MASTER

Relaxation scheme for macroscopic traffic flow models

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Relaxation Scheme For Macroscopic Traffic Flow Models

by

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Abstract

Effective traffic flow models are very crucial in the study and management of real life traffic situations. In this thesis we present a study of the macroscopic type traffic flow models. Different traffic flow models including single and two equation systems have been studied. The relaxation systems for one dimensional conservation laws is presented. With these relaxation systems we present the derivation of the relaxation schemes for conservation law problem. The relaxation schemes includes the Upwind scheme and the Monotone Upwind Scheme for Conservation Laws (MUSCL). These schemes are implemented along side the TVD Runge-Kutta time discretization scheme to solve numerically, Riemann problem of some of the traffic flow models.
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Dedication

This work is respectfully dedicated
to
My Parents
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Chapter 1

Introduction

In recent times, there has been a growing interest in the study of traffic flow models describing different features and dynamics in vehicular traffic flow. Traffic flow modeling and optimization have traditionally been studied under three main approaches; the Microscopic, Kinetic and Macroscopic models.

The first and most basic of the models, the microscopic or car following model, models the actual response of individual vehicles to their predecessor by ordinary differential equations based on Newton’s law. These models have been investigated by soo many authors ([27], [28], [29]).

Kinetic models present an intermediate step between the other two types of traffic flow models mentioned above. They are based on Boltzmann type kinetic equations. On the one hand they can be more fundamentally justified than the standard macroscopic models, leading to a better justification of the macroscopic models and potentially to more accurate results. On the other side, compared to microscopic models, computational time is strongly decreased [4]. This may make the kinetic models applicable to the description of real life situations and traffic control problems.

Prigogine et. al. with ([30], [31], [32], [33]) originally started the modeling of traffic flow with Kinetic models. In their works they introduced a kinetic term to account for the slowing down interactions [4].

The macroscopic models after its introduction in the 1950th by the work of Ligthill and Whitham [13] have seen an extensive attention over the years. These models describe traffic flow by the use of the relationship between the averaged quantities like density and speed. Most of the proposed models ([22], [14], [15]) are based on fluid dynamic equations. After the earlier works by Ligthill and Whitham [13], Newell [New61] extended this model to certain
non-equilibrium cases, and Payne [21] and Whitham [22] developed a second-order system of equations also derived from the equations of gas dynamics which is capable of describing types of non-equilibrium flow. There are other models dealing with individual cars [26] or probability functions ([2], [3]). It is however worth mentioning that some of these macroscopic models including the those by Payne [21] and Whitham [22] have been subjected to considerable controversy and critics concerning their validity and applicability to traffic flow [23]. This led to the development of the models by Aw and Rascle [12] as well as that by Zhang [35].

1.1 Content of thesis

In chapter 2, we present an overview of one dimensional macroscopic traffic flow models. We also discuss in this same chapter some of the solution types opened to some of the models depending on the Riemann data available. We devote the chapter 3 to discussing what relaxation system is. In the chapter 3 we present as well the relaxation schemes, the first order upwind scheme and the Monotone Upwind Scheme for Conservation Laws (MUSCL) due Jin and Xin [5]. For time discretisation, the second order TVD Runge-Kutta splitting scheme introduced by Jin [5] is used and the details of it is also presented in chapter 3.

In chapter 4, we present the numerical solution of the one-dimensional traffic flow models, Lighthill-Whitham-Richards (LWR), Aw-Rascle (AR) and Aw-Rascle-Zhang (ARZ) Models by implementing the relaxation schemes presented in chapter 3 in Matlab. Errors for using the MUSCL and the Upwind schemes are computed for comparing the two schemes. Finally, in chapter 5, we present a brief discussion of the simulations and make some recommendations for future works.
Chapter 2

One Dimensional Traffic Flow Models

We devote this chapter to traffic flow models. As noted in the previous chapter there are three main groups of traffic flow models. In this work we shall consider the macroscopic traffic flow models. In the macroscopic models we can also look at the single equation models and the two(multi)-equation models.

The rest of the chapter begins with the single equation models, followed by the two-equation models. Some of the specific models we shall discussed includes, the Lighthill-Whitham-Richards (LWR), the Aw-Rascle, the Aw-Rascle-Zhang as well as an Aw-Rascle type model formulated by Borsche, Kimathy and Klar.

2.1 Macroscopic Equations

In this section we describe some of the macroscopic traffic flow models which form the main focus of this work. The models include both single and two (multi)-equation models.

2.2 Single equation Models

Here we discuss some single equation models. These models are made of single equations and form the basis of many other models.

2.2.1 Lighthill-Whitham-Richards (LWR) Model

The LWR model formulated by Lighthill,Whitham and Richards in [13] and [14] is a macroscopic model which forms part or forms the base for many other traffic flow models. The
model is given by;

\[ \partial_t \rho + \partial_x (\rho V(\rho)) = 0, \quad V(\rho) = u_{\text{max}} \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right) \quad 0 \leq \rho \leq \rho_{\text{max}} \] (2.1)

where \( u_{\text{max}} \) and \( \rho_{\text{max}} \) are the maximum velocity and density respectively. Maximum velocity is the velocity of traffic when the density is zero and the maximum density is the density at which there is a traffic jam or the speed is equal to zero.

By using the transformation \( u = v_{\text{max}}(1 - 2\rho/\rho_{\text{max}}) \), equation (2.1) can be simplified to the Burger’s equation;

\[ \partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0 \] (2.2)

This simplification makes it easy to analysis the solution types of the model.

The flux \((f = \rho V(\rho))\) of the model is concave and is shown in Figure 2.1 whiles that of the burgers equation is convex.

![Figure 2.1: (a) Velocity function and (b) Flux function of the LWR model](image)

**2.2.2 Elementary Waves of LWR Model**

Here we discuss the various solution types of the LWR model. With a given initial data;

\[ \rho_0(x) = \begin{cases} 
\rho_l & \text{for } x < 0 \\
\rho_r & \text{for } x > 0 
\end{cases} \] (2.3)

The Riemann problem of equation (2.1) and 2.3 with the concave flux function, there are two main solution types. These are the shock and the rarefaction waves depending on the riemann data.
The Shock wave
The shock wave of the problem (2.1) and (2.3) is of the form

\[ \rho(x, t) = \begin{cases} 
\rho_l & \text{for } x < st \\
\rho_r & \text{for } x > st 
\end{cases} \]  

(2.4)

where the shock speed \( s \) satisfies the Rankine-Hugoniot condition and is given by:

\[ s = \frac{u_l + u_r}{2} = v_{\text{max}} \left( 1 - \frac{\rho_l + \rho_r}{\rho_{\text{max}}} \right) \]

This shock occurs when the following condition is satisfied

\[ 0 < \rho_l < \rho_r < \rho_{\text{max}} \]

This is equivalent to when \( u_l > u_r \) for the case of using the burgers equation.

The Rarefaction wave
When the condition

\[ 0 < \rho_r < \rho_l < \rho_{\text{max}} \]

or equivalently \( u_l > u_r \) for using the burgers equation is satisfied, the riemann problem (2.1) and (2.3) produces the rarefaction wave.

\[ \rho(x, t) = \begin{cases} 
\rho_l & \text{for } x < f'(\rho_l)t \\
\rho(\xi) & \text{for } x \in [f'(\rho_l)t, f'(\rho_r)t] \\
\rho_r & \text{for } x > f'(\rho_l)t 
\end{cases} \] 

(2.5)

where the \( \rho(\xi) \) satisfies for \( \xi := \frac{\xi}{t} \) the condition

\[ f'(\rho(\xi)) = \xi \]

In the next chapter we present numerical simulation of different other scenarios of data that may produce solution involving different forms of these wave types.

