of the dissipative energy per unit crack extension remains uncovered. This implicates that the delamination toughness for systems with non-negligible interface strength can be determined from a critical steady-state 4PB-test moment only by successive guess and test of the interface toughness in the traction-separation description in a numerical 4PB-test environment.

A known phenomenon in the determination of interfacial adhesion energy is that the adhesion energy increases for increasing mode II loading, due to the additional resistance provided by e.g. frictional forces in this mode. Hence, for these systems the adhesion energy is dependent on the mode-mixity, or phase angle. The effect of mode-mixity on the fracture energy may be an interesting continuation of research, especially since the mode-mix is affected by both residual stresses and plasticity near the crack tip. For the latter both the yield stress and the hardening rate play a role.

Further improvements to the analytical elasto-plastic model may be achieved by inclusion of an additional top layer on top of the film. As with the elastic three-layer system the required bending moment for steady-state delamination can be significantly lower than in the absence of this layer, which makes more specimens suitable for the 4PB-test.
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A Derivation of the ERR of film-substrate systems with residual film stress in a 4PB-test setup

In figure A.1 a bilayer system in a four point bend setup is sketched with a delaminating crack evolving along the interface between the film and the substrate. The delaminating crack is driven by a couple \( M \) and a uniform residual stress \( \sigma_R \) (where a positive sign denotes tension, and a negative sign denotes compression).

Whether the curvature is positive (which is in correspondence with the deformed configuration sketched in figure A.1) or negative depends on the signs of moment \( M \) and the residual stress \( \sigma_R \): A positive bending moment \( M \) (which corresponds to the direction of \( M \) in figure A.1) and/or a negative residual stress \( \sigma_R \) provide the system with a positive curvature, whereas a negative moment \( M \) and/or a positive residual stress \( \sigma_R \) provide the system with a negative curvature.

The deformed situation as sketched in the figure shows the case of a partly delaminated top layer. It is noted that the curvature the upstream and downstream section are different.

![Figure A.1: Bilayer system subjected to couple \( M \) and residual stress \( \sigma_R \) in the top layer.](image)

Only one half of beam beam needs to be considered in the delamination process, since the system and the loading are symmetric. In the lower figure of Figure A.1, the right half of the beam is drawn. For the downstream direction the location of the neutral axis is known to be positioned \( h_2/2 \) from the substrate bottom. For the upstream direction, the location of the neutral axis is unknown, but said to be located at distance \( a \) from the substrate bottom. In the location of the neutral axis a downward positive \( y \) – direction is specified.

**Location of neutral line**

As a first requirement, the integral of *elemental forces* over the entire upstream cross-section needs to be equal to zero, which leads to

\[
\int_1 \sigma_{u1} \mathrm{d}A + \int_2 \sigma_{u2} \mathrm{d}A = 0, \quad (A.1)
\]

with \( \sigma_{u1} \) and \( \sigma_{u2} \) the normal stresses in the longitudinal direction of the film and the substrate, respectively, and \( A \) the upstream cross-section. The constitutive relations for the (elastic) film and substrate under a state of bending with a residual stress \( \sigma_R \) present in the film are
where $\bar{\sigma}_i$ with $i = 1, 2$ are the plane-strain moduli of the film and substrate, as given by $\bar{E}_i = E_i/(1-\nu^2)$ with $E_i$ the Young’s modulus and $\nu$ the Poisson’s ratio. Further, $\kappa_u$ is the curvature of the upstream cross-section, generated by both the couple $M$ and the uniform residual stress $\sigma_R$. Substituting (A.2) into (A.1) leads to

$$\bar{E}_1 \int_{y=-b-h_1}^{b} (-\kappa_u y + \frac{\sigma_R}{E_1}) \, dy + \bar{E}_2 \int_{y=-b}^{a} -\kappa_u y \, dy = 0,$$

where $h_1$ is the thickness of the film, and $a$ and $b$ indicate the position of the neutral axis, as shown in figure A.1. As a second requirement, the curvature $\kappa_u$ needs to be expressed in terms of couple $M$ and the residual stress $\sigma_R$. This is done by equating the integral of elemental moments over the entire cross-section to the external moment $M$:

$$-\int_{-b}^{-b-h_2} \sigma_{u1} y \, dy - \int_{-b}^{a} \sigma_{u2} y \, dy = M.$$  

(4.4)

With the constitutive relations given by equation (A.2), this expression turns into

$$-\bar{E}_1 \int_{y=-b-h_1}^{b} (-\kappa_u y + \frac{\sigma_R}{E_1}) \, dy + \bar{E}_2 \int_{y=-b}^{a} \kappa_u y^2 \, dy = M.$$  

(4.5)

Carrying out the integration of both (4.3) and (4.5), observing that $b = h_2 - a$, yields both upstream bilayer curvature $\kappa_u$ and values for $a$ and $b$:

$$\kappa_u = \frac{12M(\bar{E}_1 h_1 + \bar{E}_2 h_2) - 6\sigma_R h_1 h_2 \bar{E}_2(h_1 + h_2)}{6\alpha},$$

with $\alpha = \bar{E}_1^2 h_1^2 + 4\bar{E}_1 \bar{E}_2 h_1 h_2 + 6\bar{E}_1 h_1^2 \bar{E}_2 h_2^2 + 4\bar{E}_1 h_1 \bar{E}_2 h_2^2 + \bar{E}_2^2 h_2^4$.

$$a = \frac{6M(\bar{E}_2 h_2^2 + \bar{E}_1 h_1(2h_2 + h_1)) + \sigma_R(\bar{E}_1 h_1^2 - \bar{E}_2 h_2^2 h_1(2h_2 + 3h_1))}{12M(\bar{E}_1 h_1 + \bar{E}_2 h_2) - 6\sigma_R \bar{E}_2 h_1 h_2(h_1 + h_2)},$$

$$b = \frac{6M(\bar{E}_2 h_2^2 - \bar{E}_1 h_1^2) - \sigma_R(\bar{E}_1 h_1^2 + \bar{E}_2 h_2^2 h_1(4h_2 + 3h_1))}{12M(\bar{E}_1 h_1 + \bar{E}_2 h_2) - 6\sigma_R \bar{E}_2 h_1 h_2(h_1 + h_2)}.$$  

(7)

**Strain energy**

With (7), the location of the neutral axis is fully determined. The upstream strain energy (per unit beam length) is obtained by integration of the strain energy density over the cross-section as:

$$U_u = \int_A u_0 dA,$$

where $u_0 = \int_0^\varepsilon \sigma d\varepsilon$.

Recognizing the strain energy density is $u_0 = E\varepsilon^2/2$ for linear elastic materials, the strain energy for the upstream section (per unit width and beam length) then is computed as

$$U_u = \frac{1}{2} \bar{E}_1 \int_{y=-b-h_1}^{b} (-\kappa_u y + \frac{\sigma_R}{E_1})^2 \, dy + \frac{1}{2} \bar{E}_2 \int_{y=-b}^{a} (-\kappa_u y)^2 \, dy.$$  

(9.1)
The strain energy downstream is computed in a similar fashion as for the upstream direction. Since the downstream cross-section is homogeneous and does not experience the residual stress $\sigma_R$ (due to the developed delamination) the neutral axis relates to a state of pure bending and is thus positioned at a distance $h_2/2$ from the bottom of the substrate, see figure A.1. Requiring the integral of elemental moments equals $M$,

$$\int_{y=-h_2/2}^{h_1/2} \sigma_{d2} y \, dy = M, \quad (A.10)$$

in combination with the constitutive equation $\sigma_{d2} = -E_2 \kappa_d y$, and solving for the downstream curvature $\kappa_d$, yields

$$\kappa_d = \frac{12M}{E_2 h_2^3}. \quad (A.11)$$

Subsequently, the strain energy density is integrated over the downstream cross-section as

$$U_d = \frac{1}{2} E_2 \int_{y=-h_2/2}^{h_2} (-\kappa_d y)^2 \, dy. \quad (A.12)$$

The decrease in strain energy $\Delta U$ during delamination over a distance $a$ can be calculated from the difference in strain energy downstream and upstream as

$$\Delta U = (U_d - U_u) a \quad (A.13)$$

**External work and energy balance**

The external work applied to the film-substrate system during delamination is obtained from the product of the couple $M$ and the energetically-conjugated rotation $\phi$. The rotation $\phi$ is determined as the product of the delamination $a$ and the difference in curvature between the downstream and upstream cross-sections, i.e.,

$$\phi = (\kappa_d - \kappa_u) a. \quad (A.14)$$

With equation (A.14), the external work may be expressed as

$$\Delta F = M(\kappa_d - \kappa_u) a \quad (A.15)$$

Recalling the derivation of the Energy Release Rate from the energy balance, as denoted in equation (2.2), the Energy Release Rate is computed as

$$G_{ss} = \frac{\partial \Delta W}{\partial a} = \frac{\partial (\Delta F - \Delta U)}{\partial a}. \quad (A.16)$$

**Results**

Working out (A.16), using the expressions for the curvature $\kappa_u$ and $\kappa_d$ from equations (A.6) and (A.11) in equations (A.15) and (A.13), thus yields the steady-state energy release rate

$$G_{ss} = 6M^2 \left( \frac{1}{E_2 h_2^3} - \frac{(E_1 h_1 + E_2 h_2)}{\alpha} \right) + \sigma_R^2 E_2 h_1 h_2 (E_1 h_1^3 + E_2 h_2^3) + 12aM E_1 h_1 E_2 h_2 (h_1 + h_2), \quad (A.17)$$

with $\alpha = E_1^2 h_1^4 + 4E_1 h_1^3 E_2 h_2 + 6E_1 h_1^2 E_2 h_2^2 + 4E_1 h_1 E_2 h_2^3 + E_2^2 h_2^4$.

It is observed that the expression for $G_{ss}$ essentially is composed of three parts, namely a part solely due to the four point bend test loading $M$, a part solely due to the residual stress $\sigma_R$ in the top layer, and a part due to the interaction of $M$ and $\sigma_R$. When the residual stress is absent, the result reduces to the known expression for the energy release rate for the case of pure bending.
Elasto-plastic pure bending of a single layer in plane strain conditions is considered using the deformation theory of plasticity. This layer is coupled to an elastic layer by requirement of compatible strains to arrive at a film-substrate system. The stress and strain profiles of the film-substrate system are used to determine the strain energy in the upstream direction, whereas the downstream strain energy is computed from the elasto-plastic single layer stress and strain distribution. Using the section curvatures, the energetically conjugated work provided during delamination of the interface crack over a unit length is computed. In combination with the change in strain energy in this same process, the energy release rate (ERR) is determined.

Stress-strain curve in the longitudinal direction

In the deformation theory of plasticity, the Hookean stress-strain relations are used to relate the multiaxial stress and strain components \[ \sigma \] and \[ \varepsilon \], respectively, in the hardening law by the equivalent stress \( \sigma_m \) and equivalent strain \( \varepsilon_m \). The latter two are, in the case of J2, or Von Mises, plasticity equal to

\[
\begin{align*}
\sigma_m &= \frac{\sqrt{2}}{2} \sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2} \\
\varepsilon_m &= \frac{1}{\sqrt{2} (1 + v)} \sqrt{(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2}
\end{align*}
\]

where the Cartesian stress and strain components are the principle stresses and strains. The relations (B.1) are used to compute a relation between the longitudinal stress \( \sigma_x \) and the longitudinal strain \( \varepsilon_x \), which are of interest in the determination of the neutral line for pure bending loading. As mentioned, the stress and strain components are coupled through the general Hookean stress strain relations. However, in these relations, the two independent material constants \( E \) and \( v \) are variable in the transition from the elastic to the plastic case. The material stiffness is equal to the secant modulus \( E_{sec} = \sigma_m / \varepsilon_m \) in the Ramberg Osgood relation (2.8), whereas the Poisson’s ratio \( v \) varies between the elastic ratio \( v_e \) and the plastic ratio \( v = 1/2 \), which represents the incompressible state in case of full plasticity. The relation for Poisson’s ratio is determined from the requirement that the total dilation (volumetric strain) with the equivalent \( v \) equals the sum of the elastic dilation and the plastic dilation components \[ \frac{E_{sec}}{E} \]. It is formulated as

\[
v = \frac{1}{2} - \left[ \frac{1}{2} - v_e \right] \frac{E_{sec}}{E}.
\]

In the case of plane strain four point bending, the region of interest solely consists of the nonzero components \( \sigma_x, \sigma_z, \varepsilon_x \) and \( \varepsilon_y \). The stress strain equations then reduce to (with \( \sigma_x \) as independent variable)

\[
\begin{align*}
\sigma_x &= \sigma_x \\
\sigma_z &= v \sigma_x \\
\varepsilon_x &= \frac{1 - v^2}{E_{sec} \sigma_x} \\
\varepsilon_y &= -\frac{1 + v^2}{E_{sec} \sigma_x}
\end{align*}
\]

Combining (B.3) with the equivalent stress and strain (B.1) yields
be computed from the outer fiber strains as be computed from equations (B.4), using the current values for $E\varepsilon\sigma$ each point differing $\sigma$ at distance $b$ from the interface. The equivalent stress $\sigma$ and strain $\varepsilon$ are linearly distributed in this layer, it is sufficient to discretize the film as one material slice with

$$
\sigma_x = \frac{\sigma_m}{\sqrt{v^2 - v + 1}} \quad \varepsilon_x = \frac{(v + 1)}{2} \left[ \frac{-v}{E_{sec}}\sigma_x + \sqrt{\frac{4\varepsilon_m^2 - 3\varepsilon_m^2}{E_{sec}}\sigma_x^2} \right] 
$$

Equations (B.4) provide the stress-strain information for positive values of $\sigma_x$ and $\varepsilon_x$ for given values of the equivalent stress $\sigma_m$ and strain $\varepsilon_m$. Their negative counterparts are found by mirroring these results.

