Lateral torsional buckling analysis of multiple laterally restrained I-beams in bending

Jovic, M.

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Author:
Marko Jović

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Graduation committee:
Prof.Ir.H.H. Snijder – Eindhoven University of Technology
Ir. B.W.E.M. van Hove – Eindhoven University of Technology
Dr.Ir.H. Hofmeyer – Eindhoven University of Technology

Eindhoven University of Technology
Department of the Built Environment
Structural Design
Lateral torsional buckling analysis of multiple laterally restrained I-beams
Preface

Eindhoven, June 2015

This report concludes the Master’s research project regarding the lateral torsional buckling analysis of multiple laterally restrained I-beams in bending, performed at the Eindhoven University of Technology for the master track Structural Design at the department of the Built Environment.

First I would like to express my gratitude towards the graduation committee: Prof.Ir.H.H. Snijder, Dr.Ir.H. Hofmeyer, and Ir. B.W.E.M. van Hove for the professional supervision during this graduation project. I am also grateful to be given the opportunity to present my graduation project to the Technical Committee of “Bouwen met Staal” (TC8-Stability).

Furthermore, I would like to thank my parents, Miodrag and Mirjam Jović, and my sister Marina Karnata, for their support both financially and mentally. Also I would like to thank a long list of friends and fellow students: Stefan Zwegers, Hamza Chakiri, Ivan Šarić, Fandi Pawiroredjo, Rob van der Aa, Jordan Dorlijn, Benny Ng, Lin Luu, Stephany Ritoi, and Aukje Goossens for their help, their support, their company, and their understanding during the time of my graduation project. And last but not least, I would like to thank my girlfriend Sandra Janssen for her patience and for keeping up with me especially during the last months of this graduation project.

Marko Jović
Lateral torsional buckling analysis of multiple laterally restrained I-beams
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Summary

Long span beams that are subjected by bending are sensitive to lateral torsional buckling, therefore for these beams the lateral torsional buckling resistance needs to be assessed in order to guarantee the structure will not fail due to lateral torsional buckling instability. The design codes provide equations to determine the lateral torsional buckling resistance for different load cases, but these equations are only validated for laterally unrestrained beams. In practice long span beams are laterally restrained to provide more resistance against lateral torsional buckling and it is assumed that any type of lateral restraints guarantee this additional resistance. This research project has its emphasise on the influence of multiple lateral restraints and the type of lateral restraints which add to the lateral torsional buckling resistance of doubly symmetrical I-beams.

In order to obtain more information regarding lateral torsional buckling and more specifically lateral torsional buckling of laterally restrained beams, a literature study has been performed. For this literature study many different methods to obtain the elastic critical lateral torsional buckling moment were studied and compared with one another. In this literature study it is concluded which method to use for comparison with the results of this research project.

For this research project only numerical research is performed to provide results regarding the lateral torsional buckling stability of multiple laterally restrained I-beams. Therefore, correct modelling using finite element method software is of utmost importance to provide reliable results for the numerical analyses. Because for this research project no experiments were conducted simultaneously to validate the results from the numerical analyses, the results from other independent experiments and finite element models were used for validation.

The results for this research project were generated by means of a parametric study using the validated finite element model. During this parametric study different beam lengths, different numbers of lateral restraints, and different positions of the lateral restraints were studied. For all these cases also the influence of different imperfection shapes has been included in the parametric study. The results from this parametric study are carefully studied and compared with the results obtained with the design codes.

The main conclusion of this research project is that a correct definition of the term lateral restraints is not provided by the design codes. In order to obtain safe and consistent results, the lateral restraints need to be applied in such a way that fork conditions are satisfied at the position of the lateral restraints.
Lateral torsional buckling analysis of multiple laterally restrained I-beams
## Nomenclature

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>height of the web</td>
<td>$mm$</td>
</tr>
<tr>
<td>$e$</td>
<td>distance from shear centre to centroid</td>
<td>$mm$</td>
</tr>
<tr>
<td>$f$</td>
<td>modification factor used with modified specific method</td>
<td>-</td>
</tr>
<tr>
<td>$f_y$</td>
<td>yield strength steel</td>
<td>$N/mm^2$</td>
</tr>
<tr>
<td>$g$</td>
<td>distance between load introduction and shear centre</td>
<td>$mm$</td>
</tr>
<tr>
<td>$h$</td>
<td>height of the cross-section</td>
<td>$mm$</td>
</tr>
<tr>
<td>$i_{yz}$</td>
<td>radius of gyration of the equivalent compression flange</td>
<td>$mm$</td>
</tr>
<tr>
<td>$k$</td>
<td>cross-sectional property, $k = 0$ for doubly symmetrical cross-sections</td>
<td>-</td>
</tr>
<tr>
<td>$k_s$</td>
<td>slenderness correction factor for moment distribution between restraints</td>
<td>-</td>
</tr>
<tr>
<td>$k_g$</td>
<td>modification factor accounting for conservatism of the equivalent compression flange method</td>
<td>-</td>
</tr>
<tr>
<td>$k_{rd}$</td>
<td>reduction factor depending on the dimensional stability of the cross-section</td>
<td>-</td>
</tr>
<tr>
<td>$k_y$</td>
<td>buckling length factor</td>
<td>-</td>
</tr>
<tr>
<td>$k_z$</td>
<td>buckling length factor</td>
<td>-</td>
</tr>
<tr>
<td>$k_w$</td>
<td>buckling length factor</td>
<td>-</td>
</tr>
<tr>
<td>$z_g$</td>
<td>distance between the load application point and shear centre</td>
<td>$mm$</td>
</tr>
<tr>
<td>$z_j$</td>
<td>cross-sectional property, $z_j = 0$ for doubly symmetrical cross-sections</td>
<td>-</td>
</tr>
</tbody>
</table>

<p>| $A$      | area of the cross-section | $mm^2$ |
| $C$      | coefficient depending on the beam length, cross-section, and applied loads | - |
| $C_1$    | coefficient depending on load type | - |
| $C_2$    | coefficient depending on the application position of the load | - |
| $C_3$    | coefficient to take account for asymmetry of the cross-section | - |
| $C_{95%}$ | Rotational spring stiffness of lateral restraint that accounts for 95% of $M_{cr}$ value | $Nm/rad$ |
| $E$      | Young’s modulus of elasticity | $N/mm^2$ |
| $G$      | shear modulus | $N/mm^2$ |
| $I_t$    | torsional constant | $mm^4$ |
| $I_w$    | warping constant | $mm^6$ |
| $I_z$    | out-of-plane moment of inertia around the z-axis | $mm^4$ |</p>
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>out-of-plane rotation coefficient</td>
<td>-</td>
</tr>
<tr>
<td>$K_{95%}$</td>
<td>Lateral spring stiffness of lateral restraint that accounts for 95% of $M_{cr}$ value</td>
<td>N/mm</td>
</tr>
<tr>
<td>$K_B$</td>
<td>stiffness of lateral restraint</td>
<td>N/mm</td>
</tr>
<tr>
<td>$K_B$</td>
<td>stiffness matrix</td>
<td>-</td>
</tr>
<tr>
<td>$K_L$</td>
<td>linear stiffness matrix</td>
<td>-</td>
</tr>
<tr>
<td>$K_G$</td>
<td>geometrical stiffness matrix</td>
<td>-</td>
</tr>
<tr>
<td>$L_g$</td>
<td>unrestrained length of the beam</td>
<td>mm</td>
</tr>
<tr>
<td>$L_i$</td>
<td>length between lateral restraints</td>
<td>mm</td>
</tr>
<tr>
<td>$L_{js}$</td>
<td>beam length between two fork supports</td>
<td>mm</td>
</tr>
<tr>
<td>$L_{lt}$</td>
<td>lateral torsional buckling length between two fork supports, a fork support and a restraint, or between two restraints</td>
<td>mm</td>
</tr>
<tr>
<td>$L_{ltb}$</td>
<td>lateral torsional buckling length of the beam</td>
<td>mm</td>
</tr>
<tr>
<td>$L_{sd}$</td>
<td>unrestrained length between two fork supports, a fork support and a restraint, or between two restraints</td>
<td>mm</td>
</tr>
<tr>
<td>$M_{b,Rd}$</td>
<td>design buckling resistance moment</td>
<td>kNm</td>
</tr>
<tr>
<td>$M_{cr}$</td>
<td>elastic critical bending moment for lateral torsional buckling</td>
<td>kNm</td>
</tr>
<tr>
<td>$M_{s,Rd}$</td>
<td>design resistance moment</td>
<td>kNm</td>
</tr>
<tr>
<td>$M_{y,Ed}$</td>
<td>maximum design moment of bending moment (y-axis)</td>
<td>kNm</td>
</tr>
<tr>
<td>$N_{y,z}$</td>
<td>elastic buckling force for buckling about z-z axis</td>
<td>N</td>
</tr>
<tr>
<td>$U$</td>
<td>strain energy</td>
<td>-</td>
</tr>
<tr>
<td>$U_g$</td>
<td>non-linear part of the strain energy</td>
<td>-</td>
</tr>
<tr>
<td>$U_l$</td>
<td>linear part of the strain energy</td>
<td>-</td>
</tr>
<tr>
<td>$W_{pl,y}$</td>
<td>plastic section modulus (y-axis)</td>
<td>mm$^3$</td>
</tr>
<tr>
<td>$W_y$</td>
<td>section modulus (y-axis)</td>
<td>mm$^3$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>coefficient which varies with the loading and support conditions</td>
<td>-</td>
</tr>
<tr>
<td>(Nethercot)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha_{LT}$</td>
<td>imperfection factor</td>
<td>-</td>
</tr>
<tr>
<td>$\alpha_{cr}$</td>
<td>critical moment multiplier</td>
<td>-</td>
</tr>
<tr>
<td>$\beta$</td>
<td>quotient of the design values of end moments</td>
<td>-</td>
</tr>
<tr>
<td>$\beta_y$</td>
<td>Wagner's factor of the section</td>
<td>-</td>
</tr>
<tr>
<td>$\gamma_{M1}$</td>
<td>partial safety factor, recommendation: $\gamma_{M1} = 1.0$</td>
<td>-</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>strain</td>
<td>-</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>factor that accounts for the effect of the bending moment distribution between lateral supports.</td>
<td>-</td>
</tr>
</tbody>
</table>

10
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>load proportionality factor used in modified Riks method</td>
</tr>
<tr>
<td>$\bar{\lambda}_{c,0}$</td>
<td>slenderness limit</td>
</tr>
<tr>
<td>$\bar{\lambda}_f$</td>
<td>slenderness of the compression flange</td>
</tr>
<tr>
<td>$\bar{\lambda}_{LT}$</td>
<td>non-dimensional slenderness for lateral torsional buckling</td>
</tr>
<tr>
<td>$\bar{\lambda}_{LT,0}$</td>
<td>maximum value for non-dimensional slenderness</td>
</tr>
<tr>
<td>$\bar{\lambda}_z$</td>
<td>corresponding slenderness for weak-axis flexural buckling</td>
</tr>
<tr>
<td>$\mu_{cr}$</td>
<td>critical load multiplier</td>
</tr>
<tr>
<td>$\Phi_{LT}$</td>
<td>factor used to determine reduction factor</td>
</tr>
<tr>
<td>$\chi$</td>
<td>reduction factor of the equivalent compression flange</td>
</tr>
<tr>
<td>$\chi_{LT}$</td>
<td>reduction factor for lateral torsional buckling</td>
</tr>
<tr>
<td>$\chi_{LT, GM}$</td>
<td>reduction factor used with general method</td>
</tr>
<tr>
<td>$\chi_{LT, MSM}$</td>
<td>reduction factor used with modified specific method</td>
</tr>
<tr>
<td>$\chi_{LT, SM}$</td>
<td>reduction factor used with specific method</td>
</tr>
</tbody>
</table>
Definitions

Coordinate system
Throughout this research project two different coordinate systems are used for different applications, therefore a clear distinction must be made between the two. Figure 1.1 shows the coordinate system as it is used in EN 1993-1-1 and other literature. Therefore, this coordinate system should be used when considering the equations from EN 1993-1-1 and other literature. Figure 1.2 shows the default coordinate system as is used in the finite element software ABAQUS. This coordinate system is used for the finite element modelling and analysis. Using the two different coordinate systems is not problematic as long as the correct directions are considered for comparison between the results from the design codes and the results from the finite element analysis.

Elastic critical lateral torsional buckling mode
Throughout this research project, two significantly different lateral torsional buckling modes are distinguished. Figure 2 shows the first positive lateral torsional buckling mode for a laterally unrestrained beam subject to an equally distributed load. For this research project only the positive lateral torsional buckling mode is considered because for this case the buckling mode directly corresponds to the load applied on the model. Figure 3 shows the first positive lateral torsional buckling mode of the laterally restrained beam with two lateral restraints. The blue lines represent the position of the lateral restraints. These lateral torsional buckling modes are determined using a linear buckling analysis.
Plastic failure mode

The plastic failure mode is the true shape at which the beam is considered to have failed due to instability. The plastic failure mode is determined using a geometric and material non-linear analysis in which system imperfections and residual stresses are included. The plastic failure mode should not be mistaken to be the same as the elastic critical lateral torsional buckling mode, even though they are related. The plastic failure mode of the non-linear model (Figure 4) should however follow the same buckling shape as the elastic critical lateral torsional buckling mode of the corresponding linear elastic model (Figure 3).
1. Introduction

1.1 Problem statement

Lateral torsional buckling is a common problem when it comes to stability of long span beams. Extensive research has been conducted on the subject resulting in numerous equations for calculating the lateral torsional buckling resistance. However, these equations only apply to laterally unrestrained beams with simple end supports, end supports satisfying fork conditions, and fixed end supports. In practice, lateral restraints are commonly used to improve the lateral torsional buckling strength of beams in bending. However, only limited studies have been conducted for these cases.

The current state of EN 1993-1-1 (Eurocode 3: Design of steel structures) provides methods to determine the lateral torsional buckling resistance of laterally unrestrained beams. And in order to determine this lateral torsional buckling resistance, the value of the elastic critical lateral torsional buckling moment is needed. How to determine the elastic critical lateral torsional buckling moment is not provided in EN 1993-1-1. However, the Dutch National Annex of EN-1993-1-1 does provide equations to determine the elastic critical lateral torsional buckling moment, and even provides a method to determine the elastic critical lateral torsional buckling moment for laterally restrained beams. The latter equations however have not been validated yet. This leads to the following problems:

Problem 1: “The current state of EN 1993-1-1 does provide rules for the lateral torsional buckling resistance for I-beams in bending; however, the determination of the elastic critical lateral torsional buckling moment is not provided, for both laterally unrestrained and laterally restrained beams.”

Problem 2: “The Dutch National Annex of EN 1993-1-1 does provide equations for determining the elastic critical lateral torsional buckling moment for both laterally unrestrained and laterally restrained beams; however, the latter have not been validated to be safe.”

Problem 3: “If the equations from The Dutch National Annex are used to determine the elastic critical lateral torsional buckling moment for laterally restraints beams, does this provide safe results in combination with the equations of EN 1993-1-1 for the lateral torsional buckling resistance?”

1.2 Goal

Multiple lateral restraints are commonly used in practice to improve the lateral torsional buckling resistance of long span beams in bending, such as the office building illustrated in Figure 5. Therefore, more knowledge about the lateral torsional buckling behaviour for these cases is desirable. EN 1993-1-1 only provides validated rules for laterally unrestrained beams. However, a simple assessment method is given for beams with restraints. This method only determines if a laterally restrained beam is susceptible to lateral torsional buckling or not, it
does not provide a value for the lateral torsional buckling resistance that accounts for the influence of lateral restraints.

![Example of laterally restrained beams in practice, office building in Heteren, the Netherlands](image)

The main goal of this research project is to obtain more knowledge about the lateral torsional buckling behaviour of multiple laterally restrained beams and about the behaviour of the lateral restraints themselves. This information is then used to validate whether the current equations in EN 1993-1-1 provide safe results for the case of laterally restrained beams. This is studied by performing numerical analyses using the finite element method (FEM). The results from these analyses are then compared with the results determined with the equations from EN 1993-1-1.

**Goal:** “Validate whether the current (and future) equations from EN 1993-1-1 provide safe results for the case of multiple laterally restrained I-beams in bending.”

### 1.3 Approach

At first, a literature study is conducted to obtain more knowledge concerning lateral torsional buckling itself. Furthermore, research articles concerning lateral torsional buckling are studied to determine what kind of research has been conducted and what results have been found so far, especially concerning laterally restrained I-beams. The determination of the elastic critical lateral torsional buckling moment $M_c$ is part of the literary study as well, given the fact that many different methods for determining this elastic critical lateral torsional buckling moment are available. These different methods are analysed and the elastic critical lateral torsional buckling moment is determined using these different methods for certain unrestrained I-beams. The results from these different methods are then compared with one another.

Next, a finite element model is built which is used to perform the numerical analyses. This FEM model needs to simulate the behaviour of a realistic I-beam and therefore a correct modelling procedure needs to be maintained. This FEM model is validated using data obtained from experiments and data from obtained by other FEM models.

After a validated FEM model is acquired, this model is then used to perform a parametric study. In this parametric study, different steel profiles and beam lengths are studied that could occur in practice. Also the number and the positions of the lateral restraints are varied.
A linear buckling analysis (LBA) is used to determine the elastic critical lateral torsional buckling moment, which is used to determine the non-dimensional slenderness $\lambda_{LT}$. Subsequently, a geometrical and material non-linear imperfection analysis (GMNIA) is used to determine the lateral torsional buckling resistance, which is used to determine the reduction factor $\chi_{LT}$.

Finally, the results from the numerical analyses are compared with the results obtained from EN 1993-1-1 and these results are the foundation from which the conclusions and recommendations follow.
2. Literature study

2.1 Lateral torsional buckling

When a perfectly straight elastic beam is subjected to bending and only the yield strength is considered, the bending moment resistance of this beam can be determined with:

\[ M_{b,Rd} = W_y f_y \]

\( Eq. 1 \)

Where \( W_y \) is the appropriate section modulus as follows;
- \( W_y = W_{pl,y} \) for Class 1 or 2 cross-sections
- \( W_y = W_{pl,y} \) for Class 3 cross-sections
- \( W_y = W_{ef,y} \) for Class 4 cross-sections

However, this value for the bending moment resistance may not be reached due to an instability phenomenon called lateral torsional buckling. Considering an I-beam in bending, the cross-section of the beam is subjected to compression stresses and tensile stresses. When the lateral stiffness of the beam is lower than the vertical stiffness (the direction of the load), the beam is able to deflect horizontally as well as vertically. The flange that is subjected to compression may buckle laterally when a certain stress value is reached. The flange that is subjected to tension will resist this lateral deflection, resulting in a smaller lateral deflection with respect to the deflection of the flange in compression. As a result of these unequal deflections, the beam will undergo a rotation resulting in a torsional deflection (Figure 6). The lateral deflection causes a lateral moment and the torsional deflection causes a torsional moment, in addition to the existing bending moment. The combination of these effects can result in instability and failure of the beam.

![Figure 6: Rotation of the beam due to lateral torsional buckling [1]](image_url)
Due to lateral torsional buckling, the bending moment resistance is reduced by a factor $\chi_{LT}$. This factor is called the reduction factor for lateral torsional buckling, or the lateral torsional buckling factor. The bending moment resistance of a beam in bending with lateral torsional buckling taken into account should be determined with:

$$M_{b, Rd} = \chi_{LT} W_y f_y$$

Eq. 2

### 2.2 Lateral torsional buckling resistance according to EN 1993-1-1

In chapter 6.3.2 of EN 1993-1-1, three methods are described to determine the ultimate lateral torsional buckling load of beams in bending. A fourth method to determine if a restrained beam is susceptible to lateral torsional buckling is given as well. First the general method (GM) is described, then the specific method (SM), the modified specific method (MSM) is described thereafter, and finally the simplified assessment method for beams with restraints. For the first three methods, the design lateral torsional buckling resistance moment should be taken as equation Eq. 2. The reduction factor $\chi_{LT}$ is a function of the imperfection factor $\alpha_{LT}$ and the non-dimensional slenderness given by [2]:

$$\bar{\lambda}_{LT} = \sqrt{\frac{W_y f_y}{M_{cr}}}$$

Eq. 3

Where, $M_{cr}$ is the elastic critical lateral torsional buckling moment of the gross cross-section, about the main axis. EN 1993-1-1 does not specify a method to determine the elastic critical lateral torsional buckling moment in this chapter; however, the required equations can be found in the Dutch National Annex. Different methods to determine the elastic critical lateral torsional buckling moment are described in section 2.4.

#### 2.2.1 General method

For members in bending of constant cross-section, the value of $\chi_{LT}$ for appropriate non-dimensional slenderness $\bar{\lambda}_{LT}$ should be determined from [2]:

$$\chi_{LT, GM} = \frac{1}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \bar{\lambda}_{LT}^2}}$$ \text{, but } \chi_{LT} \leq 1,0$$

Eq. 4

$$\Phi_{LT} = 0,5 \left\{ 1 + \alpha_{LT} \left( \bar{\lambda}_{LT} - 0,2 \right) + \bar{\lambda}_{LT}^2 \right\}$$

Eq. 5
The imperfection factor $\alpha_{LT}$ corresponding to the appropriate buckling curve may be obtained from the Dutch National Annex which are given in Table 1.

Table 1: Recommended values for imperfection factor [2]

<table>
<thead>
<tr>
<th>Buckling curve</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Imperfection factor $\alpha_{LT}$</td>
<td>0.21</td>
<td>0.34</td>
<td>0.49</td>
<td>0.76</td>
</tr>
</tbody>
</table>

Recommendations for buckling curves are given in Table 2.

Table 2: Recommended values for lateral torsional buckling curves for cross-sections using Eq. 4[2]

<table>
<thead>
<tr>
<th>Cross-section</th>
<th>Limits</th>
<th>Buckling curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rolled I-sections</td>
<td>$h/b \leq 2$</td>
<td>a</td>
</tr>
<tr>
<td></td>
<td>$h/b &gt; 2$</td>
<td>b</td>
</tr>
<tr>
<td>Welded I-sections</td>
<td>$h/b \leq 2$</td>
<td>c</td>
</tr>
<tr>
<td></td>
<td>$h/b &gt; 2$</td>
<td>d</td>
</tr>
<tr>
<td>Other cross-sections</td>
<td>-</td>
<td>d</td>
</tr>
</tbody>
</table>

Values of the reduction factor $\chi_{LT}$ for the appropriate non-dimensional slenderness $\bar{\lambda}_{LT}$ may be obtained from Figure 7.

Figure 7: Buckling curves
2.2.2 Specific method

For rolled or equivalent welded sections in bending the value of $\chi_{LT}$ for the appropriate non-dimensional slenderness may be determined from [2]:

$$\chi_{LT, SM} = \frac{1}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \beta \lambda^2_{LT}}}, \text{ but } \chi_{LT, SM} \leq 1,0 \text{ and } \chi_{LT} \leq \frac{1}{\lambda^2_{LT}}$$ \hspace{1cm} \text{Eq. 6}

$$\Phi_{LT} = 0,5 \left[ 1 + \alpha_{LT} \left( \bar{\lambda}_{LT} - \bar{\lambda}_{LT, 0} \right) + \beta \lambda^2_{LT} \right]$$ \hspace{1cm} \text{Eq. 7}

For the parameters $\bar{\lambda}_{LT, 0}$ and $\beta$ the following recommended values are given:

$\bar{\lambda}_{LT, 0} = 0,4$ (maximum value)

$\beta = 0,75$ (minimum value)

Recommendations for buckling curves are given in Table 3.