Many other researchers have proposed different models of this kind by giving a different velocity function which achieves the convexity of the flow flux. Some of these includes;
2.2.3 Greenberg Model

Greenberg ([15]) proposed a similar model with a different velocity function in which velocity is set to be inversely related to density. The model assumes that velocity of the flow can be very large for a low density.

\[ \partial_t \rho + \partial_x (\rho V(\rho)) = 0, \quad V(\rho) = u_{\text{max}} \ln \left( \frac{\rho_{\text{max}}}{\rho} \right) \quad 0 \leq \rho \leq \rho_{\text{max}} \]  

(2.6)

The velocity and flux are shown in Figure 2.2

![Figure 2.2: (a) Velocity function and (b) Flux function of the Greenberg model](image)

2.2.4 Underwood Model

The Underwood model [16] is given by;

\[ \partial_t \rho + \partial_x (\rho V(\rho)) = 0, \quad V(\rho) = u_{\text{max}} \exp \left( \frac{-\rho}{\rho_{\text{max}}} \right) \quad 0 \leq \rho \leq \rho_{\text{max}} \]  

(2.7)

The velocity and flux are shown in Figure 2.3

There are several other complicated velocity functions in literature which satisfy the convexity requirement of the flux. Kühne and Rödiger [17] proposed one such expressions for velocity as;

\[ V(\rho) = u_{\text{max}} \left( 1 - \left( \frac{\rho}{\rho_{\text{max}}} \right)^{n_1} \right)^{n_2} \quad 0 \leq \rho \leq \rho_{\text{max}} \]

where \( n_1 \) and \( n_2 \) are fitted to spacial traffic flow situation. Other models includes the Drew
Figure 2.3: (a) Velocity function and (b) Flux function of the Underwood model

model [19] given by

\[ V(\rho) = u_{\text{max}} \left( 1 - \left( \frac{\rho}{\rho_{\text{max}}} \right)^{(n+1)/2} \right) \]

and Pipes-Munjal Models [20] given by

\[ V(\rho) = u_{\text{max}} \left( 1 - \left( \frac{\rho}{\rho_{\text{max}}} \right)^n \right) \]

2.3 Two equation Models

In this section we discuss some of the traffic flow models which are made of two different equations.

2.3.1 Payne-Whitham (P-W) Model

The Payne-Whitham model formulated by Payne [21] and Whitham [22] is a two equation model which is based on the properties of the flow of gas particles.

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2 + p(\rho)) &= 0
\end{align*}
\]

(2.8)

where \( p(\rho) \) is the pressure of the flow. This model has been criticized by Daganzo [23] as failing in some properties of traffic flows.
2.3.2 Daganzo’s Criticism of P-W model

Daganzo in [23] realized that there are differences in the properties of car traffic and that of fluid flows. In particular, there is no conservation of momentum in the traffic flow. The following were outlined in [23] as reasons why P-W model is not a good representation of traffic flow;

- A fluid particle responds to stimuli from the front and from behind, but a car is an anisotropic particle that mostly responds to frontal stimuli
- The width of a traffic shock only encompasses a few vehicles, and
- Unlike molecules, vehicles have personalities (e.g., aggressive and timid) that remain unchanged by motion

It was further asserted that these failures is highly undesirable because it means that the future conditions of a traffic element are, in part, determined by what is happening behind [23].

2.3.3 Aw-Rascle Model

Aw and Rascle [12] proposed a new two equation traffic flow model that seeks to improve the P-W models and addresses the concerns raised by Daganzo [23].

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0, \\
\frac{\partial}{\partial t}(u + p(\rho)) + u \frac{\partial}{\partial x}(u + p(\rho)) &= 0
\end{align*}
\] (2.9)

With a suitable choice of pressure function \( p(\rho) \), the model satisfies the following properties [12]:

A. The system must be hyperbolic

B. When solving the Riemann Problem with arbitrary bounded nonnegative Riemann data \((\rho, u)\) in a suitable region \( \mathcal{R} \) of the plane, the density and the velocity must remain nonnegative and bounded from above.

C. In solving the same Riemann Problem with arbitrary data \( U_\pm := (\rho_\pm, u_\pm) \) all waves connecting any state \( U := (\rho, u) \) to its left (i.e. behind it) must have a propagation speed (eigenvalue or shock speed) at most equal to the velocity \( u \).
D. The solution to the Riemann problem must agree with the qualitative properties that each driver practically observes every day. In particular, braking produces shock waves whose propagation speed can be either negative or nonnegative, whereas accelerating produces rarefaction waves which in any case satisfy Principle C.

E. Near by the vacuum, the solution to the Riemann Problem must be very sensitive to the data. In other words, there must be no continuous dependence with respect to the initial data at $\rho = 0$

The property C, addresses the first point of Daganzo’s criticism [23]: a car traveling at a velocity $u$ receives no information from the rear.

The model (2.9) in conservative form, is given as:

$$
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho(u+p(\rho))) + \partial_x (\rho u(u+p(\rho))) &= 0
\end{align*}
$$

(2.10)

where the conservative variables are $\rho$ and $y := \rho(u+p(\rho))$. Even though $(\rho, y)$ is the simplest and most natural pair of conservative variables, there is no obvious physical interpretation of $y$ (the "momentum").

Now we present some of the basic properties of the model. Setting $U := (\rho, u)$, $Y := (\rho, \rho(u+p(\rho)))$ and $F := (\rho u, \rho u(u+p(\rho)))$

$$
DY(\rho, u) = \begin{pmatrix}
1 & 0 \\
(u+p'(\rho)+\rho p'(\rho)) & \rho
\end{pmatrix},
$$

$$
DF(\rho, u) = \begin{pmatrix}
u & \rho \\
(u+u'(\rho)+\rho p'(\rho)) & \rho(2u+p(\rho))
\end{pmatrix}.
$$

(2.11)

Since the determinant of $DY$ exist, $\rho > 0$, there exist $(DY)^{-1}$ and is given by:

$$
DY^{-1}(\rho, u) = \begin{pmatrix}
1 & 0 \\
-\frac{1}{\rho}(u+p'(\rho)+\rho p'(\rho)) & \frac{1}{\rho}
\end{pmatrix}.
$$

(2.12)
hence the system \([2.10]\) can be written in vector form as;

\[
DY \partial_t U + DF \partial_x U = 0 \tag{2.13a}
\]

\[
\Rightarrow \partial_t U + (DY)^{-1} DF \partial_x U = 0 \tag{2.13b}
\]

\[
\partial_t U + A(U) \partial_x U = 0 \tag{2.13c}
\]

where the jacobian of the system \([2.9]\):

\[
A(U) = (DY)^{-1} DF(\rho, u) = \begin{pmatrix}
    u & \rho \\
    0 & u - \rho p'(\rho)
\end{pmatrix}. \tag{2.14}
\]

If \(\lambda\) is an eigenvalue of the system (or \(A(U)\)) then it satisfies;

\[
|A - \lambda I| = 0
\]

which implies the eigen values of the system satisfy;

\[
(u - \lambda)(u - \rho p'(\rho) - \lambda) = 0
\]

and therefore we get;

\[
\lambda_1 = u - \rho p'(\rho) \quad \text{and} \quad \lambda_2 = u \tag{2.15}
\]

The right eigenvectors of \(\lambda_1\) and \(\lambda_2\) are respectively;

\[
r^1 = \begin{pmatrix}
    1 \\
    -p'(\rho)
\end{pmatrix}, \quad r^2 = \begin{pmatrix}
    1 \\
    0
\end{pmatrix} \tag{2.16}
\]

This shows that the system \([2.9]\) is strictly hyperbolic except for \(\rho = 0\) \([12]\) where the two eigenvalues coalesce.

