MASTER

Modelling and analyzing the thermal effect on wafer lenses

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MODELLING AND ANALYZING THE THERMAL EFFECT ON WAFER LENSES

By

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Not I But Jesus
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Nomenclature

\( x, y, z \) \hspace{1cm} \text{Cartesian coordinates (Eulerian description)}

\( X, Y, Z \) \hspace{1cm} \text{Cartesian coordinates (Lagrangian description)}

\( r, \theta, z \) \hspace{1cm} \text{Cylindrical coordinates (Eulerian description)}

\( e_1, e_2, e_3 \) \hspace{1cm} \text{Unit Cartesian vectors (Eulerian description)}

\( E_A, E_B, E_C \) \hspace{1cm} \text{Unit Cartesian vectors (Lagrangian description)}

\( e_r, e_\theta, e_z \) \hspace{1cm} \text{Unit cylindrical vectors (Eulerian description)}

\( \mathcal{F} \) \hspace{1cm} \text{Deformation gradient tensor}

\( \mathcal{C} \) \hspace{1cm} \text{Right Cauchy-Green strain tensor}

\( \mathcal{B} \) \hspace{1cm} \text{Left Cauchy-Green strain tensor}

\( \mathcal{E} \) \hspace{1cm} \text{Lagrangian deformation tensor}

\( E \) \hspace{1cm} \text{Elasticity Modulus}

\( \mu \) \hspace{1cm} \text{Shear Modulus}

\( \nu \) \hspace{1cm} \text{Poisson’s ratio}

\( \sigma_{ij} \) \hspace{1cm} \text{Cauchy stresses}

\( \sigma_{ij,j} = \sum_{j=1}^{N} \frac{\partial \sigma_{ij}}{\partial x_j} \) (summation convention)

\( \sigma_x, \sigma_y, \sigma_z \) \hspace{1cm} \text{Normal components of stress parallel to } x, y, \text{ and } z \text{ axes}

\( \sigma_r, \sigma_\theta, \sigma_z \) \hspace{1cm} \text{Normal stress components in cylindrical coordinates}

\( \sigma_{xy}, \sigma_{xz}, \sigma_{yz} \) \hspace{1cm} \text{Shearing stress components in rectangular coordinates}

\( \sigma_{r\theta}, \sigma_{\theta z}, \sigma_{rz} \) \hspace{1cm} \text{Shearing stress components in cylindrical coordinates}
$\varepsilon_x, \varepsilon_y, \varepsilon_z$ Normal components of strain parallel to $x, y, \text{ and } z$ axes

$\varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz}$ Shearing strain components in rectangular coordinates

$\varepsilon_r, \varepsilon_\theta, \varepsilon_z$ Normal strain components in cylindrical coordinates

$\varepsilon_r \theta, \varepsilon_\theta z, \varepsilon_r z$ Shearing strain components in cylindrical coordinates

$\lambda$ Lamé's constant, Eigenvalue

$T$ Temperature

$\alpha$ Linear coefficient of thermal expansion, Angle

$\beta$ Volume coefficient of thermal expansion

$\rho$ Density
Abstract

A wafer lens comprises a polymer with a shape of lens and a cylindrical glass substrate on which the polymer is bonded. When the lens is subjected to a uniform temperature change, the thermal property mismatch between glass and polymer induces thermal stress on the lens which affects its performance. This research focuses on adequate modelling and analyzing the thermal stress effect on the lens. The thermal effect accounts for the change in optical lens shape, the refractive index and other optical parameters. In this study we solved a mathematical model problem using a bilayer material system subjected to a uniform temperature change. We describe the thermal stresses and deformations due to thermal property mismatch. We also solved thermal stress problem in a composite long cylinder formed from two different materials. A software package, called Comsol Multiphysics, is used to perform numerical simulations for the actual physical model, the wafer lens model. The numerical result from the software was first verified with the analytic solution for the bilayer material system. From the simulation result for the wafer lens, we develop a program that predict the new lens shape, change in refractive index and other optical parameters. The results of this project will be given as input for ray tracing software that calculate the new optical properties of the lens.
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Chapter 1

Introduction

1.1 Background

Anteryon BV is engaged in the production of optical lenses by means of the replication technique. The lens consists of a flat glass substrate containing a polymer pellet with a certain form for the lens to give a desired effect. In a high volume production environment, optical components or optical systems require low-cost materials and low-cost fabrication techniques. Polymers are widely used to fabricate lenses because of numerous advantageous characteristics, including low cost, ease of manufacture, and high transmission in the visible spectral region [8]. However, there are some issues that must be considered when using polymers as an optical component. The principal disadvantage of polymers is their relatively high coefficient of thermal expansion. The thermal expansion coefficient of polymers is approximately 5-10 times higher than that of glass. Lenses may encounter a wide range of temperatures both in production and utilization stage. Refractive optics have three important thermally sensitive properties:

- Surface curvatures,
• Refractive indices,

• Inter-element spacings and lens thicknesses.

This shows that the lens may perform poorly at temperature extremes unless special care is taken in their design.

1.2 The wafer lens

Wafer Optics is a new technology enabling high volume production of integrated optics using 8 inch wafer level packaging. Using its Wafer Optics production technology, Anteryon BV produces lens modules on wafer scale. The scalability of the company’s

Figure 1.1: Camera modules formed from lens modules

processes allows it to ramp production quickly for high volume applications such as mobile handsets. Lens wafers are made on glass substrates carrying large arrays of lenses which are formed by the Anteryon’s replication technology. Wafer scale
manufacturing process enables the production of a large volume of optical elements on a single glass wafer (see figure 1.3). These micro optical elements can be refractive such as (a)spherical, (a)cylindrical or free-form lenses or diffractive optical elements so called DOE’s.
For a better understanding of the thermal behaviour of a wafer lens, we first discuss the form and production of the lens.

1.2.1 Shape of the lens

Since both geometries of the lens part and the substrate part are axisymmetric, the wafer lens is an axisymmetric in its shape (see figure 1.4). The height of the lens, which is called buffer layer, is approximately 7-10 times smaller than the height of the glass substrate (see figure 1.7).

![Figure 1.4: Schematic representation of a lens](image)

1.2.2 Production of lenses

The production of wafer lenses at Anteryon BV involves several steps [16]. First, a liquid monomer (a UV curable lacquer) is placed in a mould as shown in figure 1.5. When the droplet is in the mould, a glass substrate will be printed on the mould so that the monomer spreads and takes the form of the mould. In order to harden and attach the monomer to the substrate, UV light is used (see figure 1.6). Once the mould is removed, the lens is further cured in an oven. After this the wafer passes
various stabilization procedures. Finally the stabilized lenses are sawed out and ready for use. Though production of lenses with the above method is relatively new and still in development, they are being fabricated in a high volume production environment. In practice approximately 4500 lenses are produced on a small circular shape wafer with a diameter of 200 mm [4].
1.2.3 Thermal effects

The thermal effect on a lens is a factor that plays a role in its ultimate performance. Here we present the main thermal effects when a lens is subjected to a uniform change of temperature.

- **Change of curvature**
  A lens has two geometric parameters that determine the radius: The height $P$ and the diameter $D$ of the lens (see figure 1.7). These geometric parameters will change when a uniform change of temperature is occurred in the lens.

![Figure 1.7: The geometric properties of the lens](image)

- **Change of refractive index**
  When the lens is subjected to a uniform change in temperature, its density will be changed. This leads to a change in the refractive index. The new refractive index as a function of temperature is

$$n(T) = n_0 + \frac{dn}{dT}(T - T_0)$$  \hspace{1cm} (1.1)

where, $n_0$ is refractive index at initial temperature, $dn/dT$ the rate of change of refractive index with temperature and $T_0$ is initial temperature. The rate of
change of refractive index with temperature is estimated as \([17]\).

\[
\frac{dn}{dT} \approx \kappa \beta
\]  

(1.2)

where \(\beta\) is the volume coefficient of thermal expansion and \(\kappa\) is a factor which is empirically found to be \(-0.56\) for a wide range of polymers.

- **Change of focal length**
  
The focal length of a lens is one of thermally sensitive optical parameters. It is related to the radius of curvature and refractive index of a lens.

### 1.3 The software environment

For the numerical simulation part, we use Comsol multiphysics. This software environment is a finite element simulation software package for various physics and engineering applications, especially coupled phenomena \([6]\). There are many application modules in the software which serve as template for various scientific problems.

The corresponding application module for the physical model we are going to consider is the solid stress-strain with thermal expansion. It combines solid, stress-strain with heat transfer by conduction, including thermal expansion in the structural loads using the temperature field. The software also provides tools for geometric modeling of the physical domain. The CAD import module allows users to import their CAD models from various CAD packages. We make the geometric model using SolidWorks and the files are imported to Comsol which enable us to do the simulation on the actual geometry of the physical domain.
1.4 Outline of the thesis

This thesis is organized as follows: in Chapter 2 we first give a brief recapitulation of
elasticity theory and thermoelastic equations. After this formulations, in Chapter 3,
we solve two model problems. We begin by solving a bilayer material system, which
is a theoretical model for the analysis of stresses in thermally mismatched structures.
This example shows the bending of the composite system caused by a uniform change
in temperature. We determine an expression for the radius of curvature and the axial
stresses. We then give a validation of our solution by comparing it with the numerical
simulation. In the second part of Chapter 3 we solve axis symmetric three dimensional
problem that serves as a prelude to the study of stresses and displacements in the
axis symmetric lens. We consider a long composite cylinder consisting of two axially
connected circular cylinders, made of different materials. The thermal stresses and
displacements due to a uniform rise in temperature are determined for both cylinders.
Chapter 4 is entirely devoted to model implementation with Comsol multiphysics.
First we implement a wedge model to study the relation between thermally induced
rate of change of refractive index of the lens materials and their volume coefficient
of expansion. We then consider three different wafer lenses for simulations and the
numerical results are presented. Finally, in Chapter 5, we survey some conclusions
that can be drawn from the analytic and numerical results. We conclude the chapter
by citing a possible direction for future work.
Chapter 2

Theory of elasticity and thermoelastic equations

To derive the stresses and displacements of our physical model we need the basic aspects of mechanics. In this Chapter, linear thermoelasticity theory is derived based on the fundamentals of continuum mechanics. The result will be used to derive deformations and stresses in our mathematical model problems.

2.1 Fundamentals of continuum mechanics

In continuum mechanics we define a continuous body $B$ as a set of material points, denoted as $X$, $Q$ etc. The region occupied by $B$ at certain time $t$ in the 3-dimensional Euclidean space $\mathbb{R}^3$ is referred to as the configuration $G$ of $B$ [15]:

$$ G := \{ \mathbf{x} \in \mathbb{R}^3 | \mathbf{x} = \mathbf{x}(X, t), X \in B, t \in \mathbb{R} \}. $$  \hspace{1cm} (2.1)
In general \( G \) is a function of time \( (G = G(t)) \). The family of configurations \( G(t) \quad t \in \mathbb{R} \) is called the \textit{motion} of \( B \) during the given time interval. The elements of \( G \), denoted by \( X \quad (X \in G \subset \mathbb{R}^3) \), are the position vectors of the material points \( X \in B \) at certain \( t \) with respect to the fixed origin \( O \) of \( \mathbb{R}^3 \). To describe the motion of \( B \) in a geometrical sense, that is by a mapping from \( \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), we label each material point \( X \in B \) with a reference position \( X \in \mathbb{R}^3 \) and we introduce the reference configuration \( G_r \) as the set of \( X \) for all \( X \in B \). The vector \( X \) does not change during the motion of \( X \) (as it is a label for \( X \)). The motion of \( X \) from \( X \) to \( x \) is described by the bijective mapping from \( \mathbb{R}^3 \rightarrow \mathbb{R}^3 \)

\[ \forall x \in G_r, t \in \mathbb{R} : x = x(X, t). \tag{2.2} \]

In this Chapter, we will use two different types of notations:

(i.) \textit{Tensor notation}. This notation is independent of any basis and is specially used for general formulations.

(ii.) \textit{Index notation}. Besides the tensor notation, we shall use the index notation. The index-notation uses components of vectors. These components always with respect to an orthonormal basis \( \{Oe_1, e_2, e_3\} \), especially used for explicit equations with respect to this basis. In this notation, automatic summation over double indices (summation convention) is applied.

\section*{2.2 Balance laws}

The theory of continuous media rests upon a system of global balance laws. In this subsection, we shall present the four mechanical balance laws. In order to formulate
global balance laws, we need the notion *material region*. Let $\mathcal{B}$ be a material body with current configuration $G(=G(t))$ and let $b$ be a subset of $\mathcal{B}(b \subset \mathcal{B})$, such that $b$ during the motion of $\mathcal{B}$ is occupied by the same set of material points. Then, $b$ is called a sub-body of $\mathcal{B}$. The configuration of any subset $b \subset \mathcal{B}$ at time $t$ will be denoted by $g(t)$ and its boundary by $\partial g(t)$. Notions as *material surface* $\mathcal{S}$ in $\mathcal{B}(\mathcal{S} \subset \mathcal{B})$ and *material curve* $\mathcal{K}(\mathcal{K} \subset \mathcal{B})$ are defined in a completely analogous way. The configuration of $\mathcal{S}$ at time $t$ is denoted by $\mathcal{S}(\mathcal{S} \subset \mathcal{B})$. The boundary $\partial \mathcal{S}$ of $\mathcal{S}$ is a material curve.

The general expression of a global balance law is of the form

$$\forall b \subset \mathcal{B} \quad \frac{d}{dt} \int_{g} \rho \Phi dV = \int_{g} \rho s dV + \int_{\partial g} t_{n} dS.$$  \hspace{1cm} (2.3)

Here, $\rho$ is the *mass density* and $\rho \Phi$ denotes the physical quantity considered in the balance, with $\Phi$ being either a scalar or a vector, $\rho s$ is the source per unit of time of $\rho \Phi$ and $t_{n}$ is the flux per unit of time over the boundary of $g(t)$, where $n$ denotes the unit outward normal to $\partial g(t)$. The field $\Phi$ can be either a function of the local coordinates $x$ and $t$ ($\Phi = \bar{\Phi}(x, t)$; Euler formulation), or of the material coordinates $X$ and $t$ ($\Phi = \tilde{\Phi}(X, t)$; Lagrangian formulation). Thus we have to distinguish between two time derivatives:

1. the local time derivative (notation $\frac{\partial}{\partial t}$)

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \bar{\Phi}(x, t)}{\partial t},$$ \hspace{1cm} (2.4)

2. the material (total) time derivative (notation $\frac{d}{dt}$ or superimposed $\cdot$)

$$\dot{\Phi} = \frac{\partial \Phi}{\partial t} = \frac{\partial \tilde{\Phi}(X, t)}{\partial t}.$$ \hspace{1cm} (2.5)
The relation between the two operators is

\[
\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + (\text{grad}\Phi, v),
\]

where \( v = \dot{x} \) is the velocity of the material point \( X \in \mathcal{B} \).

The four global conservation laws are derived from the general form given in equation 2.3. We now proceed by postulating them.

1. **Conservation of mass**: This asserts that the mass of a body, or of any portion of the body, is invariant under motion, that is, remains constant in every configuration. Thus, the material derivative is zero,

\[
\forall b \subset \mathcal{B} \quad \frac{d}{dt} \int g \rho dV = 0,
\]

yielding the local balance equation of mass or the *equation of continuity*

\[
\frac{d\rho}{dt} + \rho \text{div} v = \frac{\partial \rho}{\partial t} + \text{div}(\rho v).
\]

2. **Conservation of momentum**: The forces and torques acting on \( b \subset \mathcal{B} \) can be due to external sources (e.g., gravity) or arising from actions of the part of \( \mathcal{B} \) outside \( b \). Therefore, two distinct types of forces are recognized in continuum mechanics:

   (a) **body forces**: \( \mathbf{b} \), acting on particles in \( \mathcal{B} \) (e.g., gravity or magnetic forces);

   \( \rho \mathbf{b} \) is a force per unit of volume (\( N/m^3 \));

   (b) **contact forces or surface tractions**: \( \mathbf{t}_n \), arising from the action of one part of a body on an adjacent part across the separating surface of \( b \);

   \( \mathbf{t}_n \) is a force per unit of area (\( N/m^2 \)) acting on a surface element with outward
normal \( n \); \( t_n \) does not only depend on its point of action \( x(x \in \partial g) \) and the time \( t \), but depends also on the normal \( n \) in \( x \) on \( \partial g \), so \( t_n = (x, t, n) \). The surface traction \( t_n \) is also called the stress vector.