**Determining the stress profile**

The location of the neutral line, and consequently the stress profile, is determined using the formulations for the stresses and strains in the longitudinal direction from equation (B.4). The solving procedure is done numerically, in an iterative fashion. The film-substrate system is discretized into strips. In the elasto-plastic requirement of compatible strains. At $y = a$, the net force on the section is determined from the sum of very material strip $i$ in $n$ using the trapezoidal rule, as

$$
F = \sum_{i=0}^{n} \sigma_{x,i}(y_{i+1} - y_i) + \frac{1}{2}(\sigma_{x,i+1} - \sigma_{x,i})(y_{i+1} - y_i). 
$$

For zero net force (B.6) needs to equate to zero. The bending moment due to the stresses $\sigma_x$ is computed from the individual contribution of the $n$ material strips, as the moment produced in the trapezoid between two consecutive points $i$ and $i + 1$. It is computed as

$$
M = \sum_{i=0}^{n} \frac{1}{2}\sigma_{x,i}(y_{i+1}^2 - y_i^2) + \frac{1}{6}(\sigma_{x,i+1} - \sigma_{x,i})(2y_{i+1}^2 - y_{i+1}y_i - y_i^2). 
$$

The contribution of the elastic film layer has to be included in equations (B.6) and (B.7). Since stresses and strains are linearly distributed in this layer, it is sufficient to discretize the film as one material slice with two material points. The distances of the outer film fibers to the neutral line can be computed as $b$ and $b + h_1$ for the bottom and top film fibers, respectively. The stresses in these consecutive points are found from the requirement of compatible strains. At $y = -b$, the strain of the top fiber of the substrate needs to equal the strain of the bottom film fiber. If residual film stresses are present, these strains differ by $\varepsilon_R = \sigma_R/E_1$. Using the section curvature from (B.5), the strains are related through

$$
\varepsilon_{1,\text{top}} = \varepsilon_b + \varepsilon_R \\
\varepsilon_{1,\text{bot}} = \varepsilon_b \left( \frac{h_1 + h_2}{h_2} \right) - \varepsilon_a \left( \frac{h_1}{h_2} \right) + \varepsilon_R, 
$$

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where $\varepsilon_{1,\text{top}}$ and $\varepsilon_{1,\text{bot}}$ are the top film strain and the bottom film strain, respectively. The corresponding stresses are computed as $\sigma = E_1 \varepsilon$. These are included in both (B.6) and (B.7). For the initially assumed substrate stress interval $\sigma_{m,t}$ and top substrate stress $\sigma_{m,b}$ a net force $F$ and net moment $M$ are computed from (B.6) and (B.7). Since no external force acts on the section, $F$ needs to vanish. To this end, the stress interval $\sigma_{m,t}$ needs to be shifted until the value of $\sigma_{m,b}$ is found for which the net force approximates zero. This is done using a Newton-Raphson scheme. Next, the value of the stress interval $\sigma_{m,t}$ needs to be updated until the net bending moment $M$ equals the applied bending moment. This is also done using a Newton-Raphson scheme. Note that for every update of the stress interval $\sigma_{m,t}$ a full optimization scheme for the determination of $\sigma_{m,b}$ needs to be executed.

### Strain energy

The strain energy density for a Ramberg-Osgood hardening material in a multi-axial stress state is given by [24]

$$u_0 = \frac{1 + v_e}{3E} \sigma_m^2 + \frac{3}{2} \frac{1 - 2v_e}{E} \sigma_p^2 + \frac{n}{n+1} \beta \sigma_m^{n+1},$$

where

$$\sigma_p = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{(1 + v)\sigma_x}{3}.$$  \hspace{1cm} (B.9)

Here, the Poisson’s ratio in front of the first two terms is the elastic Poisson’s ratio, whereas the values of the equivalent stress $\sigma_m$ and the hydrostatic pressure $\sigma_p$ are determined using the current Poisson’s ratio. The strain energy is computed from the integral of the strain energy density over the section height. The contribution of the elasto-plastic layer to the strain energy per unit layer length and width then is

$$U = \int_{-b-h_1}^{a} u_0 dy$$  \hspace{1cm} (B.10)

$$U = \sum_{i=0}^{n} u_{0,i}(h_{i+1} - h_i) + \frac{1}{2}(u_{0,i+1} - u_{0,i})(h_{i+1} - h_i).$$  \hspace{1cm} (B.11)

For the strain energy density in the elastic film $u_0 = E_1 \varepsilon^2/2$ is used. Using equation (B.11) the strain energy of the film-substrate layer under bending with residual film stress can be computed.

### Energy release rate

The energy release rate is computed from equation (A.16). In terms of section curvatures and strain energies it is

$$G_{ss} = (\kappa_d - \kappa_u) M - (U_d - U_u),$$  \hspace{1cm} (B.12)

where $\kappa$ and $U$ are the section curvature and strain energy respectively. The subscripts $d$ and $u$ refer to the downstream and upstream section. The upstream quantities are computed from the procedures presented above. The downstream curvature and strain energy are computed similarly. Disregarding any film in the above procedure yields the stress and strain profile of the downstream direction. The procedure is further simplified since the neutral line is known be to located at the center of the substrate, that is, $a = b = h_2/2$. 

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Appendix 1: 
Film-substrate systems 
with additional top layer 
and residual film stress

Equilibrium and energy release rate

P.J.J. Forschelen

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Eindhoven, March 2015
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Chapter 1

Introduction

In this appendix an expression for the Energy Release Rate (ERR) is derived for residually stressed film-substrate systems with an additional top layer in a four-point bend (4PB) test environment.

In Figure 1.1 a three-layer system is shown in a 4PB-test environment. The additional top layer is denoted by the subscript \( d \), whereas the film and the substrate are denoted by the subscripts 1 and 2 respectively. A symmetrical pre-crack is present throughout the thickness of the superlayer and the film. The film layer is loaded with an initial residual stress \( \sigma_R \).

Figure 1.1: Film-substrate system with additional top layer subjected to couple \( M \) and residual stress \( \sigma_R \) in film.

With increasing external loading \( M \) in the 4PB-test, the energy available for crack progression is steadily increased. When it reaches a critical level, the cracking process is initiated and progresses in a steady state fashion: without further increase of the external loading (but with further increase in applied displacements at the beam ends) the crack propagates. At this loading level the energy generated by the specimen is said to equal the interfacial toughness of the cracking interface.

The determination of the ERR is done by a global energy balance approach, in which the cracking process is characterized by the transition of the uncracked into the cracked state. The energetic quantities affected by this transition are coupled with the energy consumed by the fracture process. Inspection of these energetic quantities in the transition of uncracked to the cracked state is used to formulate a delamination criterion. In the global energy approach, the transition process is captured by comparison of the uncracked and cracked section in terms of internal energetic quantities, or strain energy, and the external energetic quantities, or work. The former components are computed from stress and strain profiles, whereas the latter components are derived from the
section curvatures and the external bending loading. Hence, equilibrium conditions are utilized to determine the stress and strain profiles (or, equivalently, the location of the neutral line) for both the cracked and uncracked sections. Once these are known, both section strain energies and section curvatures are readily computed, from which the ERR can be extracted.

In Chapter 2 the upstream (uncracked) section is examined. Force and moment equilibrium are executed to find the stress and strain profiles. Then, the section strain energies are computed. Subsequently, in Chapter 3 the downstream (cracked) section is analyzed in terms of equilibrium. After finding the stress and strain profiles, the strain energy of this section are determined.

In Chapter 4 the results from the earlier chapters are combined to arrive at an expression for the energy release rate.
Chapter 2

Analysis of upstream direction

Since the three-layer specimen which is to be examined is symmetrical, only half of the system is analyzed. In Figure 2.1 this system is sketched.

Figure 2.1: Half of film-substrate system with additional top layer. In the upstream direction the ordinates are zero at the film-substrate interface. Positive values of \( y \) are directed downward.

The origin of the coordinate system is positioned at the film-substrate interface and defined downward positive. In earlier analysis of the film-substrate system subjected to \( \sigma_R \) and \( M \) it was located at the neutral line, defined somewhere in the substrate. This will prove very cumbersome in the elaboration of the integrals (due to the additional nonzero bounds in these integrals) in both force and moment equilibrium later on.
CHAPTER 2. ANALYSIS OF UPSTREAM DIRECTION

2.1 Equilibrium

Since no external force \( F \) acts on the upstream section, the integral of elemental forces needs to vanish:

\[
F = \int \sigma dA = b \left( \int_{-h_1}^{-h_2} \sigma_{ud} dy + b \int_0^{h_2} \sigma_{ud} dy \right) = 0.
\]  

(2.1)

In equation (2.1), the subscripts 1, 2, \( d \) correspond to the film, substrate, and top layer, respectively (see Figure 2.1). Subscript \( u \) refers to the upstream direction. It is seen the integral of elemental forces vanishes when the sum of the individual layer forces vanishes. Since the analysis is performed for a specimen of unit width, \( b \) will be omitted in further elaborations. The layer stresses are obtained from the strain distribution as shown in Figure 2.1 and expressed in terms of section curvature \( \kappa_u \) and elastic moduli as

\[
\sigma_{u1} = -\bar{E}_1 \kappa_u y + \bar{E}_1 \varepsilon_0 + \sigma_R, \\
\sigma_{u2} = -\bar{E}_2 \kappa_u y + \bar{E}_2 \varepsilon_0, \\
\sigma_{ud} = -\bar{E}_d \kappa_u y + \bar{E}_d \varepsilon_0, 
\]

(2.2)

In the first equation of (2.2) the residual stress \( \sigma_R \) is included. The elastic moduli are denoted as the plane strain moduli, or \( \bar{E}_i = E_i / (1 - v_i^2) \). Since the origin is not located at the neutral line, some mismatch strain \( \varepsilon_0 \) is present, hence it is included in the formulation of the stresses \( \sigma_i \). Inserting equations (2.2) into force equilibrium (2.1) and solving for \( \varepsilon_0 \) yields

\[
\varepsilon_0 = \frac{\left( -\bar{E}_1 h_1^2 + \bar{E}_2 h_2^2 - \bar{E}_d h_d^2 - 2 \bar{E}_d h_d h_1 \right) \kappa_u - 2 \sigma_R h_1}{2 \left( \bar{E}_1 h_1 + \bar{E}_2 h_2 + \bar{E}_d h_d \right)}.
\]  

(2.3)

The next step is to require moment equilibrium for the upstream section, from which a value for the curvature \( \kappa_u \) can be derived. The integral of elemental moments needs to equate to the external bending moment \( M \) as

\[
M = -\int \sigma y dy = -\int_{-h_1}^{h_2} \sigma_{ud} y dy - \int_{-h_1}^{0} \sigma_{u1} y dy - \int_{0}^{h_2} \sigma_{u2} y dy.
\]  

(2.4)

Bending moment \( M \) is said to be positive if the top layer is on the convex side, hence the negative signs in equation (2.4). Again, this integral can be split up into three parts, each part corresponding to one of the specimen layers. Using the expression for the initial strain \( \varepsilon_0 \) (2.3) in combination with the values for the stresses \( \sigma_i \) from (2.2), the integrals in the moment equilibrium (2.4) can be elaborated. Solving for curvature \( \kappa_u \) yields

\[
\kappa_u = \frac{12 M \left( \bar{E}_1 h_1 + \bar{E}_2 h_2 + \bar{E}_d h_d \right) - 6 \sigma_R h_1 \left( \bar{E}_2 h_2 - \bar{E}_d h_d \right) h_1 + \bar{E}_2 h_2^2 - \bar{E}_d h_d^2}{\gamma}
\]

(2.5)

with

\[
\gamma = \left( \bar{E}_1 h_1 + \bar{E}_2 h_2 + \bar{E}_d h_d + 2 \bar{E}_d h_d \right) h_1 + \bar{E}_2 h_2^2 - \bar{E}_d h_d^2.
\]
CHAPTER 2. ANALYSIS OF UPSTREAM DIRECTION

\[ \gamma = E_1^2 h_1^4 + 4E_1 h_1^3 (E_2 h_2 + E_d h_d) + h_2^2 ((6E_1 h_2^2 + 12E_d h_2 h_d) E_2 + 6E_1 E_d h_d^2) \\
+ h_1 (4h_2 (3h_d (h_2 + h_d) E_d + E_1 h_2^2) E_2 + 4E_1 E_d h_d^2) \\
+ E_2^2 h_2^4 + 4E_2 h_2 E_d h_d \left( h_2^2 + \frac{3}{2} h_2 h_d + h_d^2 \right) + E_d^2 h_d^4. \] (2.6)

It is clear from (2.5) the curvature \( \kappa_u \) is of similar format as for the film-substrate specimen. However, it is of larger proportions. In order to maintain legibility, in addition to \( \gamma \) some auxiliary parameters are introduced:

\[ A = E_1 h_1 + E_2 h_2 + E_d h_d \]
\[ B = (E_2 h_2 - E_d h_d) h_1 + E_2 h_2^2 - E_d h_d^2 \]
\[ C = - [E_d (2h_d h_1 + h_d^2) + E_1 h_1^2 - E_2 h_2^2] \]
\[ F = 3E_d h_1 h_d (h_1 + h_d) + E_1 h_1^2 + E_2 h_2^2 + E_d h_d^2 \] (2.7)

Using equations (2.6) and (2.7), the section curvature \( \kappa_u \) and initial strain \( \varepsilon_0 \) become

\[ \kappa_u = \frac{12MA - 6\sigma_R h_1 B}{\gamma}, \]
\[ \varepsilon_0 = \frac{6MC}{\gamma} - 3\sigma_R h_1 \left( \frac{CB}{\gamma A} + \frac{1}{3A} \right). \] (2.8)

With equations (2.8), the stress and strain profile for a three-layer system with residual film stress \( \sigma_R \) is fully determined. The location for the neutral line can be found from the condition that the strain is zero.

