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Derivation and identification of the model of a mechanical manipulator

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SUMMARY

Knowledge of the behaviour of a mechanical manipulator in the form of a mathematical model can be useful for simulation and control purposes. By means of simulations the system behaviour can be studied, so it is not necessary to perform experiments with the system itself in view of considering safety or cost matters. In control applications the model can serve as a basis for the determination of an input signal, that causes manipulator parts to follow a desired trajectory.

For both applications it is needed that the model is not only a qualitative description of the manipulator but also a quantitative one, which implies a realistic representation of the system. This means that the model quantities determining the characteristic behaviour of the system have to get a numerical value. This is called model identification.

In this paper some background information is given concerning the derivation and the structure of mechanical manipulator models. The derivation of a model of a xy-table is discussed. This so-called xy-table is an experimental design of a mechanical manipulator, which is developed within the framework of the research work of T. Heeren [ref.1]. However, the main topics of this report are the discussion of an identification method, using measurement data and its application to the model of the xy-table.
NOTATION

A, a  scalar numbers
a     column matrix
A     matrix
I     unit matrix
O     zero matrix
a_i   i-th element of column a
A_{ij} element of matrix A on row i, column j

\(a^T, A^T\)  transposition
\(A^{-1}\)   inversion

\(A(x,y)\)  function arguments of A between round brackets, separated by commas

\([A B]\)  matrix combination

\(\dot{a}\)  first time derivative of \(a\)
\(\ddot{a}\)  second time derivative of \(a\)
\(\hat{a}\)  estimate of \(a\)

\([\text{ref.r}]\)  referred is to reference number r
CHAPTER 1: INTRODUCTION

The main topics of this report are the derivation and identification of a model of the experimental design of a mechanical manipulator. The manipulator in question is the so called xy-table, developed within the framework of the research work of T. Heeren. [ref.1].

In the context of this report a model is a set of mathematical equations, which gives an input–output relation that must be close to the input–output relation of the real manipulator. This mathematical description of the system creates the possibility to calculate the response, due to a certain input signal. The model can be used in off–line applications to predict the systems response for a certain input signal, and to study the effect of control algorithms on the manipulators behaviour. In this case the model is used as a simulation model. The model can also be used in on–line applications. In that case the model is used as a basis for a control law, which produces an input signal that causes manipulator parts follow a desired trajectory.

The simulation model and the control model do not have to be the same. A simulation model can be very complex and comprehensive to obtain a good response prediction. It also does not matter whether the computing effort, necessary to calculate the systems response, becomes relatively large. The control model however cannot be to complex, because the on–line computing time is limited. The available computing time to calculate the desired input signal, depends on the computing speed of the computer and the used sample frequency of the control algorithm.

The model contains a number of variables and constants. The model variables are the so called degrees of freedom, their first and second time derivatives, the inputs and the outputs. The outputs of the system are in general functions of the degrees of freedom, their time derivatives and the constants. The constants are called parameters. The parameter values have to be determined such, that the input–output relation, given by the model, is close to the input–output relation in reality. This is called identification. The parameters represent physical properties of the system components like inertia, stiffness, friction, damping, gravitation etc., but also geometrical properties like angles and lengths. These physical and geometrical properties determine, in combination with the system structure, the characteristic behaviour of
the system.

Chapter 2 of this report goes further into the matter of model derivation and identification. The structure of manipulator models and the importance of parameter determination are discussed. A method concerning model identification is presented. Next to a short description of the xy-table, the derivation of a model is given in chapter 3. The identification of the model derived in chapter 3 is discussed in chapter 4. Chapter 5 is the final chapter of this report. It contains the conclusions and some recommendations with respect to the matters discussed in this report.
CHAPTER 2:
DERIVATION AND IDENTIFICATION OF MANIPULATOR MODELS

2.1 Model derivation

The model exists of two sets of mathematical equations. One set contains the so called equations of motion, which determine the characteristic behaviour of the system. The equations of motion are in fact force or moment balances associated with the degrees of freedom used in the model. This set of equations is referred to as the system model. The other set of equations contains equations which couple quantities of the system to be measured with model quantities. This part of the model is called the measurement model.

The equations of motion are in general non-linear second order differential equations, which can be written in the following matrix notation:

\[ M(q, \dot{q}, t) \ddot{q} + h(q, \dot{q}, \dot{\theta}, t) + H(q, \dot{q}, \dot{\theta}, t) u(t) = 0 \]  

The \( n \)-dimensional column \( q \) accommodates the \( n \) chosen degrees of freedom. The \( n \times n \) mass–matrix \( M \) accommodates inertia terms. \( M \) is positive definite which physically implies that inertia is associated with all degrees of freedom and that the manipulators kinetic energy is always larger than or equal to zero and can only equal zero when the velocities of the modelled manipulator parts equal zero. The \( n \)-dimensional column \( h \) accommodates all terms which are not caused by inertia or by the actuators. The column \( h \) consists therefore of e.g. stiffness, damping, friction and gravitation terms. The \( m \)-dimensional column \( u(t) \) accommodates the adjustable system inputs. The actuator currents can be chosen as inputs when a linear–static relation exists between these currents and the torques or forces supplied by the actuators. In case the number of degrees of freedom is larger than the number of system inputs the manipulator can be called flexible. This implies that not all degrees of freedom can directly be influenced by the actuators. The \( n \times m \) amplification matrix \( H \) relates the system inputs to actuator forces or torques. The \( p \)-dimensional column \( \theta \) accommodates the model parameters. The model parameters are assumed to be constant. The inputs, the degrees of freedom and their first and second time derivatives can vary in time and are called the model variables.
The measurement model consists of —possible non-linear— algebraic equations. These can also be written in a matrix notation:

\[
\mathbf{z} = \mathbf{G}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \mathbf{t}) \mathbf{q} + \mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{t}) + \mathbf{R}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{t}) \mathbf{u}(t)
\]

The 1-dimensional column \( \mathbf{z} \) accommodates quantities to be measured. These can be displacements, velocities and accelerations of manipulator parts, but also actuator currents. The output of the system is measured in most cases. The measurements are coupled with model quantities by means of the \( 1 \times n \) matrix \( \mathbf{G} \), the \( 1 \times 1 \) column \( \mathbf{g} \) and the \( 1 \times m \) matrix \( \mathbf{R} \).

