MASTER

Dynamic response calculation of axi-symmetrical structures, applied to carillon bells

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Dynamic response calculation of
axi-symmetrical structures,
applied to carillon bells

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Summary

The sound spectrum of a swinging bell or carillon bell strongly depends on the participation of its particular modes of vibration or eigenmodes, in the dynamic response of the bell. Each excitated eigenmode radiates a sound field, called a partial. The composition of all the audible partials is denoted as the sound spectrum. The intensity at which each eigenmode is present in the dynamic response after the bell has been struck by a clapper, decreases owing to damping. Distinguished is structural damping, occurring in the bell material itself, and acoustical damping, mainly a result of the radiation of sound through the medium surrounding the bell and for some part by friction between this medium and the bell. In this essay the major principles of both types of damping are described. The geometry of a bell is of major importance in acoustical damping and has to be subject of further research.

In order to be able to make some predictions about the sound spectrum of a bell and its time dependent behaviour, even before physical realisation of the bell has taken place, a mathematical model is needed to compute its response to a stroke with a clapper or to a prescribed input force. In the former case the contact force between bell and clapper during collision has to be computed as well.

Hertz contact law is used to solve simultaneously the contact force (input) and the displacement of the bell (dynamic response) to it, both being mutually dependent. A bell is regarded as a slightly damped system and its response is written modally, using eigenmodes of the undamped system. This results in uncoupled equations of motion for the modes of vibration of the examined structure. Further research is necessary to determine the mimimum number of modes, required to describe the response within a desired limit of accuracy.

The resons calculation has been implemented in an existing F.E.M. program (DYNOPT).
Samenvatting

Het klankspectrum van een klok wordt in grote mate bepaald door de aanwezigheid van verschillende eigenmodes in de dynamische respons van die klok. Elke eigenmode straalt een geluidsvedl uit, een zogenaamde partiaal. Het samenstel van alle hoorbare partialen wordt aangeduid als het klankspectrum. De participatie van elke mode in de dynamische respons na aanslag met een klepel, neemt af in de tijd tengevolge van demping. Onderscheid wordt daarbij gemaakt tussen materiaaldemping, tengevolge van wrijving in het klokmateriaal zelf en acoustische demping, tengevolge van wrijving tussen klok en omringend medium en afstraling van trillingsenergie in dat medium. Bij modes met hogere frequentie daalt de participatie sneller dan bij modes die een bijbehorende lagere frequentie hebben. De wezenlijke rol die de geometrie van een klok bij dit fenomeen speelt, zal nog nader onderzocht moeten worden.

Om voorspellingen te kunnen doen over het klankspectrum van een klok en het gedrag daarvan in de tijd, nog voordat de klok werkelijk is gegoten, moeten we beschikken over een mathematisch model waarmee de respons op het aanslaan met een klepel of op een voorgeschreven ingangskracht kan worden berekend. In het eerste geval moet de nog niet bekende contactkracht tussen klok en klepel ook bepaald worden. De wet van Hertz is gebruikt om de contactkracht tijdens de botsing (input) en de daardoor veroorzaakte verplaatsing van de klok (respons) te bepalen. Beiden beinvloeden elkaar wederzijds. Een klok wordt beschouwd als een zwak–gedempt systeem en de respons wordt modaal beschreven met eigenmodes van het ongedempte systeem. Dit leidt tot een ontkoppeld stelsel bewegingsvergelijkingen, die de afzonderlijke eigenmodes van de klok beschrijven en waaruit de participatie factoren zijn te berekenen. Nader onderzoek is nog nodig naar het in de responsberekening te gebruiken minimum aantal modes, vereist om een gewenste nauwkeurigheid te bereiken.

De hier gepresenteerde responsberekening is zodanig opgezet, dat deze geïmplementeerd kon worden in een bestaand eindige elementen programma (Dynopt).
1. Introduction

Goal in the STW–project "Modelling and optimizing of the sound spectrum of bells" is to formulate a mathematical model that enables us to compute and to optimize the sound spectrum of bells.

The composition of tones, each tone appearing with its own intensity, audible after a bell has been struck by a clapper is called its sound spectrum. A tone of certain height, expressed in its frequency, owes its appearance in the spectrum to the fact that a bell vibrates (among others) in a structural mode of vibration (eigenmode) with a frequency (eigenfrequency) of this value. This eigenmode radiates a sound field that is perceived as a tone. Each eigenmode of a bell can be described by the vibration pattern of the mouth of the bell (node mode) and a matching pattern for vibration of the waist of the bell (construction mode). Fig. 1.1 shows some of these patterns.

![Vibration patterns of a bell](image)

**Construction modes (waist)**

**Node modes (mouth)**

Fig. 1.1: Vibration patterns of a bell.

(mouth = horizontal section, waist = vertical section)

Since a bell is a continuous structure, an infinite number of different modes of vibration can occur simultaneously in the dynamic response of a bell. The most important tones, a
bell founder speaks of partials, with their name, partial code and frequency ratio, are listed in table 1.1. The code of a partial denotes the vibration pattern of the eigenmode by which it is generated.

<table>
<thead>
<tr>
<th>partial name</th>
<th>partial code</th>
<th>frequency ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>minor third</td>
<td>major</td>
</tr>
<tr>
<td></td>
<td>third</td>
<td></td>
</tr>
<tr>
<td>Hum note</td>
<td>H − 2</td>
<td>1</td>
</tr>
<tr>
<td>Fundamental</td>
<td>F − 2</td>
<td>2</td>
</tr>
<tr>
<td>Third</td>
<td>I − 3</td>
<td>2.4</td>
</tr>
<tr>
<td>Fifth</td>
<td>II − 3</td>
<td>3</td>
</tr>
<tr>
<td>Nominal</td>
<td>I − 4</td>
<td>4</td>
</tr>
<tr>
<td>Twelfth</td>
<td>I − 5</td>
<td>6</td>
</tr>
<tr>
<td>Double Octave</td>
<td>I − 6</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 1.1 Important partials with name, code and frequency ratio.

For the sound spectrum only the modes within the audible frequency range of 20–15,000 [Hz] are important, while for the response calculation (chapter 2) modes of a wider frequency range must be taken in account.

The medium surrounding the bell, in most cases just air, damps the amplitude of the vibration (acoustical damping). Together with the damping occurring within the bell material itself (structural damping) it causes a decrease in the sound intensity of the considered tone. Measurements (van Heuven [1949]) show a faster decay of sound intensity for the higher tones than for lower ones. Damping is the cause of the time dependent behaviour of the sound spectrum of a bell. To determine this time dependent behaviour of the sound spectrum, we first need a model that enables us to compute the structural dynamic response to a stroke with a clapper. In this essay a model is described to calculate the dynamic response of a bell. It is implemented in an existing FEM program (DYNOPT), suited for the analysis of axi-symmetrical structures.
2. Response calculation

Which particular modes of vibration actually participate in the dynamic response of the bell after it has been struck by a clapper, depends on the course of the contact force during the collision and on the place of the contact point on the sound bow or waist of the bell. A mode of vibration having a nodal circle on the sound bow or waist of the bell (see fig. 1.1) just at the place where it is hit by the clapper will not be excited by the clapper. The course of the contact force during the collision between bell and clapper is unknown. Before the dynamic response of the bell can be calculated, the collision between bell and clapper has to be analysed for a bell and clapper with arbitrary geometry.

2.1 Response to a collision between bell and clapper.

2.1.1. Model to describe the collision between bell and clapper.

In the first attempt to describe the collision between a bell and a clapper, Schepens [1986] used a model based on preservation of momentum during the collision. The collision was characterised by a coefficient $K$

$$K = \frac{\text{change of momentum during expansion period}}{\text{change of momentum during compression period}}$$

For a completely inelastic collision we find $K=0$ and for a completely elastic collision we find $K=1$. All values of $K$ have to lie in the interval $[0, 1]$. However, in measurements of collision between bell and clapper values of $K>1$ occurred. Therefore this model to characterize the collision between bell and clapper was rejected.

Another attempt to describe the collision was made by Sonnemans [1988] who used a model based on preservation of energy during the collision. The vibration energy of each of the seven structural eigenmodes radiating the seven most important partials (see table 1.1) of the bell were measured, before and after the collision. The vibration energy that each of the most important eigenmodes received during the collision was used to characterize the collision. From this research it was concluded that the course of the contact force during collision depends on:

* State of deformation (the phase of the dynamic response) of bell and clapper at the moment of collision, if the bell is not at rest
* Material properties of bell and clapper
* Geometry of bell and clapper
* Place of contact point between bell and clapper on the sound bow or waist of the bell
* Kinetic energy of the clapper before collision
In our case the first mentioned factor can be disregarded as we will analyse only one collision between the clapper and a bell at rest. This is the usual situation for a carillon bell, where the bell is suspended rigidly and the clapper strikes the bell only once.

The influence of the geometry and the place of contact prevent the general application of a once measured course of the contact force (for one particular combination of bell and clapper), to arbitrary combinations of geometries of bell and clapper.

Roozen-Kroon [1989] proposed to use a model based on Hertz law of contact.

\[ F(t) = k \cdot \alpha^{3/2}(t) \]  

(2.1.1.1)

where

- \( F(t) \) contact force between the bodies in contact
- \( \alpha(t) \) compression between the bodies in contact
- \( k \) Hertz constant depending on material and geometry properties

The compression \( \alpha(t) \) between the bell and the clapper is assumed to be the difference between the displacement of the contact point on the clapper and the displacement of the contact point on the bell. Expressions for the displacements of the contact point on the colliding bodies are formulated as a function of the unknown contact force. Using Hertz law to relate the compression of two colliding bodies to the occurring contact force, an implicit equation for the unknown contact force is obtained. This equation can be solved at discrete times.

In the case of collision between bell and clapper, the following assumptions are made:

* Only elastic deformation occurs in bell during collision.
* During the collision between a clapper and a bell, the relative tangential movement of the clapper along the bell surface together with its related friction forces will be disregarded.
* Thermal losses caused by friction of the bell and the clapper with surrounding air, and friction in bearings will be disregarded.
* Imperfections in material of bell and clapper won't be taken in account (the material is assumed to be homogeneous and isotropic).
* The collision is assumed to be perpendicular to the \( \phi \)-direction (see fig. 2.1), therefore the component \( F_\phi(t) = 0 \).
* The compression between the bell and the clapper in the contact point is assumed to be directed perpendicular to the bell surface:

\[ \alpha(t) = W_{\text{clapp}}(t) - W_{\text{bell}}(t) \]  

(2.1.1.2)

with

- \( \alpha(t) \) compression between bell and clapper
- \( W_{\text{clapp}}(t) \) displacement of the contact point on the clapper
$W_{\text{bell}}(t)$ \quad \text{displacement of the contact point on the bell}

* Hertz law of contact relates compression and contact force as:
\[ F_n(t) = k \cdot \alpha^{3/2}(t) \]  \hspace{1cm} (2.1.1.3)

with $F_n(t)$ \quad \text{component of the contact force $F(t)$, perpendicular to the bell wall in the contact point}

Now the Hertz constant and expressions for the displacements of the contact point on bell and clapper as a function of the unknown contact force, have to be determined.
2.1.2 Determination of the Hertz constant.

Hertz law of contact relates the contact force, occurring between two compressed elastic bodies and their approach as:
\[
F(t) = k \cdot a(t)^{3/2}
\]  
(2.1.2.1)

Where

- \(k\) = Hertz constant
- \(a\) = approach
- \(F\) = contact force
- \(F\) and \(a\) are measured along the normal to the tangent plane in the contact point.

\(R_1 = R_w\) in the \(R-Z\) plane
\(R_1' = R_c\) in the \(R-Q\) plane
\(R_2 = R_{ball}\)
\(R_2' = R_{ball}\)

Fig. 2.1.2.1: Principal radii of curvature in the contact of bell and clapper.

Goldsmith [1960] states that the Hertz constant is:
\[
k = \frac{4}{3} \frac{q_k}{(\delta_1 + \delta_2) \cdot \sqrt{A+B}}
\]  
(2.1.2.2)

where \(q_k\) depends on the value of \(A/B\) and further
\[
A + B = \frac{1}{2} \left\{ \frac{1}{R_1} + \frac{1}{R_1'} + \frac{1}{R_2} + \frac{1}{R_2'} \right\}
\]  
(2.1.2.3)

(Radii as shown in fig. 2.1.2.1).
\[
\delta_i = \frac{1 - \nu_i^2}{\pi \cdot E_i}
\]  
(2.1.2.4)

\(E_i = \) Young's modulus
\(\nu_i = \) Poisson's ratio

The value of \(q_k\) depends on the ratio \(A/B\). For the type of contact we are dealing with, the contact between a surface of rotation and a ball, is given:
\[
A = \frac{1 - R_{ball}/R_c}{1 + R_{ball}/R_w}
\]  
(2.1.2.5)

At this stage information is needed about the radii of curvature of the bell in the contact point. The localisation of the contact point on the bell and the determination of radii of curvature at that place, is explained in appendix F.
Parameter $q_k$ can be evaluated by means of Legendre polynomials. As this demands great computational effort and because Goldsmith shows a table of values for $q_k$, at discrete values of $A/B$ (see table (2.1.2.1)), interpolation will be used.

<table>
<thead>
<tr>
<th>$A/B$</th>
<th>$q_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
<td>0.3180</td>
</tr>
<tr>
<td>0.7041</td>
<td>0.3215</td>
</tr>
<tr>
<td>0.4903</td>
<td>0.3322</td>
</tr>
<tr>
<td>0.3333</td>
<td>0.3505</td>
</tr>
<tr>
<td>0.2174</td>
<td>0.3819</td>
</tr>
<tr>
<td>0.1325</td>
<td>0.4300</td>
</tr>
<tr>
<td>0.0718</td>
<td>0.5132</td>
</tr>
<tr>
<td>0.0311</td>
<td>0.6662</td>
</tr>
<tr>
<td>0.00765</td>
<td>1.1450</td>
</tr>
</tbody>
</table>

Table (2.1.2.1): Values of $q_k(A/B)$ as a function of $A/B$ according to Goldsmith.

At first view of the $q_k$ versus $A/B$ values, the describing function seems to be:

$$ q_k(A/B) = b + a \cdot f(A/B) $$

where

$$ f(A/B) \approx \log(A/B) \text{ or } f(A/B) \approx B/A. $$

First $(A/B)$ is renamed as $x$ and $q_k(A/B)$ as $q_k(x)$. An approximation $\tilde{q}_k(x)$ is introduced:

$$ \tilde{q}_k(x) = b + a \cdot f(x) $$

First $(A/B)$ is renamed as $x$ and $q_k(A/B)$ as $q_k(x)$. An approximation $\tilde{q}_k(x)$ is introduced:

The constants $a$ and $b$ will be computed for the chosen function $f(x)$ by minimizing the sum of squares of the deviations, for a given number of points $n$:

$$ S = \sum_{i=1}^{n} [q_k(x_i) - \tilde{q}_k(x_i)]^2 = \sum_{i=1}^{n} [b + a \cdot f(x_i) - q_k(x_i)]^2 $$

$$ \frac{\partial S}{\partial a} = 0 \text{ and } \frac{\partial S}{\partial b} = 0 $$

From (2.1.2.7) is solved:

$$ b = \frac{1}{n} \cdot \sum_{i=1}^{n} q_k(x_i) - a \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} f(x_i) $$

$$ = \bar{q}_k - a \cdot \bar{f} $$

Where $\bar{q}_k$ and $\bar{f}$ are average values of $q_k(x_i)$ and $f(x_i)$.

$$ a = \frac{\sum_{i=1}^{n} \{\bar{q}_k - q_k(x_i)\} \cdot f(x_i)}{\sum_{i=1}^{n} \{\bar{f} - f(x_i)\} \cdot f(x_i)} $$

Of the investigated functions, $f(x) = 1/x$ proved to give the smallest relative deviations:
Deviations occurring within a certain range of A/B tend to decrease when only points in this range were used, to fit the function on. Now an estimation is made of the borders of the range within which the approximate expression (2.1.2.6) has to be applicable.

Bells usually will be excited in the neighbourhood of their soundbow, so therefore we assume:

\[0.8 \cdot R_0 \leq R_c \leq 1.0 \cdot R_0\]

The radius of the clapper ball varies in practice as:

\[0.05 \cdot R_0 \leq R_{ball} \leq 0.20 \cdot R_0\]

which results in:

\[0.05 \leq \frac{R_{ball}}{R_c} \leq 0.38\]  \hspace{1cm} (2.1.2.9)

Radius of curvature of the bell profile will vary as:

\[R_{ball} \leq R_w \leq \infty\] so:

\[0 \leq \frac{R_{ball}}{R_w} \leq 1\]  \hspace{1cm} (2.1.2.10)

Substitution of (2.1.2.9) and (2.1.2.10) in (2.1.2.5) results in:

\[1 - \left(\frac{R_{ball}}{R_c}\right)_{\max} \leq \frac{A}{B} \leq \frac{1}{\left(\frac{R_{ball}}{R_w}\right)_{\min}}\]

\[0.405 \leq \frac{A}{B} \leq 0.950\]  \hspace{1cm} (2.1.2.11)

In this way, an approximation of \(q_k(A/B)\) within the range \(0.3333 \leq \frac{A}{B} \leq 1.0\) from table (2.1.2.1) is justified. The approximate function \(\tilde{q}_k(A/B)\), evaluated for the number of points \(n=4\), and its relative deviations, are given in table (2.1.2.2).
\[ \tilde{q}_k(A/B) = 0.299416 + 0.016694 \cdot B/A \]

Table (2.1.2.2): Approximate function \( \tilde{q}_k(A/B) \) and its relative deviations.

If a collision response calculation is executed for an axi–symmetrical body, and a value \( A/B \) for the contact falls outside the range we proposed here, the parameters \( a \) and \( b \) have to be adjusted.
2.1.3 Displacement of the contact point on clapper.