Appendix 1: Film-substrate systems with additional top layer and residual film stress
2.2 Strain energy

The upstream strain energy (per unit beam length) is obtained by integration of the strain energy density over the cross-section as:

\[ U_u = \int_A u_0 \, dA, \]

where \( u_0 = \int \sigma \, d\varepsilon. \) (2.9)

Recognizing the strain energy density is \( u_0 = \sigma^2 / 2E \) for linear elastic materials, the strain energy for the upstream section (per unit width and beam length) then is computed as

\[ U_u = \frac{1}{2} \bar{E} \int_{-h_1-h_d}^{-h_1} \sigma_{ud}^2 \, dy + \frac{1}{2} \bar{E}_1 \int_{-h_1}^{0} \sigma_{u1}^2 \, dy + \frac{1}{2} \bar{E}_2 \int_{0}^{h_2} \sigma_{u2}^2 \, dy \] (2.10)

In the elaboration of (2.10), equations (2.2) are used for the stresses. The upstream strain energy per unit length and width then is formulated in terms of \( \kappa_u \) and \( \varepsilon_0 \) as

\[ U_u = \frac{1}{2} \left( \left( h_1^2 + \frac{3}{12} h_1^2 h_d \right) \kappa_u^2 + 2 \left( h_1 + \frac{1}{2} h_d \right) \varepsilon_0 \kappa_u + \varepsilon_0^2 \right) \bar{E}_d h_d \]

\[ + \frac{1}{6} \left( h_1^2 \kappa_u^2 + 3h_1 \varepsilon_0 \kappa_u + 3 \varepsilon_0^2 \right) \bar{E}_1 + 3 \sigma_R h_1 \left( 2 \varepsilon_0 + h_1 \kappa_u \right) + 3 \frac{\sigma_R^2 h_1}{\bar{E}_1} \] (2.11)

where the three individual lines in (2.11) correspond to the three subsequent components of equation (2.10). Equation (2.11) is elaborated with the use of the expressions for \( \kappa_u \) and \( \varepsilon_0 \) as shown in (2.8). In elaboration the parameters are collected with respect to the loading \( M \) and \( \sigma_R \). The upstream strain energy then becomes

\[ U_u = U_{u,M} + U_{u,\sigma_R} + U_{u,M\sigma_R} \]

\[ = \frac{24FA^2 - 18C^2 A}{\gamma^2} M^2 \]

\[ + 6 \left( \frac{1}{12} \bar{E}_1 h_1^2 \gamma^2 + \frac{1}{2} h_1 A^2 B \gamma + \frac{1}{12} \bar{E}_2 h_2 + \frac{1}{12} \bar{E}_d h_d \right) \gamma^2 \]

\[ + \frac{-\frac{1}{2} ABC \gamma + AB^2 \left( FA - \frac{3}{2} C^2 \right) \bar{E}_1 h_1 + \frac{1}{12} A^2 \gamma^2 h_1 \sigma_R^2}{\bar{E}_1 A^2 \gamma^2} \]

\[ + 0. \] (2.12)

It is noted the interaction component \( U_{u,M\sigma_R} \) is always zero. The format of the upstream strain energy as presented in equation (2.12) is not elaborated in terms of the original system parameters yet, because in the determination of the ERR some components cancel out. It is after the combination of all components in Chapter 4 that the auxiliary variables (2.7) are eliminated from the results.
Chapter 3

Analysis of downstream direction

The downstream section consists, in contrary to the film-substrate downstream analysis, of two loaded beam portions: the first, denoted with subscript \( a \), is the substrate, which has to equilibrate the external bending moment \( M \); the second, denoted with the subscript \( b \), is the delaminated bilayer above the crack front. The delaminated beam above the substrate has to equilibrate the residual stress still active in the film layer. In Figure 3.1 this is illustrated by the two strain profiles.

Figure 3.1: Half of film-substrate system with additional top layer. In the downstream direction the ordinates are zero at center of the substrate for bottom part. The upper part, the delaminated bilayer, has its origin located at the film-top layer interface. Positive values of \( y \) for both cases are directed downward.
CHAPTER 3. ANALYSIS OF DOWNSTREAM DIRECTION

3.1 Equilibrium

3.1.1 Substrate part

First, the substrate beam is analyzed. Since the location of the neutral line is trivial, in the determination of the substrate curvature, $\kappa_{d,a}$, only moment equilibrium is required. The downstream substrate stresses $\sigma_{d2}$ are expressed in terms of $\kappa_{d,a}$ as

$$\sigma_{d2} = -\bar{E}_2 \kappa_{d,a} y$$  \hspace{1cm} (3.1)

The integral of elemental moments is computed as

$$M = -\int_{-h_z/2}^{h_z/2} \sigma_{d2} y dy$$  \hspace{1cm} (3.2)

Inserting (3.1) into (3.3) and solving for $\kappa_{d,a}$ yields

$$\kappa_{d,a} = \frac{12M}{\bar{E}_2 h_z^3}$$  \hspace{1cm} (3.3)

3.1.2 Delaminated bilayer part

After delamination of the two top layers (film and top layer) from the substrate, residual film stress $\sigma_R$ remains active. Since no external forces act on this part of the system, both the integral of elemental forces and moments needs to vanish. Similar to the earlier equilibrium analyses, the layer stresses are expressed in terms of section curvature $\kappa_{d,b}$ and initial strain $\varepsilon_{0,b}$. The stresses $\sigma_{d1}$ (film) and $\sigma_{dd}$ (superlayer) are written as

$$\sigma_{d1} = -\bar{E}_1 \kappa_{d,b} y + \bar{E}_1 \varepsilon_{0,b} + \sigma_R,$$
$$\sigma_{dd} = -\bar{E}_d \kappa_{d,b} y + \bar{E}_d \varepsilon_{0,b}.$$  \hspace{1cm} (3.4)

Force equilibrium is formulated as the integral of elemental forces for each layer:

$$F = \int \sigma dA$$
$$= \int_0^{h_1} \sigma_{d1} dy + \int_{-h_d}^0 \sigma_{dd} dy$$  \hspace{1cm} (3.5)

Inserting equations (3.4) into (3.5) and solving for the initial strain $\varepsilon_{0,b}$ yields

$$\varepsilon_{0,b} = \frac{(\bar{E}_1 h_1^2 + \bar{E}_d h_d^2) \kappa_{d,b} - 2h_1 \sigma_R}{2(\bar{E}_1 h_1 + \bar{E}_d h_d)}$$  \hspace{1cm} (3.6)

In order to fully determine the stress and strain profile, it is required the integral of elemental moments vanishes also. It is formulated in terms of the layer stresses as

$$M = -\int \sigma y dy$$
$$= -\int_0^{h_1} \sigma_{d1} y dy - \int_{-h_d}^0 \sigma_{dd} y dy$$  \hspace{1cm} (3.7)
$$= 0$$
CHAPTER 3. ANALYSIS OF DOWNSTREAM DIRECTION

Inserting stresses $\sigma_{d1}$ and $\sigma_{dd}$ from equation (3.4) and the expression for the initial strain $\varepsilon_{0,b}$ from (3.6) into (3.7) and solving for curvature $\kappa_{d,b}$ yields

$$
\kappa_{d,b} = \frac{6\sigma_R \tilde{E}_d h_1 h_d (h_1 + h_d)}{\beta} \tag{3.8}
$$

with

$$
\beta = \tilde{E}_1^2 h_1^4 + 4\tilde{E}_1 \tilde{E}_d h_1 h_d \left(\frac{3}{2} h_1 h_d + h_1^2 + h_d^2\right) + \tilde{E}_d^2 h_d^4 \tag{3.9}
$$

Using equation (3.8), the strain $\varepsilon_{0,b}$ from equation (3.6) can be rewritten as

$$
\varepsilon_{0,b} = -\frac{\sigma_R h_1 \left(\tilde{E}_1 h_1^3 + 3\tilde{E}_d h_1 h_d^2 + 4\tilde{E}_d h_d^3\right)}{\beta} \tag{3.10}
$$
CHAPTER 3. ANALYSIS OF DOWNSTREAM DIRECTION

3.2 Strain energy

3.2.1 Substrate part

The downstream strain energy (per unit beam length) of the substrate beam part is obtained by integration of the strain energy density over the cross-section as

\[
U_{d,a} = \frac{1}{2\bar{E}_2} \int_{-h_z/2}^{h_z/2} \sigma_{d2}^2 dy
\]

(3.11)

Inserting the downstream substrate stresses from (3.1), equation (3.11) is found to be

\[
U_{d,a} = \frac{6M^2}{\bar{E}_2h_z^3}
\]

(3.12)

3.2.2 Delaminated bilayer part

The strain energy per unit length and width, \(U_{d,b}\), produced by the delaminated bilayer is computed from the integral of the strain energy density over the section bilayer height. It is formulated as

\[
U_{d,b} = \frac{1}{2\bar{E}_1} \int_{0}^{h_1} \sigma_{d1}^2 dy + \frac{1}{2\bar{E}_d} \int_{-h_d}^{0} \sigma_{dd}^2 dy
\]

(3.13)

Using expressions (3.4) for the stresses, the section strain energy \(U_{d,b}\) is expressed in terms of curvature \(\kappa_{d,b}\) and initial strain \(\varepsilon_{0,b}\) as

\[
U_{d,b} = \frac{1}{6} \left( h_1^2 \kappa_{d,b}^2 - 3h_1\varepsilon_{0,b}\kappa_{d,b} + 3\varepsilon_{0,b}^2 \right) h_1\bar{E}_1 - 3\sigma_R h_1 \left( -2\varepsilon_{0,b} + h_1\kappa_{d,b} \right) + \frac{3\sigma_R^2}{\bar{E}_1} 
\]

\[
+ \frac{1}{6} \left( h_1^2 \kappa_{d,b}^2 + 3h_d\varepsilon_{0,b}\kappa_{d,b} + 3\varepsilon_{0,b}^2 \right) \bar{E}_dh_d,
\]

(3.14)

where individual lines correspond two the two components form equation (3.13), respectively. The section strain energy is further elaborated using (3.8) and (3.10) to

\[
U_{d,b} = \frac{\sigma_R^2 \bar{E}_d h_1 h_d \left( \bar{E}_1 h_1^3 + \bar{E}_d h_d^3 \right)}{2\beta \bar{E}_1}
\]

(3.15)

3.2.3 Total downstream section

The total section strain energy for the downstream section is computed from the sum of part \(a\) and \(b\) as

\[
U_d = U_{d,a} + U_{d,b}
\]

\[
= \frac{6M^2}{\bar{E}_2h_z^3} + \frac{\sigma_R^2 \bar{E}_d h_1 h_d \left( \bar{E}_1 h_1^3 + \bar{E}_d h_d^3 \right)}{2\beta \bar{E}_1}
\]

(3.16)

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Chapter 4

Energy release rate

The steady state energy release rate (ERR), \( G_{ss} \), can be deduced analytically by consideration of the energy balance. The ERR is given by

\[
G = \frac{d\Delta F}{da} - \frac{d\Delta U}{da},
\]

(4.1)

where \( \Delta F \) denotes the produced work by external moment \( M \) during a unit delamination \( da \), and \( \Delta U \) is the change in strain energy during this delamination. The change in strain energy is given by

\[
\Delta U = (U_d - U_u) a,
\]

(4.2)

where the downstream and upstream direction refer to the cracked and uncracked section respectively. Both the downstream and upstream section strain energies were derived in Chapter 3 and Chapter 2, respectively. The change strain energy due to delamination \( a \) then, from equation (3.16) and (2.12), is

\[
\frac{\Delta U}{a} = \frac{6M^2}{E_2h_2} \gamma^2 \epsilon_1 h_1 h_2 (\epsilon_1 h_3^3 + \epsilon_2 h_3^3) + \frac{\sigma_R^2}{2\beta E_1} \frac{24FA^2 - 18C^2A}{M^2} \gamma^2
\]

(4.3)

The work done by the bending moment during the delamination process can be calculated from the change in curvature in the downstream and upstream direction. It is defined as

\[
\Delta F = M \phi = (\kappa_{d,a} - \kappa_u) Ma.
\]

(4.4)

Using the expressions for the curvatures \( \kappa_{d,a} \), from (3.3), and \( \kappa_u \), from (2.8) (1), the conjugated work becomes

\[
\Delta F = \left( \frac{12M}{E_2 h_2^3} - \frac{12MA - 6\sigma_R h_1 B}{\gamma} \right) Ma
\]

(4.5)

Equations (4.3) and (4.5) are inserted in (4.1) to find an expression for the ERR. After combining, the auxiliary parameters \( A, B, C, F \) from equation (2.7) are replaced with their original components, and the result is laboriously simplified to the final expression for the ERR:
CHAPTER 4. ENERGY RELEASE RATE

\[ G_{ss} = 6M^2 \left( \frac{1}{E_2h_2^3} - \frac{(E_1h_1 + E_2h_2 + E_dh_d)}{\gamma} \right) \]

\[ + \frac{\sigma_R^2 h_1}{2E_1} \left( \frac{-3E_1h_1 (E_2h_2 - E_dh_d) + E_2h_2^3 - E_dh_d^3)^2 + \gamma (E_2h_2 + E_dh_d)}{(E_1h_1 + E_2h_2 + E_dh_d) \gamma} - \frac{E_dh_d (E_1h_1^3 + E_dh_d)}{\beta} \right) \]

\[ + 6\sigma_R M h_1 \left( \frac{h_1 (E_2h_2 - E_dh_d) + E_2h_2^3 - E_dh_d^3}{\gamma} \right), \]

with

\[ \beta = E_1^2 h_1^4 + 4E_1 h_1^3 E_dh_d + 6E_1 h_1^2 E_dh_d^2 + 4E_1 h_1 E_dh_d^3 + E_d^2 h_d^4 \]

\[ \gamma = E_1^2 h_1^4 + 4E_1 h_1^3 (E_2h_2 + E_dh_d) + h_1^3 \left( (6E_1 h_2^2 + 12E_dh_2h_d) E_2 + 6E_1 E_dh_d^2 \right) \]

\[ + h_1 (4h_2 (3h_d (h_2 + h_d) E_d + E_1 h_2^4) E_2 + 4E_1 E_d h_d^4) + E_2^2 h_2^4 + 4E_2 h_2 E_dh_d \left( h_2^2 + \frac{3}{2} h_2 h_d + h_d^2 \right) + E_d^2 h_d^4. \]

(4.6)

(4.7)

Appendix 1: Film-substrate systems with additional top layer and residual film stress
Appendix 2:  
Film-substrate systems with residual stress and elasto-plastic behaviour under bending

Elasto-plastic bending and energy release rate

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Eindhoven, March 2015
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Appendix 2: Film-substrate systems with residual stress and elasto-plastic behaviour under bending
Chapter 1

Introduction

In this appendix a formulation for the Energy Release Rate (ERR) is derived for film-substrate systems, built up from a residually stressed elastic layer and an elasto-plastic layer, in a four-point bend (4PB) test setup.