The input—output relation determined by the model has to be such, that response within the working area of the system can be described. The working area denotes the amplitude range and frequency range of the response. In general the frequencies to be predicted are committed to an upper boundary and the amplitudes to be predicted are committed to a lower boundary. This results in the fact that not all physical properties of the manipulator parts have to be taken into account to obtain an acceptable input—output relation. This can lead to a more simple mathematical description of the manipulator.

The complexity of the model, primarily determining the computing effort for simulation or control purposes, depends for a great deal on the number of degrees of freedom used in the model. In general there are as many equations of motion as there are degrees of freedom. The amount of computing effort can increase because more equations have to be solved when more degrees of freedom are introduced in the model.

The number of degrees of freedom can be reduced on basis of assumptions with respect to the physical properties of the manipulator parts. It takes of course a minimum number of degrees of freedom to describe a system's behaviour, because some manipulator parts have to make prescribed motions in space. More degrees of freedom are introduced by e.g. describing elasticity. The number of degrees of freedom, used to describe elasticity, can be reduced by assuming that some manipulator components are rigid. These assumptions can only be made when one is sure that frequencies and amplitudes in the system output, caused by the elasticity of these components, fall beyond the chosen working area of the system. This can be the case when friction and damping in the system are large enough to cause the unwanted responses to disappear fast, when the forces in the system are to small to cause any
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deformation of system parts at all or when the bandwidth of the controller does not cover any of the eigen-frequencies.

As soon as choices are made with respect to the number of degrees of freedom and with respect to the geometrical and physical properties which are to be modelled, the model can be derived by setting up moment or force balances of the manipulator. This results in equation 2.1. Methods that lead to equations of motion will not be discussed. For this matter the reader is referred to [ref.3]. Equation 2.2 can be derived by relating the quantities to be measured to the model quantities.

2.2 Model identification

Equations 2.1 and 2.2 form the model of the manipulator and give an input–output relation of the manipulator. The model parameter values have to be determined such, that the input–output relation given by the model is close to the input–output relation in reality. This is called model identification. The parameter values can be obtained by means of experiments with parts of the system which are to be modelled. In general this leads to great experimental efforts, because some physical properties, especially damping and friction, are hard to determine. Other properties, like masses and lengths, can be determined with a rather good accuracy by means of measurements.

Another possible way of identification is to confront the model with reality. This holds that the model parameters will be reconstructed by means of measurement data, acquired from the system. This is called estimation. When they come up for the constraint of observability, the measurements contain essentially the input–output relation of the manipulator. Observability implies in this context that both model variables and parameter can be reconstructed from the measurements. So observability implies also identifiability. This matter is not further discussed. A more extensive discussion of this matter is found in [ref.2].

When the measurements are reliable and the model is reasonably well, the parameters derived by means of an estimation method will lead to an input–output relation which is assumed to be better than an input–output relation derived by means of experiments with separate parts of the manipulator. This is due to the fact that the
effective influence of all system parts together is taken into account. This set of parameters incorporates the global effective influence of the physical properties of the system parts on the manipulators response.

Example:
Consider the system shown in figure 2.1. The mass $M$ is driven by means of a motor and a rigid belt. Coulomb friction is assumed in the motor, in both belt wheel bearings and in the sliding surface between the mass $M$ and its supporting underground. In the figure are stated: the angular displacement $\varphi$ of the belt wheel rigidly connected with the motor axis and the actuator torque $T$. These are also the quantities to be measured. The equation of motion can be written as:

$$m \ddot{\varphi} + w \text{sign}(\varphi) - T = 0$$

The parameter $m$ denotes an inertia term associated with $\varphi$, $w$ denotes a friction term and $T$ denotes the torque supplied by the motor. The estimated value of the friction term $w$ shall incorporate the sum of friction terms due to friction in the actuator, the bearings and the sliding surface. The estimated inertia term $m$ shall incorporate the mass $M$, the belt wheel radius and the moment of inertia of the motor. Possible neglected friction or inertia terms are also held by the estimates $m$ and $w$, because their effect is held by the measurements.

![Figure 2.1: a simple mechanical system](image)

Due to model errors it is in general not possible to fit the model exactly to the measurements. This means that must be searched for estimates which fulfill the model equations the best. This can be achieved by the use of an optimal estimation technique. The estimation method discussed in this chapter is based on the minimization of a cost function in a least-squares sense which is analogous to the
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optimal control concept. For estimation purposes the model will be rewritten in a compact form and the augmented state \( z \) and augmented input \( p \) are introduced as:

\[
x^T = [q \quad \dot{q} \quad \theta] ; \quad p^T = [u \quad \dot{u}] ;
\]

The model becomes:

\[
(2.3) \quad \dot{x} = Ax + Bp \\
(2.4) \quad Ez = f(x) + F(x)p
\]

with:

\[
A = \begin{bmatrix}
0 & I & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} ; \quad B = \begin{bmatrix}
0 & 0 \\
0 & I \\
0 & 0
\end{bmatrix}
\]

\[
E = \begin{bmatrix}
0 \\
I
\end{bmatrix} ; \quad f(x) = \begin{bmatrix}
h(x) \\
g(x)
\end{bmatrix} ; \quad F(x) = \begin{bmatrix}
H(x) & M(x) \\
R(x) & G(x)
\end{bmatrix}
\]

Equation 2.3 is added to the original model. It denotes that the first time derivative of \( q \) is equal to \( \dot{q} \), the first time derivative of \( \dot{q} \) is equal to \( \ddot{q} \) and the first time derivative of \( \theta \) equals \( \dot{\theta} \). This seems rather trivial but its significance is explained later.

