MASTER

Mathematical modeling of blood flow through a deformable thin-walled vessel
an analytical and quantitative investigation of the advection and diffusion contribution
in the global and local equations of motion

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An analytical and quantitative investigation of the advection and diffusion contribution in the global and local equations of motion

R.G.P. (Rob) Ritzen
Mathematical modeling of blood flow through a deformable thin-walled vessel

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Master's thesis

Industrial and Applied Mathematics
Computational Science and Engineering
Abstract

In hemodynamics, the interaction of the blood flow through an arterial vessel and the geometrical and mechanical properties of the vessel is studied. Very often, numerical methods are used to determine important flow characteristics, such as the pressure distribution, in estimating the behavior of blood flow. This is a very useful method, often based on measurements of patient specific parameters, in predicting and preventing complications in cardiovascular surgery or in the application of cardiovascular equipment. Mathematical investigation of the order of magnitude of the flow characteristics can help to reduce the computational costs in the numerical analysis of the blood flow.

In this project, the flow of blood through a thin-walled, linearly elastic deformable vessel is modeled. The assumption of linearly elastic behavior of the vessel wall gives rise to an affine constitutive relationship between the cross-sectional surface of the vessel and the pressure distribution. The blood, as a fluid, is regarded as an incompressible Newtonian fluid. We distinguish between the global one-dimensional equations of motion and the local three-dimensional equations of motion, describing the relation between flow characteristics like the volumetric flow rate, the pressure distribution and the axial velocity. Of special interest is the order of relevance of the contributions of radial diffusion, i.e. the shear stress at the wall of the vessel, and nonlinear advection, which for the global equations of motion are assumed to be of the forms as proposed by Bessems in [1]. In contrast with most studies in this area, where periodic solutions are studied, we focus on the transient solutions by imposing a single-pulse inlet flow. Moreover, the solutions obtained from the global and local equations of motion are as far as possible analytical and they are compared which each other, based on plots of the solutions for a set of representative arteries, both on quantitative and qualitative features.

A parameter of great relevance in this study is the dimensionless compliance (in this report denoted by $\delta$), which equals the ratio of the characteristic speed of the blood flow and the elastic wave speed of the wall (for flexural waves). This parameter is always less than one, and often much smaller than one. For arteries having a small compliance, i.e. for a compliance much smaller than one, we observe that radial diffusion is of important influence. For arteries with larger values of the compliance, close to but still less than one, we still observe the effect of the radial diffusion term, however the flow is now dominated by the effects of the spatial and time derivatives of the volumetric flow rate and the pressure distribution. We expect that the influence of the nonlinear advection term is almost always small compared to the effect of all other terms in the equations of motion. However, we can not draw any conclusions about this statement based on the results in this project, due to the asymptotic assumptions we make in deriving the solutions for the nonlinear advection contribution, which lead to large variances in the qualitative and quantitative aspects of the solution. This holds specifically if the compliance is not too small (i.e. close to one).
Mathematical modeling of blood flow through a deformable thin-walled vessel
“Just as the body cannot exist without blood, so the soul needs the matchless and pure strength of faith”

Mahatma Gandhi, 1869-1948
4 Mathematical modeling of blood flow through a deformable thin-walled vessel
Dankwoord

Dit verslag is het resultaat van mijn afstudeerproject voor de opleiding Industrial and Applied Mathematics met specialisatie Computational Science and Engineering aan de Technische Universiteit Eindhoven. Ik sluit een bijzondere studietijd af en hoop hiermee een goede basis te leggen voor de toekomst. Met veel plezier maak ik gebruik van de gelegenheid een aantal mensen te bedanken die een belangrijke rol hebben gespeeld in de totstandkoming van dit resultaat.

Allereerst wil ik de afstudeercommissie, bestaande uit Fons van de Ven, Frans van de Vosse, Adrian Muntean (plaatsvervanger van leerstoelhouder Mark Peletier) en Arris Tijsseling, bedanken voor hun eindeloze wijsheid en de bereidwilligheid mijn afstudeerverslag te bestuderen en te beoordelen. Een bijzonder woord van dank gaat hierbij uit naar Fons van de Ven. Fons, ik heb genot van de wijze waarop en het geduld waarmee jij mij hebt begeleid bij het tot stand brengen van dit eindresultaat. Ik kijk met genoegen terug op de vele gesprekken die we hebben gevoerd over dit afstudeerproject en de adviezen die je mij hierbij hebt gegeven. Natuurlijk denk ik hierbij ook terug aan de gesprekken die we hebben gehad aan de bar bij GEWIS, vaak onder het genot van een biertje, over de wijze waarop wij, ieder op onze eigen manier, genieten van het leven. Nogmaals heel erg bedankt!

Uiteraard wil ik ook graag mijn studiegenoten bedanken. Een speciaal woord van dank is voor de mensen van dispuut GELIMBO, met wie ik tijdens mijn studie een bijzondere vriendschap heb opgebouwd en met wie ik heel veel bijzondere momenten heb mogen delen. Verder ben ik dank verschuldigd aan mijn vrienden van het “HBO-schakelprogramma”, de collega’s van Muziekbouw Frits Philips, mijn musicalvrienden (in het bijzonder de mensen van Moulin Rouge de Musical) en iedereen die afgelopen tijd aan mij heeft gevraagd “Rob, ben je al klaar met afstuderen?”.

Tenslotte wil ik mijn familie bedanken. Zij hebben mij gevormd tot wie ik nu ben en tijdens mijn studie hebben ze mij op alle fronten gesteund, mijn keuzes gerespecteerd en mij het vertrouwen gegeven om het afstuderen tot een goed einde te brengen. In het bijzonder dank ik: Laura, voor het “groene rakkertje”, die mij heel veel vervelende treinreizen naar Valkenburg heeft bespaard; Pap, voor de geweldige manier waarop je chef was van klusteam Euterpestraat; Mandy, voor alle opmerkingen die je er zonder bij na te denken uitgoot en waar ik altijd zo hard om moet lachen; en tenslotte Mam, vooral voor wie je bent, maar natuurlijk ook “veur ut doon van de wesj”. Jullie zijn top!

Rob Ritzen
16 augustus 2012
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8 Mathematical modeling of blood flow through a deformable thin-walled vessel
# Nomenclature

## Greek symbols

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<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\alpha$</td>
<td>Womersley number.</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Dimensionless compliance parameter (Mach number).</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Quotient of $R_a$ and $L$.</td>
</tr>
<tr>
<td>$\varepsilon_{ij}$</td>
<td>Components of the deformation tensor (Appendix A).</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Characteristic.</td>
</tr>
<tr>
<td>$\Gamma_a$</td>
<td>Local nonlinear advection contribution.</td>
</tr>
<tr>
<td>$\gamma_a$</td>
<td>Global nonlinear advection contribution.</td>
</tr>
<tr>
<td>$\lambda_k$</td>
<td>Positive zeros of $J_0$.</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Dynamic viscosity.</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Kinematic viscosity. (Only in Section 4.2.3 and Appendix A, $\nu$ represents the Poisson ratio)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density.</td>
</tr>
<tr>
<td>$\sigma_k$</td>
<td>Positive zeros of $J_2$.</td>
</tr>
<tr>
<td>$\sigma_{ij}$</td>
<td>Components of the stress tensor (Appendix A).</td>
</tr>
<tr>
<td>$\tau_w$</td>
<td>Shear force at the wall.</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Pulsating (radial) blood frequency.</td>
</tr>
</tbody>
</table>