The displacement of the contact point on the clapper will be determined by solving the equation of motion of the clapper. The following assumptions are made (see fig. 2.1.3.1):

* The clapper will be regarded as rigid
* The distance between the pivot point of the clapper and the contact point \( L = |oc| \) is regarded constant during collision
* The contact force \( F(t) \) acts in a direction \( \perp \) to \( |oc| \) during collision (This is the same direction as the initial clapper velocity \( \vec{v}_0 \))
* In position \( \alpha = \alpha_0 \) the clapper is about to hit the bell
* The clapper in a carillon is driven by a driving force \( F_{dr} \). The force \( F_{dr} \) is regarded constant during collision (and acting under a constant angle \( \gamma \) with the horizontal axis during collision)

\[ \text{Fig. 2.1.3.1 Forces acting on the clapper during collision} \]

The equation of motion for the clapper is:

\[
-F(t) \cdot L - M \cdot g \cdot L \cdot \sin(0_0 + \phi) + F_{dr} \cdot L \cdot \cos(0_0 + \phi - \gamma) = J_0 \cdot \ddot{\alpha} \tag{2.1.3.1}
\]

where \( \alpha = 0_0 + \phi \)
\( \ddot{\alpha} = \ddot{\phi} \tag{2.1.3.2} \)

When rotations about the position \( \alpha = 0_0 \) are small, linearisation is permitted:

\[
\cos(\phi) \approx 1 \\
\sin(\phi) \approx \phi \tag{2.1.3.3}
\]

Substitution of the equations (2.1.3.2) and (2.1.3.3) in (2.1.3.1) results after rearranging in:
\[ \ddot{\phi} + \left\{ M \cdot g \cdot \frac{L_{\text{aw}}}{J_0} \cdot \cos(\alpha_0) + F_{\text{fr}} \cdot \frac{L_{\text{dr}}}{J_0} \cdot \sin(\alpha_0 - \gamma) \right\} \cdot \phi = \]
\[ = -M \cdot g \cdot \frac{L_{\text{aw}}}{J_0} \cdot \sin(\alpha_0) + F_{\text{fr}} \cdot \frac{L_{\text{dr}}}{J_0} \cdot \cos(\alpha_0 - \gamma) - \frac{L}{J_0} \cdot F(t) \] \hspace{1cm} (2.1.3.4)

Substitution of the expressions
\[ M \cdot g \cdot \frac{L_{\text{aw}}}{J_0} \cdot \cos(\alpha_0) + F_{\text{fr}} \cdot \frac{L_{\text{dr}}}{J_0} \cdot \sin(\alpha_0 - \gamma) = A \] \hspace{1cm} (2.1.3.5)
\[ M \cdot g \cdot \frac{L_{\text{aw}}}{J_0} \cdot \sin(\alpha_0) - F_{\text{fr}} \cdot \frac{L_{\text{dr}}}{J_0} \cdot \cos(\alpha_0 - \gamma) = B \] \hspace{1cm} (2.1.3.6)
in equation (2.1.3.1) leads to the following form of the equation of motion of the clapper:
\[ \ddot{\phi} + A \cdot \phi = -B - \frac{L}{J_0} \cdot F(t) \] \hspace{1cm} (2.1.3.7)

The solution of this inhomogeneous ordinary differential equation of second order is derived in appendix A. The displacement of the contact point on the clapper in a direction perpendicular to the bell wall in the contact point is:
\[ W_{\text{clapp}}(t) = \cos(\beta_1) \cdot L \cdot \phi(t) = \]
\[ = W_{\text{clapp}}(t_b) \cdot \cos(\sqrt{A}(t-t_b)) + \frac{1}{\sqrt{A}} \cdot W_{\text{clapp}}(t_b) \cdot \sin(\sqrt{A}(t-t_b)) + \]
\[ - \cos(\beta_1) \cdot L \cdot \frac{B}{A} \cdot [1 - \cos(\sqrt{A}(t-t_b))] + \]
\[ + \frac{\cos(\beta_1)}{\sqrt{A}} \cdot \frac{L^2}{J_0} \cdot \cos(\sqrt{A}t) \cdot \int_{t=t_b}^{t} \sin(\sqrt{A} \tau) \cdot F(\tau) \, d\tau + \]
\[ - \frac{\cos(\beta_1)}{\sqrt{A}} \cdot \frac{L^2}{J_0} \cdot \sin(\sqrt{A}t) \cdot \int_{t=t_b}^{t} \cos(\sqrt{A} \tau) \cdot F(\tau) \, d\tau \] \hspace{1cm} (2.1.3.8)

Where initial velocity and pendulum frequency of the clapper are:
\[ \dot{W}_{\text{clapp}}(t) = L \cdot \dot{\phi}(t=t_b) \] \hspace{1cm} [m/s] \hspace{1cm} (2.1.3.9)
\[ \omega_{\text{pend}} = \sqrt{A} \] \hspace{1cm} [1/s] \hspace{1cm} (2.1.3.10)

Since we want to use Hertz law to solve the contact force \( F(t) \) we obtain, using equation (2.1.3.8) to describe the displacement of the contact point on the clapper, an implicit relation from which the contact force has to be solved. It will clearly be impossible to derive an analytical expression for the course of the contact force as a function of time.
In order to be able to solve the implicit relation for the contact force \( F(t) \), this implicit relation will be discretised, assuming a piecewise linear course of the contact force:

\[
F(\tau) = F_{i-1} + \frac{F_i - F_{i-1}}{\Delta t} \cdot \{\tau - (i-1) \cdot \Delta t\} \quad (2.1.3.11)
\]

for \((i-1) \cdot \Delta t \leq \tau \leq i \cdot \Delta t\)

where \( F_i = F(t_b + i \cdot \Delta t) \)

![Diagram showing piecewise linear course of contact force during collision.](Fig. 2.1.3.2)

The discretisations of the integrals in (2.1.3.8), using (2.1.3.11) for the unknown course of the contact force, are given in appendices B and C. Substituting these discretisations in (2.1.3.8) together with:

\[
t_b = n_b \Delta t
\]

\[
t = n \Delta t
\]

leads to the discretised form of equation (2.1.3.8):

\[
W_{\text{clapp}}(n \Delta t) =
\]

\[
\frac{1}{\sqrt{A}} \cdot \hat{W}_{\text{clapp}}(n_b \Delta t) \cdot \sin(\sqrt{A}(n-n_b) \Delta t) + \]

\[
- \cos(\beta_1) \cdot \frac{L B}{A} \cdot [1 - \cos(\sqrt{A}(n-n_b) \Delta t)] +
\]

\[
+ \cos(\beta_1) \cdot \frac{L^2}{A J_0} \left[ \cos(\sqrt{A}(n-n_b) \Delta t) + \frac{-1}{\sqrt{A} \cdot \Delta t} \cdot [\sin(\sqrt{A}(n-n_b) \Delta t) - \sin(\sqrt{A}(n-n_b-1) \Delta t)] \right] \cdot F_{n_b} +
\]

\[
- \cos(\beta_1) \cdot \frac{L^2}{A J_0} \left[ 1 - \frac{\sin(\sqrt{A} \cdot \Delta t)}{\sqrt{A} \cdot \Delta t} \right] \cdot F_n +
\]

\[
- \cos(\beta_1) \cdot \frac{2L^2}{\Delta t J_0 A \sqrt{A}} \cdot [1 - \cos(\sqrt{A} \cdot \Delta t)] \sum_{i=n_b+1}^{n-1} \sin(\sqrt{A}(n-i) \Delta t) \cdot F_i \quad (2.1.3.13)
\]
Equation (2.1.3.13) will be abbreviated to:

\[
W_{\text{clapp}}(n\Delta t) = W_{\text{clf}}(n) + W_{\text{cl}}(n_b) \cdot F_{n_b} - W_{\text{cl}}(n) \cdot F_n - \sum_{i=n_b}^{n-1} W_{\text{cl}}(i) \cdot \sin(\sqrt{K}(n-i)\Delta t) \cdot F_i
\]  

(2.1.3.14)

The summons in (2.1.3.14) vanishes when \( n - n_b \leq 2 \). Expression (2.1.3.14) will be used in the Hertz relation to solve \( F_n(t) = F(t)\cos(\beta t) \).
2.1.4 Displacement of contact point on the bell.

The displacement of the contact point on the bell will be computed using a FEM program Dynopt (van Asperen [1984]), able to analyse the eigenfrequencies and eigenmodes of axi-symmetrical structures. In order to obtain an expression of the displacement of the contact point on the bell as a function of the unknown course of the contact force, a number of assumptions has been made.

\[ W_{\text{bell}}(t) = q_r(c,t) \cdot \cos(\beta) + q_\phi(c,t) \cdot 0 + q_z(c,t) \cdot \sin(\beta) = n(c)^t \cdot q(c,t) \]

Where:

- the normal vector in \( c \) is: \( n(c)^t = [\cos(\beta), 0, \sin(\beta)] \) (2.1.4.2)
- the coordinates of the contact point: \( c^t = [r_c, \phi_c, z_c] \) (2.1.4.3)

Contact point \( c \) and normal \( n(c) \) are determined in appendix F.

\* The displacement of the contact point on the bell in the direction of the normal to the bell surface in this point is (see fig. 2.1.4.1):

\[ W_{\text{bell}}(t) = q_r(c,t) \cdot \cos(\beta) + q_\phi(c,t) \cdot 0 + q_z(c,t) \cdot \sin(\beta) = n(c)^t \cdot q(c,t) \]

Where:

- the normal vector in \( c \) is: \( n(c)^t = [\cos(\beta), 0, \sin(\beta)] \) (2.1.4.2)
- the coordinates of the contact point: \( c^t = [r_c, \phi_c, z_c] \) (2.1.4.3)

Contact point \( c \) and normal \( n(c) \) are determined in appendix F.

\* The displacement \( q(c,t) \) of the contact point \( c \) on the bell can be written as a linear combination of the displacements of the nodal circles \( q_e(\phi_c,t) \) of the circular element on which the contact point \( c \) lies:

\[ q(c,t) = \begin{bmatrix} q_r(c,t) \\ q_\phi(c,t) \\ q_z(c,t) \end{bmatrix} = \Phi(r_c,z_c) \cdot q_e(\phi_c,t) \]

Where:

- \( q_e(\phi_c,t) \) are the displacements of the element nodal circles (3n_e x 1)
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$n_e$ is the number of nodal circles of the element

$$\Phi(r_c, z_c) = \begin{bmatrix} N^t & 0^t & 0^t \\ 0^t & N^t & 0^t \\ 0^t & 0^t & N^t \end{bmatrix} (\xi_c, \eta_c) \quad (3 \times 3n_e) \quad (2.1.4.5)$$

The element interpolation functions $N^t = [N_1, \ldots, N_n]$ are described in local coordinates: $N = N(\xi, \eta)$. The local coordinates $(\xi_c, \eta_c)$ of the contact point, corresponding with the global ones $(r_c, \phi_c, z_c)$, have to be determined (appendix F).

* Between element nodal displacements and construction nodal displacements $q(\phi, t)$ the following relation exists:

$$q_e(\phi, t) = \begin{bmatrix} q_{re}(\phi, t) \\ q_{\phi e}(\phi, t) \\ q_{ze}(\phi, t) \end{bmatrix} = L_e \cdot q(\phi, t) \quad (3n_e \times 1) \quad (2.1.4.6)$$

where:

$L_e$ is the location matrix connecting the displacements of the element nodal circles to the construction nodal circles, $L_e (3n_e \times 3n)$

$n =$ number of construction nodal circles

* Since the FEM program Dynopt uses axi–symmetrical Fourier elements to model axi–symmetrical structures (like bells are), the displacements of construction nodal circles are described using a Fourier series in $\phi$ :

$$q(\phi, t) = \begin{bmatrix} q_r(\phi, t) \\ q_\phi(\phi, t) \\ q_z(\phi, t) \end{bmatrix} = \begin{bmatrix} q_{r0}(t) \\ q_{\phi0}(t) \\ q_{z0}(t) \end{bmatrix} + \sum_{m=1}^{\infty} \begin{bmatrix} q_{\ell m}(t)\cos(m\phi) \\ q_{\phi m}(t)\sin(m\phi) \\ q_{z m}(t)\cos(m\phi) \\ q_{\phi m}(t)\cos(m\phi) \\ q_{z m}(t)\sin(m\phi) \end{bmatrix} \quad (2.1.4.7a)$$

$$q_0(t) = \begin{bmatrix} q_{r0}(t) \\ q_{\phi0}(t) \\ q_{z0}(t) \end{bmatrix}, \quad q_{\ell m}(t) = \begin{bmatrix} q_{\ell m}(t) \\ q_{\phi m}(t) \\ q_{z m}(t) \end{bmatrix}, \quad q_{\phi m}(t) = \begin{bmatrix} q_{\phi m}(t) \\ q_{\phi m}(t) \\ q_{z m}(t) \end{bmatrix}, \quad q_{z m}(t) = \begin{bmatrix} q_{z m}(t) \\ q_{z m}(t) \\ q_{z m}(t) \end{bmatrix}, \quad (3n \times 1) \quad (2.1.4.7b)$$

The Fourier coefficients $q_{r0}(t), q_{\phi m}(t), q_{z m}(t)$ represent respectively the breathing mode, symmetrical and anti metrical mode. The radial components occurring at nodal circles, are shown in fig. 2.1.4.2. The Fourier coefficients are calculated in the FEM program Dynopt (van Asperen [1984]) from a set equations of motion for a free vibrating undamped system, derived for the eigenmodes within each different node mode group:

$$M \cdot q_0(t) + K_0 \cdot q_0(t) = f_0(t) \quad (2.1.4.8a)$$

$$M \cdot q_0(t) + K_0 \cdot q_0(t) = f_0(t) \quad (2.1.4.8b)$$
Before the equations of motion are solved, they are partituted in Dynopt to reduce
the dimension of the eigenvalue problem related to it:

\[
\begin{bmatrix}
\mathbf{M}_{uu} & \mathbf{M}_{uk} \\
\mathbf{M}_{ku} & \mathbf{M}_{kk}
\end{bmatrix}
\begin{bmatrix}
\ddot{\mathbf{q}}_u(t) \\
\ddot{\mathbf{q}}_k(t)
\end{bmatrix}
+
\begin{bmatrix}
\mathbf{K}_{uu} & \mathbf{K}_{uk} \\
\mathbf{K}_{ku} & \mathbf{K}_{kk}
\end{bmatrix}
\begin{bmatrix}
\mathbf{q}_u(t) \\
\mathbf{q}_k(t)
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{f}_u(t) \\
\mathbf{f}_k(t)
\end{bmatrix}
\quad j \in [s,a]
\]  

(2.1.4.9a)

Where

\[
\begin{align*}
\mathbf{q}_u(t) &= \text{unknown construction nodal displacements} \\
\mathbf{q}_k(t) &= \text{known construction nodal displacements} \\
\mathbf{f}_u(t) &= \text{known construction nodal forces} \\
\mathbf{f}_k(t) &= \text{unknown construction nodal forces}
\end{align*}
\]  

The partitution (2.1.4.9) distorts the previous used description (2.1.4.7). Since this
transformation is only executed in the program and for reasons of clarity, the latter
description is used in the rest of the response calculation.

(2.1.4.9d) contains the contact force that has to be calculated, distributed over the
construction nodal circles. In Dynopt however the eigenmodes of the free vibrating
undamped system are computed from:

\[
-\omega^2 \mathbf{M} + \mathbf{K} \mathbf{x} = 0. \quad \text{nundof equations}
\]  

(2.1.4.11)

Nundof solutions for the eigenfrequencies and eigenmodes are found:

\[
\omega^2 = [\omega_1, \ldots, \omega_{\text{nundof}}] \quad \text{and} \quad \mathbf{X}_m = [x_1, \ldots, x_{\text{nundof}}]_m \quad (\text{nundof} \times \text{nundof})
\]

Important note:
The number of components of each eigenmode in dynopt is filled out to
becomes: \[x_{m} = [x_1, \ldots, x_{\text{nundof}}]_m \quad (3n \times \text{nundof})
\]  

(2.1.4.12)

As a direct consequence of the use of a Fourier series to describe the displacements of
the construction nodal circles in the \(\phi\)-direction, the nodal forces exciting the model
must also be expressed in a Fourier series:

\[
\begin{bmatrix}
\mathbf{f}_{x}(\phi,t) \\
\mathbf{f}_{\phi}(\phi,t) \\
\mathbf{f}_{z}(\phi,t)
\end{bmatrix}
= \begin{bmatrix}
\mathbf{f}_{x0}(t) \\
\mathbf{f}_{\phi0}(t) \\
\mathbf{f}_{z0}(t)
\end{bmatrix}
+ \sum_{m=1}^{\infty}
\begin{bmatrix}
\mathbf{f}_{xm}(t)\cos(m\phi) \\
\mathbf{f}_{\phi m}(t)\sin(m\phi) \\
\mathbf{f}_{zm}(t)\cos(m\phi)
\end{bmatrix}
\]

(2.1.4.13a)

where Fourier coefficients are:
\[ f_0(t) = \begin{bmatrix} f_{r0}(t) \\ f_{\phi 0}(t) \\ f_{z0}(t) \end{bmatrix}, \quad f_{n}(t) = \begin{bmatrix} f_{rn}(t) \\ f_{\phi n}(t) \\ f_{zn}(t) \end{bmatrix}, \quad f_{m}(t) = \begin{bmatrix} f_{rm}(t) \\ f_{\phi m}(t) \\ f_{zm}(t) \end{bmatrix}, \quad (3n \times 1) \quad (2.1.4.13b) \]

* The nodal forces \( f(\phi,t) \) have to be expressed in the unknown contact force \( F(t) \) before the equations of motion (2.1.4.8) can be solved.