The estimation model can be written in the following form ( \( ^\sim \) denotes estimated):

\[
\hat{x}^T = [\hat{q} \quad \hat{\dot{q}} \quad \hat{\theta}] ; \quad \hat{p}^T = [\hat{u} \quad \hat{\dot{u}}] ;
\]

\[
(2.5) \quad \dot{\hat{x}} = A\hat{x} + B\hat{p} + \xi \\
(2.6) \quad E\hat{x} = f(\hat{x}) + F(\hat{x})\hat{p} + \zeta
\]

The residuals \( \xi \) en \( \zeta \) are due to model errors. \( \xi \) Can also differ from zero because there is no reason to maintain the physical meaning of the model variables \( q, \dot{q} \) and \( \ddot{q} \) as displacements, velocities and accelerations, in case of model errors. This results in the fact that the model parameters \( \theta \) loose their physical meaning as well. Their estimates may be allowed time-varying and the parameter estimates may be considered as variables. This however can only be tolerated to a certain extent in our
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2.6

case, because the major goal is to find an input--output relation that is close to the input--output relation in reality. The wanted parameters don't have to represent the physical significance of a modelled system component, but they must indeed represent the global effective physical influence of the manipulator parts on the input--output relation. When the residuals \( \xi \) and \( \zeta \) become unacceptable large and the measurements are reliable, it is sensible to modify the model in the hope a better input--output relation can be derived.

As mentioned before the optimal estimation technique is optimal in a least-squares sense. The residuals \( \xi \) en \( \zeta \) will be minimized in a least-squares sense. This can be described in terms of an optimal control problem:

Find \( \hat{p}(\tau) \, \hat{z}(\tau) \, (t_0 < \tau < t) \) such, that the cost functional:

\[
J[\hat{p}(\tau); \hat{z}(\tau)] = \\
\int_{t_0}^{t} \left[ \frac{1}{2} \xi^T(\tau) W(\tau) \xi(\tau) + \frac{1}{2} \zeta^T(\tau) V(\tau) \zeta(\tau) \right] d\tau + \\
\frac{1}{2} (\hat{z}(\tau=t_0) - q_0)^T R_0 (\hat{z}(\tau=t_0) - q_0)
\]

is minimized under the constraints:

\[
\dot{\hat{z}}(\tau) = A\hat{z}(\tau) + B\hat{p}(\tau) + \xi(\tau) \\
E \hat{z}(\tau) = f((\dot{\hat{z}}(\tau)) + F((\dot{\hat{z}}(\tau))\hat{p}(\tau) + \zeta(\tau)
\]

By means of the weighing matrices \( W \) and \( V \) the confidence in the different model equations can be expressed. The matrix \( R_0 \) tells, if there is knowledge of the initial state in the form of \( q_0 \). \(-R_0 \) and \( R \) are not the same matrices.—The weighing matrices must be non-negative definite to guarantee the existence of a minimum. The matrix \( W \) must be positive definite in order to keep the physical coupling of the state estimates and to guarantee a unique solution. The matrix \( V \) may be semi-positive definite as long as the matrix \( F^T VF \) remains invertible in view of solution to the estimation problem [ref.2]. Applied to non-linear models the method seems suited for off-line identification (smoothing). A linear variant may be useful for on-line applications (filtering). The difference between smoothing and filtering is that estimates derived by means of smoothing are based on measurements over the time interval \([t_0,t_e]\) and estimates derived by means of filtering are based on measurements over the time interval \([t_0,t]\). Filtering can be considered as a recursive form of
Solutions of both variants will not be discussed in this paper. The linear variant can be used within the software packet ASP. This software packet is used to perform simulations with models of non-linear mechanical systems. For more detailed information with respect to solutions of the estimation problem the reader is referred to [ref.2].

2.3 Optimal estimation by means of a filter algorithm

This section describes the use of the optimal estimation method in a filter algorithm. This filter algorithm is called Optimal Filter and can be used as a utility within the software packet ASP. Yet, on-line applications are not available, among other facts mainly due to the amount of computing effort needed to solve the problem.

Model identification by means of the optimal estimation method is not very easy at first sight. This is due to the fact that there is no direct indication whether the model quality is good enough or not, or in other words which residual magnitude is realizable. Another problem is the choice of the weighing matrices \( W \) and \( V \).—The elements of \( R_0 \) are chosen infinitely large, so the initial state is considered as good. Wrong initial conditions are corrected by means of \( W \).—The elements of \( W \) and \( V \) may be chosen rather arbitrarily, as long as \( W \) is positive definite and \( V \) is semi-positive without affecting the solution procedure. Given a model, the intention is to choose the elements of \( W \) and \( V \) such, that the residuals \( \xi \) and \( \zeta \) are small enough.

The matrix \( W \) is being used to indicate whether the physical significance of the estimates as being positions, velocities, accelerations and constant parameters have to be maintained or not. The matrix \( V \) is being used to indicate the confidence put in the knowledge of the system. This knowledge is held by the model and the measurements. It seems sensible to make the knowledge of the system as big as possible. The model is based on assumptions with respect to the geometrical and the physical properties of the manipulator. The measurements have to be such, that the constraint of observability is satisfied. Next to this requirement the measurements have to be accurate enough so they can be relied on.
Murphy's law plays an important role in the choice of $W$ and $V$. The residuals held by $\xi$ and $\zeta$ will not be equal to zero due to model errors. When very much confidence is put in e.g. the physical significance of constant parameters the residual in the column $\xi$, associated with this feature, will be small. The consequence however is that the other residuals in $\xi$ and the residuals in $\zeta$ will probably be larger.

It seems sensible to spread the model errors over the residuals in $\xi$ and $\zeta$. This means that the residuals on the different model equations become of relative similar order of magnitude. This does not mean that all residuals become of relative equal magnitude since the model can have such a structure that e.g. the estimate of a certain measurement is the measurement itself. This results in the fact that the residual associated with this quantity is very small. Nevertheless a method to choose the weighing factors can be as follows.