In Chapter 2, the concepts of J2-plasticity, or Von Mises plasticity, are presented in the environment of the 4PB-test. A distinction is made between bilayer specimens in state of plane stress and plane strain, and its implications with respect to the following analysis is emphasized. Then, the Ramberg-Osgood relation is adopted as the hardening law for the elasto-plastic layer and projected onto the effective stress and strain as defined in the yield criterion. For both plane stress and plane strain conditions this result is elaborated to find a relation between the stress and strain in the longitudinal direction, which is the direction parallel to the neutral line. This relation is to be used in the equilibrium conditions to find the neutral line, and consequently the stress and strain profile at the cracked and uncracked section due to the external moment loading and the internal residual stress loading.

Next, in Chapter 3, the analysis of inelastic beams (of single layer), representative for plane stress conditions, is considered. A general procedure for the determination of the neutral line (hence, the stress and strain profile) is presented for beams with arbitrary material characteristics. Using this procedure, the stress-strain relation derived in Chapter 2 is used to formulate expressions for the stress and strain profile. Then, the strain energy of the elasto-plastic beam section is derived. The results found in this chapter are representative for the downstream (cracked) section in delamination analysis of Chapter 6, and will be the starting point for the analysis of the bilayer beam with elasto-plastic layer in Chapter 4.

In Chapter 4, the inelastic analysis of single-layered beams is extended for the case of an additional (elastic) layer. By requirement of compatible strains, the relations for the stress and strain profiles are extended from the elasto-plastic beam presented in Chapter 3. In the compatibility relations, an additional extension regarding residual stresses is included.

Chapter 5 is considered with the analysis of bilayer systems in state of plane strain. A numerical procedure is presented for the use of the stress-strain relation established in Chapter 2 in the determination of the stress and strain profile. Then, the determination of strain energies in a numerical fashion is discussed.

Having obtained stress and strain profiles, and strain energies for both bilayer systems in state of plane stress and plane strain, an energy balance is set up in Chapter 6. From this balance an ERR is deduced which represents a limit case useful for the determination of the delamination toughness for bilayer systems with an elasto-plastic layer.
Chapter 2

Plasticity in 4PB-test

In the analysis of delamination behaviour of bilayer systems the stress-strain relation of the elasto-plastic material for the longitudinal direction are required. In this chapter the determination of the $\sigma_x - \varepsilon_x$-curve is elaborated for materials yielding according to the Von Mises criterion and hardening according to the Ramberg-Osgood hardening law. This is done for both the case of plane stress and plane strain.

2.1 Von Mises plasticity

J2-plasticity is characterized by its invariance with respect to pressure loading ($\sigma_p$). Material yield is said to be dependent only on deviatoric stress components. The yield criterion is defined using an equivalent stress, the Von Mises stress, which equals the tensile stress in case of uniaxial tension. The criterion is expressed as

$$
\sigma_m = \frac{1}{\sqrt{2}} \sqrt{2 \left( (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\tau_{xy}^2 + 6\tau_{yz}^2 + 6\tau_{xz}^2 \right)}.
$$

or, in terms of principle stresses,

$$
\sigma_m = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}.
$$

The corresponding equivalent strain $\varepsilon_m$ is defined, in terms of principle strains, as

$$
\varepsilon_m = \frac{1}{\sqrt{2(1 + v)}} \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}.
$$

It is noted the Poisson’s ratio $v$ appearing in (2.3) is not constant throughout the loading process. Only in the elastic regime it is equal to $v_e$, where the subscript $e$ denotes elastic. In case of full plasticity it equals $v_p = 0.5$. Between the two stages it varies between $v_e < v < 0.5$. 

Appendix 2: Film-substrate systems with residual stress and elasto-plastic behaviour under bending
2.2 Stress-strain relations

As seen from equations (2.2) and (2.3), the equivalent stress $\sigma_m$ and strain $\varepsilon_m$ are formulated in terms of multi-axial stress and strain components. In order to set up a relation between $\sigma_x$ and $\varepsilon_x$ (to be used in the next chapters), relations between the stress and strain components are required. The multi-axial stress and strain components are related through adapted general Hookean stress-strain relations. From the deformation theory of plasticity [1], the stresses and strains for nonlinear (yet elastic) materials are related through

$$\varepsilon_x = \frac{1}{E_{sec}} [\sigma_x - v(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E_{sec}} [\sigma_y - v(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E_{sec}} [\sigma_z - v(\sigma_x + \sigma_y)]$$

with

$$E_{sec} = \frac{\sigma_m}{\varepsilon_m}$$

$$G_{sec} = \frac{E_{sec}}{2(1 + v)}$$

in which $E_{sec}$ and $v$ are not constant, but variable. $E_{sec}$ is the secant modulus of the (non-linear) stress-strain relation, whereas the Poisson’s ratio $v$ varies between the elastic ratio $v_e$ and the plastic ratio $v_p = 1/2$, which represents the incompressible state in case of full plasticity. The relation for Poisson’s ratio is computed from the requirement that the total dilation (volumetric strain) with the equivalent $v$ equals the sum of the elastic dilation and the plastic dilation components. It is formulated as

$$v = \frac{1}{2} - \frac{1}{2} \cdot \frac{E_{sec}}{E}$$

(2.5)

It can be seen (2.5) is a function of $E_{sec}$, but $E_{sec}$ itself is a function of the stress level ($E_{sec} = \sigma_m/\varepsilon_m$), thus the equivalent Poisson’s ratio cannot be determined directly. Analogously to the formulations of the strains in terms of stresses, the stress components can be interrelated as

$$\sigma_x = \frac{E_{sec}}{(1 + v)(1 - 2v)} [(1 - v)\varepsilon_x + v(\varepsilon_y + \varepsilon_z)]$$

$$\sigma_y = \frac{E_{sec}}{(1 + v)(1 - 2v)} [(1 - v)\varepsilon_y + v(\varepsilon_z + \varepsilon_x)]$$

$$\sigma_z = \frac{E_{sec}}{(1 + v)(1 - 2v)} [(1 - v)\varepsilon_z + v(\varepsilon_x + \varepsilon_y)]$$

$$\tau_{yz} = \frac{E_{sec}}{2(1 + v)} \gamma_{yz}$$

$$\tau_{zx} = \frac{E_{sec}}{2(1 + v)} \gamma_{zx}$$

$$\tau_{xy} = \frac{E_{sec}}{2(1 + v)} \gamma_{xy}.$$

(2.6)

Relations (2.4) and (2.6) are used to determine the stress and strain components in the longitudinal direction for the analysis of elasto-plastic bending Chapters 3 to 5.
2.3 Von Mises plasticity in 4PB-test

In a four-point bend test environment, between the supports the cross-sections are subject to pure bending. Hence, no shear stresses are present. Furthermore, the vertical stress component $\sigma_y$ is absent in such loading conditions. Dependent on the specimen width $b$, two limit cases can be distinguished: plane stress and plane strain. The former represents the case in which the specimen width $b$ is small in comparison to the other geometric bilayer parameters (Figure 2.1, top figure). This situation is representative for beams. The latter limit case represents specimens with widths going to infinity (Figure 2.1, bottom figure), which is regularly assumed to be useful for specimens with widths several times larger than their thickness (e.g. plates).

Figure 2.1: Plane stress (top) and plane strain (bottom) conditions in a 4PB-test.

2.3.1 Plane stress conditions

When the beam width $b$ is very small, the stress $\sigma_z$ in the thickness direction vanishes. With the absence of vertical stress $\sigma_y$ and shear stresses $\sigma_{xy}, \sigma_{yz}, \sigma_{xz}$, the equivalent Von Mises stress $\sigma_m$, using equation (2.1), reduces to

$$\sigma_m = \sigma_x. \quad (2.7)$$

The equivalent strain $\varepsilon_m$ is found from the equivalent stress $\sigma_m$ as

$$\varepsilon_m = \varepsilon_x = \frac{\sigma_x}{E_{sec}}, \quad (2.8)$$

which is found from the first equation of (2.4). It is concluded from (2.7) the equivalent stress $\sigma_m$ equals the longitudinal stress $\sigma_x$. From (2.8) it follows that the equivalent strain $\varepsilon_m$ is computed directly from the hardening law and is independent of the Poisson’s ratio. Essentially, in plane stress the loading condition reduces to uni-axial loading. Hence, the (equivalent) stress-strain relation (to be defined in the next sections) can be used directly to compute the longitudinal strains $\varepsilon_x$. It is this fact that makes the plane stress case particularly applicable to analytical analysis in the coming chapters.
2.3.2 Plane strain conditions

In the case of plane strain four-point bending, the region of interest, between the inner supports, solely consists of the nonzero components $\sigma_x, \sigma_z, \varepsilon_x$ and $\varepsilon_y$. The equations (2.4) and (2.6) then reduce to (with $\sigma_x$ as independent variable)

\begin{align*}
\sigma_x &= \sigma_x \\
\sigma_z &= v \sigma_x \\
\varepsilon_x &= \frac{1-v^2}{E_{se}} \sigma_x \\
\varepsilon_y &= -\frac{1+v^2}{E_{se}} \sigma_x
\end{align*} \tag{2.9}

These four components are inserted in equations (2.2) and (2.3) to find the equivalent stress and strain for plane strain four point bending. They read

\begin{align*}
\sigma_m &= \sqrt{v^2 - v + 1} \sigma_x \\
\varepsilon_m &= \frac{1}{1+v} \sqrt{\varepsilon_x^2 + \varepsilon_x \frac{v(v+1)}{E_{se}} \sigma_x + \frac{2v^2(v+1)^2}{E_{se}^2} \sigma_x^2} \\
\varepsilon_m &= \frac{1}{1+v} \sqrt{\varepsilon_x^2 + \varepsilon_x \sigma_x + \frac{2v^2(v+1)^2}{E_{se}^2} \sigma_x^2} \tag{2.10}
\end{align*}

It is seen from (2.10) the to be defined hardening law, which is defined as the $\sigma_m \varepsilon_m$-relation, does not coincide with a $\sigma_x \varepsilon_x$-relation as it did in the plane stress case. Additionally, it is not possible to write down such a relation in a compact manner. Therefore, a $\sigma_x \varepsilon_x$-relation is set up numerically in the following chapters.
2.4 Ramberg-Osgood hardening law

The hardening law adopted in this chapter is the Ramberg-Osgood constitutive relation. The Ramberg-Osgood equation for uni-axial tension is defined as:

$$\varepsilon = \frac{\sigma}{E} + \alpha \frac{\sigma_0}{E} \left( \frac{\sigma}{\sigma_0} \right)^n. \quad (2.11)$$

This equation, however, is not symmetrical for negative values of stress $\sigma$ in the case power $n$ is even. Therefore the equation is altered into the form:

$$\varepsilon = \frac{\sigma}{E} + \frac{\alpha}{E} \left( \frac{|\sigma|}{\sigma_0} \right)^{n-1} \sigma. \quad (2.12)$$

In this format, the strain $\varepsilon$ is always positive with positive stress, and always negative with negative stress, invariant of the value of power $n$.

For the purpose of clear analysis, a new parameter $\beta$ is introduced as

$$\beta = \frac{\alpha}{E} \frac{\sigma_0^{n-1}}{\sigma_0}. \quad (2.13)$$

Using (2.13), (2.12) is turned into

$$\varepsilon = \frac{\sigma}{E} + \beta |\sigma|^{n-1} \sigma. \quad (2.14)$$

Using the deformation theory of plasticity for the multi-axial loading case, the uni-axial tensile stress-strain relation is defined as the equivalent or universal stress-strain relation as shown below (Figure 2.2). This assumption is based on the fact the equivalent stress and strain in a multi-axial stress state reduce to the uni-axial components in the case of uni-axial loading.

![Uni-axial and Universal Stress-Strain Curves](image)

Figure 2.2: Equivalence between the uni-axial stress-strain curve and the universal stress-strain curve.

The universal stress-strain curve describes the relation between the equivalent stress $\sigma_m$ and the equivalent strain $\varepsilon_m$. These are, in the case of Von Mises plasticity, the Von Mises stress and strain. Thus, the relation between $\sigma_m$ and $\varepsilon_m$ is defined as

$$\varepsilon_m = \frac{\sigma_m}{E} + \beta |\sigma_m|^{n-1} \sigma_m. \quad (2.15)$$

From the equivalent values $\sigma_m$ and $\varepsilon_m$, the orthogonal stress and strain components may be derived, as will be shown in the next subsections.
2.4.1 Plane stress

In the case of plane stress, the equivalent stress $\sigma_m$ and equivalent strain $\varepsilon_m$ were shown to equal $\sigma_x$ and $\varepsilon_x$ (section 2.3.1), respectively. Hence, the $\sigma_x \varepsilon_x$-curve can be defined directly as

$$\varepsilon_x = \frac{\sigma_x}{E} + \beta |\sigma_x|^{n-1} \sigma_x$$

(2.16)

Equation (2.16) is used in the analytical analysis of bilayer systems with an elasto-plastic layer in the coming chapters.

2.4.2 Plane strain

From (2.10) it can be concluded no explicit relation between the longitudinal stress $\sigma_x$ and strain $\varepsilon_x$ for plane strain cases is possible. To set up this relation, a numerical approximation is needed.

For an assumed interval $a < \sigma_m < b$ the equivalent strains $\varepsilon_m$ are computed from (2.15). Using these stresses and strains, the secant modulus $E_{sec}$ is

$$E_{sec} = \frac{\sigma_m}{\varepsilon_m}$$

(2.17)

The equivalent Poisson’s ratio then is found as

$$v = \frac{1}{2} \left[ \frac{1}{2} - v' \right] - \frac{E_{sec}}{E}. $$

(2.18)

Now, the normal stresses in the longitudinal direction $\sigma_x$ can be found from (2.10) (1) as

$$\sigma_x = \frac{\sigma_m}{\sqrt{v^2 - v + 1}}$$

(2.19)

The accompanied strains $\varepsilon_x$ are solved from (2.10) (2). The correct solution is found from the solution where the root part is positive. Rewriting (2.10) (2) and solving for $\varepsilon_x$ gives

$$a \varepsilon_x^2 + b \varepsilon_x + c = 0$$

$$\varepsilon_x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

with

$$a = 1$$

$$b = v(v + 1) \sigma_x$$

$$c = \frac{v^2(v + 1)^2}{E_{sec}^2} \sigma_x^2 - (v + 1) \varepsilon_m^2.$$

(2.20)

Now the $\sigma_x \varepsilon_x$-curve can be set up for a given range of equivalent stresses $\sigma_m$.