## Roman symbols (1)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Cross-sectional area.</td>
</tr>
<tr>
<td>$A_0$</td>
<td>Undeformed cross-sectional area.</td>
</tr>
<tr>
<td>$C$</td>
<td>Compliance.</td>
</tr>
<tr>
<td>$c_0$</td>
<td>Unperturbed wave speed in thin-walled elastic vessel wall.</td>
</tr>
<tr>
<td>$c$</td>
<td>Actual wave speed in thin-walled elastic vessel wall.</td>
</tr>
<tr>
<td>$D$</td>
<td>Plate constant.</td>
</tr>
<tr>
<td>$E$</td>
<td>Young’s modulus.</td>
</tr>
</tbody>
</table>
### Roman symbols (2)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>Thickness of the vessel wall.</td>
</tr>
<tr>
<td>$L$</td>
<td>Typical length scale in axial direction.</td>
</tr>
<tr>
<td>$L_v$</td>
<td>Length of an arterie.</td>
</tr>
<tr>
<td>$M_z$</td>
<td>Bending moment.</td>
</tr>
<tr>
<td>$N_{\theta}$</td>
<td>Normal force in axial direction.</td>
</tr>
<tr>
<td>$N_{\theta_0}$</td>
<td>Normal force in azimuthal direction.</td>
</tr>
<tr>
<td>$n$</td>
<td>Outward normal vector.</td>
</tr>
<tr>
<td>$p$</td>
<td>Pressure.</td>
</tr>
<tr>
<td>$p_0$</td>
<td>Reference pressure.</td>
</tr>
<tr>
<td>$Q_0$</td>
<td>Maximum of the initial volumetric flow rate.</td>
</tr>
<tr>
<td>$Q_z$</td>
<td>Shear force.</td>
</tr>
<tr>
<td>$q$</td>
<td>Volumetric flow rate.</td>
</tr>
<tr>
<td>$q_i$</td>
<td>Initial volumetric flow rate.</td>
</tr>
<tr>
<td>$R$</td>
<td>Internal radius.</td>
</tr>
<tr>
<td>$R_a$</td>
<td>Typical length scale in radial direction.</td>
</tr>
<tr>
<td>$\Delta R$</td>
<td>Relative change of internal radius.</td>
</tr>
<tr>
<td>$Re$</td>
<td>Reynolds number.</td>
</tr>
<tr>
<td>$r$</td>
<td>Radial coordinate.</td>
</tr>
<tr>
<td>$St$</td>
<td>Strouhal number.</td>
</tr>
<tr>
<td>$T$</td>
<td>Typical time scale.</td>
</tr>
<tr>
<td>$t$</td>
<td>Time variable.</td>
</tr>
<tr>
<td>$U$</td>
<td>Radial displacement of the vessel wall.</td>
</tr>
<tr>
<td>$V$</td>
<td>Characteristic axial velocity.</td>
</tr>
<tr>
<td>$v$</td>
<td>Velocity vector with components $(v_r, v_\theta, v_z)$.</td>
</tr>
<tr>
<td>$\bar{v}$</td>
<td>Mean axial velocity.</td>
</tr>
<tr>
<td>$w$</td>
<td>Alternative notation for $v_z$.</td>
</tr>
<tr>
<td>$x$</td>
<td>Position vector.</td>
</tr>
<tr>
<td>$z$</td>
<td>Axial coordinate.</td>
</tr>
</tbody>
</table>
### Miscellaneous

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_i$</td>
<td>Contour element.</td>
</tr>
<tr>
<td>$\mathcal{D}$</td>
<td>Domain.</td>
</tr>
<tr>
<td>$H$</td>
<td>Heaviside function.</td>
</tr>
<tr>
<td>$I_n$</td>
<td>Modified Bessel function of first kind of order $n$.</td>
</tr>
<tr>
<td>$i$</td>
<td>Imaginary unit.</td>
</tr>
<tr>
<td>$J_n$</td>
<td>Bessel function of first kind of order $n$.</td>
</tr>
<tr>
<td>$K_n$</td>
<td>Modified Bessel function of second kind of order $n$.</td>
</tr>
<tr>
<td>$\mathcal{L}$</td>
<td>Laplace transform.</td>
</tr>
<tr>
<td>$\mathcal{L}^{-1}$</td>
<td>Inverse Laplace transform.</td>
</tr>
<tr>
<td>$O$</td>
<td>Asymptotic order symbol (“big-O”).</td>
</tr>
<tr>
<td>$\approx$</td>
<td>Equality up to leading order.</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

This report is entitled “Mathematical modeling of blood flow through a deformable vessel: An analytical and quantitative investigation of the advection and diffusion contribution in the global and local equations of motion” and it is the result of a final project in order to obtain a master’s degree in Industrial and Applied Mathematics. In this first chapter we introduce the problem of the project in full detail and shortly recapitulate the results obtained.

1.1 Background of the problem

Fluid mechanics, one of the many specialisms in physics where mathematics plays an important role, finds its applications in many practical situations. A prominent example of a problem, coming from a subspecialism of this area of research called cardiovascular fluid mechanics, is the modeling of blood flow through a deformable vessel. The wall of a blood vessel deforms under influence of the local pressure and the shear stress at the wall. Therefore, this modeling project must be captured under the name fluid-structure interactions. Since the shape of the wall is essentially unknown, this class of problems is more complex then the purely elastic or purely fluid ones. The study of the flow of blood through an arterial vessel, or more specific, of the relationship of flow characteristics as pressure and flow wave forms with the geometrical and mechanical properties of the vessel, such as its radius, wall thickness, and elastic properties (stiffness/compliance), is captured by the name hemodynamics. The blood flow through the human circulatory system, consisting of arteries and veins, is driven by pressure waves generated by the heart. Changes in the geometry of the arteries due to stresses on the vessel wall result in changes of pressure and flow-rate of the blood flow. Drastic changes in for instance the blood pressure can be indications for diseases like aortic stenoses.

Often studies of the blood flow through the human circulatory system are based on physical models in which more or less severe approximations are made. Such approximations can be based on a chosen shape of the velocity profile in the vessel; see e.g. Bessems et al. [10], or on an empirical relation between the pressure gradient and the flow rate; see, e.g. Stergiopulos et al. [14]. Often these physical approaches lead to one-dimensional models relating the pressure, flow rate and cross-sectional area to each other; Stergiopulos [14], Bessems [10], [11]. Formally, these 1-D models are obtained by integration of the 3-D hydrodynamical (Navier-Stokes) equations over the cross-section of the vessel, see [10]. Computer simulations are the common tool to obtain explicit numerical results, as done by Stergiopulos [14], Giannopapa [9], Van Geel [13], Olufsen et al. [6] and Bessems [1]. In most of these papers, periodic flow processes are consid-
ered. However, in many of these processes, not only the periodic part of the flow is relevant, but also the transient one. For instance, in a so-called lock-exchange problem, in which, by suddenly opening a valve, blood flows initially into an empty vessel, the transient problem is of main interest. Therefore, in this project we will focus on the transient flow, rather than the periodic flow.

We are convinced that an analytical mathematical model can be of great use to support the physical modeling mentioned above. Therefore, we will try to solve the 3-D hydrodynamical equations for the flow in an elastically deformable vessel by as much as possible analytical means. For this, we will first construct a mathematical model, based on certain, physically motivated, assumptions, and then try to solve the thus obtained system of partial differential equations and initial and boundary conditions in a consistent and accurate way. We will do this with great accuracy, but, on the other hand, we must realize that the desired accuracy to support the physical models, is not too high. This holds the more as the experimental accuracy in blood-flow measurements is at most of the order of 10 percent.

1.2 Problem description

The basic problem that we consider in this project is the sudden inlet flow of blood into a semi-infinite circular cylindrical deformable vessel. Blood is modeled as an incompressible Newtonian fluid. The vessel is a linearly elastic, thin-walled cylinder. The flow of blood is described by the classical hydrodynamical equations: equation of continuity (or incompressibility condition), and equations of motion (Navier-Stokes equations). The wall is treated as a thin-walled, linearly elastic tube under (space and time dependent) internal pressure. The interaction between fluid (blood) and solid (wall) is via the normal stress of the fluid on the wall. The problem is essentially 3-D, but we assume it to be rotationally symmetric.