Suppose:

$W$ has elements $w_{ii}$, $i = 1...2n+p$

$V$ has elements $v_{cc}$, $c = 1...n+m$

$d_e$ is the second time derivative of degree of freedom $e$, $e = 1...n$

then all quantities are taken relatively into the same account when:

$w_{ii}(x_i)^2$

$v_{cc}(d_c)^2$, $c = 1...n$

$v_{cc}(x_{c-n})^2$, $c = n+1...n+m$

have similar order of magnitude. This magnitude can be chosen equal to one.

The use of this method as basis for the choice of weighing factors is coupled with a few disadvantages. There must be some knowledge with respect to the order of magnitude of the model quantities. This is also handy with respect to initial estimates of the model quantities. Indications with regard to the magnitude of the model variables can be derived by studying the algebraic equations of the measurement model by means of the measurements which are to be used for the purpose of estimation. Indications with respect to the magnitude of the model parameters can be derived by means of simple experiments with parts of the manipulator, by means of calculating parameter values with the help of formulas and by means of looking up the values of
physical properties of manipulator components which are manufactured in the industry. These are rough but useful indications to get a picture of the order of magnitude.

Finally the off-line estimation procedure can be performed as follows. Choose the matrices $W$ and $V$ in such a way as discussed above. The weighing associated with the model parameters can be chosen relatively smaller so more freedom is assigned to the physical significance of the parameters as being constant physical properties. This results in the fact that the parameters can move to a more suited value than their initial condition. After a few estimation cycles over the same time interval the parameters are not constant, but their value at the end—of the time interval is equal to the value at the beginning of the time interval. Now their weighing can be strengthened so the residuals associated with them can decrease. Probably other residuals will increase but only to a certain extent. When the parameters can be considered constant the total estimation result can be studied. When all residuals are relatively small enough the result can be accepted and the identification of the model has succeeded.
CHAPTER 3:
DERIVATION OF A XY–TABLE MODEL

3.1 Introduction

This chapter describes the derivation of a xy–table model. There are no demands concerning a working area, as mentioned in chapter 2, but nevertheless the model is useful to obtain a global indication of the systems behaviour. For a detailed description of the xy–table the reader is referred to [ref.1]. This chapter only discusses some important modifications of the original experimental design. The measurements that can be obtained from the system will also be paid attention to, since these are important in view of the identification.

3.2 Derivation of a model

A top view of the xy–table is shown in figure 3.1. The elasticity of the xy–table is

![Figure 3.1: top view of the xy–table](image-url)
increased by means of an torsion elastic spindle. This results in the fact that the y-slideway can make a larger rotation in the xy-plane than was possible in the original experimental design.

The maximum static elastic deformations due to the elastic connections between the belts and the slides associated with the belt wheels 1, 2 and 3, and due the spindle are about 1 mm, 1 mm, 3 mm and 30 mm respectively. The eigen-frequencies due to these elastic components are about 11 Hz, 11 Hz, 25 Hz and 4 Hz respectively. All drives and transmissions are assumed to be rigid, except for the spindle, on basis of the above data. This results in the fact that the system has three mutual independent degrees of freedom. Friction is assumed to be coulomb friction. This means that the force on a body due to friction is opposite to the sign of the speed of the body. Static friction is not taken into account in the model.

Figure 3.2 shows the chosen model structure of the xy-table. The various quantities are defined as:

- $\varphi_1$: angular displacement of belt wheel 1
- $\varphi_2$: angular displacement of belt wheel 2
- $\varphi_3$: angular displacement of belt wheel 3
- $x_1$: position of x-slide 1 on slideway 1
- $x_2$: position of x-slide 2 on slideway 2
- $y$: position of the end-effector on the y-slideway
- $b$: distance between the slideways 1 and 2
- $c$: length of the y-slideway
- $r_x$: radius of the belt wheels 1 en 2
- $r_y$: radius of belt wheel 3
- $m_1$: mass of x-slide 1
- $m_2$: mass of x-slide 2
- $m_e$: mass of the end-effector (including the accelerometers)
- $m_y$: mass of the y-slideway (including the y-motor)
- $J_1$: moment of inertia associated with $\varphi_1$ (x-motor, transmission, spindle, belt, belt wheels)
- $J_2$: moment of inertia associated with $\varphi_2$ (spindle, belt, belt wheels)
- $J_3$: moment of inertia associated with $\varphi_3$ (y-motor, belt, belt wheels)
- $W_1$: friction torque associated with $\varphi_1$
- $W_2$: friction torque associated with $\varphi_2$
The experimental design incorporates the possibility to measure various manipulator quantities. The position of the end-effector in the horizontal xy-plane can be measured by means of an optical measurement system. For this purpose a reflector is mounted to the end-effector. This end-effector position is referred to as reflector position. The end-effector position can also be expressed in terms of angular positions of the motor axes. This so-called motor position of the end-effector equals the reflector position in case no clearances and no elastic deformations occur in the system. Furthermore, the currents of both motors are measured. These currents are measures for the torques supplied by the motors.

Next to the six measurements mentioned above the possibility is created to
measure end-effector accelerations. For this purpose two accelerometers are attached to the end-effector. These meters measure accelerations of the end-effector in two mutual perpendicular directions. The orientation of the acceleration measurement is the same as the orientation of the optical measurement system in case no clearances and elastic deformations occur. By attaching the accelerometers the mass of the end-effector is increased. This is taken into account in the model of the xy-table. The accelerometers are added to the system to increase the amount of measurement information. Measurement of accelerations supplies information with regard to forces due to inertia, which by definition appear explicitly in the equations of motion.

Measuring the reflector and motor positions has an accuracy of about 0.05 mm. The signals supplied by the accelerometers and amplifiers belonging to them, have a noise of about 0.05 m/s² which results in a relative inaccuracy of about 1%. The non-linearity of the accelerometers is smaller than 2% of the nominal signal. The measured currents have an unknown, but for this application an insignificant noise level. The measurements are assumed to be accurate enough for our case on the basis of the above data.