Adapted plane stress approximation

As seen from the above, the relation between $\sigma_x$ and $\varepsilon_x$ for plane strain conditions can practically only be determined in a numerical fashion due to the variable nature of $v$. However, a first approximation for the plane strain can be made by making use of the plane stress relation (2.16) and the assumption of a constant Poisson’s ratio $v$. Through manipulation of both $\beta$ and $E$ in
equation (2.16), a reasonable approximation is possible. From the first equation in (2.10), the equivalent stress $\sigma_m$ is estimated to be

$$\sigma_x = \frac{\sigma_m}{\sqrt{v_e^2 - v_e + 1}}. \quad (2.21)$$

Additionally, the elastic modulus $E$ is replaced with the plane strain modulus $\tilde{E} = E/(1 - v_e^2)$, as is done in elastic plane strain analysis. Elaboration then gives an approximate relation between $\sigma_x$ and $\varepsilon_x$:

$$\varepsilon_x = \frac{\sigma_x}{\tilde{E}} + \tilde{\beta} |\sigma_x|^{n-1} \sigma_x$$

with

$$\tilde{\beta} = \beta (v_e^2 - v_e + 1)^{\frac{\sigma}{2}} \quad (2.22)$$

Equation (2.22) can be used in the analytical result for the ERR in Chapter 6.
Chapter 3

Analysis of elasto-plastic beam

In the following derivations the plane stress assumption is obeyed. Since in this framework sub-
scripts regarding stress or strain components are superfluous, these will be omitted.

3.1 General procedure

As a preliminary from statics, both force equilibrium and moment equilibrium need to be satisfied
in the section plane everywhere along the neutral line, regardless of material characteristics. These
two requirements from statics are used to establish stress and strain profiles of a prismatic beam
with elasto-plastic material behaviour subjected to bending. The procedure shown next is ana-
logous to that of Timoshenko [2]. In this section no specific constitutive law is specified, though
the solving procedure is presented. This is done assuming a rectangular cross-section, although
the procedure is valid for arbitrary section shape.

\[
\int \sigma \, dA = \int_{-h_2}^{h_1} \sigma dy = 0,
\]

where the width \( b \) of the beam is constant along the layer height (rectangular section) and
subscripts 1 and 2 denote the top and the bottom of the beam, respectively. The positive \( y \)-
direction is said to be towards the top of the beam, as shown in Figure 3.1. Assuming plane
sections remain plane and perpendicular to the neutral line after deformation, the strains are
linearly related to the height of the cross-section as

Figure 3.1: Rectangular beam section subjected to pure bending (left) and the resulting stress
profile for elasto-plastic bending (right).

In the case axial forces being absent, no net force acts on the section, and the integral of
elemental forces has to equal zero:

\[
\int \sigma \, dA = \int_{-h_2}^{h_1} \sigma dy = 0,
\]

Appendix 2: Film-substrate systems with residual stress and elasto-plastic behaviour under
bending
Here, $\rho$ is the radius of curvature of the neutral line. The curvature direction is said to be positive if the convex side is the top face of the beam. Force equilibrium (3.1) is transformed, using the last line of (3.2), into

$$\int_{-h_2}^{h_1} \sigma dy = \rho b \int_{\varepsilon_2}^{\varepsilon_1} \varepsilon d\varepsilon = 0.$$  \hspace{1cm} (3.3)

Since both the width and the curvature are nonzero in a nontrivial solution, for force equilibrium it is required that the integral of the $\sigma\varepsilon$-diagram vanishes, or geometrically, that the area beneath the stress-strain curve equals zero (Figure 3.2):

$$\int_{\varepsilon_2}^{\varepsilon_1} \sigma d\varepsilon = 0.$$  \hspace{1cm} (3.4)

Figure 3.2: Integral of $\sigma\varepsilon$-diagram needs to vanish for force equilibrium.

Now a strain difference $\varepsilon_t$, defined as the difference between the outer strains $\varepsilon_1$ and $\varepsilon_2$, is introduced:

$$\varepsilon_t = \varepsilon_1 - \varepsilon_2 = \kappa h_1 - \kappa(-h_2) = \kappa h = \frac{h}{\rho}.$$  \hspace{1cm} (3.5)

The neutral line can be found by solving the integral with an assumed value for the strain difference $\varepsilon_t$. Essentially, the strain difference is shifted along the horizontal axis of the $\sigma\varepsilon$-curve (Figure 3.2) until the compression part equals the tension part. Alternatively, the location of the neutral line can be found by translation of a defined stress difference $\sigma_t = \sigma_1 - \sigma_2$. It is noted the former is more stable than the latter in a numerical environment, since with the drop in current Elastic modulus $E_{\text{tan}} = d\sigma/d\varepsilon$, the ability of the cross-section to equilibrate increasing $\sigma$, deteriorates greatly. The neutral axis can be computed as

$$\frac{h_1}{h_2} = \frac{\varepsilon_1}{\varepsilon_2} = \left| \frac{\varepsilon_1}{\varepsilon_2} \right|.$$  \hspace{1cm} (3.6)
The second requirement is moment equilibrium. In case an external moment $M$ acts on the beam, the integral of elemental moments of the cross-section needs to equal the external moment $M$. Analogous to equation (3.3), the integral component is rewritten in terms of stress and strain as

$$\int \sigma y dA = \int_{-h_2}^{h_1} \sigma y_0 dy = \rho^2 b \int_{\varepsilon_2}^{\varepsilon_1} \sigma \varepsilon d\varepsilon = M. \quad (3.7)$$

This integral can also be written down, using (3.5), as

$$M = \frac{bh^2}{\varepsilon_1} \int_{\varepsilon_2}^{\varepsilon_1} \sigma \varepsilon d\varepsilon. \quad (3.8)$$

Solving equation (3.4) and equation (3.8) together yields the stress and strain profiles for the observed cross-section, loaded with external bending moment $M$. The next step is to adopt a constitutive law, from which the integrals can be elaborated and the stress and strain profiles can be determined explicitly.

Appendix 2: Film-substrate systems with residual stress and elasto-plastic behaviour under bending
3.2 Neutral line and stress distribution

\[ \int_{\varepsilon_2}^{\varepsilon_1} \sigma \, d\varepsilon \quad (3.9) \]

Since the Ramberg-Osgood constitutive law is non-invertible, the integral (3.9) cannot be integrated directly. Using a substitution, it can be reworked into a workable integral. For this substitution, the derivative of (2.14) is needed. This equation contains absolute value signs. The derivative of an absolute value is:

\[ |x| = \sqrt{x^2} \]
\[ \frac{d|x|}{dx} = \frac{d\sqrt{x^2}}{dx} = \frac{1}{2\sqrt{x^2}} \cdot 2x = \frac{x}{|x|} \quad (3.10) \]

The derivative of (2.14), using (3.10) and the chain rule, then is

\[ \frac{d\varepsilon}{d\sigma} = \frac{1}{E} + \beta |\sigma|^{n-1} + (n-1) \beta \frac{|\sigma|}{\sigma} |\sigma|^{n-2} \sigma \]
\[ = \frac{1}{E} + \beta |\sigma|^{n-1} + (n-1) \beta |\sigma|^{n-1} \]
\[ = \frac{1}{E} + n \beta |\sigma|^{n-1} \quad (3.11) \]

Solving for \( d\varepsilon \) and inserting in (3.9) gives

\[ \int_{\sigma_2}^{\sigma_1} \left( \frac{1}{E} + n \beta |\sigma|^{n-1} \right) \, d\sigma \]
\[ = \int_{\sigma_2}^{\sigma_1} \left( \frac{\sigma}{E} + n \beta |\sigma|^{n-1} \right) \, d\sigma \quad (3.12) \]

Note that the integration bounds have to be changed according to (2.14) in the substitution process. In the further elaboration it will be assumed that the lower and upper bound have negative and positive values, respectively. This assumption makes elaboration of (3.12) much easier, but requires careful inspection of the signs of the stresses/strains at the outer fibers of the elasto-plastic material. The integral of equation (3.12) can be split up into two parts:

\[ \int_{\sigma_2}^{\sigma_1} \left( \frac{\sigma}{E} + n \beta |\sigma|^{n-1} \right) \, d\sigma = \int_{0}^{\sigma_1} \left( \frac{\sigma}{E} + n \beta |\sigma|^{n-1} \right) \, d\sigma + \int_{0}^{\sigma_2} \left( \frac{\sigma}{E} + n \beta |\sigma|^{n-1} \right) \, d\sigma \quad (3.13) \]

In the second part of the after the equality, the integral for positive values of stress, the absolute value signs can be dropped and the integration can be carried out as

\[ \int_{0}^{\sigma_1} \left( \frac{\sigma}{E} + n \beta |\sigma|^{n-1} \right) \, d\sigma = \int_{0}^{\sigma_1} \left( \frac{\sigma}{E} + n \beta \sigma^{n} \right) \, d\sigma \]
\[ = \left[ \frac{\sigma^2}{2E} + \frac{n}{n+1} \beta \sigma^{n+1} \right]_{0}^{\sigma_1} \]
\[ = \frac{\sigma_1^2}{2E} + \frac{n}{n+1} \beta \sigma_1^{n+1}. \quad (3.14) \]

Since power \( n \) is always positive, the lower bound terms drop from the result. The other part of the integral, the negative part, can be solved using the positive part. Since the \( \sigma\varepsilon \)-curve is
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symmetric in the origin, an integral from a negative value \( \sigma_2 \) to the origin can be computed as an integral from the origin to a positive value \(-\sigma_2\). Thus,

\[
\int_{\sigma_2}^{0} \left( \frac{\sigma}{E} + n \beta |\sigma|^{n-1} \right) d\sigma = \int_{0}^{-\sigma_2} \left( \frac{\sigma}{E} + n \beta \sigma^n \right) d\sigma \\
= \left[ \frac{\sigma^2}{2E} + \frac{n}{n+1} \beta \sigma^{n+1} \right]_0^{-\sigma_2} \\
= \frac{\sigma_2^2}{2E} + \frac{n}{n+1} \beta (-\sigma_2)^{n+1} .
\]

(3.15)

The total integral, using (3.14) and (3.15), then becomes

\[
\int_{\varepsilon_1}^{\varepsilon_2} \sigma d\varepsilon = \frac{\sigma_1^2 - \sigma_2^2}{2E} + \frac{n}{n+1} \beta \left[ \sigma_1^{n+1} - (-\sigma_2)^{n+1} \right]
\]

(3.16)

### 3.2.1 Determination of neutral line for given stress range

The determination of the position of the neutral in case of pure bending in a monolayer is trivial. However, in the extension of this monolayer system to a bilayer system the procedure presented here will be used. A loading parameter \( \sigma_t \) is introduced (see Figure 3.2), which represents the difference in stresses in the outer fibers of the beam. Using \( \sigma_t \), the negative stress \( \sigma_2 \) can be written as

\[
\sigma_t = \sigma_1 - \sigma_2 \\
\sigma_2 = \sigma_1 - \sigma_t \\
\sigma_2 = \sigma_t - 2 \sigma_1 \sigma_t + \sigma_t^2
\]

(3.17)

The second and third line from (3.17) are inserted in (3.16) and give

\[
\int_{\varepsilon_2}^{\varepsilon_1} \sigma d\varepsilon = \frac{2 \sigma_1 \sigma_t - \sigma_t^2}{2E} + \frac{n}{n+1} \beta \left[ \sigma_1^{n+1} - (-\sigma_1 + \sigma_t)^{n+1} \right]
\]

(3.18)

The neutral line can be found from the condition that (3.16) equals 0 (see (3.4)). Assuming a value for the stress difference \( \sigma_t \) leaves one unknown, namely \( \sigma_1 \). Unfortunately, no exact value for \( \sigma_1 \) can be derived. A numerical solution, using a Newton-Raphson scheme, is proposed:

\[
\sigma_{1,i+1} = \sigma_{1,i} - \frac{f(\sigma_{1,i})}{f'(\sigma_{1,i})}
\]

with

\[
f(\sigma_{1,i}) = \frac{2 \sigma_1 \sigma_t - \sigma_t^2}{2E} + \frac{n}{n+1} \beta \left[ \sigma_1^{n+1} - (-\sigma_1 + \sigma_t)^{n+1} \right],
\]

\[
f'(\sigma_{1,i}) = \frac{\sigma_t}{E} + n \beta [\sigma_1^n + (-\sigma_1 + \sigma_t)^n].
\]

(3.19)

From a successive use of (3.19) a converged value of \( \sigma_1 \) is obtained. Automatically, \( \sigma_2 \) is known from (3.17). From these stresses the corresponding strains \( \varepsilon_1 \) and \( \varepsilon_2 \) are computed from (2.14). Since plane sections are assumed to remain plane and perpendicular to the neutral line after deformation, the ratio of strains equals the ratio of distances from the neutral line, or

\[
\frac{h_1}{h_2} = \left| \frac{\varepsilon_1}{\varepsilon_2} \right|.
\]

(3.20)

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The distance from the neutral line to the outer fiber with compressive stresses is

\[
\frac{h - h_2}{h_2} = \frac{\varepsilon_1}{\varepsilon_2},
\]

\[
h_2 = \frac{h}{1 + \frac{\varepsilon_1}{\varepsilon_2}}.
\]

(3.21)

3.2.2 Bending moment M from stress distribution

In the previous section a stress interval \(\sigma_t\) was assumed from which the neutral line was determined. Assuming the same \(\sigma_t\), in this section the resulting bending moment from the stress profile is derived. Using equation (3.8), the bending moment \(M\) per unit width can be computed as

\[
M = \frac{h^2}{\varepsilon_t} \int_{\xi_2}^{\xi_1} \sigma \varepsilon \, d\xi.
\]

(3.22)