* The components of the contact force \( F(t) \) are:

\[
F(t) = \begin{bmatrix} F_r(t) \\ F_\phi(t) \\ F_z(t) \end{bmatrix} = F(t) \begin{bmatrix} \cos(\beta_2) \\ 0 \\ \sin(\beta_2) \end{bmatrix}, \quad F(t) = \|F(t)\| \quad (2.1.4.14)
\]

The contact force \( F(t) \) with magnitude \( F(t) \) has no tangential component \( F_\phi(t) \) since the collision is assumed to be perpendicular to the \( \phi \)-direction.

* In the vicinity of the contact point a force distribution for the contact force is assumed on the bell wall. In the \( r-z \) plane the contact force is described as a point force:

\[
p(r,z) = \frac{1}{r_c} \delta(r-r_c, z-z_c) \quad (2.1.4.15)
\]

In the \( r-z \) plane the equivalent element nodal forces are calculated for the distributed force. Since the contact area between the clapper and the bell is much smaller than the characteristic element size, the precise form of the force distribution in the \( r-z \) plane is not important. For simplicity the dirac function was chosen.

In the \( \phi \)-direction a force distribution \( \text{shape}(\phi) \) of a half wave form of the cosine is used. This form can be described accurately with fewer terms of the Fourier series than the dirac function \( \delta(\phi) \), or any other possible force distribution.

\[
\text{shape}(\phi) = \begin{cases} 2\cos[i \cdot (\phi-\phi_c)] & \phi_c-\pi/2i \leq \phi \leq \phi_c+\pi/2i \\ 0 & \text{elsewhere} \end{cases} \quad (2.1.4.17)
\]

The distributions \( p(r,z) \) and \( \text{shape}(\phi) \) are scaled in such a way that the integral of the force distribution \( p(r,z) \cdot \text{shape}(\phi) \) over the surface \( A_e \) of the element is equal to 1.

\[
\int_{A_e} \int_{\phi=\phi_c-\pi/2i}^{\phi_c+\pi/2i} p(r,z) \cdot \text{shape}(\phi) \cdot r(z) d\phi \cdot dz = 1
\]

The value of \( i \) will be chosen so that the force is distributed over a small part of the circumference of the bell. (For instance, for \( i=18 \) over 1/36th part.) The influence of parameter \( i \) is discussed in paragraph 2.3.

* When the contact point on the bell doesn't coincide with a nodal circle, equivalent
element nodal forces have to be computed (Przemieniecki [1968]). For the nodal circle \( j \) of the element \( e \), the equivalent nodal forces are:

\[
\mathbf{t}_j(\phi_c, t) = \int_{A_e} F(t) \cdot N_j(\xi, \eta) \cdot p(r, z) \cdot \text{shape}(\phi) \, dA = \\
= F(t) \begin{bmatrix}
\cos(\beta_2) \\
0 \\
\sin(\beta_2)
\end{bmatrix} \cdot \int_{\phi=\phi_c-\pi/2i}^{\phi=\phi_c+\pi/2i} \text{shape}(\phi) \, d\phi \cdot \int_{z_e} N_j(\xi, \eta) \cdot p(r, z) \cdot r(z) \, dz = \\
= F(t) \begin{bmatrix}
\cos(\beta_2) \\
0 \\
\sin(\beta_2)
\end{bmatrix} \cdot N_j(\xi_c, \eta_c) = F(t) \cdot \mathbf{V}_e^t 
\]

\[(2.1.4.18)\]

\[
\text{since} \quad \int_{\phi=\phi_c-\pi/2i}^{\phi=\phi_c+\pi/2i} \frac{1}{2} \cos[i \cdot (\phi-\phi_c)] \, d\phi = 1 
\]

\[(2.1.4.19a)\]

and

\[
\int_{z_e} N_j(\xi, \eta) \cdot \frac{1}{r_c} \cdot \delta(r-r_c, z-z_c) \cdot r(z) \, dz = N_j(\xi_c, \eta_c) 
\]

\[(2.1.4.19b)\]

Notice that \( N_j(\xi_c, \eta_c) = 1 \) when the contact point \( c \) is lying on nodal circle \( j \). (The interpolation functions aren't given an upper index \( e \), since the functions are identical for all elements of the same type (see appendix F).

The element nodal forces are composed as:

\[
f^e(\phi_c, t) = F(t) \cdot \begin{bmatrix}
V_1r \ldots V_{ne}r; \ V_1\phi \ldots V_{ne}\phi; \ V_{1z} \ldots V_{nez}
\end{bmatrix} = F(t) \cdot \mathbf{V}_e^t 
\]

\[(2.1.4.20)\]

where:

\( \mathbf{V} (3ne \times 1) \) contains relative equivalent construction nodal forces. \[(2.1.4.21)\]

Only the \( N_j(\xi, \eta) \) are non zero for nodal points \( j \) on the same element side as the contact point is lying on. If \( c \) is sited on element side 1–7–8 of a QUAX8 element (see appendix F), \( f^e(\phi_c, t) \) becomes:

\[
f^e(\phi_c, t) = \\
F(t) \begin{bmatrix}
N_1\cos(\beta_2), \ldots, N_T\cos(\beta_2), N_8(\beta_2) ; 0, \ldots, 0 ; N_1\sin(\beta_2), \ldots, N_T\sin(\beta_2), N_8\sin(\beta_2)
\end{bmatrix}(\xi_c, \eta_c) 
\]

In analogy with the nodal displacements the element nodal forces can be connected to the construction nodal forces using the location matrix:

\[
\mathbf{f}^e(\phi_c, t) = \mathbf{L}_e^e \cdot \mathbf{f}(\phi_c, t) 
\]

\[(2.1.4.22a)\]

\[
\mathbf{f}(\phi_c, t) = F(t) \cdot \mathbf{V} (3n \times 1) 
\]

\[(2.1.4.22b)\]

\[
\mathbf{V} = \mathbf{L}_e^e \cdot \mathbf{V} 
\]

\[(2.1.4.22c)\]

where: \( \mathbf{V} (3n \times 1) \) is the column with relative equivalent construction nodal forces.
\[ f(\phi, t) = f(\phi_c, t) \cdot \text{shape}(\phi) \quad (3n \times 1) \]

Using (2.1.4.22b) and (2.1.4.13) the construction nodal forces are written as:

\[
\begin{bmatrix}
  f_r(t) \\
  f_\phi(t) \\
  f_z(t)
\end{bmatrix} =
\begin{bmatrix}
  f_{r0}(t) \\
  f_{\phi0}(t) \\
  f_{z0}(t)
\end{bmatrix} + \sum_{m=1}^{\infty} \begin{bmatrix}
  f_{r(m)}(t) \cos(m\phi) + f_{\phi(m)}(t) \sin(m\phi) \\
  f_{\phi(m)}(t) \sin(m\phi) + f_{z(m)}(t) \cos(m\phi) \\
  f_{z(m)}(t) \cos(m\phi) + f_{\phi(m)}(t) \sin(m\phi)
\end{bmatrix}
\]

(2.1.4.23)

The Fourier coefficients in equation (2.1.4.23) are:

\[
f_0(t) = \begin{bmatrix}
  f_{r0}(t) \\
  f_{\phi0}(t) \\
  f_{z0}(t)
\end{bmatrix} = \frac{F(t)}{2\pi} \cdot \int_{-\pi}^{\pi} \text{shape}(\phi) d\phi = F(t) \cdot \text{SFC}_0 \cdot \text{shape}(\phi) \quad (2.1.4.24a)
\]

\[
f_{m}^{(r)}(t) = \begin{bmatrix}
  f_{r,m}^{(r)}(t) \\
  f_{\phi,m}^{(r)}(t) \\
  f_{z,m}^{(r)}(t)
\end{bmatrix} = \frac{F(t)}{2\pi} \cdot \int_{-\pi}^{\pi} \begin{bmatrix}
  v_r(\text{SFC})_m^{(r)} \\
  v_\phi(\text{SFC})_m^{(r)} \\
  v_z(\text{SFC})_m^{(r)}
\end{bmatrix} \cdot \text{shape}(\phi) d\phi = F(t) \cdot \begin{bmatrix}
  v_r(\text{SFC})_m^{(r)} \\
  v_\phi(\text{SFC})_m^{(r)} \\
  v_z(\text{SFC})_m^{(r)}
\end{bmatrix} \quad (2.1.4.24b)
\]

\[
f_{m}^{(\phi)}(t) = \begin{bmatrix}
  f_{r,m}^{(\phi)}(t) \\
  f_{\phi,m}^{(\phi)}(t) \\
  f_{z,m}^{(\phi)}(t)
\end{bmatrix} = \frac{F(t)}{2\pi} \cdot \int_{-\pi}^{\pi} \begin{bmatrix}
  v_r(\text{SFC})_m^{(\phi)} \\
  v_\phi(\text{SFC})_m^{(\phi)} \\
  v_z(\text{SFC})_m^{(\phi)}
\end{bmatrix} \cdot \text{shape}(\phi) d\phi = F(t) \cdot \begin{bmatrix}
  v_r(\text{SFC})_m^{(\phi)} \\
  v_\phi(\text{SFC})_m^{(\phi)} \\
  v_z(\text{SFC})_m^{(\phi)}
\end{bmatrix} \quad (2.1.4.24c)
\]

* Substituting equations (2.1.4.24abc) into equation (2.1.4.23) we obtain the construction nodal forces as a function of the magnitude \( F(t) \) of the contact force:

\[
f(\phi, t) = F(t) \cdot \begin{bmatrix}
  (\text{SFC})_0^{(r)} \\
  \sum_{m=1}^{\infty} \begin{bmatrix}
    (\text{SFC})_{r,m}^{(r)}(t) \cos(m\phi) + (\text{SFC})_{\phi,m}^{(r)}(t) \sin(m\phi) \\
    (\text{SFC})_{\phi,m}^{(r)}(t) \sin(m\phi) + (\text{SFC})_{z,m}^{(r)}(t) \cos(m\phi) \\
    (\text{SFC})_{z,m}^{(r)}(t) \cos(m\phi) + (\text{SFC})_{\phi,m}^{(r)}(t) \sin(m\phi)
  \end{bmatrix} \\
  (\text{SFC})_0^{(\phi)}
\end{bmatrix} \cdot \text{shape}(\phi) \quad (2.1.4.25)
\]

* Without loss of generality, the \( \phi \)-coordinate of contact point \( \text{c} \) can be chosen equal to zero:

\[
\phi_c = 0 \quad (2.1.4.26)
\]

If \( \phi_c = 0 \), only eigenmodes symmetric with respect to the \( \phi \)-axis are excited, thus reducing the number of modes necessary to describe the displacement of the bell.

Since the \( \phi \)-coordinate of the contact point was chosen \( \phi_c = 0 \), the antimetric eigenmodes will not be excited by the collision between bell and clapper. Therefore the antimetric shape factors \( \text{SFC}_m^{(a)} \) will be equal to zero.
Substituting \((SFC)_{m} = 0\) in expression (2.1.4.25) for the construction nodal forces, we obtain the following expression for the equivalent construction nodal forces:

\[
f(t) = F(t) \cdot \chi \cdot ((SFC)_0 + \sum_{m=1}^{\infty} (SFC)_m(t) \cos(m\phi))
\]

(2.1.4.27)

Here the choice of a cosine form for the force distribution shape(\(\phi\)) becomes obvious. The constructions nodal forces \(f(\phi,t)\) can be approximated by a limited number of terms in the Fourier series of equation (2.1.4.27). For the value of \((SFC)_0\) and \((SFC)_m\) we find

\[
(SFC)_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{shape}(\phi) \, d\phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \cos(i\phi) \, d\phi = \frac{1}{2\pi}
\]

(2.1.4.28a)

\[
(SFC)_m = \frac{1}{4\pi} \int_{-\pi}^{\pi} \cos(m\phi) \cdot \text{shape}(\phi) \, d\phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(m\phi) \cdot \frac{1}{2} \cos(i\phi) \, d\phi =
\]

\[
= \frac{1}{4\pi} \left[ \frac{\sin((m+1)\phi)}{m+1} + \frac{\sin((m-1)\phi)}{m-1} \right] \bigg|_{-\pi/2i}^{\pi/2i} = \frac{1}{\pi} \cdot \frac{1}{1-(\pi/2i)^2} \cdot \cos\left(\frac{m\pi}{2i}\right) \quad \text{for } m \neq i
\]

(2.1.4.28b)

\[
(SFC)_m = \frac{1}{4\pi} \int_{-\pi}^{\pi} \sin(m\phi) \cdot \text{shape}(\phi) \, d\phi = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(m\phi) \cdot \frac{1}{2} \cos(i\phi) \, d\phi = 0 \quad \forall m
\]

(2.1.4.28c)

For increasing values of \(m\), \((SFC)_m\) decreases (see paragraph 2.3). The correct value of \(m\) has to be determined in practice, using a convergence criterion.

Damping is introduced in the equations of motion for the undamped vibrating system. A bell is regarded as a weak damped system, which means that eigenmodes of the damped system and eigenmodes of the undamped system are almost alike. Responsvectors of the damped system are modally written in eigenmodes of the undamped system:

\[
g_0(t) = X_0 \cdot s_0(t) = \sum_{i=1}^{ne} x_{0,i} \cdot s_{0,i}(t)
\]

(2.1.4.29a)

\[
g_m(t) = X_m \cdot s_m(t) = \sum_{i=1}^{ne} x_{m,i} \cdot s_{m,i}(t)
\]

(2.1.4.29b)

\[
g_m(t) = X_m \cdot s_m(t) = \sum_{i=1}^{ne} x_{m,i} \cdot s_{m,i}(t)
\]

(2.1.4.29c)

where
\( x_{k,m}^i \) is the eigenmode of the axi-symmetrical structure characterized by node mode \( m \), construction mode \( i \)

\( k=s \) symmetrical node mode, \( k=a \) anti metrical node mode

\( s_{k,m}^i(t) \) is the corresponding mode participation factor, i.e. the time dependent amplitude of an eigenmode in the dynamic response of an axi-symmetric structure

\( n_e = \) number of construction modes \( \leq \) nundof, used in the description of the response vector of each node mode. In paragraph 2.3 a criterion is discussed on what basis a range can be chosen \( i = 1,...,n_e \) sufficient for a response description within certain limits of accuracy.

The type of damping that is introduced is viscous damping. The main characteristic of weak damping is:

\[
\begin{align*}
X_m^i \cdot B_m^i \cdot X_m \approx r^i b_{m,i}^i
\end{align*}
\]

This means that in computing the eigenvalues, changed with respect to the undamped system, only the parameters on the diagonal are taken in account.

Equations of motion now become:

\[
\begin{align*}
M \cdot \dddot{q}_0(t) + B_0 \cdot \dot{q}_0(t) + K_0 \cdot q_0(t) &= f_0(t) \quad (2.1.4.31a) \\
M \cdot \dddot{q}_m^a(t) + B_m^a \cdot \dot{q}_m^a(t) + K_m^a \cdot q_m^a(t) &= f_m^a(t) \quad (2.1.4.31b) \\
M \cdot \dddot{q}_m^s(t) + B_m^s \cdot \dot{q}_m^s(t) + K_m^s \cdot q_m^s(t) &= f_m^s(t) \quad (2.1.4.31c)
\end{align*}
\]

Since the eigenmodes of the undamped system meet the next equation:

\[
\begin{align*}
(x_{k,m}^s)^t M x_{k,m}^s = (\omega_{k,m}^s)^2 \\
(x_{k,m}^a)^t M x_{k,m}^a = m_{k,m} \\
\frac{b_{k,m}^s}{2m_{k,m}} \omega_{k,m}^s = \zeta_{k,m}^s
\end{align*}
\]

Substituting (2.1.4.29) into (2.1.4.31) using (2.1.4.24), (2.1.4.30) and (2.1.4.32) results in:

\[
\begin{align*}
\ddot{s}_{0,i}(t) + 2\zeta_{0,i} \omega_{0,i} \cdot \dot{s}_{0,i}(t) + (\omega_{0,i})^2 \cdot s_{0,i}(t) &= F(t) \cdot (x_{0,i})^t \cdot y \cdot (SFC)_0 \\
\ddot{s}_{m,i}^s(t) + 2\zeta_{m,i}^s \omega_{m,i}^s \cdot \dot{s}_{m,i}^s(t) + (\omega_{m,i}^s)^2 \cdot s_{m,i}^s(t) &= F(t) \cdot (x_{m,i}^s)^t \cdot y \cdot (SFC)_m^a \\
\ddot{s}_{m,i}^a(t) + 2\zeta_{m,i}^a \omega_{m,i}^a \cdot \dot{s}_{m,i}^a(t) + (\omega_{m,i}^a)^2 \cdot s_{m,i}^a(t) &= F(t) \cdot (x_{m,i}^a)^t \cdot y \cdot (SFC)_m^s
\end{align*}
\]

The solution of equations (2.1.4.33a,b,c) is presented by the so-called Duhamel integral:

\[
\begin{align*}
s_{k,m}^i(t) &= e^{-\zeta_{m,i}^k \omega_{m,i}^k t} \left\{ s_{m,i}(t_b) \cos(\omega_{m,i}^k(t-t_b)) + \frac{\ddot{s}_{m,i}^k(t_b) + \zeta_{m,i}^k}{\omega_{m,i}^k} \sin(\omega_{m,i}^k(t-t_b)) \right\} + \\
&\quad + e^{-\zeta_{m,i}^a \omega_{m,i}^a t} \left\{ s_{m,i}(t_b) \cos(\omega_{m,i}^a(t-t_b)) + \frac{\ddot{s}_{m,i}^a(t_b) + \zeta_{m,i}^a}{\omega_{m,i}^a} \sin(\omega_{m,i}^a(t-t_b)) \right\}
\end{align*}
\]
$$+\frac{1}{\omega_{m,i}} \int_{t_a}^{t_b} F(t) e^{-\zeta_{m,i}^k (t-\tau)} \sin(\omega_{m,i}^k t-\tau) d\tau \quad k \in [s,a]$$

Since \((SFC)_{x}^{a} = 0\) the anti-metrical modes are not excited.
2.2 Response to a prescribed input.