### 2.3.4 Elementary Waves of AR Model

Aw and Rascle in \([12]\) assumed for analysis of the model the following;

\[
p(\rho) \sim \rho^\gamma, \quad \text{near by} \quad \rho = 0, \gamma > 0 \quad \text{and} \quad \forall \rho, \quad pp''(\rho) + 2p'(\rho) > 0. \tag{2.17}
\]

An eigenvalue \(\lambda_k\) is genuinely nonlinear \([24]\) if the function \(\nabla \lambda_k(U) \cdot r^k(U)\) never vanishes and linearly degenerate if it vanishes for all \(U\), where \(\nabla\) is the gradient operator. The
waves associated with linearly degenerate eigenvalues is contact discontinuities and that of genuinely nonlinear is either a rarefaction or a shock wave depending on the data. Under the assumptions \((2.17)\), \(\lambda_1\) is genuinely nonlinear and hence admits either a shock wave or rarefaction \([12]\). On the other hand the \(\lambda_2\) is is linearly degenerate and therefore admit a contact discontinuity\([12]\).

With the assumption \((2.17)\), another feature of the model is;

\[
\lambda_1 \leq \lambda_2 = u. \tag{2.18}
\]

which means all the waves propagates at a speed at most equal to the velocity \(u\) which satisfy the property C.

The scalar function \(w\) of \(U := (\rho, u)\) is a \(k\)-Riemann Invariant \((k\text{-RI Lax})\) if \(\nabla w \cdot r^k \equiv 0\) \([24]\). The Riemann invariants in the sense of Lax, associated with \(\lambda_1\) \((1\text{-RI Lax})\) and \(\lambda_2\) \((2\text{-RI Lax})\) are respectively

\[
w(U) = u + p(\rho) \quad \text{and} \quad z(U) = u. \tag{2.19}
\]

Therefore, depending of the data, the waves of the first family is either rarefaction or shock and that of the second family is a contact discontinuity. As will be seen in section 4, depending on the data, the AR-model may have solution of the following kinds in space

- A 1-shock wave followed by a 2-contact discontinuity
- A 1-rarefaction wave followed by a 2-contact discontinuity
- A 1-rarefaction wave followed by a Vacuum and then a 2-contact discontinuity
- Isolated 1-rarefaction wave
- Isolated 2-contact discontinuity

To discuss all the various wave forms of the model we solve a Riemann problem with initial data

\[
u_o(x) = \begin{cases} u_l & \text{for } x < 0 \\ u_r & \text{for } x > 0 \end{cases} \tag{2.20}
\]

where \(u_l = (\rho_l, u_l), u_r = (\rho_r, u_r) \in \{(\rho, u) : \rho > 0, u > 0\}\) are constant vectors.
The 1-Rarefaction wave
A 1-rarefaction wave of the problem (2.10) with initial data (2.20) connecting $u_l$ and $u_r$ is a continuous solution of the form;

$$u(x,t) = \begin{cases} u_l & \text{for } \frac{x}{t} < \lambda_1(u_l) \\ \bar{u}(\frac{x}{t}) & \text{for } \frac{x}{t} \in [\lambda_1(u_l), \lambda_1(u_r)] \\ u_r & \text{for } \frac{x}{t} > \lambda_1(u_r) \end{cases}$$

(2.21)

where $t > 0$ and $\bar{u}(\frac{x}{t})$ satisfies;

$$\bar{u} = \frac{1}{(\nabla \lambda_1(\bar{u}))^r_1(\bar{u})}, \quad \frac{x}{t} = \lambda_1(\bar{u})$$

$$\bar{u}(\lambda_1(u_l)) = u_l, \quad \bar{u}(\lambda_1(u_r)) = u_r$$

(2.22)

Setting $\xi = \frac{x}{t}$, then $(\rho(\xi), u(\xi))$ satisfies the ODE

$$\xi Y_\xi + F(U)_\xi = 0$$

or

$$(-\lambda_1 DY + DF) \begin{pmatrix} \rho_\xi \\ u_\xi \end{pmatrix} = 0$$

where $DY$ and $DF$ are as defined in (2.11)

With $\lambda_1$, the eigenvector $(\rho_\xi, u_\xi)$ satisfies

$$(-\lambda_1 DY + DF) \begin{pmatrix} \rho_\xi \\ u_\xi \end{pmatrix} = 0$$

(2.23)

From (2.11) we have

$$(-\lambda_1 DY + DF) = -(u - \rho p'(\rho)) \begin{pmatrix} 1 \\ (u + p'(\rho) + \rho p'(\rho)) \end{pmatrix}.$$ 

$$+ \begin{pmatrix} u \\ (u + p'(\rho) + \rho p'(\rho)) \end{pmatrix}.$$ 

(2.24)

= \begin{pmatrix} \rho p'(\rho) \\ \rho p'(\rho)(u + p(\rho) + \rho p'(\rho)) \end{pmatrix}.$$ 

$$\rho(u + p(\rho) + \rho p'(\rho))$$

hence, using (2.23) and (2.24) gives

$$\begin{pmatrix} \rho p'(\rho) \\ \rho p'(\rho)(u + p(\rho) + \rho p'(\rho)) \end{pmatrix} \begin{pmatrix} \rho_\xi \\ u_\xi \end{pmatrix} = 0$$

(2.25)
which implies
\[ \rho p'(\rho)\rho + \rho u\xi = 0 \]
\[ \Rightarrow \frac{dp}{u} = -\frac{1}{p'(\rho)} \]
which upon integration give the 1-rarefaction wave as
\[ R1 : u - u_t = p(\rho) - p(\rho_l) \quad (2.26) \]

It can be noted that satisfying equation (2.26) is equivalent to satisfying;
\[ u - p(\rho) = u_t - p(\rho_t) \quad \Rightarrow \quad w(u) = w(u_t) \quad (2.27) \]
where \( w(u) \) is the Riemann invariant of \( \lambda_1 \).
Hence the 1-rarefaction wave satisfies the following;
\[ w(u) = w(u_t), \quad \frac{x}{t} = \lambda_1(u) \]
\[ \text{and} \quad \lambda_1(u) > \lambda_1(u_t) \quad (2.28) \]

**The 1-Shock wave**

A 1-shock wave of the Riemann problem (2.10) with initial data (2.20) is a jump discontinuity that has the form;
\[ u(x,t) = \begin{cases} u_l & \text{for } x < s_1t \\ u_r & \text{for } x < s_1t \end{cases} \quad (2.29) \]
where \( s_1 \) is the speed of the shock. The 1-shock wave satisfies the Rankine-Hugoniot condition
\[ [F(Y)] = s_1[Y] \]

Now, applying the Rankine-Hugoniot condition to equation (2.10) gives;
\[ [\rho u] = s_1[\rho] \quad \text{and} \quad [u(\rho(u + p(\rho)))] = s_1[\rho(u + p(\rho_t))] \quad (2.30) \]
Setting \( u = u_r \) and eliminating \( s_1 \) from the simplification of equation (2.30) we obtain;
\[ S1 : u + p(\rho) = u_t + p(\rho_t) \]
which defines the 1-shock wave.
The 1-shock waves of speed $s_1$ are called admissible in the sense of Lax if they satisfy:

\[
\begin{align*}
    s_1 &< \lambda_1(u_l) \\
    \lambda_1(u_r) &< s_1 < \lambda_2(u_r)
\end{align*}
\]  

This implies, the shock wave is a Lax shock of the first family if and only if the corresponding points belongs to the following curve:

\[
\begin{align*}
    \rho(u + p(\rho)) &= \rho(u_l + p(\rho_l)) = \rho w(u_l) \\
    \rho &> \rho_l
\end{align*}
\]  

\[(2.32)\]

The 2-Contact Discontinuity

A 2-contact discontinuity of the Riemann problem \[(2.10)\] with initial data \[(2.20)\] connecting a state $u_l$ and $u_r$ has the form.

\[
u(x,t) = \begin{cases} 
    u_l & \text{for } x < s_2 t \\
    u_r & \text{for } x < s_2 t
\end{cases}
\]  

\[(2.33)\]

where the speed $s_2 = \lambda_2 = u$, i.e exactly the speed of the corresponding cars. Classically, these contact discontinuities satisfy both the $z = z(u_r)$ and the Rankine-Hugoniot relation \[12\].