One of the main goals of this project is to estimate the order of magnitude of the diffusion and advection terms in the equations of motion for the dynamical blood flow through the vessel and to indicate under what conditions these effects can, or can not, be neglected. We distinguish between two different types of equations of motion:

1. LOCAL 3-D EQUATIONS OF MOTION. A set of partial differential equations, based on the classical Navier-Stokes equations, describing the relation between the velocity vector and the pressure gradient.

2. GLOBAL 1-D EQUATIONS OF MOTION. A set of partial differential equations describing the relationship between the flow rate, the cross-sectional area and the pressure gradient. These equations are the integrated version of the local equations of motion.

We will solve the global equations of motion for three cases: (i) we neglect both the advection and viscous diffusion term, (ii) we neglect the advection term, (iii) we neglect the viscous diffusion term. The local equations of motion will be solved up to leading order of the dimensionless compliance parameter, which is equal to the ratio of the characteristic flow velocity of the blood and the elastic wave speed of the vessel wall. We will try to find as much as possible fully analytical solutions for all these cases. This will finally result in a classification of the order of relevance of the diffusion and advection for a range of representative human arteries.

1.3 Problem approach

To answer the questions in the previous section, we construct a (simplified) model and we try to solve this with mathematical tools. We model the flow of blood through an arterie as the flow
through a semi-infinite vessel and we assume the flow to be rotationally symmetric. For the vessel we assume a state of plane strain (no axial elongation). Moreover, we take the vessel "thin-walled", meaning that the thickness of the vessel wall is much smaller than the internal radius of the vessel. We describe the vessel wall as a linearly elastic medium. Thus, we consider the vessel wall as a linearly elastic thin-walled tube, and this enables us to derive a constitutive equation for the radial displacement of the wall in terms of the blood pressure. On top of this, we assume here, following Bessems [1], that also the wavelength of the deformations of the vessel is much larger than the internal radius of the vessel.

To describe the flow of a fluid through a vessel we need a balance of mass (equation of continuity) and a balance of momentum (equation of motion). We assume the blood to be an incompressible Newtonian fluid. With this assumption, the balance of mass and the balance of momentum turn into the well-known Navier-Stokes equations as the governing equations of motion and, with some dimensional analysis, we arrive at a dimensionless system of equations describing the relation between the pressure gradient, the volumetric flow rate and the velocity vector. We refer to this set of equations as the local 3-D equations of motion. The global 1-D equations of motion can be obtained by integrating the local ones over the cross-section of the vessel.

Next, we try to solve both the local and global equations of motion. We focus on transient solutions, which means that the (time-dependent) solution is depending on the initial state of the flow, and thus depends on the initial conditions; this in contrast to periodic solutions. To this end, we will prescribe a single pulse inlet-flow function as initial boundary condition. Furthermore, we try to solve the equations analytically as far as possible.

In solving the global equations of motion we distinguish three cases: (i) we neglect both the advection and viscous diffusion term, (ii) we neglect the advection term, (iii) we neglect the viscous diffusion term. For the first case the system of equations of motion reduces to a wave-equation-like system. For this reason, it seems reasonable to use the method of characteristics to solve the system. As a basis for investigating the effect in cases (ii) and (iii), we follow Bessems et al. [10] in using an expression for both the advection and the diffusion term in the equations of motion. Another way to solve these kind of problems, is to use Laplace transformations. We opt for this method in solving case (ii). It will turn out that the characteristics found in (i) also pop up here. In case (iii), we switch back to characteristics. However, because of the non-linearity in the pressure distribution, we have to use an asymptotic expression for the pressure function and we solve the system up to leading order.

To solve the local equations of motion we introduce asymptotic expansions for the pressure-, flow rate- and velocity-function. This enables us to solve the system up to leading order, again completely analytically. To this end, we assume a so-called Fourier-Bessel series for the velocity-function and we use the method of Laplace transformations to solve the system. At the end, we will compare the solutions found here with the ones found for the global equations of motion, and we hope to find some similarity.

1.4 Outline of the report

As basis structure for this report we follow the classification as described in the problem description (Section 1.2). We started with a general introduction, in which we stated the formulation and the goal of this project and in which we shortly introduced the aspects related to the problem, that will be discussed in the core chapters of this report. In the second chapter a short introduction of the main biological aspects, related to the problem, is presented. We start with a description of the human circulation system, which is the complex network of the heart, arteries and veins and which transports blood through the body. This is followed by a section about the structure of a
vessel wall and a physiological description of blood as a fluid.

In the third chapter the equations of motion for the flow through a tube will be formulated and used as a model for blood flow through an artery or a vein. Starting from the full Navier-Stokes equations as a description of the flow of a fluid, we arrive via a dimensional analysis based on a set of data from the real human circulatory system at two nonlinear partial differential equations depending on the flow velocity, the blood pressure, the flow rate and some flow characteristic parameters. We will refer to these equations as the local equations of motion. To arrive at the so-called global equations of motion, we integrate the local equations of motions over the cross-section of vessel.

In Chapter 4 we try to find analytical solutions for the global equations of motion, where we distinguish the three different cases as denoted with (i)-(iii) in Section 1.2. In Chapter 5 we introduce asymptotic expansions for the flow quantities and solve the local equations of motion up to leading order of the dimensionless compliance parameter. With the solution obtained here we estimate flow parameters for the advective flow and the diffusive flow and compare them with the parameters used for the global solutions in Chapter 4.

Numerical results from Chapters 4 and 5 will be presented graphically in Chapter 6. To this end, we have to choose an expression for the inlet flow function. The parameters, based on real measurements and needed to make the plots, are borrowed from literature and tabled in the beginning of Chapter 6. Finally, in Chapter 7 we recapitulate all results from the previous three chapters and combine them to come to a conclusion and preferably also to a classification for the relevance of the advection and diffusion contribution in some representative human arteries.

Some notes on the elastic behavior of the vessel wall can be found in Appendix A. The other appendices mainly contain extensive computations to support the subresults of the core chapters.

1.5 Results and conclusions

From the expressions obtained for the volumetric flow rate, the pressure distribution and the axial flow velocity, we made plots for the different cases studied for the global equations of motion and for the zeroth order approximations obtained for the local equations of motion. Here, we choose a specific expression for the inlet flow function and we opt for three representative human arteries and corresponding parameters on which the plots are based. Next to this, we listed the percentage differences between the solutions from the global equations of motion and the local ones. We discussed in detail the plots and the table and based on this discussion we drew our conclusions.

Summarizing, we state that, whenever the dimensionless compliance parameter is “large”, i.e. still less than, but close to one, the effect of the shear stress at the wall is noticeable, specifically in the local 3-D model, however the flow is mainly dominated by the time and spatial derivatives of the volumetric flow function and the pressure distribution. For small values of the dimensionless compliance parameter, the radial diffusion term is of significant order. Although we strongly believe that the influence of the advection term is almost always negligible with respect to the other terms in the equations of motion in all cases, we can not conclude this from the results obtained. Here, the neglect of higher order terms of the dimensionless compliance parameter causes large fluctuations in the amplitude and large phase differences for the solution from the global equations of motion.
Biological background

As we are going to study the flow of blood through a vessel, it is important to get insight into the biological structure of the blood flow through the human body. Therefore we provide some biological background of the problem in this chapter. First the system of arteries and veins, through which the blood will flow, will be described. Secondly, blood as a fluid will be described. Finally, since we are dealing with a fluid-solid interaction problem, we describe the mechanical properties of a vessel wall. The text of this chapter is mainly based on [3] and [12]. The figures are taken from [5].