Table 3: Recommendation for the section of lateral torsional buckling curves for cross-sections using Eq. 6[2]

<table>
<thead>
<tr>
<th>Cross-section</th>
<th>Limits</th>
<th>Buckling curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rolled I-sections</td>
<td>$h/b \leq 2$</td>
<td>b</td>
</tr>
<tr>
<td></td>
<td>$h/b &gt; 2$</td>
<td>c</td>
</tr>
<tr>
<td>Welded I-sections</td>
<td>$h/b \leq 2$</td>
<td>c</td>
</tr>
<tr>
<td></td>
<td>$h/b &gt; 2$</td>
<td>d</td>
</tr>
</tbody>
</table>

2.2.3 Modified specific method

For taking into account the moment distribution between the lateral restraints of members, the reduction factor $\chi_{LT, SM}$ may be modified as [2]:

$$\chi_{LT, MSM} = \frac{\chi_{LT, SM}}{f}$$ \hspace{1cm} \text{Eq. 8}

$$f = 1 - 0,5 \left( 1 - \kappa \right) \left[ 1 - 2,0 \left( \bar{\lambda}_{LT} - 0,8 \right)^2 \right], \text{ but } f \leq 1,0$$ \hspace{1cm} \text{Eq. 9}
The correction factor $k_c$ is given in Table 4.

**Table 4: Correction factors $k_c$ [2]**

<table>
<thead>
<tr>
<th>Moment distribution</th>
<th>$k_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi = 1$</td>
<td>1,0</td>
</tr>
<tr>
<td>$-1 \leq \psi \leq 1$</td>
<td>$\frac{1}{1,33 - 0,33\psi}$</td>
</tr>
<tr>
<td></td>
<td>0,94</td>
</tr>
<tr>
<td></td>
<td>0,90</td>
</tr>
<tr>
<td></td>
<td>0,91</td>
</tr>
<tr>
<td></td>
<td>0,86</td>
</tr>
<tr>
<td></td>
<td>0,77</td>
</tr>
<tr>
<td></td>
<td>0,82</td>
</tr>
</tbody>
</table>

### 2.2.4 Simplified assessment method for beams with restraints

Members with discrete lateral restraints to the compression flange are not susceptible to lateral torsional buckling if the length between restraints or the resulting slenderness $\lambda_j$ of the compression flange satisfies [2]:

$$\bar{\lambda}_j = \frac{k_i L_c}{i_{f,z} \lambda_1} \leq \bar{\lambda}_0 \frac{M_{c,rd}}{M_{y,ed}}$$  

Eq. 10

Where:

$$M_{c,rd} = W_y \frac{f_y}{\gamma_{st1}}$$  

Eq. 11

$$\lambda_1 = \pi \sqrt{\frac{E}{f_y}}$$  

Eq. 12

$M_{y,ed}$ = the maximum design value of the bending moment within the restraint spacing

$L_c$ = the length between the discrete lateral restraints

$k_i$ = a slenderness correction factor for moment distribution between restraints (Table 4)

$i_{f,z}$ = the radius of gyration of the equivalent compression flange composed of compression flange plus 1/3 of the compressed part of the web area, about the minor axis of the section
$\lambda_{c0} =$ slenderness limit of the equivalent compression flange defined above (recommended value according to Dutch National Annex: 0.4.)

If the slenderness of the compression flange exceeds the limit $\lambda_f$ as given above, the design resistance moment may be taken as:

$$M_{b,Rd} = k_p \chi M_{c,Rd}, \quad \text{but} \quad M_{b,Rd} \leq M_{c,Rd} \quad \text{Eq. 13}$$

Where;

$\chi =$ the reduction factor of the equivalent compression flange determined with $\lambda_f$ and buckling curve $c$

$k_p =$ the modification factor accounting for the conservatism of the equivalent compression flange method

The value $k_p = 1.10$ is recommended for the modification factor.

The first three methods given in EN 1993-1-1 only apply to laterally unrestrained beams in bending. The fourth method determines if a laterally restrained beam is susceptible to lateral torsional buckling or not and provides a new value for $M_{b,Rd}$ if the limit value $\lambda_f$ is exceeded.

### 2.3 Proposal for amended rules in EN 1993-1-1 by Taras, Greiner and Unterweger

This section covers the newest amendments proposed for EN 1993-1-1. These were proposed at the EN 1993-1-1 Evaluation Group meeting in October 2012, in Brussels by Taras, Greiner and Unterweger from the Graz University of Technology.

The first amendment applies to the non-dimensional slenderness treated in section 2.2 of this document, in which a new value for the appropriate section modulus $W_y$ is proposed specifically class 3 cross-sections.[3]

$$W_y = W_{d0} - (W_{d1} - W_{d2})c/t_{ref} \quad \text{Eq. 14}$$

Where;

$$\frac{c}{t_{ref,y}} = \max \left[ \frac{c/t_j - 10\varepsilon}{4\varepsilon}; \frac{c/t_w - 83\varepsilon}{41\varepsilon} \right] \leq 1 \quad \text{Eq. 15}$$

$$\frac{c}{t_{ref,c}} = \max \left[ \frac{c/t_j - 10\varepsilon}{6\varepsilon}; 0 \right] \leq 1 \quad \text{Eq. 16}$$
\( c = \) height of the web  
\( t_f = \) thickness of the flange  
\( t_w = \) thickness of the web  
\( e = \sqrt{235/f_y} \)

The second amendment applies to the general method treated in section 2.2.1 of this document, in which a new equation for the reduction factor \( \chi_{LT} \) is proposed. The new equation is as follows:

\[
\chi_{LT} = \frac{\phi}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \phi^2 \lambda^2_{LT}}} \leq 1.0
\]

Eq. 17

\[
\Phi_{LT} = 0.5 \left[ 1 + \phi \left( \frac{\lambda^2_{LT}}{\lambda^2_{z}} \alpha_{LT} \left( \frac{\lambda^2}{\lambda^2_{LT}} - 0.2 \right) + \lambda^2_{LT} \right) \right]
\]

Eq. 18

Where:

\( \alpha_{LT} = \) imperfection factor  
\( \lambda_{LT} = \) non-dimensional slenderness for lateral torsional buckling  
\( \phi = \) factor that accounts for the effect of the bending moment distribution between lateral supports.  
\( \lambda_{z} = \) corresponding slenderness for weak-axis flexural buckling

\[
\lambda_{z} = \sqrt{\frac{Af_y}{N_{crt}}}
\]

Eq. 19

\( A = \) area of the cross-section  
\( f_y = \) yield strength of steel  
\( N_{crt} = \) elastic buckling force for buckling about z-z axis

The new values for the imperfection factor \( \alpha_{LT} \) can be found in Table 5 and the values for factor \( \phi \) can be found in Table 6.

<table>
<thead>
<tr>
<th>Table 5: Imperfection factor ( \alpha_{LT} ) [3]</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cross-section</strong></td>
</tr>
</tbody>
</table>
| **Bolted J-section** | h/b>1.2 | 0.12: \( \sqrt{W_{d,LT}} / W_{d,t} \leq 0.34 \)
| | h/b\leq1.2 | 0.16: \( \sqrt{W_{d,LT}} / W_{d,t} \leq 0.49 \)
| **Welded J-section** | - | 0.21: \( \sqrt{W_{d,LT}} / W_{d,t} \leq 0.64 \)

\(^{1}\) Note that these values of \( \alpha_{LT} \) converge towards the value \( \alpha \) for weak-axis column buckling, see Tables 6.1 and 6.2
Lateral torsional buckling analysis of multiple laterally restrained I-beams

Table 6: Factor \( \varphi \) [3]

<table>
<thead>
<tr>
<th>LOAD CASE</th>
<th>( \varphi )</th>
<th>( k_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M^\text{uniform} )</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>( M )</td>
<td>( 1.25 - 0.1 \cdot \varphi - 0.15 \cdot \varphi^2 )</td>
<td>( \frac{1}{1.33 - 0.33 \cdot \varphi} )</td>
</tr>
<tr>
<td>( M_0 )</td>
<td>1.05</td>
<td>0.94</td>
</tr>
<tr>
<td>( M_b )</td>
<td>( \frac{M_b}{M_h} &lt; 2.0 )</td>
<td>( \frac{M_b}{M_h} &gt; 2.0 )</td>
</tr>
<tr>
<td>( M_a )</td>
<td>( \frac{M_a}{M_h} &lt; 1.47 )</td>
<td>( \frac{M_a}{M_h} &gt; 1.47 )</td>
</tr>
<tr>
<td>( M_c )</td>
<td>( \frac{M_c}{M_h} &lt; 1.0 )</td>
<td>( \frac{M_c}{M_h} &gt; 1.0 )</td>
</tr>
<tr>
<td>( M_{12} )</td>
<td>( \frac{M_{12}}{M_h} &lt; 1.0 )</td>
<td>( \frac{M_{12}}{M_h} &gt; 1.0 )</td>
</tr>
</tbody>
</table>

The third amendment is the elimination of the specific method and the modified specific method from EN 1993-1-1, which was treated in section 2.2.2 and 2.2.3 respectively.

2.4 Determining the elastic critical lateral torsional buckling moment for laterally unrestrained beams

In order to determine the reduction factor \( \chi_{LT} \) according to EN 1993-1-1, the elastic critical lateral torsional buckling moment \( M_{cr} \) has to be determined first. Numerous different methods for determining the elastic critical lateral torsional buckling moment are available in literature. Also the Dutch National Annex of EN 1993-1-1 provides a method. This section describes the different methods found in literature.

2.4.1 Boundary conditions

Before determining the elastic critical lateral torsional buckling moment of a beam subjected to bending, a few boundary conditions have to be established first. In the following calculations for the elastic critical lateral torsional buckling moment, fork supports are considered. These fork supports prevent the cross-section from translations in the \( z \)-axis and
y-axis as well as rotation around the x-axis (Figure 8). These fork supports are made in practice by welding stiffening plates at the end of the beams.

![Figure 8: Fork supports realized in practice for beams subjected to lateral torsional buckling](image)

### 2.4.2 Timoshenko

In the book “Theory of elasticity”[4], Timoshenko describes the general governing differential equation for lateral torsional buckling for a perfectly straight, simply supported beam subjected to pure bending (Figure 9). The following equations are rewritten with the terminology used in EN 1993-1-1:

![Figure 9: Beam subjected to pure bending](image)

\[
\frac{EI_y}{dx} \frac{d^4 \phi}{dx^4} - GI_t \frac{d^2 \phi}{dx^2} - \frac{M^2}{EI_z} \phi = 0
\]

**Eq. 20**

Solving this differential equation results in the closed equation for the elastic critical lateral torsional buckling moment described as:

\[
M_{cr} = \frac{\pi}{L_b} \sqrt{EI_z \left( GL_t + \frac{\pi^2 EI_z}{I_w^2} \right)}
\]

**Eq. 21**

Where:

- \( L_b \) = unbraced length of the beam
- \( E \) = modulus of elasticity
- \( G \) = shear modulus
- \( I_t \) = torsional constant
- \( I_w \) = warping constant
- \( I_z \) = out-of-plane moment of inertia around the z-axis
2.4.3 Clark and Hill

The equation for the elastic critical lateral torsional buckling moment derived by Timoshenko only applies for beams in pure bending. Many have researched the influence of different load and support conditions on the lateral torsional buckling stiffness of a beam in bending. This resulted in the addition of numerous factors to the equation for the elastic critical lateral torsional buckling moment. Clark and Hill have derived an equation based on the differential equation Eq. 20 by Timoshenko. This resulted in a closed equation for the elastic critical lateral torsional buckling moment with the addition of certain factors that account for different boundary conditions, different load types, the position of the loads, and the geometry of the cross-section. The elastic buckling theory has been confirmed by tests for several types of loading and cross-sections. Clark and Hill published the following equation for the elastic critical lateral torsional buckling moment [5]:

\[
M_{cr} = \frac{C_1 \pi^2 EI_1}{(KL)^2} \left[ C_2 g + C_3 k + \left( C_2 g + C_3 k \right)^2 + \frac{I_m}{I_z} \left( 1 + \frac{G I_1 (KL)^2}{\pi^2 EI_w} \right) \right]
\]

Eq. 22

Where;

- \( K \) = out-of-plane rotation coefficient, \( 1 \) = free out-of-plane rotation, \( 0.5 \) = restraint out-of-plane rotation
- \( g \) = distance between load introduction and shear centre (negative if load below shear centre, positive if above shear centre, and zero if load applied in shear centre)
- \( k = \varepsilon + \frac{1}{2} \int \left( x^2 + y^2 \right) dA = 0 \) for double symmetric cross-sections
- \( \varepsilon \) = distance from shear centre to centroid (positive if shear centre lies between centroid and compression flange, negative otherwise)

When considering doubly symmetrical cross-sections, the coefficient \( k \) will become zero, resulting in \( C_3 \) not taking part in the equation. If the transverse loads are applied at the shear centre, the value for \( g \) will become zero as well. This results in the simplified equation of the elastic critical lateral torsional buckling moment:

\[
M_{cr} = \frac{C_1 \pi^2 EI_1}{(KL)^2} \sqrt{ \frac{I_m}{I_z} \left( 1 + \frac{G I_1 (KL)^2}{\pi^2 EI_w} \right) }
\]

Eq. 23

The values \( C_1, C_2 \) and \( C_3 \) are dependent on the type of loads, the position of the loads, and the supports of the beam. The values of these coefficients are documented in Table 16 of Appendix A.
2.4.4 Nethercot

Nethercot was also among the many researchers that studied the influence of different load and support conditions on the lateral torsional buckling stiffness of beams in bending. Instead of using coefficients $C_1$, $C_2$ and $C_3$ to take into account the influence of the load and support conditions, the factor $\alpha$ is introduced. The following equation for the elastic critical lateral torsional buckling moment is introduced in [6]:

$$M_n = \frac{\pi \alpha}{L_b} \sqrt{EI_G I_t} \sqrt{1 + \frac{\pi^2 E I_w}{L_b^2 G I_t}}$$

*Eq. 24*

Where:

$\alpha = \text{coefficient which varies with the loading and support conditions}$

It should be noted that when $\alpha = 1$ the load and support conditions of a perfectly straight beam in pure bending are described, the same case described by Timoshenko. This equation is rewritten in the article with the addition of a factor $R$ for an easier use of the tables provided for determining the factor $\alpha$ (Table 17 to Table 19, in Appendix B)

$$M_n = \alpha \gamma \sqrt{EI_G I_t}$$

*Eq. 25*

Where:

$$\gamma = \frac{\pi}{L_b} \sqrt{1 + \frac{\pi^2}{R^2}}$$

*Eq. 26*

$$R^2 = \frac{L_b^2 G I_t}{E I_w}$$

*Eq. 27*

The values for factor $\alpha$ are derived from results obtained by experimental research. For many different boundary conditions and load conditions, the values of the elastic critical lateral torsional buckling moment were determined using factor $\alpha$ from Table 17 to Table 19 (Appendix B). These values were compared with the theoretical values given by Timoshenko. The maximum error was determined, which is included in the tables in Appendix B. The maximum error did not exceed a deviation of 5%.

2.4.5 Dutch National Annex EN 1993-1-1

According to the Dutch National Annex of EN 1993-1-1, the elastic critical lateral torsional buckling moment should be determined as follows [2]:
Lateral torsional buckling analysis of multiple laterally restrained I-beams

\[ M_{cr} = k_{nd} \frac{C}{L_S} \sqrt{EI/GL_t} \]  \hspace{1cm} Eq. 28

Where;
\( k_{nd} \) = reduction factor depending on the dimensional stability of the cross-section
\( k_{nd} = 1 \) for standard rolled sections
\( C \) = coefficient depending on the beam length, cross-section, and the applied loads
\( L_S \) = beam length between supports
\( E \) = modulus of elasticity
\( I_z \) = moment of inertia around the z-axis
\( G \) = shear modulus
\( I_t \) = torsional moment of inertia

The coefficient \( C \) is dependent on the type and position of the loads, and the area of the cross-section and should be determined as follows:

\[ C = \pi C_1 \frac{I_z}{L_{tip}} \left[ \sqrt{1 + \left( \frac{\pi^2 S^2}{L_{tip}^2} \right)^2 (C_2^2 + 1)} \right] \frac{\pi CS}{L_{tip}} \]  \hspace{1cm} Eq. 29

Where;
\[ S = \sqrt{\frac{EI}{GL_t}} \]  \hspace{1cm} Eq. 30

And for I-beams:
\[ S = \frac{h}{2} \sqrt{\frac{EI}{GL_t}} \]  \hspace{1cm} Eq. 31

\( C_1 \) = coefficient depending on the load type
\( C_2 \) = coefficient depending on the application position of the load

Where;
\( C_2 = 0 \) if the loading is applied in the centre of gravity
\( C_2 \) has a negative value if the load is applied in the centre of gravity of the top flange
\( C_2 \) has a positive value if the load is applied in the centre of gravity of the bottom flange
If the load is applied between the centres of gravity of the top and bottom flange, the value of \( C_2 \) should be determined using linear interpolation. If the load is applied between the centre of gravity of the top flange and a maximum value of 0.1h above the flange, the value of \( C \) should be determined using linear extrapolation.

The values \( C_1 \) and \( C_2 \) are dependent on the load types and the position of the loads and can be determined using the tables given in the Dutch National Annex of EN 1993-1-1. Copies of these tables are given in Appendix C.
2.4.6 Koleková and Baláž

Researchers Koleková and Baláž have studied the determination of the elastic critical lateral torsional buckling moment as well. They have found that the Clark-Mrazík 3-factor approximation formula for the elastic critical lateral torsional buckling moment has the best form and it is much more convenient than 1-factor formulae. They have also shown that using the values for the factors $C_1$, $C_2$, and $C_3$, which are taken from the tables of EN 1993-1-1, leads to incorrect values for the elastic critical lateral torsional buckling moment in many cases. Therefore, Koleková and Baláž have developed a new general formula for determining the elastic critical lateral torsional buckling moment, together with new tables containing more accurate values for the factors $C_1$, $C_2$, and $C_3$.

In an independent study by J. Fruchtengarten, many different formulae were evaluated in the frame of a parametric study by comparing them with exact results of the computer program PEFSYS. It was concluded that the proposal of Koleková and Baláž gives the most exact results. These formulae of Koleková and Baláž and tables have been fully accepted for the National Annex of EN 1993-1-1 for numerous different countries. The proposed equation is as follows [7]:

$$M_y = \mu_y \cdot \frac{\pi \sqrt{EI/GI}}{L_y}$$  \hspace{1cm} \text{Eq. 32}

Where:

$$\mu_y = \frac{C_1}{k_y} \left[ 1 + \kappa_{aw}^2 + \left( C_2 \xi_y - C_3 \xi_z \right)^2 - \left( C_2 \xi_y - C_3 \xi_z \right) \right]$$  \hspace{1cm} \text{Eq. 33}

And the three non-dimensional parameters are:

$$\kappa_{aw} = \frac{\pi}{k_y} \frac{EI}{GL_y}$$  \hspace{1cm} \text{Eq. 34}

$$\xi_y = \frac{\pi z_y}{k_y L_y} \sqrt{\frac{EI}{GL_y}}$$  \hspace{1cm} \text{Eq. 35}

$$\xi_z = \frac{\pi z_z}{k_y L_y} \sqrt{\frac{EI}{GL_y}}$$  \hspace{1cm} \text{Eq. 36}

$$C_i = C_{i,0} + \left( C_{i,1} - C_{i,0} \right) \kappa_{aw}$$  \hspace{1cm} \text{Eq. 37}

$L_y$ = length of the beam
$z_y$ = distance between the load application point and the shear centre
$z_z$ = cross-sectional property, $z_i = 0$ for doubly symmetric cross-sections
$k_y$ = buckling length factor, if $k_y = 1 \rightarrow$ the beam is simply supported on both ends
    if $k_y = 0.7 \rightarrow$ the beam is fixed on one end and simply supported on the other end
    if $k_y = 0.5 \rightarrow$ the beam is fixed on both ends
    if $k_y = 2 \rightarrow$ it is a cantilever beam
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\[ k_z = \text{buckling length factor, if } k_z = 1 \rightarrow \text{both ends are restrained against lateral movement and free to rotate in plane} \]
\[ \quad \text{if } k_z = 0.5 \rightarrow \text{both ends are restrained against lateral movement and restrained against rotation in plane} \]

\[ k_w = \text{buckling length factor, if } k_w = 1 \rightarrow \text{both ends are restrained against rotation about longitudinal axis and free to warp} \]
\[ \quad \text{if } k_w = 0.5 \rightarrow \text{both ends are restrained against rotation about longitudinal axis and restrained against warping} \]

The three factors \( C_1 \), \( C_2 \), and \( C_3 \) depend on the load conditions, end restraint conditions, and the shape of the cross-section. The factor \( C_1 \) depends on bending moment distribution, boundary conditions and parameter \( \kappa_{ir} \). In the article a linear interpolation between the values \( C_{1,0} = C_1(\kappa_{ir} = 0) \) and \( C_{1,1} = C_1(\kappa_{ir} = 1) \) is proposed as the factor \( C_1 \) (Table 26, Table 27 of Appendix D). For \( \kappa_{ir} \geq 1 \) Koleková and Baláž propose to use an approximation \( C_1(\kappa_{ir} \geq 1) = C_{1,1} = C_1(\kappa_{ir} = 1) \). For many load cases the difference between the values \( C_{1,0} = C_1(\kappa_{ir} = 0) \) and \( C_{1,1} = C_1(\kappa_{ir} = 1) \) is negligible, that is why many authors of various publications do not even inform which value of \( C_1 \) is used.

For the case of EN 1993-1-1, it is advised to use the values of \( C_1 \) proposed by Koleková and Baláž or the approximation \( C_1 = (k_z)^2 \) can be used, where \( k_z \) is the correction factor for the relevant moment distribution (Table 4).

The factor \( C_2 \) depends on the cross-section monosymmetry factor \( \psi_j \). Different values are given in the tables for different cross-sections. However, for the case that the load is applied in the shear centre of the cross-section, the value for \( z_j = 0 \, mm \), which results that the parameter \( \zeta_j \) is zero, in which case \( C_2 \) does not take part in the equation.

The factor \( C_3 \) also depends on the cross-section monosymmetry factor \( \psi_j \). However, for doubly symmetrical cross-sections, the value \( z_j = 0 \, mm \), which results that the parameter \( \zeta_j \) is zero, in which case \( C_3 \) does not take part in the equation.

### 2.5 Determining the elastic critical lateral torsional buckling moment for laterally restrained beams

#### 2.5.1 Flint and Nethercot

Flint was the first author to present results for lateral torsional buckling of beams with intermediate elastic supports. Neglecting the effects of warping he presented an analysis for a beam having a lateral restraint at mid-span. For the case of a beam supported by a lateral restraint at mid-span (Figure 10), a simple design relationship between the stiffness of the restraint and the increase of stability was proposed in the following equation [8]:

---

32
\[ c = \sqrt{1 + \lambda} \]  \hspace{1cm} \textit{Eq. 38}

Where;

\[ c = \frac{\text{critical load for laterally restrained beam}}{\text{critical load for laterally unrestrained beam}} \]

\[ \lambda = \frac{K_B L^3}{48EI_c} \text{ (relative stiffness)} \]  \hspace{1cm} \textit{Eq. 39}

\( K_B \) = spring stiffness of lateral restraint

Figure 10: Top view and front view of a laterally restrained I-beam at mid-span by Flint and Nethercot

Nethercot continued on the work of Flint. After conducting a parametric study, using a two-dimensional finite element model, numerous values for \( c \) have been determined for different load cases with the lateral restraint applied to the top flange and with the lateral restraint applied to the shear centre of the cross-section. These values are found in Table 20 of Appendix B.

Figure 11: Relationship between increased stability and restraint stiffness for a beam with a lateral restraint at mid-span
The relationship between the critical load and the stiffness of the lateral restraint is shown in Figure 11. For small values of the lateral restraint stiffness, an increase in $\lambda$ corresponds with an increase in stability. However, when a certain value for the stiffness is reached $(\lambda_c)$, further increase of $\lambda$ does not correspond with an increase of stability. This is the value at which the critical load is equal to the load corresponding with the second buckling mode (two half waves).

It should be noted that the parameter $a$ in Table 20 is the relation between the actual value of the spring stiffness of the lateral restraint $\lambda$ and the spring stiffness at which the second buckling mode is reached. The relation is as follows:

$$a = \frac{\lambda}{\lambda_c}$$  \hspace{1cm} Eq. 40

When $a = 0$, the value for $c = 1$, meaning that the critical load of the laterally restrained beam is equal to the critical load of the laterally unrestrained beam, in which case the beam is laterally unrestrained. When $a = 1$, the highest value for the critical load is obtained, in which case the spring stiffness of the lateral restraint is equal the spring stiffness at which the second buckling mode is reached.