The model equations are of the form (2.1) and (2.2) presented and discussed in chapter 2 of this report. The chosen degrees of freedom in the model are \( \varphi_1, \varphi_2 \) and \( \varphi_3 \). The motor currents \( I_x \) and \( I_y \) are chosen as system inputs. This implies:

\[
q^T = [\varphi_1 \varphi_2 \varphi_3], \quad u^T = [I_x I_y],
\]

The quantities to be measured are defined as:

- end-effector positions: \( x_r, y_r \) [m]
- end-effector accelerations: \( a_{xr}, a_{yr} \) [m/s²]
- motor positions: \( x_m, y_m \) [m]
- motor currents: \( I_x, I_y \) [A]

This results in the column \( z \):

\[
z^T = [x_r \ y_r \ a_{xr} \ a_{yr} \ x_m \ y_m \ I_x \ I_y]
\]

Concluding the model describes a non-linear elastic system and consists of an 3*3 mass-matrix \( M \), a 3-dimensional column \( h \), a 3*2 matrix \( H \), a 8*3 matrix \( G \), a
8-dimensional column $g$ and a $8 \times 2$ matrix $R$. The detailed description of these model columns and matrices is found in appendix A.
CHAPTER 4:
IDENTIFICATION OF THE XY–TABLE MODEL

4.1 Introduction

The main topic of estimating all model parameters together is weakened, because observability can be endangered since some parameters appear in the same combination. It would also take a too large amount of computing effort to estimate a large number of parameters. Therefore is chosen to determine some parameters a priori and to estimate some others. Physical properties like masses and lengths can be determined with a rather good accuracy by means of simple measurement methods. In that way the following parameter values are determined:

\[ m_e = 2.3 \text{ kg} \]
\[ m_y = 8.5 \text{ kg} \]
\[ m_1 = m_2 = 3.8 \text{ kg} \]
\[ r_x = r_y = 1.0 \cdot 10^{-2} \text{ m} \]
\[ c = 1.25 \text{ m} \]
\[ b = 1.0 \text{ m} \]

The values of the amplification factors \( K_x \) and \( K_y \) are determined by means of information supplied by the motor fabricator. \( K_y \) is equal to \( 5.6 \cdot 10^{-2} \text{ Nm/A} \) and \( K_x \) is, due to the transmission ratio of 60/13 between the x–motor and the spindle, equal to \( 2.58 \cdot 10^{-1} \text{ Nm/A} \). Some additional assumptions with respect to physical properties of the system are that the moment of inertia \( J_2 \) and the friction moment \( W_2 \) are assumed to be equal to zero. Reasons for these assumptions are the lack of a motor, the lack of a transmission and presence of the least heaviest part of the spindle in drive 2.

The remaining parameters \( k, J_1, J_3, W_1 \) and \( W_3 \) shall be determined by means of the optimal estimation method discussed in chapter 2. The estimation procedure is performed off–line by means of a filter algorithm called Optimal Filter. This Optimal Filter is a utility of the software packet ASP. This software package is used to perform simulations with models of non–linear mechanical systems.

The estimation model contains the augmented state \( \mathbf{x} \) and the augmented input \( \mathbf{p} \):
\[ z^T = [\varphi_1 \varphi_2 \varphi_3 \dot{\varphi}_1 \dot{\varphi}_2 \dot{\varphi}_3 \ k \ J_1 \ J_3 \ W_1 \ W_3] \]

\[ p^T = [I_x \ I_y \ \dot{\varphi}_1 \ \dot{\varphi}_2 \ \dot{\varphi}_3] \]

The three degrees of freedom, the five parameters and the eight measurements lead to an 11x11 matrix \( W \) and an 11x11 matrix \( V \). First choices of the elements of these matrices are determined by means of the method discussed in chapter 2 of this report.

Next to the choice of weighing factors, the estimation method requires initial estimates of the augmented state elements. Initial values for the degrees of freedom and their first time derivatives can be obtained by means of the measurement data in combination with the measurement model. The initial parameter values are obtained by means of simple experiments with the system. This matter is discussed in appendix B and results in the initial parameter values:

\[ k = 4.6 \cdot 10^{-1} \text{ Nm/rad} \]
\[ W_1 = 3.2 \cdot 10^{-1} \text{ Nm} \]
\[ W_3 = 7.0 \cdot 10^{-2} \text{ Nm} \]
\[ J_1 = 1.28 \cdot 10^{-3} \text{ kgm}^2 \]
\[ J_3 = 6.0 \cdot 10^{-5} \text{ kgm}^2 \]

The subject of observability is discussed in [ref.2]. To prove the observability of the xy-table estimation model a rather ad-hoc method is used. Observability means that both model variables and model parameters can be reconstructed by means of measurements. It can be proved that the model variable \( q \) can be reconstructed in case the reflector positions, and the motor positions are measured. The variables \( \dot{q} \) en \( \ddot{q} \) can be determined out of \( q \) by means of differentiating. This holds that all model variables can be determined without interference of the model since the inputs are also measured.

To find an optimal set of parameters by means of optimal estimation the parameters are introduced in an augmented state description, so they are made time-varying. The identifiability of the unknown parameters can be affected. By studying the system model it can be seen that the parameters can be uniquely determined because they are all separately associated with the different model variables. This means that on every point of time a new set of equations is generated which can lead to a unique set of estimates.
4.2 Estimation tactics and results

Two sets of measurement data, used for identification purposes, are obtained from the xy-table. The first set of measurement data is obtained from letting the end-effector follow a straight line in the xy-plane. The other set is obtained from letting the end-effector perform a circle in the xy-plane. Examining the measurement data of the linear trajectory, depicted in figure 4.1, it can be seen that the line is followed six times away and back with no significant difference in resemblance of the measurement data. The only thing notable is the acceleration measurements which contain components of higher order.