Considering $\lambda_2$, this condition could be obtained by finding the 2-rarefaction curve as done in the case of 1-rarefaction. Since the second characteristic field is linearly degenerate, the 2-rarefaction curve generate a type of solution called contact discontinuity.

\section*{2.3.5 AR-Type Model}

Borsche, Kimathi and Klar \[1\] has derived an Aw-Rascle-type macroscopic model from kinetic derivation. The model in Eulerian coordinates is of the form

\[
\begin{align*}
    \partial_t \rho + \partial_x (\rho u) &= 0 \\
    \partial_t (\rho u) + \partial_x (\rho u^2) - \rho a(\rho) \partial_x (u) &= 0
\end{align*}
\]  

\[(2.34)\]

where

\[
a(\rho) = v_{ref} \left( \frac{1}{\rho} - 1 \right)^{-1}
\]

Putting equation \[(2.34)\] into the $\rho v - \text{plane/M-plane form gives};
\[ \frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho u) = 0 \] \quad (2.35) \]

\[ \frac{\partial}{\partial t} u + (u - \rho (1/1 - \rho)) \frac{\partial}{\partial x} u = 0 \]

Comparing the equation (2.35) to the AR-model (2.9) in similar form as in [12], implies the pressure function for this model due Borsche, Kimathi and Klar [1] is given by;

\[ p'(\rho) = \frac{1}{1 - \rho} \implies p(\rho) = -\ln(1 - \rho) \]

This model will be solved for the various solution types just as the AR model.

### 2.4 Aw-Rascle-Zhang (ARZ) Model

This model is very similar with AR models discussed above. The basic model is of the form

\[ \frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho v) = 0, \]

\[ \frac{\partial}{\partial t} v + (v + \rho V_e'(\rho)) \frac{\partial}{\partial x} v = 0 \] \quad (2.36)

where \( v \) is the velocity at location \( x \) and time \( t \) and \( V_e \) is the equilibrium (maximum) velocity some of which proposed by researchers are given in section 2.2.1 above.

The model (2.36) in conservative form, is given as;

\[ \frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho v) = 0, \]

\[ \frac{\partial}{\partial t} y + \frac{\partial}{\partial x} p(\rho) = 0 \] \quad (2.37)

where

\[ y := \rho v - \rho V_e(\rho) \quad \text{and} \quad p(\rho) = \rho v(v - V_e(\rho)) = vy \]

The \( y \) in this case is the relative flow, that is, the difference between the actual flow \( (q = \rho v) \) and equilibrium flow \( (q_e = \rho V_e) \). See [25] for a graphical explanation of the physical meaning of \( y \) variable. The \( p \) variable is the relative pressure, that is the flux of relative flow.

Now setting \( U := (\rho, u), Y := (\rho, \rho v - \rho V_e(\rho)) \) and \( F := (\rho u, \rho v(v - V_e(\rho))) \) and following the steps as used in section 2.3.3 gives the following eigenvalues;

\[ \lambda_1 = v - \rho V_e'(\rho) \quad \text{and} \quad \lambda_2 = v \] \quad (2.38)
The right eigenvectors of \( \lambda_1 \) and \( \lambda_2 \) are respectively

\[
\begin{align*}
\mathbf{r}^1 &= \begin{pmatrix} -\rho \\ -y \end{pmatrix}, & \mathbf{r}^2 &= \begin{pmatrix} 1 \\ v - V_e(\rho) - \rho V_e'(\rho) \end{pmatrix}
\end{align*}
\]

(2.39)

The eigenvalue \( \lambda_1 \) is genuinely nonlinear and \( \lambda_2 \) is linearly degenerate. Hence, as was in the case of AR-models the \( \lambda_1 \) is associated with either a shock wave or a rarefaction wave and \( \lambda_2 \) is associated with only contact discontinuity.

### 2.4.1 Elementary Waves of ARZ Model

The elementary waves of the ARZ model includes the 1-rarefaction wave and 1-shock wave associated with the 1-wave (or \( \lambda_1 \)) and a 2-contact discontinuity discontinuity associated with the 2-wave (\( \lambda_2 \))

Figure 2.4: 1-Rarefaction Wave, 1-Shock Wave and 2-Contact discontinuity in \((\rho, v)\) plane

**The 1-Shock Wave**

The 1-shock wave of the system (2.39) with initial data (2.20), connecting the states \( \mathbf{u}_l \) and \( \mathbf{u}_o \) has the form of equation (2.29) and satisfies the conditions

\[ v_r - V_e(\rho_r) = v_l - V_e(\rho_l) \quad \text{and} \quad \rho_l < \rho_r \]

This defines the shock wave with speed \( s_3 \) satisfying the Rankine Hugoniot condition.
The 1-Rarefaction Wave
The rarefaction wave is a function of parameter $\xi = x/t$ associated with the 1-wave. It is a self-similar solution connecting the left state ($u_l$) to an intermediate state ($u_o$) and satisfies the condition;

$$v(\xi) - V_e(\rho(\xi)) = v_o - \rho V_e(\rho_o)$$

(2.40)

Since it is associated with the eigenvalue $\lambda_1$, it also satisfy the condition;

$$\lambda_1 = v(\xi) - \rho(\xi)V_e'(\rho(\xi)) = \xi$$

(2.41)

It follows from conditions (2.40) and (2.41) that in the rarefaction wave, $v(\xi)$ and $y(\xi)$ must be increasing functions, and $\rho(\xi)$ must be a decreasing function of the parameter $\xi$, which represents the slope of the characteristics.

The 2-Contact Discontinuity
A 2-contact discontinuity of the Riemann problem (2.39) with initial data (2.20) connecting a state $u_l$ and $u_r$ has the form of (2.33). Where the speed $s_4 = \lambda_2 = v$, (i.e exactly the speed of the corresponding cars) and satisfies the Rankine-Hugoniot relation.

The 2-contact discontinuity propagates discontinuities of the relative speed or flow, but the speed itself is equal on both sides of the discontinuity [25]. That is, it satisfies $v_l = v_r$.

For a detailed discussion and analysis of the various general analytical solutions of the ARZ models see [25].

We present in the next chapter some numerical schemes for conservation laws which will be used for solving different Riemann problems of all the problems discussed in this chapter.
Chapter 3

Numerical Methods

This chapter is devoted to the analysis of one-dimensional relaxation scheme formulated by Jin and Xin [5]. This scheme as described in [5] makes use of the Monotone Upwind Scheme for Conservation Laws (MUSCL) and the TVD Runge-Kutta splitting scheme by Jin [6]. We begin with the formulation of the one-dimensional relaxation system in Section 3.1. The Relaxation schemes, the Upwind and MUSCL are described in Sections 3.2.1 and 3.2.2 respectively. In Section 3.2.3 the time discretization with TVD Runge-Kutta method is given.