The relationship between $c$ and $\lambda$ was analysed for different loading conditions and was compared with the results obtained by Flint. It was discovered that Flint’s approximate expression, Eq. 38, becomes increasingly optimistic in its predictions of $c$ once $\lambda$ exceeds the value of 1.5. However, good estimates of $c$ may conveniently be obtained with the following equations [9]:

For load case with equal end moments:

$$c = \sqrt{1 + \frac{\lambda}{1 + 0.02\lambda}}$$  \hspace{1cm} Eq. 41

For load case with central load:

$$c = \sqrt{1 + \frac{\lambda}{1 + 0.04\lambda}}$$  \hspace{1cm} Eq. 42

For load case with uniform load:

$$c = \sqrt{1 + \frac{\lambda}{1 + 0.015\lambda}}$$  \hspace{1cm} Eq. 43
2.5.2 Bijlaard and Steenbergen

At the Delft University of Technology, Bijlaard and Steenbergen have studied the effects of lateral restraints on a beam subjected to pure bending using the equations given by EN 1993-1-1 at that time. The following general equation was given [10]:

\[ M_{cr} = C_1 \frac{\pi^2 E I_z}{L_{LT}^2} \left\{ \frac{I_m}{I_z} + \frac{L_{LT}^2 G I_t}{\pi^2 E I_z} + \left( C_2 z_g \right)^2 - C_2 z_g \right\} \quad \text{Eq. 44} \]

Where:
- \( z_g \) = distance between the load application point and the shear centre
- \( L_{LT} \) = lateral torsional buckling length of the beam
- \( L_b \) = unrestrained length of the beam

For a beam subjected to pure bending where factor \( C_1 = 1 \) and factor \( C_2 = 0 \) and \( L_{LT} = L_b \), the following equation is valid:

\[ M_{cr} = \frac{\pi^2 E I_z}{L_{LT}^2} \left\{ \frac{I_m}{I_z} + \frac{L_{LT}^2 G I_t}{\pi^2 E I_z} \right\} \quad \text{Eq. 45} \]

When an intermediate lateral restraint is applied at mid-span so that \( L_{LT} = 0.5L_b \) and irrelevant of the position above the shear centre, the elastic critical lateral torsional buckling moment is as follows:

\[ M_{cr} = \frac{\pi^2 E I_z}{(0.5L_b)^2} \left\{ \frac{I_m}{I_z} + \frac{(0.5L_b)^2 G I_t}{\pi^2 E I_z} \right\} \quad \text{Eq. 46} \]

In order to verify beams with lateral restraints at various locations between the supports, the individual parts of the beam between a fork support and a lateral restraint (Figure 12, between A and B, and between C and D) should be considered as well as the individual parts of the beam between the two lateral restraints (Figure 12, between B and C).

![Figure 12: Subdivision of a beam for code checking according to paper of Bijlaard and Steenbergen [10]](image)
For the individual beam parts; AB, BC, and CD the appropriate non-uniform moment distributions should be determined. To determine the elastic critical lateral torsional buckling moment for each part, the length of that part must be multiplied by a factor, which is composed from a parametric study using a numerical model.

\[ L_{LT} = (1,4 - 0,8\beta)L_{\text{beam part}} \quad \text{Eq. 47} \]

Where;
\[ 1,0 \leq \frac{L_{LT}}{L_{\text{beam part}}} \]

and the ratio of the end moment for the individual parts;
\[ \beta = \frac{M_A}{M_B}, \text{ or } \beta = \frac{M_B}{M_C}, \text{ or } \beta = \frac{M_C}{M_D} \quad \text{Eq. 48} \]

### 2.5.3 Dutch National Annex EN 1993-1-1

The equations proposed by Bijlaard and Steenbergen have been introduced in the Dutch National Annex; however, with a slight difference in the notation of some parameters. The following additions should be used with the equations discussed in section 2.4.5.

\[ k_{rd} = \text{reduction factor for the deformability of the cross-section, for standard I-beams } k_{rd} = 1 \]
\[ L_{lip} = \text{lateral torsional buckling length between two fork supports, a fork support and a restraint,} \]

or between two restraints

- between two fork supports
  \[ L_{lip} = L_{si} \]

- between one fork support and a restraint, or between two restraints
  \[ L_{lip} = (1,4 - (0,8\beta))L_{si} \]; however, \( 1,0 \leq \frac{L_{lip}}{L_{si}} \leq 1,4 \)

\[ L_{si} = \text{unrestrained length between two fork supports, between a fork support and a restraint,} \]

or between two restraints

\[ L_{i} = \text{the beam length between the fork supports} \]
\[ \beta = \text{the quotient of the design value of the end moment around the y-axis with the lowest absolute value } M_{y,1,Ed} \text{ and the design value of the end moment around the y-axis with the highest absolute value } M_{y,2,Ed} \text{ by the loads} \]
\[ \beta = \frac{M_{y,1,Ed}}{M_{y,2,Ed}} \quad \text{Eq. 49} \]

Figure 13 shows the lateral torsional buckling length \( L_{lip} \) in relation with the entire length of the beam. Using Eq. 28 to Eq. 30 and Table 22 from Appendix C, the new values for \( C_1 \) and \( C_2 \) are found that are used to determine the elastic critical lateral torsional buckling moment for this case with Eq. 49.
The Centre Technique Industriel de la Construction Métallique (CTICM) [11] in France have developed a computer program which enables the designer to quickly calculate the elastic critical lateral torsional buckling moment in a matter of seconds. The elastic critical lateral torsional buckling moment is determined using an iterative calculation process in which a linear eigenvalue analysis is performed. The behaviour of the beam is treated using the finite element method and the discretisation of the beam can be varied from 100 elements up to 300 elements.

### 2.5.4 LTBeam (CTICM)

In order to determine the elastic critical lateral torsional buckling moment, the following assumptions have been made:

- The beam is straight and prismatic. It is discretised in small elements, which allows for adopting simplified assumptions concerning their behaviour.
- The material is elastic, isotropic and homogeneous.
- Cross-sections are assumed without deformations and local buckling is not considered.
- Strains are small, but displacements and rotations may be moderately large.
- Shear strains in the middle plane of the cross-section walls are negligible (Vlassov’s assumption).
- The direction of the forces does not change when the beam twists.

#### Strain Energy

The strain energy $U$ of the beam when lateral torsional buckling just occurs may be split into two parts. First, a linear part $U_L$ which is a function of the geometrical properties, the mechanical properties, and the material properties of the beam. Second, a non-linear part $U_{nc}$ which is a function of the internal forces in the beam. The total strain energy of the beam may be expressed by [12]:

![Figure 13: Subdivision of a beam for code checking according to the Dutch National Annex](image-url)
Lateral torsional buckling analysis of multiple laterally restrained I-beams

\[ U = U_L + U_g \]

Eq. 50

Considering the beam is discretised in \( n \) elements it yields:

\[ U = \sum_{e=1}^{n} \left( U'_L + U'_G \right) \]

Eq. 51

where \( U'_L \) and \( U'_G \) are the individual strain energies for each element \( e \). These individual strain energies are expressed as:

\[ U'_L = \frac{1}{2} \int_0^L \left[ EI \left( \frac{\partial^2 v}{\partial x^2} \right)^2 + EI_u \left( \frac{\partial^2 \theta}{\partial x^2} \right)^2 + GI \left( \frac{\partial \theta}{\partial x} \right)^2 \right] dx \]

Eq. 52

\[ U'_G = \frac{1}{2} \int_0^L \left[ M_y(x) \frac{\partial v}{\partial x} - \beta \frac{\partial \theta}{\partial x} \right] dx - \frac{1}{2} \int_0^L V_z(x) \left( \theta_x \frac{\partial v}{\partial x} + z \theta_x \frac{\partial \theta}{\partial x} \right) dx \]

\[ + \frac{1}{2} \sum_i F_i \theta_i^2 + \frac{1}{2} \sum_j \left( q_j \frac{\partial \theta}{\partial x} \right)^2 \int_0^L dx \]

Eq. 53

Where:
\( \beta \) = Wagner’s factor of the section. \( \beta = 0 \) for double symmetric cross-sections

\( M_{y(x)} \) and \( V_{z(x)} \) are the bending moment distribution and the shear force distribution along the beam respectively. Furthermore, along each element, the following transversal loads may act (Figure 14):

- Point loads \( F_i \) applied at abscissas \( x_{F_i} \) along the element, in the direction of the \( z \)-axis and at distances \( z_{F_i} \) from the shear centre \( S \).
- Distributed loads \( q_j \) applied between abscissas \( x_{q_{1j}} \) and \( x_{q_{2j}} \) along the element, in the direction of the \( x \)-axis and at distances \( z_{q_j} \) from the shear centre \( S \).

\[ \text{Figure 14: Notations and coordinate system for the element} \]

Degrees of freedom

Degrees of freedom are the displacements and rotations of the nodes at the ends of the element. Considering an element in the three-dimensional space, seven degrees of freedom may be defined for each node (Figure 15).
The vector of displacements at an end node $i$ is defined by:

\[
\begin{pmatrix}
u_i, w_i, \theta_x i, \theta_y i, \theta_z i, \theta'_x i
\end{pmatrix}
\]

Where:

$u_i$ = axial displacements of centroid $G_i$
$w_i, \theta_x i, \theta_y i, \theta_z i$ = displacements of the shear centre $S_i$
$\theta'_x i$ = displacements which are linked to warping of the cross-section

In the context of lateral torsional buckling of a beam with a doubly symmetrical cross-section about the plane of bending $xz$, an interaction between the in-plane and out-of-plane displacements is not present. Therefore, only the following four degrees of freedom may be considered here; $(v_i, \theta_x i, \theta_z i, \theta'_x i)$.

**Stiffness matrix of an element**

The stiffness matrix of an element is obtained from the second derivation of the strain energy $U$ with respect to the displacements of the system. If the stiffness matrix is called $K$ and if $a_i$ is any degree of freedom, the element $K_{ij}$ of the matrix $K$ is obtained by:

\[
K_{ij} = \frac{\partial^2 U}{\partial a_i \partial a_j}
\]

Eq. 54

Accounting for the split of the strain energy into a linear part and a geometrical part such as defined earlier; the matrix $K$ may be expressed by:

\[
K = K_L + K_G
\]

Eq. 55

Where:
Lateral torsional buckling analysis of multiple laterally restrained I-beams

\[ K_\text{l} = \text{linear stiffness matrix} \]
\[ K_\text{c} = \text{geometrical stiffness matrix} \]

The elements of these matrices are respectively obtained from:
\[ K_{ij} = \frac{\partial^2 U}{\partial a_i \partial a_j}, \quad \text{and} \quad K_{ij} = \frac{\partial^2 U}{\partial a_i \partial a_j} \]

The matrix \( K_\text{c} \) is a linear function of the bending moments in the beam element and of the transverse loads applied on it. If \( K_\text{c}(\overline{M}) \) denotes the matrix calculated for a given loading generating the bending distribution, \( \overline{M} \) and if this loading varies proportionally to a factor \( \alpha \) (load multiplier) at each load level, the stiffness matrix is expressed by:

\[ K = K_\text{l} + \alpha \cdot K_\text{c}(\overline{M}) \quad \text{Eq. 56} \]

**Elastic critical lateral torsional buckling moment**

When lateral torsional buckling occurs, the differential behaviour of the beam is governed by the relationship:

\[ \text{Determinant}\left| K_\text{l} + \alpha \cdot K_\text{c}(\overline{M}) \right| = 0 \quad \text{Eq. 57} \]

Where;
\[ K_\text{l} = \text{a so-called "linear" matrix, function of dimensions (beam and cross-sections) and material properties} \]
\[ K_\text{c} = \text{a so-called "geometrical" matrix, function of certain dimensions and of the bending distribution \( \overline{M} \) in the beam resulting from loading data} \]
\[ \mu_i = \text{the so-called "critical" value of a multiplier \( \mu \) of the bending distribution \( \overline{M} \) along the beam, that is of the loading applied to the beam} \]

The matrices \( K_\text{l} \) and \( K_\text{c} \) are obtained by the second derivation of the linear and non-linear parts of the total strain energy \( U \) of the beam at buckling level versus the displacements associated to lateral torsional buckling.

Solving equation Eq. 57 leads to several solutions called “eigenvalues”. LTBeam only keeps the smallest positive value, which is the mechanical solution. Once the eigenvalue for \( \alpha \) is known, the elastic critical lateral torsional buckling moment can be determined with:

\[ M_i = \alpha \cdot \overline{M}_{\text{max}} \quad \text{Eq. 58} \]

**2.5.4.2 User interface**

The program provides a database of different steel profiles, saving the designer time to input the material and geometrical properties of the beam. Of course a manual input is possible as well (Figure 16).
The next step is to choose the support conditions at the end of the beam. The program also allows the designer to apply one, two, or a continuous lateral restraint at different heights to the beam (Figure 17).

After that, the load conditions must be chosen. Many types of load combinations can be chosen. The point of application is located in the shear centre by default; however, the distance from the shear centre can be varied as well. The program also calculates the maximum bending moment for the chosen loads (Figure 18).

The final step is to calculate the elastic critical lateral torsional buckling moment with just a mouse click. As a result, the value of the elastic critical lateral torsional buckling moment is given as well as a graphical example of the deformed shape (Figure 19).

![Figure 16: Input of the beam](image-url)
Lateral torsional buckling analysis of multiple laterally restrained I-beams

Figure 17: Input of the restraints

Figure 18: Input of the loads
2.6 Lateral torsional buckling analysis of laterally restrained beams by R.H.J. Bruins

At Eindhoven University of Technology, R.H.J. Bruins has performed a lateral torsional buckling analysis of laterally restrained beams in bending by means of a finite element study. This study emphasised on a single lateral restraint on different locations of the beam. The results from the numerical analysis were compared to methods used in EN 1993-1-1 for determining the ultimate lateral torsional buckling load [13][14].

The finite element model consisted of 4-node shell elements based on Mindlin-Reissner shell theory. Using only shells element would cause the fillet radius between the web and the flange to be ignored. This was compensated using elastic-plastic rectangular hollow section beam elements at the flange-web intersection. Residual stresses and geometrical imperfections were also taken into account.

After performing a parametric study in which the position of the lateral restraint was varied along the longitudinal direction of the beam as well as the height of the beam, the results of this study were plotted in a graph (Figure 20). These results only apply to that specific load case.

For these calculations it was most desirable to use an infinitely stiff lateral restraint in the finite element model. However, infinitely stiff lateral restraints do not exist in practice. Therefore, certain stiffness requirements for the lateral restraints need to be set. These stiffness requirements are based on the spring stiffness needed to allow a maximum of 5% reduction of the elastic critical lateral torsional buckling moment when applying infinitely stiff lateral restraints. Or in other words, the stiffness of the lateral restraints is determined in
such a way that the value of the elastic critical lateral torsional buckling moment is 95% of value for the elastic critical lateral torsional buckling moments with infinitely stiff lateral restraints. The 95% spring stiffness for the lateral restraints was referred to as $K_{95\%}$.

The non-dimensional slenderness $\lambda_{LT}$, the reduction factor $\chi_{LT}$, and the ultimate lateral torsional buckling load $M_{b,ud}$ for certain beams with different load cases were determined using a geometrically and materially non-linear analysis with imperfections (GMNIA). These results were compared to the values of $\lambda_{LT}$, $\chi_{LT}$, and $M_{b,ud}$ obtained from the equations of EN 1993-1-1. After this comparison it was concluded that the results from the general method (2.2.1) gives relatively good results for the ultimate lateral torsional buckling load for beams with lateral restraints. However, the specific method (2.2.2) and the modified specific method (2.2.3) described in EN 1993-1-1 give unsafe overestimations (in some cases 10% overestimation) for the ultimate lateral torsional buckling load compared with the results from the numerical analysis.

### 2.7 Comparison of the different methods for determining the elastic critical lateral torsional buckling moment for laterally unrestrained beams

In this section the different methods of determining the elastic critical lateral torsional buckling moment are compared by means of examples. In these examples the elastic critical lateral torsional buckling moment is determined for a laterally unrestrained I-beam with fork supports. The input data for the beam are found in Table 7.
Table 7: Data of an IPE 300 profile

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>210000</td>
<td>N/mm²</td>
</tr>
<tr>
<td>Iₐ</td>
<td>6037800</td>
<td>mm⁴</td>
</tr>
<tr>
<td>Iₜ</td>
<td>198700</td>
<td>mm⁴</td>
</tr>
<tr>
<td>Iₜₑ</td>
<td>1263323⋅10⁵</td>
<td>mm⁶</td>
</tr>
<tr>
<td>G</td>
<td>80769</td>
<td>N/mm²</td>
</tr>
<tr>
<td>Iₜₚ</td>
<td>7000</td>
<td>mm</td>
</tr>
<tr>
<td>h</td>
<td>300</td>
<td>mm</td>
</tr>
</tbody>
</table>

First, the elastic critical lateral torsional buckling moment is determined for a laterally unrestrained beam subjected to an equally distributed load applied in the shear centre of the cross-section (Figure 21).

2.7.1 Example according to Clark and Hill

For this example Eq. 22 is used to determine the elastic critical lateral torsional buckling moment. The value for  is equal zero because the load is applied in the shear centre, the value for  is equal to zero because the cross-section is doubly symmetrical, and the value for  is equal to one because the beam is fork supported. The values for factors  and  are found in Table 16 of Appendix A, which are 1,13 and 0,45 respectively. The elaboration of Eq. 22 is as follows:

\[
M_{cr} = \frac{1,13 \cdot \pi^2 \cdot 210000 \cdot 6037800}{(1 \cdot 7000)^2} \sqrt{\frac{1263323 \cdot 10^5}{6037800} \left(1 + \frac{80769 \cdot 198700(1 \cdot 7000)^2}{\pi^4 \cdot 210000 \cdot 1263323 \cdot 10^5}\right)} = 83,57 \text{kNm}
\]

2.7.2 Example according to Nethercot

For this example Eq. 24 to Eq. 27 are used to determine the elastic critical lateral torsional buckling moment. The value for factor  is found in Table 18 of Appendix B. Because the load is applied in the shear centre of the cross-section, the value for  is equal to 1,123. The elaboration of the equations is as follows:

\[
R^2 = \frac{7000^2 \cdot 80789 \cdot 198700}{210000 \cdot 1263323 \cdot 10^5} = 29,64
\]
\[ \gamma = \frac{\pi}{7000} \sqrt{1 + \frac{\pi^2}{29.64^2}} = 0.00045 \]
\[ M_{cr} = 1,123 \cdot 0.00045 \sqrt{210000 \cdot 6037800 \cdot 80769 \cdot 198700} = 83.01 \text{ kNm} \]

### 2.7.3 Example according to Dutch National Annex

For this example Eq. 28 to Eq. 31 are used to determine the elastic critical lateral torsional buckling moment. The values for factors \( C_1 \) and \( C_2 \) are found in Table 21. The value for \( C_1 \) is equal to 1.13 and the value for \( C_2 \) is equal to zero because the load is applied in the shear centre of the cross-section. The elaboration of the equations is as follows:

\[ S = \frac{300}{2} \sqrt{\frac{210000 \cdot 6037800}{80769 \cdot 198700}} = 1333.27 \]
\[ C = \frac{\pi \cdot 1.13 \cdot 7000}{7000} \left[ 1 + \left( \frac{\pi^2 \cdot 1333.27^2}{7000^2} \right) \right] + \frac{\pi \cdot 1333.27}{7000} = 4.137 \]
\[ M_{cr} = \frac{4.137}{7000} \sqrt{210000 \cdot 6037800 \cdot 80769 \cdot 198700} = 84.31 \text{ kNm} \]

### 2.7.4 Example according to Koleková and Baláž

For this example Eq. 32 to Eq. 37 are used to determine the elastic critical lateral torsional buckling moment. The value for factor \( \zeta_j \) is equal to zero because the cross-section is doubly symmetrical and the value for \( z_g \) is equal to zero because the load is applied in the shear centre of the cross-section. The values for \( C_{1,0} \) and \( C_{1,1} \) are found in Table 27, which are 1,127 and 1,132 respectively. The elaboration of the equations is as follows:

\[ \kappa_{wt} = \frac{\pi}{7000} \sqrt{\frac{210000 \cdot 1263323 \cdot 10^3}{80769 \cdot 198700}} = 0.577 \]
\[ C_1 = 1,127 + (1,132 - 1,127) \cdot 0.577 = 1,299 \]
\[ \mu_{cr} = \frac{1.299}{1} \sqrt{1 + 0.577^2} = 1,3045 \]
\[ M_{cr} = \frac{1,3045}{7000} \sqrt{210000 \cdot 6037800 \cdot 80769 \cdot 198700} = 83.51 \text{ kNm} \]

### 2.7.5 Example according to LTBeam

For this example the computer program LTBeam is used. The same data is entered as in the examples above. The value for the elastic critical lateral torsional buckling moment determined using LTBeam is equal to:
\[ M_{cr} = 83,581 \text{ kNm} \]
2.7.6 Results from the examples

The results for the elastic critical lateral torsional buckling moment using the different methods are summarized in Table 8. The values do not differ much from each other. For these examples it is assumed that the values for the elastic critical lateral torsional buckling moment according to Koleková and Baláž are the most accurate. Therefore the error is based on the difference between their values and those of the others.

Table 8: Values for the elastic critical lateral torsional buckling moment for the load case given in Figure 21

<table>
<thead>
<tr>
<th>Method</th>
<th>$M_{cr}$</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clark &amp; Hill</td>
<td>83,57 kNm</td>
<td>0,072 %</td>
</tr>
<tr>
<td>Nethercot</td>
<td>83,01 kNm</td>
<td>-0,6%</td>
</tr>
<tr>
<td>Dutch National Annex</td>
<td>84,31 kNm</td>
<td>0,96%</td>
</tr>
<tr>
<td>Koleková &amp; Baláž</td>
<td>83,51 kNm</td>
<td>0%</td>
</tr>
<tr>
<td>LTBeam</td>
<td>83,581 kNm</td>
<td>0,085%</td>
</tr>
</tbody>
</table>

The tables below display the values for the elastic critical lateral torsional buckling moment for the same laterally unrestrained IPE 300 beam with fork supports only with different load cases and different load application points.

Table 9: Values for the elastic critical lateral torsional buckling moment for load case given in Figure 22

<table>
<thead>
<tr>
<th>Method</th>
<th>$M_{cr}$</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clark &amp; Hill</td>
<td>64,75 kNm</td>
<td>-2,72%</td>
</tr>
<tr>
<td>Nethercot</td>
<td>66,01 kNm</td>
<td>-0,83%</td>
</tr>
<tr>
<td>Dutch National Annex</td>
<td>67,05 kNm</td>
<td>0,74%</td>
</tr>
<tr>
<td>Koleková &amp; Baláž</td>
<td>66,56 kNm</td>
<td>0%</td>
</tr>
<tr>
<td>LTBeam</td>
<td>66,66 kNm</td>
<td>0,15%</td>
</tr>
</tbody>
</table>

Figure 22: Laterally unrestrained I-beam subjected to an equally distributed load applied in the centre of the top flange

Figure 23: Laterally unrestrained I-beam subjected to a point load applied at mid-span in the centre of the top flange
Table 10: Values for the elastic critical lateral torsional buckling moment for the load case given in Figure 23

<table>
<thead>
<tr>
<th>Method</th>
<th>( M_c )</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clark &amp; Hill</td>
<td>72.37 kNm</td>
<td>-6.14%</td>
</tr>
<tr>
<td>Nethercot</td>
<td>75.91 kNm</td>
<td>-1.54%</td>
</tr>
<tr>
<td>Dutch National Annex</td>
<td>76.21 kNm</td>
<td>-1.15%</td>
</tr>
<tr>
<td>Koleková &amp; Baláž</td>
<td>77.10 kNm</td>
<td>0%</td>
</tr>
<tr>
<td>LTBeam</td>
<td>75.895 kNm</td>
<td>-1.56%</td>
</tr>
</tbody>
</table>

2.8 Determination of the elastic critical lateral torsional buckling moment for a laterally restrained beam

Section 2.7 compared the different methods to determine the elastic critical lateral torsional buckling moment for laterally unrestrained beams. This section will determine the elastic critical lateral torsional buckling moment for a laterally restrained beam. For this determination the equations from the Dutch National Annex discussed in section 2.5.3 are used. This will be done for a simply supported IPE 500 beam subjected to an equally distributed load and with 4 lateral restraints applied at the top flange (Figure 24). The cross-sectional and material properties for this beam are given in Table 11.