The requirements concerning observability are satisfied when the reflector position, the motor position and the motor currents are measured. So first estimation results are obtained by taken only these measurements into account. This means that the weighing factors associated with the acceleration measurements are chosen equal to zero. Next, the influence of acceleration measurements is examined by taking into account these measurements too.

It is important to mention that it is rather difficult to judge the estimation results obtained by means of the optimal filter. The method introduces the parameters as state variables which indeed converge to certain values in the sense that they decline from these values and the return back with. The declension depends on the weighing associated with the parameter but also on the quality of the model. All values within this range can be accepted as sensible estimates and lead to similar results concerning physical significance and resemblance to reality. This can be shown when the weighing is strengthened and the parameter estimates remain constant. The parameter values mentioned below can be seen as average values.

The tactics used to identify the model are the following—the linear trajectory is used for identification purposes, the circular trajectory for verification. First, identification is performed by means of measurement data over the time interval 1.7 till 2.5 seconds. No changes in sign of the velocities occur in this time interval. The weighing associated with the parameters is chosen very small, so the parameter values can converge to the most suited value. Second, the length of the time interval is enlarged till 3.5 seconds, and the weighing of the parameters is strengthened. This leads to indications whether the model can be used for both positive and negative
identification of the xy–table model

*figure 4.1: measurement data linear trajectory*
velocities of the slides, and also whether the parameters can be considered as constants.

The following weighing matrices are used:

\[
W = \begin{bmatrix}
10^{-4} & 10^{-4} & 10^{-5} & 10^{-5} \\
10^{-4} & 10^{-5} & W_{77} & W_{88} \\
10^{-5} & W_{99} & W_{1010} & W_{1111} \\
W_{77} & W_{88} & W_{99} & W_{1010} \\
W_{1010} & W_{1111} & W_{1010} & W_{1111}
\end{bmatrix}
\]

in case 1:

- \( W_{77} = 10^{-2} \)
- \( W_{88} = 10^{2} \)
- \( W_{99} = 10^{4} \)
- \( W_{1010} = 10^{-2} \)
- \( W_{1111} = 10^{-2} \)

in case 2:

- \( W_{77} = 10 \)
- \( W_{88} = 10^{5} \)
- \( W_{99} = 10^{7} \)
- \( W_{1010} = 10 \)
- \( W_{1111} = 10 \)

\[
V = \begin{bmatrix}
10^{-4} & 10^{-4} & 10^{-4} & 10^{-4} & 10^{-4} \\
10^{-4} & 10 & 0 & 0 & 0 \\
10^{-4} & 0 & 10 & 0 & 0 \\
10^{-4} & 0 & 0 & 10 & 10 \\
10^{-4} & 0 & 0 & 10 & \cdot 10^{-2}
\end{bmatrix}
\]
The estimated parameter values obtained in case 1 and used as constant parameters in case 2 are:

- \( k = 27.5 \cdot 10^{-2} \text{ Nm/rad} \)
- \( J_1 = 2.33 \cdot 10^{-3} \text{ kgm}^4 \)
- \( J_3 = 2.37 \cdot 10^{-4} \text{ kgm}^4 \)
- \( W_1 = 4.79 \cdot 10^{-1} \text{ Nm} \)
- \( W_3 = 1.49 \cdot 10^{-1} \text{ Nm} \)
It is clear that the estimated values differ strongly from the initial values. This could mean that the parameters incorporate physical properties of the system parts from which the effective influence could not be determined a priori. This was expected as discussed in chapter 2 of this report.

In figure 4.2 the estimated reflector position and acceleration are compared with the real measurements. The resemblance to reality concerning the reflector position is very good, except for \( y_r \) for positive values of \( \hat{\varphi}_2 \). The residuals on the position measurements \( x_r \) and \( x_m \) are smaller than the deformation, so the estimate of \( k \) is assumed to be sensible. Another reason to trust the estimate of \( k \) is the fact that the estimate of \( \hat{\varphi}_2 \) shows a component with a frequency of about 4 Hz which exists in reality. This frequency of 4 Hz exists also in the estimate of \( ax_r \), but is not found in the real measurement of \( ax_r \). This could be due to the neglectance of damping associated with \( \varphi_2 \). The residuals on the equations and the current measurements are extremely small in comparison with the other residuals. These residuals cannot be increased by
means of another choice of the associated weighing factors, which implies that the chosen model can always be adapted to this set of position and input measurements. In other words: this set of measurements will always lead to a set of parameters which result in very small residuals on the equations and on the input measurements. The cause of this could be that non–conflicting information concerning system dynamics is used to find the estimates. The estimates of $q$, $\dot{q}$ and $\ddot{q}$ are namely deducted from the position measurements and the estimate of $u$ is $u$ measured.

Examining the physical significance, it is clear that the part of the model describing the motion of the end–effector in $x$–direction is good for both positive and negative values of $\dot{\varphi}_1$ and $\dot{\varphi}_2$. However, for positive values of $\dot{\varphi}_3$ the physical significance of the part of the model describing the motion of the end–effector in $y$–direction is fairly bad.

The estimate of the end–effector acceleration is good in a qualitative sense. There is only a global resemblance to reality which implies that the acceleration measurements contain a lot of information which cannot be described by means of this model. This becomes clear when identification is attempted by means of the acceleration measurements. The physical significance of the model and the resemblance with reality become much worse. The residuals associated with the three system equations and the input measurements increase tremendously. Since the results of the estimates obtained, taking the acceleration measurements into account, are worse compared to the results obtained in case 1, no parameter values are depicted. Notable is that the estimate of $k$ increases, may be due the higher order components in the acceleration measurements.