Conservation of momentum states that the rate of change of momentum equals the resultant force acting on the body. This resultant force is the sum of body forces \( b \) and surface tractions \( t_n \). In other words

\[
\frac{d}{dt} \int_g \rho \mathbf{v} dV = \int_g \rho \mathbf{b} dV + \int_{\partial g} \mathbf{t}_n dS, \quad \forall \mathbf{b} \subset \mathcal{B}. \tag{2.9}
\]

To convert this expression to its local version, we need to express \( t_n \) in \( n \), according to \( t_n(x, t, n) = T(x, t)n \). In continuum mechanics, this relation is referred to as Cauchy’s stress law. The tensor \( T \) is called the Cauchy stress tensor. The components of \( T \), denoted by \( \sigma_{ij} \),

\[
\sigma_{ij} = (\mathbf{e}_i, T \mathbf{e}_j), \quad i, j \in (1, 2, 3) \tag{2.10}
\]

are simply called the stresses; \( \sigma_{ij} \) represents the force per unit of area in \( \mathbf{e}_i \) direction acting on a surface element with outward normal in \( \mathbf{e}_j \) direction. In component form, Cauchy’s law reads

\[
t_i = (t_n(x, t, n), \mathbf{e}_i) = \sigma_{ij} n_j. \tag{2.11}
\]

From this relation, it follows immediately that

\[
\sigma_{ij} = (t_n(x, t, \mathbf{e}_j), \mathbf{e}_i) = (t_n(x, t, -\mathbf{e}_j), -\mathbf{e}_i), \tag{2.12}
\]

in accordance with the basic mechanical principle ‘action is reaction’. The local balance law of momentum, or equation of motion, states

\[
\forall x \in G \quad \text{div} \mathbf{T}^T + \rho \mathbf{b} = \rho \dot{\mathbf{v}} \quad \leftrightarrow \quad \sigma_{ij,j} + \rho b_i = \rho \dot{v}_i, \tag{2.13}
\]
and the associated discontinuity condition expresses the continuity of the stress vector across a material discontinuity surface $\Sigma$,
\[
\|T n\| = 0 \iff \|\sigma_{ij} n_j\| = 0, \text{ at } \Sigma. \tag{2.14}
\]

In case of a static situation, when $\rho \dot{v} = 0$, the equation of motion reduces to the \textit{equation of equilibrium}:
\[
\text{div}T + \rho \dot{b} = \rho \ddot{v} \iff \sigma_{ij,j} + \rho b_i = 0. \tag{2.15}
\]

3. \textit{Conservation of moment of momentum}: Moment of momentum is the phrase used to designate the moment of the linear momentum with respect to some point. This vector quantity is also frequently called the \textit{angular momentum}. The principle of angular momentum states that the rate of change of moment of momentum of a body with respect to a given point is equal to the moment of the surface and body forces with respect to that point:
\[
\frac{d}{dt} \int_g (x \times v) \rho dV = \int_g (x \times v) \rho dV + \int_{\partial g} (x \times t(n)) dS, \quad \forall b \subset B. \tag{2.16}
\]

We can also write this in component form as
\[
\frac{d}{dt} \int_g \rho \varepsilon_{ijk} x_j v_k dV = \int_g \rho \varepsilon_{ijk} x_j b_k dV + \int_{\partial g} \rho \varepsilon_{ijk} x_j (t(n))_k dS, \quad \forall b \subset B. \tag{2.17}
\]

with $\varepsilon_{ijk}$ the permutation symbol:
\[
\varepsilon_{ijk} = \begin{cases} 
1, & \text{if the numerical values of } ijk \text{ appear as in sequence 12312} \\
-1, & \text{if the numerical values of } ijk \text{ appear as in sequence 32132} \\
0, & \text{if numerical values of } ijk \text{ appear in any other sequence.}
\end{cases}
\]

Using the equation of motion in the localized version of 2.17, we have
\[
\varepsilon_{ijk} \sigma_{kj} = 0 \quad \Rightarrow \quad \sigma_{ji} = \sigma_{ij} \quad \iff \quad T^T = T \tag{2.18}
\]
which shows that the stress tensor is symmetric.
4. Conservation of energy: The balance law of energy is based on the first law of thermodynamics, which postulates that the change per unit of time of the energy contained in \( b \) is equal to the mechanical work \( P \) done on \( b \) (power of the mechanical forces) plus the heat \( Q \) supplied per unit of time. We split the total energy in \( b \) into the kinetic energy \( K \) (or energy of motion) and the internal energy \( W \) (being the remaining part of the total energy). The kinetic energy is due to the motion of \( B \), whereas the internal energy can contain the potential energy of the deformations (e.g., elastic energy), but also thermal and/or electromagnetic energy. The considerations above bring us to the following energy postulate,

\[
\dot{K} + \dot{W} = P + Q, \quad \forall b \subset B.
\] (2.19)

By definition, we have

\[
K(b) = \int_{g(t)} \frac{1}{2} \rho(v, v) dV,
\] (2.20)

\[
W(b) = \int_{g(t)} \rho E_{int} dV,
\] (2.21)

\[
P(b) = \int_{g(t)} (b, v) \rho dV + \int_{\partial g(t)} (t(n), v) dS,
\] (2.22)

\[
Q(b) = \int_{g(t)} \rho r dV - \int_{\partial g(t)} (q, n) dS,
\] (2.23)

with \( E_{int} \) the internal energy density per unit of mass, \( q \) the heat flux vector and \( \rho r \) the heat source. Thus, we have the equation of conservation of energy

\[
\frac{d}{dt} \int_{g(t)} \left( \frac{1}{2}(v, v) + E_{int} \right) \rho dV = \int_{g(t)} ((b, v) + r) \rho dV + \int_{\partial g(t)} ((t_n, v) - (q, n)) dS.
\]

Using the equation of motion, the local energy balance can be reduced to

\[
\rho \dot{E}_{int} = tr(TD) + \rho r - \text{div} q \quad \leftrightarrow \quad \rho \dot{E}_{int} = \sigma_{ij} D_{ij} + \rho r - q_{i,i},
\] (2.24)
where \( tr(.) \) stands for 'trace', and \( D \) is the rate of deformation tensor, having

\[
D_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)
\]  

(2.25)
as components.

### 2.3 Theory of elasticity

The notion *elasticity* is characterized by the following two conditions:

(i) The stress at the actual time \( t \) is only related to the strain at the same time \( t \),

(ii) The material has the property for complete recovery to a "natural" shape upon removal of the applied forces.

To describe the deformation i.e the change in volume and shape, we define the deformation gradient tensor \( \mathcal{F} \) as

**Definition 2.1.**

\[
\mathcal{F} = \frac{\partial x}{\partial X},
\]  

(2.26)

where \( x = x(X,t) \) and \( X \) are the current and the reference configuration, respectively.

A solid elastic body has a natural reference configuration \( G_r \), namely the undeformed state of \( B \). In this state the stresses are zero and the body always returns to this state after removal of the forces on \( B \). Hence in an elastic body no permanent deformations or residual stresses can occur. The bijective relation between the current
and the initial configuration leads to the existence of the inverse of the deformation tensor \( \mathcal{F} \) which is

\[
\mathcal{F}^{-1} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}}.
\] (2.27)

If \( \mathcal{F} = \mathcal{I} \), the body is undeformed, but the reciprocal statement (i.e. if the body is undeformed, \( \mathcal{F} = \mathcal{I} \)) is not generally true. For \( \mathcal{F} \) still contains rigid body rotations of \( \mathcal{B} \). Therefore, a more adequate measure for the real deformations is a rotation invariant deformation tensor.

**Definition 2.2.** The *Lagrangian deformation tensor* \( \mathcal{E} \) is defined as

\[
\mathcal{E} := \frac{1}{2}(\mathcal{F}^T \mathcal{F} - \mathcal{I}).
\] (2.28)

Note that this tensor is symmetric and rotation invariant. An important property of this tensor is that \( \mathcal{E} = 0 \), if and only if \( \mathcal{B} \) is undeformed.

**Definition 2.3.** The *displacement vector* \( \mathbf{u} \) for a point \( \mathcal{P} \in \mathcal{B} \) is defined as

\[
\mathbf{u} := \mathbf{x} - \mathbf{X}.
\] (2.29)

Note that this displacement \( \mathbf{u} \) can again be either a function of \( \mathbf{X} \) and \( t \) or a function of \( \mathbf{x} \) and \( t \).

Introducing the component notation \( \mathbf{X} = X_A \mathbf{e}_A, \ A \in (1,2,3) \), and \( \mathbf{x} = x_i \mathbf{e}_i, \ i \in (1,2,3) \), we can write (2.29) as

\[
u_A = x_i \delta_{iA} - X_A.
\] (2.30)

The deformation gradient \( \mathcal{F} \) can be expressed in the displacement \( \mathbf{u} \) as defined in (2.29), by use of \( \mathbf{x} = \mathbf{X} + \mathbf{u} \), which with (2.1) yields

\[
\mathcal{F} = \mathcal{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \quad \leftrightarrow \quad F_{iA} = \delta_{iA} + u_{iA}.
\] (2.31)
Substitution of the above expression in 2.2 gives

\[ E = \frac{1}{2} \left( \left( \frac{\partial u}{\partial X} + I \right)^T \left( \frac{\partial u}{\partial X} + I \right) - I \right) \],
\[ = \frac{1}{2} \left( \frac{\partial u}{\partial X} + \left( \frac{\partial u}{\partial X} \right)^T + \left( \frac{\partial u}{\partial X} \right)^T \left( \frac{\partial u}{\partial X} \right) \right). \] (2.32)

Or, in components,

\[ E_{AB} = \frac{1}{2} \left( u_{A,B} + u_{B,A} + u_{C,A} u_{C,B} \right), \] (2.34)

where we have introduced the notation \( A = \frac{\partial}{\partial X_A} \). Likewise we will use \( i = \frac{\partial}{\partial x_i} \).

**Definition 2.4.** The *spatial velocity gradient* \( \mathcal{L} \) is defined by

\[ \mathcal{L} := \frac{\partial v}{\partial x} = \text{grad} \ v \quad \leftrightarrow \quad L_{ij} := \frac{\partial v_i}{\partial x_j} = v_{i,j}, \] (2.35)

with \( v = v(x,t) \) the velocity; note that \( v = \dot{u} \). The material time derivative of the deformation gradient \( \mathbf{F} \) can be expressed in the velocity gradient \( \mathcal{L} \) by

\[ \dot{\mathbf{F}} = \frac{d}{dt} \frac{\partial x}{\partial X} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial X} = \mathcal{L} \mathbf{F} \] (2.36)

or

\[ \mathcal{L} = \dot{\mathbf{F}} \mathbf{F}^{-1} \] (2.37)

In the previous section we stated the balance laws. These laws form an incomplete system of equations for the unknown field variables introduced there. To make the system complete, in this case for a linearly elastic medium, it has to be supplemented by a set of constitutive equations. These constitutive equations relate the stresses to the deformations. We introduce stress tensor \( \mathbf{T} \) by means of Cauchy’s stress formula

\[ t_n = \mathbf{T} n, \] (2.38)
Figure 2.1: Cauchy stress in Cartesian coordinate

whose components are

\[ \sigma_{ij} = (e_i, t_n(x, t, e_j)). \tag{2.39} \]

For a visualization of the stress components in Cartesian components, see Figure 2.1.

The Cauchy stress tensor can be represented by a matrix.

\[ \mathbf{T} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \tag{2.40} \]

The eigenvalues of this matrix are called the *principal stresses*. The eigenvectors are called *principal stress directions*.

The stress vector on a surface through a material point can be written as the sum of two vectors, a vector perpendicular to the surface and a vector tangential to the surface, according to

\[ t_n = (n, Tn)n + (Tn - (n, Tn)n) = \sigma_n n + t_s, \tag{2.41} \]
with

$$\sigma_n = (n, Tn), \quad \text{and} \quad t_s = Tn - \sigma_n n = t_n - \sigma_n n. \quad (2.42)$$

These vectors are called normal stress vector and shear stress vector, respectively. Their respective magnitudes are called normal stress and shear stress.

**Definition 2.5.** A linear elastic medium is a medium in which the stress tensor $T$ and the internal energy density $E_{int}$ only depend on the Lagrangian deformation tensor $E$,

$$T = T(E). \quad (2.43)$$

### 2.4 Linear elasticity

With linear elasticity we mean a linear relationship between the stresses and the strains. For small deformations the elements of $E$ become small, i.e. the norm of $E$ is of order $\varepsilon$, with $0 < \varepsilon << 1$. Then under the neglect of terms of $O(\varepsilon^2)$, the constitutive equation for linear elastic behavior can be written as

$$T = \mathcal{C}(E) \quad \leftrightarrow \quad \sigma_{ij} = C_{ijkm} \varepsilon_{km}. \quad (2.44)$$

This equation is the most fundamental form of Hooke’s law; the 4-tensor $\mathcal{C}$ is called the elasticity tensor. When assuming the displacement gradients to be small everywhere compared with unity the distinction between the Eulerian and Lagrangian descriptions is negligible. We can now introduce an infinitesimal strain tensor as

$$\mathcal{E}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \quad (2.45)$$
In order to gather more information about the tensor $C$ in 2.44, we have to introduce the elastic energy density. The energy balance 2.24 combined with the neglect of thermal effects

$$\dot{E}_{int} = \frac{1}{\rho} \sigma_{ij} D_{ij}. \quad (2.46)$$

For small-deformation theory we may take $D_{ij} = \dot{\varepsilon}_{ij}$, so

$$\dot{E}_{int} = \frac{1}{\rho} \sigma_{ij} \dot{\varepsilon}_{ij}. \quad (2.47)$$

We introduce the elastic energy density $W$ per unit of volume by

$$W = \frac{1}{\rho} \dot{E}_{int}. \quad (2.48)$$

Note that $W$ is a function of the deformation only ($W = W(\varepsilon_{ij})$), so we obtain from 2.47

$$\sigma_{ij} = \frac{\partial E_{int}}{\partial \varepsilon_{ij}} = \frac{\partial W}{\partial \varepsilon_{ij}}. \quad (2.49)$$

An expansion of $W$ in terms of $\varepsilon_{ij}$, substituted in 2.47 gives

$$\sigma_{ij} = \frac{\partial W(0)}{\partial \varepsilon_{ij}} + \frac{\partial^2 W(0)}{\partial \varepsilon_{ij} \partial \varepsilon_{km}} \varepsilon_{km} + \ldots \quad (2.50)$$

Since there are no residual stresses in the undeformed state of the material, we get

$$\sigma_{ij} = \frac{\partial^2 W(0)}{\partial \varepsilon_{ij} \partial \varepsilon_{km}} \varepsilon_{km} = C_{ijkm} \varepsilon_{km}. \quad (2.51)$$

Changing the order of the partial derivatives gives $C_{ijkm} = C_{kmij}$. If a body’s elastic proper- ties are the same in every set of reference axes, we call the elastic material isotropic. For such materials, the constitutive equation has only two independent
elastic constants. The most general fourth-order isotropic tensor can be written in terms of Kronecker deltas as

\[ C_{ijkm} = \lambda \delta_{ij} \delta_{km} + \mu (\delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}) + \beta (\delta_{ik} \delta_{jm} - \delta_{im} \delta_{jk}), \]  

(2.52)

where \( \lambda, \mu \) and \( \beta \) are scalars. From the symmetry of tensor \( C \), it follows that \( \beta = -\beta = 0 \). Substitution finally gives Hooke’s law:

\[ T = \lambda tr(E) I + 2\mu E \leftrightarrow \sigma = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}, \]  

(2.53)

The Lamé constants, \( \lambda \) and \( \mu \), are material coefficients. Also other material constants can be introduced:

**Definition 2.6.** The *elasticity modulus*, or *Young’s modulus*, \( E \), relates the elongation in a certain direction to the tensile stress in that direction. The Young’s modulus is related to the Lamé parameters according to

\[ E = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}. \]  

(2.54)

**Definition 2.7.** *Poisson’s ratio*, \( \nu \), is defined as the measure for lateral contraction, and is related to \( \lambda \) and \( \mu \) according to

\[ \nu = \frac{\lambda}{2(\lambda + \mu)}. \]  

(2.55)

**Definition 2.8.** The *shear modulus*, \( G \), is defined as the ratio between shear stress and shear, and satisfies the relations

\[ G = \frac{E}{2(1 + \nu)} = \mu. \]  

(2.56)

**Definition 2.9.** The *bulk modulus*, \( K \), describes the resistance against compression and

\[ K = \frac{1}{3} (2\mu + 3\lambda) = \frac{E}{2(1 - 2\nu)}. \]  

(2.57)
2.5 Linear thermoelasticity

Consider a homogeneous isotropic linearly elastic medium, experiencing a relatively small change in temperature from a uniform constant reference temperature \( T_0 \) to a, possibly, non-uniform and time-dependent temperature \( T(x, t) \). Let \( \Delta T = \Delta T(x, t) \) denote the temperature difference, so that

\[
T(x, t) = T_0 + \Delta T(x, t).
\]  

(2.58)

where we require

\[
\frac{\Delta T(x, t)}{T_0} << 1.
\]  

(2.59)

The most striking effect of a rise in temperature in a thermoelastic medium is an expansion, which in any \( P \in B \) is identical in all directions; hence, a dilatation with \( \varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} \neq 0 \), and \( \varepsilon_{12} = \varepsilon_{23} = \varepsilon_{13} = 0 \). This dilation is proportional to the change in temperature \( T \). In linear theory the total deformation \( \varepsilon_{ij} \) may be written as the sum (this is so because in linear theory superposition of solutions is allowed)

\[
\varepsilon_{ij} = \varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)},
\]  

(2.60)

where \( \varepsilon_{ij}^{(1)} \) are the deformations due to the stresses in \( B \), following from Hooke’s law (6.37)

\[
\varepsilon_{ij}^{(1)} = \frac{1}{2\mu} \sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu} \delta_{ij} \sigma_{kk},
\]  

(2.61)

where as \( \varepsilon_{ij}^{(2)} \) is the dilation due to the thermal expansion, thus

\[
\varepsilon_{ij}^{(2)} = \alpha \Delta T \delta_{ij},
\]  

(2.62)
The coefficient of proportionality \( \alpha \) is called the *linear thermal expansion coefficient*; it is measured in \( K^{-1} \). The total deformation is known as Hooke-Duhammel’s law and given by

\[
\varepsilon_{ij} = \frac{1}{2\mu} \sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu} \delta_{ij}\sigma_{kk} + \alpha \Delta T \delta_{ij},
\]

(2.63)

The inverse of (2.63) reads

\[
\sigma_{ij} = \lambda \delta_{ij}\varepsilon_{kk} + 2\mu \varepsilon_{ij} - 3\lambda + 2\mu \alpha \Delta T \delta_{ij}
\]

(2.64)

In practical problems, also the temperature distribution \( \Delta T(x, t) \) is unknown. In these cases, \( \Delta T(x, t) \) must first be calculated, before we can apply the equations given above.

### 2.6 Two dimensional formulation of thermoelastic equations

Sometimes simplifications are necessary to be able to handle the problem analytically. When we assume specific body geometry and loading patterns the equations of elasticity can be reduced to a two-dimensional form. In plane stress problems, the geometry of the body is that of a thin plate with one dimension very much smaller than the other two. In plane strain problems, the geometry is that of a prismatic body having one dimension very much larger than the other two and having the loads perpendicular to and distributed uniformly with respect to this large dimension \([2, 14]\).

A state of plane strain is defined by the equation

\[
u_z = 0.
\]

(2.65)
A state of plane stress is defined by the equations

\[ \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0. \] (2.66)

The eight unknowns quantities \( \sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}, u_x, u_y \), should satisfy in either case the following eight equations:

3 Strain(deformation)-displacement relations

\[ \varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right). \] (2.67)

3 Hooke-Duhamel equations

\[ \varepsilon_{xx} = \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) + \alpha \Delta T, \quad \varepsilon_{yy} = \frac{1}{E} (\sigma_{yy} - \nu \sigma_{xx}) + \alpha \Delta T, \quad \varepsilon_{xy} = \frac{1}{2G} \sigma_{xy}, \] (2.68)

where \( \nu \) is Poisson’s ratio and \( G = \frac{E}{2(1+\nu)} \).

2 Equilibrium equations

\[ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \] (2.69)

provided that for the plane-strain formulation the constants \( E, \nu, \) and \( \alpha \) are replaced by \( E_1, \nu_1, \) and \( \alpha_1 \) respectively, where

\[ E_1 = \frac{E}{1 - \nu^2}, \quad \nu_1 = \frac{\nu}{1 - \nu}, \quad \alpha_1 = \alpha(1 + \nu). \] (2.70)

while \( G = \frac{E_1}{2(1+\nu_1)} \) remains unchanged.