After the calculation of the dynamic response to a stroke with a clapper had been implemented, it was decided to implement the response calculation to a prescribed input as well. This part of the programme was developed for two reasons:

- The wish to be able to compute the response of a bell on an arbitrary input force, prescribed as a function of time.
- Creating the possibility to compare the results of the collision calculations with more sophisticated numerical integration techniques.

In the last case the course of the contact force during collision as computed with our collision calculation, serves as input for numerical integration routines of the NAG library [NAG Mk13].

Resuming the equations of motion for each node mode:

\[
M \ddot{q}_m(t) + B_m \dot{q}_m(t) + K_m q_m(t) = \sum_{i=1}^{n_{undof}} X_{i,m} \cdot s_i(t) = F_m(t) \cdot \text{(SFC)} = \chi_m, \quad m = 0, \ldots, \infty
\]  

(2.2.1)

The modal description of the displacements for this node mode only:

\[
q_m(t) = \sum_{i=1}^{n_{undof}} X_{i,m} \cdot s_i(t) = X_m \cdot \chi_m \quad (3n \times 1)
\]  

(2.2.2)

Where

\[
n_{undof} = \text{number of unknown degrees of freedom of the structure} = 3n - \text{number of boundary conditions}
\]  

(2.2.3)

is the matrix containing eigenmodes as computed by Dynopt from the free vibrating undamped system:

\[
M \ddot{q}_m(t) + K_m q_m(t) = 0
\]  

(2.2.4)

Important note:

When it comes to solution of the equations of motion, following partition is executed:

\[
\begin{bmatrix}
M_{uu} & M_{uk} \\
M_{ku} & M_{kk}
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_u(t) \\
\ddot{q}_k(t)
\end{bmatrix}
+ \begin{bmatrix}
K_{uu} & K_{uk} \\
K_{ku} & K_{kk}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_u(t) \\
\dot{q}_k(t)
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  

(2.2.5)

Where \(q_u(t)\) = unknown construction nodal displacements \((n_{undof} \times 1)\)

\(q_k(t)\) = known construction nodal displacements \((n_{bound} \times 1)\)

Dynopt solves the set of \(n_{undof}\) equations:

\[
M_{uu} \ddot{q}_u(t) + K_{uu} q_u(t) = 0  
\]  

(2.2.6)
Assuming the structure is to be described adequately by a weak damped system, we find:

\[ X^t_m B_m \cdot X_m = r_{i,m_j} b_{i,m_j} \quad i = 1, \ldots, \text{neigen} \]  

(2.2.7)

Since weak damping is assumed, the off-diagonal terms of the matrix \( X^t_m B_m X_m \) are small with respect to the terms on the diagonal. The off-diagonal terms will be assumed to be zero, so that matrix \( X^t_m B_m X_m \) is a diagonal matrix.

Substituting (2.2.2) in (2.2.1), and premultiplying by \( X_m^t \) results in:

\[ X_m^t M_m X_m \cdot \hat{g}_m (t) + X_m^t B_m X_m \cdot \hat{g}_m (t) + X_m^t K_m X_m \cdot \hat{g}_m (t) = X_m^t \cdot \chi(SFC)_m \cdot F_n (t) \]

(2.2.8)

Using damping ratio: \( \xi_{i,m} = \frac{b_{i,m}}{b_{i,m \text{ critical}}} = \frac{b_{i,m}}{2 m_{i,m} \omega_{i,m}} \quad i = 1, \ldots, \text{neigen} \)

(2.2.9)

undamped natural frequency: \( \omega_{i,m} = \sqrt{\frac{k_{i,m}}{m_{i,m}}} \quad [s^{-1}] \)

(2.2.10)

and

\[ C_{i,m} = \frac{1}{m_{i,m}} \cdot X_m^t \cdot \chi(SFC)_m \]

(2.2.11)

in (2.2.8) results in:

\[ \hat{g}_{i,m} (t) + 2 \xi_{i,m} \cdot \omega_{i,m} \cdot \hat{g}_{i,m} (t) + \omega_{i,m}^2 \cdot \hat{g}_{i,m} (t) = C_{i,m} \cdot F_n (t) \]

(2.2.12)

Equations (2.2.12) now have to be transformed to a form applicable for numerical integration routines.

Where \( y(2 \cdot i - 1) = \hat{g}_{i,m} (t) \)

\[ y(2 \cdot i) = \hat{g}_{i,m} (t) \quad i = 1, \ldots, \text{neigen} \]

(2.2.13)

we can introduce for each node mode \( m \) a response solution column matrix:

\[ y^t_m = [s_{i,m} (t) \quad \hat{g}_{i,m} (t) \quad \hat{g}_{i,m} (t) \quad \hat{g}_{i,m} (t)] \]

\[ = [y_1 \quad y_2 \quad \ldots \quad y(2 \cdot \text{neigen} - 1) \quad y(2 \cdot \text{neigen})] \]

(2.2.14)

Substituting (2.2.14) in (2.2.12) results in:

\[
\dot{y} = \begin{bmatrix}
0 & -2 \xi_{1,m} \cdot \omega_{1,m} & 1 \\
-\omega_{1,m}^2 & -2 \xi_{1,m} \cdot \omega_{1,m} & 1 \\
-\omega_{ne,m}^2 & -2 \xi_{ne,m} \cdot \omega_{ne,m} & 1 \\
-\omega_{ne,m}^2 & -2 \xi_{ne,m} \cdot \omega_{ne,m} & 1
\end{bmatrix} \cdot y + \begin{bmatrix}
0 \\
C_{1,m} \\
0 \\
C_{ne,m}
\end{bmatrix} \cdot F_n (t)
\]

(2.2.15)
The computed contact force $F_n(t)$ or a prescribed input can be used as input for the response calculation. The place of application of the force on the bell influences $v$, the values of $y$ that are non-zero depend on the angle $\beta$ (see paragraph 2.1.4). Parameter $i$ has influence on $C_{isn}$ through $(SFC)_m$.

When a check on the already computed response to a collision is executed, NAG routines calls in values of contact forces at times lying between two discrete times at which forces are computed. These routines use a much smaller stepsize $\Delta t$ than we did in our calculation of the collision. The size of each step in numerical integration is decreased to gain a solution within a relative accuracy, given at the call of a routine. In that case linear interpolation will be used (fig. 2.2.1) since this course also was assumed in the collision calculation.
2.3  Criterion for the number of eigenmodes necessary in the response calculation.

Resuming the equations of motion for the forced vibration of a damped system

\[ M \ddot{q}_m(t) + B \dot{q}_m(t) + K q_m(t) = f_m(t) = (SFC)_m \cdot y \cdot F_n(t) \]  

(2.3.1)

The response vector \( q_m(t) \) for each node mode \( m \) is written modally, using the eigenvectors of the undamped system

\[ q_m(t) = \sum_{i=1}^{n_m} x_{mi} \cdot s_{mi}(t) = X_m \cdot s_m(t) \]  

(2.3.2)

The assumption that the system is slightly damped justifies the use of the eigenmodes of the undamped system in equation (2.3.2). The eigenmodes of the undamped system are orthogonal with respect to the mass matrix \( M \) and the stiffness matrix \( K \):

\[ X^t \cdot M \cdot X = \delta_{mi} \]

\[ X^t \cdot K \cdot X = \delta_{mi} \cdot \omega_{mi}^2 \]

and thus we find

\[ K \cdot X = M \cdot X \cdot \omega_{mi}^2 \]  

(2.3.3)

For the damping matrix of a slightly damped system we find

\[ X^t \cdot B \cdot X \approx \delta_{mi} \cdot \zeta \cdot \omega_{mi} \]

thus

\[ B \cdot X = M \cdot X \cdot \zeta \cdot \omega_{mi} \]  

(2.3.4)

Substitution of (2.3.3) and (2.3.4) in equation (2.3.1), using relation (2.3.2) gives

\[ M \cdot X \cdot [s_m(t) + \delta_{mi} \cdot \zeta \cdot \omega_{mi} \cdot s_m(t) + \delta_{mi} \cdot \omega_{mi}^2 \cdot s_m(t)] = f_m(t) \]  

(2.3.5)

This equation can also be expressed as

\[ M \cdot \sum_{i=1}^{n_m} \left[ s_{mi}(t) + 2 \zeta \omega_{mi} \cdot s_{mi}(t) + \omega_{mi}^2 \cdot s_{mi}(t) \right] \cdot x_{mi} = f_m(t) \]  

(2.3.6)

The problem now is to determine how many node modes \( m = \text{MFIRST}, \ldots, \text{MLAST} \) and how many eigenmodes \( i = 1, \ldots, n_m \) (for each node mode \( m \)) should be chosen in the calculation of the dynamic response.

In section 2.1.4 for the shape function \( (SFC)_m \) was derived

\[ (SFC)_m = \frac{1}{\pi} \cdot \frac{1}{1 - (m/j)^2} \cdot \cos\left(\frac{m \cdot \pi}{2j}\right) \]

for \( m \neq j \)  

(2.3.7)

\[ = \begin{cases} \frac{1}{4} & \text{for } m \neq j, \\ \frac{1}{2\pi} & \text{for } m = 0 \end{cases} \]

and for the chosen shape \( \phi = \frac{1}{2} \cdot \cos(\phi) \)

\[ -\frac{\pi}{2j} \leq \phi \leq \frac{\pi}{2j} \]

\( (SFC)_m \) can be regarded as the fraction of the normal component \( F_n(t) \) of the excitation.
force $F(t)$ that excites the equation of motion for node mode $m$ (2.3.5). When a bell is
excitated with a clapper the node mode $m=0$ will not be excitated.

The choice of the range of node mode $m = \text{MFIRST}, \ldots, \text{MLAST}$ will influence both the
computed contact force $F_n(t)$ and the displacements of the bell, since the contact force and
the displacements of bell and clapper are related by Hertz law of contact.

In figure 2.3.1 the fractions $(SFC)_m$ of the contact force $F_n(t)$ are visualized as a function of
node mode number $m$, for different values of the parameter $j$ in expression (2.3.7)

![Figure 2.3.1](image)

Figure 2.3.1 Value of the fractions $(SFC)_m$ as a function of the node mode number $m$, for different values of force distribution parameter $j$.

It is obvious that the distribution of the contact force over a smaller circumferential angle
$\phi$, $\frac{-\pi}{2J} \leq \phi \leq \frac{\pi}{2J}$, causes higher node modes $m$ to contribute more to the course of the
contact force $F_n(t)$. The precise effect of the chosen number of node modes has to be
subject of further investigations.

Once the value MLAST for the node mode number $m$ has been chosen, it is important to
determine the relative contribution of the $p$ highest construction modes to the kolom with
the equivalent global nodal forces for node mode $m$, $f_m(t)$, for every node mode number $m$.
The relative contribution will be used as a criterion to determine whether the highest $p$
construction modes or node mode $m$ are relevant in the calculation of the contact force.

Consider
node mode : $m$

number of construction modes for node mode $m$ : $n_c$

number of highest construction modes considered : $p$

The contribution of the highest $p$ construction modes to the kolom $f_m(t)$ is $h_m(t)$

$$h_m(t) = M \cdot \sum_{i=n_c-p+1}^{n_c} \left[ \frac{1}{\zeta_m} \cdot \left( \frac{\Delta_m}{\Delta_i} \right)^2 \cdot \left( \frac{\omega_m}{\omega_i} \right) \cdot \tilde{g}_m(t) \right]$$

The criterion for the number of construction modes is defined as the relative contribution of the highest $p$ construction modes to the kolom $f_m(t)$

$$\frac{\| h_m(t) \|}{\| f_m(t) \|} \leq \epsilon_{\text{crimod}}$$

Using the criterion defined in equation (2.3.9) it is possible to investigate whether it is necessary to increase the number of construction modes for node mode $m$, or whether the last $p$ construction modes don't contribute significantly to the kolom $f_m(t)$. 
Fig. 4.1 Reference calculation of the contact force, the displacement of the contact point on the bell and the clapper, and the velocity of the contact point on the bell and the clapper, for a maximum number of node modes and construction modes.
Fig. 4.3.A Influence of the percentage of critical damping on the contact force and the participation factor of the eigenmode, corresponding to the hum note, to the dynamic response.
Fig. 4.2  

A Contribution of node mode $m$ ($m \in \{1, 2, \ldots, 8\}$) to the length of the nodal force vector $f_i^m(t)$.

B Contribution of the $10\%$ of the highest construction modes to the length of the nodal force vector $f_i^m(t)$, for node mode $m$ ($m \in \{1, 2, \ldots, 8\}$).
Fig. 4.3.B  Influence of the percentage of critical damping on the participation factors of the eigenmodes, corresponding to the nominal and the twelfth, to the dynamic response.
Fig. 4.4.A Influence of the place of the contact point on the bell on the participation factors of the eigenmodes, corresponding to the hum note and the fundamental, to the dynamic response.
Fig. 4.4.B Influence of the place of the contact point on the bell on the participation factors of the eigenmodes, corresponding to the third and the fifth, to the dynamic response.
Fig. 4.4.C Influence of the place of the contact point on the bell on the participation factors of the eigenmodes, corresponding to the nominal and the twelfth, to the dynamic response.
Fig. 4.5.A Influence of the velocity of the contact point on the clapper, at the start of the collision, on the contact force and the participation factor of the eigenmode, corresponding to the hum note, to the dynamic response.
Fig. 4.5.B  Influence of the velocity of the contact point on the clapper, at the start of the collision, on the participation factors of the eigenmodes, corresponding to the fifth and the nominal, to the dynamic response.
Fig. 4.5.C  Influence of the velocity of the contact point on the clapper, at the start of the collision, on the participation factors of the eigenmodes, corresponding to the twelfth and the double octave, to the dynamic response.
Appendix A  Solving the equation of motion of the clapper

As stated in expression (2.1.3.9) the equation of motion of the clapper is the following inhomogeneous differential equation of second order, having constant coefficients:

\[ \ddot{\phi}(t) + A \cdot \dot{\phi}(t) = -B - \frac{L}{f_0} \cdot F(t) \]  \hspace{1cm} (A.1)

The complementary solution of this equation is, using initial condition
\( \phi(t=0) = 0 \):
(Rotation of the clapper is described by (A.1) from the moment when it is about to hit the bell. From there \( \phi(t) \) is measured.)

\[ \phi(t) = c \cdot \sin(\sqrt{A} \cdot t) \]  \hspace{1cm} (A.2a)

The particular solution is derived, using the method of varying constants:

\[ \phi(t) = c(t) \cdot \sin(\sqrt{A} \cdot t) \]  \hspace{1cm} (A.2b)
\[ \ddot{\phi}(t) = \ddot{c}(t) \cdot \sin(\sqrt{A} \cdot t) + \sqrt{A} \cdot \dot{c}(t) \cdot \cos(\sqrt{A} \cdot t) \]  \hspace{1cm} (A.3a)
\[ \ddot{\phi}(t) = \ddot{c}(t) \cdot \sin(\sqrt{A} \cdot t) + 2\sqrt{A} \cdot \dot{c}(t) \cdot \cos(\sqrt{A} \cdot t) - A \cdot \dot{c}(t) \cdot \sin(\sqrt{A} \cdot t) \]  \hspace{1cm} (A.3b)

Substituting (A.2b) and (A.3ab) in (A.1) results in:

\[ \dot{c}(t) \cdot \sin(\sqrt{A} \cdot t) + 2\sqrt{A} \cdot \dot{c}(t) \cdot \cos(\sqrt{A} \cdot t) = -B - \frac{L}{f_0} \cdot F(t) \]  \hspace{1cm} (A.4)

Again the complementary solution and, by varying constants, a particular solution of (A.4) must be be determined:

\[ \dot{c}(t) \cdot \sin(\sqrt{A} \cdot t) + 2\sqrt{A} \cdot \dot{c}(t) \cdot \cos(\sqrt{A} \cdot t) = 0 \]
\[ \int \frac{d[\dot{c}(t)]}{\dot{c}(t)} = -2\sqrt{A} \int \frac{\cos(\sqrt{A} \cdot t)}{\sin(\sqrt{A} \cdot t)} \cdot dt \]
\[ \exp \ln |\dot{c}(t)| = u_0 - 2 \cdot \exp \ln |\sin(\sqrt{A} \cdot t)| \]

The complementary solution is:

\[ \dot{c}(t) = e^{u_0} \cdot e^{-2 \cdot \exp \ln |\sin(\sqrt{A} \cdot t)|} = u \cdot \frac{1}{\{ \sin(\sqrt{A} \cdot t) \}^2} \]  \hspace{1cm} (A.5)

The method of varying constants is used to determine \( u(t) \):

\[ \dot{c}(t) = u(t) \cdot \frac{1}{\{ \sin(\sqrt{A} \cdot t) \}^2} \]
\[ \dot{c}(t) = \frac{1}{\{ \sin(\sqrt{A} \cdot t) \}^2} \cdot \dot{u}(t) - \frac{\cos(\sqrt{A} \cdot t)}{\{ \sin(\sqrt{A} \cdot t) \}^3} \cdot u(t) \]  \hspace{1cm} (A.6)

Substitution of (A.5) and (A.6) in (A.4) results in:
\[ u(t) = -B \cdot \sin(\sqrt{\lambda} \cdot t) - \frac{L}{\gamma_0} \cdot \sin(\sqrt{\lambda} \cdot t) \cdot F(t) \]

\[ u(t) = u(0) - B \cdot \int_{\tau=0}^{t} \sin(\sqrt{\lambda} \cdot \tau) \cdot d\tau - \frac{L}{\gamma_0} \cdot \int_{\tau=0}^{t} \sin(\sqrt{\lambda} \cdot \tau) \cdot F(\tau) d\tau \]