3.1 Relaxation System

We consider the 1-D system of conservation laws given by:

\[ \frac{\partial}{\partial t} u + \frac{\partial}{\partial x} F(u) = 0, \quad (x, t) \in \mathbb{R}^1 \times \mathbb{R}, \ u \in \mathbb{R}^n \]  

(3.1)

where \( F(u) \in \mathbb{R}^n \) is a smooth vector-valued function. The relaxation system for (3.1) as formulated in [5] is given by:

\[ \frac{\partial}{\partial t} u + \frac{\partial}{\partial x} v = 0, \]

\[ \frac{\partial}{\partial t} v + A \frac{\partial}{\partial x} u = -\frac{1}{\epsilon} (v - F(u)); \quad \epsilon > 0 \]  

(3.2)

where

\[ A = \text{diag}\{a_1, a_2, \ldots, a_n\} \]  

(3.3)

is a positive diagonal matrix to be chosen. As shown in [5], with a small \( \epsilon \), applying the
Chapman Enskog expansion in the relaxation system (3.2) gives the approximation of $u$ as:

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} F(u) = \epsilon \frac{\partial}{\partial x} \left( A - F'(u) \frac{\partial}{\partial x} u \right)$$  (3.4)

where $F'$ is the Jacobian matrix of the vector of flux function $F$. Equation (3.4) gives the first order behavior of the relaxation system (3.2). For equation (3.4) to be dissipative, it necessary the following is satisfied

$$A - F'(u)^2 \geq 0 \quad \text{for all } u$$  (3.5)

For $u$ varying in a bounded domain, equation (3.5) can be satisfied by choosing a sufficiently large $A$ [5]. Because of the CFL condition on numerical stability, it is however appropriate to obtain the smallest $A$ which satisfies the criterion (3.5).

### 3.2 The Relaxation Scheme

This section is devoted to discussing the relaxing schemes. These comprises of the discretization of the relaxation system in section 3.1 which depends on $\epsilon$ and the artificial variable $v$. For a sufficiently small $\epsilon$, it is expected that solving (3.2) properly, one can obtain good approximations to the original conservation law (3.1). An extensive study on numerical schemes for nonlinear hyperbolic conservation laws with stiff source terms that share the same relaxation limit as that of (3.2) for semidiscrete schemes have been done by Jin and Levermore [7], for implicit time discretizations by Jin [6] and for the Broadwell model of the nonlinear Boltzmann equation by Caflisch, Jin and Russo [8].

According to Jin and Xin [5], adopting the approach in these works, a good numerical discretization for (3.2) should possess a discrete analogy of the continuous zero relaxation limit, in the sense that the zero relaxation limit of the numerical discretization (let $\epsilon \to 0$ for a fixed mesh) should be a consistent and stable discretization to (3.1).

For the discretization we set $x_{j+1/2}$ as the spatial grid points with mesh height $h_j = x_{j+1/2} - x_j - 1/2$ and $t_n$ as the uniformly spaced discrete time levels with time step size of $k = t_{n+1} - t_n$ for $n = 0, 1, 2, \cdots$. we denote by $w_j^n$ the approximate cell averages of a quantity $w$ in the cell $[x_{j-1/2}, x_{j+1/2}]$ at $t_n$, and by $w_{j+1/2}^n$ the approximate point value of $w$ at $x = x_{j+1/2}$ and $t = t_n$.

Jin and Xin [5] made use of the method of lines and conveniently treated the spacial and time discretization of (3.2) separately.

We now present the relaxation scheme as outline in the work by Jin and Xin in [5].
The spatial discretization of (3.2) is written as;

\[
\frac{\partial}{\partial t} u_j + \frac{1}{h_j} (v_{j+1/2} - v_{j-1/2}) = 0,
\]

\[
\frac{\partial}{\partial t} v_j + \frac{1}{h_j} A(v_{j+1/2} - v_{j-1/2}) = -\frac{1}{\epsilon} (v_j - F_j)
\] (3.6)

where the averaged quantity \( F_j \) is defined by;

\[
F_j = \frac{1}{h_j} \int_{x_{j-1/2}}^{x_{j-1/2}} F(u)dx = F \left( \frac{1}{h_j} \int_{x_{j-1/2}}^{x_{j-1/2}} u dx \right) + O(h^2)
\]

\[
= F(u_j) + O(h^2)
\] (3.7)

for \( h = \max_j h_j \).

Thus for sufficiently accurate spatial discretization, we have with an accuracy of \( O(h^2) \),

\[
\frac{\partial}{\partial t} u_j + \frac{1}{h_j} (v_{j+1/2} - v_{j-1/2}) = 0,
\]

\[
\frac{\partial}{\partial t} v_j + \frac{1}{h_j} A(v_{j+1/2} - v_{j-1/2}) = -\frac{1}{\epsilon} (v_j - F(u_j)).
\] (3.8)

The point value quantities \( u_{j+1/2} \) and \( v_{j+1/2} \) are defined by upwind schemes. The relaxation system (3.1) has two characteristic variables

\[
v \pm A^{1/2} u
\]

which travel with the frozen characteristic speeds \( \pm A^{1/2} \) respectively and therefore the upwind approximation scheme is applied to \( v \pm A^{1/2} u \) respectively.

### 3.2.1 The Upwind Scheme

Applying a first order upwind scheme to \( v \pm A^{1/2} u \) gives respectively

\[
(v + A^{1/2} u)_{j+1/2} = (v + A^{1/2} u)_j, \quad \text{and}
\]

\[
(v - A^{1/2} u)_{j+1/2} = (v - A^{1/2} u)_j
\] (3.9)

By solving (3.9) for the unknowns \( u_{j+1/2} \) and \( v_{j+1/2} \) gives
\[ u_{j+1/2} = \frac{1}{2}(u_j + u_{j+1}) - \frac{1}{2}A^{-\frac{1}{2}}(v_{j+1} - v_j), \]
\[ v_{j+1/2} = \frac{1}{2}(v_j + v_{j+1}) - \frac{1}{2}A^\frac{1}{2}(u_{j+1} - u_j), \]

Now, by plugging (3.10) into (3.8) give the first order semi-discrete upwind approximation to the relaxation system (3.2) as;

\[ \frac{\partial}{\partial t} u_j + \frac{1}{2h_j} (v_{j+1} - v_{j-1}) - \frac{1}{2h_j} A^{\frac{1}{2}} (u_{j+1} - 2u_j + u_{j-1}) = 0, \]
\[ \frac{\partial}{\partial t} v_j + \frac{1}{2h_j} A (u_{j+1} - u_{j-1}) - \frac{1}{2h_j} A^{\frac{1}{2}} (v_{j+1} - 2v_j + v_{j-1}) = -\frac{1}{\epsilon} (v_j - F(u_j)). \]

### 3.2.2 The MUSCL Scheme

To obtain a second order scheme, Jin and Xin [5] applied the van Leer’s MUSCL scheme [10] which uses the piecewise linear interpolation. The MUSCL applied to the \( p \)-th components of \( v \pm A^{\frac{1}{2}}u \) respectively gives;

\[ (v + \sqrt{a_p}u)_{j+1/2} = (v + \sqrt{a}u)_j + \frac{1}{2}h_j \sigma_j^+ \]
\[ (v - \sqrt{a_p}u)_{j+1/2} = (v - \sqrt{a}u)_j - \frac{1}{2}h_j \sigma_j^- \]

where \( u \) and \( v \) are the \( p \)-th \( (1 \leq p \leq n) \) component of \( u \) and \( v \) respectively and \( \sigma_j^\pm \) is the slope of \( v \pm \sqrt{a_p}u \) on the \( j \)th cell. In the Sweby’s notation [11] we have

\[ \sigma_j^\pm = \frac{1}{h_j} (v_{j+1} \pm \sqrt{a_p}u_{j+1} - v_j \mp \sqrt{a_p}u_j) \phi(\theta_j^\pm), \]
\[ \theta_j^\pm = \frac{v_j \pm \sqrt{a_p}u_j - v_{j-1} \mp \sqrt{a_p}u_{j-1}}{v_{j+1} \pm \sqrt{a_p}u_{j+1} - v_j \mp \sqrt{a_p}u_j} \]