2.1 The human circulatory system

The human circulatory system enables the blood to flow through the whole body and to transport all kinds of substances over large distances via a complex network of arteries and veins. The human circulation system, as well as the circulation system for other vertebrates, can be split into two main separated parts, the pulmonary circulation and the systemic circulation. In the pulmonary circulation deoxygenated blood will be pumped from the right part of the heart via the pulmonary arterie to the lungs. In the lungs the red blood cells will react with the oxygen and all waste materials will be transported to the air. Via the pulmonary veins the oxygenated blood will go back to the left part of the heart. In the systemic circulation oxygenated blood is transported away from the heart to the rest of the body, where oxygen is taken up by the organs. The deoxygenated blood is then returned to the right atrium of the heart via the inferior and superior vena cava.

For a schematic representation of the human circulatory system see Figure 2.1 and remark that left and right are taken with respect to the body itself (so opposite in the figure).

2.1.1 The heart

The heart is situated in the chest of the human body and consists of two parts separated by an internal wall. Each of these two parts has an atrium and a ventricle, so the heart has four chambers in total. Deoxygenated blood from the body enters the right atrium via the inferior vena cava and the superior vena cava. From the right atrium the blood flows to the right ventricle where it will be pumped into the pulmonary arterie. Now the pulmonary circulation starts. After this circulation oxygenated blood reenters the heart via the pulmonary veins into the left atrium. From the left atrium the blood flows via the left ventricle into the aorta en starts the systemic circulation.
Figure 2.1: Schematic representation of the human circulatory system. Oxygenated and deoxygenated blood is transported through the red and blue part of the circulatory system, respectively.
Both the right and left atrium and the right and left ventricle are separated by the atrioventricular (tricuspid or mitral) valve. At the beginning of the pulmonary artery as well as at the beginning of the aorta there are semilunar valves in order to regulate the inflow of blood into the pulmonary and systemic circulation.

The heart can be regarded as a hollow muscle. Along the wall of the heart blood is transported via the coronary arteries and the coronary veins in order to transport oxygen to the heart muscle tissue and expel waste materials from the heart muscle tissue. The coronary veins end directly into the right ventricle.

Three different main stages can be distinguished in the function of the heart during a cycle. The contraction and the relaxation of the cardiac muscles is called systole and diastole, respectively. During the first phase of the cycle, blood is pumped from the atrium into the ventricle under influence of the pressure increase in the atrium due to the systole of the atrium. The ventricle is now completely relaxed. Next, a contraction in the ventricle takes place. The pressure in the ventricle increases until it exceeds the pressure in the aorta. Blood now flows directly into the systemic circulation. Finally the pressure in both the ventricle and the aorta drops down and also the ventricle is in diastole. These three stages are represented in Figure 2.3. Clearly, the phases are not equal in length.

### 2.1.2 The systemic circulation

We can distinguish three different parts in the systemic circulation, namely the arterial system, the capillary system and the venous system.

The arterial system transports oxygenated blood from the left ventricle into the systemic circulation towards tissues. This starts with transport via arteries. One of the most important arteries in the arterial system is the aorta. This main artery is able to buffer blood by use of internal valves to regulate the pulsating blood flow. The blood pressure in the arterial system is relatively high. In general, the arteries are named after the tissue they are leading to. The pressure in the arterial system can be kept at high level because at its end the arteries strongly divide into two
Figure 2.3: The heart cycle for a heart having rate frequency of 75 beats per minute.

or more smaller branches and so increase the total resistance to the blood. These branches are called small arteries or arterioles. The smooth muscle cells (see Section 2.2) in the wall of these arterioles are able to change the diameter of the vessel and, while doing so, meeting the needs of the specific part of the body. In the arterial system, viscous forces play an important role in the blood flow since there are significant variations in the characteristic velocities and length scales of the arteries and arterioles.

From the arterioles blood flows into the capillary system. This is a network of very small vessels situated around a certain tissue. The diameter of these vessels is even that small, that modeling the blood as a homogenous fluid is not possible and so differs strongly from modeling the blood flow in the arterial and the venous system. In the capillary system, oxygen is exchanged from the red blood cells to the tissue with all other kinds of nutrients and waste materials. The blood flow in the capillary system is also called micro circulation.

Finally, in the venous system deoxygenated blood is transported back from the capillary system to the right atrium. The blood from the capillaries is collected in the small veins, also called venules. These venules merge into veins, which are generally called after the tissue they are coming from. The blood flows via the vena cava back to the heart. The vena vaca is, just like the aorta, able to store blood and so enables the heart to regulate the flow into the arteries. The total volume of the venous system is much larger than the volume of the arterial system and consequently the characteristic velocity of the blood is lower than the one in the arterial system. Moreover, the pressure in the venous system is that low that gravitational forces can play a role in the flow of the blood.

The changes in blood pressure and velocity due to changes in characteristic lengths (diameter) of the different types of vessels are graphically shown in Figure 2.4.

### 2.2 The vessel wall

The vessel wall plays an important role in the pressure distribution of the blood in the vessel, since the vessel wall deforms under influence of the internal pressure. The deformation of the vessel wall is determined by the mechanical properties of the vessel wall.

We distinguish three different layers in the vessel wall (see Figure 2.5). The first one is the...
Figure 2.4: Changes of the blood pressure and velocity due to changes in characteristic lengths of the vessel.

Figure 2.5: Schematic representation of an artery wall, a vein wall and a capillary wall.
innermost layer, which is called the *tunica intima*. The tunica intima is divided into three different structures. The *endothelial cells* cover the inner surface of the vessel wall and are therefore in direct contact with the blood that flows through the vessel. These endothelial cells play an important role in the growth and regeneration of the vessel. The *internal elastic membrane* consists of collagenous bundles and elastic fibrils. These elastic fibrils are build of *elastin*, which is a biological material that provides the elastic behavior of the vessel wall. The collagenous bundles are build of *collagen*, a protein that gives strength to the vessel wall and which in turn is build of three chains of amino-acids. Since this membrane is relatively small, it does not have significant influence on the mechanical properties of the vessel wall. The endothelial cells and the internal elastic membrane are separated by a connective tissue.

The second layer of the vessel wall is the middle one, called the *tunica media*. The tunica media is the thickest one of the three layers and consists of two typical structures, namely *elastic fibrils* and *involuntary muscle fibrils*, or also called *smooth muscle cells*. Especially for small arteries, the smooth muscle cells are dominating the mechanical properties of the vessel wall and are able to regulate the local blood flow. The smooth muscle cells are placed in the large network of elastic fibrils. This structure gives the vessel wall strength and elasticity. In large arteries, such as the aorta, the smooth muscle cells are almost absent and the tunica media mainly consists of elastic fibrils. Therefore, these kind of vessels are often referred to as *elastic arteries*. The opposite is true for arteries situated towards the periphery, very often referred to as *muscular arteries*.

Finally, the last and outermost layer is the *tunica adventitia*. The thickness of this layer can differ from one to the other artery. It consist of two different structures, as can be seen in Figure 2.5. The *external elastic membrane* consist just as the internal elastic membrane of collagenous bundles and elastic fibrils. The connective tissue serves to connect the vessel to the surrounding tissue.

As becomes clear from the above classification, the geometry of a vessel wall is very complex. The elasticity of the vessel wall is determined by all constituents of the wall and is represented by the so-called *Young’s modulus* (see Table 2.1). This parameter clearly varies a lot. In the rest of this report we will assume the vessel wall to be thin, which means that the ratio between the diameter of the vessel and the thickness of the vessel wall is small. In Appendix A we will show that this assumption enables us to regard the vessel wall as a homogeneous medium and to use linear elasticity theory in modeling the behavior of the vessel wall. In this case we use an overall and commonly accepted Young’s modulus equal to $0.5 \cdot 10^6 \text{Pa}$.