These stresses and strains are accompanied by a stress component \( \sigma_{zz} \) given by

\[ \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) - \alpha E \Delta T = \frac{1}{1 + \nu_1}(\nu_1(\sigma_{xx} + \sigma_{yy}) - \alpha_1 E_1 \Delta T) \] (2.71)

for the case of plane strain, and by a strain component \( \varepsilon_{zz} \) given by

\[ \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) + \alpha \Delta T \] (2.72)

for the case of plane stress.


2.7 Thermoelastic equations in cylindrical coordinates

Equations of motion for the static condition in three-dimensional cylindrical coordinates \((r, \theta, z)\) in the absence of body forces are [5, 7].

\[
\begin{align*}
\frac{\partial \sigma_{rr}}{\partial r} + r^{-1} \frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{z r}}{\partial z} + r^{-1} (\sigma_{rr} - \sigma_{\theta \theta}) &= 0, \\
\frac{\partial \sigma_{\theta r}}{\partial r} + r^{-1} \frac{\partial \sigma_{\theta \theta}}{\partial \theta} + \frac{\partial \sigma_{z \theta}}{\partial z} + 2r^{-1} \sigma_{\theta \theta} &= 0, \\
\frac{\partial \sigma_{z r}}{\partial r} + r^{-1} \frac{\partial \sigma_{z \theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + r^{-1} \sigma_{zz} &= 0,
\end{align*}
\]

(2.73)

where

\[
T = \begin{pmatrix}
\sigma_{rr} & \sigma_{\theta r} & \sigma_{z r} \\
\sigma_{\theta r} & \sigma_{\theta \theta} & \sigma_{z \theta} \\
\sigma_{z r} & \sigma_{z \theta} & \sigma_{zz}
\end{pmatrix}
\]

(2.74)

is the stress tensor in cylindrical coordinate system.

The stress-strain relations are

\[
\begin{align*}
\sigma_{rr} &= (2\mu + \lambda)\varepsilon_{rr} + \lambda(\varepsilon_{\theta \theta} + \varepsilon_{zz}) - \beta(T - T_0), \\
\sigma_{\theta \theta} &= (2\mu + \lambda)\varepsilon_{\theta \theta} + \lambda(\varepsilon_{rr} + \varepsilon_{zz}) - \beta(T - T_0), \\
\sigma_{zz} &= (2\mu + \lambda)\varepsilon_{zz} + \lambda(\varepsilon_{rr} + \varepsilon_{\theta \theta}) - \beta(T - T_0), \\
\sigma_{r z} &= 2\mu \varepsilon_{rz}, \\
\sigma_{\theta z} &= 2\mu \varepsilon_{\theta z}, \\
\sigma_{r \theta} &= 2\mu \varepsilon_{r \theta},
\end{align*}
\]

(2.75)

where

\[
E = \begin{pmatrix}
\varepsilon_{rr} & \varepsilon_{r \theta} & \varepsilon_{r z} \\
\varepsilon_{\theta r} & \varepsilon_{\theta \theta} & \varepsilon_{\theta z} \\
\varepsilon_{z r} & \varepsilon_{z \theta} & \varepsilon_{zz}
\end{pmatrix}
\]

(2.76)

is the strain tensor in cylindrical coordinate system, \(\beta = (3\lambda + 2\mu)\alpha\) where \(\alpha\) is linear coefficient of thermal expansion and \(T_0\) and \(T\) are initial and final temperature.
respectively.

The strain-displacement relations in cylindrical coordinates are

\[ \varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \]
\[ \varepsilon_{\theta\theta} = r^{-1}(u_r + \frac{\partial u_\theta}{\partial \theta}), \]
\[ \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \]
\[ \varepsilon_{r\theta} = \frac{1}{2}(r^{-1}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - r^{-1}u_\theta), \]
\[ \varepsilon_{\theta z} = \frac{1}{2}(\frac{\partial u_\theta}{\partial z} + r^{-1}\frac{\partial u_z}{\partial \theta}), \]
\[ \varepsilon_{rz} = \frac{1}{2}(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}), \]

where \( u_r, u_\theta \) and \( u_z \) are components of displacement.

In particular, when a model can be regarded as axisymmetric stress and deformation in a solid of revolution, the displacement component \( u_\theta \) vanishes and \( u_r, u_z \) are independent of \( \theta \). The stress components are also independent of \( \theta \), two of them, \( \sigma_{r\theta} \) and \( \sigma_{\theta z} \), being zero. Therefore, we have the following reduced thermoelastic relations [7].

Equilibrium equations:

\[ \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \]
\[ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0. \]  

(2.78)

The stress-strain relations:

\[ \sigma_{rr} = (2\mu + \lambda)\varepsilon_{rr} + \lambda(\varepsilon_{\theta\theta} + \varepsilon_{zz}) - \beta(T - T_0), \]
\[ \sigma_{\theta\theta} = (2\mu + \lambda)\varepsilon_{\theta\theta} + \lambda(\varepsilon_{rr} + \varepsilon_{zz}) - \beta(T - T_0), \]
\[ \sigma_{zz} = (2\mu + \lambda)\varepsilon_{zz} + \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta}) - \beta(T - T_0), \]
\[ \sigma_{rz} = 2\mu\varepsilon_{rz}. \]  

(2.79)
The strain-displacement relations:

\[ 
\begin{align*}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, \\
\varepsilon_{\theta\theta} &= \frac{1}{r} \frac{\partial u_r}{\partial r}, \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z}, \\
\varepsilon_{rz} &= \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right). 
\end{align*} \]  

(2.80)

2.8 Problem formulation for the lens model

First from the actual physical phenomenon and the geometry of the lens we assume the following conditions

- The lens is axisymmetric,
- The two parts of the lens (polymer and glass) are isotropic,
- The distribution of temperature is uniform,
- The deformation is small and

![Figure 2.2: Cross sectional view of the lens in cylindrical coordinate system](image-url)
The lens is free from any external force.

Based on the above assumptions and the thermoelasticity we derive before, we have the following thermoelastic equations.

- **Equilibrium equations:**

  \[
  \frac{\partial \sigma_{rr}^i}{\partial r} + \frac{\partial \sigma_{rr}^i}{\partial z} + \frac{(\sigma_{rr}^i - \sigma_{\theta\theta}^i)}{r} = 0, \quad \frac{\partial \sigma_{rz}^i}{\partial r} + \frac{\partial \sigma_{zz}^i}{\partial z} + \frac{\sigma_{rz}^i}{r} = 0, \quad (2.81)
  \]

  where

  \[
  T^i = \begin{pmatrix} 
  \sigma_{rr}^i & \sigma_{r\theta}^i & \sigma_{rz}^i \\
  \sigma_{\theta r}^i & \sigma_{\theta\theta}^i & \sigma_{\theta z}^i \\
  \sigma_{z r}^i & \sigma_{z\theta}^i & \sigma_{zz}^i 
  \end{pmatrix}
  \]

  is the stress tensor and \( i \) refers to the two regions.

- **Stress-strain relations:**

  \[
  \begin{align*}
  \sigma_{rr}^i &= (2\mu^i + \lambda^i)\varepsilon_{rr}^i + \lambda^i(\varepsilon_{\theta\theta}^i + \varepsilon_{zz}^i) - \beta^i(T - T_0), \\
  \sigma_{\theta\theta}^i &= (2\mu^i + \lambda^i)\varepsilon_{\theta\theta}^i + \lambda^i(\varepsilon_{rr}^i + \varepsilon_{zz}^i) - \beta^i(T - T_0), \\
  \sigma_{zz}^i &= (2\mu^i + \lambda^i)\varepsilon_{zz}^i + \lambda^i(\varepsilon_{rr}^i + \varepsilon_{\theta\theta}^i) - \beta^i(T - T_0), \\
  \sigma_{rz}^i &= 2\mu^i\varepsilon_{rz}^i,
  \end{align*}
  \]

  where \( \lambda^i \) and \( \mu^i \) are Lamè constants and \( \beta^i = \frac{3\lambda^i + 2\mu^i}{3} \).

  \[
  \mathcal{E}^i = \begin{pmatrix} 
  \varepsilon_{rr}^i & \varepsilon_{r\theta}^i & \varepsilon_{rz}^i \\
  \varepsilon_{\theta r}^i & \varepsilon_{\theta\theta}^i & \varepsilon_{\theta z}^i \\
  \varepsilon_{z r}^i & \varepsilon_{z\theta}^i & \varepsilon_{zz}^i 
  \end{pmatrix}
  \]

  is strain tensor

- **Strain-displacement relations:**

  \[
  \begin{align*}
  \varepsilon_{rr}^i &= \frac{\partial u^i}{\partial r}, \quad \varepsilon_{\theta\theta}^i = \frac{u^i}{r}, \quad \varepsilon_{zz}^i = \frac{\partial w^i}{\partial z}, \quad \varepsilon_{rz}^i = \frac{1}{2} \left( \frac{\partial u^i}{\partial z} + \frac{\partial w^i}{\partial r} \right),
  \end{align*}
  \]

  where \( u^i \) and \( w^i \) are radial and vertical displacements, respectively.

Since the lens is free from external force we have
• Boundary conditions:

\[ T^i n = 0, \quad n \text{ is normal vector on } \partial \Omega^i. \]

and in the interface between the two parts of the lens we have

• Interface continuity conditions:

\[ \sigma_{zz}^1 = \sigma_{zz}^2, \quad \sigma_{rz}^1 = \sigma_{rz}^2, \quad z = 0, \]

\[ u^1 = u^2, \quad w^1 = w^2, \quad z = 0. \]

Finally, the solution is regular at \( r=0 \).
Chapter 3

Mathematical Modelling

3.1 Bilayer material system

3.1.1 Introduction

The study of thermal mismatch induced stresses and their role is a topic relevant to composite materials. A bilayer material system is a useful theoretical model for the analysis of stresses in thermally-mismatched structures and it has received considerable attention in the literature for theoretical modeling, numerical simulation and experimental analysis [9, 10, 11]. Here we derive equations with which we can predict the stresses and the bending effect on a thermally loaded composite system.

3.1.2 Pure bending of the system and axial stresses

We consider a simply supported system made of two bonded layers of polymer and glass, initially at reference temperature $T_0$. The temperature is slowly raised to a constant uniform temperature $T$, where the temperature change is $\Delta T = T - T_0$. 
The origin \((y = 0)\) is taken at the bonding surface of the continuous body as shown in the figure 3.1. We also assume

- The length is considerably longer than the width and the thickness,
- The deformation is small and has linear elastic behavior,
- Plane cross-sections remains plane before and after deformation.

This means that the thermal load produces only pure bending (no twisting and buckling) effect on the system [13]. Here,

\[
\sigma_{xx}^k \neq 0, \quad \text{and all the remaining} \quad \sigma_{ij}^k = 0 \quad k = 1, 2. \tag{3.1}
\]

and

\[
u^1_x = u^2_x, \quad u^1_y = u^2_y \quad \text{at} \quad y = 0. \tag{3.2}\]

Note that the boundary conditions are trivially satisfied at lateral surface.

### 3.1.3 Method of Solution I

As shown in the figure 3.1, consider two line elements of the system, \(\overline{CD}\) and \(\overline{AB}\), which are straight and along the axial direction with equal lengths before the deformation. Element \(\overline{CD}\) lies on the neutral axis, while element \(\overline{AB}\) is at a distance \(y\) from the neutral axis. The elongation of \(\overline{CD}\) and \(\overline{AB}\) elements may be written as:

\[
\overline{CD}' = (1 + \varepsilon_0) \overline{CD},
\]

\[
\overline{AB}' = (1 + \varepsilon) \overline{AB},
\]

\[
\frac{\overline{AB}'}{\overline{CD}'} = \frac{(r_y + y)}{r_y}. \tag{3.3}
\]
where $\varepsilon$ and $\varepsilon_0$ are strains of $AB$ and $CD$ elements, respectively, and $r_y$ is the radius of curvature of the system axis at $y = 0$ in the $xy$-plane. Dividing the second equation of (3.3) by the first equation and using the last of equation (3.3) gives

$$\frac{r_y + y}{r_y} = \frac{1 + \varepsilon}{1 + \varepsilon_0} \quad (3.4)$$

then, using small deformation theory, it yields

$$\varepsilon = \varepsilon_0 + \frac{y}{r_y} + \frac{y}{r_y} \varepsilon_0 \approx \varepsilon_0 + \frac{y}{r_y}. \quad (3.5)$$

The axial strains of each layer shown in Figure 1 are expressed by ($i = 1, 2$)

$$\varepsilon_{xx}^i = \frac{\sigma_{xx}}{E_i} + \alpha_i \Delta T, \quad (3.6)$$

$$\varepsilon_{xx}^i = \varepsilon_0 + \frac{y}{r_y}, \quad (3.7)$$
where \( \varepsilon_0 \) and \( r_y \) denote the strain and the radius of curvature at the bonding surface \( y = 0 \), respectively. From (3.6), we find

\[
\sigma_{xx}^i = E_i \varepsilon_{xx}^i - E_i \alpha_i \Delta T, \tag{3.8}
\]

\[
\sigma_{xx}^i = E_i \varepsilon_0 + E_i \frac{y}{r_y} - E_i \alpha_i \Delta T. \tag{3.9}
\]

Since external forces do not act on the continuous body, from equilibrium requirements, we have

\[
\int_{-h_1}^{0} \sigma_{xx}^1 dy + \int_{0}^{h_2} \sigma_{xx}^2 dy = 0, \tag{3.10}
\]

\[
\int_{-h_1}^{0} \sigma_{xx}^1 ydy + \int_{0}^{h_2} \sigma_{xx}^2 ydy = 0. \tag{3.11}
\]

Substituting the stresses \( \sigma_{xx}^1 \) and \( \sigma_{xx}^2 \) from (3.8) into (3.10), we get

\[
2(E_2 h_2 + E_1 h_1) \varepsilon_0 + (E_2 h_2^2 - E_1 h_1^2) \frac{1}{r_y} = 2 \int_{-h_1}^{0} E_1 \alpha_1 \Delta T dy + 2 \int_{0}^{h_2} E_2 \alpha_2 \Delta T dy. \tag{3.12}
\]

Similarly, substituting the stresses \( \sigma_{xx}^1 \) and \( \sigma_{xx}^2 \) from (3.8) into (3.11), we have

\[
3(E_2 h_2^2 - E_1 h_1^2) \varepsilon_0 + 2(E_2 h_2^3 + E_1 h_1^3) \frac{1}{r_y} = 6 \int_{-h_1}^{0} E_1 \alpha_1 \Delta T dy + 6 \int_{0}^{h_2} E_2 \alpha_2 \Delta T dy. \tag{3.13}
\]

Performing the integration, we find

\[
2(E_2 h_2 + E_1 h_1) \varepsilon_0 + \frac{1}{r_y} (E_2 h_2^2 - E_1 h_1^2) = -2 \Delta T (E_1 \alpha_1 h_1 - E_2 \alpha_2 h_2), \tag{3.14}
\]

\[
3(E_2 h_2^2 - E_1 h_1^2) \varepsilon_0 + \frac{2}{r_y} (E_2 h_2^3 + E_1 h_1^3) = -3 \Delta T (E_1 \alpha_1 h_1^2 - E_2 \alpha_2 h_2^2). \tag{3.15}
\]

Solving (3.14) for \( \varepsilon_0 \) and \( \frac{1}{r_y} \) we have

\[
\frac{1}{r_y} = \frac{6 \Delta T}{D} \frac{(E_2 \alpha_2 h_2^2 - E_1 \alpha_1 h_1^2)(E_1 h_1 + E_2 h_2) - (E_1 \alpha_1 h_1 + E_2 \alpha_2 h_2)(E_2 h_2^2 - E_1 h_1^2)}, \tag{3.16}
\]
\[ \varepsilon_0 = \frac{2\Delta T}{D} (2(E_1\alpha_1 h_1 + E_2\alpha_2 h_2)(E_1 h_1^2 + E_2 h_2^2) - \frac{3}{2} (E_2\alpha_2 h_2^2 - E_1\alpha_1 h_1^2)(E_2 h_2^2 - E_1 h_1^2)), \] 

(3.17)

where

\[ D = [(E_2 h_2^2 - E_1 h_1^2)^2 + 4E_1 E_2 h_1 h_2(h_1 + h_2)^2]. \]

We conclude that the stresses \( \sigma_{xx}^1 \) and \( \sigma_{xx}^2 \) are given by (3.8).

### 3.1.4 Method of Solution II

From Hooke’s Duhamel equations for layer 1, we have

\[ \varepsilon_{xx}^1 = \frac{1}{E_1}(\sigma_{xx}^1 - \nu_1 \sigma_{yy}^1) + \alpha_1 \Delta T, \varepsilon_{yy}^1 = \frac{1}{E_1}(\sigma_{yy}^1 - \nu_1 \sigma_{xx}^1) + \alpha_1 \Delta T, \varepsilon_{xy}^1 = \frac{1}{2G_1} \sigma_{xy}^1, \]

(3.18)

where \( \nu_1 \) is Poisson’s ratio and \( G_1 = \frac{E_1}{2(1+\nu_1)}. \)

From compatibility, the necessary and sufficient condition for the existence of single-valued continuous function \( u_x^1 \) and \( u_y^1 \) in terms of strains is

\[ \frac{\partial^2 \varepsilon_{xx}^1}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}^1}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}^1}{\partial x \partial y}. \]

(3.19)

Then from (3.18) we have

\[ \nabla^2 (\sigma_{xx}^1 + \sigma_{yy}^1) = 0, \]

(3.20)

and from equilibrium equations we have also

\[ \frac{\partial \sigma_{xx}^1}{\partial x} + \frac{\partial \sigma_{xy}^1}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}^1}{\partial x} + \frac{\partial \sigma_{yy}^1}{\partial y} = 0. \]

(3.21)

These equilibrium equations are satisfied if the stresses are derived from a stress function \( \phi \) (Airy’s stress function), in the following manner

\[ \sigma_{xx}^1 = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{xy}^1 = -\frac{\partial^2 \phi}{\partial x \partial y}, \quad \sigma_{yy}^1 = \frac{\partial^2 \phi}{\partial x^2}. \]

(3.22)
Substitution of the stresses into compatibility equation (3.20), we have

$$\nabla^4 \phi = 0,$$  \hspace{1cm} (3.23)

or, alternatively

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0.$$  \hspace{1cm} (3.24)

Now to determine \( \phi \), we introduce the slow variable \( X = \varepsilon x \) and we write

$$\phi = \phi(X, y; \varepsilon).$$  \hspace{1cm} (3.25)

From (3.24) we find

$$\varepsilon^4 \frac{\partial^4 \phi}{\partial X^4} + 2 \varepsilon^2 \frac{\partial^4 \phi}{\partial X^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0.$$  \hspace{1cm} (3.26)

Using the regular expansion for \( \phi \)

$$\phi(X, y; \varepsilon) = \phi_0(X, y) + \varepsilon^2 \phi_1(X, y) + O(\varepsilon^4),$$  \hspace{1cm} (3.27)

we obtain to leading order

$$\frac{\partial^4 \phi_0(X, y)}{\partial y^4} = 0.$$  \hspace{1cm} (3.28)

This shows that

$$\phi_0(X, y) = A_0(X)y^3 + B_0(X)y^2 + C_0(X)y + D_0(X),$$  \hspace{1cm} (3.29)

which implies

$$\sigma_{xx}^1(X, y) \approx \frac{\partial^2 \phi_0}{\partial y^2} = 6A_0(X)y + 2B_0(X),$$  \hspace{1cm} (3.30)

$$\sigma_{xy}^1(X, y) \approx -\frac{\partial^2 \phi_0}{\partial x \partial y} = -\varepsilon \left[3 \frac{dA_0}{dX} y^2 + 2 \frac{dB_0}{dX} y + \frac{dC_0}{dX}\right],$$  \hspace{1cm} (3.31)

$$\sigma_{yy}^1(X, y) \approx \frac{\partial^2 \phi_0}{\partial x^2} = \varepsilon^2 \left[ \frac{d^2 A_0}{dX^2} y^3 + \frac{d^2 B_0}{dX^2} y^2 + \frac{d^2 C_0}{dX^2} y + \frac{d^2 D_0}{dX^2} \right].$$  \hspace{1cm} (3.32)
from the assumptions given in (3.1) and equation (3.30) we have,

$$\frac{dA_0(X)}{dX} = 0, \quad \frac{dB_0(X)}{dX} = 0, \quad \frac{dC_0(X)}{dX} = 0,$$

(3.33)

and

$$\frac{d^2D_0(X)}{dX^2} = 0.$$

(3.34)

Therefore,

$$\phi_0 = A_0y^3 + B_0y^2 + C_0y + D_{01}X + D_{02},$$

(3.35)

where $A_0, B_0, C_0, D_{01}$ and $D_{02}$ are constants.