In this work we use the slope-limiter introduced by van Leer [10] as;

\[ \phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|} \]

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To achieve a TVD scheme we use a more general condition for $\phi$ proposed in [11] as
\[
0 \leq \frac{\phi(\theta)}{\theta} \leq 2 \quad \text{and} \quad 0 \leq \phi(\theta) \leq 2
\]

Another simple slope in practice is the so-called \textit{minmod} slope given by;
\[
\phi(\theta) = \max(0, \min(1, \theta))
\]

Solving equation (3.12) for $u_{j+1/2}$ and $v_{j+1/2}$ gives
\[
\begin{align*}
  u_{j+1/2} &= \frac{1}{2}(u_j + u_{j+1}) - \frac{1}{2\sqrt{a_p}}(v_{j+1} - v_j) + \frac{1}{4\sqrt{a_p}}(h_j\sigma_j^+ + h_{j+1}\sigma_{j+1}^-)
  \\
  v_{j+1/2} &= \frac{1}{2}(v_j + v_{j+1}) - \frac{\sqrt{a_p}}{2}(u_{j+1} - u_j) + \frac{1}{4}(h_j\sigma_j^- - h_{j+1}\sigma_{j+1}^-)
\end{align*}
\]

Plugging equation (3.14) into equation (3.8) gives in componentwise the MUSCL scheme for the relaxation system (3.2) as;
\[
\begin{align*}
  \frac{\partial}{\partial t} u_j + \frac{1}{2h_j}(v_{j+1} - v_j) - \frac{\sqrt{a_p}}{2h_j}(u_{j+1} - 2u_j + u_{j-1}) &- \frac{1}{4h_j}(h_{j+1}\sigma_{j+1}^- - h_j(\sigma_j^+ + \sigma_j^-) + h_{j-1}\sigma_{j-1}^+) = 0, \\
  \frac{\partial}{\partial t} v_j + \frac{a_p}{2h_j}(u_{j+1} - u_{j-1}) - \frac{\sqrt{a_p}}{2h_j}(v_{j+1} - 2v_j + v_{j-1}) &+ \frac{\sqrt{a_p}}{4h_j}(h_{j+1}\sigma_{j+1}^- + h_j(\sigma_j^- - \sigma_j^+) - h_{j-1}\sigma_{j-1}^+) = -\frac{1}{\epsilon}(v_j - F^{(p)}(u_j)).
\end{align*}
\]

where $F^{(p)}(u_j)$ is the $p$-th component of $\mathbf{F}$.

It can be observed from equation (3.15) that when $\sigma^\pm = 0$, the MUSCL reduces to the first order upwind scheme (3.11).

### 3.2.3 The Time Discretization

Jin and Xin [5] made use of the second order TVD Runge-Kutta splitting scheme introduced by Jin in [6]. The splitting scheme takes two implicit stiff source steps and two explicit convection steps alternatively.

We reproduce the scheme as outlined in [5] as follows. Denote
\[
D_+ w_j = \frac{1}{h_j}(w_{j+1/2} + w_{j-1/2})
\]
Then applying the second order TVD Runge-Kutta splitting scheme to (3.2) gives;

\[
\begin{align*}
\mathbf{u}^* &= \mathbf{u}^n \quad \text{(3.16a)} \\
\mathbf{v}^* &= \mathbf{v}^n + \frac{k}{\epsilon} (\mathbf{v}^* - \mathbf{F}(\mathbf{u}^*)) \quad \text{(3.16b)} \\
\mathbf{u}^{(1)} &= \mathbf{u}^* - kD_+ \mathbf{v}^* \quad \text{(3.16c)} \\
\mathbf{v}^{(1)} &= \mathbf{v}^* - kA D_+ \mathbf{u}^* \quad \text{(3.16d)} \\
\mathbf{u}^{**} &= \mathbf{u}^{(1)} \quad \text{(3.16e)} \\
\mathbf{v}^{**} &= \mathbf{v}^{(1)} + \frac{k}{\epsilon} (\mathbf{v}^{**} - \mathbf{F}(\mathbf{u}^{**})) - 2\frac{k}{\epsilon} (\mathbf{v}^* - \mathbf{F}(\mathbf{u}^*)) \quad \text{(3.16f)} \\
\mathbf{u}^{(2)} &= \mathbf{u}^{**} - kD_+ \mathbf{v}^{**} \quad \text{(3.16g)} \\
\mathbf{v}^{(2)} &= \mathbf{v}^{**} - kAD_+ \mathbf{u}^{**} \quad \text{(3.16h)} \\
\mathbf{u}^{n+1} &= \frac{1}{2} (\mathbf{u}^n + \mathbf{u}^{(2)}) \quad \text{(3.16i)} \\
\mathbf{v}^{n+1} &= \frac{1}{2} (\mathbf{v}^n + \mathbf{v}^{(2)}) \quad \text{(3.16j)}
\end{align*}
\]

It can be observed in equations (3.16b) and (3.16f) that the scheme will break down if \( k = O(\epsilon) \). In order to avoid this breakdown in this work, we chose \( k \) such that \( k >> \epsilon \).

The outline in section 3.2.3 summarizes how the upwind and MUSCL schemes can be implemented in the TVD Runge-Kutta. The derivation of the schemes shows that they work for the conservation laws. These schemes where implemented for the Navier Stokes equations by the authors in [5]. In the next chapter, we present a numerical implementation of the schemes above for different macroscopic traffic flow models.
Chapter 4

Numerical Implementation and Examples

In this chapter we present the numerical implementation of the schemes described in chapter 3. The schemes are used to solve numerically Riemann problems of some of the traffic flow models discussed in chapter 3.2. This is to demonstrate the performance of the schemes in solving the problems numerically. We begin by discussing the parameter estimates and boundary conditions used in the numerical simulations.

4.1 Parameters and Boundary Conditions

Let $u$ and $v$ be the $p$-th components of $u$ and $v$ respectively and $\epsilon$ a small positive parameter called the relaxation rate. The choice of $a$, the positive entry of the $p$-th row of the matrix $A$ is such that the following is satisfied

$$-\sqrt{a} \leq f'(u) \leq \sqrt{a} \quad \text{for all } u \quad (4.1)$$

Given the initial data of $u$, that of $v$ are chosen to satisfy the local equilibrium $v = f(u)$. The initial condition is therefore given as

$$u(x,0) = u_o(x), \quad v(x,0) = v_o(x) \equiv f(u_o(x)) \quad (4.2)$$

where $f(u)$ is the $p$-th component of the flux vector. Condition (4.2) implies that the characteristic speed $\lambda = f'(u)$ for the original equation (3.1) is bounded by those $\pm \sqrt{a}$ of the relaxation system [5]. This condition is referred to as the
We make use of the Neumann boundary condition for both $u$ and $v$. Let $\Omega$ be the domain of integration and $\partial \Omega$ the boundary. If $u|_{\partial \Omega}$ is given by $\frac{\partial}{\partial x} u = 0$, then by satisfying the local equilibrium condition $v = f(u)$, the boundary condition of $v$ is estimated as:

$$\frac{\partial}{\partial x} v = f'(u) \frac{\partial}{\partial x} u, \quad \Rightarrow \frac{\partial}{\partial x} v = 0 \text{ on } \partial \Omega$$

The boundary condition on $v$ can be derived in a similar manner from that on $u$ for all other forms of boundary condition.