<table>
<thead>
<tr>
<th>Constituent</th>
<th>Young’s modulus [MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastin</td>
<td>$\pm 0.5$</td>
</tr>
<tr>
<td>Collagen</td>
<td>$\pm 0.5 \cdot 10^3$ (stretched)</td>
</tr>
<tr>
<td>Smooth muscle cells</td>
<td>$\pm 0.1$ (relaxed)</td>
</tr>
<tr>
<td></td>
<td>$\pm 2.0$ (activated)</td>
</tr>
</tbody>
</table>

Table 2.1: The Young’s modulus for the three main constituents of the vessel wall.

### 2.3 Blood

Blood takes care of the transport of, amongst others, oxygen en waste products in the human body. It consists of a fluid called *blood plasma* and some solid substances. These substances can be divided into *platelets*, *red* and *white blood cells*. 

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The blood plasma, which constitutes approximately 55% of the blood, consists of water and dissolved proteins. The plasma transports many substances such as glucose, vitamins, oxygen, minerals, carbon dioxide and of course platelets and blood cells. Some of these substances can dissolve easily into the plasma (for example glucose), while others need to be coupled to a plasma-protein to dissolve into the blood plasma. Blood plasma plays an important role in regulating the internal human body climate via a homeostatic regulation system. On average, the plasma has a constant temperature of $38\,^\circ\text{C}$ and a pH-value of about $7.4$.

The platelets and the red and white blood cells arise from a blood stem cell. These stem cells can be found mainly in the red bone marrow of for example the shoulder bones and ribs.

The red blood cells (erythrocytes) look like small circular plates that are less thick towards the center of the plate. These cells only live for 4 months. Consequently, the red bone marrow produces red blood cells continuously (approximately 2 million cells per second). The red blood cells contain hemoglobin, which is essential in the transport of oxygen and carbon dioxide and gives the blood its red color. People who suffer from anemia have an insufficient level of hemoglobin, probably because they have a shortage of iron minerals in their nutrition.

The white blood cells (leukocytes) can change their form thus enabling them to diffuse through a vessel wall. The red bone marrow produces three types of white blood cells: lymphocytes, granulocytes and monocytes. Lymphocytes can make antibodies against different types of foreign material, granulocytes are able to encapsulate bacteria, while monocytes have the same function as granulocytes, but these cells can also remove dead cells after an infection. People who suffer from leukemia have an overdosis of white blood cells because of a malfunction of the stem cells.
Finally, the platelets (thrombocytes) are parts of fragmented cells. The platelets play an important role in blood clotting. For example, they can react on substances coming from an injured vessel wall. This reaction leads to a mesh of fibrin onto which red blood cells collect and clot. People who suffer from hemophilia have a genetic disorder in the level of platelets. The composition of blood is summarized in Figure 2.6.

In the main body of this report we will regard the blood as a whole (plasma and cells) and model it as an incompressible Newtonian fluid. The material parameters of this fluid are given in Table 2.2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Notation</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>$\rho$</td>
<td>$\text{kg/m}^3$</td>
<td>$1.05 \cdot 10^3$</td>
</tr>
<tr>
<td>Kinematic viscosity</td>
<td>$\nu$</td>
<td>$\text{m}^2/\text{s}$</td>
<td>$3.5 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>Dynamic viscosity</td>
<td>$\eta$</td>
<td>$\text{kg/(m \cdot s)}$</td>
<td>$3.675 \cdot 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 2.2: Values for the material parameters of blood, regarded as a Newtonian fluid.
Chapter 3

Mathematical model

In this chapter the flow of blood through a deformable (elastic) thin-walled vessel is investigated. As in many (bio)mechanical problems it is necessary to make simplifying assumptions and to find the equations of motion in order to analyse the different types of behavior in a given configuration. In this chapter the equations of motion for the blood flow and the relevant boundary conditions are formulated. The equations are scaled in order to estimate the relative relevance of the different terms in these equations. This leads to some, generally valid, simplification of the system of equations. Finally, the local 3-D equations are integrated over the cross-section to obtain the global 1-D equations.

3.1 Geometry of the model

For describing the blood flow through a (deformable or corrugated) vessel a cylindrical coordinate system \((r, \theta, z)\) is introduced. A schematic representation of the configuration is given in Figure 3.1. We assume the flow to be rotationally symmetric, i.e. \(v_\theta = 0\) and the flow variables are independent of \(\theta\). This leads us to the following set of unknowns:

\[
\begin{align*}
    v_r &= v_r(r, z, t), \\
    v_z &= v_z(r, z, t), \\
    p &= p(r, z, t),
\end{align*}
\]

(3.1)

for \(0 \leq r \leq R(z, t), t > 0\) and \(z \in [0, \infty)\). Here, \(v_r\) is the radial velocity, \(v_z\) the axial velocity, \(p\) the pressure and \(R\) the current radius of the vessel at time \(t\) and axial position \(z\). If \(R_a\) is the typical length scale in radial direction and \(L\) the typical length scale in axial direction, then we assume \(R_a\) to be much smaller than \(L\). For later purposes we define the parameter \(\varepsilon\) to be equal to \(R_a/L\). This means that \(\varepsilon \ll 1\).

Figure 3.1: Schematic representation of a thin-walled vessel
3.2 Mechanical model

The flow of a fluid is described by the equation of continuity and the momentum equation. The fluid (i.e. blood) is modeled here as an incompressible Newtonian fluid. Using the corresponding constitutive equation in the momentum equation, or equation of motion, this equation becomes the Navier-Stokes equation. These equations reduce for our rotationally symmetric configuration to:

\[
\begin{align*}
\frac{1}{r} \frac{\partial}{\partial r} \left( r v_r \right) + \frac{\partial v_z}{\partial z} &= 0, \quad (3.2) \\
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + v \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_r}{\partial r} \right) + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} \right], \quad (3.3) \\
\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + v \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{\partial^2 v_z}{\partial z^2} \right]. \quad (3.4)
\end{align*}
\]

Equation (3.2) states the conservation of mass (equation of continuity) for an incompressible fluid, and (3.3) and (3.4) represent the \( r \)- and \( z \)-component of the momentum equation, respectively.

3.2.1 Boundary conditions

Two boundary conditions at the wall \( r = R(z, t) \) are given by

\[
(v, t) = 0, \quad (v, n) = \frac{\partial R}{\partial t}, \quad (3.5)
\]

where \( t \) is the axial tangential vector and \( n \) the outward normal vector to the wall. If we define \( \gamma \) to be the angle between the radial direction and the normal vector to the wall, then \( n \) and \( t \) can be expressed as

\[
\begin{align*}
  n &= \sin(\gamma) e_z + \cos(\gamma) e_r, \\
  t &= \cos(\gamma) e_z - \sin(\gamma) e_r.
\end{align*} \quad (3.6)
\]

Substitution of (3.6) into (3.5) yields

\[
\begin{align*}
(v, t) &= v_z \cos(\gamma) - v_r \sin(\gamma) = 0, \quad (3.7) \\
(v, n) &= v_z \sin(\gamma) + v_r \cos(\gamma) = \frac{\partial R}{\partial t}. \quad (3.8)
\end{align*}
\]

From (3.7) and (3.8) we find

\[
\begin{align*}
  v_r &= \frac{\partial R}{\partial t} \cos(\gamma), \\
  v_z &= \frac{\partial R}{\partial t} \sin(\gamma). \quad (3.9)
\end{align*}
\]

Since

\[
\tan(\gamma) = \frac{\partial R}{\partial z} = O \left( \frac{R_a}{L} \right) = O(\varepsilon) \ll 1, \quad (3.10)
\]

by assumption, we have \( \gamma = O(\varepsilon) \) and then, neglecting all \( O(\varepsilon) \)-effects, we obtain

\[
\begin{align*}
  v_r(R, z, t) &= \frac{\partial R}{\partial t}, \\
  v_z(R, z, t) &= 0. \quad (3.11)
\end{align*}
\]

These are the boundary conditions we shall use in the sequel.