From (3.35) we find

$$\sigma^1_{xx} = 6A_0y + 2B_0, \quad \sigma^1_{xy} = 0, \quad \sigma^1_{yy} = 0,$$

(3.36)

where $A_0$ and $B_0$ are constants. A similar expression holds for the second layer.

Therefore,

$$\sigma^i_{xx} = 6A_iy + 2B_i, \quad i = 1, 2,$$

(3.37)

where $A_i$ and $B_i$ are constants. To determine these four constants we need four equations.

From equilibrium requirements, we have

$$\int_{-h_1}^{0} \sigma^1_{xx}dy + \int_{h_2}^{h_2} \sigma^2_{xx}dy = 0, \quad \int_{-h_1}^{0} \sigma^1_{xy}dy + \int_{h_1}^{0} \sigma^2_{xx}dy = 0,$$

(3.38)

which yields the following two equations

$$2B_1h_1 - 6A_1\frac{h_1^2}{2} + 2B_2h_2 + 6A_2\frac{h_2^2}{2} = 0,$$

(3.39)

$$-2B_1\frac{h_1^2}{2} + 6A_1\frac{h_1^3}{3} + 2B_2\frac{h_2^2}{2} + 6A_2\frac{h_2^3}{3} = 0.$$

(3.40)
From the compatibility condition (3.2)

\[ u_x^1 = u_x^2 \quad \forall x \quad \text{and at } y = 0, \quad (3.41) \]

we have

\[ \frac{\partial u_x^1(x, y)}{\partial x} = \frac{\partial u_x^2(x, y)}{\partial x} \quad \forall x \quad \text{and at } y = 0. \quad (3.42) \]

Therefore,

\[ \varepsilon_{xx}^1 = \varepsilon_{xx}^2 \quad \forall x \quad \text{and at } y = 0 \quad (3.43) \]

Substituting (3.18) and (3.36) into (3.43) gives

\[ \frac{2B_1 + \alpha_1 E_1 \Delta T}{E_1} = \frac{2B_2 + \alpha_2 E_2 \Delta T}{E_2}. \quad (3.44) \]

The fourth and last equation follows from the deformation-displacement equations, viz.,

\[ \varepsilon_{xx}^1 = \frac{\partial u_x^1}{\partial x}, \quad \varepsilon_{yy}^1 = \frac{\partial u_y^1}{\partial y}, \quad \varepsilon_{xy}^1 = \frac{1}{2} \left[ \frac{\partial u_x^1}{\partial y} + \frac{\partial u_y^1}{\partial x} \right]. \quad (3.45) \]

Substituting (3.18) and (3.36) into (3.45) gives

\[ \frac{\partial u_x^1}{\partial x} = \frac{2B_1}{E_1} + 6 \frac{A_1 y}{E_1} + \alpha_1 \Delta T, \quad \frac{\partial u_y^1}{\partial y} = -\frac{\nu}{E_1} (2B_1 + 6A_1 y) + \alpha_1 \Delta T, \quad \frac{\partial u_y^1}{\partial x} + \frac{\partial u_x^1}{\partial y} = 0. \quad (3.46) \]

By integration, we have

\[ u_x^1 = \frac{2B_1}{E_1} x + 6 \frac{A_1 y}{E_1} x + \alpha_1 \Delta Tx + f_1(y), \quad (3.47) \]

\[ u_y^1 = -\frac{\nu}{E_1} (2B_1 + 6A_1 y) y + \alpha_1 \Delta Ty + f_2(x). \quad (3.48) \]

Then from the third equation of (3.46), \( f_1(y) \) and \( f_2(x) \) should satisfy

\[ 6 \frac{A_1 x}{E_1} + \frac{df_1(y)}{dy} + \frac{df_2(x)}{dx} = 0, \quad (3.49) \]
\[
\frac{df_1(y)}{dy} = -\frac{df_2(x)}{dx} - 6\frac{A_1x}{E_1}. \tag{3.50}
\]

Because the left-hand side depends on \(y\) only and the right-hand side depends on \(x\) only, we conclude that

\[
\frac{df_1(y)}{dy} = -\omega, \tag{3.51}
\]
\[
\frac{df_2(x)}{dx} = -6\frac{A_1x}{E_1} + \omega, \tag{3.52}
\]

where \(\omega\) is constant.

This gives

\[
f_1(y) = -\omega y + k_1, \quad f_2(x) = -3\frac{A_1x^2}{E_1} + \omega x + k_2, \tag{3.53}
\]
\[
u^1_x = \frac{2B_1}{E_1}x + 6\frac{A_1y}{E_1}x + \alpha_1\Delta Tx - \omega y + k_1, \tag{3.54}
\]
\[
u^1_y = -\frac{\nu}{E_1}(2B_1 + 6A_1y)y + \alpha_1\Delta Ty - 3\frac{A_1x^2}{E_1} + \omega x + k_2. \tag{3.55}
\]

Therefore, the angle of rotation of any vertical line element in the beam is

\[
\beta = \frac{\partial u^1_x}{\partial y} = 6\frac{A_1x}{E_1} - \omega, \tag{3.56}
\]

and all the longitudinal lines of the beam will have curvature

\[
\frac{1}{r_y} = \frac{\partial^2 u^1_y}{\partial x^2} = -6\frac{A_1}{E_1}.
\]

Similarly, for layer 2, we have angle of rotation and curvature

\[
\beta = \frac{\partial u^2_x}{\partial y} = 6\frac{A_2x}{E_2} - \omega, \quad \frac{1}{r_y} = \frac{\partial^2 u^2_y}{\partial x^2} = -6\frac{A_2}{E_2},
\]

respectively. But, the longitudinal lines of the two layers 1 and 2 have the same radius of curvature. Therefore,

\[
\frac{A_1}{E_1} = \frac{A_2}{E_2}. \tag{3.57}
\]
Solving (3.39), (3.40), (3.44) and (3.57) simultaneously we have

\[
B_1 = \frac{E_1 \Delta T}{D} [2(E_1 \alpha_1 h_1 + E_2 \alpha_2 h_2)(E_1 h_1^3 + E_2 h_2^3) - \frac{3}{2} (E_2 \alpha_2 h_2^2 - E_1 \alpha_1 h_1^2)(E_2 h_2^2 - E_1 h_1^2)]
- \frac{1}{2} E_1 \alpha_1 \Delta T,
\]

\[
B_2 = \frac{E_2 \Delta T}{D} [2(E_1 \alpha_1 h_1 + E_2 \alpha_2 h_2)(E_1 h_1^3 + E_2 h_2^3) - \frac{3}{2} (E_2 \alpha_2 h_2^2 - E_1 \alpha_1 h_1^2)(E_2 h_2^2 - E_1 h_1^2)]
- \frac{1}{2} E_2 \alpha_2 \Delta T,
\]

\[
A_1 = \frac{E_1 \Delta T}{D} [2(E_2 \alpha_2 h_2^2 - E_1 \alpha_1 h_1^2)(E_1 h_1 + E_2 h_2) - (E_1 \alpha_1 h_1 + E_2 \alpha_2 h_2)(E_2 h_2^2 - E_1 h_1^2)],
\]

\[
A_2 = \frac{E_2 \Delta T}{D} [(E_2 \alpha_2 h_2^2 - E_1 \alpha_1 h_1^2)(E_1 h_1 + E_2 h_2) - (E_1 \alpha_1 h_1 + E_2 \alpha_2 h_2)(E_2 h_2^2 - E_1 h_1^2)],
\]

where

\[
D = [(E_2 h_2^2 - E_1 h_1^2)^2 + 4 E_1 E_2 h_1 h_2 (h_1 + h_2)^2],
\]

and

\[
\sigma_{xx}^i = 6 A_i y + 2 B_i,
\]

where \( A_i \) and \( B_i \) are given above.

### 3.1.5 Numerical Solution

The result from the expressions of axial stress and strain in (3.6) and (3.8) is compared with the numerical result from Comsol. We consider a system layered with polymer (PMMA) and silica glass. The material parameters and layer thicknesses are given in table 3.1. Figure 3.2 shows the simulation result for longitudinal normal stress.

<table>
<thead>
<tr>
<th>Layer</th>
<th>Young’s Modulus</th>
<th>Poisson’s Ratio</th>
<th>Thickness</th>
<th>CTE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top layer</td>
<td>73.1e9[Pa]</td>
<td>0.17</td>
<td>0.2[m]</td>
<td>0.55e-6[1/K]</td>
</tr>
<tr>
<td>Bottom layer</td>
<td>3e9[Pa]</td>
<td>0.40</td>
<td>0.2[m]</td>
<td>70e-6[1/K]</td>
</tr>
</tbody>
</table>

Table 3.1: Material properties and layer thicknesses
The numerical result shows good agreement with analytic solution. For instance, the distribution of axial strain for the bottom layer at the point $x = 0$ estimated by numerically matched with the analytic result with relative error 0.002. Figure 3.3 shows the axial strains and the error of the numerical result from the analytic.
3.2 Linearly elastic composite cylinder

3.2.1 Introduction

As we presented in Chapter 1, the physical domain of the lens consists of two solid parts, a polymer with a shape of lens having a circular base and a glass substrate whose shape is cylindrical. These two axisymmetric bodies are bonded together at the interface so that their axes coincide.

The solution of the thermal stress problem in a long composite cylinder serves as a prelude to the study of stresses and deflections which occur in axisymmetric composite bodies due to a uniform rise in temperature. In this part of the work, we have made
a practical formulation for the solution of the linear thermoelastic problem of a long composite cylinder. The composite cylinder consists of two solid circular cylinders, made of dissimilar materials but having the same diameter which are stuck together. The interface between the two cylinders is a plane perpendicular to their axes, and the lengths of both parts of the composite cylinder can vary. The composite cylinder is assumed to be free of stress initially. We used the linear theory of thermoelasticity to calculate the stresses and displacements.

### 3.2.2 Composite infinite cylinder

We consider a linearly elastic, infinite cylinder with radius $R$ which occupies the region $G \subset \mathbb{R}^3$, where $G = \{x \in \mathbb{R}^3, 0 \leq r < R\}$. We also assume that it is formed from two circular cylinders made of distinct materials which are perfectly bonded in the axial direction at its section $z = 0$. The lateral surface of the cylinder (i.e. $r = R$) is free of stress. The stresses vanish for $z \to \pm \infty$.

![Two semi-infinite cylinders bonded together](image)

**Figure 3.4**: Two semi-infinite cylinders bonded together
3.2.3 Method of solution

First we will solve the two semi-infinite cylinders using prescribed rotationally symmetric normal stresses $\zeta(r)$ and shear stresses $\tau(r)$ on their bonded surface $z = 0$. Then, using interface continuity conditions, the two stresses $\zeta(r)$, $\tau(r)$ and all other displacement and stress components can be obtained.

For a layer of two circular cylinders, the model can be regarded as axisymmetric.

Hence, in cylindrical coordinates $(r, \theta, z)$, the component $u_\theta$ vanishes and $u_r$, $u_z$ are independent of $\theta$. The stress components are also independent of $\theta$, two of them, $\sigma_{r\theta}$ and $\sigma_{\theta z}$, being zero. Hence, the unknowns are $u_r^{(k)}$, $u_z^{(k)}$, $\varepsilon_{rr}^{(k)}$, $\varepsilon_{rz}^{(k)}$, $\varepsilon_{\theta\theta}^{(k)}$, $\sigma_{rr}^{(k)}$, $\sigma_{r\theta}^{(k)}$, $\sigma_{\theta z}^{(k)}$, $\sigma_{\theta\theta}^{(k)}$, where $k = 1, 2$ representing the two semi infinite cylinders.

Note that for clarity we denote $u_r(r, z) = u(r, z)$ and $u_z(r, z) = w(r, z)$. The relations between deformations and displacements are given by

$$
\varepsilon_{rr} = \frac{\partial u}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u}{r}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}, \quad \varepsilon_{r z} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right). \tag{3.58}
$$

The relations between stresses and deformations are given by the Hooke Duhamel law,

$$
\sigma_{rr} = (2\mu + \lambda)\varepsilon_{rr} + \lambda(\varepsilon_{\theta\theta} + \varepsilon_{zz}) - \beta(T - T_0),
$$
$$
\sigma_{\theta\theta} = (2\mu + \lambda)\varepsilon_{\theta\theta} + \lambda(\varepsilon_{rr} + \varepsilon_{zz}) - \beta(T - T_0),
$$
$$
\sigma_{zz} = (2\mu + \lambda)\varepsilon_{zz} + \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta}) - \beta(T - T_0),
$$
$$
\sigma_{rz} = 2\mu\varepsilon_{rz}, \tag{3.59}
$$

where

$$
T = \begin{pmatrix}
\sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\
\sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\
\sigma_{z r} & \sigma_{z\theta} & \sigma_{zz}
\end{pmatrix} \quad \tag{3.60}
$$
and

\[ E = \begin{pmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{\theta r} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ \varepsilon_{z r} & \varepsilon_{z\theta} & \varepsilon_{zz} \end{pmatrix} \] (3.61)

are stress and strain tensor in cylindrical coordinate system, respectively,

\[ \beta = (3\lambda + 2\mu)\alpha \] where \( \lambda \) and \( \mu \) are Lame’ constants, \( \alpha \) is the linear coefficient of thermal expansion and \( T_0 \) and \( T \) are initial and final temperature, respectively.

The equations of equilibrium are given by

\[ \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{r\theta}}{\partial z} + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} = 0, \]
\[ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0. \] (3.62)

The boundary conditions for the stresses are given by

\[ \sigma_{kr}^{(1)} = 0, \quad \sigma_{rr}^{(1)} = 0 \quad r = R \]
\[ \sigma_{kr}^{(2)} = \zeta(r), \quad \sigma_{rr}^{(2)} = \tau(r) \quad \text{at} \quad z = 0 \]
\[ \sigma_{zz}^{(1)} \to 0, \quad \sigma_{rz}^{(1)} \to 0, \quad \text{for} \quad z \to \infty \]
\[ \sigma_{zz}^{(2)} \to 0, \quad \sigma_{rz}^{(2)} \to 0, \quad \text{for} \quad z \to -\infty \] (3.63)

and solution is regular at \( r = 0 \). We introduce the following dimensionless parameters, denoted by a bar:

\[ \sigma_{ij}^{(k)} = G^{(k)}\alpha^{(k)}(T - T_0)\tilde{\sigma}_{ij}^{(k)}, \quad r = R\tilde{r}, \quad z = R\tilde{z}, \]

where \( G = 2E/(2(1 + \nu)) \) with Young’s modulus \( E \) and Poisson ratio \( \nu \),

\[ u^{(k)} = \alpha^{(k)}(T - T_0)\tilde{u}^{(k)}, \quad w^{(k)} = \alpha^{(k)}(T - T_0)\tilde{w}^{(k)} \]

With these dimensionless parameters the equilibrium equations are given by (3.62)
(we omit the bars) and the boundary condition turn into

\[
\begin{align*}
\sigma_{zz}^{(k)} &= 0, & \sigma_{rr}^{(k)} &= 0 & r &= 1 \\
\sigma_{zz}^{(k)} &= \frac{\zeta(r)}{G^{(k)}\alpha^{(k)}(T - T_0)}, & \sigma_{rz}^{(k)} &= \frac{\tau(r)}{G^{(k)}\alpha^{(k)}(T - T_0)} & \text{at } z &= 0 \\
\sigma_{zz}^{(1)} &\to 0, & \sigma_{rz}^{(1)} &\to 0, & \text{for } z &\to \infty \\
\sigma_{zz}^{(2)} &\to 0, & \sigma_{rz}^{(2)} &\to 0, & \text{for } z &\to -\infty.
\end{align*}
\]

Finally the interface conditions are

\[
\begin{align*}
\alpha^{(1)}(T - T_0)G^{(1)}\sigma_{zz}^{(1)} &= \alpha^{(2)}(T - T_0)G^{(2)}\sigma_{zz}^{(2)}, \\
\alpha^{(1)}(T - T_0)G^{(1)}\sigma_{rz}^{(1)} &= \alpha^{(2)}(T - T_0)G^{(2)}\sigma_{rz}^{(2)}, \\
\alpha^{(1)}(T - T_0)w^{(1)} &= \alpha^{(2)}(T - T_0)w^{(2)}, \\
\alpha^{(1)}(T - T_0)u^{(1)} &= \alpha^{(2)}(T - T_0)u^{(2)}.
\end{align*}
\]