Substituting (A.7) in (A.5), and replacing \( \tau \) by \( \sigma \) gives:

\[ \dot{c}(t) = \frac{1}{\sin^2(\sqrt{\lambda} \cdot t)} \left\{ u(0) - B \cdot \int_{\sigma=0}^{t} \sin(\sqrt{\lambda} \cdot \sigma) \cdot d\sigma - \frac{L}{\gamma_0} \cdot \int_{\sigma=0}^{t} \sin(\sqrt{\lambda} \cdot \sigma) \cdot F(\sigma)d\sigma \right\} \]

\[ c(t) = \int_{\tau=0}^{t} \dot{c}(\tau) d\tau =
\]

\[ = c(0) + u(0) \cdot \int_{\tau=0}^{t} \frac{1}{\sin^2(\sqrt{\lambda} \cdot \tau)} \cdot d\tau - B \cdot \int_{\tau=0}^{t} \frac{1}{\sin^2(\sqrt{\lambda} \cdot \tau)} \left\{ \int_{\sigma=0}^{\tau} \sin(\sqrt{\lambda} \cdot \sigma) d\sigma \right\} d\tau +
\]

\[ - \frac{L}{\gamma_0} \cdot \int_{\tau=0}^{t} \frac{1}{\sin^2(\sqrt{\lambda} \cdot \tau)} \left\{ \int_{\sigma=0}^{\tau} \sin(\sqrt{\lambda} \cdot \sigma) \cdot F(\sigma)d\sigma \right\} d\tau \]

\[ \text{(A.8)} \]

To solve (A.8\textsuperscript{b,c}) next expression is useful:

\[ \int_{\tau=0}^{t} \int_{\sigma=0}^{\tau} g(\tau) f(\sigma)d\sigma d\tau = \int_{\tau=0}^{t} \frac{d}{d\tau} \left\{ G(\tau) \int_{\sigma=0}^{\tau} f(\sigma)d\sigma \right\} d\tau - \int_{\tau=0}^{t} G(\tau) \frac{d}{d\tau} \left\{ \int_{\sigma=0}^{\tau} f(\sigma)d\sigma \right\} d\tau \]

where \( g(\tau) = \frac{d}{d\tau} G(\tau) \)

\[ = G(t) \int_{\sigma=0}^{t} f(\sigma)d\sigma - G(0) \int_{\sigma=0}^{t} f(\sigma)d\sigma - \int_{\tau=0}^{t} G(\tau) \cdot f(\tau) \cdot d\tau \]

\[ \text{(A.9)} \]

In (A.8\textsuperscript{b,c}):

\[ g(\tau) = \frac{1}{\sin^2(\sqrt{\lambda} \cdot \tau)} \implies G(\tau) = -\frac{1}{\sqrt{\lambda}} \cdot \frac{1}{\tan(\sqrt{\lambda} \cdot \tau)} \]

\[ \text{(A.10)} \]

In (A.8\textsuperscript{b}):

\[ f(\sigma) = \sin(\sqrt{\lambda} \cdot \sigma) \]

\[ \text{(A.11)} \]

(A.8\textsuperscript{c}):

\[ f(\sigma) = \sin(\sqrt{\lambda} \cdot \sigma) \cdot F(\sigma) \]

\[ \text{(A.12)} \]

Substitution of (A.10) and (A.11) in (A.8\textsuperscript{b}), using (A.9) results in:
The second integral will be treated according to 'L Hopital:

\[
\lim_{\tau \to 0} \frac{\sin(\sqrt{A} \cdot \sigma)}{\tan(\sqrt{A} \cdot \tau)} = \lim_{\tau \to 0} \frac{\sin(\sqrt{A} \cdot \tau)}{\cos^2(\sqrt{A} \cdot \tau)} = 0
\]

Thus remains: \( (A.8b) = \frac{1}{A} \cdot \frac{1-cos(\sqrt{A}t)}{\sin(\sqrt{A} \cdot \tau)} \) (A.13)

Substitution of (A.10) and (A.12) in (A.8c), using (A.9) results in:

\[
\int_{\tau=0}^{t} \frac{1}{\sin^2(\sqrt{A} \cdot \tau)} \left\{ \int_{\sigma=0}^{\tau} \sin(\sqrt{A} \cdot \sigma)F(\sigma)d\sigma \right\} d\tau = -\frac{1}{\sqrt{A}} \tan(\sqrt{A} \cdot t) \int_{\sigma=0}^{t} \sin(\sqrt{A} \cdot \sigma)F(\sigma)d\sigma + \\
+ \frac{1}{\sqrt{A}} \tan(\sqrt{A} \cdot \tau) \int_{\sigma=0}^{\tau} \sin(\sqrt{A} \cdot \sigma) \cdot F(\sigma)d\sigma + \frac{1}{\sqrt{A}} \int_{\tau=0}^{t} \cos(\sqrt{A} \cdot \tau) \cdot F(\tau)d\tau
\]

'L Hopitals principle is used again on the second integral:

\[
\lim_{\tau \to 0} \frac{1}{\tan(\sqrt{A} \cdot t)} \int_{0}^{\tau} \sin(\sqrt{A} \cdot \sigma)F(\sigma)d\sigma = \lim_{\tau \to 0} \frac{\sin(\sqrt{A} \cdot \tau) \cdot F(\tau)}{\cos^2(\sqrt{A} \cdot \tau)} = 0 \text{ since } F(\tau) \text{ is bounded.}\]

Now is found:

\[
(A.8c) = -\frac{1}{\sqrt{A}} \tan(\sqrt{A} \cdot t) \int_{\sigma=0}^{t} \sin(\sqrt{A} \cdot \sigma) \cdot F(\sigma)d\sigma + \frac{1}{\sqrt{A}} \int_{\tau=0}^{t} \cos(\sqrt{A} \cdot \tau) \cdot F(\tau)d\tau \quad (A.14)
\]

Substitution of (A.10), (A.13) and (A.14) in (A.8) gives:

\[
c(t) = c(0) - \frac{u(0)}{\sqrt{A}} \frac{1}{\tan(\sqrt{A} \cdot \tau)} \int_{\tau=0}^{t} \frac{B}{A} \cdot \frac{1-cos(\sqrt{A} \cdot t)}{\sin(\sqrt{A} \cdot t)} + \\
- \frac{L}{J} \left\{ -\frac{1}{\sqrt{A}} \cdot \frac{\cos(\sqrt{A} \cdot t)}{\sin(\sqrt{A} \cdot t)} \int_{\tau=0}^{t} \sin(\sqrt{A} \cdot \tau) \cdot F(\tau)d\tau + \frac{1}{\sqrt{A}} \int_{\tau=0}^{t} \cos(\sqrt{A} \cdot \tau) \cdot F(\tau)d\tau \right\}
\]
Hence

\[
\phi(t) = c_0 \sin(\sqrt{A} \cdot t) - \frac{b}{\sqrt{A}} \cdot \{1 - \cos(\sqrt{A} \cdot t)\} + \frac{1}{\sqrt{A}} \cdot \frac{L}{J_0} \cdot \left[ \cos(\sqrt{A} \cdot t) \cdot \int_{\tau=0}^{t} \sin(\sqrt{A} \cdot \tau) \cdot F(\tau) d\tau - \int_{\tau=0}^{t} \cos(\sqrt{A} \cdot \tau) \cdot F(\tau) d\tau \right]
\]

Substitution of (A.14) in (A.15) gives the solution to the equation of motion:

\[
\phi(t) = c_0 \sin(\sqrt{A} \cdot t) - \frac{b}{\sqrt{A}} \cdot \{1 - \cos(\sqrt{A} \cdot t)\} + \frac{1}{\sqrt{A}} \cdot \frac{L}{J_0} \cdot \left[ \cos(\sqrt{A} \cdot t) \cdot \int_{\tau=0}^{t} \sin(\sqrt{A} \cdot \tau) \cdot F(\tau) d\tau - \int_{\tau=0}^{t} \cos(\sqrt{A} \cdot \tau) \cdot F(\tau) d\tau \right]
\]

The constant \(c_0\) is determined using the initial condition \(\dot{\phi}(0) = 0\):

\[
\dot{\phi}(t) = \sqrt{A} \cdot c_0 \cos(\sqrt{A} \cdot t) - \frac{b}{\sqrt{A}} \cdot \sin(\sqrt{A} \cdot t) - \frac{L}{J_0} \cdot \left[ \sin(\sqrt{A} \cdot t) \cdot \int_{\tau=0}^{t} \sin(\sqrt{A} \cdot \tau) \cdot F(\tau) d\tau + \cos(\sqrt{A} \cdot t) \cdot \int_{\tau=0}^{t} \sin(\sqrt{A} \cdot \tau) \cdot F(\tau) d\tau \right]
\]

\[
\dot{\phi}(t=0) = \dot{\phi}(0) = \sqrt{A} \cdot c_0 \Rightarrow c_0 = \frac{1}{\sqrt{A}} \cdot \dot{\phi}(0)
\]

The displacement of the contact point on the clapper in the direction perpendicular to the bell wall in the contact point is:

\[
W_{clapp}(t) \approx L \cdot \phi(t) \cdot \cos\beta_1 = \cos\beta_1 \cdot \left[ \frac{1}{\sqrt{A}} \cdot \frac{L}{J_0} \cdot \dot{\phi}(0) \sin(\sqrt{A} \cdot t) - \frac{b}{\sqrt{A}} \cdot \{1 - \cos(\sqrt{A} \cdot t)\} + \frac{1}{\sqrt{A}} \cdot \frac{L}{J_0} \cdot \left[ \cos(\sqrt{A} \cdot t) \cdot \int_{\tau=0}^{t} \sin(\sqrt{A} \cdot \tau) \cdot F(\tau) d\tau \right] \right]
\]
\[ -\sin(\sqrt{A} \cdot t) \cdot \left[ \cos(\sqrt{A} \cdot \tau) \cdot F(\tau) \, d\tau \right]_{\tau=0}^{t} \]  

(A.17)

Where \( L \cdot \phi(0) = v_0 \) [m/s]: startvelocity at which clapper hits bell.

\[
\frac{b}{A} = \tan \alpha_0 \quad \alpha = m \cdot g \cdot \frac{L_{zw}}{J_0} \cdot \cos \alpha_0 \left[ \frac{1}{J^2} \right]
\]

For the displacement and the velocity of the contact point on the clapper in the direction perpendicular to the bell wall in the contact point we can write:

\[
W_{clapp}(t) = W_{clapp}(t_a) \cdot \cos \sqrt{A} \cdot (t-t_a) + \frac{1}{\sqrt{A}} \cdot \frac{b}{A} \cdot W_{clapp}(t_a) \cdot \sin \sqrt{A} \cdot (t-t_a)
\]

\[
- \cos \beta_1 \cdot \frac{b}{A} \left[ 1 - \cos \sqrt{A} \cdot (t-t_a) \right]
\]

\[
\cos \beta_1 \cdot \frac{1}{\sqrt{A}} \cdot \frac{L^2}{J_0} \cdot \left[ \cos(\sqrt{A} \cdot t) \cdot \left[ \int_{\tau=0}^{t} \sin(\sqrt{A} \cdot \tau) \cdot F(\tau) \, d\tau \right] \right.
\]

\[
- \sin(\sqrt{A} \cdot t) \cdot \left[ \int_{\tau=0}^{t} \cos(\sqrt{A} \cdot \tau) \cdot F(\tau) \, d\tau \right]
\]

(A.18)

\[
\dot{W}_{clapp}(t) = -\sqrt{A} \cdot W_{clapp}(t_a) \cdot \sin \sqrt{A} \cdot (t-t_a) + \dot{W}_{clapp}(t_a) \cdot \cos \sqrt{A} \cdot (t-t_a)
\]

\[- \sqrt{A} \cdot \cos \beta_1 \cdot L \cdot \frac{b}{A} \cdot \sin \sqrt{A} \cdot (t-t_a)
\]

\[- \cos \beta_1 \cdot \frac{L^2}{J_0} \cdot \left[ \sin(\sqrt{A} \cdot t) \cdot \left[ \int_{\tau=t_a}^{t} \sin(\sqrt{A} \cdot \tau) \cdot F(\tau) \, d\tau \right] \right.
\]

\[+ \cos(\sqrt{A} \cdot t) \cdot \left[ \int_{\tau=t_a}^{t} \cos(\sqrt{A} \cdot \tau) \cdot F(\tau) \, d\tau \right] \]  

(A.19)

The discretised form of equations (A.18) and (A.19), using the results of the appendices B and C are:

\[
W_{clapp}(n \cdot \Delta t) = W_{clapp}(n_b \cdot \Delta t) \cdot \cos(n-n_b) \cdot \sqrt{A} \cdot \Delta t
\]

\[+ \frac{1}{\sqrt{A}} \cdot W_{clapp}(n_b \cdot \Delta t) \cdot \sin(n-n_b) \cdot \sqrt{A} \cdot \Delta t
\]

\[- \cos \beta_1 \cdot L \cdot \frac{b}{A} \cdot \left[ 1 - \cos(n-n_b) \cdot \sqrt{A} \cdot \Delta t \right]
\]

\[+ \cos \beta_1 \cdot \frac{1}{\sqrt{A}} \cdot \frac{L^2}{J_0} \left[ \cos(n-n_b) \cdot \sqrt{A} \cdot \Delta t + \frac{1}{\sqrt{A} \cdot \Delta t} \cdot \left( -\sin(n-n_b) \cdot \sqrt{A} \cdot \Delta t \right) \right] \]
\[ + \sin(n-n_b-1) \cdot \sqrt{\alpha} \cdot \Delta t} \right] \cdot F_{n_b} \]

\[ - \cos \beta_1 \cdot \frac{1}{\sqrt{\alpha}} \cdot \frac{L^2}{J_0} \cdot \left[ 1 - \sin \frac{\sqrt{\alpha} \cdot \Delta t}{\sqrt{\alpha} \cdot \Delta t} \right] \cdot F_n \]

\[ + \cos \beta_1 \cdot \frac{2}{n_b \cdot \sqrt{\alpha}} \cdot \frac{L^2}{J_0} \cdot \left\{ 1 - \cos \sqrt{\alpha} \cdot \Delta t \right\} \cdot F_n \]

\[ \cdot \sum_{j=n_b+1}^{n-1} \left( \frac{-\sin(n-j) \cdot \sqrt{\alpha} \cdot \Delta t}{\sqrt{\alpha} \cdot \Delta t} \right) \cdot F_j \]  

where \( W_{\text{clapp}}(0 \cdot \Delta t) = 0 \)

\[ W_{\text{clapp}}(0 \cdot \Delta t) = \cos \beta_1 \cdot L \cdot \phi(0) = \cos \beta_1 \cdot v_0 \text{ (clapper !)} \]

and:

\[ W_{\text{clapp}}(n \cdot \Delta t) = -\sqrt{\alpha} \cdot W_{\text{clapp}}(n_b \cdot \Delta t) \cdot \sin(n-n_b) \cdot \sqrt{\alpha} \cdot \Delta t \]

\[ + W_{\text{clapp}}(n_b \cdot \Delta t) \cdot \sin(n-n_b) \cdot \sqrt{\alpha} \cdot \Delta t \]

\[ - \cos \beta_1 \cdot L \cdot \frac{b}{\sqrt{\alpha}} \cdot \sin(n-n_b) \cdot \sqrt{\alpha} \cdot \Delta t \]

\[ - \cos \beta_1 \cdot \frac{1}{\sqrt{\alpha}} \cdot \frac{L^2}{J_0} \cdot \left[ \sin(n-n_b) \cdot \sqrt{\alpha} \cdot \Delta t + \frac{1}{\sqrt{\alpha} \cdot \Delta t} \right. \]

\[ \cdot \left\{ \cos(n-n_b) \cdot \sqrt{\alpha} \cdot \Delta t - \cos(n-n_b-1) \cdot \sqrt{\alpha} \cdot \Delta t \right\} \right] \cdot F_{n_b} \]

\[ - \cos \beta_1 \cdot \frac{1}{\sqrt{\alpha}} \cdot \frac{L^2}{J_0} \cdot \left[ 0 + \frac{1}{\sqrt{\alpha} \cdot \Delta t} \cdot \left\{ 1 - \cos \sqrt{\alpha} \cdot \Delta t \right\} \right] \cdot F_n \]

\[ - \cos \beta_1 \cdot \frac{2}{n_b \cdot \Delta t} \cdot \frac{L^2}{J_0} \cdot \left\{ 1 - \cos \sqrt{\alpha} \cdot \Delta t \right\} \cdot \sum_{j=n_b+1}^{n-1} F_j \cdot \cos(n-j) \cdot \Delta t \]  

(A.20)
Appendix B Solution of \[ \int_{\tau=t_b}^{t} \sin(\sqrt{A} \cdot \tau) \cdot F(\tau) \, d\tau \]

In order to be able to solve the integral \[ \int_{\tau=t_b}^{t} \sin(\sqrt{A} \cdot \tau) \cdot F(\tau) \, d\tau \] the (not analytically known) function \( F(\tau) \) is assumed to be stepwise linear:

\[ F(\tau) = F_{i-1} + \frac{1}{\Delta t} (F_{i} - F_{i-1}) \cdot [(\tau - (i-1) \cdot \Delta t)] \quad \tau \in [(i-1) \cdot \Delta t, i \cdot \Delta t] \]

with:

\[ t_b = n_b \cdot \Delta t, \quad t = n \cdot \Delta t, \quad F_1 = F(i \cdot \Delta t) \]