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3.3 Scaling

Let us introduce the following dimensionless variables:

\[ \hat{\tau} := \frac{r}{R_a}, \quad \hat{\zeta} := \frac{\zeta}{L}, \quad \hat{t} := \frac{t}{T}, \]

(3.12)

where \( R_a \), the typical length scale in radial direction is the inner radius of the undeformed vessel, \( L \) the typical length scale in \( z \)-direction and \( T \) a typical time scale depending on the frequency \( \omega \) of the pulsating blood flow through \( T = \omega^{-1} \). Also we define the dimensionless velocities and pressure by

\[ \hat{v}_r := \frac{v_r}{V_r}, \quad \hat{v}_z := \frac{v_z}{V_z}, \quad \hat{p} := \frac{p}{P_c}, \]

(3.13)

with \( V_r, V_z \) and \( P_c \) specified further on. Introducing the dimensionless variables into (3.2), we obtain

\[ \frac{1}{\hat{\tau}} \frac{\partial}{\partial \hat{\tau}} (\hat{\tau} R_a V_r \hat{v}_r) + \frac{V_z}{L} \frac{\partial \hat{v}_z}{\partial \hat{\zeta}} = 0. \]

(3.14)

To make the two terms in (3.14) of the same order of magnitude we take \((L V_r / R_a V_z) = 1\), which leads to

\[ V_z = V, \quad V_r = \frac{R_a}{L} V_z = \varepsilon V, \]

(3.15)

or

\[ v_r = \varepsilon V \hat{v}_r, \quad v_z = V \hat{v}_z, \]

(3.16)

and so (3.2) becomes in dimensionless form

\[ \frac{1}{\hat{\tau}} \frac{\partial}{\partial \hat{\tau}} (\hat{\tau} \hat{v}_r) + \frac{\partial \hat{v}_z}{\partial \hat{\zeta}} = 0. \]

(3.17)

Here, \( V \) is the characteristic speed for the axial flow through the vessel (e.g. the averaged axial velocity). Substituting the dimensionless variables into (3.3) and (3.4) and multiplying by \((R_a^2 / \varepsilon V)\) and \((\varepsilon V / V)\), respectively, we arrive at

\[ \frac{R_a^2}{TV} \frac{\partial \hat{v}_r}{\partial \hat{\tau}} + \frac{R_a \varepsilon V}{v} \frac{\partial \hat{v}_r}{\partial \hat{\tau}} + \frac{R_a V}{v} \frac{\partial \hat{v}_z}{\partial \hat{\tau}} = - \frac{1}{\rho} \frac{\partial}{\partial \hat{\tau}} (\hat{\tau} \frac{\partial \hat{v}_r}{\partial \hat{\tau}}) + \frac{1}{\rho} \frac{\partial}{\partial \hat{\tau}} \left( \hat{\tau} \frac{\partial \hat{v}_z}{\partial \hat{\tau}} \right) + \varepsilon^2 \frac{\partial^2 \hat{v}_r}{\partial \hat{\zeta}^2} = \hat{v}_r. \]

(3.18)

\[ \frac{R_a^2}{vT} \frac{\partial \hat{v}_z}{\partial \hat{\tau}} + \frac{\varepsilon V R_a}{v} \frac{\partial \hat{v}_r}{\partial \hat{\tau}} + \frac{\varepsilon V R_a}{v} \frac{\partial \hat{v}_z}{\partial \hat{\tau}} = - \frac{1}{\rho} \frac{\partial}{\partial \hat{\tau}} (\hat{\tau} \frac{\partial \hat{v}_z}{\partial \hat{\tau}}) + \frac{1}{\rho} \frac{\partial}{\partial \hat{\tau}} \left( \hat{\tau} \frac{\partial \hat{v}_r}{\partial \hat{\tau}} \right) + \varepsilon^2 \frac{\partial^2 \hat{v}_z}{\partial \hat{\zeta}^2}. \]

(3.19)

We introduce the following two dimensionless numbers:

\[ Re = \frac{R_a V}{v}, \quad \alpha = Ra \sqrt{\frac{1}{TV}} = Ra \sqrt{\frac{\omega}{v}}, \]

(3.20)

referred to as the Reynolds number and the Womersley number, respectively.

We now have two possibilities to scale the pressure (gradient):

1. We assume that the axial pressure gradient scales with the axial viscous term, suggesting that \((R_a^2 P_c / \rho \varepsilon V L)\) should be chosen equal to 1, yielding

\[ P_c = p^{(v)} = \frac{\rho \varepsilon V L}{R_a^2} \quad \Rightarrow \quad p = \frac{\rho V L}{R_a^2} \hat{p}. \]

(3.21)

The justification for this choice is the requirement that the advection terms in (3.19) remain of \( \mathcal{O}(1) \), or \( \varepsilon V R_a / v = VL / v = \mathcal{O}(1) \), for \( \varepsilon \ll 1 \).
2. We assume that the axial pressure gradient scales with the axial advection term, suggesting that \((\varepsilon V R_a/\nu) = (R_a^2 P_c/\rho v V L)\) should be chosen, yielding

\[
P_c = P^{(c)} = \rho V^2 \quad \Rightarrow \quad p = \rho V^2 \hat{P}.
\] (3.22)

The justification for this choice is the requirement that, after dividing (3.19) by \(\varepsilon V R_a/\nu\), the viscous term remains of \(O(1)\), or \(v/\varepsilon V R_a = v/\nu V L = O(1)\), for \(\varepsilon \ll 1\).

Hence, the choice for the two options 1. and 2. is determined by the order of magnitude of \(\varepsilon \text{Re}\). Here, and in first instance, we opt for the first choice. With the use of \(\hat{P}\) and the two dimensionless numbers \(\text{Re}\) and \(\alpha\), the equations of motion become

\[
\frac{1}{r} \frac{\partial}{\partial r}(\hat{r} \hat{v}_r) + \frac{\partial \hat{v}_z}{\partial z} = 0,
\] (3.23)

\[
\varepsilon^2 \alpha^2 \frac{\partial \hat{\nu}_r}{\partial t} + \varepsilon \text{Re} \hat{\nu}_r \frac{\partial \hat{v}_r}{\partial r} + \varepsilon^2 \text{Re} \hat{v}_z \frac{\partial \hat{v}_z}{\partial z} = -\frac{\partial \hat{P}}{\partial r} + \varepsilon \frac{\partial}{\partial r} \left( \frac{\hat{r} \hat{v}_r}{r} \right) + \varepsilon^2 \frac{\partial^2 \hat{v}_z}{\partial z^2},
\] (3.24)

\[
\alpha^2 \frac{\partial \hat{\nu}_r}{\partial t} + \varepsilon \text{Re} \hat{\nu}_r \frac{\partial \hat{v}_r}{\partial r} + \varepsilon \text{Re} \hat{v}_z \frac{\partial \hat{v}_z}{\partial z} = -\frac{\partial \hat{P}}{\partial r} + \frac{1}{\varepsilon \text{Re}} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\hat{r} \hat{v}_r}{r} \right) + \varepsilon^2 \frac{\partial^2 \hat{v}_z}{\partial z^2} \right],
\] (3.25)