The monolayer height is \(h\), the difference in strains between the outer fibers is \(\varepsilon_t\) and the integral represents a scaled version of the bending moment \(M\). The strain difference can be worked out into terms of stress as

\[
\varepsilon_t = \varepsilon_1 - \varepsilon_2
\]

\[
= 2\varepsilon_1
\]

\[
= 2\left(\frac{\sigma_1}{E} + \beta \sigma_n\right).
\]

(3.23)

Since the monolayer has a symmetric stress and strain profile, the outer strains are equal in absolute value. The integral in equation (3.22) is split up in two parts, again for simplicity in the solving process. It is split as

\[
\int_{\xi_2}^{\xi_1} \sigma \varepsilon \, d\xi = \int_{\xi_2}^{\xi_0} \sigma \varepsilon \, d\xi + \int_{\xi_0}^{\xi_1} \sigma \varepsilon \, d\xi
\]

(3.24)

The latter part of equation (3.24) is now elaborated. The product \(\sigma \varepsilon\) inside the integral in (3.22), for positive values of \(\sigma\), is

\[
\sigma \varepsilon = \sigma \left(\frac{\sigma}{E} + \beta \sigma^n\right) = \frac{\sigma^2}{E} + \beta \sigma^{n+1}.
\]

(3.25)

As was done in section 3.2, the integral is transformed to an integral in terms of stress \(\sigma\) only. Using (3.11) and (3.25), the latter part of (3.24) turns into

\[
\int_{\xi_0}^{\xi_1} \sigma \varepsilon \, d\xi = \int_{\xi_0}^{\sigma_1} \left(\frac{\sigma^2}{E} + \beta \sigma^{n+1}\right) \left(\frac{1}{E} + n\beta \sigma^{n-1}\right) \, d\sigma
\]

\[
= \int_{\xi_0}^{\sigma_1} \left(\frac{\sigma^2}{E^2} + n\beta^2 \sigma^{2n} + (n + 1) \frac{\beta}{E} \sigma^{n+1}\right) \, d\sigma.
\]

(3.26)

Performing the integration and inserting the bounds, yields

\[
\int_{\sigma_0}^{\sigma_1} \left(\frac{\sigma^2}{E^2} + n\beta^2 \sigma^{2n} + (n + 1) \frac{\beta}{E} \sigma^{n+1}\right) \, d\sigma = \left[\frac{\sigma^3}{3E^2} + \frac{n\beta^2}{2n + 1} \sigma^{2n+1} + \left(\frac{n + 1}{n + 2}\right) \frac{\beta}{E} \sigma^{n+2}\right]_0^{\sigma_1}
\]

\[
= \frac{\sigma_1^3}{3E^2} + \frac{n\beta^2}{2n + 1} \sigma_1^{2n+1} + \left(\frac{n + 1}{n + 2}\right) \frac{\beta}{E} \sigma_1^{n+2} - \frac{\sigma_0^3}{3E^2} - \frac{n\beta^2}{2n + 1} \sigma_0^{2n+1} - \left(\frac{n + 1}{n + 2}\right) \frac{\beta}{E} \sigma_0^{n+2}.
\]

(3.27)
Again it is noted the power \( n \) is always positive and the lower bound contribution is zero. Since the stress distribution is symmetric around the center line of the beam, also its resultant bending moments are symmetric. Thus:

\[
M = 2 \frac{h^2}{\varepsilon^2} \left( \frac{\sigma_1^3}{3E^2} + \frac{n\beta^2}{2n+1} \sigma_1^{2n+1} + \left( \frac{n+1}{n+2} \right) \frac{\beta}{E} \sigma_1^{n+2} \right)
\]  

(3.28)

Inserting (3.23) into (3.28) yields the final function for the bending moment \( M \) in terms of the outer fiber stress \( \sigma_1 \) for a Ramberg-Osgood hardening material:

\[
M = \frac{1}{2} \frac{h^2}{(\frac{2h}{h} + \beta \sigma_1^2)^2} \left( \frac{\sigma_1^3}{3E^2} + \frac{n\beta^2}{2n+1} \sigma_1^{2n+1} + \left( \frac{n+1}{n+2} \right) \frac{\beta}{E} \sigma_1^{n+2} \right)
\]  

(3.29)

### 3.2.3 Stress distribution from bending moment \( M \)

As in inverse problem of 3.2.2, often the stress distribution due to an external loading \( M \) needs to be known. The solution reduces to finding an inverse function of (3.29), where \( \sigma_1 \) needs to be isolated and expressed in terms of \( M \). This is, however, not possible analytically. Therefore a Newton-Raphson procedure is proposed. Three parameters \( A, B, \) and \( C \) are defined as

\[
A = \frac{\sigma_1^2}{E^2} + \frac{2\beta}{E} \sigma_1^{n+1} + \beta^2 \sigma_1^{2n},
\]

\[
B = \frac{\sigma_1^2}{E^2} + (n+1) \frac{\beta}{E} \sigma_1^{n+1} + n\beta^2 \sigma_1^{2n},
\]

\[
C = \frac{\sigma_1^3}{3E^2} + \left( \frac{n+1}{n+2} \right) \frac{\beta}{E} \sigma_1^{n+2} + \frac{n\beta^2}{2n+1} \sigma_1^{2n+1}.
\]  

(3.30)

The Newton-Raphson elements then are formulated in terms of (3.30) as

\[
f(\sigma_1) = \frac{1}{2} \frac{h^2}{(A)} \left[ \frac{C}{A} \right] - M
\]

\[
f'(\sigma_1) = \frac{1}{2} \frac{h^2}{B} \left[ \frac{1}{A} - \frac{C}{\sigma_1 A^2} \right]
\]  

(3.31)

A starting value for \( \sigma_1 \) can be obtained from consideration of a linear elastic case:

\[
\sigma_{1,0} = \frac{6M}{h^2}.
\]  

(3.32)

The successive approximation of \( \sigma_1 \) can now be computed from

\[
\sigma_{1,i+1} = \sigma_{1,i} - \frac{C}{B} \left[ 1 - \frac{2C}{\sigma_{1,i} A} \right]
\]  

(3.33)
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3.3 Strain energy of beam section

The strain energy in a monolayer is computed from the volume integral of the strain energy density \( u_0 \), where

\[
 u_0 = \int_0^\varepsilon \sigma d\varepsilon. \tag{3.34}
\]

This integral was seen in the integration of the \( \sigma \varepsilon \)-diagram in section 3.2. From (3.16) it can be concluded that

\[
 u_0 = \begin{cases} \frac{\sigma^2}{2E} + \frac{n\beta}{n+1}\sigma^{n+1} & \text{for } \sigma \geq 0, \\ \frac{\sigma^2}{2E} + \frac{n\beta}{n+1}(-\sigma)^{n+1} & \text{for } \sigma < 0. \end{cases} \tag{3.35}
\]

The strain energy per unit beam length and width \( U \) is determined from the strain energy density as

\[
 U = \int_{-h_2}^{h_1} u_0 dy. \tag{3.36}
\]

Using equation (3.2), (3.36) can be transformed to

\[
 U = \rho \int_{\varepsilon_2}^{\varepsilon_1} u_0 d\varepsilon = \rho \left( \int_0^0 u_0 d\varepsilon + \int_0^{\varepsilon_1} u_0 d\varepsilon \right). \tag{3.37}
\]

The integral in (3.37) is split in order to account for the different formulas for \( u_0 \) in the positive and negative stress range. The last, or positive, part is elaborated. It is formulated in terms of stress \( \sigma \) using (3.11) and becomes

\[
 \int_0^{\varepsilon_1} \left( \frac{\sigma^2}{2E} + \frac{n\beta}{n+1}\sigma^{n+1} \right) d\varepsilon = \int_0^{\sigma_1} \left( \frac{\sigma^2}{2E} + \frac{n\beta}{n+1}\sigma^{n+1} \right) \left( \frac{1}{E} + n\beta\sigma^{-1} \right) d\sigma
\]

\[
 = \int_0^{\sigma_1} \left( \frac{\sigma^2}{2E^2} + \frac{n\beta}{2E} \sigma^{n+1} + \frac{n\beta}{(n+1)E} \sigma^{n+1} + \frac{n^2\beta^2}{n+1} \sigma^{2n} \right) d\sigma \tag{3.38}
\]

\[
 = \int_0^{\sigma_1} \left( \frac{\sigma^2}{2E^2} + \frac{1}{2} + \frac{1}{n+1} \right) \frac{n\beta}{E} \sigma^{n+1} + \frac{n^2\beta^2}{n+1} \sigma^{2n} d\sigma.
\]

Integration of the last line of (3.38) and filling in the bounds yields

\[
 \int_0^{\sigma_1} \left( \frac{\sigma^2}{2E^2} + \frac{1}{2} + \frac{1}{n+1} \right) \frac{n\beta}{E} \sigma^{n+1} + \frac{n^2\beta^2}{n+1} \sigma^{2n} d\sigma
\]

\[
 = \left[ \frac{\sigma^3}{6E^2} + \left( \frac{n}{n+2} \right) \left( \frac{1}{2} + \frac{1}{n+1} \right) \frac{\beta}{E} \sigma^{n+2} + \left( \frac{1}{2n+1} \right) \left( \frac{n^2}{n+1} \right) \beta^2 \sigma^{2n+1} \right]_0^{\sigma_1} \tag{3.39}
\]

\[
 = \frac{\sigma_1^3}{6E^2} + \left( \frac{n(n+3)}{2(n+2)(n+1)} \right) \frac{\beta}{E} \sigma_1^{n+2} + \left( \frac{n^2}{2n^2+3n+1} \right) \beta^2 \sigma_1^{2n+1}.
\]

Again, it is assumed that the power \( n \) is always positive. Now the last integral of (3.37) is known. In the case of a monolayer, the stress profile is symmetric and therefore the first integral from (3.37) is equal to the last integral, since the strain energy and the strain energy density are positive quantities. Using (3.5) for \( \rho, \varepsilon_1 = 2\varepsilon_1 \), and (2.14), the strain energy for a monolayer loaded with bending moment \( M \) is equal to

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\[ U = \frac{2h}{2\varepsilon_t} \left[ \frac{\sigma_1^3}{6E^2} + \left( \frac{n(n + 3)}{2(n + 2)(n + 1)} \right) \frac{\beta}{E} \sigma_1^{n+2} + \left( \frac{n^2}{2n^2 + 3n + 1} \right) \beta^2 \sigma_1^{2n+1} \right] \]

\[ = \frac{h}{2\varepsilon_t} \frac{\sigma_1^3}{6E^2} + \left( \frac{n(n + 3)}{2(n + 2)(n + 1)} \right) \frac{\beta}{E} \sigma_1^{n+2} + \left( \frac{n^2}{2n^2 + 3n + 1} \right) \beta^2 \sigma_1^{2n+1} \]  

(3.40)

From equation (3.40) the strain energy per unit beam length and width can be computed if the outer positive fiber stress \( \sigma_1 \) is known. More general, the case of nonequal top and bottom stresses is considered. Since the strain energy (density) is a positive quantity, the strain energy due to the negative stresses (first integral of (3.37)) can be evaluated by considering the integral from 0 to \(-\sigma_2\):

\[ \int_{\varepsilon_2}^{0} u_0 d\varepsilon = \int_{0}^{-\varepsilon_2} u_0 d\varepsilon \]

\[ = \int_{0}^{-\varepsilon_2} \left( \frac{\sigma^2}{2E^2} + \left( \frac{1}{2} + \frac{1}{n + 1} \right) \frac{n\beta}{E} \sigma_n^{n+1} + \frac{n^2\beta^2}{n + 1} \sigma_2^{2n+1} \right) d\sigma \]

\[ = \frac{(-\varepsilon_2)^3}{6E^2} + \left( \frac{n(n + 3)}{2(n + 2)(n + 1)} \right) \frac{\beta}{E} (\varepsilon_2)^{n+2} + \left( \frac{n^2}{2n^2 + 3n + 1} \right) \beta^2 (\varepsilon_2)^{2n+1}. \]  

(3.41)

Hence, the strain energy per unit beam length and width for an elasto-plastic rectangular beam with outer stresses positive valued \( \sigma_1 \) and negative valued \( \sigma_2 \) is

\[ U = \frac{h}{\varepsilon_t} \left[ \frac{\sigma_1^3 + (\varepsilon_2)^3}{6E^2} + \left( \frac{n(n + 3)}{2(n + 2)(n + 1)} \right) \frac{\beta}{E} (\sigma_1^{n+2} + (\sigma_2)^{n+2}) + \left( \frac{n^2}{2n^2 + 3n + 1} \right) \beta^2 (\sigma_1^{2n+1} + (\sigma_2)^{2n+1}) \right] \]  

(3.42)

Appendix 2: Film-substrate systems with residual stress and elasto-plastic behaviour under bending
Chapter 4

Analysis of bilayer beam with elasto-plastic layer

In this chapter the single layer system from Chapter 3 is extended with an elastic top layer as shown in Figure 4.1. The top layer has a thickness $h_f$, whereas the bottom layer subscripts remain unchanged with respect to the beam in Chapter 3. In addition, the presence of residual stresses in the (elastic) film is considered in the compatibility of the two layers.

4.1 Neutral line and stress distribution

4.1.1 Compatibility of layer strains

In case of the addition of a elastic layer on top of the considered elasto-plastic layer, strains need to be compatible. As was defined in Chapter 3, the positive outer strain is labeled 1, and the negative outer strain is labeled 2. The curvature of the section is constant throughout the section height, i.e. the section is plane. The film strains are labeled with subscripts 3 (bottom) and 4 (top). For positive bending (convex side up), the compatibility relations are

---

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Figure 4.2: Strain profile of bilayer system with strain mismatch in the film for the case of positive bending (convex side up). Caution is needed with respect to the indices: 1 relates to the positive stresses, whereas 2 relates to the negative stresses.

\[
\varepsilon_3 = \varepsilon_1
\]
\[
\varepsilon_4 = \varepsilon_2 + \varepsilon_1 \left( \frac{h + h_f}{h} \right) = \varepsilon_2 + (\varepsilon_1 - \varepsilon_2) \left( \frac{h + h_f}{h} \right)
\]
\[
= \varepsilon_1 \left( \frac{h + h_f}{h} \right) + \varepsilon_2 \left( 1 - \frac{h + h_f}{h} \right) \tag{4.1}
\]

In case a residual stress is present in the elastic film, its corresponding residual strain \( \varepsilon_R \) needs to be incorporated in the compatibility equations (4.1) (see jump in Figure 4.2). With the introduction of parameter \( \gamma \), these relations turn into

\[
\varepsilon_3 = \varepsilon_1 + \varepsilon_R
\]
\[
\varepsilon_4 = \varepsilon_1 \gamma + \varepsilon_2 \left( 1 - \gamma \right) + \varepsilon_R \tag{4.2}
\]

with \( \gamma = \frac{h + h_f}{h} \).