Substitution of this approximation in the integral leads to:

\[ \int_{\tau=t_b}^{t} \sin(\sqrt{A} \cdot \tau) \cdot F(\tau) \, d\tau = \]

\[ = \sum_{i=n_b+1}^{n} \int_{\tau=(i-1)\Delta t}^{i\Delta t} \sin(\sqrt{A} \cdot \tau) \cdot \left\{ F_{i-1} + \frac{F_i - F_{i-1}}{\Delta t} \cdot (\tau - (i-1) \cdot \Delta t) \right\} \, d\tau \]

\[ = \sum_{i=n_b+1}^{n} \int_{\tau=(i-1)\Delta t}^{i\Delta t} \sin(\sqrt{A} \cdot \tau) \cdot \left\{ i \cdot F_{i-1} - (i-1) \cdot F_i + \frac{F_i - F_{i-1}}{\Delta t} \cdot \tau \right\} \, d\tau \]

\[ = \sum_{i=n_b+1}^{n} \left\{ \int_{\tau=(i-1)\Delta t}^{i\Delta t} \sin(\sqrt{A} \cdot \tau) \, d\tau + \int_{\tau=(i-1)\Delta t}^{i\Delta t} \frac{F_i - F_{i-1}}{\Delta t} \cdot \tau \cdot \sin(\sqrt{A} \cdot \tau) \, d\tau \right\} \]

\[ = \sum_{i=n_b+1}^{n} \left[ i \cdot F_{i-1} - (i-1) \cdot F_i \right] \cdot \left[ \frac{-1}{\sqrt{A}} \cos(\sqrt{A} \cdot \tau) \right]_{\tau=(i-1)\Delta t}^{i\Delta t} + \]

\[ \frac{F_i - F_{i-1}}{\Delta t} \left[ \frac{-\tau}{\sqrt{A}} \cos(\sqrt{A} \cdot \tau) + \frac{1}{A} \sin(\sqrt{A} \cdot \tau) \right]_{\tau=(i-1)\Delta t}^{i\Delta t} \]
\[ i = n_{b+1} \]
\[ \frac{1}{A} \cdot F_{i-1} \cdot \cos((i-1)\sqrt{A} \cdot \Delta t) - \frac{1}{\sqrt{A}} \cdot F_{1} \cdot \cos(i\sqrt{A} \cdot \Delta t) + \]
\[ -\Delta t \sum_{i=n_{b+1}}^{n} \left[ F_{1} \cdot \sin(i\sqrt{A} \cdot \Delta t) + F_{i-1} \cdot \sin((i-1)\sqrt{A} \cdot \Delta t) - F_{1} \cdot \sin((i-1)\sqrt{A} \cdot \Delta t) - F_{i-1} \cdot \sin(i\sqrt{A} \cdot \Delta t) \right] \]

Substitution of \( i - 1 \) by \( j \) leads to:
\[ \sum_{j=n_{b}}^{n-1} \frac{1}{\sqrt{A}} \cdot F_{j} \cdot \cos(j\sqrt{A} \cdot \Delta t) + \frac{1}{\Delta t} \cdot F_{j} \cdot \sin(j\sqrt{A} \cdot \Delta t) - \frac{1}{\Delta t} \cdot F_{j} \cdot \sin((j+1)\sqrt{A} \cdot \Delta t) \]
\[ + \sum_{i=n_{b+1}}^{n} \frac{1}{\sqrt{A}} \cdot F_{i} \cdot \sin(i\sqrt{A} \cdot \Delta t) + \frac{1}{\Delta t} \cdot F_{i} \cdot \sin(i\sqrt{A} \cdot \Delta t) - \frac{1}{\Delta t} \cdot F_{i} \cdot \sin((i-1)\sqrt{A} \cdot \Delta t) \]
\[ = \frac{1}{\sqrt{A}} \cdot \left[ F_{n} \cdot \cos(n\sqrt{A} \cdot \Delta t) - F_{n} \cdot \cos(n\sqrt{A} \cdot \Delta t) \right] + \]
\[ -\Delta t \sum_{j=n_{b}}^{n-1} \left[ F_{2} \cdot \sin(j\sqrt{A} \cdot \Delta t) + \frac{1}{\Delta t} \cdot F_{j} \cdot \sin(j\sqrt{A} \cdot \Delta t) \right] \]
\[ - \frac{1}{\sqrt{A}} \cdot F_{n} \cdot \sin((n-1)\sqrt{A} \cdot \Delta t) - \sum_{i=n_{b+1}}^{n-1} \left[ F_{1} \cdot \sin((i-1)\sqrt{A} \cdot \Delta t) + \frac{1}{\Delta t} \cdot F_{i} \cdot \sin((i-1)\sqrt{A} \cdot \Delta t) \right] \]
\[ = \frac{1}{\sqrt{A}} \cdot F_{n} \cdot \left[ \cos(n\sqrt{A} \cdot \Delta t) + \frac{1}{\sqrt{A} \Delta t} \cdot \sin(n\sqrt{A} \cdot \Delta t) - \sin((n+1)\sqrt{A} \cdot \Delta t) \right] \]
\[ - \frac{1}{\sqrt{A}} \cdot F_{n} \cdot \left[ \cos(n\sqrt{A} \cdot \Delta t) - \frac{1}{\sqrt{A} \Delta t} \cdot \sin(n\sqrt{A} \cdot \Delta t) - \sin((n-1)\sqrt{A} \cdot \Delta t) \right] \]
\[ + \frac{1}{\Delta t} \sum_{i=n_{b+1}}^{n-1} \left[ -\sin((i-1)\sqrt{A} \cdot \Delta t) + 2\sin(i\sqrt{A} \cdot \Delta t) - \sin((i+1)\sqrt{A} \cdot \Delta t) \right] \]

The final solution is:
\[ = \frac{1}{\sqrt{A}} \cdot F_{n} \cdot \left[ \cos(n\sqrt{A} \cdot \Delta t) + \frac{1}{\sqrt{A} \Delta t} \cdot \sin(n\sqrt{A} \cdot \Delta t) - \sin((n+1)\sqrt{A} \cdot \Delta t) \right] \]
\[ - \frac{1}{\sqrt{A}} \cdot F_{n} \cdot \left[ \cos(n\sqrt{A} \cdot \Delta t) - \frac{1}{\sqrt{A} \Delta t} \cdot \sin(n\sqrt{A} \cdot \Delta t) - \sin((n-1)\sqrt{A} \cdot \Delta t) \right] \]
\[ + \frac{1}{\Delta t} \cdot \sum_{i=n_{b+1}}^{n-1} \left[ \cos(i\sqrt{A} \cdot \Delta t) \right] \]
Appendix C  Solution of $\int_{\tau=t_b}^{t} \cos(\sqrt{A} \cdot \tau) \cdot F(\tau) \, d\tau$

In order to be able to solve the integral $\int_{\tau=t_b}^{t} \cos(\sqrt{A} \cdot \tau) \cdot F(\tau) \, d\tau$ the (not analytically known) function $F(\tau)$ is assumed to be stepwise linear:

$$F(\tau) = F_{i-1} + \frac{1}{\Delta t} (F_i - F_{i-1}) \cdot [\tau - (i-1) \cdot \Delta t] \quad \tau \in [(i-1) \cdot \Delta t, i \cdot \Delta t]$$

with:

$$t_b = n_b \cdot \Delta t, \quad t = n \cdot \Delta t, \quad F_i = F(i \cdot \Delta t)$$

Substitution of this approximation in the integral leads to:

$$\int_{\tau=t_b}^{t} \cos(\sqrt{A} \cdot \tau) \cdot F(\tau) \, d\tau =$$

$$= \sum_{i=n_b+1}^{n} \int_{\tau=(i-1) \Delta t}^{i \Delta t} \cos(\sqrt{A} \cdot \tau) \cdot \left\{ F_{i-1} + \frac{F_i - F_{i-1}}{\Delta t} (\tau - (i-1) \Delta t) \right\} \, d\tau$$

$$= \sum_{i=n_b+1}^{n} \int_{\tau=(i-1) \Delta t}^{i \Delta t} \cos(\sqrt{A} \cdot \tau) \cdot \left[ i \cdot F_{i-1} - (i-1) \cdot F_i + \frac{F_i - F_{i-1}}{\Delta t} \right] \, d\tau$$

$$= \sum_{i=n_b+1}^{n} \left\{ \int_{\tau=(i-1) \Delta t}^{i \Delta t} \left[ i \cdot F_{i-1} - (i-1) \cdot F_i \right] \cos(\sqrt{A} \cdot \tau) \, d\tau + \right.$$

$$\left. \int_{\tau=(i-1) \Delta t}^{i \Delta t} \left[ \frac{F_i - F_{i-1}}{\Delta t} \right] \cdot \tau \cdot \cos(\sqrt{A} \cdot \tau) \, d\tau \right\}$$

$$= \sum_{i=n_b+1}^{n} \left[ i \cdot F_{i-1} - (i-1) \cdot F_i \right] \cdot \left[ \frac{1}{\sqrt{A}} \sin(\sqrt{A} \cdot \tau) \right]_{\tau=(i-1) \Delta t}^{\tau=i \Delta t} +$$

$$\frac{F_i - F_{i-1}}{\Delta t} \cdot \left[ \frac{\tau \sin(\sqrt{A} \cdot \tau) + \frac{1}{A} \cos(\sqrt{A} \cdot \tau)}{\sqrt{A}} \right]_{\tau=(i-1) \Delta t}^{\tau=i \Delta t}$$
\[
\begin{align*}
&= \frac{1}{\sqrt{A}} \sum_{i=1}^{n} \left( -F_{i-1}\sin(i\sqrt{A} \cdot \Delta t) + F_{i}\sin(i\sqrt{A} \cdot \Delta t) \right) \\
&+ \frac{1}{A\Delta t} \sum_{i=1}^{n} \left( F_{i}\cos(i\sqrt{A} \cdot \Delta t) + F_{i-1}\cos((i-1)\sqrt{A} \cdot \Delta t) \right) \\
&- F_{1}\cos(i\sqrt{A} \cdot \Delta t) - F_{-1}\cos(i\sqrt{A} \cdot \Delta t) \\
\end{align*}
\]

Substitution of (i-1) by j leads to :
\[
\begin{align*}
&= \frac{1}{\sqrt{A}} \sum_{j=1}^{n-1} \left( -F_{j}\sin(j\sqrt{A} \cdot \Delta t) + \frac{1}{A\Delta t} F_{j}\cos(j\sqrt{A} \cdot \Delta t) - \frac{1}{A\Delta t} F_{j+1}\cos((j+1)\sqrt{A} \cdot \Delta t) \right) \\
&+ \frac{1}{A\Delta t} \sum_{i=1}^{n} \left( F_{i}\sin(i\sqrt{A} \cdot \Delta t) + \frac{1}{A\Delta t} F_{i}\cos(i\sqrt{A} \cdot \Delta t) - \frac{1}{A\Delta t} F_{i+1}\cos((i+1)\sqrt{A} \cdot \Delta t) \right) \\
\end{align*}
\]

Substitution of (i-1) by j leads to :
\[
\begin{align*}
&= \frac{1}{\sqrt{A}} \left[ F_{n}\sin(n\sqrt{A} \cdot \Delta t) - F_{nb}\sin(nb\sqrt{A} \cdot \Delta t) \right] + \\
&\frac{1}{A\Delta t} \left[ F_{n}\cos(n\sqrt{A} \cdot \Delta t) + \sum_{i=1}^{n-1} F_{i}\cos(i\sqrt{A} \cdot \Delta t) + \\
F_{nb}\cos(nb\sqrt{A} \cdot \Delta t) + \sum_{j=1}^{n-1} F_{j}\cos(j\sqrt{A} \cdot \Delta t) - \\
F_{n}\cos((n-1)\sqrt{A} \cdot \Delta t) - \sum_{i=1}^{n-1} F_{i}\cos((i-1)\sqrt{A} \cdot \Delta t) + \\
-F_{nb}\cos((nb+1)\sqrt{A} \cdot \Delta t) - \sum_{j=1}^{n-1} F_{j}\cos((j+1)\sqrt{A} \cdot \Delta t) \right] \\
\end{align*}
\]

The final solution is :
\[
\begin{align*}
&= \frac{1}{\sqrt{A}} F_{nb} \left[ \sin(nb\sqrt{A} \cdot \Delta t) - \frac{1}{\sqrt{A}\Delta t} \left[ \cos(nb\sqrt{A} \cdot \Delta t) - \cos((nb+1)\sqrt{A} \cdot \Delta t) \right] \right] \\
&+ \frac{1}{\sqrt{A}} F_{n} \left[ \sin(n\sqrt{A} \cdot \Delta t) + \frac{1}{\sqrt{A}\Delta t} \left[ \cos(n\sqrt{A} \cdot \Delta t) - \cos((n-1)\sqrt{A} \cdot \Delta t) \right] \right] \\
&+ \frac{1}{A\Delta t} \sum_{i=nb+1}^{n} \left[ -\cos((i-1)\sqrt{A} \cdot \Delta t) + 2\cos(i\sqrt{A} \cdot \Delta t) - \cos((i+1)\sqrt{A} \cdot \Delta t) \right] \\
\end{align*}
\]
Appendix D  Solution of \[ \int_{\tau=t_b}^{t} F(\tau) \cdot e^{-\zeta \omega (t-\tau)} \sin(\varpi(t-\tau)) \, d\tau \]

In order to be able to solve the integral \[ \int_{\tau=t_b}^{t} F(\tau) \cdot e^{-\zeta \omega (t-\tau)} \sin(\varpi(t-\tau)) \, d\tau \] the (not analytically known) function \( F(\tau) \) is assumed to be stepwise linear:

\[
F(\tau) = F_{i-1} \pm \frac{1}{\Delta t} (F_i - F_{i-1}) \cdot [\tau - (i-1) \cdot \Delta t] \quad \tau \in [(i-1) \cdot \Delta t, i \cdot \Delta t]
\]

\[
= i \cdot F_{i-1} - (i-1) \cdot F_i + \frac{F_i - F_{i-1}}{\Delta t} \cdot \tau
\]

with:

\[
t_b = n_b \cdot \Delta t \\
t = n \cdot \Delta t \\
F_1 = F(i \cdot \Delta t)
\]

Substitution of this approximation in the integral leads to:

\[
\int_{\tau=t_b}^{t} F(\tau) \cdot e^{-\zeta \omega (t-\tau)} \sin(\varpi(t-\tau)) \, d\tau = \\
\sum_{i=n_b+1}^{n} \int_{\tau=(i-1)\Delta t}^{i \Delta t} \left[ i \cdot F_{i-1} - (i-1) \cdot F_i + \frac{F_i - F_{i-1}}{\Delta t} \cdot \tau \right] \cdot e^{-\zeta \omega (t-\tau)} \sin(\varpi(t-\tau)) \, d\tau
\]

\[
= \sum_{i=n_b+1}^{n} \frac{1}{2} \int_{\tau=(i-1)\Delta t}^{i \Delta t} \left[ i \cdot F_{i-1} - (i-1) \cdot F_i + \frac{F_i - F_{i-1}}{\Delta t} \cdot \tau \right] \cdot \left[ e^{(\varpi - \zeta \omega)(n\Delta t - \tau)} - e^{-(\varpi + \zeta \omega)(n\Delta t - \tau)} \right] \, d\tau
\]

(D.1)

Substitute:

\[
\varpi - \zeta \omega = A \\
-(\varpi + \zeta \omega) = B
\]

(D.2)

\[
= \sum_{i=n_b+1}^{n} \left\{ \frac{1}{2} \int_{\tau=(i-1)\Delta t}^{i \Delta t} \left[ i \cdot F_{i-1} - (i-1) \cdot F_i \right] \left[ e^{A(n\Delta t - \tau)} - e^{B(n\Delta t - \tau)} \right] \, d\tau + \frac{1}{2} \int_{\tau=(i-1)\Delta t}^{i \Delta t} \left[ F_i - F_{i-1} \right] \cdot \left[ e^{A(n\Delta t - \tau)} - e^{B(n\Delta t - \tau)} \right] \, d\tau \right\}
\]
\[
\sum_{i=n_b+1}^{n} \frac{1}{2j} \left[ j \cdot F_{i-1} - (i-1) \cdot F_i \right] \left[ e^{A(n-i)\Delta t} + e^{A(n-i+1)\Delta t} \right] + \\
\sum_{i=n_b+1}^{n} \frac{1}{2j} \left[ j \cdot F_{i-1} - (i-1) \cdot F_i \right] \left[ e^{B(n-i)\Delta t} - e^{B(n-i+1)\Delta t} \right] + \\
\sum_{i=n_b+1}^{n} \frac{1}{2j} \left[ \frac{F_i-F_{i-1}}{\Delta t} \right] \left[ i\Delta t \cdot e^{A(n-i)\Delta t} + (i-1)\Delta t \cdot e^{A(n-i+1)\Delta t} \right] - \frac{-e^{A(n-i)\Delta t} + e^{A(n-i+1)\Delta t}}{A^2} + \\
\sum_{i=n_b+1}^{n} \frac{1}{2j} \left[ \frac{F_i-F_{i-1}}{\Delta t} \right] \left[ i\Delta t \cdot e^{B(n-i)\Delta t} - (i-1)\Delta t \cdot e^{B(n-i+1)\Delta t} \right] + \frac{e^{B(n-i)\Delta t} - e^{B(n-i+1)\Delta t}}{B^2}.
\]

\[
= \sum_{i=n_b+1}^{n} \frac{1}{2j} \left[ -F_{i-1} \cdot e^{A(n-i)\Delta t} + F_{i-1} \cdot e^{A(n-i+1)\Delta t} \right] + \\
\sum_{i=n_b+1}^{n} \frac{1}{2j} \left[ \frac{F_i-F_{i-1}}{\Delta t} \right] \left[ -e^{A(n-i)\Delta t} + e^{A(n-i+1)\Delta t} \right] + \frac{e^{B(n-i)\Delta t} - e^{B(n-i+1)\Delta t}}{B^2}.
\]

\[
= \sum_{i=n_b+1}^{n} \frac{1}{2j} \left[ \frac{F_{i-1}}{\Delta t} \cdot e^{B(n-i)\Delta t} - \frac{B \cdot e^{A(n-i)\Delta t}}{AB} \right] - \\
\sum_{i=n_b+1}^{n} \frac{1}{2j} \left[ \frac{F_{i-1}}{\Delta t} \cdot e^{B(n-i+1)\Delta t} - \frac{B \cdot e^{A(n-i+1)\Delta t}}{AB} \right] + \\
\sum_{i=n_b+1}^{n} \frac{1}{2j} \left[ \frac{F_{i-1}}{\Delta t} \cdot e^{B(n-i)\Delta t} - \frac{B^2 \cdot e^{A(n-i)\Delta t}}{A^2B^2} \right] - \\
\sum_{i=n_b+1}^{n} \frac{1}{2j} \left[ \frac{F_{i-1}}{\Delta t} \cdot e^{B(n-i+1)\Delta t} - \frac{B^2 \cdot e^{A(n-i+1)\Delta t}}{A^2B^2} \right].
\]

where:

\[
D_1 = \frac{1}{2j} \cdot \frac{A \cdot e^{B(n-i)\Delta t} - B \cdot e^{A(n-i)\Delta t}}{AB}
\]

\[
= \frac{e^{-\zeta \omega(n-i)\Delta t}}{\omega^2} \cdot \left[ \zeta \cdot \omega \cdot \sin(\omega(n-i)\Delta t) + \omega \cdot \cos(\omega(n-i)\Delta t) \right].
\]
and

$$E_i = \frac{1}{2}j \frac{A^2eB(n-i)\Delta t - B^2eA(n-i)\Delta t}{A^2B^2}$$

$$= \frac{e^{-\zeta\omega(n-i)\Delta t}}{\omega^2} \left[ (1-2\zeta^2) \cdot \sin(\omega(n-i)\Delta t) - 2\zeta\sqrt{1-\zeta^2} \cdot \cos(\omega(n-i)\Delta t) \right]$$  \hspace{1cm} (D.5)