As said before, option 1. is justified if \(\varepsilon \text{Re} = O(1)\). However, for larger arteries, e.g. the aorta, \(\varepsilon \text{Re} \gg 1\), and then we have to use option 2. In that case (3.25) becomes

\[
\text{Sr} \frac{\partial \hat{\nu}_r}{\partial t} + \hat{v}_r \frac{\partial \hat{v}_r}{\partial r} + \hat{v}_z \frac{\partial \hat{v}_z}{\partial z} = -\frac{\partial \hat{P}}{\partial r} + \frac{1}{\varepsilon \text{Re}} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\hat{r} \hat{v}_r}{r} \right) + \varepsilon^2 \frac{\partial^2 \hat{v}_z}{\partial z^2} \right],
\] (3.26)

where \(\text{Sr}\) is the Strouhal number defined as

\[
\text{Sr} = \frac{\alpha^2}{\varepsilon \text{Re}} = \frac{\omega L}{V},
\] (3.27)

which is always \(O(1)\), for \(\varepsilon \ll 1\), in all our applications. Considering (3.24), we note that if \(\varepsilon \text{Re} = O(1)\) (option 1) and \(\alpha = O(1)\), then \(\partial \hat{P}/\partial r = O(\varepsilon)\); this holds for not too large arteries. On the other hand, for larger arteries such as the aorta, \(\varepsilon \text{Re}\) becomes \(O(\varepsilon^{-1})\), and also \(\alpha = O(\varepsilon^{-1})\) then. In that case we have to apply option 2, and then again we conclude that \(\partial \hat{P}/\partial r = O(\varepsilon)\) (note for instance that \(\varepsilon^2 \alpha^2 \rightarrow \varepsilon^2 \text{Sr} = O(\varepsilon^2)\)). Hence, for all types of arteries we have

\[
\frac{\partial \hat{P}}{\partial r} = O(\varepsilon^2) = 0,
\] (3.28)

for \(\varepsilon \rightarrow 0\).

Equation (3.23) is not influenced at all by the scaling, so it keeps its original form. Finally (3.25) shows different behavior for either small or larger arteries: for the small ones the advective terms can be neglected (diffusion dominated flow), whereas for the larger ones the viscous term is negligible (advection dominated flow). To keep our analysis general (for large and small arteries), we neglect neither the advection nor the viscous term. However, we see that the contribution of the term \(\partial^2 \hat{v}_z/\partial z^2\) in the viscous term is always \(O(\varepsilon^2)\), compared to the radial diffusion term and therefore we will consistently neglect this term, unless explicitly stated otherwise.

Summarizing, accounting for the results of our scaling, we arrive at the following reduced, again
dimensional, system of equations, as we will use it in the rest of this report,

\[
\frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{\partial v_z}{\partial z} = 0, \tag{3.29}
\]

\[
\frac{\partial p}{\partial r} = 0 \quad \Rightarrow \quad p = p(z, t), \tag{3.30}
\]

\[
\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right). \tag{3.31}
\]

To conclude this section, we take a short look at the boundary conditions, especially \([3.11]\) for \(v_r\), which with \(v_r = \varepsilon V \hat{v}_r\), \(R = R_a \hat{R}\) and \(t = T \hat{t}\) becomes

\[
\hat{v}_r = \frac{R_a}{\varepsilon V T} \frac{\partial \hat{R}}{\partial \hat{t}} = \frac{L V T}{\partial R / \partial t}, \tag{3.32}
\]

for \(\hat{r} = \hat{R}\). This would suggest a time scale \(T = L/V\), while up to here we have used \(T = \omega^{-1}\). However, at this point there seems to be no need to change this time scale, because their appears to be never a ‘large order of magnitude’ difference between these two time scales, since \(Sr = \omega L/V\) is always strictly of \(O(1)\), for \(\varepsilon \ll 1\).

### 3.4 One-dimensional equations of motion

In the previous section the local equations of motion were presented. In this section the global version of these equations is derived. To do so, the equations are integrated over the cross-sectional surface, resulting in equations only depending on \(z\) and \(t\).

1. For the equation of continuity:

\[
\int_0^{2\pi} \int_0^{R(z,t)} \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{\partial v_z}{\partial z} \right\} r \, dr \, d\theta =
\]

\[
2\pi \int_0^{R(z,t)} \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{\partial v_z}{\partial z} \right\} r \, dr =
\]

\[
2\pi \left[ r v_z \right]_0^{R(z,t)} + 2\pi \frac{\partial}{\partial z} \int_0^{R(z,t)} r v_z \, dr = 2\pi \left[ r v_z \frac{\partial R}{\partial z} \right]_0^{R(z,t)} =
\]

\[
\frac{\pi}{\partial t} (R^2(z,t)) + \frac{\partial}{\partial z} (A(z,t) \overline{v}_z(z,t)) =
\]

\[
\frac{\partial}{\partial t} (A(z,t)) + \frac{\partial}{\partial z} (A(z,t) \overline{v}_z(z,t)) = 0, \tag{3.33}
\]

where in the third step we have used the boundary conditions \([3.11]\). Here, \(A = \pi R^2\) is the cross-sectional area at \((z, t)\) and \(\overline{v}_z\) the mean velocity at \((z, t)\), defined as

\[
\overline{v}_z(z,t) = \frac{2\pi}{A(z,t)} \int_0^{R(z,t)} r v_z(r,z,t) \, dr. \tag{3.34}
\]
2. For the left-hand side of the $z$-component of the Navier-Stokes equation:

$$2\pi \int_0^{R(z,t)} \left\{ \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right\} r \, dr =$$

$$2\pi \int_0^{R(z,t)} \left\{ r \frac{\partial v_z}{\partial t} + \frac{\partial}{\partial r} \left( r v_r v_z \right) - v_z \frac{\partial}{\partial r} \left( r v_r \right) + r v_z \frac{\partial v_z}{\partial z} \right\} dr =$$

$$2\pi \int_0^{R(z,t)} \left\{ \frac{\partial}{\partial t} \int_0^{R(z,t)} r v_z \, dr - \left[ r v_z \frac{\partial R}{\partial t} \right]_{R(z,t)} + 2\pi \left[ r v_z v_r \right]_{R(z,t)} \right\} =$$

$$2\pi \left\{ \frac{\partial}{\partial t} \int_0^{R(z,t)} r v_z^2 \, dr - \left[ r v_z^2 \frac{\partial R}{\partial z} \right]_{R(z,t)} \right\} =$$

$$\frac{\partial}{\partial t} \left(A(z,t) \bar{v}_z(z,t)\right) + \frac{\partial \gamma_a}{\partial z}, \quad (3.35)$$

where in the second step the equation of continuity $\frac{\partial}{\partial t} \left( \int_A \rho \, dz \right) = 0$ is used. Here, following [10], we have introduced $\gamma_a$ by

$$\gamma_a = \gamma_a(z,t) = 2\pi \int_0^{R(z,t)} r v_z^2 \, dr. \quad (3.36)$$

Hence, $\gamma_a$ represents the contribution of the nonlinear advection term.

For the right-hand side of the $z$-component of the Navier-Stokes equation:

$$2\pi \int_0^{R(z,t)} \left\{ - \frac{1}{\rho} \frac{\partial p}{\partial z} + v_r \frac{\partial}{\partial r} \left( \frac{\partial v_z}{\partial r} \right) \right\} r \, dr =$$

$$- \frac{\pi}{\rho} \frac{\partial p}{\partial z} R^2(z,t) + 2\pi v \left[ r \frac{\partial v_z}{\partial r} \right]_{R(z,t)} =$$

$$- \frac{1}{\rho} \frac{\partial p}{\partial z} A(z,t) - \tau_w(z,t), \quad (3.37)$$

where

$$\tau_w(z,t) = -2\pi v R(z,t) \frac{\partial v_z}{\partial r} \bigg|_{R(z,t)}, \quad (3.38)$$

the shear force per unit of length on the fluid at the wall of the vessel. Just as the nonlinear advection term in $\frac{\partial}{\partial t} \left( \int_A \rho \, dz \right)$, the shear term $\tau_w$ can not directly be expressed in terms of the global quantities $p(z,t)$ and $A(z,t)$, because for this we need to know the precise distribution of $v_z(r, z, t)$ over the cross-section of the flow. We come back to this subject in Chapter 5 of this report.