In terms of stresses \( \sigma_1 \) and \( \sigma_2 \) these equations read

\[
\varepsilon_3 = \frac{\sigma_1}{E} + \beta \sigma_1^\nu + \varepsilon_R
\]
\[
\varepsilon_4 = \left( \frac{\sigma_1}{E} + \beta \sigma_1^\nu \right) \gamma + \left( \frac{\sigma_2}{E} - \beta (-\sigma_2)^\nu \right) \left( 1 - \gamma \right) + \varepsilon_R \tag{4.3}
\]

4.1.2 Integration of \( \sigma\varepsilon \)-curve

With the addition of an elastic layer on top of the substrate, force equilibrium has to be extended to incorporate the effect of film stresses. Analogous to section 3.2, the integration of the \( \sigma\varepsilon \)-diagram is required. It is extended for the elastic film as

\[
\int_{\varepsilon_2}^{\varepsilon_1} \sigma \, d\varepsilon = \frac{\sigma_1^2 - \sigma_2^2}{2E} + \frac{n}{n + 1} \beta \left[ \sigma_1^{n+1} - (-\sigma_2)^{n+1} \right] + \frac{E_f}{2} \left( \varepsilon_1^2 - \varepsilon_2^2 \right), \tag{4.4}
\]
where the last part is the contribution of the film. Here, $E_f$ is the elastic modulus. Since the stresses are known to be linearly distributed in the film (Figure 4.2), the integral of the $\sigma$-curve for the elastic top layer is computed as

$$(\varepsilon_4 - \varepsilon_3) \left( \frac{\sigma_4 + \sigma_3}{2} \right) = (\varepsilon_4 - \varepsilon_3) \frac{E_f}{2} (\varepsilon_4 + \varepsilon_3) = \frac{E_f}{2} (\varepsilon_4^2 - \varepsilon_3^2).$$

4.1.3 Determination of neutral line for given stress range

As was done in section 3.2.1, the location of the neutral line is solved using loading parameter $\sigma_t = \sigma_1 - \sigma_2$ (Figure 4.1). Since no analytical solution for a value of $\sigma_1$ for which equation (4.4) vanishes is possible, a numerical approximation is used. This approximation consists of components similar to those used in section 3.2.1. The top layer strains in terms of $\sigma_1$ and $\sigma_t$ and their derivatives with respect to $\sigma_1$ are

$$\varepsilon_3 = \frac{\sigma_1}{E} + \beta \sigma_1^n + \varepsilon_R$$
$$\varepsilon_4 = \left( \frac{\sigma_1}{E} + \beta \sigma_1^n \right) \gamma + \left( \frac{\sigma_1 - \sigma_t}{E} - \beta (\sigma_t - \sigma_1)^n \right) (1 - \gamma) + \varepsilon_R$$

and

$$\varepsilon_3' = \frac{1}{E} + n \beta \sigma_1^{n-1}$$
$$\varepsilon_4' = \left( \frac{1}{E} + n \beta \sigma_1^{n-1} \right) \gamma + \left( \frac{1}{E} + n \beta (\sigma_t - \sigma_1)^{n-1} \right) (1 - \gamma).$$

For a given stress interval $\sigma_t$ the outer positive fiber stress is found by successive use of:

$$\sigma_{1,i+1} = \sigma_{1,i} - \frac{f(\sigma_{1,i})}{f'(\sigma_{1,i})}$$

where

$$f(\sigma_{1,i}) = \frac{2\sigma_1 \sigma_t - \sigma_1^2}{2E} + \frac{n}{n + 1} \beta \left[ \sigma_1^{n+1} - (-\sigma_1 + \sigma_t)^{n+1} \right] + \frac{E_f}{2} (\varepsilon_4^2 - \varepsilon_3^2),$$

$$f'(\sigma_{1,i}) = \frac{\sigma_1}{E} + n \beta \sigma_1^n + (-\sigma_1 + \sigma_t)^n + E_f (\varepsilon_4' \varepsilon_4 - \varepsilon_3' \varepsilon_3).$$

Here, an initial value for $\sigma_1$ can be approximated as half the stress interval $\sigma_t$: $\sigma_{1,0} = \sigma_t/2$. The location of the neutral line can be computed using (equation (3.21))

$$h_2 = \frac{h}{1 + \frac{\varepsilon_4}{\varepsilon_3}}.$$ 

4.1.4 Bending moment $M$ from stress distribution

Contribution of elasto-plastic substrate

In the analysis of bending of an elasto-plastic monolayer in section 3.2.2 the bending moment for a given stress difference $\sigma_t$ between the upper and lower fiber was derived. There, the neutral line was located at the center of the beam and the therefore the two bending moment contributions (from the beam portions above and below the neutral line) were identical. Now, however, those contributions differ. Since bending moment contributions above and below the neutral line bear the same sign, the part due to the negative stresses can be considered using the result for the positive stresses, or equation (3.27), and the total bending moment in the case of outer fiber stresses $\sigma_1$ and $\sigma_2$ is
CHAPTER 4. ANALYSIS OF BILAYER BEAM WITH ELASTO-PLASTIC LAYER

\[
M_{\text{substrate}} = \frac{h^2}{\varepsilon_i^2} \left( \frac{\sigma_i^3 + (-\sigma_2)^3}{3E^2} + \frac{n\beta^2}{2n+1} (\sigma_1^{2n+1} + (-\sigma_2)^{2n+1}) + \left( \frac{n+1}{n+2} \right) \frac{\beta}{E} (\sigma_1^{n+2} + (-\sigma_2)^{n+2}) \right)
\]  
(4.9)

Contribution of elastic film

![Figure 4.3: Bending moment contribution due to linearly varying stress in elastic film.](image)

The bending moment caused by the stresses in the elastic film is determined using Figure 4.3. It is known the stresses vary linearly and their resultant force is decomposed in two parts: one part due to the stresses in the rectangular area, and one part due to the stresses in the triangular area. For these force components, labeled $F_1$ and $F_2$ respectively, the location with respect to the neutral line is known. They produce the bending moment:

\[
M_{\text{film}} = F_1 \left( h_3 + \frac{1}{2} (h_4 - h_3) \right) + F_2 \left( h_3 + \frac{2}{3} (h_4 - h_3) \right).
\]  
(4.10)

The forces $F_1$ and $F_2$ are

\[
F_1 = \sigma_3 (h_4 - h_3),
\]

\[
F_2 = \frac{1}{2} (\sigma_4 - \sigma_3) (h_4 - h_3).
\]  
(4.11)

Combining (4.10) and (4.11) gives the bending moment contribution of the elastic film in terms of stresses $\sigma_4$ and $\sigma_3$ as

\[
M_{\text{film}} = \frac{1}{2} \sigma_3 (h_4^2 - h_3^2) + \frac{1}{6} (\sigma_4 - \sigma_3) (2h_4^2 - h_4 h_3 - h_3^2).
\]  
(4.12)

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Total bending moment

The total bending is the sum of \( M_{\text{substrate}} \) and \( M_{\text{film}} \), or \((4.9)\) plus \((4.12)\). It can be computed if the stress distribution of the bilayer system is known. The total equation is

\[
M = \frac{1}{2} \sigma_3 (h_2^4 - h_3^4) + \frac{1}{6} (\sigma_4 - \sigma_3) (2h_4^2 - h_4h_3 - h_3^2) \\
+ \frac{h^2}{\varepsilon_t^2} \left( \frac{\sigma_1^1 + (-\sigma_2)^3}{3E^2} + \frac{n\beta^2}{2n+1} (\sigma_1^{2n+1} + (-\sigma_2)^{2n+1}) \right) \left( \frac{n+1}{n+2} \right) \beta \left( \sigma_1^{n+2} + (-\sigma_2)^{n+2} \right) .
\]

\( (4.13) \)

4.1.5 Stress distribution from bending moment \( M \)

Often the external bending moment is given and the stress profile is wanted. These can be derived from an inverse form of \((4.13)\). In order to derive such an inverse form, some variables need to be eliminated first. It is observed the heights \( h_3 \) and \( h_4 \) are

\[
h_3 = h_1, \quad h_4 = h_1 + h_f.
\]

where \( h_f \) is the film thickness. Then, the stresses in the film, with the use of \( \sigma_t \), are related to those of the substrate, through

\[
\sigma_3 = E_f \varepsilon_3 = \left( \frac{\sigma_1}{E} + \beta \sigma_1^n + \varepsilon_R \right) E_f, \\
\sigma_4 = E_f \varepsilon_4 = \left( \left( \frac{\sigma_1}{E} + \beta \sigma_1^n \right) \gamma + \left( \frac{\sigma_1 - \sigma_t}{E} + \beta (\sigma_t - \sigma_1)^n \right) (1 - \gamma) + \varepsilon_R \right) E_f .
\]

Using \((4.14)\) and \((4.15)\), the bending moment distribution from the film can be fully expressed in terms of \( \sigma_1 \) and \( \sigma_t \). The contribution of the substrate, in terms of these stresses, is

\[
M_{\text{substrate}} = \frac{h^2}{\varepsilon_t^2} \left( \frac{\sigma_1^1 + (\sigma_t - \sigma_1)^3}{3E^2} + \frac{n\beta^2}{2n+1} (\sigma_1^{2n+1} + (\sigma_t - \sigma_1)^{2n+1}) \right) \left( \frac{n+1}{n+2} \right) \beta \left( \sigma_1^{n+2} + (\sigma_t - \sigma_1)^{n+2} \right) .
\]

\( (4.16) \)

Now the total bending moment \( M \) is expressed in terms of \( \sigma_1 \) and \( \sigma_t \). To solve for \( \sigma_1 \), a numerical method has to be applied. A Newton-Raphson scheme is used. A complication here is that the value \( \sigma_t \) influences the position of the neutral line (in the monolayer the neutral line was fixed at the center of the beam), and hence the value of the bending moment due to this stress difference. Thus for every update (main iteration) of \( \sigma_t \), \( h_1 \) and \( h_2 \) need to be updated. These result in a new value for \( \sigma_1 \), which is used to update \( \sigma_t \). Consequently, in order to determine the stress profile from an external bending moment, two Newton-Raphson schemes are needed.

The procedure to obtain the stress profile is shown below.
CHAPTER 4. ANALYSIS OF BILAYER BEAM WITH ELASTO-PLASTIC LAYER

1. First, an estimate for the starting value for \( \sigma_t \) is needed. The stress interval is approximated to be twice the outer fiber stress in case of linear elastic bending, with a maximum value of twice the yielding stress to prevent a too large interval:

\[
\sigma_{t,0} = \frac{6M}{h^2} = \frac{12M}{h^2} \leq 2\sigma_y
\]  

(4.17)

2. Now the location of the neutral line needs to be found using a Newton-Raphson scheme. As was shown in section 4.1.3:

(a) guess a starting value for \( \sigma_1 \) as \( \sigma_{1,0} = \sigma_t / 2 \)

(b) perform the iterative procedure:

\[
\sigma_{1;i+1} = \sigma_{1;i} - \frac{f(\sigma_{1;i})}{f'(\sigma_{1;i})}
\]

where

\[
f(\sigma_{1;i}) = \frac{2\sigma_1 - \sigma_t^2}{2E} + \frac{n}{n+1}\beta \left[ \sigma_1^{n+1} - (\sigma_1 - \sigma_t)^{n+1} \right] + \frac{E_f}{2} (\varepsilon_4^2 - \varepsilon_3^2),
\]

\[
f'(\sigma_{1;i}) = \frac{\sigma_t}{E} + n\beta \left[ \sigma_1^n + (\sigma_1 - \sigma_t)^n \right] + E_f (\varepsilon_4^2 - \varepsilon_3^2)
\]

(c) compute location of neutral line:

\[
h_1 = h - \frac{h}{1 + \frac{h_2}{\varepsilon_t}}
\]

(4.19)

3. With the updated location of the neutral line known, the stress range \( \sigma_t \) can be updated:

4. Compute new value for \( \sigma_t \):

\[
\sigma_{t;i+1} = \sigma_{t;i} - \frac{f(\sigma_{t;i})}{f'(\sigma_{t;i})}
\]

where

\[
f(\sigma_{t;i}) = M_{\text{film}} + M_{\text{substrate}} - M
\]

\[
f'(\sigma_{t;i}) = M_{\text{film}}' + M_{\text{substrate}}'
\]

with

\[
M_{\text{film}}' = \frac{1}{6} \sigma_4' (2h_1^2 - h_4 h_3 - h_3^2),
\]

\[
M_{\text{substrate}}' = \frac{h_4^2}{\varepsilon_t} \left( A - 2B \varepsilon_4' \varepsilon_1' \right)
\]

in which

\[
\sigma_4' = \left( \frac{1}{E} + n\beta (\sigma_t - \sigma_1)^{n-1} \right) (1 - \gamma) E_f
\]

\[
\varepsilon_t' = \frac{1}{E} - n\beta (\sigma_t - \sigma_1)^{n-1}
\]

\[
A = \frac{(\sigma_t - \sigma_1)^2}{E^2} + n\beta^2 (\sigma_t - \sigma_1)^{2n} + (n + 1) \frac{\beta}{E} (\sigma_t - \sigma_1)^{n+1}
\]

\[
B = \frac{\sigma_1^3 + (\sigma_t - \sigma_1)^3}{3E^2} + \frac{n\beta^2}{2n + 1} \left( \sigma_1^{2n+1} + (\sigma_t - \sigma_1)^{2n+1} \right)
\]

\[+ \left( \frac{n + 1}{n + 2} \right) \frac{\beta}{E} (\sigma_1^n + (\sigma_t - \sigma_1)^n)
\]