Substitution of equation (D.4) and (D.5) in equation (D.3) results in:

$$\int_{t}^{\infty} \frac{F(\tau) \cdot e^{-\zeta\omega(\tau-t)\sin(\omega(\tau-t))}}{\tau} d\tau =$$

$$= \sum_{i=nb+1}^{n} F_i \cdot D_1 - F_i \cdot D_{i-1} + \frac{1}{\Delta t} \left[ \sum_{i=nb+1}^{n} F_i \cdot E_i + \sum_{i=nb+1}^{n} F_i \cdot E_{i-1} \right]$$

$$= F_n \cdot D_n - F_{nb} \cdot D_{nb} + \frac{1}{\Delta t} \left[ \sum_{i=nb+1}^{n} F_i \cdot E_i + \sum_{i=nb+1}^{n} 2F_i \cdot E_i - \sum_{j=nb+1}^{n-1} F_j \cdot E_{j+1} \right]$$

$$= F_n \cdot D_n - F_{nb} \cdot D_{nb} + \frac{1}{\Delta t} \left[ F_n \cdot E_n + F_{nb} \cdot E_{nb} + \sum_{i=nb+1}^{n-1} F_i \cdot E_i \right]$$

$$= F_n \left[ D_n + \frac{1}{\Delta t} (E_n - E_{n-1}) \right] - F_{nb} \left[ D_{nb} + \frac{1}{\Delta t} (E_{nb} - E_{nb+1}) \right]$$

$$\frac{1}{\Delta t} \sum_{i=nb+1}^{n-1} F_i \cdot [E_{i+1} - 2E_i + E_{i+1}]$$
The solution for the integral \( \int_{t=t_b}^{t} F(\tau) \cdot e^{-\zeta \omega(t-\tau)} \sin(\omega(t-\tau)) \, d\tau \) is:

\[
F_n \left[ \frac{\bar{\omega}}{\omega} + \frac{1}{\omega^2 \Delta t} \left[ 2 \zeta \sqrt{1-\zeta^2} \left[ -1 + e^{-\zeta \omega \Delta t \cos(\omega \Delta t)} \right] - (1-2\zeta^2) e^{-\zeta \omega \Delta t \sin(\omega \Delta t)} \right] \right] - \\
F_n \left[ \frac{\bar{\omega}}{\omega} + \zeta \cdot \omega \cdot \sin(\omega (n-n_b) \Delta t) + \omega \cdot \cos(\omega (n-n_b) \Delta t) \right] - \\
F_n \left[ \frac{\bar{\omega}}{\omega^2 \Delta t} \cdot e^{-\zeta \omega (n-n_b) \Delta t} \cdot \omega \cdot \cos(\omega (n-n_b) \Delta t) \right] - \\
\left[ (1-2\zeta^2) \sin(\omega (n-n_b-1) \Delta t) - 2\zeta \sqrt{1-\zeta^2} \cos(\omega (n-n_b-1) \Delta t) \right] + \\
\left[ (1-2\zeta^2) \sin(\omega (n-n_b-1) \Delta t) - 2\zeta \sqrt{1-\zeta^2} \cos(\omega (n-n_b-1) \Delta t) \right] - \\
\sum_{i=n_b+1}^{n-1} \frac{1}{\omega^2 \Delta t} \cdot F_i \cdot e^{-\zeta \omega (n-i) \Delta t} \cdot \left[ (1-2\zeta^2) \sin(\omega (n-i+1) \Delta t) - 2\sin(\omega (n-i) \Delta t) + e^{\zeta \omega \Delta t \sin(\omega (n-i) \Delta t)} \right] + \\
\sum_{i=n_b+1}^{n-1} \frac{1}{\omega^2 \Delta t} \cdot F_i \cdot e^{-\zeta \omega (n-i) \Delta t} \cdot \left[ (1-2\zeta^2) \cos(\omega (n-i+1) \Delta t) - 2\cos(\omega (n-i) \Delta t) + e^{\zeta \omega \Delta t \cos(\omega (n-i) \Delta t)} \right] (D.6)
\]

For \( n - n_b \leq 2 \) the summons will vanish, and equation (D.6) becomes:

\[
\int_{t=t_0}^{t} F(\tau) \cdot e^{-\zeta \omega(t-\tau)} \sin(\omega(t-\tau)) \, d\tau
=/
\]

\[
F_n \left[ \frac{\bar{\omega}}{\omega^2} + \frac{1}{\omega^2 \Delta t} \left[ 2 \zeta \sqrt{1-\zeta^2} \left[ -1 + e^{-\zeta \omega \Delta t \cos(\omega \Delta t)} \right] - (1-2\zeta^2) e^{-\zeta \omega \Delta t \sin(\omega \Delta t)} \right] \right] - \\
F_n \left[ \frac{\bar{\omega}}{\omega^2 \Delta t} \cdot e^{-\zeta \omega \Delta t} \cdot \omega \cdot \sin(\omega \Delta t) + \omega \cdot \cos(\omega \Delta t) \right] - \\
\left[ -2\zeta \sqrt{1-\zeta^2} \right] + \\
\left[ (1-2\zeta^2) \sin(\omega \Delta t) - 2\zeta \sqrt{1-\zeta^2} \cos(\omega \Delta t) \right] (D.7)
\]
Appendix E  Solution of  \[ \int_{\tau=t_b}^{t} F(\tau) \cdot e^{-\zeta \omega (t-\tau)} \cos(\tilde{\omega}(t-\tau)) \, d\tau \]

In order to be able to solve the integral \[ \int_{\tau=t_b}^{t} F(\tau) \cdot e^{-\zeta \omega (t-\tau)} \cos(\tilde{\omega}(t-\tau)) \, d\tau \] the (not analytically known) function \( F(\tau) \) is assumed to be stepwise linear:

\[
F(\tau) = F_{i-1} + \frac{1}{\Delta t} (F_i - F_{i-1}) \cdot [\tau - (i-1) \cdot \Delta t] \quad \tau \in [(i-1) \cdot \Delta t, i \cdot \Delta t]
\]

\[
= i \cdot F_{i-1} - (i-1) \cdot F_i + \frac{F_i - F_{i-1}}{\Delta t} \cdot \tau
\]

with:

\[ t_b = n_b \cdot \Delta t \]
\[ t = n \cdot \Delta t \]
\[ F_i = F(i \cdot \Delta t) \]

Substitution of this approximation in the integral leads to:

\[
\int_{\tau=t_b}^{t} F(\tau) \cdot e^{-\zeta \omega (t-\tau)} \cos(\tilde{\omega}(t-\tau)) \, d\tau =
\]

\[
= \Sigma_{i=n_b+1}^{n} \int_{\tau=(i-1)\Delta t}^{i \Delta t} \left[ i \cdot F_{i-1} - (i-1) \cdot F_i + \frac{F_i - F_{i-1}}{\Delta t} \cdot \tau \right] \cdot e^{-\zeta \omega (n \Delta t - \tau)} \cos(\tilde{\omega}(t-\tau)) \, d\tau
\]

\[
= \Sigma_{i=n_b+1}^{n} \frac{1}{2} \int_{\tau=(i-1)\Delta t}^{i \Delta t} \left[ i \cdot F_{i-1} - (i-1) \cdot F_i + \frac{F_i - F_{i-1}}{\Delta t} \cdot \tau \right] \cdot \left[ e^{(\tilde{\omega} - \zeta \omega)(n \Delta t - \tau)} + e^{-(\tilde{\omega} + \zeta \omega)(n \Delta t - \tau)} \right] \, d\tau
\]

Substitute:

\[ \tilde{\omega} - \zeta \omega = A \]
\[ -(\tilde{\omega} + \zeta \omega) = B \]

\[
= \Sigma_{i=n_b+1}^{n} \left\{ \frac{1}{2} \left[ i \cdot F_{i-1} - (i-1) \cdot F_i \right] \int_{\tau=(i-1)\Delta t}^{i \Delta t} \left[ e^{A(n \Delta t - \tau)} + e^{B(n \Delta t - \tau)} \right] \, d\tau + \right\}
\]

\[
= \frac{1}{2} \left[ \frac{F_i - F_{i-1}}{\Delta t} \right] \cdot \int_{\tau=(i-1)\Delta t}^{i \Delta t} r \cdot \left[ e^{A(n \Delta t - \tau)} + e^{B(n \Delta t - \tau)} \right] \, d\tau
\]
\[
\begin{align*}
\sum_{i=nb+1}^{n} \left[ \frac{1}{2} \left[ F_{i} - F_{i-1} \right] \left( -i \Delta t \cdot e^{A(n-i)\Delta t} + (i-1) \Delta t \cdot e^{A(n-i+1)\Delta t} \right) - e^{A(n-i)\Delta t} + e^{A(n-i+1)\Delta t} \right] + \\
\frac{1}{A^2} \sum_{i=nb+1}^{n} \left[ -i \Delta t \cdot e^{B(n-i)\Delta t} + (i-1) \Delta t \cdot e^{B(n-i+1)\Delta t} \right] + \\
\frac{-e^{B(n-i)\Delta t} + e^{B(n-i+1)\Delta t}}{B^2} \\
\sum_{i=nb+1}^{n} \left[ \frac{1}{2} \left[ F_{i} - F_{i-1} \right] \left( -e^{B(n-i)\Delta t} + e^{B(n-i+1)\Delta t} \right) \right] + \\
\frac{-e^{B(n-i)\Delta t} + e^{B(n-i+1)\Delta t}}{B^2}
\end{align*}
\]

\[
\sum_{i=nb+1}^{n} \left[ \frac{1}{2} \left[ F_{i} - F_{i-1} \right] \left( -e^{A(n-i)\Delta t} + e^{A(n-i+1)\Delta t} \right) \right] + \\
\frac{-e^{A(n-i)\Delta t} + e^{A(n-i+1)\Delta t}}{A^2}
\]

\[
\sum_{i=nb+1}^{n} \left[ \frac{1}{2} \left[ F_{i} - F_{i-1} \right] \left( -e^{A(n-i)\Delta t} + e^{A(n-i+1)\Delta t} \right) \right] + \\
\frac{-e^{A(n-i)\Delta t} + e^{A(n-i+1)\Delta t}}{A^2}
\]

\[
\sum_{i=nb+1}^{n} \left[ \frac{1}{2} \left[ F_{i} - F_{i-1} \right] \left( -e^{B(n-i)\Delta t} + e^{B(n-i+1)\Delta t} \right) \right] + \\
\frac{-e^{B(n-i)\Delta t} + e^{B(n-i+1)\Delta t}}{B^2}
\]

\[
\sum_{i=nb+1}^{n} \left[ \frac{1}{2} \left[ F_{i} - F_{i-1} \right] \left( -e^{A(n-i)\Delta t} + e^{A(n-i+1)\Delta t} \right) \right] + \\
\frac{-e^{A(n-i)\Delta t} + e^{A(n-i+1)\Delta t}}{A^2}
\]

\[
\sum_{i=nb+1}^{n} \left[ \frac{1}{2} \left[ F_{i} - F_{i-1} \right] \left( -e^{B(n-i)\Delta t} + e^{B(n-i+1)\Delta t} \right) \right] + \\
\frac{-e^{B(n-i)\Delta t} + e^{B(n-i+1)\Delta t}}{B^2}
\]

\[
\sum_{i=nb+1}^{n} \left[ \frac{1}{2} \left[ F_{i} - F_{i-1} \right] \left( -e^{A(n-i)\Delta t} + e^{A(n-i+1)\Delta t} \right) \right] + \\
\frac{-e^{A(n-i)\Delta t} + e^{A(n-i+1)\Delta t}}{A^2}
\]

\[
\sum_{i=nb+1}^{n} \left[ \frac{1}{2} \left[ F_{i} - F_{i-1} \right] \left( -e^{B(n-i)\Delta t} + e^{B(n-i+1)\Delta t} \right) \right] + \\
\frac{-e^{B(n-i)\Delta t} + e^{B(n-i+1)\Delta t}}{B^2}
\]

where:

\[
D_i = \frac{1}{2} \frac{A e^{B(n-i)\Delta t} + B e^{A(n-i)\Delta t}}{AB}
\]

\[
= \frac{e^{-\zeta \omega(n-i)\Delta t}}{\omega^2} \left[ -\zeta \omega \cos(\omega(n-i)\Delta t) + \omega \sin(\omega(n-i)\Delta t) \right]
\]

(E.4)
\[ E_i = \frac{1}{2} \frac{A^2 e^B(n-i)\Delta t}{A^2 B^2} + B^2 e^A(n-i)\Delta t \]

\[ = e^{-\zeta \omega (n-i)\Delta t} \cdot \left[ -(1-2\zeta^2) \cdot \cos(\omega(n-i)\Delta t) - 2\sqrt{1-\zeta^2} \cdot \sin(\omega(n-i)\Delta t) \right] \] (E.5)

Substitution of equation (E.4) and (E.5) in equation (E.3) results in:

\[
\int_{\tau=t_b}^{t} F(\tau) \cdot e^{-\zeta \omega (t-\tau)} \cos(\omega(t-\tau)) \, d\tau =
\]

\[ = \sum_{i=n_b+1}^{n} -F_i \cdot D_i + F_{i-1} \cdot D_{i-1} + \frac{1}{\Delta t} \left[ [F_i - F_{i-1}] \cdot [E_i - E_{i-1}] \right] + \]

\[ \frac{1}{\Delta t} \sum_{i=n_b+1}^{n} [F_{i-1} \cdot E_i + F_i \cdot E_{i-1}] \]

\[ = -F_n \cdot D_n + F_{n_b} \cdot D_{n_b} - \frac{1}{\Delta t} \left[ F_n \cdot E_n + F_{n_b} \cdot E_{n_b} + \sum_{i=n_b+1}^{n-1} 2F_i \cdot E_i - \sum_{j=n_b+1}^{n-1} F_j \cdot E_{j+1} \right] - \]

\[ \sum_{i=n_b+1}^{n} F_i \cdot E_{i-1} \]

\[ = -F_n \cdot D_n + F_{n_b} \cdot D_{n_b} - \frac{1}{\Delta t} \left[ F_n \cdot E_n + F_{n_b} \cdot E_{n_b} - F_{n_b} \cdot E_{n_b+1} - F_n \cdot E_{n-1} - \right] \]

\[ \sum_{i=n_b+1}^{n-1} F_i \cdot [E_{i-1} - 2E_i + E_{i+1}] \]

\[ = F_n \left[ -D_n + \frac{1}{\Delta t} (E_n - E_{n-1}) \right] + F_{n_b} \left[ D_{n_b} + \frac{1}{\Delta t} (E_{n_b+1} - E_{n_b}) \right] - \]

\[ \frac{1}{\Delta t} \sum_{i=n_b+1}^{n-1} F_i \cdot [-E_{i-1} + 2E_i - E_{i+1}] \]
The solution for the integral \( \int_{\tau=t_b}^{t} F(\tau) \cdot e^{-\zeta\omega(t-\tau)} \sin(\tilde{\omega}(t-\tau)) \, d\tau \) is:

\[
F_n \left[ \frac{\zeta \omega}{\omega^2} + \frac{1}{\omega^2 \Delta t} \left[ (1-2\zeta^2)\left[ 1 - e^{-\zeta\omega \Delta t \cos(\tilde{\omega}\Delta t)} - 2\sqrt{1-\zeta^2} e^{-\zeta\omega \Delta t \sin(\tilde{\omega}\Delta t)} \right] \right] \right.
\]

\[
F_{n_b} \cdot \frac{e^{-\zeta\omega(n-n_b)\Delta t}}{\omega^2} \cdot \left[ -\zeta \cdot \omega \cdot \cos(\tilde{\omega}(n-n_b)\Delta t) + \tilde{\omega} \cdot \sin(\tilde{\omega}(n-n_b)\Delta t) \right] +
\]

\[
\frac{F_{n_b}}{\omega^2 \Delta t} \cdot e^{-\zeta\omega(n-n_b)\Delta t} \cdot \left[ (1-2\zeta^2)\cos(\tilde{\omega}(n-n_b-1)\Delta t) - 2\sqrt{1-\zeta^2} \sin(\tilde{\omega}(n-n_b-1)\Delta t) \right] +
\]

\[
\frac{i}{\omega^2 \Delta t} \sum_{i=n_b+1}^{n-1} F_i \cdot e^{-\zeta\omega(n-i)\Delta t} \cdot (1-2\zeta^2) \cdot \left[ e^{-\zeta\omega \Delta t \cos(\tilde{\omega}(n-i+1)\Delta t)} - 2\cos(\tilde{\omega}(n-i)\Delta t) \right.
\]

\[
+ e^{\zeta\omega \Delta t} \cos(\tilde{\omega}(n-i-1)\Delta t) \right] -
\]

\[
\frac{i}{\omega^2 \Delta t} \sum_{i=n_b+1}^{n-1} F_i \cdot e^{-\zeta\omega(n-i)\Delta t} \cdot 2\sqrt{1-\zeta^2} \cdot \left[ e^{-\zeta\omega \Delta t \sin(\tilde{\omega}(n-i+1)\Delta t)} - 2\sin(\tilde{\omega}(n-i)\Delta t) \right.
\]

\[
+ e^{\zeta\omega \Delta t} \sin(\tilde{\omega}(n-i-1)\Delta t) \right]
\]