![Figure 24: Laterally restrained IPE 500 beam subjected to an equally distributed load applied at the centroid of the top flange](image)

Table 11: Data of an IPE 500 profile

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E )</td>
<td>210000</td>
<td>N/mm²</td>
</tr>
<tr>
<td>( I_s )</td>
<td>21376135</td>
<td>mm⁴</td>
</tr>
<tr>
<td>( I_t )</td>
<td>717342</td>
<td>mm⁴</td>
</tr>
<tr>
<td>( I_w )</td>
<td>( 1.24937 \times 10^{12} )</td>
<td>mm⁶</td>
</tr>
<tr>
<td>( G )</td>
<td>80769</td>
<td>N/mm²</td>
</tr>
<tr>
<td>( I_p )</td>
<td>25000</td>
<td>mm</td>
</tr>
<tr>
<td>( h )</td>
<td>500</td>
<td>mm</td>
</tr>
</tbody>
</table>

To determine the elastic critical lateral torsional buckling moment for this case Eq. 28 to Eq. 30 and Table 22 from Appendix C are used. The elaboration is as follows:
\[
S = \frac{EI}{\sqrt{GL_i}} = \sqrt{\frac{210000 \cdot 1.24937 \cdot 10^{12}}{80769 \cdot 717342}} = 2128 \text{ N/mm}^4
\]

\[L_{\text{kip}} = (1,4 - (0,8\beta))L_d\]

\[\beta = \frac{M_{a}}{M_{b}} = \frac{\phi \cdot q L^2}{\phi \cdot q L^2} = 1 \text{ (for governing middle section of the beam)}\]

\[L_{\text{kip}} = (1,4 - (0,8\beta))L_d = (1,4 - (0,8\cdot1)) \cdot 5000 = 3000 \text{ mm}; \text{ however, } 1,0 \leq L_{\text{kip}} / L_d \leq 1,4
\]

\[L_{\text{kip}} / L_d = 0,6 < 1; \text{ therefore, } L_{\text{kip}} = L_d = 5000 \text{ mm}\]

\[B' = \frac{8M_{a}}{8[M] + qL^2} = \frac{8 \cdot \phi \cdot q L^2}{8 \cdot \phi \cdot q L^2 + qL_d} = \frac{48 \cdot q L^2}{48 \cdot q L^2 + q \cdot \frac{L_d}{2}} = \frac{48 \cdot q L^2}{48 \cdot q L^2} = \frac{48}{50} = 0,96\]

Determining the values for \(C_1\) and \(C_2\) using \(\beta = 1, B' = 0,96\) and Table 22 from Appendix C gives:

\[C_1 = 1,009 \text{ (determined iteratively)}\]

\[C_2 = -0,016 \text{ (determined iteratively)}\]

\[C = \frac{\pi C_1 L_s}{L_{\text{kip}}} \left[1 + \left(\frac{\pi^2 S^2}{I_{\text{kip}}} (C_2^2 + 1)\right)\right] + \frac{\pi C_S}{L_{\text{kip}}} = \frac{\pi \cdot 1,009 \cdot 25000}{5000} \left[1 + \left(\frac{\pi^2 \cdot 2128^2}{5000^2} (-0,016^2 + 1)\right)\right] + \frac{\pi \cdot -0,016 \cdot 2128}{5000} = 26,465\]

\[M_{cr} = k_{cr}C \sqrt{\frac{EI}{L_s}} = 1 \cdot \frac{26,465}{25000} \sqrt{210000 \cdot 21376135 \cdot 80769 \cdot 717342} = 535,01 \cdot 10^6 \text{ Nmm} = 539,87 \text{ kNm}\]

Unfortunately this value cannot be compared with other values from literature. For this case the value will be compared with a value obtained from a linear buckling analysis. The elastic critical lateral torsional buckling moment determined with the LBA gives a value of \(M_{cr,\text{LBA}} = 641,45 \text{ kNm}\). The difference between the two values is 15,8\%. This difference may seem too high and therefore inaccurate, but this difference can be explained.

Considering the method from the Dutch National Annex, for this method the governing beam segment is isolated from the structure and then the elastic critical lateral torsional buckling moment is determined for this isolated beam segment, which is again considered as a simply supported beam. This method could be considered too conservative, because in reality the adjacent beam segments that are neglected could presumably provide a certain lateral torsional buckling resistance to the beam. The value determined from the LBA does include the additional resistance of the adjacent beam segments and is therefore higher.
If the isolated beam segment is studied using the LBA, another remarkable result is obtained. The isolated part is modelled as a simply supported beam with a length of 5000 mm subjected to an equally distributed load applied to the centroid of the top flange. The additional bending moments at the beam-ends are not taken into account for this comparison. The elastic critical lateral torsional buckling moment determined with the LBA for this isolated segment gives a value of $M_{cr} = 405,125 \text{kNm}$. This value is considerably lower when compared to the value obtained with the Dutch National Annex.

The same isolated beam segment is again studied using the LBA, but this time the equally distributed load is applied at the centroid of the web. Now, the elastic critical lateral torsional buckling moment determined with the LBA for this isolated segment gives a value of $M_{cr} = 579,02 \text{kNm}$. This value is a little higher, but much closer to the value obtained with the Dutch National Annex. The higher value can be explained by the fact that the bending moments at the beam-ends are not taken into account. Dutch National Annex apparently determines the elastic critical lateral torsional buckling moment for the isolated segment by applying the load at the centroid of the web, which gives a higher value when compared to applying the load at the centroid of the top flange, and then reducing this value by means of the additional bending moments at the beam ends. The values for $C_1$ and $C_2$ determined with Table 22 from Appendix C take the effects of the load type and the position of the load into account, but for this case these do not reduce the value for the elastic critical lateral torsional buckling moment significantly. This is advantageous for this case, but additional resistance presumably provided by the adjacent beam segments are still not taken into account.
3. Finite element model and analysis

This chapter will discuss the modelling method for obtaining the finite element model that is used for the analyses. For this research project the finite element software ABAQUS 6.12 is used. In Appendix F the Python script is shown that generates the FEM model which is described in this chapter in an instance.

3.1 Element type

The S8R thick shell element type is used for modelling the finite element model. This is an 8-node shell element with six degrees of freedom per node, three displacement and three rotation degrees of freedom (Figure 25). The S8R element is based on a first-order shear deformation theory. This element also uses a reduced integration rule, which reduces the processing time for the analysis. Due to the reduced number of integration points, hourglass modes can occur. An hourglass stabilisation control feature is built in the software, which automatically checks for possible hourglass modes.

![Figure 25: 8-node shell element](image)

![Figure 26: Mesh density of FEM model](image)

3.2 Mesh

Meshing divides the three-dimensional model into the shell elements described in section 3.1. For the mesh density the division of the flange is set at 16 elements and the division of the web is set at 32 elements (Figure 26). This division is only valid for steel profiles with a height-to-width ratio of approximately 2. For other steel profiles with other height-to-width ratios, the division of elements is chosen in such a way that the element width is almost equal for the both the flange and the web. The length of the element is chosen in such a way that a length-to-width ratio of approximately 5 is obtained. This mesh density is chosen based on a mesh density study to determine which mesh density provides the most accurate results. Because the chosen mesh density also influences the residual stress pattern applied in the model, this mesh density study is treated in section 3.9.
3.3 Geometrical error

Using shell elements to model the cross-section of an I-shaped beam causes certain geometrical errors. These geometrical errors particularly affect the torsional properties of the cross-section because the fillet radius at the web-flange intersections is not modelled. Another modelling error exists in the web-flange intersections in which a small area of the web is taken into account twice (Figure 27). There are ways to compensate for these errors using certain compensation methods that account for the additional torsional properties for the fillet radius at the web-flange intersections.

![Figure 27: Real beam cross-section compared with finite element model with discretisation errors](image)

However, for this research project it is chosen not to use these compensation methods, but to use a finite element model including the previous mentioned errors. Therefore, special attention has to be given when comparing the results from the finite element analyses with the results from the design codes. These models represent imaginary cross-sections, which are not used in practice and the cross-sectional properties for these imaginary cross-sections are determined by hand, instead of using the values found in a steel profile catalogue.

3.4 Cross-sectional properties

The cross-sectional properties such as the weak axis moment of inertia, the torsional constant, the warping constant, and the plastic section modulus are determined with the following equations and Figure 28.

\[
I_z = \frac{2b^3t_f + h^3t_w}{12} \quad \text{Eq. 59}
\]

\[
I_t = \frac{2b^3t_f + ht^3}{3} \quad \text{Eq. 60}
\]

\[
I_w = \frac{t_f h^2 b^3}{24} \quad \text{Eq. 61}
\]

\[
W_{pl,y} = b h t_y + 0.25 t_w (h - t_f)^2 \quad \text{Eq. 62}
\]

![Figure 28: Cross-sectional geometry of the imaginary profile](image)
3.5 Material properties

The material properties of steel grade S235 are used throughout this research project. For the purpose of linear elastic analysis the Young’s modulus $E = 2.1 \times 10^5 \text{ N/mm}^2$ and the Poisson’s ratio $\nu = 0.3$ is used. For the non-linear plastic analysis the yield strength $f_y = 235 \text{ N/mm}^2$ is added to the material properties, resulting in a bilinear stress-strain relationship of the material (Figure 29).

![Bilinear stress-strain relationship](image)

Figure 29: Bilinear stress-strain relationship.

3.6 Boundary conditions

Fork supports have been briefly discussed in section 2.4.1. This section will discuss how these fork supports are modelled in the finite element method software. At one end of the beam, at the centroid of the web, the displacement in the $x$-direction, $y$-direction, $z$-direction, and the rotation around the $x$-axis is set to zero (Figure 30). At the other end the same boundary conditions apply except for the displacement in the $x$-direction, which is not set to zero. With these boundary conditions a simply supported beam is simulated.

In order to comply with the fork conditions, the distortions of the cross-section at the ends of the beam have to be prevented, but leaving the flanges free to warp in the $xz$-plane. These fork conditions are modelled using kinematic coupling constraints in ABAQUS. Kinematic coupling constrains the motion of the coupling nodes to the rigid motion of a reference node. The constraint can be applied to user-specific degrees of freedom at the coupling nodes with respect to the global or local coordinate system. Figure 30 shows how the kinematic coupling constraints are applied to the FEM model. The nodes N1, N2, and N2 represent the reference nodes and the thick lines adjacent to these reference nodes represent the coupling nodes. The constrained degrees of freedom are written at the specific reference node where $U$ stands for the constrained displacement for the corresponding direction and $UR$ stands for the constrained rotation around the corresponding axis.
3.7 Loads

Throughout this research project, a couple of different load cases were considered. The most basic load case considered is the constant bending moment. This bending moment can easily be applied at any specific node and subsequently the magnitude and the direction are entered. Concentrated loads operate identically.

In order to model an equally distributed load, a different approach is required. ABAQUS does provide the option ‘line load’ to model equally distributed loads; however, this option does not work using shell elements. Therefore, the equally distributed load is modelled using multiple concentrated loads (Figure 30). These are applied on the nodes along the length of the beam and the amount of concentrated loads used depends on the density of the mesh.

3.8 Lateral restraints

The lateral restraint is the part of the model that, as the word itself describes, restrains the model to deflect laterally at a certain point. These lateral restraints are modelled by means of applying a boundary condition at the node that is not allowed to deflect laterally. These boundary conditions are applied at the centroid of the top flange and the centroid of the bottom flange (Figure 31). Modelling the lateral restraints as boundary conditions gives them an infinite stiffness; however, in reality the lateral restraints do have a certain finite stiffness. Therefore other methods for modelling lateral restraints by using lateral and rotational springs are also studied. This will be discussed in more detail in section 6.3.
3.9 Residual stresses

Due to the fabrication process of rolled steel beams, compressive and tensile stresses are present in the web and flanges of the beam, which reduce the lateral torsional buckling resistance of the beam significantly. These residual stresses have a linear distribution pattern over the web and flanges, as seen in Figure 32, and the value of the these stresses are $\sigma_{\text{res}} = 0.3 f_y$, where $f_y$ is the yield strength.

![Figure 32: Linear residual stress pattern](image)

![Figure 33: Web and flanges divided into multiple surfaces with discrete residual stresses](image)

3.9.1 Applying residual stresses on the finite element model

Applying the residual stresses on the finite element model proved to be a little more complex than Figure 32 suggests. ABAQUS does allow applying initial stresses to a certain surface of the model; however, it only allows applying a single value for the initial stress to that surface, not a linear distribution of the stress as is desired. Therefore, the linear stress distribution is approximated using a discrete pyramid-shaped stress distribution for which the mean values match the linear distribution of Figure 32.

First, the flanges and the web are divided into multiple surfaces where the initial stresses are applied. Subsequently, the appropriate initial stress value is applied for the correct orientation (Figure 33). Dividing the web and flanges into more and thinner surfaces, allows for a finer discrete initial stress distribution. This has no effect on the linear distribution of the initial stress mean values (Figure 34); however, it does affect the value for the lateral torsional buckling resistance followed by the GMNIA analysis. Therefore, a study is performed to analyse the effect of the discrete initial stress pattern density on the lateral torsional buckling resistance.
3.9.2 Discrete initial stress pattern density study

In order to apply a finer discrete pyramid-shaped stress distribution, the web and flanges are divided in thinner surfaces. First, the web and the flanges are divided into 4 surfaces (Figure 34.1). Furthermore, they are divided into 8 surfaces (Figure 34.2), 16 surfaces (Figure 34.3), 32 surfaces (Figure 34.4), 64 surfaces, and 128 surfaces. Dividing the web and flanges of the model into multiple surfaces has a direct impact on the density of the discrete initial stress distribution, as well as the mesh density of the model.

The value of the residual stress is determined with the linear relation

\[ y = 0.3 f_y - \left( \frac{0.3 f_y}{w} \right) x, \]

where \( w \) is the width of the web or the flange. For this research project, only beams with yield strength of 235 N/mm² are considered.

The results of the GMNIA analysis with the different residual stress distributions (black marks) and without residual stresses (red marks) are given in Figure 35. These values are valid for one specific case, namely an IPE240 with a length of 4800 mm subjected to an equally distributed load. In this graph it is visible that using a finer discrete stress distribution eventually does not increase the accuracy of the results with more than 0.15%. Therefore, the discrete stress distribution with 16 surfaces (Figure 34.3) is chosen to be accurate enough for the purpose of this research project.
3.10 System imperfections

System imperfections are small out-of-plane displacements and rotations along the length of the beam. In order to perform a non-linear analysis, it is necessary to apply this initial imperfection to the FEM model. For this research project, only the first positive lateral torsional buckling mode is considered for the imperfection shape. However, a study on the influence of higher order lateral torsional buckling modes combined as an imperfection shape has also been studied. This is discussed in section 6.2.

The first lateral torsional buckling mode is obtained by performing a linear buckling analysis. The results from this linear buckling analysis are written to a node file, where all the displacements of every node in the model are saved. Subsequently, a scaled version of the buckling mode is applied to the non-linear model as the initial imperfection.

This research project considers two main types of imperfection shapes, namely the first positive lateral torsional buckling mode of the laterally unrestrained beam (Figure 36) and the first positive lateral torsional buckling mode of the laterally restrained beam (Figure 37).
The amplitude $e_0$ of the imperfection is determined with the factor $L^*/1000$, where $L^*$ is the effective lateral torsional buckling length of the beam. For the laterally unrestrained beam $L^*$ is the total length of the beam, for the laterally restrained beam $L^*$ is the length between the lateral restraints. The value of the amplitude $e_0$ is used as the scale factor for the lateral torsional buckling mode determined from the linear buckling analysis. The lateral torsional buckling mode with scale factor $e_0$ is then applied on the non-linear model as the initial imperfection shape.

3.11 Solving methods

3.11.1 Linear buckling analysis

To determine the elastic critical lateral torsional buckling load of the beam, a linear buckling analysis (LBA) is performed. During this analysis, the bifurcation point is determined by solving an eigenvalue problem. This eigenvalue problem is solved when the stiffness matrix of the model becomes singular and provides nontrivial solutions.

Performing the linear buckling analysis with ABAQUS, the linear perturbation step is used with the buckling procedure. The software gives the option to choose from two different methods to solve the eigenvalue problem, namely the Lanczos and the subspace iteration method. Both methods provide the option to determine multiple eigenvalues; however, for this research project only the first positive buckling mode is required, therefore only the first positive eigenvalue needs to be determined. For this reason the Lanczos method is used, because this method provides the option to enter the minimum value required for the eigenvalue. For the case of the first positive eigenvalue, this minimum value needs to be at least zero; otherwise the solver would also determine negative eigenvalues.

To eventually determine the elastic critical lateral torsional buckling load of the beam, the applied load needs to be multiplied by the eigenvalue resulting from the linear buckling analysis and subsequently, the elastic critical lateral torsional buckling moment can be determined.

For the linear buckling analysis, linear elastic material properties are applied to the model, which have been discussed in section 3.5.

3.11.2 Geometrical and material non-linear imperfection analysis

To determine the lateral torsional buckling resistance of the beam, a geometrical and material non-linear imperfection analysis (GMNIA) is performed. This is considered to be the most accurate method to determine the true lateral torsional buckling resistance of beam.
Because of buckling instability, the load-displacement response shows a negative stiffness and the structure must release strain energy to remain in equilibrium (Figure 38).

![Figure 38: Possible non-linear buckling load-displacement behaviour.](image)

Therefore it is important that a solution method is chosen that can predict the load-displacement response after buckling has occurred.

The modified Riks method or arc-length method is an algorithm that provides effective solutions for such cases. The modified Riks method uses a tangent line of a function to intersect with an arc, situated at the end of every step. From this point on, the curve will converge over the arc-length until it reaches an intersection of the arc with the function. At this point, the step is completed and the process will continue with the next step (Figure 39).

![Figure 39: Graphical example of the modified Riks method.](image)

In addition to the linear elastic model, the non-linear plastic model also includes the plastic material properties as is discussed in section 3.5, the residual stresses as is discussed in section 3.9, and the system imperfections as is discussed in section 3.10.

The results from the modified Riks method are given in a LPF vs. arc-length graph, where LPF stands for the load proportionality factor $\lambda$ (Figure 39). The applied load on the model needs to be multiplied with the LPF to determine the actual failure load. To determine the lateral torsional buckling load, the maximum value of the LPF needs to be determined at the top of the graph, which indicates the point right before instability occurs (Figure 40).
3.12 Validation of the finite element model

In order to confirm if the above-described FEM model gives reliable results, it needs to be validated. This validation can be acquired by comparing the results from the FEM model with the results from practical experiments, or by comparing the results with other FEM models from independent researches. Both validation methods have been used for this research project and are explained in the following section.

3.12.1 Using experimental results for validation

The experimental results from the dissertation of H. Barnard from the University of Johannesburg[15] are used for validation of the FEM model. The research of Barnard regards the elastic and inelastic lateral torsional buckling strength of hot rolled type 3CR12 (corrosion resisting) steel beams. Barnard studies both doubly symmetrical I-beams and singly symmetric channels sections. For the purpose of validation only the doubly symmetrical I-beams are considered, which are designated as 203 203×133×25 kg/m.

The beam test setup is shown in Figure 41. The test setup consists of a single span test beam with cantilevers on both sides when viewed in the vertical bending plane. The cantilevers serve as moment lever arms in order to subject the test length to equal and opposite moments between the intermediate supports. A constant bending moment therefore exists over the test section.
The beam is supported by a pin and roller system in the vertical plane. The test setup forms a lateral continuous beam with three spans in the horizontal plane. The intermediate supports are knife edged which prevent lateral movement as well as twisting of the beam section. The cantilever end tips are also restrained from lateral movement and twisting but are free regarding vertical in-plane movement in order to apply the concentrated load to the cantilevers (Figure 42).

The lateral torsional buckling resistance is determined for beam test lengths ranging from 800 mm to 5600 mm. An important remark is that these beams have been heat treated in order to relieve the material from residual stresses, and therefore residual stresses are not implemented in the FEM model.

The beam lengths, cross-sectional properties, and stress-strain relationships have been measured for all test specimens and well documented, which greatly contributes to the accuracy of the FEM model when compared to the real test specimens. The greatest challenge lies in the modelling of the supports and restraints. Unfortunately the imperfections are not measured for the test specimens and therefore it is assumed that the imperfection shape is the first positive lateral torsional buckling mode of the FEM model with amplitude $e_0 = L/1000 \text{ mm}$. Figure 43 illustrates the FEM model that is used for the LBA to obtain the
imperfection shape and for the GMNIA to determine the plastic lateral torsional buckling resistance of the beam.

![Figure 43: FEM model of the Barnard test beam setup.](image)

The results from the experiments and the GMNIA are given in Table 12. Here it can be seen that the difference in results is less than 5%, from which can be resulted that the finite element model gives accurate results compared with the results from the experiments. The small difference in results can be explained by the fact that the test specimens, unlike the FEM model, do have the fillet radius present at the flange-web intersections that contributes to the torsional stiffness of the cross-section.

<table>
<thead>
<tr>
<th>Beam length [mm]</th>
<th>$M_{b,cr}$ GMNIA [kNm]</th>
<th>$M_{b,cr}$ Barnard [kNm]</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>2400</td>
<td>86,21</td>
<td>89,63</td>
<td>3,81 %</td>
</tr>
<tr>
<td>3200</td>
<td>76,86</td>
<td>78,15</td>
<td>1,88 %</td>
</tr>
<tr>
<td>4000</td>
<td>69,10</td>
<td>68,06</td>
<td>-1,53 %</td>
</tr>
<tr>
<td>4800</td>
<td>61,81</td>
<td>63,17</td>
<td>2,15 %</td>
</tr>
<tr>
<td>5600</td>
<td>51,54</td>
<td>50,88</td>
<td>-1,30 %</td>
</tr>
</tbody>
</table>

3.12.2 Using other FEM results for validation

Many other independent researchers have conducted studies regarding lateral torsional buckling, also with the help of FEM software to generate the results. Naturally these FEM models have also been validated themselves; therefore, using these results for comparison is also a valid method for validation of the FEM model used for this research. The results from R.H.J. Bruins from the Eindhoven University of Technology[14] and A. Taras from the Graz University of Technology[16] are used in particular.

Both researchers have studied the lateral torsional buckling stiffness of an IPE240 beam for different load cases and elaborated these results and compared them with buckling curves from EN-1993-1-1. These buckling curves will be used for comparison and validation.
First, the results from Bruins are used for validation. He studied three different load cases with three different beam lengths. The first load case he studied is a simply supported beam with a concentrated load applied at mid-span of the beam and applied at the centroid of the top flange. The second load case is a simply supported beam with an equally distributed load applied at the centroid of the top flange. The third load case is similar to the first one with a concentrated load at mid-span of the beam applied at the centroid of the flange, but at one end the beam is clamped, resulting in a fixed support. For the purpose of validation, only the first and second load case will be used as comparison. Further three beam lengths were studied, namely 3600 mm, 5400 mm, and 7200 mm. The model Bruins used also includes residual stresses and a system imperfection with amplitude of $L/1000 \text{ mm}$. It is also important to mention that all these load cases are laterally unrestrained and that Bruins used compensation elements for modelling the fillet radius.

Figure 44 shows the results from the comparison of both FEM models. The first notable difference is that the results from Bruins (the blue marks) has higher values for both $\lambda_{LT}$ and $\chi_{LT}$, meaning a higher value for $M_c$ and $M_{b,\text{rd}}$. This difference can be explained by the fact that the compensation elements modelled by Bruins contribute to the lateral torsional buckling strength.