3.2.4 Equations for the displacements

We focus on the right semi-infinite cylinder; the derivation for the left semi-infinite cylinder follows the same way and we drop \(k\) for brevity. First, we write the displacements \(u(r,z)\) and \(w(r,z)\) in terms of the potentials \(V(r,z)\) and \(\varphi(r,z)\) according to [1]

\[
u = \frac{1}{2} r \frac{\partial V}{\partial z} + \frac{\partial \varphi}{\partial r} + r, \quad w = -2(1 - \nu)V - \frac{1}{2} r \frac{\partial V}{\partial r} + \frac{\partial \varphi}{\partial z} + z. \quad (3.66)
\]

We shall express the deformations and the stresses in terms of \(V\) and \(\varphi\). For the deformations we have

\[
\begin{align*}
\varepsilon_{rr} &= \frac{1}{2} \frac{\partial V}{\partial z} + \frac{r}{2} \frac{\partial^2 V}{\partial r \partial z} + \frac{\partial^2 \varphi}{\partial r^2}, \\
\varepsilon_{rz} &= \frac{r}{2} \frac{\partial^2 V}{\partial z^2} - \frac{r}{4} \Delta V + (\nu - 1) \frac{\partial V}{\partial r} + \frac{\partial^2 \varphi}{\partial r \partial z}, \\
\varepsilon_{zz} &= \frac{(2\nu - 2)}{2} \frac{\partial V}{\partial z} - \frac{r}{2} \frac{\partial^2 V}{\partial r \partial z} + \frac{\partial^2 \varphi}{\partial z^2}, \\
\varepsilon_{\theta \theta} &= \frac{1}{2} \frac{\partial V}{\partial z} + \frac{1}{r} \frac{\partial \varphi}{\partial r}.
\end{align*}
\]

(3.67)
For the stresses we have

\[
\begin{align*}
\sigma_{rr} &= (1 - 2\nu) \frac{\partial V}{\partial z} + r \frac{\partial^2 V}{\partial r \partial z} + \frac{2\nu}{1 - 2\nu} \Delta \varphi, \\
\sigma_{\theta\theta} &= (1 - 2\nu) \frac{\partial V}{\partial z} + \frac{2\nu}{1 - 2\nu} \Delta \varphi, \\
\sigma_{rz} &= r \frac{\partial^2 V}{\partial z^2} + 2(\nu - 1) \frac{\partial V}{\partial r} - \frac{r}{2} \Delta V + 2 \frac{\partial^2 \varphi}{\partial r \partial z}, \\
\sigma_{zz} &= 2(\nu - 2) \frac{\partial V}{\partial z} - r \frac{\partial^2 V}{\partial r \partial z} + 2 \frac{\partial^2 \varphi}{\partial z^2} + \frac{2\nu}{1 - 2\nu} \Delta \varphi.
\end{align*}
\]

(3.68)

With these expressions for the stresses we can write the equations of equilibrium as

\[
\begin{align*}
\frac{r}{2} \frac{\partial \Delta V}{\partial z} + \frac{2(1 - \nu)}{1 - 2\nu} \frac{\partial \Delta \varphi}{\partial r} &= 0, \\
(2\nu - 3) \Delta V + \frac{2(1 - \nu)}{1 - 2\nu} \frac{\partial \Delta \varphi}{\partial z} &= 0.
\end{align*}
\]

(3.69)

We see that the equations of equilibrium are satisfied if \( V(r, z) \) and \( \varphi(r, z) \) obey the Laplace equations

\[
\Delta V = \Delta \varphi = 0.
\]

(3.70)

We note that the equations of equilibrium are also satisfied if \( \Delta \varphi = C \). So, \( \varphi \) is not uniquely determined. However, the stresses are uniquely determined. The boundary conditions can be formulated in terms of \( V \) and \( \varphi \).

\[
\begin{align*}
(1 - 2\nu) \frac{\partial V}{\partial z} + r \frac{\partial^2 V}{\partial r \partial z} + \frac{\partial^2 \varphi}{\partial r^2} &= 0 \quad r = 1, \\
(1 - 2\nu) \frac{\partial V}{\partial z} + \frac{2\partial \varphi}{\partial r} &= 0 \quad r = 1, \\
\frac{r}{2} \frac{\partial^2 V}{\partial z^2} + 2(\nu - 1) \frac{\partial V}{\partial r} + 2 \frac{\partial^2 \varphi}{\partial r \partial z} &= \frac{\tau(r)}{G\alpha(T - T_0)} \quad z = 0, \\
2(\nu - 2) \frac{\partial V}{\partial z} - r \frac{\partial^2 \varphi}{\partial r \partial z} + 2 \frac{\partial^2 \varphi}{\partial z^2} &= \frac{\zeta(r)}{G\alpha(T - T_0)} \quad z = 0, \\
V(r, z), \varphi(r, z) &\text{ bounded at } \ r = 0, \\
V(r, z), \varphi(r, z) &\to 0, z \to \infty.
\end{align*}
\]
3.2.5 Solution to the Laplace equations

We solve the Laplace equations (3.70) by a separation of variables. We focus on the equation for $V$. The solution for $\varphi$ can be obtained in exactly the same way. We introduce the separation

$$V(r, z) = R(r)Z(z).$$  \hspace{1cm} (3.71)

We have

$$\frac{d^2 R(r)}{dr^2}Z(z) + \frac{1}{r} \frac{dR(r)}{dr}Z(z) + R(r) \frac{d^2 Z(z)}{dz^2} = 0.$$  \hspace{1cm} (3.72)

Dividing by $R(r)Z(z)$, we have

$$\frac{1}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} + \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = 0,$$  \hspace{1cm} (3.73)

$$\frac{1}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} = -\lambda^2,$$  \hspace{1cm} (3.74)

$$\frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} + \lambda^2 R(r) = 0,$$  \hspace{1cm} (3.75)

where $\lambda$ is complex number. So, for $Z$ we have

$$\frac{d^2 Z(z)}{dz^2} - \lambda^2 Z(z) = 0,$$  \hspace{1cm} (3.76)

with the solution

$$Z(z) = \exp(\pm \lambda z).$$  \hspace{1cm} (3.77)

Since $V(r, z) \to 0$ as $z \to \infty$, we have

$$Z(z) = \exp(-\lambda z), \quad \lambda \in \mathbb{C}, \quad Re\lambda > 0.$$  \hspace{1cm} (3.78)

So, we get

$$V(r, z) = R(r)\exp(-\lambda z), \quad \lambda \in \mathbb{C}, \quad Re\lambda > 0.$$  \hspace{1cm} (3.79)
For $R(r)$ we have
\[
\frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} + \lambda^2 R(r) = 0.
\] (3.80)
The solution to this equation is given by
\[
R(r) = AJ_0(\lambda r) + CY_0(\lambda r),
\] (3.81)
where $J_0$ and $Y_0$ are Bessel functions of the first and the second kind of order zero, respectively. Since for $r = 0$, $R(r)$ is bounded, we obtain $C = 0$. Therefore, we get
\[
V(r, z) = AJ_0(\lambda r) \exp(-\lambda z), \quad \lambda \in \mathbb{C}, \quad Re \lambda > 0.
\] (3.82)
In a similar way, we obtain
\[
\varphi(r, z) = BJ_0(\lambda r) \exp(-\lambda z), \quad \lambda \in \mathbb{C}, \quad Re \lambda > 0.
\] (3.83)
Substitution of $V(r, z)$ and $\varphi(r, z)$ into the boundary condition $\sigma^{(1)}_{rr} = 0$ at $r = 1$ yields
\[
(1 - 2\nu) \frac{\partial V}{\partial z} + r \frac{\partial^2 V}{\partial r \partial z} + \frac{\partial^2 \varphi}{\partial r^2} = 0,
\] (3.84)
or
\[
A((2\nu - 1)\lambda J_0(\lambda) + \lambda^2 J_1(\lambda)) + B(2\lambda J_1(\lambda) - 2\lambda^2 J_0(\lambda)) = 0.
\] (3.85)
Substitution of $V(r)$ and $\varphi(r)$ into the boundary condition $\sigma^{(1)}_{rz} = 0$ at $r = 1$ yields
\[
r \frac{\partial^2 V}{\partial z^2} + (2\nu - 2) \frac{\partial V}{\partial r} + 2 \frac{\partial^2 \varphi}{\partial r \partial z} = 0,
\] (3.86)
or
\[
A(\lambda^2 J_0(\lambda) + (2 - 2\nu)\lambda J_1(\lambda)) + B(2\lambda J_1(\lambda) = 0.
\] (3.87)
So we obtain a system of equations which has a non-trivial solution if the determinant
\[
((2\nu - 1)\lambda J_0(\lambda) + \lambda^2 J_1(\lambda))(2\lambda^2 J_1(\lambda))
- (2\lambda \lambda J_1(\lambda) - 2\lambda^2 J_0(\lambda))(\lambda^2 J_0(\lambda) + (2 - 2\nu)\lambda J_1(\lambda)) = 0,
\] (3.88)
or

\[ \lambda^2 J_1^2(\lambda) - (2 - 2\nu)\lambda J_1^2(\lambda) + \lambda^2 J_0^2(\lambda) = 0. \]  \hfill (3.89)

We obtain

\[ \Lambda(\lambda) := \lambda^2(J_0^2(\lambda) + J_1^2(\lambda)) - 2(1 - \nu)\lambda^2 J_1^2(\lambda) = 0, \]  \hfill (3.90)

This equation has a countable number of roots, which are the eigenvalues of the problem. Following the same argument, we can get eigenvalues of the problem related to the left semi-infinite cylinder.

Let \( \lambda_n^{(k)} \) be the eigenvalues (n=1,2,3,...), where \( k = 1, 2 \), represents the two semi-infinite cylinder. We can write \( V \) and \( \varphi \) for each semi infinite cylinders in terms of the eigenfunctions as follows:

\[ V^{(1)}(r, z) = \sum_{n=1}^{\infty} A_n^{(1)} J_0(\lambda_n^{(1)} r) \exp(-\lambda_n^{(1)} z), \]
\[ \varphi^{(1)}(r, z) = \sum_{n=1}^{\infty} B_n^{(1)} J_0(\lambda_n^{(1)} r) \exp(-\lambda_n^{(1)} z), \quad z > 0; \]
\[ V^{(2)}(r, z) = \sum_{n=1}^{\infty} A_n^{(2)} J_0(\lambda_n^{(2)} r) \exp(-\lambda_n^{(2)} |z|), \]
\[ \varphi^{(2)}(r, z) = \sum_{n=1}^{\infty} B_n^{(2)} J_0(\lambda_n^{(2)} r) \exp(-\lambda_n^{(2)} |z|), \quad z < 0. \]  \hfill (3.91)

### 3.2.6 The roots of the characteristic equations

We now examine (3.90) in greater detail. To start our analysis of the characteristic equation we will show that this equation has no purely real roots \( \lambda \) (Keep in mind that \( Re\lambda > 0 \) and \( 0 < \nu < 0.5 \)). Assume that \( \lambda \) is a real number. The derivative of \( \Lambda \) is equal to

\[ \Lambda'(\lambda) = 2\lambda(J_0^2(\lambda) + J_1^2(\lambda)) + 2\lambda^2(J_0(\lambda)J_0'(\lambda) + J_1(\lambda)J_1'(\lambda)) - 4(1 - \nu)J_1(\lambda)J_1'(\lambda). \]  \hfill (3.92)
Using the relations
\[ J'_0(\lambda) = -J_1(\lambda), \]
\[ J'_1(\lambda) = -\frac{1}{\lambda} J_1(\lambda) + J_0(\lambda), \]
for derivatives of the Bessel functions \( J_0 \) and \( J_1 \), we can write (3.92) as follows:
\[ \Lambda'(\lambda) = 2\lambda J_0^2(\lambda) - 4(1-\nu)J_1(\lambda) J_0(\lambda) + 4\left(\frac{1-\nu}{\lambda}\right) J_1^2(\lambda). \]
We multiply the previous equation by \( \lambda \):
\[ \lambda \Lambda'(\lambda) = 2\lambda^2 J_0^2(\lambda) - 4(1-\nu)\lambda J_1(\lambda) J_0(\lambda) + 4(1-\nu)J_1^2(\lambda). \]
Using the fact that \((1-\nu)^2 < (1-\nu)\), we can deduce the following:
\[ \lambda \Lambda'(\lambda) > 2\lambda^2 J_0^2(\lambda) - 4(1-\nu)\lambda J_1(\lambda) J_0(\lambda) + 4(1-\nu)^2 J_1^2(\lambda) \]
\[ = \lambda^2 J_0^2(\lambda) + (\lambda J_0(\lambda) - 2(1-\nu)J_1(\lambda))^2 \]
\[ > 0. \]
So, the derivative of \( \Lambda \) is strictly positive for \( \lambda > 0 \). We also know that \( \Lambda(0) = 0 \). From this, we can conclude that the characteristic equation has no purely real roots \( \lambda > 0 \).

Note that if \( \lambda \) is the root of (3.90) then so also is the complex conjugate \( \bar{\lambda} \) (\( \eta \), say).

3.2.7 Infinite sets of linear equations

Since (3.87) and (3.89) are homogenous, the constants \( A^{(1)} \), \( B^{(1)} \) are not independent and we can therefore write, from (3.89) for \( \lambda^{(1)} \neq 0 \),
\[ B^{(1)} = -\omega^{(1)} A^{(1)}, \]
where

\[
\omega^{(1)} = \frac{(1 - \nu^{(1)})}{\lambda^{(1)}} + \frac{J_0(\lambda^{(1)})}{2J_1(\lambda^{(1)})}.
\]

Similarly, following the same argument for the left semi-infinite cylinder, we have also

\[
B^{(2)} = \omega^{(2)}A^{(2)},
\]

where

\[
\omega^{(2)} = \frac{(1 - \nu^{(2)})}{\lambda^{(2)}} + \frac{J_0(\lambda^{(2)})}{2J_1(\lambda^{(2)})}.
\]

Let \( k = 1, 2 \) again represents the two semi infinite cylinders, then in the last section it was indicated that each eigenvalue \( \lambda_n^{(k)} \) and \( \tilde{\lambda}_n^{(k)} \), \( n = 1, 2, ... \) has a positive real part [12]. As a consequence, from (3.66), (3.68) and (3.91) we can express the stresses and the displacements at the interface \( z = 0 \) for each semi infinite cylinder as,

\[
\sigma_{zz}^{(k)} = \sum_{n=1}^{\infty} A_n^{(k)} ((4 - 2\nu^{(k)})\lambda_n^{(k)}J_0(\lambda_n^{(k)}r) - r(\lambda_n^{(k)})^2J_1(\lambda_n^{(k)}r) - \omega_n^{(k)}2(\lambda_n^{(k)})^2J_0(\lambda_n^{(k)}r)) + \\
\sum_{n=1}^{\infty} \tilde{A}_n^{(k)} ((4 - 2\nu^{(k)})\eta_n^{(k)}J_0(\eta_n^{(k)}r) - r(\eta_n^{(k)})^2J_1(\eta_n^{(k)}r) - \Omega_n^{(k)}2(\eta_n^{(k)})^2J_0(\eta_n^{(k)}r)),
\]

\[
\sigma_{rz}^{(k)} = \sum_{n=1}^{\infty} A_n^{(k)} ((\lambda_n^{(k)})^2rJ_0(\lambda_n^{(k)}r) + (2 - 2\nu^{(k)})\lambda_n^{(k)}J_1(\lambda_n^{(k)}r) - \omega_n^{(k)}2(\lambda_n^{(k)})^2J_1(\lambda_n^{(k)}r)) + \\
\sum_{n=1}^{\infty} \tilde{A}_n^{(k)} ((\eta_n^{(k)})^2rJ_0(\eta_n^{(k)}r) + (2 - 2\nu^{(k)})\eta_n^{(k)}J_1(\eta_n^{(k)}r) - \Omega_n^{(k)}2(\eta_n^{(k)})^2J_1(\eta_n^{(k)}r)),
\]

\[
u^{(k)} = \sum_{n=1}^{\infty} A_n^{(k)} (-\frac{1}{2}\lambda_n^{(k)}rJ_0(\lambda_n^{(k)}r) + \omega_n^{(k)}\lambda_n^{(k)}J_1(\lambda_n^{(k)}r)) + r + \\
\sum_{n=1}^{\infty} \tilde{A}_n^{(k)} (-\frac{1}{2}\eta_n^{(k)}rJ_0(\eta_n^{(k)}r) + \Omega_n^{(k)}\eta_n^{(k)}J_1(\eta_n^{(k)}r)),
\]

\[
w^{(k)} = \sum_{n=1}^{\infty} A_n^{(k)} (-2(1 - \nu^{(k)})J_0(\lambda_n^{(k)}r) + \frac{1}{2}r\lambda_n^{(k)}J_1(\lambda_n^{(k)}r) + \omega_n^{(k)}\lambda_n^{(k)}J_0(\lambda_n^{(k)}r)) + \\
\sum_{n=1}^{\infty} \tilde{A}_n^{(k)} (-2(1 - \nu^{(k)})J_0(\eta_n^{(k)}r) + \frac{1}{2}r\eta_n^{(k)}J_1(\eta_n^{(k)}r) + \Omega_n^{(k)}\eta_n^{(k)}J_0(\eta_n^{(k)}r)).
\]
Here we use $\omega_n^{(k)} = \omega^{(k)}(\lambda_n^{(k)})$ and $\Omega_n^{(k)} = \omega^{(k)}(\eta_n^{(k)}) = \omega^{(k)}(\lambda_n^{(k)})$.