(4.20)

5. Go back to step 2 until a converged value for \( \sigma_t \) is found.
4.2 Strain energy

Just as was done in section 4.1.4 with the bending moment $M$, the strain energy due to the film and substrate are determined separately. It is assumed the stress distribution is known. The contribution of the elasto-plastic substrate then is, according to (3.42):

$$U_{\text{substrate}} = \frac{h}{\varepsilon_t} \left[ \frac{\sigma_1^3 + (\sigma_2^2)^3}{6E^2} + \left( \frac{n(n+3)}{2(n+2)(n+1)} \right) \frac{\beta}{E} \left( \sigma_1^{n+2} + (-\sigma_2)^{n+2} \right) \right]
+ \left( \frac{n^2}{2n^2 + 3n + 1} \right) \beta^2 \left( \sigma_1^{2n+1} + (-\sigma_2)^{2n+1} \right)$$

(4.21)

The contribution of the elastic film is now elaborated. For linear elastic materials, the strain energy density $u_0$ is

$$u_0 = \frac{\sigma \varepsilon}{2} = \frac{E \varepsilon^2}{2}$$

(4.22)

The strain energy is computed as

$$U_{\text{film}} = \int_{h_3}^{h_4} u_0 dy = \int_{h_3}^{h_4} E_f \frac{\varepsilon^2}{2} dy.$$

(4.23)

Using (3.2) and (3.5), (4.23) becomes

$$U_{\text{film}} = \frac{h_f}{\varepsilon_4 - \varepsilon_3} \int_{\varepsilon_3}^{\varepsilon_4} E_f \frac{\varepsilon^2}{2} d\varepsilon.$$

(4.24)

Elaborating yields

$$U_{\text{film}} = \frac{h_f}{\varepsilon_4 - \varepsilon_3} \int_{\varepsilon_3}^{\varepsilon_4} E_f \frac{\varepsilon^2}{2} d\varepsilon
= \frac{h_f}{\varepsilon_4 - \varepsilon_3} \frac{E_f}{2} \left[ \varepsilon^3 \right]_{\varepsilon_3}^{\varepsilon_4}
= \frac{E_f h_f}{6} \frac{\varepsilon_4^{3} - \varepsilon_3^{3}}{\varepsilon_4 - \varepsilon_3}$$

(4.25)

The total strain energy for a bilayer system with an elastic film and elastoplastic substrate then is

$$U = \frac{h}{\varepsilon_t} \left[ \frac{\sigma_1^3 + (\sigma_2^2)^3}{6E^2} + \left( \frac{n(n+3)}{2(n+2)(n+1)} \right) \frac{\beta}{E} \left( \sigma_1^{n+2} + (-\sigma_2)^{n+2} \right) \right]
+ \left( \frac{n^2}{2n^2 + 3n + 1} \right) \beta^2 \left( \sigma_1^{2n+1} + (-\sigma_2)^{2n+1} \right)
+ \frac{E_f h_f}{6} \frac{\varepsilon_4^{3} - \varepsilon_3^{3}}{\varepsilon_4 - \varepsilon_3}$$

(4.26)
Chapter 5

Analysis of plane strain bilayer with elasto-plastic layer

5.1 Determination of the neutral line and stress profile

Figure 5.1: Numerical procedure to determine neutral line. 1) establish interval $\sigma_t$ and determine $\sigma_m\varepsilon_m$-curve, 2) compute corresponding $\sigma_x\varepsilon_x$-curve, 3) discretize section into $n$ parts and determine net force and equilibrium.

In section 2.4.2 a procedure was presented for the determination of the $\sigma_x\varepsilon_x$-curve for plane strain problems. In this section the same interval $a < \sigma_m < b$ is assumed, for which data points for the $\sigma_x\varepsilon_x$-curve are computed at $n$ points. These can be viewed as stresses in the elasto-plastic layer in some loading situation. The corresponding curvature is determined from the outer strains as

$$\kappa = \frac{\varepsilon_1 - \varepsilon_2}{h}, \quad (5.1)$$

where the subscripts 1 and 2 are used for the top and bottom (which are positive and negative in case of positive bending), respectively. The location of the longitudinal strains from the neutral line in the section are found from

Appendix 2: Film-substrate systems with residual stress and elasto-plastic behaviour under bending
Knowing the stresses and strains (and their location with respect to the neutral line), the net force on the section is determined from the sum of every material strip in set $n$ (see material strips in right figure of Figure 5.1) using the trapezoidal rule (see area of trapezoid in Figure 4.3) as

$$F = \sum_{i=0}^{n} \sigma_{x;i}(h_{i+1} - h_i) + \frac{1}{2}(\sigma_{x;i+1} - \sigma_{x;i})(h_{i+1} - h_i)$$  \hspace{1cm} (5.3)

For zero net force, (5.3) needs to equate to zero. The bending moment due to the stresses $\sigma_x$ is again computed from the material strips $i$. To this end, the bending moment as formulated in (4.12) (see moments produced in trapezoid in Figure 4.3) is used

$$M = \sum_{i=0}^{n} \frac{1}{2} \sigma_{x;i}(h_{i+1}^2 - h_i^2) + \frac{1}{6}(\sigma_{x;i+1} - \sigma_{x;i})(2h_{i+1}^2 - h_{i+1}h_i - h_i^2).$$  \hspace{1cm} (5.4)

The individual layer strain components need to be compatible (see subsection 4.1.1), thus the top layer strains are computed from the bottom layer strains (which are computed from the assumed stresses) as

$$\varepsilon_3 = \varepsilon_1 + \varepsilon_R$$

$$\varepsilon_4 = \varepsilon_1 \left( \frac{h + h_f}{h} \right) - \varepsilon_2 \left( \frac{h_f}{h} \right) + \varepsilon_R$$  \hspace{1cm} (5.5)

where $\varepsilon_R$ denotes the strain mismatch between the top and bottom layer, and the subscripts 1 and 2 denote the elasto-plastic strains, and the 3 and 4 the elastic strains. In the case the top layer is elasto-plastic and the bottom layer elastic, the compatibility equations are

$$\varepsilon_3 = \varepsilon_2 - \varepsilon_R$$

$$\varepsilon_4 = -\varepsilon_1 \left( \frac{h}{h_f} \right) + \varepsilon_2 \left( \frac{h + h_f}{h_f} \right) - \varepsilon_R.$$  \hspace{1cm} (5.6)

Equations (5.3) and (5.4) are used in the iterative procedure to find the neutral line. This procedure consists of two optimizations: First, the assumed stress difference $\sigma_t$ needs to be shifted until force equilibrium is satisfied (thus $\sigma_1$ is determined). Next, the value of $\sigma_t$ itself needs to be updated to satisfy moment equilibrium. Each optimization step where $\sigma_t$ is updated, a full optimization scheme is needed to find $\sigma_1$ for zero net force. The two optimization processes can be performed using two Newton-Raphson procedures interactively, where the main optimization loop updates $\sigma_t$ and the suboptimization is used for updating $\sigma_1$.

In the downstream section analysis no determination of the neutral line is required, since it is always located at the center of the layer. The optimization process consists only of updating $\sigma_t$ to find a net bending moment of $M$. 

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5.2 Strain energy

For a multiaxial Ramberg-Osgood material, the strain energy density $u_0$ is defined as

$$u_0 = \frac{1 + \nu}{3E} \sigma_m^2 + \frac{3}{2} \frac{1 - 2\nu}{E} \sigma_p^2 + \frac{n}{n+1} \beta \sigma_m^{n+1},$$

where

$$\sigma_p = \frac{\sigma_x + \sigma_y + \sigma_z}{3}.$$ (5.7)

It is noted that in the formulation of the strain energy density the used values for $\nu$ and $E$ are those in the elastic case, for the first two components of (5.7) are the elastic contributions to $u_0$. The relation between the individual stress (or strain) components, however, are still dominated by the variable value for $\nu$ and $E$. The third component in (5.7) denotes the plastic contribution to the strain energy density.

From the strain energy density $u_0$ the strain energy is computed over the section height as

$$U = \sum_{i=0}^n u_{0,i}(h_{i+1} - h_i) + \frac{1}{2}(u_{0,i+1} - u_{0,i})(h_{i+1} - h_i).$$ (5.8)

Equation (5.8) is again computed for every material strip $i$ over the section height. In the upstream direction, the total strain energy is computed from the sum of equation (6.8) and the elastic top layer contribution (computed from equation (4.24)). The downstream strain energy is computed from (6.8) also.
Chapter 6

Energy release rate of bilayer with elasto-plastic layer

In this chapter the results from the previous chapters are combined to determine an expression for the ERR of a bilayer, both for plane stress and plane strain, in a 4PB-test, with a residually stressed elastic top layer and an elasto-plastic bottom layer.

6.1 Energy releate rate for bilayer beam

The energy release rate is given by

\[ G = \frac{d\Delta F}{da} - \frac{d\Delta U}{da}, \] (6.1)

where \( \Delta F \) denotes the produced work by external moment \( M \) during a unit delamination \( da \), and \( \Delta U \) is the change in strain energy during this delamination. The change in strain energy is given by

\[ \Delta U = (U_{\text{downstream}} - U_{\text{upstream}}) a, \] (6.2)

where the downstream and upstream direction refer to the cracked and uncracked section respectively. It is observed the strain energy derived in section 3.3 represents the cracked section state, whereas the strain energy of the bilayer system in section 4.2 relates to the uncracked section. Using those expressions for the strain energies, the change in strain energy per unit delamination \( a \) can be formulated as

\[
\frac{\Delta U}{a} = \frac{h}{\varepsilon_{d1}} \left[ \frac{\sigma_{d1}^3}{6E} + \left( \frac{n(n+3)}{2(n+2)(n+1)} \right) \frac{\beta}{E} \sigma_{d1}^{n+2} + \left( \frac{n^2}{2n^2 + 3n + 1} \right) \beta^2 \sigma_{d1}^{2n+1} \right] - \frac{h}{\varepsilon_{u1} - \varepsilon_{u2}} \left[ \frac{\sigma_{u1}^3}{6E} + \left( \frac{n(n+3)}{2(n+2)(n+1)} \right) \frac{\beta}{E} \sigma_{u1}^{n+2} + (-\sigma_{u2})^{n+2} \right] + \left( \frac{n^2}{2n^2 + 3n + 1} \right) \beta^2 (\sigma_{u1}^{2n+1} + (-\sigma_{u2})^{2n+1}) - \frac{E_f h_f}{6} \left( \frac{\varepsilon_{u4}^3 - \varepsilon_{u4}^3}{\varepsilon_{u4} - \varepsilon_{u4}} \right) \] (6.3)

The subscripts \( d \) and \( u \) refer to downstream and upstream, respectively. The change in strain energy \( \Delta U \) can be computed after computation of the stress profiles in the downstream and upstream direction, since then the stresses \( \sigma \) and strains \( \varepsilon \) at the outer fibers are known.

The work done by the bending moment during the delamination process can be calculated from the change in curvature in the downstream and upstream direction. It is defined as

\[ \Delta F = M\phi = (\kappa_{\text{downstream}} - \kappa_{\text{upstream}}) Ma \] (6.4)

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After evaluation of the stress profiles in these sections, the curvatures are determined from (3.5) as

$$\kappa_d = \frac{2\varepsilon_{d1}}{h},$$
$$\kappa_u = \frac{\varepsilon_{u1} - \varepsilon_{u2}}{h}. \quad (6.5)$$

Combining (6.5) and (6.4) gives

$$\Delta F = M\phi = \left(\frac{2\varepsilon_{d1}}{h} - \frac{\varepsilon_{u1} - \varepsilon_{u2}}{h}\right) Ma \quad (6.6)$$

The energy release rate now yields

$$G = \left(\frac{2\varepsilon_{d1}}{h} - \frac{\varepsilon_{u1} - \varepsilon_{u2}}{h}\right) M - \frac{h}{\varepsilon_{d1}} \left[\frac{\sigma_{d1}^3}{6E^2} + \left(\frac{n(n+3)}{2(n+2)(n+1)}\right) \frac{\beta}{E} \sigma_{d1}^{n+2} + \left(\frac{n^2}{2n^2+3n+1}\right) \beta^2 \sigma_{d1}^{2n+1}\right]$$
$$+ \frac{h}{\varepsilon_{u1} - \varepsilon_{u2}} \left[\frac{\sigma_{u1}^3}{6E^2} + \left(\frac{n(n+3)}{2(n+2)(n+1)}\right) \frac{\beta}{E} \left(\sigma_{u1}^{n+2} + (-\sigma_{u2})^{n+2}\right)\right]$$
$$+ \left(\frac{n^2}{2n^2+3n+1}\right) \beta^2 \left(\sigma_{u1}^{2n+1} + (-\sigma_{u2})^{2n+1}\right) + \frac{Eh_f}{6} \left(\frac{\varepsilon_{u4}^3 - \varepsilon_{u3}^3}{\varepsilon_{u4} - \varepsilon_{u3}}\right) \quad (6.7)$$

This result is expressed in terms of stress and strain determined from the stress and strain profiles as determined from the equilibrium conditions. For the plane stress case, equation (6.7) provides analytical results for the limit case in which Small Scale Yielding (SSY) condition at the crack tip is satisfied. For specimens in a state of plane strain, equation (6.7) can provide an approximate result by adjustment stress-strain relation as indicated in section 2.4.2.

6.2 Energy releate rate for plane strain bilayer

With the stress and strain profiles known form Chapter 5, the section curvatures and strain energies can be computed in the upstream and downstream direction. Then, the energy release rate (ERR) is known also. The energetically conjugated work, done by the bending moment loading $M$, is computed from the product of the difference in curvature in the downstream and upstream direction (as was done in 6.1) and the loading $M$. The strain energies in the subsequent sections are found from the previous section (5.2). The energy release rate then is found from

$$G = \frac{d ((\kappa_d - \kappa_u)Ma) - d ((U_d - U_u)a)}{da} \quad (6.8)$$
$$G = (\kappa_d - \kappa_u)M - (U_d - U_u)$$
Bibliography


Appendix 2: Film-substrate systems with residual stress and elasto-plastic behaviour under bending