In order to make a more reliable comparison between the two FEM models, the same compensation elements that are used by Bruins are added to the FEM model. For the case of an IPE240 cross-section, Bruins has documented the element type and cross-sectional properties of the compensation elements, which makes it possible to recreate the exact same FEM model. The comparison of the results from both FEM models with the compensation elements included is given in Figure 45. This shows that the marks are much closer together meaning that differences between the values of both $M_c$ and $M_{b,\text{rd}}$ are much smaller compared with the results of Figure 44.
Taras has performed extensive research concerning lateral torsional buckling and has also studied the influence of omitting the fillet radius at the flange web intersections specifically. The comparison is made between the results of an IPE240 FEM model with and without the fillet radius. The results from this comparison are given in Figure 46. The results of the FEM model without the fillet radius are used for validation of the FEM model used for this research. The model Taras used concerns a simply supported beam with a constant bending moment and laterally unrestrained. Residual stresses and a system imperfection with amplitude $e_0 = L/1000 \text{ mm}$ are also included.

Figure 45: Comparison of FEM model with results of R. Bruins (compensation elements included).

Figure 46: Influence of omitting the fillet radius on the shape of the buckling curve[16].
Figure 47 shows the results of both FEM models without the fillet radius. The results of Taras are displayed as a continuous line without marks along the line. This is done because it is unknown for which beam lengths the marks from Figure 46 stand for, meaning that comparing the position of both marks would be useless. Therefore, for this case the position of the black marks need to be compared with only the blue line, which shows very little difference between the results.

The conclusion that can be drawn here is that the FEM model that is used for this research project does give satisfying results when comparing with other independent studies, especially for the last case in which it is compared with a model without the fillet radius. This is a promising result because for the continuation of this research the results from the FEM model will only be compared with analytical results in which the cross-sections also do not include the fillet radius.

Figure 47: Comparison of FEM model and results of A. Taras.
Lateral torsional buckling analysis of multiple laterally restrained I-beams \
4. Parametric study

Now that the finite element model has been verified, it can be used to perform the parametric study. The parametric study will generate the results that are used to compare with the results from EN-1993-1-1 and the Dutch National Annex. This chapter will discuss the various parameters that are varied to obtain a great range of results. These results are elaborated in chapter 5 and the conclusions and recommendations based from these results are made in chapter 7.

4.1 Variable parameters

4.1.1 Beam lengths

The beam length is one of the most logical parameters to study. For this research, the effects of the lateral restraints are of greatest interest and therefore it is a wiser decision to look at the results of varying the lengths between the lateral restraints. This of course automatically varies the total length of the beam as well. The length between the lateral restraints will be varied in such a way that every segment, the length between the lateral restraints or between a lateral restraint and a support, will have the same length. The maximum length of the beam that will be studied is 28 metres; this value is based on the maximum beam length for rolled sections available in the Netherlands.

4.1.2 Amount of lateral restraints

The variation of the amount of lateral restraints is also an interesting parameter to study. For this study the amount of lateral restraints will be varied from a minimum of two lateral restraints to a maximum of five lateral restraints. One lateral restraint will not be studied, because an extensive study on this case has already been conducted by R. Bruins.

The variation in beam lengths, more specifically the lengths of the beam segments, are directly associated with the amount of lateral restraints. For the case of two lateral restraints, meaning three beam segments, the maximum beam segment length is \(28/3 = 9\frac{1}{3} \, \text{m}\). It is chosen to only use integers for the beam lengths, meaning the maximum length will be 9 \(\text{m}\) or 9000 \(\text{mm}\). The minimum length is chosen to be 2000 \(\text{mm}\) with increments of 1000 \(\text{mm}\). Figure 48 to Figure 51 below display the all the possible variations based on the beam lengths and the amount of lateral restraints.
4.1.3 Height position of lateral restraints

The height position of the lateral restraint also has an effect on the load bearing resistance of the beam, which makes it an interesting parameter to study. The lateral restraint can be applied at the top flange, in the middle of the web, at the bottom flange, or somewhere in between these positions. Another interesting variation is to apply a lateral restraint at both the top and bottom flange.
The study of Bruins has shown at what position of the beam, a single lateral restraint has the greatest influence on the lateral torsional buckling stability. It was concluded that the ideal position of the lateral restraint was at mid-span of the beam applied at the centroid of the compression flange of the beam. Therefore, this position is certainly considered as well for this parametric study. The three different studied positions are illustrated in Figure 52.

4.1.4 Steel profiles

Different types of steel profiles have different bending and buckling behaviours, which makes it interesting to study these differences. For this research only I-shaped steel profiles will be analysed. This may seem as a limitation, yet there is much variation in I-shaped steel profiles only and not to mention the different dimensions available.

The most commonly used steel profiles in practice are the IPE and the HEA steel profiles. For the IPE steel profile, it is also interesting to choose the dimensions of the profile in such a way that two different buckling curves can be compared (buckling curve a and buckling curve b, for rolled I-sections). Therefore, the dimensions should be chosen in a way that one profile meets $h/b \leq 2$ requirement (buckling curve a) and one that meets $h/b > 2$ requirement (buckling curve b). The steel profiles that meet these requirements are the IPE 240 and the IPE 500 steel profile. The HEA steel profiles with the same dimensions of the previous mentioned IPE profiles both meet the $h/b \leq 2$ requirements so they can only be compared with one buckling curve. Note that these steel profiles are modelled without the fillet radius, as is discussed in section 3.3. Therefore only the height, width, flange thickness, and web thickness are used for the mentioned steel profiles. The visual differences between the steel profiles are displayed in Figure 53.
4.1.5 System imperfections

It is still not clear what kind of imperfection mode should be used according to the EN-1993-1-1 when considering laterally restrained beams. Different types of buckling modes can be used for the shape of the initial imperfection. For this parametric study only two imperfection shapes will be considered, namely the first positive lateral torsional buckling mode of the laterally unrestrained beam, and the first positive lateral torsional buckling mode for the laterally restrained beam. For both shapes, the amplitude $e_0 = L^*/1000 \text{ mm}$ is used. However, for the laterally unrestrained beam, $L^*$ is the total length of the beam, and for the laterally restrained beam, $L^*$ is the length of a segment between a support and a lateral restraint, or between two restraints. See section 3.10 for more details.

Note that these imperfections, whether based on the laterally restrained beam or laterally unrestrained beam, are always applied to a non-linear FEM model that is laterally restrained. Thus, one variant is the laterally restrained beam with its own lateral torsional buckling mode as imperfection shape (Figure 54.1), and the second variant is the laterally restrained beam with an imperfection shape of the laterally unrestrained beam (Figure 54.2). These two types of imperfections will be referred to as IMP restrained and IMP unrestrained respectively in the results of chapter 5. The imperfection shapes of all the different lateral restraint setups are found in Appendix E.

![Figure 54: Two imperfection modes studied.](image)

4.2 Static parameters

4.2.1 Load case

Normally, the load cases should belong to the variable parameters. For the case with lateral restraints, the variation between different load cases is perhaps a less interesting parametric study to perform. A constant bending moment never occurs in practice and the point loads that would occur in practice will be situated at the position of the lateral restraints, which is

70
also a less interesting load case to study. Therefore it is best to mainly focus on the load case with an equally distributed load, which is applied at the centroid of top flange (Figure 55).

![Equally distributed load applied at the centroid of the top flange.](image)

**Figure 55: Equally distributed load applied at the centroid of the top flange.**

### 4.2.2 Steel grade

For this parametric study only the S235 steel grade will be used. More information about the material properties of this steel grade can be found in section 3.5.

### 4.2.3 Residual stresses

For all GMNIA calculations the same residual stress patterns and values are used. More information about the applied residual stresses can be found in section 3.9.
5. Elaboration of results

This chapter covers the elaboration of the results from the parametric study explained in chapter 4. The results are plotted in the form of buckling curves that show the relationship between the non-dimensional slenderness $\overline{\lambda}_{LT}$ and the reduction factor $\chi_{LT}$.

The black and the blue marks represent the results from the parametric study. The values for $\overline{\lambda}_{LT}$ are determined using the LBA (section 3.11.1) and the values for $\chi_{LT}$ are determined using the GMNIA (section 3.11.2). In the graphs the distinction is made for the two different imperfection types. The black marks represent the results from the GMNIA using the first positive lateral torsional buckling shape of the laterally restrained beam as imperfection (IMP restrained) and the blue marks represent the results from the GMNIA using the first positive lateral torsional buckling shape of the laterally unrestrained beam as imperfection (IMP unrestrained).

The red marks (EN-1993-1-1) represent the results obtained from the design codes. The values for $\overline{\lambda}_{LT}$ are determined using the equations of the Dutch National Annex (section 2.5.3) and the values for $\chi_{LT}$ are determined with the equations from EN-1993-1-1 (section 2.2.1).

Every mark stands for a result for a specific beam length, or rather a specific beam segment length. Taking Figure 56 as an example, the marks at the far left stand for a beam segment length of 2000 mm and the marks at the far right stand for a beam segment length of 9000 mm. The marks in between stand for the remaining studied beam segment lengths (section 4.1.2).

In order to make the right conclusions, the results need to be interpreted correctly. Only the results representing the same beam segment lengths should be compared. Taking Figure 56 as an example again, the first black, blue, and red marks all represent the results for a beam segment of 2000 mm.
segment length of 2000 mm. Comparing the position of the black and blue marks with the position of the red mark, the conclusion can be made that a lower value for $\lambda_{LT}$ is found and a higher value for $\chi_{LT}$ is found. This directly means that a higher value for the elastic critical lateral torsional buckling moment $M_{cr}$ and a higher value for the lateral torsional buckling resistance $M_{b,Rd}$ is found for the results from the GMNIA compared with the results from the design codes. Comparing the position of the black and the blue marks, the conclusion can be made that the beam with IMP unrestrained gives a higher value for $\chi_{LT}$ than a beam with IMP restrained.

5.1 Case 1: Lateral restraints applied at the centroid of the top flange

The first case that is examined consists of laterally restrained beams where the lateral restraints are applied at the centroid of the top flange.

5.1.1 IPE 240

Figure 57 to Figure 60 show the results of an IPE 240 beam with the lateral restraints applied at the centroid of the top flange. The first thing that strikes is that the results from the LBA give lower values for the non-dimensional slenderness $\bar{\lambda}_{LT}$, which means that higher values for the elastic critical lateral torsional buckling moment $M_{cr}$ are obtained with the LBA, compared with the values given by the Dutch National Annex.

![Figure 57: Results IPE 240 – 2 lateral restraints – top.](image-url)
Figure 58: Results IPE 240 – 3 lateral restraints – top.

Figure 59: Results IPE 240 – 4 lateral restraints – top.
Another striking aspect is that for some lengths the results from the GMNIA lie below the buckling curve. This is not considered a problem, because if these values below the buckling curve are compared with the corresponding values according to EN-1993-1-1, it still has higher values for the reduction factor $\chi_{LT}$. However, for some cases (Figure 59 and Figure 60) the value for $\chi_{LT}$ determined with the GMNIA are far lower than those from EN-1993-1-1. These values are considered unsafe.

If the buckling behaviour is studied of these unsafe cases, it is found that the failure mode of the laterally restrained beam does not follow the lateral torsional buckling mode of a laterally restrained beam (lateral torsional buckling occurring between the lateral restraints), but more of a laterally unrestrained beam (lateral torsional buckling occurring between the supports). This is due to the fact that the bottom flange is not restrained from lateral movement. The imperfection shape of the laterally unrestrained beam encourages the beam to also fail with the buckling mode of the laterally unrestrained beam. The lateral restraints applied at the top flange therefore do not act like lateral restraints at all, but act like hinges at which the beam is able to rotate (Figure 61).
5.1.2 IPE 500

Figure 62 to Figure 65 show the results of an IPE 500 beam with the lateral restraints applied at the centroid of the top flange. Also for this cross-section the values for $\bar{\lambda}_{LT}$ are lower compared to the values from the Dutch National Annex. It is remarkable that this cross-section is not negatively affected by the fact that the bottom flange is free to displace laterally. This could be explained by the fact that the IPE 500 cross-section is more slender compared to the IPE 240 cross-section and therefore more sensitive to lateral torsional buckling. For all cases, the results of the GMNIA are considered safe compared with the values from the design codes.

*Figure 62: Results IPE 500 – 2 lateral restraints – top.*

*Figure 63: Results IPE 500 – 3 lateral restraints – top.*
Figure 64: Results IPE 500 – 4 lateral restraints – top.

Figure 65: Results IPE 500 – 5 lateral restraints – top.
5.1.3 HEA 240

Figure 66 to Figure 70 show the results of a HEA 240 beam with the lateral restraints applied at the centroid of the top flange. These figures show more unsafe results for the value of $\chi_{LT}$ determined with the GMNIA. The same unexpected buckling behaviour occurs as described in section 5.1.1. Another remarkable aspect is that the torsional stiffness of this cross-section is so large that lateral torsional buckling did not occur until a beam segment length of 4000 mm. Instead, local buckling behaviour occurred.

---

Figure 66: Results HEA 240 – 2 lateral restraints – top.

Figure 67: Results HEA 240 – 3 lateral restraints – top.
Figure 68: Results HEA 240 – 4 lateral restraints – top.

Figure 69: Results HEA 240 – 5 lateral restraints – top.
5.1.4 HEA500

Figure 70 to Figure 73 show the results of a HEA 500 beam with the lateral restraints applied at the centroid of the top flange. These figures show safe results for the values for both the non-dimensional slenderness $\bar{\lambda}_{LT}$ and reduction factor $\chi_{LT}$. Also this cross-section is not negatively affected by the fact that the bottom flange is free to displace laterally.

![Figure 70: Results HEA 500 – 2 lateral restraints – top.](image)

![Figure 71: Results HEA 500 – 3 lateral restraints – top.](image)
5.1.5 Discussion

The case with the lateral restraint applied at the centroid of the top flange shows satisfactory results for the non-dimensional slenderness \( \bar{\lambda}_{LT} \); however, the results for reduction factor \( \chi_{LT} \) for some cases are unsafe. The fact that for these cases a laterally restrained beam shows a failure mode of a laterally unrestrained beam is concerning and unexpected.
5.2 Case 2: Lateral restraints applied at the centroid of the web

This section covers the case with the lateral restraint is applied at the centroid of the web.

5.2.1 IPE 500

Figure 74 shows the results for an IPE 500 beam with the lateral restraints applied at the centroid of the web. Comparing the GMNIA results with the results from the design codes shows that the results from the GMNIA are significantly lower for both the non-dimensional slenderness $\bar{\lambda}_{LT}$ and the reduction factor $\chi_{LT}$. Looking at the lateral torsional buckling behaviour of the beams studied with the GMNIA, it reveals that the failure modes are similar to the failure modes of a laterally unrestrained beam. The lateral restraints act as a hinge at which the beam is able to rotate.

From this can be concluded that applying the lateral restraints at the centroid of the web does not contribute to the lateral torsional buckling resistance of the beam and therefore may not be considered as lateral restraints at all. These beams must be considered as laterally unrestrained beams and therefore must also be checked using the equations for laterally unrestrained beams.

![Figure 74: Results IPE 500 – 2 lateral restraints – middle.](image)
5.3 Case 3: Lateral restraints applied at the centroid of the top and bottom flange

This case follows from the unexpected results from section 5.1, where the lateral restraints are only applied at the centroid of the top flange. By applying the lateral restraints at the centroid of both the top and bottom flange, fork conditions are obtained. This means that the cross-section is not able to rotate at the position of the lateral restraints. These fork conditions at the position of the lateral restraints ensure that the failure mode is identical to the elastic critical lateral torsional buckling mode of the same laterally restrained beam.

Figure 75 shows an example of an IPE 240 beam with an IPE 80 beam connected at the top of the beam which functions as a lateral restraint. This structure may be considered as an IPE 240 beam with lateral restraints applied at the top and bottom because the IPE 80 beam restrains the top flange of the IPE 240 beam from lateral displacement due to axial stiffness and also restrains the IPE 240 beam from rotation due to the bending stiffness, assuming the connection between the beams is rigid.

For this case however, the properties of the lateral restraints are assumed to have an infinite stiffness, while in reality it does have a certain finite stiffness. This is examined in section 6.3.

5.3.1 IPE 240

Figure 76 to Figure 79 show the results of an IPE 240 beam with the lateral restraints applied at the centroid of the top and bottom flange. For all cases, the results from the GMNIA give higher values for both the non-dimensional slenderness $\bar{\lambda}_{LT}$ and the reduction factor $\chi_{LT}$ compared with the results from the design codes.

Another striking aspect is the fact that for all cases the imperfection shape of the laterally unrestrained beam gives higher values for $\chi_{LT}$ compared with the results of the laterally restrained beam. This can be explained due to the fact that the imperfection shape of the laterally unrestrained beam is different from the failure mode of the laterally restrained beam. This gives the beam a higher lateral torsional buckling resistance compared to the beam that has an imperfection shape identical to its failure mode.
Figure 76: Results IPE 240 – 2 lateral restraints – top and bottom.

Figure 77: Results IPE 240 – 3 lateral restraints – top and bottom.
Figure 78: Results IPE 240 – 4 lateral restraints – top and bottom.

Figure 79: Results IPE 240 – 5 lateral restraints – top and bottom.
5.3.2 IPE 500

Figure 80 to Figure 83 show the results of an IPE 500 beam with the lateral restraints applied at the centroid of the top and bottom flange. For all cases, the results from the GMNIA give higher values for both the non-dimensional slenderness $\lambda_{LT}$ and the reduction factor $\chi_{LT}$ compared with the results from the design codes. Also for this cross-section, the imperfection shape of the laterally restrained beam is governing, similar to the results of the IPE 240 beam.

Figure 80: Results IPE 500 – 2 lateral restraints – top and bottom.

Figure 81: Results IPE 500 – 3 lateral restraints – top and bottom.
Lateral torsional buckling analysis of multiple laterally restrained I-beams

Figure 82: Results IPE 500 – 4 lateral restraints – top and bottom.

Figure 83: Results IPE 500 – 5 lateral restraints – top and bottom.
5.3.3 HEA240

Figure 84 to Figure 87 show the results of a HEA 240 beam with the lateral restraints applied at the centroid of the top and bottom flange. Also for this cross-section, similar results are obtained as the previous mentioned cross-sections.

Figure 84: Results HEA 240 – 2 lateral restraints – top and bottom.

Figure 85: Results HEA 240 – 3 lateral restraints – top and bottom.
Lateral torsional buckling analysis of multiple laterally restrained I-beams

Figure 86: Results HEA 240 – 4 lateral restraints – top and bottom.

Figure 87: Results HEA 240 – 5 lateral restraints – top and bottom.
5.3.4 HEA500

Finally, Figure 88 to Figure 91 show the results of a HEA 500 beam with the lateral restraints applied at the centroid of the top and bottom flange. Also for this cross-section, similar results are obtained as the previous mentioned cross-sections.

Figure 88: Results HEA 500 – 2 lateral restraints – top and bottom.

Figure 89: Results HEA 500 – 3 lateral restraints – top and bottom.
5.3.5 Discussion

The case with the lateral restraints applied at the centroid of the top and bottom flange gives promising results, for both the non-dimensional slenderness $\overline{\lambda}_{LT}$ and reduction factor $\chi_{LT}$. It also provides consistent lateral torsional buckling failure modes. Furthermore, it shows that the imperfection shape of the laterally restrained beam is governing.
6. Sensitivity studies for validation of results

The results discussed in chapter 5 show that changing a small detail, in this case the position of the lateral restraints, can have a great influence on the lateral torsional buckling behaviour of the beam and therefore on the outcome of the results. This raised numerous questions about the potential influence of other changes to the FEM model on the results. In this chapter three significantly different changes in the FEM model are studied and the results are discussed.

6.1 Different position of boundary conditions

The boundary conditions used throughout this research project are based on a simply supported beam with fork conditions at the beam-ends. The displacements in all directions and the rotation around the x-axis are all restrained at the centroid of the web, see section 3.6 which discusses these applied boundary conditions in more detail. Another method is to apply the boundary conditions in such a way that the FEM model is simply supported with fork conditions at the beam-ends, but the displacement in the x-direction and y-direction are restrained at the bottom flange and the displacement in the z-direction is restrained at the top and bottom flange. The rotation around the x-axis is still restrained at the centroid of the web (Figure 92).

![Figure 92: IPE 240 beam with different position of the boundary conditions.](image)

Figure 93 shows the results of the GMNIA with the boundary conditions according to Figure 92 (blue marks) compared to the boundary conditions according to section 3.6 (black marks). For this case only the IPE 240 beam with 2 lateral restraints applied at the centroid of the top and bottom flange is studied. Residual stresses are applied and the imperfection for both models is the imperfection shape of the laterally restrained beam. The results of the two different types boundary conditions are almost identical, with the actual difference ranging from 0,01% to 0,9%. From this can be concluded that this change in boundary conditions has no significant influence on the lateral torsional buckling resistance of the beam.
6.2 Influence of higher order lateral torsional buckling modes combined as imperfection shape

Throughout this research project only the first positive lateral torsional buckling mode is considered for the imperfection shape. More specifically, the first positive lateral torsional buckling mode of the laterally restrained beam as imperfection shape and the first positive lateral torsional buckling mode of the laterally unrestrained beam as imperfection shape were considered. This section studies the influence of higher order lateral torsional buckling modes as imperfection shape on the lateral torsional buckling resistance of the beam.

The combination of the first five positive lateral torsional buckling modes is used as the imperfection shape for this study. Figure 95.1 shows the imperfection shape of the first five positive lateral torsional buckling modes combined for the laterally restrained beam with two lateral restraints and Figure 94.2 shows the imperfection shape of the first five positive lateral torsional buckling modes combined for the laterally unrestrained beam.

Figure 93: Results IPE 240 - 2 lateral restraints – top and bottom – position boundary conditions.

Figure 94: Imperfection shapes of the first five positive lateral torsional buckling modes combined.
Note that the node with the maximum lateral deflection of the combined imperfection shape is not positioned at mid-span of the beam, as is the case with the imperfection shape of the first positive lateral torsional buckling mode. The value $e_0 = L^* / 1000 \, mm$ is still used for the amplitude of the imperfection shape, where $L^*$ is the effective lateral torsional buckling length (section 3.10).

Figure 95 shows the results of the GMNIA with the combined lateral torsional buckling modes as imperfection shape (green and purple marks) compared with the results of the GMNIA with only the first positive lateral torsional buckling mode as imperfection shape (black and blue marks). These results are for the case with two lateral restraints applied at the centroid of the top and bottom flange and with residual stresses applied to the model. The results show that for the imperfection shape of the laterally restrained beam, the combined imperfection shape (green marks) gives higher values compared with the imperfection shape of only the first positive lateral torsional buckling mode (black marks). This is due to the fact that the combined imperfection shape less resembles the failure mode, creating more resistance against failure. For the imperfection shape of the laterally unrestrained beam, the exact opposite occurs because the combined imperfection shape (purple marks) now resembles the failure mode more closely compared to the imperfection shape of only the first lateral torsional buckling mode of the laterally unrestrained beam (blue marks).

The same study as mentioned above is performed again, only for this case a laterally restrained beam with five lateral restraints applied at the centroid of the top and bottom flange is studied. Figure 96.1 shows the imperfection shape of the first five positive lateral torsional buckling modes combined for the laterally restrained beam with five lateral restraints and Figure 96.2 shows the imperfection shape of the first five positive lateral torsional buckling modes combined for the laterally unrestrained beam.
6.2.1 Discussion

The results from Figure 95 and Figure 97 show that the imperfection shape of the first five positive lateral torsional buckling modes combined does have an influence on the lateral torsional buckling resistance of the beam. For the case of the laterally restrained beam it gives higher values and for the case for the laterally unrestrained beam it gives lower values. From this behaviour can be concluded that the imperfection shape that resembles the failure mode the most is governing.
6.3 Influence of lateral restraints with spring stiffness

In section 5.3 infinitely stiff lateral restraints applied at the centroid of the top and bottom flange are used to model the structure given in Figure 75. In reality the IPE 80 beam used as a lateral restraint does have a certain stiffness due to cross-sectional properties of the beam. In this section the influence of this finite stiffness is studied and compared with the case with infinite stiff lateral restraints.