From the specific form of the above equations, it follows directly that the resulting stresses and displacements are real functions. For simplicity, we denote:

\[
S_{n0}^{(1)} = A_n^{(1)}(4 - 2\nu^{(1)})\lambda_n^{(1)} - \omega_n^{(1)}2(\lambda_n^{(1)})^2)J_0(\lambda_n^{(1)}r), \quad S_{n1}^{(1)} = -A_n^{(1)}(\lambda_n^{(1)})^2rJ_1(\lambda_n^{(1)}r),
\]

\[
S_{n0}^{(2)} = A_n^{(2)}(-4 - 2\nu^{(2)})\lambda_n^{(2)} - \omega_n^{(2)}2(\lambda_n^{(2)})^2)J_0(\lambda_n^{(2)}r), \quad S_{n1}^{(2)} = A_n^{(2)}(\lambda_n^{(2)})^2rJ_1(\lambda_n^{(2)}r),
\]

\[
Y_{n0}^{(1)} = A_n^{(1)}((\lambda_n^{(1)})^2r)J_0(\lambda_n^{(1)}r), \quad Y_{n1}^{(1)} = A_n^{(1)}((2 - 2\nu^{(1)})\lambda_n^{(1)} - \omega_n^{(1)}2(\lambda_n^{(1)})^2)J_1(\lambda_n^{(1)}r),
\]

\[
Y_{n0}^{(2)} = A_n^{(2)}((\lambda_n^{(2)})^2r)J_0(\lambda_n^{(2)}r), \quad Y_{n1}^{(2)} = A_n^{(2)}((2 - 2\nu^{(2)})\lambda_n^{(2)} + \omega_n^{(2)}2(\lambda_n^{(2)})^2)J_1(\lambda_n^{(2)}r),
\]

\[
H_{n0}^{(1)} = A_n^{(1)}(-2(1 - \nu^{(1)}) + \omega_n^{(1)}\lambda_n^{(1)})J_0(\lambda_n^{(1)}r), \quad H_{n1}^{(1)} = A_n^{(1)}(\frac{1}{2}\lambda_n^{(1)})rJ_1(\lambda_n^{(1)}r),
\]

\[
H_{n0}^{(2)} = A_n^{(2)}(-2(1 - \nu^{(2)}) - \omega_n^{(2)}\lambda_n^{(2)})J_0(\lambda_n^{(2)}r), \quad H_{n1}^{(2)} = A_n^{(2)}(\frac{1}{2}\lambda_n^{(2)})rJ_1(\lambda_n^{(2)}r),
\]

\[
F_{n0}^{(1)} = A_n^{(1)}(-\frac{1}{2}\lambda_n^{(1)})rJ_0(\lambda_n^{(1)}r), \quad F_{n1}^{(1)} = A_n^{(1)}(\omega_n^{(1)}\lambda_n^{(1)})J_1(\lambda_n^{(1)}r),
\]

\[
F_{n0}^{(2)} = -A_n^{(2)}(-\frac{1}{2}\lambda_n^{(2)})rJ_0(\lambda_n^{(2)}r), \quad F_{n1}^{(2)} = A_n^{(2)}(\omega_n^{(2)}\lambda_n^{(2)})J_1(\lambda_n^{(2)}r).
\]

From the interface conditions $\alpha^{(1)}(T - T_0)G^{(1)}\sigma_{zz}^{(1)} = \alpha^{(2)}(T - T_0)G^{(2)}\sigma_{zz}^{(2)}$ and $\alpha^{(1)}(T - T_0)G^{(1)}\sigma_{rz}^{(1)} = \alpha^{(2)}(T - T_0)G^{(2)}\sigma_{rz}^{(2)}$, we obtain the following equations

\[
\alpha^{(1)}G^{(1)} \sum_{n=1}^{\infty} \left\{ (S_{n0}^{(1)} + S_{n0}^{(1)} + S_{n1}^{(1)} + S_{n1}^{(1)}) \right\} = \alpha^{(2)}G^{(2)} \sum_{n=1}^{\infty} \left\{ (S_{n0}^{(2)} + S_{n0}^{(2)} + S_{n1}^{(2)} + S_{n1}^{(2)}) \right\},
\]

\[
\alpha^{(1)}G^{(1)} \sum_{n=1}^{\infty} \left\{ (Y_{n0}^{(1)} + Y_{n0}^{(1)} + Y_{n1}^{(1)} + Y_{n1}^{(1)}) \right\} = \alpha^{(2)}G^{(2)} \sum_{n=1}^{\infty} \left\{ (Y_{n0}^{(2)} + Y_{n0}^{(2)} + Y_{n1}^{(2)} + Y_{n1}^{(2)}) \right\}.
\]

\[\text{(3.94)}\]

From the interface conditions $\alpha^{(1)}(T - T_0)R u^{(1)} = \alpha^{(2)}(T - T_0)R u^{(2)}$, $\alpha^{(1)}(T - T_0)R w^{(1)} = \alpha^{(2)}(T - T_0)R w^{(2)}$, we obtain the following equations, holding for every $r \in [0, 1]$,

\[
\alpha^{(1)} \sum_{n=1}^{\infty} \left\{ (H_{n0}^{(1)} + H_{n1}^{(1)} + H_{n1}^{(1)}) \right\} = \alpha^{(2)} \sum_{n=1}^{\infty} \left\{ (H_{n0}^{(2)} + H_{n0}^{(2)} + H_{n1}^{(2)} + H_{n1}^{(2)}) \right\},
\]

\[
\alpha^{(1)} \sum_{n=1}^{\infty} \left\{ (F_{n0}^{(1)} + F_{n0}^{(1)} + F_{n1}^{(1)} + F_{n1}^{(1)}) + r \right\} = \alpha^{(2)} \sum_{n=1}^{\infty} \left\{ (F_{n0}^{(2)} + F_{n0}^{(2)} + F_{n1}^{(2)} + F_{n1}^{(2)}) + r \right\}.
\]

\[\text{(3.95)}\]
for brevity, we drop \( k = 1, 2 \). The \( r \)-dependent functions \( Y_{n0}, F_{n0}, Y_{n1}, F_{n1}, r \) from (3.94) and (3.95) are expanded in series of Bessel function \( J_1(\mu_s r) \), where \( \mu_s \) is a zero of \( J_1(\mu) \),

\[
Y_{n0} = \sum_{s=1}^{\infty} X_{sn} J_1(\mu_s r),
\]

\[
Y_{n1} = \sum_{s=1}^{\infty} W_{sn} J_1(\mu_s r),
\]

\[
F_{n0} = \sum_{s=1}^{\infty} R_{sn} J_1(\mu_s r),
\]

\[
F_{n1} = \sum_{s=1}^{\infty} L_{sn} J_1(\mu_s r),
\]

\[
r = \sum_{s=1}^{\infty} T_s J_1(\mu_s r),
\]

where

\[
X_{sn} = 2 \frac{J_2^2(\mu_s)}{J_1^2(\mu_s)} \int_0^1 r Y_{n0}(r) J_1(\mu_s r) \, dr,
\]

\[
W_{sn} = \frac{2}{J_1^2(\mu_s)} \int_0^1 r Y_{n1}(r) J_1(\mu_s r) \, dr,
\]

\[
R_{sn} = \frac{2}{J_1^2(\mu_s)} \int_0^1 r F_{n0}(r) J_1(\mu_s r) \, dr,
\]

\[
L_{sn} = \frac{2}{J_1^2(\mu_s)} \int_0^1 r F_{n1}(r) J_1(\mu_s r) \, dr,
\]

\[
T_s = 2 \frac{J_2^2(\mu_s)}{J_1^2(\mu_s)} \int_0^1 r^2 J_1(\mu_s r) \, dr.
\]

Similarly, \( S_{n0}, H_{n0}, S_{n1}, H_{n1} \) are expanded in series of Bessel functions \( J_0(\mu_s r) \) as

\[
S_{n0} = \sum_{s=1}^{\infty} U_{sn} J_0(\mu_s r),
\]

\[
S_{n1} = \sum_{s=1}^{\infty} V_{sn} J_0(\mu_s r),
\]

\[
H_{n0} = \sum_{s=1}^{\infty} P_{sn} J_0(\mu_s r),
\]

\[
H_{n1} = \sum_{s=1}^{\infty} Q_{sn} J_0(\mu_s r),
\]

(3.98)
The selection of the eigenfunctions, \( J_1(\mu_s r) \) and \( J_0(\mu_s r) \), is based on the nature of the functions to be expanded. In fact, we use \( J_1(\mu_s r) \) and \( J_0(\mu_s r) \) for odd and even functions, respectively. These eigenfunctions are also orthogonal on \([0,1]\) with respect to weight function \( r \). Finally, we have

\[
\begin{align*}
\alpha^{(1)}G^{(1)} \sum_{n=1}^{\infty} \left\{ (U_{sn}^{(1)} + \tilde{U}_{sn}^{(1)} + V_{sn}^{(1)} + \tilde{V}_{sn}^{(1)}) \right\} - \alpha^{(2)}G^{(2)} \sum_{n=1}^{\infty} \left\{ (U_{sn}^{(2)} + \tilde{U}_{sn}^{(2)} + V_{sn}^{(2)} + \tilde{V}_{sn}^{(2)}) \right\} &= 0, \\
\alpha^{(1)} \sum_{n=1}^{\infty} \left\{ (P_{sn}^{(1)} + \tilde{P}_{sn}^{(1)} + Q_{sn}^{(1)} + \tilde{Q}_{sn}^{(1)}) \right\} - \alpha^{(2)} \sum_{n=1}^{\infty} \left\{ (P_{sn}^{(2)} + \tilde{P}_{sn}^{(2)} + Q_{sn}^{(2)} + \tilde{Q}_{sn}^{(2)}) \right\} &= 0, \\
\alpha^{(1)}G^{(1)} \sum_{n=1}^{\infty} \left\{ (W_{sn}^{(1)} + \tilde{W}_{sn}^{(1)} + X_{sn}^{(1)} + \tilde{X}_{sn}^{(1)}) \right\} - \alpha^{(2)}G^{(2)} \sum_{n=1}^{\infty} \left\{ (W_{sn}^{(2)} + \tilde{W}_{sn}^{(2)} + X_{sn}^{(2)} + \tilde{X}_{sn}^{(2)}) \right\} &= 0, \\
\alpha^{(1)} \sum_{n=1}^{\infty} \left\{ (R_{sn}^{(1)} + \tilde{R}_{sn}^{(1)} + L_{sn}^{(1)} + \tilde{L}_{sn}^{(1)} + T_s) \right\} - \alpha^{(2)} \sum_{n=1}^{\infty} \left\{ (R_{sn}^{(2)} + \tilde{R}_{sn}^{(2)} + L_{sn}^{(2)} + \tilde{L}_{sn}^{(2)} + T_s) \right\} &= 0.
\end{align*}
\]

For \( s = 1, 2, 3...S \) where \( S \) is the total number of terms we want to truncate. Thus, we have exactly \( 4S \) equations for \( 4S \) unknowns \( \alpha^{(k)}_n \) and \( \beta^{(k)}_n \), \( k = 1, 2 \), from the complex coefficients \( A^{(k)}_n = \alpha^{(k)}_n + i\beta^{(k)}_n \) and \( \bar{A}^{(k)}_n = \alpha^{(k)}_n - i\beta^{(k)}_n \). Then, the displacement potentials \( V^{(k)} \) and \( \varphi^{(k)} \) become

\[
\begin{align*}
V^{(k)}(r, z) &= \sum_{n=1}^{\infty} \alpha^{(k)}_n R^{(k)}_n(r, z) + \beta^{(k)}_n Q^{(k)}_n(r, z), \\
R^{(k)}_n(r, z) &= p + q, \quad Q^{(k)}_n(r, z) = i(p - q).
\end{align*}
\]
where
\[ p \equiv p_n^{(k)}(r, z) = J_0(\lambda_n^{(k)} r) \exp(-|\lambda_n^{(k)} z|), \]
\[ q \equiv q_n^{(k)}(r, z) = J_0(\eta_n^{(k)} r) \exp(-|\eta_n^{(k)} z|). \] (3.102)

Similarly,
\[ \varphi^{(k)}(r, z) = \sum_{n=1}^{\infty} \alpha_n^{(k)} F_n^{(k)}(r, z) + \beta_n^{(k)} G_n^{(k)}(r, z), \] (3.103)
\[ F_n^{(k)}(r, z) = f + g, \quad G_n^{(k)}(r, z) = i(f - g) \]
where
\[ f \equiv f_n^{(k)}(r, z) = -\omega_n^{(k)} J_0(\lambda_n^{(k)} r) \exp(-|\lambda_n^{(k)} z|), \]
\[ g \equiv g_n^{(k)}(r, z) = -\Omega_n^{(k)} J_0(\eta_n^{(k)} r) \exp(-|\eta_n^{(k)} z|). \] (3.104)

Finally, all displacement and stress components can be obtained from \( V^{(k)} \) and \( \varphi^{(k)} \).

### 3.2.8 Numerical results and discussions

To illustrate the foregoing analysis, numerical evaluation is carried out for the following values on the assumption that the materials are polymer for the right cylinder (\( k = 1 \)) and glass for the left cylinder (\( k = 2 \)):
\[
\begin{align*}
\alpha^1 &= 75e - 6, \quad \alpha^2 = 7.2e - 6, \quad \nu^1 = 0.3, \quad \nu^2 = 0.3, \\
E^1 &= 1.54e + 9, \quad E^2 = 72.9e + 9, \\
T_0 &= 273.15K, \quad T = 333.15K, \quad R = 6e - 3[m], \quad z/R = 8.
\end{align*}
\] (3.105)

First it has been done by taking 51 terms in the series. However, in order to verify the convergence of the series calculation is also done by taking 55 terms in the series. The results of 51 terms and 55 terms are shown in table 3.2. It may be observed from table 3.2 that the convergence is fairly good considering 51 terms in the series.

Figure 3.5 and 3.6 shows the distribution of the displacement \( u^1 \) and \( w^1 \) along \( z = 0 \) at \( z/R = 1 \) for the right cylinder, respectively. Note that the non-dimensional displacements are defined as \( u^{(k)} = \alpha^{(k)}(T - T_0)R\bar{u}^{(k)} \) and \( w^{(k)} = \alpha^{(k)}(T - T_0)R\bar{w}^{(k)} \).

Figure (3.7) and Figure (3.8) are the distribution of normal stress (\( \sigma_{zz}^1 \)) and shear...
Table 3.2: Distribution of displacements \((u^1, w^1)\) in terms of \(\alpha^1(T - T_0)R\) along \(z = 0\) for \(z/R = 1\)

<table>
<thead>
<tr>
<th>(r/R)</th>
<th>(u^1) number of terms used</th>
<th>(w^1) number of terms used</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2212958</td>
<td>0.2212990</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4376646</td>
<td>0.4376698</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6449324</td>
<td>0.6449378</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8412369</td>
<td>0.8412417</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0292281</td>
<td>1.0292333</td>
</tr>
</tbody>
</table>

Figure 3.5: The distribution of radial displacement \((u^1)\) along \(z = 0\)

stress \((\sigma^1_{rz})\) along \(z = 0\) for the polymer region, respectively. The non-dimensional stresses are defined as \(\sigma^{(k)}_{zz} = G^{(k)}\alpha^{(k)}(T - T_0)\tilde{\sigma}^{(k)}_{zz}(k = 1, 2)\) and \(\sigma^{(k)}_{rz} = G^{(k)}\alpha^{(k)}(T - T_0)\tilde{\sigma}^{(k)}_{rz}(k = 1, 2)\). As shown in the figures, The stresses are maximum around the edge and near to the interface. We needed 51 terms to obtain a convergent solution
of the system of equations. This shows that the solution is converging slowly. A reason for this may be attributed to the following state of affairs. The problem we have is one which gives rise to an eigenvalue equation. From these eigenvalues, the Bessel functions $J_0, J_1$ of complex argument are computed, and we hope that these functions, which are basic to the analysis, will decrease rapidly in magnitude as $n$ increases. But this is not the situation; for instance,

$$|J_0(\lambda_1)| = 0.8595, \quad |J_0(\lambda_{16})| = 0.7981, \quad |J_0(\lambda_{76})| = 0.797901, \quad |J_0(\lambda_{201})| = 0.797889,$$

$$|J_1(\lambda_1)| = 0.8973, \quad |J_1(\lambda_{16})| = 0.7983, \quad |J_1(\lambda_{76})| = 0.797910, \quad |J_1(\lambda_{201})| = 0.797887.$$  

The values of the successive expansion coefficients $\alpha_n, \beta_n$ are not also rapidly decreasing, which shows an extremely slow convergence.

In spite of the limitation we find, we try to show how a particular method can be
Figure 3.7: The distribution of normal stress \( \sigma_{zz}^{1} \) along \( z = 0 \).

utilized to obtain displacements and thermal stresses in axisymmetric composite problem. There are alternative approaches to solve the slow convergence behavior. For instance, we can use another complete set of orthogonal functions which produce a much faster decay of \( \alpha, \beta \) than that produced by the Fourier-Bessel expansions.
Figure 3.8: The distribution of shear stress ($\sigma_{rz}$) along $z = 0$. 
Chapter 4

Model implementation

4.1 Introduction

In chapter 1 we already mentioned that a software package Comsol is used for the numerical simulation of the models. In this part of the work, we present the numerical results of the models. In the first part, we introduce the numerical result of a wedge model. The aim of this wedge model is to study the relation between thermally induced rate of change of refractive index of the lens materials (polymers) and their volume coefficient of expansion. In the second part, we present the numerical result of the lens model for three different wafer lenses.

4.2 Wedge model

The study of properties of the lens materials is a first step in the study of thermal analysis of the lens. One of these optical properties is the refractive index. Here we determine the factor $\kappa$ which relate the volume coefficient of thermal expansion and change of refractive index of polymers. For this model implementation, we use a wedge array which is designed by Anteryon BV for experiment to measure refractive index. It is made of thin glass plate coated with polymer. In its top part, there are about 13
wedge-shaped profiles positioned side by side as shown in figure 4.1. It is clear that the

wedge angle (top angle) will also change when the lens undergoes temperature caused
deformation. Hence in addition to the parameter \(dn/dT\), the thermal property of lens
material can also be characterized by the wedge angle change. Currently, Anteryon
BV is implementing both measurement and analytic methods to determine the change
in the wedge angle and refractive index when the wedge is subjected to a uniform
temperature change. First we give a brief overview about these methods.

4.2.1 Existing methods for refractive index and wedge angle

Measurement:

First the top angle \(a_{top} = (a_1 + a_2)/2\) is measured by measuring the reflection angles
\(a_1\) and \(a_2\) (see figure 4.2). Similarly, the refraction angle \(a_{refr} = (a_3 + a_4)/2\) is
determined by measuring angles \(a_3\) and \(a_4\). To find the new refractive index, Snell’s
law of refraction will be used. With refractive index of air \((n_{air})\) the refractive index
becomes:

\[
\begin{align*}
n_{wedge} &= \frac{\sin(a_{top} + a_{refr})}{\sin(a_{top})} n_{air} \\
&= \frac{\sin(a_{top} + a_{refr})}{\sin(a_{top})} n_{air} \\
&= \frac{\sin(a_{top} + a_{refr})}{\sin(a_{top})} n_{air}
\end{align*}
\]  (4.1)
Analytic (Existing model):

When a thin layer of polymer is attached to a flat glass substrate, with a lower expansion coefficient and higher stiffness, the expansion of the polymer in the x,y-direction is assumed to be equal to the expansion of the glass in the same direction [4, 16]. The expansion coefficient in the z direction is [4, 16]

\[
\alpha_{z\text{thinfilm}} = \frac{1 + \nu}{1 - \nu} \alpha_{iso} - \frac{2\nu}{1 - \nu} \alpha_{sub} \tag{4.2}
\]

where \(\alpha_{z\text{thinfilm}}\) is the coefficient of expansion in the z direction, \(\alpha_{iso}\) is the isotropic coefficient of expansion of the polymer, \(\alpha_{sub}\) is the expansion coefficient of glass and \(\nu\) is the Poisson ratio of the polymer. Using this equations for \(\alpha_{z\text{thinfilm}}\) the new wedge angle \(a_{\text{top}}(T)\) and volume \(V(T)\) are estimated with:

\[
a_{\text{top}}(T) = \arctan \left( \frac{h(T)}{b(T)} \right) \tag{4.3}
\]
where $l(T) = l_0(\alpha_{\text{sub}}(T-T_0))$, $b(T) = b_0(\alpha_{\text{sub}}(T-T_0))$ and $h(T) = h_0(\alpha_{\text{thin film}}(T-T_0))$. The index change will be determined from the change in the volume using a linear relation between volume thermal expansion coefficient and rate of change of
refractive index with temperature \( (dn/dT) \). We present this relation below.