Examining the acceleration measurements, it is clear that e.g. some dynamics of higher order are not modelled. Some not modelled system dynamics might be the connection between the slides and the belts which can cause vibrations. It is also found that the rotations of the belt wheels are not purely centered due to bent axes to which the belt wheels are fixed. When the major peaks in both acceleration measurements are counted (figure 4.2), it can be shown that they could be due to the non–centrical rotations of the belt wheels. The end–effector is displaced over a distance of 40 mm in both directions. Associated with a belt wheel radius of 10 mm, this results in 6.4 rotations of the belt wheels, thus 6.4 peaks in the acceleration measurements.
Identifying the model over the time interval 2.6 till 3.5 seconds results in the following parameter values:

\[ k = 25.2 \cdot 10^{-2} \text{ Nm/rad} \]
\[ J_1 = 1.85 \cdot 10^{-3} \text{ kgm}^4 \]
\[ J_3 = 1.84 \cdot 10^{-4} \text{ kgm}^4 \]
\[ W_1 = 4.52 \cdot 10^{-1} \text{ Nm} \]
\[ W_3 = 1.68 \cdot 10^{-1} \text{ Nm} \]

Two important remarks can be made. First, the parameter values concerning the motion of the end-effector in x-direction differ from the results obtained before. Especially the estimate of \( J_1 \), which is decreased with about 20%. This was not expected as mentioned before. Second, and more important is that the estimation result concerning the motion of the end-effector leads not to a better physical significance and resemblance to reality. This means that the chosen model is not fit to describe the entire behaviour of the end-effector in y-direction. Identification by means of different time intervals lead to the same results. The parameter estimates have always similar values and the physical significance and the resemblance to reality are always of similar order. There is no clear indication whether the parameter estimates depend on the direction of motion of the end-effector. The only important conclusion is that the results concerning the motion of the end-effector in y-direction is always worse for positive values of \( \varphi_3 \).

In order to verify the estimation results some simulations are performed. The model, with the parameters obtained in case 1, is offered the measured input signals obtained from the circular trajectory over a time interval with a negative \( \varphi_3 \). In figure 4.4 the measured reflector position is compared with the simulated reflector position. It is clear that the resemblance to reality is rather well, but the difference with reality is in terms of centimeters.

The final conclusions that can be drawn after identification of this model by means of the optimal filter are the following. The reproducibility of the estimates is only good to a certain extent. Estimating by means of measurement data over different time intervals lead to different parameter values in the sense that they can differ from each other in about 20%. This results in the fact that the accuracy of the description of the motion of the end-effector is obtained in terms of centimeters, except for the specific intervals used to obtain the parameter estimates.
At first sight it is hard to say whether the model is fit for control purposes or not. The model gives a global indication of the system behaviour, but cannot describe the motion of system parts in terms of millimeters. The results, obtained by estimating over different time intervals, are indeed of similar order, and therefore useful in on-line applications. This can be assumed because the parameters have already a value which results in a fairly good description of reality. By means of a the matrix $W$ the parameters can be given some freedom, so they can be adapted a little bit, if necessary, to obtain a better system description. Neglecting the tremendous amount of computation effort, one might conclude that the control effort can decrease.

\[\text{figure 4.4: simulation with input data of circular trajectory}\]
CHAPTER 5:
CONCLUSIONS AND RECOMMENDATIONS

The derivation and identification of a model of the xy-table, by means of optimal filtering, is possible and succeeded to a certain extent. It is clear that the model is not complete. The resemblance to reality is achieved in order of centimeter. The use in control applications is not yet studied. Neglecting the enormous amount of computing effort, the control effort might decrease because the model can describe the motion of the end-effector fairly well.

Model identification is useful because the estimates of the model parameters incorporate the global physical influence of the modelled system parts. It is found that the model gives a better description of reality when the parameters are estimated in an optimal sense than when the parameters are obtained after simple calculations or experiments.

The Optimal Filter is suited for estimation purposes and it seems that, given a model, satisfying results can be obtained. The estimation model of the xy-table, presented in this report, requires reconstruction of eleven augmented state elements which results in a computing time of about forty minutes—per estimation run over the chosen time intervals—on an APOLLO work station. Because some experience is needed to judge the estimation results and since the influence of the weighing factors have to be studied, it might take days to find a proper optimal set of parameters. It seems that practical application of the filter in combination with this model is not possible in the near future.

Measurement of accelerations is very useful, because these measurements contain much information about system dynamics, which are difficult to deduce from position measurements or from studying the motions of the system parts. Examining acceleration measurements can help developing a system model. The use of acceleration measurements for identification purposes can however lead to bad results. This is due to the fact that the measurements can contain system dynamics which are not taken into account by the model. Thus the additional information given by the acceleration measurements can lead to a kind of conflict situation during the identification procedure. In that case the model can be modified, so a better description of reality can be achieved. Another possibility is that the acceleration measurements will not be
taken into account, so the identification can lead to a model which is suited to describe the systems behaviour the best it can.

Due to numerical problems it is yet not possible to solve the optimal estimation problem by means of a smoothing algorithm. It is therefore unknown whether better results can be obtained, since smoothing takes into account the measurements of the whole identification time interval.

In order to obtain a better simulation model, recommendations are: the introduction of friction associated with $\varphi_2$, the introduction of a description of the non-centrical rotation of the belt-wheels and a description of the elastic slide-belt connections. This will of course lead to a tremendous increase of identification effort, but this more comprehensive model might lead to a better input-output description. The cause of the structural difference in the behaviour of the end-effector motion over the $y$-slide, for positive and negative velocities, must also be investigated.