The CFL number as used in the simulation is defined by:

$$\text{CFL number} = \max_i a_i \frac{dt}{dx}$$

where $a_i = \lambda_i$ for the respective models and we choose for the value of $\epsilon$ what is proposed by Jin and Xin in [5] as $\epsilon = 10^{-8}$ or $\epsilon = 10^{-4}$ when appropriate for a particular model.

### 4.2 Numerical Examples

In this section we investigate the macroscopic equations for different scenarios. With the relaxation schemes described in chapter 3 we consider the Riemann problem with a given initial data.

For the two equation models, AR and ARZ we pick:

$$\mathbf{u}(x, 0) = \begin{cases} \mathbf{u}_l & \text{for } x < x_o \\ \mathbf{u}_r & \text{for } x > x_o \end{cases} \quad (4.3)$$

where

$$\mathbf{u}_{l/r} = \begin{pmatrix} \rho_{l/r} \\ u_{l/r} \end{pmatrix}.$$ 

and for the single equation model we take:

$$\rho(x, 0) = \begin{cases} \rho_l & \text{for } x < x_o \\ \rho_r & \text{for } x > x_o \end{cases} \quad (4.4)$$

We now take particular examples and analyse the results of the simulation.
4.2.1 Numerical Examples for LWR

In this section we present numerical simulation of some specific traffic flow problems. We set $\rho_{\text{max}} = 1$ and $v_{\text{max}} = 1$ for the simulations in this section.

Scenario 1

In this scenario we look at the situation with data satisfying the condition

$$0 < \rho_l < \rho_r < \rho_{\text{max}}$$

and here we choose the following data; $\rho_l = 0.1$, $\rho_r = 0.7$

This particular data results in a shock with speed $s > 0$ and as such the shock wave moves forward (right). The simulation to this problem is as shown in Figure 4.1(a) In Table 4.1 the errors for using both the MUSCL and the Upwind schemes are given.
Scenario 2
In this scenario we look at the situation with data satisfying the condition

\[ 0 < \rho_l < \rho_r = \rho_{\text{max}} \]

and here we choose the following data; \( \rho_l = 0.4 \), and \( \rho_r = 1 \)

This according to [34] produces a shock wave with a negative shock speeds \( (s < 0) \). That is, the shock wave moves backward and this situation models the instance where cars encounter a bumper-to-bumper traffic jam and apply breaks instantaneously[34]. At this instance the velocity decreases to zero (0) whiles density increases to maximum \( (\rho_{\text{max}}) \). An output of the simulation to this scenario is shown in Figure 4.1(b) and 4.3(a) and the errors for using both the MUSCL and the Upwind schemes are given in Table 4.1

Figure 4.1: Shock solutions for LWR with \( \Delta x = 0.005 \)
**Scenario 3**

This scenario looks at the situation in which the data available satisfies the condition

\[ 0 < \rho_r < \rho_l = \rho_{\text{max}} \]

and for the simulation, we choose the following data; \( \rho_l = 1 \) and \( \rho_r = 0.5 \).

This scenario produces a rarefaction wave. It models the situation where car start to move after the light turns green. Here the density decreases as the cars spreads out over time. The numerical output for this case is as shown in Figure 4.2(a) and the errors for using both the MUSCL and the Upwind schemes are in Table 4.1.

![Solution of LWR for scenario 3](image1)

![Solution of LWR for scenario 4](image2)

**Figure 4.2:** Rarefaction solutions for LWR with \( \Delta x = 0.005 \)
Scenario 4
Here we consider the situation in which the data available satisfies the condition
\[
\rho_r = 0 \quad \text{and} \quad \rho_l = \rho_{\text{max}}
\]
and here we chose the following data for the simulation; \(\rho_l = 1\) and \(\rho_r = 0\)
This situation produces a rarefaction solution and it models the real life situation where cars start to move into an open road ahead of them after the light turns green. That is, the density of cars in the road ahead is zero. Figure 4.2(b) and 4.3(a) show the simulation for this scenario and in Table 4.1 we give the errors for using both the MUSCL and the Upwind schemes for this scenario.
The evolution of density \((\rho)\) for the LWR model is for scenarios 2 (shock wave) and 4 (rarefaction wave) is shown in Figure 4.3.

(a) Evolution of Density for Scenario 2 using the MUSCL

(b) Evolution of Density for scenario 4 using the MUSCL

Figure 4.3: Evolution of Density for LWR model: Shock and Rarefaction Waves
Now we present a comparison of the first order upwind scheme with the Runge-Kutta and the MUSCL with the Runge-Kutta for the LWR model. Here we use $\Delta x = 0.002$ and the data in the scenarios above to compute the errors of the two schemes as shown in Table 4.1. Figure 4.4 gives the comparisons of the simulations.

Figure 4.4: Comparison of the MUSCL and Upwind Schemes for LWR with $\Delta x = 0.002$

<table>
<thead>
<tr>
<th>scenario</th>
<th>MUSCL</th>
<th>Upwind</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td>1.0190</td>
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<tr>
<td>2</td>
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<td>0.9612</td>
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<td>3</td>
<td>0.6724</td>
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</tr>
<tr>
<td>4</td>
<td>1.1191</td>
<td>3.6728</td>
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</tbody>
</table>

Table 4.1: One norm of errors for LWR-model using the MUSCL and Upwind schemes
4.2.2 Numerical Examples for AR and AR-Type

We present in this section numerical simulations of specific examples of Riemann problems of these models as discussed in [12].

Scenario 1
The data in this scenario satisfy the following conditions;

\[ \rho_l > 0, \quad \rho_r > 0 \quad \text{and} \quad 0 \leq u_r \leq u_l \]

We choose for this scenario

\[ \rho_l = 0.5, \quad u_l = 1 \quad \text{and} \quad \rho_r = 0.5, \quad u_r = 0 \]

and \( x_o = 0.5 \). In conservative form, using the variable \((\rho, y = \rho(u - p(\rho)))\) where \( p(\rho) = \rho^\gamma, \quad \gamma = 2 \) and \( p(\rho) = -\ln(1 - \rho) \), for Aw-Rascle and AR-Type respectively. The exact solution of the AR / AR-Type models is given by a shock-wave followed by a contact-discontinuity, see [12]. The numerical results are as shown in Figure 4.5 and the errors for using both the MUSCL and the Upwind schemes for this simulation are given in Table 4.2.
(a) MUSCL and Upwind Schemes of AR-Type for scenario 1 with $\Delta x = 0.002$

(b) Evolution of Density of AR-Type for scenario 1 using the MUSCL

(c) MUSCL and Upwind Schemes of AR for scenario 1

(d) Evolution of Density of AR for scenario 1 using the MUSCL

Figure 4.5: Density $\rho$ for Riemann problem of scenario 1
Scenario 2  
In this scenario we consider a data set that satisfy the following conditions;

$$\rho_l > 0; \rho_r > 0 \text{ and } u_l \leq u_r \leq u_l + P(\rho_l)$$

and with $x_o = 0.5$ we choose

$$\rho_l = 0.5, u_l = 0 \text{ and } \rho_r = 0.8, u_r = 0.5$$

For the Aw-Rascle model, the exact solution in this scenario is given by a rarefaction wave followed by a contact-discontinuity. The output of the result are as shown in Figure 4.6. Table 4.2 gives the errors for using both the MUSCL and the Upwind schemes for this scenario.

Figure 4.6: Density $\rho$ at time $t = 0.4$ for Riemann problem of scenario 2
**Scenario 3**

The data in this scenario satisfy the following conditions:

\[ \rho_l > 0; \quad \rho_r > 0 \text{ and } u_l + P(\rho) \leq u_r \]

In this scenario we choose \( x_o = 0.25 \) and

\[ \rho_l = 0.5, u_l = 0 \text{ and } \rho_r = 0.1, u_r = 1 \]

To this scenario the Aw-Rascle model gives an exact solution which is given by a rarefaction wave followed by a fake vacuum wave, followed by a contact-discontinuity. The result to this problem is as shown in Figure 4.7 and in Table 4.2 the errors for using both the MUSCL and the Upwind schemes are computed.