**Relation between volume thermal expansion coefficient and rate of change of refractive index with temperature**

It is known that the change in refractive index of a polymer with temperature is due to the temperature-caused density change and temperature change itself as [17]

\[
\frac{dn}{dT} = -\left(\frac{\rho \partial n}{\partial \rho}\right)_T \beta + \left(\frac{\partial n}{\partial T}\right)_\rho
\]

where \( n \) is refractive index, \( \rho \) is the density, \( \beta \) is the volume coefficient of thermal expansion of a polymer, \( dn/dT \) is the temperature-caused index change, i.e. thermo-optic coefficient, \( (\delta n/\delta T)_\rho \) is the index change under constant density, while \( (\rho \delta n/\delta \rho)_T \) is a constant for a given polymer based on the Lorentz-Lorenz equation

\[
\left(\frac{\rho \partial n}{\partial \rho}\right)_T = (1 - L_0)(n^2 + 2)(n^2 - 1) \quad (4.5)
\]

where \( L_0 \) is the strain polarity constant related to effect of density change on atomic polarizability of the material. With equation (4.5), refractive index \( n \), polarity constant \( L_0 \), and \( (\delta n/\delta T)_\rho \) value are required for calculating the rate of change of refractive index of a polymer from its volume coefficient of thermal expansion. Instead of measuring all these parameters, which is, in fact, more difficult than measuring the rate of change of refractive index directly, the relationship between the rate of change of refractive index and coefficients of thermal expansion of many polymers has been investigated to see whether an empirical expression of (4.5) can be established and used to estimate the rate of change of refractive index of a polymer. Accordingly, Zhang [17] found empirically a linear relationship between rate of change of refractive index with temperature \( (dn/dT) \) and volume coefficient of thermal expansion \( (\beta) \) of
a wide range of polymers as

\[ \frac{dn}{dT} \approx \kappa \beta, \quad (4.6) \]

where \( \kappa \) is a multiplying factor. This relationship can be used to estimate the \( \frac{dn}{dT} \) value of a given polymer using its \( \beta \) value, which can be easily measured from:

\[ \beta = \frac{\Delta V}{V_0} \frac{1}{\Delta T} \quad (4.7) \]

where \( \Delta V/V_0 \) and \( \Delta T \) are the fractional change in volume and the change in temperature, respectively. Meanwhile, the density-unrelated index change \( (dn/dT)_\rho \) is too small for polymers and a chemical structure-related variation is insignificant from one polymer to another one. Zhang [17] also empirically found the multiplying factor \( \kappa \) to be \(-0.56\). However, this factor does not agree with experimentally found results for those polymers which are currently used by Anteryon BV as lens materials. We try to determine the corresponding factors for these polymers (polymer A, B, and BA15) by matching the experiment and numerical results.

**4.2.2 Model implementation with Comsol**

We assume the wedge is initially at reference temperature \( T_0 \). The temperature is slowly raised to a constant uniform temperature \( T \), where the temperature change is \( \Delta T = T - T_0 \). We also assume the deformation is small and has linear elastic behavior. Accordingly, the problem formulation for this model is equivalent to that of the lens model given in section 2.8. When we perform the numerical simulation, we consider borosilicate glass for the substrate part and three different polymers: polymer A, polymer B and polymer BA15 for the top part of the wedge. The numerical simulation using Comsol generally involves these steps:
1. Choice of the representative module,

We select solid stress-strain with thermal expansion from the structural mechanics module. It combines solid, stress-strain with heat transfer by conduction, including thermal expansion in the structural loads using the temperature field.

2. Create the geometry of the model,

The required geometry (see figure 4.1) was made in SolidWorks and imported to Comsol for simulation.

3. Set the material properties,

The selected three polymers have temperature dependent input parameters: Thermal expansion coefficient, Young’s modulus, and Poisson ratio. First, we take specific temperature intervals and material properties of the polymers corresponding to the selected temperature intervals were entered. For the substrate part, we use Borosilicate glass whose thermal expansion coefficient, Young’s modulus and Poisson ratio are $7.2e-6[1/K], 72.9e+9[Pa]$ and 0.208 respectively.

4. Set the boundary conditions and initial conditions,

The free boundary conditions are specified through the Physics $\rightarrow$ Boundary settings menu of Comsol.

5. Mesh the geometry,

We use the pre-defined quadratic Lagrange elements. We refined the mesh for one level refinement (see figure 4.5).

6. Choose a solver and solve the problem,

We select the stationary solver with statics analysis. The transient analysis
Figure 4.5: The corresponding meshed domain for the solution to the wedge model. It takes very long processing time and doesn’t show a significant difference when the result is compared to that of statics analysis. We also set a relative tolerance value 1e-9 for the solver.

7. Post-process the results to find the information that is required

Figure 4.6 shows the numerical solution of the wedge model for polymer B and \( \Delta T = 58^\circ C \). It is the boundary plot of the total displacement of the wedge whose top part is polymer BA15.

4.2.3 Numerical results and discussions

To determine the rate of change of refractive index with temperature from the numerical simulation result, first we find the multiplying factors (\( \kappa \)) for the three polymers. These values are estimated by matching the numerical result with the experiment result done in Anteryon BV. The result is shown in table 4.1. With the volume change in the polymer part (Top part of the wedge) found from the numerical result and the relations given in 4.6 and 4.7, we find the rate of change of refractive index. The input parameters and the results are given in the following tables (table 4.2, 4.3)
Figure 4.6: Numerical solution of the wedge model for polymer B and $\Delta T = 58^\circ C$

<table>
<thead>
<tr>
<th>polymer type</th>
<th>polymer A</th>
<th>polymer B</th>
<th>polymer BA15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>-0.7135</td>
<td>-0.856</td>
<td>-0.220</td>
</tr>
</tbody>
</table>

Table 4.1: Empirically determined multiplying factor $\kappa$ for polymer A, B, and BA15 and figures (figure 4.7, 4.8 and 4.9). Note that the initial volume of the lens part of the wedge before deformation and initial temperature are $T_0 = 293.15(K)$ and $V_0 = 1.052616e - 8(m^3)$, respectively. It is clearly observed from the figures that a linear relationship exists between the rate of change of refractive index and temperature for the three polymers. Further it may be observed from figure 4.9 that
the agreement between the numerical and the measured results is not very good for large value of temperature change. One possible reason for this is the nonlinearity behaviour of the material properties (\( \alpha, E, \nu \)) of polymer BA15 with temperature. The values of these input parameters may affect the linear thermoelasticity behaviour of the polymer at high temperature. Finally, we want to state that a good agreement is
\[
T(K) \quad E \quad \alpha(1/K) \quad \nu \quad \Delta V \quad dn/dT \sim -0.856dV/VdT
\]

<table>
<thead>
<tr>
<th>T(K)</th>
<th>E</th>
<th>\alpha(1/K)</th>
<th>\nu</th>
<th>\Delta V</th>
<th>\frac{dn}{dT}</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>294.75</td>
<td>1.900e+9</td>
<td>45.25e-6</td>
<td>0.30</td>
<td>1.6401e-012</td>
<td>-8.3222e-005</td>
<td></td>
</tr>
<tr>
<td>302.65</td>
<td>1.897e+9</td>
<td>47.25e-6</td>
<td>0.30</td>
<td>1.0133e-011</td>
<td>-8.6701e-005</td>
<td></td>
</tr>
<tr>
<td>312.65</td>
<td>1.857e+9</td>
<td>49.75e-6</td>
<td>0.30</td>
<td>2.1819e-011</td>
<td>-9.0965e-005</td>
<td></td>
</tr>
<tr>
<td>325.55</td>
<td>1.793e+9</td>
<td>51.818e-6</td>
<td>0.30</td>
<td>3.7650e-011</td>
<td>-9.4494e-005</td>
<td></td>
</tr>
<tr>
<td>341.55</td>
<td>1.709e+9</td>
<td>54.980e-6</td>
<td>0.30</td>
<td>5.9435e-011</td>
<td>-9.9847e-005</td>
<td></td>
</tr>
<tr>
<td>359.15</td>
<td>1.613e+9</td>
<td>58.881e-6</td>
<td>0.30</td>
<td>8.6417e-011</td>
<td>-1.0646e-004</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.3: Parameter values and results for polymer B

Figure 4.8: Change in refractive index with temperature for polymer B

shown between the existing model and the numerical result in all the three polymers.

In the determination of the top angle (wedge angle), we select three points each from the three edges and follow their destination after deformation. From their new position we calculate the new top angle using trigonometric law. Figure 4.10 and figure 4.11 shows the change in the top angle for polymer B and BA15, respectively.
We observe that the existing model currently used by Anteryon BV assumes the linear expansion coefficient of the polymer in the x,y directions to be equal to linear expansion coefficient of the glass (substrate part of the wedge), which is very approximate. Further more; the existing model does not consider the stiffness property of the lens materials. These and the nonlinearity behaviour of the material properties
Figure 4.10: Change in wedge angle with temperature for polymer B

$(\alpha, E, \nu)$ of polymer BA15 with temperature are some of possible reasons for the difference observed between the measurement, the numerical and the existing model results especially in the wedge angle comparison.
Figure 4.11: Change in wedge angle with temperature for polymer BA15
4.3  Lens model

For the numerical solution of the problem formulated for the wafer lens model in section 2.8, we select the spherical lenses: AC069, AC322 and AC400. These lenses are originally designed for experiment. Table 4.5 shows the geometric properties of the lenses.

<table>
<thead>
<tr>
<th>Lens</th>
<th>Buffer layer(µm)</th>
<th>$F_0$(mm)</th>
<th>$D_0$(mm)</th>
<th>$P_0$(mm)</th>
<th>glass substrate thickness(mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC069</td>
<td>65</td>
<td>17</td>
<td>5.4</td>
<td>0.398</td>
<td>1.5</td>
</tr>
<tr>
<td>AC322</td>
<td>65</td>
<td>9.3</td>
<td>2.8</td>
<td>0.198</td>
<td>1.5</td>
</tr>
<tr>
<td>AC400</td>
<td>65</td>
<td>23</td>
<td>4.8</td>
<td>0.232</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Table 4.5: Geometric properties of the three lenses

4.3.1  Model implementation with Comsol

We assume the lens is initially at reference temperature $T_0$. The temperature is slowly raised to a constant uniform temperature $T$, where the temperature change is $\Delta T = T - T_0$. We also assume the deformation is small and has linear elastic behavior. When we perform the numerical simulation, we consider again borosilicate glass for the substrate part and three different polymers: polymer A, polymer B and polymer BA15 for the top part of the lens. The numerical simulation using Comsol involves those steps we mentioned in the wedge model.

1. The representative module,

   We select solid stress-strain with thermal expansion from the structural mechanics module. It combines solid, stress-strain with heat transfer by conduction, including thermal expansion in the structural loads using the temperature field.
2. The geometry,

The geometry is drawn using SolidWorks and exported to Comsol for simulation (see figure 1.4).

3. The material properties,

The selected three polymers have temperature dependent input parameters: Thermal expansion coefficient, Young’s modulus, and Poisson ratio. We decide to simulate the model from $20^\circ C$ to $100^\circ C$ with $1^\circ C$ temperature interval so that we are able to express the results as a function of temperature. Therefore, the input parameters are first linearly interpolated and their values are loaded in the software as function of temperature. For the substrate part, we use Borosilicate glass whose thermal expansion coefficient, Young’s modulus and Poisson ratio are $7.2e^{-6}[1/K]$, $72.9e+9[Pa]$ and $0.208$ respectively. Note that the initial temperature is assumed to be $T_0 = 20^\circ C$.

4. The boundary conditions and initial conditions,

The free boundary conditions are specified through the Physics $\rightarrow$ Boundary settings menu of Comsol.

5. Meshing

We use the pre-defined quadratic Lagrange elements. We refined the mesh for one level refinement. Here, we restrict the refinement levels to one, since more levels of refinement makes the mesh too complicated to handle (see figure 4.12).

6. Solver’s setting,

We select the parametric solver with statics analysis. We use the temperature
7. Results

Figure 4.14 and figure 4.13 show two results of our implementation of the lens model. Figure 4.13 shows the total displacement corresponding to final temperature value (373.15[K]) and figure 4.14 shows the displacement of the lens curve in the z-direction for temperature profile from 293.15 [K] to 373.15 [K] for the lens AC322 and polymer type B.
Figure 4.13: Numerical solution of lens AC322 and polymer type B
Figure 4.14: The displacement of the lens curve in the z-direction for the lens AC322 and polymer type B
4.3.2 Discussions of the results

From the simulation result, we develop a Matlab program that predict the new lens shape, change in refractive index and other optical parameters. The predicted refractive index and lens shape will be given as input for ray tracing software (like ZEMAX) that calculate the new optical properties of the lens. We also compare the the numerical result with existing model currently used by Anteryon BV (see section 4.2.1) in the program. We use three lens types: (1) AC069, (2) AC322 and (3) AC400, three polymer types: (1) polymer A, (2) polymer B and (3) polymer BA15 and one glass substrate: Borosilicate glass. The program can be executed for the possible temperature range from 20 to 100 degree centigrade. Here we present how the change in refractive index at a point on the lens curve and the new lens shape are determined in the program using the models.

Determination of the change in refractive index and description of the new lens shape

- Change of refractive index

  - From existing model

    As we have stated in the wedge model, this is the model that Anteryon BV currently implementing to predict the change in the optical parameters. First the effective constrained linear thermal expansion coefficient of polymer materials (see section 4.2.1) is estimated as [3]

    \[
    \alpha_{z\text{thin}film} = \frac{1 + \nu}{1 - \nu} \alpha_{iso} - \frac{2\nu}{1 - \nu} \alpha_{sub},
    \]

    (4.8)

    where \( \alpha_{z\text{thin}film} \) is the coefficient of expansion in the z direction, \( \alpha_{iso} \) is the isotropic coefficient of expansion of the polymer, \( \alpha_{sub} \) is the expansion
coefficient of glass and \( \nu \) is the Poisson ratio of the polymer. Then the new volume is estimated by 4.4 and the refractive index change is determined from the change in the volume using a linear relation between volume thermal expansion coefficient and rate of change of refractive index with temperature.

- From the numerical result of the lens model

The volumetric infinitesimal strain \( \varepsilon_{kk} \) is a measure of volume changes. If the infinitesimal volume for undeformed and deformed solid are given by \( dV_0 \) and \( dV \), respectively, then

\[
\varepsilon_{kk} = \frac{dV - dV_0}{dV_0}.
\]

Note that \( \varepsilon_{kk} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \).

If the refractive index of the material at a point in the undeformed solid is \( n_0 \), its refractive index \( n \) in the deformed solid is related as

\[
\frac{n - n_0}{T - T_0} = \kappa \frac{\varepsilon_{kk}}{T - T_0},
\]

where \( \kappa \) is the factor that relate volume thermal expansion coefficient and change of refractive index with temperature as given in 4.6 (see table 4.1).

- New lens shape

The description of the three lenses before deformation is shown in table 4.6. We have said that the new lens description is an input for the ray tracing software. This ray tracing software needs the original form of the lens description. Because of this restriction, we describe the new shape using the magnification factor (scale) and polynomials with the same form. Note that in these descriptions only the coefficients of the polynomial terms are changing.
Table 4.6: Lens curve description

<table>
<thead>
<tr>
<th>Lens type</th>
<th>Lens curve description</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC069</td>
<td>( f(x) = (5.42244660e+1)x^2 + (5.54013510e-2)x^4 )</td>
</tr>
<tr>
<td>AC322</td>
<td>( f(x) = (1.00729830e+2)x^2 + (3.31125570e-1)x^4 - (1.83780000e-3)x^6 )</td>
</tr>
<tr>
<td>AC400</td>
<td>( f(x) = (4.01485720e+1)x^2 + (7.30742400e+4)x^4 )</td>
</tr>
</tbody>
</table>

- **Magnification factor (scale)**

First we determine the scale that the lens is expanding or contracting in each direction, with these magnification factors we describe the lens shape as

\[ \alpha_y = \left\{ \frac{1 + \nu}{1 - \nu} \alpha_{iso} - \frac{2\nu}{1 - \nu} \alpha_{sub} \right\} \Delta T, \]

and

\[ \alpha_x = \alpha_{sub} \Delta T, \]

where \( \alpha_{iso} \) is the isotropic coefficient of expansion of the polymer, \( \alpha_{sub} \) is the expansion coefficient of glass and \( \nu \) is the Poisson ratio of the polymer. With these factors, the polynomial describing the lens shape is

\[ y = a_2 \frac{1 + \alpha_y}{(1 + \alpha_x)^3} x^2 + a_4 \frac{1 + \alpha_y}{(1 + \alpha_x)^3} x^4 + a_6 \frac{1 + \alpha_y}{(1 + \alpha_x)^6} x^6, \]  \hspace{1cm} (4.9)

where \( a_6 = 0 \) for lens type AC069 and AC400.
From the numerical result of the lens model, we determine the scales from the numerical result as

$$\alpha_y = \frac{P - P_0}{P_0}, \quad \alpha_x = \frac{D - D_0}{D_0},$$

where $P_0$ and $D_0$ denote the height and diameter of the lens before deformation, respectively. Similarly, $P$ and $D$ denote the height and diameter of the lens after deformation, respectively. Using these magnification factors and the polynomial given in equation 4.9, the lens shape is described.

<table>
<thead>
<tr>
<th>Polynomial curve fit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Keeping the original form of the polynomial, we use</td>
</tr>
<tr>
<td>$y = a_2x^2 + a_4x^4$ for lens type AC069 and AC400 and</td>
</tr>
<tr>
<td>$y = a_2x^2 + a_4x^4 + a_6x^6$ for lens type AC322 to describe the new shape of the lens after deformation, where $a_2, a_4$ and $a_6$ are constants.</td>
</tr>
</tbody>
</table>

The numerical results obtained from the lens model have been compared with that of the existing model. These are shown in figure 4.15, 4.16, 4.17, 4.18, 4.19 and 4.20. The comparison has been made using the $L^2$ error norm ($\sqrt{\sum_{i=1}^{K} (U_n(i) - U_e(i))^2}$) and the maximum absolute error (peak to valley error, PV) for temperature range from $20^\circ C$ to $100^\circ C$. Some of possible reasons for the error are:
• The existing model assumes the linear expansion coefficient of the polymer in the x,y directions to be equal to linear expansion coefficient of the glass (substrate part of the wedge), which is very approximate.