The influence of the weighing matrices on the condition of the numerical problem must be studied. Some choices can lead to a failure of the solution routines. The method described in this report is useful, but it seems sensible to derive a more mathematical supported method. It is also found that use of a sign function —used to describe friction— is only useful when a more or less smooth sign function is used. This means that no slopes with infinitive time derivatives are desirable.
APPENDIX A:
DETAILS OF THE XY-TABLE MODEL

This appendix gives a detailed description of the model matrices. Referred to figure 3.2.

\[ q^T = [\varphi_1 \quad \varphi_2 \quad \varphi_3] \quad \text{;} \quad u^T = [I_x \quad I_y] \]

For \(x_1, x_2, x_3\) are valid:

\[ x_1 = \varphi_1 r_x \]
\[ x_2 = \varphi_2 r_x \]
\[ x_3 = \varphi_3 r_y \]

The angle \(\alpha\) is introduced as:

\[ \alpha = \arctan\left(\frac{x_1 - x_2}{b}\right) \]

The angle \(\alpha\) is assumed to be small, so the following definitions become valid:

\[ \alpha \approx \frac{x_1 - x_2}{b} \quad ; \quad \cos\alpha \approx 1 \quad ; \quad \sin\alpha \approx \alpha \]

The equations of motion are derived with help of the method of Lagrange [ref.6] and the software packet MAPLE. This results in the following elements of \(M\) and \(h\):

\[ M_{11} = J_1 + [m_1 + \frac{1}{3}m_y(c_b)^2 + m_e(\frac{\varphi_2}{b} r_y)^2] r_x^2 \]

\[ M_{12} = M_{21} = \left[\frac{1}{2}m_y(c_b)^2 - \frac{1}{3}m_y(c_b)^2 + m_e(\frac{\varphi_2}{b} r_y) - m_e(\frac{\varphi_3}{b} r_y)^2\right] r_x^2 \]

\[ M_{13} = M_{31} = 0 \]

\[ M_{22} = [m_2 + m_y - m_y(c_b)^2 + \frac{1}{3}m_y(c_b)^2 + m_e - 2m_e(\frac{\varphi_3}{b} r_y) + m_e(\frac{\varphi_3}{b} r_y)^2] r_x^2 + J_2 \]

\[ M_{23} = M_{32} = [m_e(\frac{\varphi_1}{b} - \frac{\varphi_2}{b} r_y)] r_x^2 \]
The quantities to be measured are:

reflector positions: $x_r$ and $y_r$ [m]
end-effector accelerations: $a_{x_r}$ and $a_{y_r}$ [m/s²]
motor positions: $x_m$ and $y_m$ [m]
motor currents: $I_x$ and $I_y$ [A]

$z^T = [x_r \ y_r \ a_{x_r} \ a_{y_r} \ x_m \ y_m \ I_x \ I_y]$
\[ g_1 = \varphi_2 r_x + \varphi_3 r_y \alpha \]
\[ g_2 = b - \varphi_3 r_y \]
\[ g_3 = \frac{2 \varphi_3 \varphi_2 r_y}{1 + \alpha^2} \]
\[ g_4 = -\frac{2 \alpha \varphi_3 \varphi_2 r_x}{1 + \alpha^2} \]
\[ g_5 = \varphi_1 r_x \]
\[ g_6 = b - \varphi_3 r_y \]
\[ g_7 = g_8 = 0 \]
\[ G_{11} = G_{12} = G_{13} = G_{21} = G_{22} = G_{23} = 0 \]
\[ G_{31} = -\frac{\varphi_3 r_x r_y}{b(1 + \alpha^2)} \]
\[ G_{32} = \frac{b r_x \varphi_3 r_x r_y}{b(1 + \alpha^2)} \]
\[ G_{33} = 0 \]
\[ G_{41} = -\frac{\alpha \varphi_3 r_x r_y}{b(1 + \alpha^2)} \]
\[ G_{42} = \frac{\alpha \varphi_3 r_x r_y - \bar{b} a r_x}{b(1 + \alpha^2)} \]
\[ G_{43} = -\frac{r_y}{1 + \alpha^2} \]

\[ G_{51} = G_{52} = G_{53} = G_{61} = G_{62} = G_{63} = G_{71} = G_{72} = G_{73} = G_{81} = G_{82} = G_{83} = 0 \]

\[ R_{11} = R_{12} = R_{21} = R_{22} = R_{51} = R_{32} = R_{41} = R_{42} = R_{51} = R_{52} = R_{61} = R_{62} = 0 \]
\[ R_{71} = 1 \]
\[ R_{72} = R_{81} = 0 \]
\[ R_{87} = 1 \]
APPENDIX B:
INITIAL ESTIMATES

To obtain rough indications of the magnitude of the unknown model parameters some simple experiments are performed.

Initial values concerning the friction parameters $W_1$ and $W_3$ are obtained by means of pulling the end-effector over the $y$-slideway, with the help of a weighing spring. Considering a static experiment, the force needed to move the end-effector is used to compensate friction. The force read of the weighing spring is about 8 N. Associated with the belt wheel radius $r_y$, this results in a friction moment $W_3$ of $8 \cdot 10^{-2}$ Nm. Assuming that friction is mainly caused by motor friction, the friction moment $W_1$ equals $3.7 \cdot 10^{-1}$ Nm. This is due to the transmission ratio of 60/13 between the $x$-motor and the spindle.

The initial values with respect to the moments of inertia $J_1$ and $J_3$ are the values of the moments of inertia of the motors, supplied by the fabricator of the motors. This results in the fact that $J_3$ equals the given numerical value of $6 \cdot 10^{-5}$ kgm$^2$ and $J_1$ equals $1.28 \cdot 10^{-3}$ kgm$^2$ due to the transmission in drive 1.

The initial value of the stiffness factor of the torsion elastic spindle is the result of some simple calculations. The elasticity is caused by a thin strip of steel which is being twisted about its longitudinal axis. The formula used to calculate the torsion stiffness $k$ reads like this [ref 8]:

$$k = \frac{GI_p}{u} \quad [\text{Nm/rad}]$$

with:
- shear modulus $G$ of steel, $8.4 \cdot 10^{10}$ N/m$^2$
- length $u$ of the strip, $15 \cdot 10^{-2}$ m
- polar moment of inertia $I_p = ab^3 \left[ \frac{1}{3} - 0.21 \frac{b}{a} \left(1 - \frac{b^4}{12a^4}\right) \right]$ m$^4$

with:
- width of the strip $a = 20 \cdot 10^{-3}$ m
- thickness of the strip $b = 5 \cdot 10^{-4}$ m

The result of above data is an elasticity factor $k$ of $4.6 \cdot 10^{-1}$ Nm/rad.
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