Figure 4.7: Density \( \rho \) for Riemann problem of scenario 3
Scenario 4
The data in this scenario satisfy the following conditions;

\[ \rho_l = 0; \quad \rho_r > 0 \]

We choose for this scenario \( x_o = 0.5 \) and

\[ \rho_l = 0, u_l = 1 \quad \text{and} \quad \rho_r = 0.5, u_r = 1 \]

Aw-Rasclie in [12] indicated that the exact solution in this scenario is a contact-discontinuity. The result to this problem is as shown in Figure 4.8. In Table 4.2, the errors for using both the MUSCL and the Upwind schemes are computed.

Figure 4.8: Density \( \rho \) at time \( t = 0.2 \) for Riemann problem of scenario 4
Scenario 5
In this scenario we seek the result to a data which satisfies the conditions;

\[ \rho_l > 0; \rho_r = 0 \]

With \( x_o = 0.5 \) we choose for this scenario;

\[ \rho_l = 0.5, u_l = 0.1 \quad \text{and} \quad \rho_r = 0, u_r = 1 \]

As indicated by Aw-Rascle in [12], the exact solution in this scenario is a rarefaction wave. The result to this problem is as shown in Figure 4.9 and the errors for using both the MUSCL and the Upwind schemes are in Table 4.2.

\[ \text{Figure 4.9: Density } \rho \text{ at time } t = 0.2 \text{ for Riemann problem of scenario 5} \]
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<th>Upwind</th>
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</tr>
</tbody>
</table>

Table 4.2: One norm of errors for AR-Type / AR-models using the MUSCL and Upwind schemes

4.2.3 Numerical Examples for ARZ

In this section we apply the schemes in chapter 3 to some specific Riemann problem of the ARZ-model for some scenarios. For the simulations, we set the following parameters,

\[ \rho_{\text{max}} = 1 \quad \text{and} \quad v_{\text{max}} = 1 \]

Scenario 1

The data in this scenario, the data satisfy the following conditions;

\[ v_r \leq v_l + V_e(\rho) \quad \text{and} \quad v_r < v_l \]

We choose for this scenario \( x_o = 0.5 \) and run the simulation for \( t = 0.8 \)

\[ \rho_l = 0.5, \quad v_l = 0.7 \quad \text{and} \quad \rho_r = 0.5, \quad v_r = 0.1 \]

Mammer, Lebacque and Haj Salem in [25], describe this scenario to produces a solution comprising of a shock wave followed by a contact discontinuity. The numerical result to this problem is as shown in Figure ??.
Scenario 2
The data in this scenario satisfy the following condition;

\[ v_r - v_l + V_e(\rho) > v_{\text{max}} \]

We choose for this scenario \( x_o = 0.5 \) and run the simulation for \( t = 0.4 \)

\[ \rho_l = 0.2, \, v_l = 0.1 \text{ and } \rho_r = 0.5, \, v_r = 0.7 \]

According to Mammer, Lebacque and Haj Salem in [25], this scenario produces a solution comprising of a rarefaction (RW-1) followed another rarefaction (RW-2) wave and then by a contact discontinuity. It can be noted that the rarefaction (RW-2) is similar to the wave described in scenario 3 of section 4.2.2 by Aw-rascle as fake vacuum wave. It occurs between
two points with $\rho = 0$ and that explains why the fake vacuum wave tag given by Aw-Rascle. The numerical result to this problem is as shown in Figure ??.

(a) Density at $t = 0.4$ for the Riemann problem of ARZ-model for scenario 2

(b) Evolution of Density of AR for scenario 2 using the MUSCL

Figure 4.11: Density at $t = 0.4$ for the Riemann problem of ARZ-model for scenario 2
Scenario 3
The data in this scenario satisfy the following conditions;

\[ y = \rho(v - V_e(\rho)) < 0 \]

This is the situation where the initial velocity is less than the equilibrium velocity. We choose for the simulation \( x_o = 0.5, t = 0.4 \) and

\[ \rho_l = 0.2, v_l = 0.5 \text{ and } \rho_r = 0.9, v_r = 0.1 \]

This scenario gives a shock wave followed by a contact discontinuity \[25\]. The output of this simulation is as shown in Figure 4.12 and the errors for using both the MUSCL

![Density of the ARZ model for scenario 3](image1)

![Evolution of Density of AR for scenario 3 using the MUSCL](image2)

![Solution of corresponding y of the ARZ model for scenario 3](image3)

Figure 4.12: Density and corresponding y for ARZ model with \( \Delta x = 0.002 \) for scenario 3

Solving the Riemann problem of AR/AR-Type and ARZ consists of considering all the possible values for \( u_l \) and \( u_r \) and then determining a valid set of waves that enable the interconnection of \( u_l \) to \( u_r \) through intermediate states. Qualitatively the only thing that makes Riemann problem solutions for the LWR different from those of the AR/AR-Type and
ARZ models is that the models AR/AR-Type and ARZ have an additional wave (2-wave) associated with their second eigen values. These 2-waves always propagates faster than the other waves.

In the next chapter we present a brief discussion and summary of our results. We will present a conclusion of this work and some directions of research for future works.
Chapter 5

Summary and Conclusions

In this chapter we summarized what we have done so far, draw conclusion from the work and present recommendations for further research work.

5.1 Conclusion of thesis

In this thesis, the one-dimensional macroscopic traffic flow models have been presented. We have presented and studied different specific models for both single and two equation models. The chapter of the thesis is devoted to a brief introduction to the various traffic flow models. In chapter two, we presented a detailed account of the different macroscopic traffic flow models. Some of the models studied includes, the Lighthill-Whitham-Richards (LWR), the Aw-Rascle, the Aw-Rascle-Zhang (ARZ) as well as an Aw-Rascle type model due Borsche, Kimathy and Klar. The various solution types of Riemann problems of these models have also been discussed.

The chapter three, was devoted to the study of the relaxation systems of the conservative laws and the derivation of the conservative schemes such the upwind and Monotone Upwind Scheme for Conservative Laws (MUSCL) by Jin and Xin [3]. We also presented in the same chapter the use of the upwind and MUSCL in the the TVD Runge Kutta scheme for time discretization.

In chapter four, we considered numerical application of the schemes presented in the chapter three. The schemes were used to solve Riemann problems of the traffic flow models presented in chapter two to find out their effectiveness in getting the various solution types for different Riemann data.

As expected, the MUSCL scheme with the Runge-Kutta was better than the Upwind scheme
in estimating shocks, rarefaction waves and the contact discontinuities when measured with the 1-norm. It was as well better in the case of solutions containing a combinations of the three wave types.

We presented in the next section, some recommendations for future research works.

## 5.2 Recommendation for Future Research

Due to time constraint, many other areas we intended to extend this work to could not be achieved. We outline the following as areas this work could be extended to cover in future works.

1. We intended to study the evolution of density and flux of car flows at road junctions.
2. The models and schemes could be implemented on a network of roads.
3. Real life traffic flow problems can also be studied and managed by implementing the models and schemes presented in this work.
4. Supply chain problems could as well be studied and managed with the models and schemes presented.
Bibliography

[1] R. Borsche, M. Kimathi and A. Klar, ”Kinetic derivation of a Hamilton-Jacobi traffic flow model,


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Declaration

I hereby declare that this thesis was done by me and that there are no sources other than those listed in the bibliography and duly cited in the work that was used.

\textit{jd.ankamah}

______________________________
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February, 2012, Kaiserslautern, Germany