6.3.1 Reduced elastic critical lateral torsional buckling moment

First an acceptable reduction needs to be assumed that accounts for the finite stiffness of the lateral restraint. Bruins[16] used a reduction of 5% to the elastic critical lateral torsional buckling moment as an acceptable reduction and determined the spring stiffness needed to account for this reduction. Or in other words, the spring stiffness of the lateral restraints is determined for which the value of the elastic critical lateral torsional buckling moment is 95% of the value for the elastic critical lateral torsional buckling moment with infinitely stiff lateral restraints. These will be referred to as $M_{cr;95\%}$ and $M_{cr;100\%}$ respectively. The 95% spring stiffness of these restraints is referred to as $K_{95\%}$. For this spring stiffness study the same principles are used as Bruins used in his study.

6.3.2 Lateral spring stiffness

To determine the lateral spring stiffness that accounts for the 5% reduction of the elastic critical lateral torsional buckling moment, the lateral restraints applied at the centroid of the top flange are replaced with lateral springs in the FEM model (Figure 98). Obtaining the value of the spring stiffness that accounts for the $M_{cr;95\%}$ requirement was an iterative process. The results from this process are given in Table 13. These results are determined for an IPE 240 beam with two lateral restraints and the imperfection shape of the laterally restrained beam.

![Figure 98: FEM model with lateral springs applied to the centroid of the top flange.](image-url)
Table 13: Results of the lateral spring stiffness study.

<table>
<thead>
<tr>
<th>Beam segment length [mm]</th>
<th>$M_{c,100%}$ [kNm]</th>
<th>$M_{c,95%}$ [kNm]</th>
<th>$K_{95%}$ [N/mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>244.85</td>
<td>232.63</td>
<td>1600</td>
</tr>
<tr>
<td>3000</td>
<td>127.85</td>
<td>121.43</td>
<td>580</td>
</tr>
<tr>
<td>4000</td>
<td>83.98</td>
<td>79.87</td>
<td>300</td>
</tr>
<tr>
<td>5000</td>
<td>62.14</td>
<td>58.96</td>
<td>175</td>
</tr>
<tr>
<td>6000</td>
<td>49.29</td>
<td>46.83</td>
<td>120</td>
</tr>
<tr>
<td>7000</td>
<td>40.89</td>
<td>38.80</td>
<td>85</td>
</tr>
<tr>
<td>8000</td>
<td>34.97</td>
<td>33.19</td>
<td>65</td>
</tr>
<tr>
<td>9000</td>
<td>30.38</td>
<td>29.05</td>
<td>52</td>
</tr>
</tbody>
</table>

Considering the IPE 80 beam as the lateral restraint and the $K_{95\%}$ value from Table 13, the length of the lateral restraint that meets the $M_{c,95\%}$ requirement can be determined using the spring stiffness relation:

$$K_{95\%} = \frac{EA}{L}$$  \hspace{1cm} Eq. 63

From this relation can be determined that a length of $L = \frac{EA}{K_{95\%}} = \frac{210000 \cdot 764}{1600} = 100275 \text{ mm}$ is sufficient for an IPE 80 beam to meet the $M_{c,95\%}$ requirement as a lateral restraint. This value is so big that can be concluded that an IPE 80 beam with a length smaller than that provides enough lateral stiffness that the elastic critical lateral torsional buckling moment is reduced with less than 5%.

### 6.3.3 Rotational spring stiffness

The lateral restraints that are applied at the centroid of the bottom flange provide fork conditions that prevent the cross-section from rotating. In order to determine the spring stiffness that accounts for the $M_{c,95\%}$ requirement, the infinitely stiff lateral restraints applied at the centroid of the bottom flange are replaced with rotational springs applied at the centroid of the top flange in the FEM model (Figure 99). The results from this are given in Table 13 and these results are also determined for an IPE 240 beam with two lateral restraints and the imperfection shape of the laterally restrained beam.

![FEM model with rotational springs applied at the centroid of the top flange.](image-url)
Table 14: Results of the rotational spring stiffness study.

<table>
<thead>
<tr>
<th>Beam segment length [mm]</th>
<th>$M_{cr,100%}$ [kNm]</th>
<th>$M_{cr,95%}$ [kNm]</th>
<th>$C_{95%}$ [Nm/rad]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>244.85</td>
<td>232.63</td>
<td>$1.4 \cdot 10^7$</td>
</tr>
<tr>
<td>3000</td>
<td>127.85</td>
<td>121.43</td>
<td>$4.75 \cdot 10^6$</td>
</tr>
<tr>
<td>4000</td>
<td>83.98</td>
<td>79.87</td>
<td>$2.5 \cdot 10^6$</td>
</tr>
<tr>
<td>5000</td>
<td>62.14</td>
<td>58.96</td>
<td>$1.5 \cdot 10^6$</td>
</tr>
<tr>
<td>6000</td>
<td>49.29</td>
<td>46.83</td>
<td>$1.1 \cdot 10^6$</td>
</tr>
<tr>
<td>7000</td>
<td>40.89</td>
<td>38.80</td>
<td>$8.5 \cdot 10^5$</td>
</tr>
<tr>
<td>8000</td>
<td>34.97</td>
<td>33.19</td>
<td>$7.0 \cdot 10^5$</td>
</tr>
<tr>
<td>9000</td>
<td>30.58</td>
<td>29.05</td>
<td>$5.5 \cdot 10^5$</td>
</tr>
</tbody>
</table>

The IPE 80 beam is considered again as the lateral restraint. In order to determine the length of the lateral restraint that meets the $M_{cr,95\%}$ requirement, the beam bending relationship displayed in Figure 100 is used.

\[ \theta_1 = \theta_2 = \frac{1}{24} \frac{ML}{EI}; \quad \theta_3 = \frac{1}{12} \frac{ML}{EI} \]

**Eq. 64**

*Figure 100: Beam bending relationship with bending moment applied at mid-span.*

The value of the rotational spring stiffness is determined using:

\[ C_{95\%} = \frac{M}{\theta} \]

**Eq. 65**

Using Eq. 64 and Eq. 65, the length of the lateral restraint is determined with:

\[ C_{95\%} = \frac{M}{\theta} = \frac{12EI}{L} \Rightarrow L = \frac{12 \cdot 210000 \cdot 801000}{1.4 \cdot 10^7} = 144180 \text{ mm}. \]

This value is also so big that can be concluded that an IPE 80 beam with a length smaller than that provides enough rotational stiffness that the elastic critical lateral torsional buckling moment is reduced with less than 5%.

### 6.3.4 Combination of lateral springs and rotational springs

The lateral spring stiffness and the rotational spring stiffness required to meet the $M_{cr,95\%}$ requirement have been determined in section 6.3.2 and in section 6.3.3. In these sections it was also determined that the IPE 80 beam shorter than 100 metres would already meet these requirements. This section will study the influence of both the lateral springs and rotational springs combined on the results of GMNIA. Therefore, the infinitely stiff lateral restraints are replaced by lateral springs and rotational springs applied at the centroid of the top flange in the FEM model (Figure 101). These results are also determined for an IPE 240 beam with two lateral restraints and the imperfection shape of the laterally restrained beam.
Figure 101: FEM model with lateral springs and rotational springs applied at the centroid of the top flange.

Figure 102 shows the results of the GMNIA values of the infinitely stiff lateral restraints (black marks) compared with the GMNIA values of the lateral and rotational springs (blue marks). The values from Table 13 and Table 14 are used for the lateral spring stiffness and the rotational spring stiffness. Note that these spring stiffnesses that already account for a 5% reduction to the elastic critical lateral torsional buckling moment. Therefore, the combination of these spring stiffnesses will reduce the value of the elastic critical lateral torsional buckling moment even further. These new reduction values are given in Table 16.

Table 15: Results of the lateral and rotational spring stiffnesses combined study.

<table>
<thead>
<tr>
<th>Beam segment length [mm]</th>
<th>$M_{c,1000}$ [kNm]</th>
<th>$M_{c,spings}$ [kNm]</th>
<th>New reduction value</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>244.85</td>
<td>207.50</td>
<td>15.25%</td>
</tr>
<tr>
<td>3000</td>
<td>127.85</td>
<td>117.84</td>
<td>7.82%</td>
</tr>
<tr>
<td>4000</td>
<td>83.98</td>
<td>77.93</td>
<td>7.20%</td>
</tr>
<tr>
<td>5000</td>
<td>62.14</td>
<td>57.76</td>
<td>7.05%</td>
</tr>
<tr>
<td>6000</td>
<td>49.29</td>
<td>45.99</td>
<td>6.69%</td>
</tr>
<tr>
<td>7000</td>
<td>40.89</td>
<td>38.23</td>
<td>6.51%</td>
</tr>
<tr>
<td>8000</td>
<td>34.97</td>
<td>32.77</td>
<td>6.30%</td>
</tr>
<tr>
<td>9000</td>
<td>30.58</td>
<td>28.66</td>
<td>6.27%</td>
</tr>
</tbody>
</table>

The thing that strikes is that the blue marks give lower values for both the non-dimensional slenderness $\bar{\lambda}_{LT}$ and the reduction factor $\chi_{LT}$. This behaviour is completely expected because reducing the stiffness of the lateral restraint would logically lead to a lower lateral torsional buckling resistance. Comparing the GMNIA values of the lateral and rotational springs with the values from the design codes, it can be concluded that even if the lateral restraints have a certain lateral spring stiffness and rotational spring stiffness that reduces the elastic critical lateral torsional buckling moment by 5%, it still provides safe results.
Figure 102: Results IPE 240 - 2 lateral restraints – rotational spring and lateral spring top.

6.3.5 Discussion

The results from Figure 102 show that giving the lateral restraints a certain lateral spring stiffness and rotational spring stiffness that account for a 5% reduction of the elastic critical lateral torsional buckling moment, it also reduces the lateral torsional buckling resistance of the structure. However, if an IPE 80 beam is used as an lateral restraint, which is the smallest IPE cross-section available in the Netherlands, the length of the beam must be around 100 metres long to account for this 5% reduction. The longest IPE beam length available in the Netherlands is 28 metres. Even if an IPE 80 beam of 28 metres would be used as a lateral restraint, which is highly unlikely, it would reduce the elastic critical lateral torsional buckling moment with only 1.5%. Therefore, the assumption made in section 5.3 that the structure given in Figure 75 can be modelled as infinitely stiff lateral restraints applied at the centroid of the top and bottom flange, is valid.

Note that this conclusion is based on another assumption made in section 5.3, namely the assumption that the connection between the IPE 240 beam and the IPE 80 beam is rigid. In reality this connection also has a certain stiffness depending on the type of connection that is used. A bolted connection is commonly used, which means that the stiffness of the connection is dependent on the bending moment capacity of this bolted connection. This will reduce the torsional spring stiffness that is determined in section 6.3.3. The exact value of this reduction is not determined in this research project, but if the stiffness of the bolted connection reduces the rotational stiffness determined in section 6.3.3 with 50%, a beam length less than 70 m would still meet the $M_{cr,95\%}$ requirement.
7. Conclusions and recommendations

This chapter summarises the conclusions and recommendations that follow from the results of this research project. The conclusions reflect upon the general lateral torsional buckling behaviour of the multiple laterally restrained beams studied in this research project and also reflect upon the design codes that are currently used for determining the elastic critical lateral torsional buckling moment and the lateral torsional buckling resistance. The recommendations that follow are based upon new questions that arose during the course of this research project, but could not be answered within the scope of this research project.

7.1 Conclusions

The correct interpretation and use of a lateral restraint is found to be very important during this research project. If a lateral restraint is not applied effectively, meaning that fork conditions are not satisfied, the GMNIA gives unsafe results for some cases when comparing them to the results of the design codes. If however the fork conditions are satisfied at the position of the lateral restraints, the GMNIA provides safe results for all cases studied in this research project when compared with the results from the design codes. For these cases the results from the design codes are even considered conservative.

Fork conditions are satisfied when the cross-section is not able to deflect laterally and also not able to rotate at the position of the lateral restraints. These fork conditions can be modelled by means of boundary conditions applied at the centroid of the top and bottom flange. In practice these fork conditions are met if the beam that functions as the lateral restraint is applied at the top flange and the connection is considered to be rigid. The assumption that the lateral restraints are infinitely stiff is valid if these fork conditions are satisfied.

The imperfection shape that resembles the failure mode of the beam the most provides the results that are governing. For the case of the multiple laterally restrained beams, the elastic critical lateral torsional buckling mode of this laterally restrained beam should be used as the imperfection shape. The value of the amplitude $e_0 = L^*/1000$, where $L^*$ is the length of the critical beam segment is a safe assumption when residual stresses are also modelled appropriately.

The Dutch National Annex that provides a method to determine the elastic critical lateral torsional buckling moment does not clearly specify the conditions of a lateral restraint. If the lateral restraints are assumed to satisfy the fork conditions, the equations from the Dutch National Annex provide safe results. These results are even considered to be too conservative because they do not account for the additional stiffness that the less loaded adjacent beam segments presumably provide.
Given the fact that the lateral restraints satisfy the fork conditions, the results for the lateral torsional buckling resistance determined with the equations from EN-1993-1-1 are considered safe and even conservative compared with the results of the GMNIA for the cases studied in this research project.

### 7.2 Recommendations

The Dutch National Annex should provide a clear definition of the term lateral restraint in which is stated that lateral restraints should satisfy fork conditions.

Experimental research can be performed to validate the results from this research project, specifically for the cross-sections and the lateral restraint setups studied in this research project.

An extension of the parametric study can be performed to obtain a broader range of results. Recommended parameters to study are:

- More types of steel profiles. The addition of single-symmetrical profiles or hollow section profiles could provide interesting results.

- Varying the beam segments with at least one segment unequal to the rest. Both a shorter and longer beam segment length can be studied and also at different positions of the beam.

A study can be performed to determine a factor that accounts for the additional stiffness that the less loaded adjacent beam segments presumably provide to the elastic critical lateral torsional buckling moment. Then, this factor can be implemented in the equations of the Dutch National Annex.
References


Lateral torsional buckling analysis of multiple laterally restrained I-beams
# Appendix A

## Tables used with Clark & Hill

Table 16: Values of coefficients in formula for elastic buckling strength of beams [5]

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Loading</th>
<th>Bending Moment Diagram</th>
<th>Condition of Restraint Against Rotation about Vertical Axis at Ends</th>
<th>Values of Coefficients</th>
<th>Sources of Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>K</td>
<td>C₁</td>
</tr>
<tr>
<td>1.</td>
<td>M</td>
<td>Simple Support</td>
<td></td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Fixed</td>
<td></td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>2.</td>
<td>M</td>
<td>Simple Support</td>
<td></td>
<td>1.0</td>
<td>1.31-1.32&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Fixed</td>
<td></td>
<td>0.5</td>
<td>1.30-1.32&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td>3.</td>
<td>M</td>
<td>Simple Support</td>
<td></td>
<td>1.0</td>
<td>1.77-1.86&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Fixed</td>
<td></td>
<td>0.5</td>
<td>1.78-1.85&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td>4.</td>
<td>M</td>
<td>Simple Support</td>
<td></td>
<td>1.0</td>
<td>2.32-2.65&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Fixed</td>
<td></td>
<td>0.5</td>
<td>2.29-2.55&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td>5.</td>
<td>M</td>
<td>Simple Support</td>
<td></td>
<td>1.0</td>
<td>2.56-2.74&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Fixed</td>
<td></td>
<td>0.5</td>
<td>2.23-2.58&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td>6.</td>
<td>W</td>
<td>Simple Support</td>
<td></td>
<td>1.0</td>
<td>1.13</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Fixed</td>
<td></td>
<td>0.5</td>
<td>0.97</td>
</tr>
<tr>
<td>7.</td>
<td>W</td>
<td>Simple Support</td>
<td></td>
<td>1.0</td>
<td>1.30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Fixed</td>
<td></td>
<td>0.5</td>
<td>0.86</td>
</tr>
<tr>
<td>8.</td>
<td>P</td>
<td>Simple Support</td>
<td></td>
<td>1.0</td>
<td>1.35</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Fixed</td>
<td></td>
<td>0.5</td>
<td>1.07</td>
</tr>
<tr>
<td>9.</td>
<td>P</td>
<td>Simple Support</td>
<td></td>
<td>1.0</td>
<td>1.70</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Fixed</td>
<td></td>
<td>0.5</td>
<td>1.04</td>
</tr>
<tr>
<td>10.</td>
<td>P/ζ</td>
<td>Simple Support</td>
<td></td>
<td>1.0</td>
<td>1.04</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.5</td>
<td>1.04</td>
</tr>
</tbody>
</table>

### Cantilever Beams

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Loading</th>
<th>Bending Moment Diagram</th>
<th>Condition of Restraint at Supported End</th>
<th>Values of Coefficients</th>
<th>Sources of Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>K</td>
<td>C₁</td>
</tr>
<tr>
<td>11.</td>
<td>P/ζ</td>
<td>Warping Restrained at Supported End</td>
<td></td>
<td>1.0&lt;sup&gt;c&lt;/sup&gt;</td>
<td>1.28-1.71&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td>12.</td>
<td>W</td>
<td>Warping Restrained at Supported End</td>
<td></td>
<td>1.0&lt;sup&gt;c&lt;/sup&gt;</td>
<td>2.05-3.42&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
</tbody>
</table>

<sup>a</sup> Minimum value applies to beams with negligible resistance to warping and can be used to obtain conservative buckling stresses for all beams.

<sup>b</sup> Average value giving buckling stresses within about 10 per cent of values computed by Peterson in all cases.

<sup>c</sup> For cantilever beams, no single combination of values of K and C₂ applies to all proportions of beams. In Case No. 11, the variation of C₁ corresponding to a single value of K is approximately a minimum for K = 1.0.
Lateral torsional buckling analysis of multiple laterally restrained I-beams
## Appendix B

### Tables used with Nethercot

**Table 17: \( \alpha \) values for symmetrical I-beams loaded with equal end moments**

<table>
<thead>
<tr>
<th>Type of support conditions</th>
<th>Formula for ( \alpha )</th>
<th>Maximum error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simply supported Type I</td>
<td>( \alpha = 1 )</td>
<td>0</td>
</tr>
<tr>
<td>Warping fixed, Type II</td>
<td>( \alpha = 1 + \frac{1.778}{R^2} - \frac{0.304}{R} )</td>
<td>1</td>
</tr>
<tr>
<td>Lateral bending fixed, Type III</td>
<td>( \alpha = 2 - \frac{0.787}{R^2} - \frac{1.134}{R} )</td>
<td>1</td>
</tr>
<tr>
<td>Fixed Type IV</td>
<td>Use ( L ) in place of ( L ) in equation (4)</td>
<td>0</td>
</tr>
<tr>
<td>Rigid central support Type V</td>
<td>Use ( \frac{L}{2} ) in place of ( L ) in equation (4)</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 18: \( \alpha \) values for simply supported symmetrical I-beams loaded in various ways**

<table>
<thead>
<tr>
<th>Type of loading</th>
<th>Formula for ( \alpha )</th>
<th>Maximum error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Beams loaded by end moments</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>( M )</td>
<td>( \alpha = 1.16 \times 0.6 - \beta ) for ( 1 \geq \beta \geq 0.8 ) and ( \alpha = 2.56 ) for ( \beta \leq 0.8 )</td>
<td>2</td>
</tr>
<tr>
<td>N.B. Terms in square brackets are used only when positive</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>(b) Beams loaded with transverse loads</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Load at top flange</td>
<td>( \alpha = A/B )</td>
<td>2</td>
</tr>
<tr>
<td>Load at shear centre</td>
<td>( \alpha = A )</td>
<td>2</td>
</tr>
<tr>
<td>Load at bottom flange</td>
<td>( \alpha = A \cdot B )</td>
<td>2</td>
</tr>
</tbody>
</table>

\( A \) | \( B \) | \( A \cdot B \) | 
--- | --- | --- | 
1.35 | 1 - \( \frac{1.779}{R^2} + \frac{2.039}{R} \) | 2               |
1.123 | 1 - \( \frac{1.522}{R^2} + \frac{1.681}{R} \) | 2               |
\( aL \) | 1 + \( a^2 \) | 1 - \( \frac{4.59a}{R^2} + \frac{5.14a}{R} \) | 5               |
Table 19: \( \alpha \) values for symmetrical I-beams loaded and supported in various ways [6]

| Type of loading | Type of supports | Formula for \( \alpha \) | Maximum error |%
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Beams loaded by end moments</td>
<td></td>
<td>a = ( \frac{1}{16} + \left( \frac{a - \frac{3}{4} \beta}{\beta} \right)^2 ) for ( 1 \geq \beta \geq 0 )</td>
<td>( \frac{3}{6} \beta )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>and use ( \frac{1}{16} ) in place of ( L ) in Equation (4)</td>
<td>N.B. terms in square brackets are used only when positive</td>
</tr>
<tr>
<td>(b) Beams loaded with transverse loads</td>
<td></td>
<td>a = ( \frac{A}{B} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>a = ( A )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>a = ( A, B )</td>
<td></td>
</tr>
<tr>
<td>Load at top flange</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Load at shear centre</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Load at bottom flange</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td></td>
<td>( 1.96 - \frac{4.88}{R^2} - \frac{5.814}{R} )</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 1.643 - \frac{4}{R^2} - \frac{5.563}{R} )</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 1.43 - \frac{4.788}{R^2} - \frac{1.55}{R} )</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 1.2 - \frac{4.106}{R^2} - \frac{1.263}{R} )</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 2.0 - \frac{0.726}{R^2} - \frac{0.955}{R} )</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 1.9 - \frac{1.184}{R^2} + \frac{0.02}{R} )</td>
<td>1</td>
</tr>
<tr>
<td>V</td>
<td></td>
<td>( 2.95 - \frac{11.984}{R^2} - \frac{12.787}{R} )</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 2.093 - \frac{3.934}{R^2} + \frac{9.792}{R} )</td>
<td>1</td>
</tr>
</tbody>
</table>

Four types of supports are described in this table with Roman numerals; these are explained in Figure 103.