• The existing model does not consider the stiffness property of the lens materials.

Further it is be observed from the figures that the error is increasing as the change in temperature increases for all lens and polymer type.

Figure 4.15: $L^2$ error norm of the existing model from numerical result of the lens model for polymer A
Figure 4.16: $L^2$ error norm of the existing model from numerical result of the lens model for polymer B

Figure 4.17: $L^2$ error norm of the existing model from numerical result of the lens model for polymer BA15
Figure 4.18: Maximum absolute error of the existing model from numerical result of the lens model for polymer A

Figure 4.19: Maximum absolute error of the existing model from numerical result of the lens model for polymer B
Figure 4.20: Maximum absolute error of the existing model from numerical result of the lens model for polymer BA15
The other main difference between the two model results is observed in the rate of change of refractive index ($dn/dT$). The result from existing model assumes a uniform $dn/dT$. But this is not the situation. In fact it is also space dependent parameter. In the numerical result of the lens model, we find $dn/dT$ which vary along the lens curve. For instance figure 4.21 shows $dn/dT$ of those points which lie on the curve of the lens. As shown in the figure, the rate is increasing from the edge point to the center of the lens curve at which the maximum occurs. In all lens type and polymer type, similar behaviour is observed.

![Graph showing rate of change of refractive index along the lens curve](image)

Figure 4.21: Rate of change of refractive index along the lens curve for the lens type AC069 and polymer type B
Chapter 5

Conclusions and Recommendations

The aim of this thesis is predicting the thermal behavior of the lens, to design a model based lens whose focal length is ideally not changing with temperature. We started the modeling aspect in Chapter 2 where the thermoelasticity equations were derived from fundamental of continuum mechanics. In Chapter 3, where we have considered the study of thermal mismatch induced stresses, we predicted the stresses and their bending effect on a composite system. The result helped us to see what kind of deformation that the uniform temperature change will induce in the lens. The thermal stress is maximum in the interface between the glass and the polymer. From this observation, we can conclude that the thickness of the buffer layer may play a great role in minimizing the effect of the stress on the lens. The numerical result showed a good agreement with this analytic solution. We have also solved a linear thermoelastic problem in axis symmetric three dimensional physical domain to have a better analytic estimation for the lens problem. In spite of the limitation we find in the solution, the result shows how a particular method can be utilized to obtain thermal stresses and displacements in axis symmetric problem.

In the first model implementation with Comsol, the wedge model, we analyzed
the rate of change of refractive index with temperature for three selected polymers. The results showed a strong relation between the rate of change of refractive index and temperature change. We also found a multiplying factor $\kappa$: -0.7135, -0.856 and -0.220 for polymer A, B and BA15, respectively.

Based on the numerical result we found from the lens model, we have been able to achieve our goal of predicting the new lens shape, change in refractive index and other optical parameters for the temperature interval $273.15[K]$ to $375.15[K]$ for the three types of polymers with three different lens shapes.

The factors we found (-0.7135, -0.856 and -0.220 for polymer A, B, BA15, respectively) are not the same and they are also far from the value $-0.56$ which is empirically estimated for wide range of polymers by Zhang [17]. Possible reasons could be:

- The conditions that Anteryon BV implemented (like wave length of the light ray) during experiment to measure the change in refractive index may differ from Zhang’s conditions.

- Optical and thermo-mechanical property difference between the polymers which are used as lens material in Anteryon BV (they are crosslinked polymers) and other common polymers especially those considered by Zhang for experiment.

This shows that it is reasonable to study the conditions implemented during experiment and optical and thermal properties of those polymers selected for the lens materials and compare with other polymers.

Finally, in solving the axis symmetric problem in chapter 3, a possible improvement (recommendation) would be consideration of a linearly elastic finite composite
cylinder with a complete set of orthogonal functions which produce a much faster decay of coefficients than produced by the Fourier-Bessel expansions.
Appendix A

Reference code

This appendix contains some of the code used to obtain most of the results described in Chapter 4. All experiments were run on a 3.0 GHz personal computer with 4.00 GB of RAM. Note that for the larger experiments, a more efficient alternative implementation was used. However, for readability, we only include the basic code. The simulation was written in Matlab, and the results were obtained in a reasonable amount of time. For the sake of completeness, we mention that the results contained herein are independent of those mentioned in any earlier publications.

function Y=main_program(lens_type,polymer_type,temperature)
%/* ---------------------------------------------------
% copyright Anteryon BV, July 2010
% Eindhoven University of Technology
% Eindhoven, The Netherlands
% distribution prohibited
% ---------------------------------------------------*/
% This program predicts the new shape, rate of change of refractive index
% and other optical parameter of a wafer lens due to deformation occurs on
% the lens when it is subjected to a uniform change in temperature. The
% analysis is based on the numerical result done on the lens using FEM.
% It also predicts the above optical parameters from the existing model
% (currently used by Anteryon BV) and gives the error of the existing model
% from the numerical analysis.
% There are 3 different lens types that the models are implemented:
There are also three different polymer types used for the lens part of the wafer lens:

(1) polymer A
(2) polymer B
(3) polymer BA15.

We use one type of glass substrate which is Borosilicate glass.

This main program gives you the required output corresponding to the input you will give to the program.

% input:

% lens_type: a number from the set {1,2,3}
% The numbers represent the lens type as
% 1: AC069
% 2: AC322
% 3: AC400

% polymer_type: a number from the set {1,2,3}
% The numbers represent the polymer type as
% 1: polymer A
% 2: polymer B
% 3: polymer BA15

% temperature: a number from the set {20,21,22,...,100} which represents the final temperatures in degree centigrade when the wafer lens is subjected to a uniform temperature from an initial temperature 20 degree centigrade.

% output:

% Rate of change of refractive index and focal length with temperature.
% Description for both the original and predicted new shape of the lens.
% Error of the existing model from the FEM.
% Plots of the lens curve from the FEM(Comsol) result, existing model result and polynomial fit.

clear all
format long
lens_type= input('For what type of lens you want to see?');
polymer_type=input('For what type of polymer?');
temperature=input('Final temperature?');
Y=out_put(lens_type,polymer_type,temperature);
function [rms] = out_put(lens_type, polymer_type, temperature)
    [new_shape_coordinates] = input4(lens_type, polymer_type, temperature);
    [dndT1] = input5(lens_type, polymer_type, temperature);
    [alphay new_height] = input3(lens_type, polymer_type);
    [nah_A069_A_top nah_A069_B_top nah_A069_BA15_top nah_A322_A_top...
     nah_A322_B_top] = data5a;
    [nah_A322_BA15_top nah_A400_A_top nah_A400_B_top...
     nah_A400_BA15_top] = data5b;
    [alphax new_diameter volume_change] = input2(lens_type, polymer_type);
    [old_volume, ~] = input1(lens_type);
    volume(1) = old_volume;
    if polymer_type == 1
        n_old = 1.5112; dndT = zeros(1, 81); dfdT = zeros(1, 100); n_new = zeros(1, 100);
        f_new = zeros(1, 100); R_new = zeros(1, 100);
        n_new(20) = n_old; f_new(20) = R_new(20) / (n_new(20) - 1);
        R_new(20) = (0.5 * new_diameter(1)^2 + new_height(1)^2) / (2 * new_height(1));
        for k = 2:81
            dndT(k) = -0.7 * (volume_change(k) - volume_change(k - 1)) / (volume_change(k - 1) + volume(1));
            end
            for temp = 21:100
                R_new(temp) = ((0.5 * new_diameter(temp - 19)^2 +...
                    + new_height(temp - 19)^2) / (2 * new_height(temp - 19)));
                n_new(temp) = n_old + dndT(temp - 19) * (temp - 20);
                f_new(temp) = R_new(temp) / (n_new(temp) - 1);
                dfdT(temp) = (1 / (n_new(temp)^2)) * (R_new(temp) - R_new(temp - 1)) -...
                    + (R_new(temp) / (n_new(temp)^2)) * dndT(temp - 19);
                dfdT(temp) = (1 / (n_new(temp)^2)) * (R_new(temp) - R_new(temp - 1)) -...
                    + (R_new(temp) / (n_new(temp)^2)) * dndT(temp - 19);
                end
    elseif polymer_type == 2
        n_old = 1.5689; dndT = zeros(1, 81); dfdT = zeros(1, 100); n_new = zeros(1, 100);
        f_new = zeros(1, 100); R_new = zeros(1, 100);
        n_new(20) = n_old; f_new(20) = R_new(20) / (n_new(20) - 1);
        R_new(20) = (0.5 * new_diameter(1)^2 + new_height(1)^2) / (2 * new_height(1));
        for k = 2:81
            dndT(k) = -0.9 * (volume_change(k) - volume_change(k - 1)) / (volume_change(k - 1) + volume(1));
            end
for temp=21:100
R_new(temp)=((0.5*new_diameter(temp-19))^2+... 
+new_height(temp-19)^2)/(2*new_height(temp-19));
n_new(temp)=n_old+dndT(temp-19)*(temp-20);
f_new(temp)=R_new(temp)/(n_new(temp)-1);
dfdT(temp)=(1/(n_new(temp)-1))*(R_new(temp)-R_new(temp-1))-... 
+(R_new(temp)/(n_new(temp)-1)^2)* dndT(temp-19);
end

else
n_old=1.5456;dndT=zeros(1,81);dfdT=zeros(1,100);n_new=zeros(1,100);
f_new=zeros(1,100);R_new=zeros(1,100);
n_new(20)=n_old;f_new(20)=R_new(20)/(n_new(20)-1);
R_new(20)=(0.5*new_diameter(1)^2+new_height(1)^2)/2*new_height(1);
for k=2:81
dndT(k)=-0.201*(volume_change(k)-volume_change(k-1))/...  
(volume_change(k-1)+volume(1));
end
for temp=21:100
R_new(temp)=((0.5*new_diameter(temp-19))^2+... 
+new_height(temp-19)^2)/(2*new_height(temp-19));
n_new(temp)=n_old+dndT(temp-19)*(temp-20);
f_new(temp)=R_new(temp)/(n_new(temp)-1);
dfdT(temp)=(1/(n_new(temp)-1))*(R_new(temp)-R_new(temp-1))-... 
+(R_new(temp)/(n_new(temp)-1)^2)* dndT(temp-19);
end
end

dndT= dndT(temperature-19);  dfdT= dfdT(temperature);

% lens curve description: magnification factor
T=temperature-19;
if lens_type==1
    x0=[-0.0027:0.00001:0.0027];
y0=(1e+3)*(0.054224466)*x0.^2+(1e+3)*(5.5401351E-05)*x0.^4;
    plot(x0,y0,'r');hold on
    hold on
    x=x0*(1+alphax(T));
y=(1e+3)*(0.054224466)*((1+alphay(T))/(1+alphax(T))^2)*x.^2+... 
(1e+3)*(5.5401351E-05)*((1+alphay(T))/(1+alphax(T))^4)*x.^4;
    plt=plot(x,y,'b');hold on
end
a=(1e+3)*(0.054224466)*((1+alphay(T))/(1+alphax(T))^2);
b=(1e+3)*(5.5401351E-05)*((1+alphay(T))/(1+alphax(T))^4);
c=0;a_0=(1e+3)*(0.054224466);b_0=(1e+3)*(5.5401351E-05);c_0=0;

elseif lens_type==2
x0=[-0.0014:0.00001:0.0014];
y0=(1e+3)*(0.10072983)*x0.^2+(1e+3)*(0.00033112557)*x0.^4+...
+(1e+3)*(-0.0000018378)*x0.^6;
plot(x0,y0,'r');hold on
x=x0*(1+alphax(T));
y=(1e+3)*(0.10072983)*((1+alphay(T))/(1+alphax(T))^2)*x.^2+...
+(1e+3)*((1e+3)*((1+alphay(T))/(1+alphax(T))^4)*x.^4+...(1e+3)*(-0.0000018378)*((1+alphay(T))/(1+alphax(T))^6)*x.^6;
plot(x,y,'b');hold on

x0=[-0.0024:0.00001:0.0024];
y0=(1e+3)*(4.0148572E-02)*x0.^2+(1e+3)*(7307424E-05)*x0.^4;
plot(x0,y0,'r');hold on
x=x0*(1+alphax(T));
y=(1e+3)*(4.0148572E-02)*((1+alphay(T))/(1+alphax(T))^2)*x.^2+...
+(1e+3)*((1e+3)*((1+alphay(T))/(1+alphax(T))^4)*x.^4;
plot(x,y,'b');hold on

end

[old_dnbydT old_dfbydT a_old b_old c_old y_old]=...
existing_model(lens_type,polymer_type,temperature);
fprintf('

');
fprintf('Rate of change of refractive index and focal length:

');
fprintf('+----------------+----------------+------------------------+
');
fprintf('| Method | dn/dT |df/dT (Rough estimation)|
');
fprintf('+----------------+----------------+------------------------+
');
fprintf('| Existing model | %12.8e | %12.8e | 
',old_dnbydT,old_dfbydT);
fprintf('| Numerical result| %12.8e | %12.8e| 
', dndT,dfdT);
fprintf('+----------------+-------------------+-------------------+

');
fprintf('Original lens %12.8e x^2 +%12.8e x^4+%12.8e x^6\n\n', a_0, b_0, c_0);
fprintf('The new lens curve description:\n\n');
fprintf('+-----------------------------------+----------------------+
');
fprintf('|Method (using magnification factor)| Polynomial |
');
fprintf('+-----------------------------------+----------------------+
');
fprintf('|Exist_model|%12.8ex^2+%12.8ex^4+%12.8ex^6|\n\n', a_old, b_old, c_old);
fprintf('|Numerical result|%12.8ex^2+%12.8e x^4+%12.8e x^6|\n\n', a, b, c);
fprintf('+-----------------+-----------------------------------+
');

[n,m]=size(y); err=sum(abs(y-y_old))/m; rms=sqrt(sum((y-y_old).^2)/m);
fprintf('Error of existing model from Numerical result %12.8e \n\n', err);
fprintf('Polynomial curve fit for the data \n');

%%%%%%%%%%%%%%%%%% lens curve description: polynomial fit %%%%%%%%%%%%%%%%%
if lens_type==1&&polymer_type==1
    x=new_shape_coordinates(1,:)+'(A069A_y_displacement(temperature))';
y=-new_shape_coordinates(2,:)+3.982406e-4+nah_A069_A_top(T);
elseif lens_type==1&&polymer_type==2
    x=new_shape_coordinates(1,:)+'(A069B_y_displacement(temperature))';
y=-new_shape_coordinates(2,:)+3.982406e-4+nah_A069_B_top(T);
elseif lens_type==1&&polymer_type==3
    x=new_shape_coordinates(1,:)+...linearA069BA15_y_displacement(temperature);
y=-new_shape_coordinates(2,:)+3.982406e-4+nah_A069_BA15_top(T);
elseif lens_type==2&&polymer_type==1
    x=new_shape_coordinates(1,:)+'(A322A_y_displacement(temperature))';
y=-new_shape_coordinates(2,:)+1.986887e-4+nah_A322_A_top(T);
elseif lens_type==2&&polymer_type==2
    x=new_shape_coordinates(1,:)+'(A322B_y_displacement(temperature))';
y=-new_shape_coordinates(2,:)+1.986887e-4+nah_A322_B_top(T);
elseif lens_type==2&&polymer_type==3
    x=new_shape_coordinates(1,:)+...linearA322BA15_y_displacement(temperature);
y=-new_shape_coordinates(2,:)+1.986887e-4+nah_A322_BA15_top(T);
elseif lens_type==3&&polymer_type==1
    x=new_shape_coordinates(1,:)+'(A400A_y_displacement(temperature))';
y=-new_shape_coordinates(2,:)+2.3216e-004+nah_A400_A_top(T);
elseif lens_type==3&&polymer_type==2
    x=new_shape_coordinates(1,:)+...
x=new_shape_coordinates(1,:)+A400B_y_displacement(temperature);  
y=-new_shape_coordinates(2,:)+(2.3216e-004)+nah_A400_B_top(T);  
elseif lens_type==3&&polymer_type==3  
    x=new_shape_coordinates(1,:)+linearA400BA15_y_displacement(temperature);  
y=-new_shape_coordinates(2,:)+(2.3216e-004)+nah_A400_BA15_top(T);  
end  
plot(x,y','.')  
if lens_type==1 || lens_type==3  
    ft = fittype({'x^4','x^2'},...  
        'dependent',{'y'},...  
        'independent',{'x'},...  
        'coefficients',{'a', 'b'});  
    [cf G]= fit(x,y,ft);  
    plot(x0*(1+alphax(T)),cf(x0*(1+alphax(T))),'g')  
else  
    ft = fittype({'x^6','x^4','x^2'},...  
        'dependent',{'y'},...  
        'independent',{'x'},...  
        'coefficients',{'a','b','c'});  
    [cf G]= fit(x,y,ft);  
    plot(x0*(1+alphax(T)),cf(x0*(1+alphax(T))),'g')  

%----------------------------------------------------- rate of change of refractive index:-----------------------------
if lens_type==1&&polymer_type==1  
    x_index=dndT1(1,:)'; y_index=dndT1(2,:)';  
elseif lens_type==1&&polymer_type==2  
    x_index=dndT1(1,:)'; y_index=dndT1(2,:)';  
elseif lens_type==1&&polymer_type==3  
    x_index=dndT1(1,:)'; y_index=dndT1(2,:)';  
elseif lens_type==2&&polymer_type==1  
    x_index=dndT1(1,:)'; y_index=dndT1(2,:)';  
elseif lens_type==2&&polymer_type==2  
    x_index=dndT1(1,:)'; y_index=dndT1(2,:)';  
elseif lens_type==2&&polymer_type==3  
    x_index=dndT1(1,:)'; y_index=dndT1(2,:)';  
elseif lens_type==3&&polymer_type==1  
    x_index=dndT1(1,:)'; y_index=dndT1(2,:)';  
elseif lens_type==3&&polymer_type==2  
    x_index=dndT1(1,:)'; y_index=dndT1(2,:)';  
else  
    x_index=dndT1(1,:)'; y_index=dndT1(2,:)';  
end
elseif lens_type==3&&polymer_type==3
    x_index=dndT1(1,:); y_index=dndT1(2,:);'
end
out1=x_index; out2=y_index';
end
Bibliography