(E.6)

For \( n - nb \leq 2 \) the summons will vanish, and equation (E.6) becomes:

\[
\int_{\tau=t_b}^{t} F(\tau) \cdot e^{-\zeta\omega(t-\tau)} \cos(\tilde{\omega}(t-\tau)) \, d\tau
\]

\[
= F_n \left[ \frac{\zeta \omega}{\omega^2} + \frac{1}{\omega^2 \Delta t} \left[ (1-2\zeta^2)\left[ 1 - e^{-\zeta\omega \Delta t \cos(\tilde{\omega}\Delta t)} - 2\sqrt{1-\zeta^2} e^{-\zeta\omega \Delta t \sin(\tilde{\omega}\Delta t)} \right] \right] \right.
\]

\[
F_{n_b} \cdot \frac{e^{-\zeta\omega\Delta t}}{\omega^2} \cdot \left[ -\zeta \cdot \omega \cdot \cos(\tilde{\omega}\Delta t) + \tilde{\omega} \cdot \sin(\tilde{\omega}\Delta t) \right] +
\]

\[
\frac{F_{n_b}}{\omega^2 \Delta t} \cdot \left[ (1-2\zeta^2) \right] +
\]

\[
\frac{F_{n_b}}{\omega^2 \Delta t} \cdot e^{-\zeta\omega\Delta t} \cdot \left[ (1-2\zeta^2)\cos(\tilde{\omega}\Delta t) + 2\sqrt{1-\zeta^2} \sin(\tilde{\omega}\Delta t) \right]
\]

(E.7)
Appendix F  Localization of the contact point on the bell: \((R_c,Z_c)\)

To begin with a part the bell profile is described by a circle. The position of the centre of curvature \(M\) depends on the position of the points \(P\), \(Q\) and \(S\) on this part of the bell profile (fig. F.1):

\[
|ON| = L_{clapp} \\
|OC| = \sqrt{(Z_{pivot} - Z_c)^2 + R_c^2} \\
|MC| = \text{radius of curvature: } R_w \\
(R_m,Z_m) = \text{centre of curvature for part of the bell profile between } P, Q \text{ and } S. \\
(R_n,Z_n) = \text{position of centre of clapper ball} \\
R_{ball} = \text{radius of clapper ball.}
\]

![Diagram](image)

**Fig. F.1:** Clapper position at initial contact with the bell.

The centre \(M = (R_m,Z_m)\) of the circle through \(P\), \(Q\) and \(S\) can be determined from:

\[
(R_m-R_p)^2 + (Z_m-Z_p)^2 = (R_m-R_q)^2 + (Z_m-Z_q)^2 \\
(R_m-R_p)^2 + (Z_m-Z_p)^2 = (R_m-R_s)^2 + (Z_m-Z_s)^2
\]

(F.1a)  
(F.1b)

From these equations \(R_m\) and \(Z_m\) can be solved (Veldpaus [1976]):

\[
R_m = R_p + \frac{1}{4A} \cdot \{(Z_q-Z_p) \cdot \{ (R_s-R_p)^2 + (Z_s-Z_p)^2 \} + \\
-(Z_s-Z_p) \cdot \{ (R_q-R_p)^2 + (Z_q-Z_p)^2 \} \} \\
Z_m = Z_p + \frac{1}{4A} \cdot \{-(R_q-R_p) \cdot \{ (R_s-R_p)^2 + (Z_s-Z_p)^2 \} + \\
(R_s-R_p) \cdot \{ (R_q-R_p)^2 + (Z_q-Z_p)^2 \} \}
\]

(F.2a)  
(F.2b)

where

\[
A = \frac{1}{2} \cdot \{(R_s-R_p) \cdot (Z_q-Z_p) - (Z_s-Z_p) \cdot (R_q-R_p)\}
\]

(F.3)

If the value of \(A\) (\(|A| = \text{area of } \Delta PQS\)) is less than a given limit, this part of the bell profile will be described by a straight line. At this point in appendix G.1, showing a flowchart of the routine to determine coordinates of the contact point and number and
The trajectory of the center N of the clapper ball is:
\[ R^2 + (Z_{\text{pivot}} - Z)^2 = L_{\text{clapp}}^2 \]  
(F.4)

The point N moves towards the bell profile until it encounters the circle:
\[ (R_m - R)^2 + (Z_m - Z)^2 = (R_w + R_{\text{ball}})^2 \] in case of positive curvature  
\[ = (R_w - R_{\text{ball}})^2 \]  
"negative"  
(F.5a)

(F.5b)

If the point M = (R_m, Z_m) lies outside the bell profile, the bell has a positive curvature; otherwise it has a negative curvature. The radius of curvature R_w is:
\[ R_w = \sqrt{[(R_m - R_p)^2 + (Z_m - Z_p)^2]/2} \]  
(F.6)

From (F.4) and (F.5) can be derived that the point (R_n, Z_n) lies on the line:
\[ R = \frac{1}{2} \cdot R_m \cdot [L_{\text{clapp}}^2 - (R \pm R_{\text{ball}})^2 - Z_{\text{pivot}}^2 + R_m^2 + Z_m^2] + \frac{1}{R_m} \cdot [Z_{\text{pivot}} - Z_n] \cdot Z \]  
(F.7)

Substitution of (F.7) in (F.4) results in the z-coordinate Z_n of the intersection of the trajectory of point N with circle (F.5):
\[ Z_n = \frac{(Z_{\text{pivot}} - V \cdot W) \pm \sqrt{(1 + V^2) \cdot L_{\text{clapp}}^2 - (W + V \cdot Z_{\text{pivot}})^2}}{1 + V^2} \]  
(F.8)

For reasons of geometry, only the negative root is of interest.

Substitution of expression (F.8) in (F.7) gives the r-coordinate R_n of the intersection. The true contact point C lies on the line through the points M and N, at a distance R_{\text{ball}} from point N. The slope of the line, perpendicular to the bell surface in the contact point is:
\[ \tan(\beta) = \frac{Z_m - Z_n}{R_m - R_n} \]  
(F.9)

In the case a circle description for the bell profile is used, the coordinates of the contact point are:
\[ R_c = R_n + R_{\text{ball}} \cdot \cos(\beta) \]
\[ Z_c = Z_n + R_{\text{ball}} \cdot \sin(\beta) \]  
(F.10)
If the surface of $\Delta PQS$ is too small, we will describe the bell profile as a line through the points $P$ and $S$.

$$|ON| = L_{clapp}$$

$$|OC| = \sqrt{(Z_{pivot} - Z_c)^2 + R_c^2}$$

$(R_n, Z_n) = \text{position of centre of clapper ball}$

$R_{ball} = \text{radius of clapper ball}$.  

$\beta = \text{angle of the normal vector to the bell wall in the contact point and the } r\text{–axis}$

$\beta_1 = \text{angle between the normal vector and the direction of the clapper velocity}$

**Fig. F.2: Clapper position at initial contact with the bell.**

In this case, independent of the precise position of $C$ on line $PS$, the slope of the line perpendicular to the bell surface in the contact point $C$ is:

$$\tan(\beta) = \frac{R_p - R_s}{Z_s - Z_p}$$  \hspace{1cm} (F.11)

The intersection between the trajectory of $N$ and a line parallel to $PS$ at a distance $R_{ball}$ of the bell wall has to be determined. $P'$ and $S'$ lie on the line parallel to $PS$:

$$R_i' = R_i - R_{ball} \cdot \cos(\beta)$$

$$Z_i' = Z_i - R_{ball} \cdot \sin(\beta), \quad \text{for } i \in \{p, s\}$$  \hspace{1cm} (F.12)

The line through the points $P'$ and $S'$ is described by:

$$Z = -\frac{Z_{s'}}{R_{p'}-R_{s'}} \cdot R + \frac{Z_{s'} \cdot R_{p'}-Z_{p'} \cdot R_{s'}}{R_{p'}-R_{s'}} = I \cdot R + K$$  \hspace{1cm} (F.13)

The coordinates of the intersection of this line with the trajectory of $N$ are:

$$R_n = \frac{(K-Z_{pivot}) \cdot I}{1+I^2} \pm \frac{1}{1+I^2} \cdot \sqrt{(1+I^2) \cdot L_{clapp}^2 - (K-Z_{pivot})^2}$$  \hspace{1cm} (F.14)

where the negative root usually will be of no interest

$$Z_n = \frac{K+Z_{pivot} \cdot I^2}{1+I^2} - \frac{1}{1+I^2} \cdot \sqrt{(1+I^2) \cdot L_{clapp}^2 - (K-Z_{pivot})^2}$$  \hspace{1cm} (F.15)

Now the coordinates of the contact point on the bell are:
In the expression for the equivalent construction nodal forces (section 2.1.3) interpolation functions \( N = N(\xi, \eta) \) are used. The local coordinates of the contact point \((\xi_c, \eta_c)\) that correspond with the global coordinates \((R_c, Z_c)\) now have to be computed.

Two axi–symmetrical element types can be used in Dynopt. For both types of elements the following relation between the global coordinates of a point \((R, Z)\) in the element and the global coordinates of the nodal circles are:

\[
Z = N_t'(\xi, \eta) \cdot Z^e
\]

\[
R = N_t'(\xi, \eta) \cdot R^e
\]

where the columns \( R^e \) and \( Z^e \) contain the \( r \)–coordinates and the \( z \)–coordinates of the nodal circles of the element respectively.

(Elements for which both the coordinates and the displacements can be written using relation (F.17) are iso–parametric elements (Van Asperen, [1984]).)

To determine the local coordinates \((\xi_c, \eta_c)\) of the contact point \((R_c, Z_c)\) on a QUAX8 element, we need the interpolation functions:

\[
N_1(\xi, \eta) = 1/4(1-\xi)(1-\eta)(\xi-\eta-1)
\]

\[
N_2(\xi, \eta) = 1/4(1-\xi^2)(1-\eta)
\]

\[
N_3(\xi, \eta) = 1/4(1+\xi)(1-\eta)(\xi-\eta-1)
\]

\[
N_4(\xi, \eta) = 1/4(1+\xi)(1-\eta^2)
\]

\[
N_5(\xi, \eta) = 1/4((1+\xi)(1+\eta)(\xi+\eta-1)
\]

\[
N_6(\xi, \eta) = 1/4(1-\xi^2)(1+\eta)
\]

\[
N_7(\xi, \eta) = 1/4(1-\xi)(1+\eta)(-\xi+\eta-1)
\]

\[
N_8(\xi, \eta) = 1/4(1-\xi)(1-\eta^2)
\]

\(-1 \leq \xi \leq 1, \quad -1 \leq \eta \leq 1\)  

Figure F.3 QUAX8 element with its interpolation functions

Since the contact point lies on the inside of the bell wall, it can directly be seen that

\[
\xi_c = -1
\]

and

\[
Z_c = N_t'(\xi, \eta) \cdot Z^e = N_7(\xi, \eta) \cdot Z_7 + N_8(\xi, \eta) \cdot Z_8 + N_1(\xi, \eta) \cdot Z_1
\]

The substitution of equations (F.18) and (F.19) in equation (F.20) gives:
If \( Z_7 = Z_4 \), the (undeformed) element (and therefore the contact surface) lies horizontal, and instead of (F.19) and (F.20) the following relations have to be used:

\[
\begin{align*}
\eta_c &= -1 \\ 
R_c &= N^t(\xi, \eta) \cdot Z^e
\end{align*}
\]

Solving the equation (F.21) results in:

\[
\begin{align*}
\eta_c &= \frac{-(Z_7-Z_4) + \sqrt{(Z_7-Z_4)^2 - 8(Z_8-Z_0)(Z_4+Z_7-2Z_8)}}{2(Z_1+Z_7-2Z_8)} \\
\end{align*}
\]

If \((Z_1+Z_7-2Z_8) = 0\), the solution to (F.21) becomes:

\[
\eta_c = 2 \cdot \frac{Z_0-Z_8}{Z_7-Z_1}
\]

The change of axes in the case of a horizontal plane of contact on the bell wall (see (F.22) and (F.23)) should prevent the occurrence of a zero denominator in equation (F.25).

The determination of the local coordinates \((\xi_c, \eta_c)\) of the contact point on a TRIAX6 element is done in an analogue way.
Since the contact point lies on the inside of the bell, the \( \xi \)-coordinate of the contact point is equal to 0
\[
\xi_c = 0 \quad \text{(F.27)}
\]
The global \( z \)-coordinate of the contact point is related to the \( z \)-coordinates of the nodal circles of the element in the following way
\[
Z_c = N^t(\xi, \eta) \cdot Z^e = N_5(\xi, \eta)Z_5 + N_6(\xi, \eta)Z_6 + N_1(\xi, \eta)Z_1
\quad \text{(F.28)}
\]
The substitution of equation (F.26) and (F.27) in equation (F.28) results in the equation
\[
2(Z_1 + Z_5 - 2Z_6)\eta^2 + (4Z_6 - 3Z_1 - Z_5)\eta + Z_4 - Z_c = 0
\quad \text{(F.29)}
\]
From equation (F.30) the local \( \eta \)-coordinate of the contact point is determined
\[
\eta_c = \frac{Z_c - Z_1}{4Z_6 - 3Z_1 - Z_5}
\quad \text{(F.30)}
\]
under the conditions
\[
\frac{Z_1 + Z_5}{2} = Z_6 \quad \text{T} \quad 3Z_1 + Z_5 \neq 4Z_6
\quad \text{(F.31)}
\]
or
\[
Z_1 \neq Z_5
\quad \text{(F.32)}
\]
If the second condition in (F.31) isn't met, the solution for \( \eta_c \) becomes
\[
\eta_c = -\frac{(4Z_6 - 3Z_1 - Z_5) + \sqrt{(4Z_6 - 3Z_1 - Z_5)^2 + 8(Z_c - Z_1)(Z_4 + Z_5 - 2Z_6)}}{4(Z_1 + Z_5 - 2Z_6)}
\quad \text{(F.33)}
\]
If the surface of the element on which the contact point lies is horizontal, a change of axes is performed.
Instead of \( \xi_c = 0 \), the \( \eta \)-coordinate is chosen equal to zero
\[
\eta_c = 0
\quad \text{(F.34)}
\]
and
\[
R_c = N^t(\xi, \eta) \cdot R^e
\quad \text{(F.35)}
\]
The local \( \xi \)-coordinate is determined in an analogue way.
Dynopt

Appendix G  Flowcharts of some important subroutines for response calculation in Dynopt

Flow of subroutine CEIGAE
June 1989, M. v.d. Sanden TUE/NNW

START
I = 0

I = I+1

/ I > MEND-3 ? yes STOP

no:

read coordinates of points 1, I+1, I+2

/ missing points ? yes STOP

no:

/ points lying on a line ? no

yes:

line:
compute BETA
compute ROOT
compute SIG

yes:

/ ROOT < 0.3 \ yes:

no:

compute (XX,YY)
compute SIG
SIGNE = 1.0-10
compute BETA

no:

/ C lying between 1, I+2 ?

yes:

/ clapper rod touches profile ? yes STOP

no:

/ point 1 is unique ? yes

no:

/ next point J > I+1 ? no STOP

yes:

I = J+1

/ C lying on conatr. node ? yes

no:

JNDFOR = 1
I = J+1

C is right:
compute SIG
compute angle AS
compute CIAP

yes:

JNDFOR = 1 2

no:

determine forces over element nodes:

/ right nodes 1, I+1, I+2 ? yes

no:

/ right nodes J-1, I+1 ? yes

no:

/ right nodes J, I+2, I+3 ? yes

STUP

determine type of element C is lying on

/ found nodes belong all to this element ? yes

no:

STOP

RETURN
Flow of subroutine COLLID
September 1989, M. v.d. Hadden

SUBROUTINE

N = 5

Set starting conditions:
WBELL(0) = 0.0, V_CLAPP(0) = 0.0
V_CLAPP(N) = V_CLAPP(I)
F(N) = 0.3

Initialize parameters
NBEGIN - NBELL

NBEGIN = N
N = N+1

if N > NEND ? yes N = N-1.RETURN

no

suppose F(N) = 0.0
compute WBELL(N), VBELL(N)
compute V_CLAPP(N)

WCLAPP(N) <- WBELL(N) ? yes

indeed F(N) > 0.0
compute WBELL(N), VBELL(N)
compute V_CLAPP(N)

NBEGIN = N
N = N+1

if N > NEND ? yes N = N-1.RETURN

no

indeed F(N) = 0.0
compute WBELL(N), VBELL(N)
compute V_CLAPP(N)

END OF COLLISION

NACT = N
FINI = 0.0
WBELL(N) - saved value
WBELL(N) = saved value
V_CLAPP(N) = saved value
compute V_CLAPP(N)
compute VBELL(N), VBELL(N)
RETURN