Figure 103: Support conditions [6]
Table 2: Values of $c$ for beams with lateral restraints [9]

<table>
<thead>
<tr>
<th>TABLE 2.—Values of $c$ for Beams Loaded by Equal End Moments and Provided With Central Lateral Restraint</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>VALUES OF $c$</strong></td>
</tr>
<tr>
<td>$R^2$</td>
</tr>
<tr>
<td>(1)</td>
</tr>
<tr>
<td>(a) Restraint Attached at Top Flange</td>
</tr>
<tr>
<td>0.0</td>
</tr>
<tr>
<td>0.2</td>
</tr>
<tr>
<td>0.4</td>
</tr>
<tr>
<td>0.6</td>
</tr>
<tr>
<td>0.8</td>
</tr>
<tr>
<td>1.0</td>
</tr>
<tr>
<td>(b) Restraint Attached at Shear Center</td>
</tr>
<tr>
<td>0.0</td>
</tr>
<tr>
<td>0.2</td>
</tr>
<tr>
<td>0.4</td>
</tr>
<tr>
<td>0.6</td>
</tr>
<tr>
<td>0.8</td>
</tr>
<tr>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 3: Values of $c$ for Beams Loaded by Central Load and Provided With Central Lateral Restraint

<table>
<thead>
<tr>
<th>Load Level</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>VALUES OF $c$</strong></td>
</tr>
<tr>
<td>$R^2$</td>
</tr>
<tr>
<td>(1)</td>
</tr>
<tr>
<td>(a) Restraint Attached at Level of Top Flange</td>
</tr>
<tr>
<td>0.0</td>
</tr>
<tr>
<td>0.2</td>
</tr>
<tr>
<td>0.4</td>
</tr>
<tr>
<td>0.6</td>
</tr>
<tr>
<td>0.8</td>
</tr>
<tr>
<td>1.0</td>
</tr>
<tr>
<td>(b) Restraint Attached at Level of Shear Center</td>
</tr>
<tr>
<td>0.0</td>
</tr>
<tr>
<td>0.2</td>
</tr>
<tr>
<td>0.4</td>
</tr>
<tr>
<td>0.6</td>
</tr>
<tr>
<td>0.8</td>
</tr>
<tr>
<td>1.0</td>
</tr>
</tbody>
</table>

---

/ Appendix B
### TABLE 4.—Values of c for Beams Loaded by Uniform Load and Provided with Central Lateral Restraint

<table>
<thead>
<tr>
<th>Load Level</th>
<th>VALUES OF c</th>
<th>Top Flange</th>
<th>Shear Center</th>
<th>Bottom Flange</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>a</td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>R²</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>1.0</td>
<td>1.0 (1)</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>0.2</td>
<td>1.95</td>
<td>1.68 (3)</td>
<td>1.55</td>
<td>1.47</td>
</tr>
<tr>
<td>0.4</td>
<td>2.97</td>
<td>2.36 (6)</td>
<td>2.02</td>
<td>1.86</td>
</tr>
<tr>
<td>0.6</td>
<td>3.89</td>
<td>3.03 (10)</td>
<td>2.48</td>
<td>2.20</td>
</tr>
<tr>
<td>0.8</td>
<td>4.74</td>
<td>3.66 (13)</td>
<td>2.92</td>
<td>2.41</td>
</tr>
<tr>
<td>1.0</td>
<td>5.60</td>
<td>4.26 (16)</td>
<td>3.29</td>
<td>2.61</td>
</tr>
</tbody>
</table>

(a) Restraint Attached at Level of Top Flange

(b) Restraint Attached at Level of Shear Center

See Fig. 3
### Appendix C

Tables used with EN 1993-1-1

*Table 21: Values for $C_1$ and $C_2$ for standard load cases [2]*

<table>
<thead>
<tr>
<th>Gevel</th>
<th>Belasting</th>
<th>$C_1$</th>
<th>$C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$M$</td>
<td>$1.75 - (1.05 \times \beta) + (0.3 \times \beta^2)$</td>
<td>$\beta$ , $1 \leq \beta \leq 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$C_1 &lt; 2.3$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$Q$</td>
<td>1.13</td>
<td>0.45</td>
</tr>
<tr>
<td>3</td>
<td>$F$</td>
<td>1.35</td>
<td>0.55</td>
</tr>
<tr>
<td>4</td>
<td>$F$</td>
<td>1.04</td>
<td>0.42</td>
</tr>
<tr>
<td>5</td>
<td>$M$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>$Q$</td>
<td>1.68</td>
<td>0.78</td>
</tr>
</tbody>
</table>
**Table 22: Values for $C_1$ and $C_2$ for load combination; equally distributed load and end moments[2]**

(3) Voor combinaties van basisbelastingsgevallen zijn de $C_1$- en $C_2$-waarden te ontlenen aan de figuren NB.32 t.m. NB.37 en tabel NB.7.

In de grafieken in figuren NB.33 en NB.34 zijn de positieve richtingen van de momenten $M$ en $M_1$ aangegeven in relatie tot de belasting $q$.

**Figuur NB.32** — Definities van $\beta$ en $B^*$ voor gelijkmatig verdeelde belasting met eindmomenten

**Figuur NB.33** — $C_1$-waarde voor gelijkmatig verdeelde belasting met eindmomenten afhankelijk van $\beta$ en $B^*$
Table 23: Values for \( C_1 \) and \( C_2 \) for load combination; point load and end moments [2]

Fig 34 — \( C_2 \)-waarde voor gelijkmatig verdeelde belasting met eindmomenten afhankelijk van \( \beta \) en \( B^* \)

\[
\beta M = M_1
\]

\( B^* = \frac{4 \times M}{(4 \times |M| + F \times L_{st})} \)

Fig 35 — Definities van \( \beta \) en \( B^* \) voor puntlast met eindmomenten

Fig 36 — \( C_1 \)-waarde voor puntlast met eindmomenten afhankelijk van \( \beta \) en \( B^* \)
Table 24: Values for $C_1$ and $C_2$ for load combination; equally distributed load and point load [2]

De waarden van $C_1$ en $C_2$ gelden alleen indien $F$ en $q$ in dezelfde richting werken.

\[
C_1 = (1.13 \times A^{\text{***}}) + (1.35 \times B^{\text{***}}) \text{ waarin: } A^{\text{***}} = \frac{q \times L_{st}}{(q \times L_{st}) + (2 \times F)}
\]

\[
C_2 = (0.45 \times A^{\text{***}}) + (0.55 \times B^{\text{***}}) \text{ waarin: } B^{\text{***}} = \frac{2 \times F}{(q \times L_{st}) + (2 \times F)}
\]

Table 25: Values for $C_1$ and $C_2$ for fixed beams [2]

<table>
<thead>
<tr>
<th>Belasting</th>
<th>$C_1$</th>
<th>$C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>q</td>
<td>2.30</td>
<td>1.55</td>
</tr>
<tr>
<td>F</td>
<td>1.70</td>
<td>1.42</td>
</tr>
</tbody>
</table>

Opmekking 1
Enige specifieke combinaties van basisbelastingsgevallen vormen de volledige buigvast ingeklemd liggers. In tabel NB.8 zijn twee combinaties gegeven. De waarden van de coefficients $C_1$ en $C_2$ zijn afkomstig uit [9], [18] en [19].

Opmekking 2
Eer willekeurig belastingsgeval is te vervangen door basisbelastingsgevallen of combinaties daarvan, zodanig dat de momentenlijn van de vervangende belasting de omhullende is van de momentenlijn van de oorspronkelijke belasting. De kipsfiselt de ligger met het willekeurige belastingsgeval kan nu worden beoordeeld aan de hand van dezelfde ligger, belast met de vervangende belasting. Hierbij moet rekening zijn gehouden met de buiging uit het vlak van het EF van de ligger veroorzaakt door tweedordworing door de belasting die door imperfecties ziptels verplaatst.
Lateral torsional buckling analysis of multiple laterally restrained I-beams
Appendix D

Tables used with Koleková & Baláž

Table 26: Values for $C_1$ and $C_3$ for beams subjected to pure bending [7]

Tab. 1: Values of factors $C_1$ and $C_3$ corresponding to various end moment ratios $\psi$, values of buckling length factor $k_z$ and cross-section parameters $\eta_f$ and $\kappa_{wz}$.

End moment loading of the simply supported beam with buckling length factors $k_y = 1$ for major axis bending and $k_w = 1$ for torsion

<table>
<thead>
<tr>
<th>Loading and support conditions.</th>
<th>Bending moment diagram.</th>
<th>Values of factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cross-section monosymmetry factor $\eta_f$</td>
<td>End moment ratio $\psi_f$. $M_f$ - side</td>
<td>$k_z$</td>
</tr>
<tr>
<td>$M_{f}^\psi / M_f$</td>
<td>$\psi_f = 0$</td>
<td>1.0</td>
</tr>
<tr>
<td>$0.7L$</td>
<td>$0.7R$</td>
<td>0.7L</td>
</tr>
<tr>
<td>$0.5$</td>
<td>$0.5$</td>
<td>0.500</td>
</tr>
<tr>
<td>$0.7L$</td>
<td>$0.7R$</td>
<td>0.7L</td>
</tr>
<tr>
<td>$0.5$</td>
<td>$0.5$</td>
<td>0.500</td>
</tr>
<tr>
<td>$0.7L$</td>
<td>$0.7R$</td>
<td>0.7L</td>
</tr>
<tr>
<td>$0.5$</td>
<td>$0.5$</td>
<td>0.500</td>
</tr>
<tr>
<td>$0.7L$</td>
<td>$0.7R$</td>
<td>0.7L</td>
</tr>
<tr>
<td>$0.5$</td>
<td>$0.5$</td>
<td>0.500</td>
</tr>
<tr>
<td>$0.7L$</td>
<td>$0.7R$</td>
<td>0.7L</td>
</tr>
<tr>
<td>$0.5$</td>
<td>$0.5$</td>
<td>0.500</td>
</tr>
<tr>
<td>$0.7L$</td>
<td>$0.7R$</td>
<td>0.7L</td>
</tr>
<tr>
<td>$0.5$</td>
<td>$0.5$</td>
<td>0.500</td>
</tr>
<tr>
<td>$0.7L$</td>
<td>$0.7R$</td>
<td>0.7L</td>
</tr>
<tr>
<td>$0.5$</td>
<td>$0.5$</td>
<td>0.500</td>
</tr>
</tbody>
</table>

1) $C_1 = C_{1,0} \cdot (1 - \frac{1}{k_w}) \cdot \kappa_{wz} \leq C_{1,1}$, $(C_1 = C_{1,0}$ for $\kappa_{wz} = 0$, $C_1 = C_{1,1}$ for $\kappa_{wz} \geq 1$)

2) $0.7L = \text{left end fixed, } 0.7R = \text{right end fixed}
### Table 27: Values for $C_1$, $C_2$ and $C_3$ for beams with various load cases [7]

**Table 2: Values of factors $C_1$, $C_2$ and $C_3$ corresponding to various transverse loading cases, values of buckling length factors $k_y$, $k_z$, $k_w$, cross-section monosymmetry factor $\psi_f$ and torsion parameter $\kappa_{wt}$.

<table>
<thead>
<tr>
<th>Loading and support conditions</th>
<th>Buckling length factors</th>
<th>Values of factors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$k_y$</td>
<td>$k_z$</td>
</tr>
<tr>
<td>1 1 1</td>
<td>1.127</td>
<td>1.132</td>
</tr>
<tr>
<td>1 1 0.5</td>
<td>1.128</td>
<td>1.231</td>
</tr>
<tr>
<td>1 0.5 1</td>
<td>0.947</td>
<td>0.977</td>
</tr>
<tr>
<td>1 0.5 0.5</td>
<td>0.947</td>
<td>0.970</td>
</tr>
<tr>
<td>1 1 1</td>
<td>1.348</td>
<td>1.363</td>
</tr>
<tr>
<td>1 1 0.5</td>
<td>1.349</td>
<td>1.452</td>
</tr>
<tr>
<td>1 0.5 1</td>
<td>1.030</td>
<td>1.087</td>
</tr>
<tr>
<td>1 0.5 0.5</td>
<td>1.031</td>
<td>1.067</td>
</tr>
<tr>
<td>1 1 1</td>
<td>1.038</td>
<td>1.040</td>
</tr>
<tr>
<td>1 1 0.5</td>
<td>1.039</td>
<td>1.148</td>
</tr>
<tr>
<td>1 0.5 1</td>
<td>0.922</td>
<td>0.940</td>
</tr>
<tr>
<td>1 0.5 0.5</td>
<td>0.922</td>
<td>0.945</td>
</tr>
</tbody>
</table>

1) $C_1 = C_{1,0} + (C_{1,1} - C_{1,0})\kappa_{wt} \leq C_{1,1}$ ,  ($C_1 = C_{1,0}$ for $\kappa_{wt} = 0$ ,  $C_1 = C_{1,1}$ for $\kappa_{wt} \geq 1$).

2) Parameter $\psi_f$ refers to the middle of the span.

3) Values of critical moments $M_{cr}$ refer to the cross section, where $M_{max}$ is located.
Appendix E

Figure 104: Imperfection shape of the laterally unrestrained beam.

Figure 105: Imperfection shape of the laterally restrained beam with two lateral restraints.
Figure 106: Imperfection shape of the laterally restrained beam with three lateral restraints.

Figure 107: Imperfection shape of the laterally restrained beam with four lateral restraints.

Figure 108: Imperfection shape of the laterally restrained beam with five lateral restraints.
Appendix F

This Python script generates the FEM model for a simply supported IPE 240 beam, subjected to an equally distributed load applied at the centroid of the top flange, with 2 lateral restraints applied at the centroid of the top and bottom flange. This script includes both elastic the model used for the LBA and the non-linear model used for the GMNIA.

```python
# coding: mbcs
from part import *
from material import *
from section import *
from assembly import *
from step import *
from interaction import *
from load import *
from mesh import *
from job import *
from sketch import *
from visualization import *
from connectorBehavior import *

height=240
width=120
tf=9.8
tw=6.2
w=width
h=height-tf

length_part=2000
length_total=length_part*3

MLM=40
MFM=1
MWN=2

### Model 1 (LBA 2 LATERAL RESTRAINTS) ###

### PART ###

mdb.models['Model-1'].ConstrainedSketch(name='__profile__', sheetSize=500.0)
mdb.models['Model-1'].sketches['__profile__'].Line(point1=(0.0, 0.0), point2=(0.0, 0.5*h))
mdb.models['Model-1'].sketches['__profile__'].VerticalConstraint(addUndoState=False, entity=db.models['Model-1'].sketches['__profile__'].geometry[3])

mdb.models['Model-1'].sketches['__profile__'].Line(point1=(-0.5*w, 0.5*h), point2=(0.5*w, 0.5*h))

mdb.models['Model-1'].sketches['__profile__'].HorizontalConstraint(
    addUndoState=False, entity=db.models['Model-1'].sketches['__profile__'].geometry[4])

mdb.models['Model-1'].sketches['__profile__'].Line(point1=(-0.5*w, -0.5*h), point2=(0.5*w, -0.5*h))

mdb.models['Model-1'].sketches['__profile__'].HorizontalConstraint(
    addUndoState=False, entity=db.models['Model-1'].sketches['__profile__'].geometry[5])

del mdb.models['Model-1'].sketches['__profile__']

### MATERIAL ###

mdb.models['Model-1'].Material(name='STEEL')
```
Lateral torsional buckling analysis of multiple laterally restrained I-beams

```python
mdb.models['Model-1'].materials['STEEL'].Elastic(table=((210000.0, 0.3), ))

### SECTION ###

mdb.models['Model-1'].HomogeneousShellSection
integrationRule=SIMPSON, material='STEEL', numIntPts=5,
poissonDefinition=DEFAULT, preIntegrate=OFF, temperature=GRADIENT,
thickness=tf, thicknessField='', thicknessModulus=NONE, thicknessType=UNIFORM, useDensity=OFF) ########CHANGE##########

mdb.models['Model-1'].HomogeneousShellSection
integrationRule=SIMPSON, material='WEB', numIntPts=5,
poissonDefinition=DEFAULT, preIntegrate=OFF, temperature=GRADIENT,
thickness=tw, thicknessField='', thicknessModulus=NONE, thicknessType=UNIFORM, useDensity=OFF) ########CHANGE##########

### ASSIGN SECTION ###

mdb.models['Model-1'].parts['BEAM'].Set(faces=
 mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['#1e'],
 ), name='Flange')

mdb.models['Model-1'].parts['BEAM'].SectionAssignment
(offset=0.0, offsetType=MIDDLE_SURFACE, region=
 mdb.models['Model-1'].parts['BEAM'].sets['Flange'], sectionName='Flange',
thicknessAssignment=FROM_SECTION)

mdb.models['Model-1'].parts['BEAM'].Set(faces=
 mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['#1'],
 ), name='Web')

mdb.models['Model-1'].parts['BEAM'].SectionAssignment
(offset=0.0, offsetType=MIDDLE_SURFACE, region=
 mdb.models['Model-1'].parts['BEAM'].sets['Web'], sectionName='Web',
thicknessAssignment=FROM_SECTION)

### PARTITION FACE ###

mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
 mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['#4'],
 ), point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
 mdb.models['Model-1'].parts['BEAM'].edges[9], MIDDLE), point2=
 mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
 mdb.models['Model-1'].parts['BEAM'].edges[11], MIDDLE))

mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
 mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['#10'],
 ), point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
 mdb.models['Model-1'].parts['BEAM'].edges[15], MIDDLE), point2=
 mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
 mdb.models['Model-1'].parts['BEAM'].edges[17], MIDDLE))

mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
 mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['#4'],
 ), point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
 mdb.models['Model-1'].parts['BEAM'].edges[10], MIDDLE), point2=
 mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
 mdb.models['Model-1'].parts['BEAM'].edges[8], MIDDLE))

mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
 mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['#20'],
 ), point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
 mdb.models['Model-1'].parts['BEAM'].edges[18], MIDDLE), point2=
 mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
 mdb.models['Model-1'].parts['BEAM'].edges[20], MIDDLE))

mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
 mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['#8'],
 ), point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
 mdb.models['Model-1'].parts['BEAM'].edges[14], MIDDLE), point2=
 mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
 mdb.models['Model-1'].parts['BEAM'].edges[12], MIDDLE))

mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
 mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['#10'],
 ), point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
 mdb.models['Model-1'].parts['BEAM'].edges[27], MIDDLE), point2=
 mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
 mdb.models['Model-1'].parts['BEAM'].edges[28], MIDDLE))
```

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mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[16], MIDDLE)
 mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['[#800 ]', ']),
    point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[24], MIDDLE), point2=
    mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[25], MIDDLE)
 mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['[#20 ]', ')),
    point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[22], MIDDLE), point2=
    mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[20], MIDDLE)
 mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['[#800 ]', ')),
    point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[24], MIDDLE), point2=
    mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[23], MIDDLE))
 mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['[#400 ]', ')),
    point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[24], MIDDLE), point2=
    mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[23], MIDDLE))
 mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['[#40000 ]', ')),
    point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[57], MIDDLE), point2=
    mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[58], MIDDLE))
 mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['[#40000 ]', ')),
    point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[59], MIDDLE), point2=
    mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[57], MIDDLE))
 mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['[^10000 ]', ')),
    point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[63], MIDDLE), point2=
    mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[62], MIDDLE))
 mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['[^44 ]', ')),
    point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[10], MIDDLE), point2=
    mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[8], MIDDLE))
 mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['[^20000 ]', ')),
    point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[67], MIDDLE), point2=
    mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[68], MIDDLE))
 mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['[^88 ]', ')),
    point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[14], MIDDLE), point2=
    mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[12], MIDDLE))
 mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['[^20000 ]', ')),
    point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[67], MIDDLE), point2=
    mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[68], MIDDLE))
 mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask(['[^88 ]', ')),
    point1=mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[14], MIDDLE), point2=
    mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[12], MIDDLE))

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mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask((
    '\#10000000'), ), ), point1=
mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[13], MIDDLE), point2=
mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[77], MIDDLE)

mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask('([#10 ]', ),
    ), point1= mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].edges[16], MIDDLE)

mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask('([#80 ]', ),
    ), point1= mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask('([#40 ]', ),
    ), point1= mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask('([#80 ]', ),
    ), point1= mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask('([#80 ]', ),
    ), point1= mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask('([#80 ]', ),
    ), point1= mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask('([#80 ]', ),
    ), point1= mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].vertices[64], point2=
    mdb.models['Model-1'].parts['BEAM'].vertices[55])

mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask('([#2 ]', ),
    ), point1= mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask('([#4 ]', ),
    ), point1= mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask('([#4 ]', ),
    ), point1= mdb.models['Model-1'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-1'].parts['BEAM'].faces.getSequenceFromMask('([#4 ]', ),
    ), point1= mdb.models['Model-1'].parts['BEAM'].InterestingPoint(126
parameter = 0.5

### PARTITION FACE LENGTH ###

```python
mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
  ['#e0000000 #37ffe ', ], ), point1=
mdb.models['Model-1'].parts['BEAM'].vertices[128], point2=
mdb.models['Model-1'].parts['BEAM'].PartitionFaceByShortestPath(faces=
  ['#fff #37ffe00 ', ], ), point1=
mdb.models['Model-1'].parts['BEAM'].vertices[5], point2=
mdb.models['Model-1'].parts['BEAM'].vertices[193])
```

### PARTITION FACE SETS ###

```python
mdb.models['Model-1'].parts['BEAM'].Set(faces=
  ['#0 #1ff8000 #c0000000 ', ], ), name='F01')
```
Lateral torsional buckling analysis of multiple laterally restrained I-beams //

```plaintext
mdb.models['Model-1'].parts['BEAM'].Set(faces=
  [{'#2400240 0:2 #40010 ', }], name='W007')
mdb.models['Model-1'].parts['BEAM'].Set(faces=
  [{'#1800180 0:2 #40010 ', }], name='W008')

### ASSEMBLY ###
mdb.models['Model-1'].parts['BEAM'].Set(faceSet=((
  ['[#0 #400 #0:2 #80000000 
  #84210842 #21084210 #84210842', 
  '#7fffff #0:174 #88880000 #88888888:5 #888888 #0:95 #22000000',
  '#22222222 #22222222 #0:288 #21084000 #42108421 #42108421 #42108421',
  '#42108421 #21084210 #84210842 #42108421 #42108421 #42108421 #42108421',
  '#0:18 #aaaaaaa0 #aaaaaaa0 #2 #aaa70 ', )]),
  setElementType=STANDARD, elemType=ElemType(elemCode=STRI65, elemLibrary=STANDARD),
  region=(
  [ 'Edges-WEB' ], ))
```

---

```plaintext
### MESH ###
```

```plaintext
### LOAD SETS ###
```

```plaintext
### ASSEMBLY ###
```

```plaintext
### ROTATE ASSEMBLY ###
```
mdb.models['Model-1'].rootAssembly.rotate(angle=-90.0, axisDirection=(0.0, 1.0, 0.0), axisPoint=(0.0, 0.0, 0.0), instanceList=('BEAM-1',))

### STEP ###
mdb.models['Model-1'].BuckleStep(blockSize=DEFAULT, eigensolver=LANCZOS, maxBlocks=DEFAULT, minEigen=5, name='LBA', numEigen=1, previous='Initial')

### BOUNDARY CONDITIONS ###
mdb.models['Model-1'].DisplacementBC(amplitude=UNSET, createStepName='Initial', directionType=UNIFORM, fieldName='', localCsys=None, name='BC-1', region=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['BC-1'], u1=SET, u2=SET, u3=SET, url=SET, ur2=UNSET, ur3=UNSET)
mdb.models['Model-1'].DisplacementBC(amplitude=UNSET, createStepName='Initial', directionType=UNIFORM, fieldName='', localCsys=None, name='BC-2', region=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['BC-2'], u1=UNSET, u2=SET, u3=SET, url=SET, ur2=UNSET, ur3=UNSET)

### LATERAL RESTRAINTS ###
mdb.models['Model-1'].DisplacementBC(amplitude=UNSET, createStepName='Initial', directionType=UNIFORM, fieldName='', localCsys=None, name='LATERAL_RESTRAINTS', region=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['LT-TOP'], u1=UNSET, u2=UNSET, u3=SET, url=UNSET, ur2=UNSET, ur3=UNSET)

### CONSTRAINTS KINEMATIC COUPLING ###
mdb.models['Model-1'].Coupling(controlPoint=ur1, surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-NO1'], couplingType=KINEMATIC, influenceRadius=WHALE_SURFACE, localCsys=None, name='KC_01', surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-SO1'], ul=ON, u2=ON, u3=ON, url=ON, ur2=ON, ur3=ON)
mdb.models['Model-1'].Coupling(controlPoint=ur1, surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-NO2'], couplingType=KINEMATIC, influenceRadius=WHALE_SURFACE, localCsys=None, name='KC_02', surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-SO2'], ul=ON, u2=ON, u3=ON, url=ON, ur2=ON, ur3=ON)
mdb.models['Model-1'].Coupling(controlPoint=ur1, surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-NO3'], couplingType=KINEMATIC, influenceRadius=WHALE_SURFACE, localCsys=None, name='KC_03', surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-SO3'], ul=ON, u2=ON, u3=ON, url=ON, ur2=ON, ur3=ON)
mdb.models['Model-1'].Coupling(controlPoint=ur1, surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-NO4'], couplingType=KINEMATIC, influenceRadius=WHALE_SURFACE, localCsys=None, name='KC_04', surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-SO4'], ul=ON, u2=ON, u3=ON, url=ON, ur2=ON, ur3=ON)
mdb.models['Model-1'].Coupling(controlPoint=ur1, surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-NO5'], couplingType=KINEMATIC, influenceRadius=WHALE_SURFACE, localCsys=None, name='KC_05', surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-SO5'], ul=ON, u2=ON, u3=ON, url=ON, ur2=OFF, ur3=ON)
mdb.models['Model-1'].Coupling(controlPoint=ur1, surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-NO6'], couplingType=KINEMATIC, influenceRadius=WHALE_SURFACE, localCsys=None, name='KC_06', surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-SO6'], ul=ON, u2=ON, u3=ON, url=ON, ur2=OFF, ur3=ON)
mdb.models['Model-1'].Coupling(controlPoint=ur1, surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-NO7'], couplingType=KINEMATIC, influenceRadius=WHALE_SURFACE, localCsys=None, name='KC_07', surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-SO7'], ul=ON, u2=ON, u3=ON, url=ON, ur2=OFF, ur3=ON)
mdb.models['Model-1'].Coupling(controlPoint=ur1, surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-NO8'], couplingType=KINEMATIC, influenceRadius=WHALE_SURFACE, localCsys=None, name='KC_08', surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-SO8'], ul=ON, u2=ON, u3=ON, url=ON, ur2=OFF, ur3=ON)
mdb.models['Model-1'].Coupling(controlPoint=ur1, surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-NO9'], couplingType=KINEMATIC, influenceRadius=WHALE_SURFACE, localCsys=None, name='KC_09', surface=mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-SO9'], ul=ON, u2=ON, u3=ON, url=ON, ur2=OFF, ur3=ON)
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name='KC 09', surface=

mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-09'], ul=ON, u2=ON, u3=ON, url=ON, ur=ON, ur3=ON)

mdb.models['Model-1'].Coupling(controlPoint=point1, name='KC_10', surface=

mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-10'], ul=ON, u2=ON, u3=ON, url=ON, ur=ON, ur3=ON)

mdb.models['Model-1'].Coupling(controlPoint=point2, name=' KC_11', surface=

mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-11'], ul=ON, u2=ON, u3=ON, url=ON, ur2=OFF, ur3=ON)

mdb.models['Model-1'].Coupling(controlPoint=point2, name=' KC_12', surface=

mdb.models['Model-1'].rootAssembly.instances['BEAM-1'].sets['KC-12'], ul=ON, u2=ON, u3=ON, url=ON, ur2=OFF, ur3=ON)

### LOADS ###

mdb.models['Model-1'].ConcentratedForce(cf2=1.0, createStepName='LBA', distributionType=UNIFORM, field='', localCsys=None, name='Load-1', region=

mdb.models['Model-1'].ConcentratedForce(cf2=0.5, createStepName='LBA', distributionType=UNIFORM, field='', localCsys=None, name='Load-2', region=

### ADD NODE FILE ###

mdb.models['Model-1'].keywordBlock.synchVersions(storeNodesAndElements=False)

mdb.models['Model-1'].keywordBlock.insert(175, '\n*NODE FILE*\n')

### JOB ###

mdb.Job(atTime=None, contactPrint=OFF, description='', echoPrint=OFF, explicitPrecisions=SINGLE, getMemoryFromAnalysis=True, historyPrint=OFF, memory=90, memoryUnits=PERCENTAGE, model='Model-1', modelPrint=OFF, numCpus=2, numDomains=2, numGPUs=0, queue=None, scratch='', type=ANALYSIS, userSubroutine='', waitHours=0, waitMinutes=0)

### Model-2 (GNIA 2 LATERAL RESTRAINTS IMPL) ###

mdb.Model(modelType=STANDARD_EXPLICIT, name='Model-2')

### PART ###

mdb.models['Model-2'].ConstrainedSketch(name='__profile__', sheetSize=500.0)

mdb.models['Model-2'].sketches['__profile__'].Line(point1=(0.0, 0.0), point2=(0.0, 0.5*h))

mdb.models['Model-2'].sketches['__profile__'].VerticalConstraint(addUndoState=False, entity= mdb.models['Model-2'].sketches['__profile__'], geometry=[])

mdb.models['Model-2'].sketches['__profile__'].Line(point1=(-0.5*w, 0.5*h), point2=(0.5*w, 0.5*h))

mdb.models['Model-2'].sketches['__profile__'].HorizontalConstraint(addUndoState=False, entity= mdb.models['Model-2'].sketches['__profile__'], geometry=[])

mdb.models['Model-2'].sketches['__profile__'].Line(point1=(0.0, 0.0), point2=(0.0, -0.5*h))

mdb.models['Model-2'].sketches['__profile__'].VerticalConstraint(addUndoState=False, entity= mdb.models['Model-2'].sketches['__profile__'], geometry=[])

mdb.models['Model-2'].sketches['__profile__'].ParallelConstraint(addUndoState=False, entity1= mdb.models['Model-2'].sketches['__profile__'], geometry=[2], entity2= mdb.models['Model-2'].sketches['__profile__'], geometry=[4])

mdb.models['Model-2'].sketches['__profile__'].Line(point1=(0.5*w, -0.5*h), point2=(0.5*w, -0.5*h))

mdb.models['Model-2'].sketches['__profile__'].HorizontalConstraint(addUndoState=False, entity= mdb.models['Model-2'].sketches['__profile__'], geometry=[])

mdb.models['Model-2'].sketches['__profile__'].PartDimensionalization(Three_D, name='BEAM', type= DEFORMABLE_BODY)

mdb.models['Model-2'].parts['BEAM'].BaseShellExtrude(depth=length_total, sketch= mdb.models['Model-2'].sketches['__profile__'])

del mdb.models['Model-2'].sketches['__profile__']
### MATERIAL ###

```plaintext
mdb.models['Model-2'].Material(name='STEEL')
```

```plaintext
mdb.models['Model-2'].materials['STEEL'].Elastic(table=((210000.0, 0.3), ))
```

```plaintext
mdb.models['Model-2'].materials['STEEL'].Plastic(table=((235.0, 0.0), ))
```

### SECTION ###

```plaintext
mdb.models['Model-2'].HomogeneousShellSection(idealization=NO_IDEALIZATION, integrationRule=SIMPSON, material='STEEL', name='FLANGE', numIntPts=5, poissonDefinition=DEFAULT, preIntegrate=OFF, temperature=GRADIENT, thicknessAssignment=UNIFORM, useDensity=OFF) ####### CHANGE #######
```

```plaintext
mdb.models['Model-2'].HomogeneousShellSection(idealization=NO_IDEALIZATION, integrationRule=SIMPSON, material='STEEL', name='WEB', numIntPts=5, poissonDefinition=DEFAULT, preIntegrate=OFF, temperature=GRADIENT, thicknessAssignment=UNIFORM, useDensity=OFF) ####### CHANGE #######
```

### ASSIGN SECTION ###

```plaintext
mdb.models['Model-2'].parts['BEAM'].Set(faces= mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask(['#1e'], ), name='Flange')
```

```plaintext
mdb.models['Model-2'].parts['BEAM'].SectionAssignment(offset=0.0, offsetType=MIDDLE_SURFACE, region= mdb.models['Model-2'].parts['BEAM'].sets['Flange'], sectionName='FLANGE', thicknessAssignment=FROM_SECTION)```

```plaintext
mdb.models['Model-2'].parts['BEAM'].Set(faces= mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask(['#1'], ), name='Web')
```

```plaintext
mdb.models['Model-2'].parts['BEAM'].SectionAssignment(offset=0.0, offsetType=MIDDLE_SURFACE, region= mdb.models['Model-2'].parts['BEAM'].sets['Web'], sectionName='WEB', thicknessAssignment=FROM_SECTION)
```

### PARTITION FACE ###

```plaintext
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask(('[#4 ]'), ), point1=mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[9], MIDDLE), point2= mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[11], MIDDLE))
```

```plaintext
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask(('[#4 ]'), ), point1=mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[9], MIDDLE), point2= mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[11], MIDDLE))
```

```plaintext
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask(('[#10 ]'), ), point1=mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[15], MIDDLE), point2= mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[17], MIDDLE))
```

```plaintext
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask(('[#4 ]'), ), point1=mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[10], MIDDLE), point2= mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[8], MIDDLE))
```

```plaintext
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask(('[#20 ]'), ), point1=mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[19], MIDDLE), point2= mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[20], MIDDLE))
```

```plaintext
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask(('[#8 ]'), ), point1=mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[14], MIDDLE), point2= mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[12], MIDDLE))
```

```plaintext
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask(('[#100 ]'), ), point1=mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[21], MIDDLE), point2= mdb.models['Model-2'].parts['BEAM'].InterestingPoint( mdb.models['Model-2'].parts['BEAM'].edges[23], MIDDLE))
```

```plaintext
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask(('[#100 ]'), )
```

```plaintext
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask(('[#100 ]'), )
```
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```
mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask([('#10 ', )],
    ), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[18], MIDDLE), point2= 
    mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[19], MIDDLE)
    mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask([('#80 ', )],
    ), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[20], MIDDLE)
    mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask([('#20 ', )],
    ), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[21], MIDDLE), point2= 
    mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[22], MIDDLE))
    mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask([('#40 ', )],
    ), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[37], MIDDLE), point2= 
    mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[38], MIDDLE)
    mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask([('#80 ', )],
    ), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[26], MIDDLE), point2= 
    mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[27], MIDDLE)
    mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask([('#6000 ', )],
    ), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[53], MIDDLE), point2= 
    mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[54], MIDDLE)
    mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask([('#8000 ', )],
    ), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[59], MIDDLE), point2= 
    mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[60], MIDDLE)
    mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask([('#10000 ', )],
    ), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[63], MIDDLE), point2= 
    mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[64], MIDDLE)
    mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask([('#40 ', )],
    ), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[30], MIDDLE), point2= 
    mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[31], MIDDLE)
    mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask([('#20000 ', )],
    ), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[67], MIDDLE), point2= 
    mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[68], MIDDLE)
```


mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[12], MIDDLE)
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask((
        '#10000000 ', ''), ), ), point1= 
mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[75], MIDDLE), point2= 
mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[77], MIDDLE)
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= 
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('[['#10 1' , ']),
    point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[18], MIDDLE), point2= 
mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[14], MIDDLE)
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= 
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('[['#80 ]', ')),
    point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[24], MIDDLE), point2= 
mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[25], MIDDLE)
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= 
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('[['#20 ]', ')),
    point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[22], MIDDLE), point2= 
mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[20], MIDDLE)
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= 
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('[['#40 ]', ')),
    point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[24], MIDDLE), point2= 
mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[23], MIDDLE)
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= 
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('[['#400 ]', ')),
    point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[34], MIDDLE), point2= 
mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[35], MIDDLE)
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= 
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('[['#80 ]', ')),
    point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[30], MIDDLE), point2= 
mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[28], MIDDLE)
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= 
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('[['#10000000 ]', ')),
    point1= mdb.models['Model-2'].parts['BEAM'].vertices[64], point2= 
mdb.models['Model-2'].parts['BEAM'].vertices[65])
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= 
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('[['#20000000 ]', ')),
    point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[91], MIDDLE), point2= 
mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[93], MIDDLE)
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= 
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('[['#2 '1'], ')),
    point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[61], MIDDLE), point2= 
mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[41], MIDDLE)
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces= 
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('[['#80000000 ]', ')),
    point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[97], MIDDLE), point2= 
mdb.models['Model-2'].parts['BEAM'].InterestingPoint(
    mdb.models['Model-2'].parts['BEAM'].edges[98], MIDDLE)
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```plaintext
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces=
mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('([#4 ])', ),
), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
  mdb.models['Model-2'].parts['BEAM'].edges[10], MIDDLE), point2=  
  mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
    mdb.models['Model-2'].parts['BEAM'].edges[8], MIDDLE)  
), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
  mdb.models['Model-2'].parts['BEAM'].edges[15], MIDDLE))  
), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
  mdb.models['Model-2'].parts['BEAM'].edges[19], MIDDLE), point2=  
  mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
    mdb.models['Model-2'].parts['BEAM'].edges[16], MIDDLE)  
), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
  mdb.models['Model-2'].parts['BEAM'].edges[24], MIDDLE), point2=  
  mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
    mdb.models['Model-2'].parts['BEAM'].edges[25], MIDDLE)  
), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
  mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('([#20 ])', ),
), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
  mdb.models['Model-2'].parts['BEAM'].edges[22], MIDDLE), point2=  
  mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
    mdb.models['Model-2'].parts['BEAM'].edges[20], MIDDLE)  
), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
  mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('([#40 ])', ),
), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
  mdb.models['Model-2'].parts['BEAM'].edges[26], MIDDLE), point2=  
  mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
    mdb.models['Model-2'].parts['BEAM'].edges[24], MIDDLE)  
), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
  mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('([#80 ])', ),
), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
  mdb.models['Model-2'].parts['BEAM'].edges[34], MIDDLE), point2=  
  mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
    mdb.models['Model-2'].parts['BEAM'].edges[35], MIDDLE)  
), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
  mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask('([#400 ])', ),
), point1= mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
  mdb.models['Model-2'].parts['BEAM'].edges[30], MIDDLE), point2=  
  mdb.models['Model-2'].parts['BEAM'].InterestingPoint(  
    mdb.models['Model-2'].parts['BEAM'].edges[28], MIDDLE)  
)
```

### BOUNDARY CONDITION SETS ###

```plaintext
mdb.models['Model-2'].parts['BEAM'].Set(name='BC-1', vertices=
  mdb.models['Model-2'].parts['BEAM'].vertices.getSequenceFromMask('  
  [#80000 ]', ),
),
)  
```

### KINEMATIC COUPLING SETS ###

```plaintext
mdb.models['Model-2'].parts['BEAM'].Set(name='KC-N01', vertices=
)```
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
mdb.models['Model-2'].parts['BEAM'].set(edges=getSequenceFromMask(
'[^0:2 *20000 ]', ),)
Lateral torsional buckling analysis of multiple laterally restrained I-beams

```python
mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask({
        '#e000000 #ffffff', }, ),
    point1=(115, 17, 5),
    point2=(120, 120, 120),
    )

mdb.models['Model-2'].parts['BEAM'].PartitionFaceByShortestPath(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask({
        '#ffffff #000000', },
    point1=(115, 17, 5),
    point2=(120, 120, 120),
    )
```

### PARTITION FACE SETS ###

```python
mdb.models['Model-2'].parts['BEAM'].Set(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask({
        '#0 #ffffff #0000 00', },
    name='W01')
```

```python
mdb.models['Model-2'].parts['BEAM'].Set(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask({
        '#00000000 00000000  #ffffff', },
    name='W02')
```

```python
mdb.models['Model-2'].parts['BEAM'].Set(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask({
        '#0 #ffffff #000000', },
    name='W03')
```

```python
mdb.models['Model-2'].parts['BEAM'].Set(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask({
        '#00000000 00000000  #ffffff', },
    name='W04')
```

```python
mdb.models['Model-2'].parts['BEAM'].Set(faces=
    mdb.models['Model-2'].parts['BEAM'].faces.getSequenceFromMask({
        '#00000000 00000000  #ffffff', },
    name='W05')
```
mdb.models['Model-2'].parts['BEAM'].Set{faces=
    '[(4200240 @0:2 @408 [)', '], }, name='W07'}
mdb.models['Model-2'].parts['BEAM'].Set{faces=
    '[(1800180 @0:2 @4010 [)', '], }, name='W08'}

### LATERAL RESTRAINTS SETS ###
mdb.models['Model-2'].parts['BEAM'].Set(name='LT-TOP', vertices=
    mdb.models['Model-2'].parts['BEAM'].vertices.getSequenceFromMask((['#9 ', '],)),
    mdb.models['Model-2'].parts['BEAM'].Set{edges=
        '[(1800180 @0:2 @4010 [)', '], }, name='W08'}

### MESH ###
mdb.models['Model-2'].parts['BEAM'].seedEdgeByNumber(constraint='FINER', edges=
    mdb.models['Model-2'].parts['BEAM'].edges.getSequenceFromMask()
    '[(0:2 b6a80000 b6b66f66 #556a5a5 #d6d6d59 #556b66d6 #255555',  
     '49200000 #4920e49 #5afe41 ]'),}, number=68M)
mdb.models['Model-2'].parts['BEAM'].seedEdgeByNumber(constraint='FINE', edges=
    mdb.models['Model-2'].parts['BEAM'].edges.getSequenceFromMask()
    '[(0:2 b6a80000 b6b66f66 #556a5a5 #d6d6d59 #556b66d6 #255555',  
     '49200000 #4920e49 #5afe41 ]'),}, number=68M)

### LOAD SETS ###
mdb.models['Model-2'].parts['BEAM'].Set(name='Q-LOAD-T1', nodes=
    mdb.models['Model-2'].nodes.getSequenceFromMask(mask=('
    '][9 @20000 @0:2 @90000000 #ffffff',  
    'fff @0:46 fffe0000 #fffe #7 #f0:89 #fffe0000',  
    'fffe #1 @0:155 #4000000 #21084241 #10842108 #8421084',  
    '21084210 #8421084 #21084241 #10842108 #8421084',  
    '21084210 #8421084 #21084241 #10842108 #8421084',  
    '0:188 #aaaaa000 #aaaaaaa2 #aaaa ]'),),
    mdb.models['Model-2'].parts['BEAM'].Set(name='Q-LOAD-T2', vertices=
    mdb.models['Model-2'].parts['BEAM'].vertices.getSequenceFromMask((
    ']) @20000 @0:2 @90000000 #ffffff',  
    'fff @0:46 fffe0000 #fffe #7 #f0:89 #fffe0000',  
    'fffe #1 @0:155 #4000000 #21084241 #10842108 #8421084',  
    '21084210 #8421084 #21084241 #10842108 #8421084',  
    '21084210 #8421084 #21084241 #10842108 #8421084',  
    '0:188 #aaaaa000 #aaaaaaa2 #aaaa ]', ),
    mdb.models['Model-2'].parts['BEAM'].Set(name='Q-LOAD-M1', nodes=
    mdb.models['Model-2'].nodes.getSequenceFromMask(mask=('
    '][30000 #400 @0:2 @800000000 #13 @f80000000 #fffe',  
    'fff #0:24 fffe0000 #fffe #1f #0:128 #fffe',  
    'fffe #174 #88888888 #fffe #88888888 #88888888 #0:05 #22200000',  
    'f22222222 #22222222 #0:288 #20840000 #842084 #2108421 #10842108',  
    '#8421084 #21084210 #8421084 #21084241 #8 ]'),),
    mdb.models['Model-2'].parts['BEAM'].Set(name='Q-LOAD-M2', vertices=
    mdb.models['Model-2'].parts['BEAM'].vertices.getSequenceFromMask((
    ']) @0 @0:2 @800000000 ],)

### ASSEMBLY ###
mdb.models['Model-2'].rootAssembly.DatumCsysByDefault(CARTESIAN)
mdb.models['Model-2'].rootAssembly.Instance(dependent=ON, name='BEAM-1', part= 139
Lateral torsional buckling analysis of multiple laterally restrained I-beams

```plaintext
mdb.models['Model-2'].parts['BEAM']

### ROTATE ASSEMBLY ###
mdb.models['Model-2'].rootAssembly.rotate(axisPoint=(0.0, 0.0, 0.0), angle=90.0, axisDirection=(0.0, 1.0, 0.0), instanceList=('BEAM-1',))

### STEP ###
mdb.models['Model-2'].StaticRiksStep(initialArcInc=0.01, minArcInc=0.05, maxNumInc=100, minArcInc=1e-18, name='Riks', nlgeom=ON, previous='Initial')

### BOUNDARY CONDITIONS ###
mdb.models['Model-2'].DisplacementBC(amplitude=UNSET, createStepName='Initial', distributionType=UNIFORM, fieldName='', localCsys=None, name='BC-1', region=mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['BC-1'], u3=SET, u2=OFF, instanceList=('BEAM-1-1',), ur1=SET, ur2=SET, ur3=SET, ur4=SET, ur5=SET, ur6=SET)

### LATERAL RESTRAINTS ###
mdb.models['Model-2'].DisplacementBC(amplitude=UNSET, createStepName='Initial', distributionType=UNIFORM, fieldName='', localCsys=None, name='LATERAL_RESTRAINTS', region=mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['LT-Top'], u1=UNSET, u2=UNSET, u3=SET, url=UNSET, ur2=UNSET, ur3=UNSET)

### CONSTRAINTS KINEMATIC COUPLING ###
```
```
mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['KC-N09'],
couplingType=KINEMATIC, influenceRadius=WHOLE_SURFACE, localCsys=None,
name='KC_09', surface=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['KC-S09'], ul=
ON, u2=ON, ur2=ON, ur3=ON, url=ON, ur2=ON, ur3=ON)

mdb.models['Model-2'].Coupling(controlPoint=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['KC-N10'],
couplingType=KINEMATIC, influenceRadius=WHOLE_SURFACE, localCsys=None,
name='KC_10', surface=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['KC-S10'], ul=
ON, u2=ON, ur2=ON, ur3=ON, url=ON, ur2=ON, ur3=ON)

mdb.models['Model-2'].Coupling(controlPoint=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['KC-N11'],
couplingType=KINEMATIC, influenceRadius=WHOLE_SURFACE, localCsys=None,
name='KC_11', surface=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['KC-S11'], ul=
ON, u2=ON, ur2=ON, ur3=OFF, ur3=ON)

mdb.models['Model-2'].Coupling(controlPoint=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['KC-N12'],
couplingType=KINEMATIC, influenceRadius=WHOLE_SURFACE, localCsys=None,
name='KC_12', surface=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['KC-S12'], ul=
ON, u2=ON, ur2=ON, url=ON, ur2=OFF, ur3=ON)

### LOADS ###

mdb.models['Model-2'].ConcentratedForce(cf2=1000, createStepName='RIKS',
distributionType=UNIFORM, field='', localCsys=None, name='Load-1', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['Q-LOAD-T1'])

mdb.models['Model-2'].ConcentratedForce(cf2=1000, createStepName='RIKS',
distributionType=UNIFORM, field='', localCsys=None, name='Load-2', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['Q-LOAD-T2'])

### PREDEFINED FIELD ###

mdb.models['Model-2'].Stress(distributionType=UNIFORM, name='F01', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['F01'],
sigma11=70.5, sigma12=0.0, sigma13=None, sigma22=0.0, sigma23=None,
sigma33=None)

mdb.models['Model-2'].Stress(distributionType=UNIFORM, name='F02', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['F02'],
sigma11=35.25, sigma12=0.0, sigma13=None, sigma22=0.0, sigma23=None,
sigma33=None)

mdb.models['Model-2'].Stress(distributionType=UNIFORM, name='F03', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['F03'],
sigma11=35.25, sigma22=0.0, sigma23=None, sigma33=None)

mdb.models['Model-2'].Stress(distributionType=UNIFORM, name='F04', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['F04'],
sigma11=0.0, sigma22=0.0, sigma33=None, sigma23=None)

mdb.models['Model-2'].Stress(distributionType=UNIFORM, name='F05', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['F05'],
sigma11=0.0, sigma22=0.0, sigma33=None, sigma23=None)

mdb.models['Model-2'].Stress(distributionType=UNIFORM, name='F06', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['F06'],
sigma11=35.25, sigma12=0.0, sigma13=None, sigma22=0.0, sigma23=None,
sigma33=None)

mdb.models['Model-2'].Stress(distributionType=UNIFORM, name='F07', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['F07'],
sigma11=35.25, sigma12=0.0, sigma13=None, sigma22=0.0, sigma23=None,
sigma33=None)

mdb.models['Model-2'].Stress(distributionType=UNIFORM, name='F08', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['F08'],
sigma11=70.5, sigma12=0.0, sigma13=None, sigma22=0.0, sigma23=None,
sigma33=None)

mdb.models['Model-2'].Stress(distributionType=UNIFORM, name='F09', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['F09'],
sigma11=70.5, sigma12=0.0, sigma13=None, sigma22=0.0, sigma23=None,
sigma33=None)

mdb.models['Model-2'].Stress(distributionType=UNIFORM, name='F10', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['F10'],
sigma11=70.5, sigma12=0.0, sigma13=None, sigma22=0.0, sigma23=None,
sigma33=None)

mdb.models['Model-2'].Stress(distributionType=UNIFORM, name='F11', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['F11'],
sigma11=70.5, sigma12=0.0, sigma13=None, sigma22=0.0, sigma23=None,
sigma33=None)

mdb.models['Model-2'].Stress(distributionType=UNIFORM, name='F12', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['F12'],
sigma11=35.25, sigma12=0.0, sigma13=None, sigma22=0.0, sigma23=None,
sigma33=None)

mdb.models['Model-2'].Stress(distributionType=UNIFORM, name='F13', region=

mdb.models['Model-2'].rootAssembly.instances['BEAM-1'].sets['F13'],
sigma11=35.25, sigma12=0.0, sigma13=None, sigma22=0.0, sigma23=None,
sigma33=None)
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