Before recapitulating the two 1-D, or global, equations of motion, we introduce the volumetric flow rate $q(z,t)$ through the vessel by

$$q(z,t) = A(z,t) \bar{v}_z(z,t) = 2\pi \int_0^{R(z,t)} r v_z(r, z, t) \, dr. \quad (3.39)$$
Then, the equations become

\[ \frac{\partial A}{\partial t} + \frac{\partial q}{\partial z} = 0, \quad (3.40) \]

\[ \frac{\partial q}{\partial t} + \frac{\partial \gamma_a}{\partial z} + A \frac{\partial p}{\rho \partial z} + \tau_w = 0, \quad (3.41) \]

We note that, assuming for a moment \( \gamma_a \) and \( \tau_w \) known, this system constitutes of two equations for three unknowns \( A, p \) and \( q \). What we still need is a constitutive equation expressing \( A \) in terms of \( p \) and representing the elasticity of the vessel wall. This relation is derived in Appendix A and describes the linear elastic behavior of the thin-walled vessel wall, and reads

\[ A(z, t) = A_0 + C(p(z, t) - p_0), \quad (3.42) \]

where \( A_0 = \pi R_a^2 \), the area of the undeformed cross section of the vessel and \( C \) the compliance of the wall. For the deformed radius of the wall we then obtain

\[ R(z, t) = \sqrt{\frac{1}{\pi} A(z, t)} = \sqrt{R_a^2 + \frac{C}{\pi} (p(z, t) - p_0)} \approx R_a \left( 1 + \frac{C}{2\pi R_a^2} (p(z, t) - p_0) \right), \quad (3.43) \]

where the latter step is in accordance with linear elasticity (here, \( \approx \) means linearization).

In Chapter 4 we will describe and define (the geometry of) the problem in full detail, such that we can pose a set of boundary and initial conditions which together with the equations of motion, as given in (3.40) and (3.41), form a well-posed mathematical description of our problem.

### 3.5 Aims and proposed solution tactics

The complexity of both the local 3-D system (3.29)-(3.31) and the global 1-D system (3.40)-(3.41) reduces drastically if both radial diffusion (represented by \( \tau_w \)) and advection (represented by \( \gamma_a \)) are neglected. However, in practical situations this is almost never allowed. On the other hand, in many cases one of the two effects may be discarded. For instance, when the radial diffusion is important (as in smaller arteries) advection is negligible and oppositely (for larger arteries). It is seldom that both effects are equally relevant. Axial diffusion is almost always negligible. In the remaining of this report we will try to estimate the order of magnitude of the two effects; radial diffusion and advection, and to indicate under what conditions these effects can, or can not, be neglected.

In the next chapter, we will analyse the global 1-D system. We will solve the three cases: (i) \( \tau_w = \gamma_a = 0 \), (ii) \( \tau_w \neq 0, \gamma_a = 0 \) (iii) \( \tau_w = 0, \gamma_a \neq 0 \). We will try to find as much as possible fully analytical solutions for these cases.

In Chapter 5 we will analyse the local 3-D system. Here, we will employ asymptotic expansions for the flow characteristics in order to find a leading order solution for the axial flow velocity. Our aim is to find expressions for \( \tau_w \) and \( \gamma_a \), which we then, at its turn, can use in the analytical 1-D results of Chapter 4. All this will finally result in a classification of the order of relevance of the diffusion and advection for a range of typical human arteries.
Mathematical modeling of blood flow through a deformable thin-walled vessel
Chapter 4

The global equations of motion

In the previous chapter we derived a general mathematical model for blood flow through a deformable vessel and we distinguished two different types of equations of motion, namely the global and the local ones. The geometry of the problem is the same as sketched in Section 3.1, in so far that we here consider a semi-infinite vessel \((z \geq 0)\). At the entrance of the vessel, the inlet flow rate is prescribed, starting at \(t = 0\). We are specially interested in the transient part of the solution. From now on we focus on thin-walled vessels only, i.e. vessels for which the ratio of wall thickness and radius is small. We investigate separately the effects of the viscous diffusion term and the advection in the global equations of motion for a thin-walled vessel. In order to estimate the importance of both effects, we start from a simple example where the effects are absent. As far as possible, we try to find analytical solutions of the governing equations of motion. The results of this chapter will graphically be presented in Chapter 5, where we impose a certain given inlet flow.

4.1 Scaling for a thin-walled vessel

The specific problem we consider in this chapter is a suddenly started inflow of a given amount of fluid into a thin-walled deformable semi-infinite vessel, which is initially already filled with fluid. The flow into the vessel starts at \(t = 0\). So, for \(t < 0\), the flow rate is zero, the pressure is equal to the reference pressure \(p_0\) and the vessel is still undeformed with area \(A_0\). The flow rate through the initial cross-section at \(z = 0\), is prescribed and given by \(q_i(t)\). For finite times, the pressure gradient goes to zero for \(z\) going to infinity. We consider here a given inlet flow rate of pulse-type (so not periodic), only unequal to zero during a finite period of time after \(t = 0\). We are especially interested in the transient behavior of the flow. This is in contrast to most of literature in which periodic behavior is studied.

We start this chapter with a rigorous scaling analysis of the global equations of motion, as stated in (3.40) and (3.41), adapted for the case of a linearly elastic thin-walled vessel. For these kind of vessels we can assume that the cross-sectional area \(A\) of the deformed vessel is an affine function of the pressure \(p\), described by the constitutive equation

\[
A(z, t) = A_0 + C(p(z, t) - p_0),
\]

where \(A_0\) is the area in the undeformed reference state, when \(p = p_0\), and \(C > 0\) is the compliance of the wall. This relation, the derivation of which is given in Appendix A, characterizes the
linear elastic behavior of the thin-walled vessel wall. Substituting (4.1) into (3.40) and (3.41), we obtain, for \( z > 0 \) and \( t > 0 \),
\[
C \frac{\partial p}{\partial t} + \frac{\partial q}{\partial z} = 0, \tag{4.2}
\]
\[
\frac{\partial q}{\partial t} + \frac{\partial \gamma_a}{\partial z} + \frac{A_0}{\rho} \left( 1 + \frac{C}{A_0} (p - p_0) \right) \frac{\partial p}{\partial z} + \tau_w = 0, \tag{4.3}
\]
where \( \gamma_a \) and \( \tau_w \) are given by (3.36) and (3.38), respectively. The boundary and initial conditions owing to the inlet problem described above are
\[
p(z, 0) = q(z, 0) = 0 \quad \text{for } z > 0, \\
q(0, t) = q_i(t) \quad \text{for } t > 0, \\
p(z, t) \to 0 \quad \text{for } z \to \infty, t > 0. \tag{4.4}
\]
To scale the dimensional equations (4.2) and (4.3), we define dimensionless variables, pressure and flow rate in the following way:
\[
t := T \hat{t}, \\
z := L \hat{z}, \\
p := p_0 + \hat{p}, \\
q := Q_0 \hat{q}. \tag{4.5}
\]
where \( Q_0 \) is the maximum of \( q_i(t) \) for \( t > 0 \). Substituting these dimensionless quantities into (4.2) and (4.3) leads us to \( C P L/Q_0 T = 1 \), and \( A_0 PT/\rho Q_0 L = 1 \), or, after elimination of \( T/L \), to \( (A_0/\rho C) (C P/\bar{Q}_0) = 1 \), yielding
\[
P = \frac{Q_0}{C} \sqrt{\frac{\rho C}{A_0}} = \frac{Q_0}{C_0} = \rho_c V, \tag{4.6}
\]
where \( c_0 = \sqrt{A_0/\rho C} \), the unperturbed wave velocity for pressure waves in the undeformed thin-walled wall and \( V = Q_0/A_0 \), the characteristic velocity of the flow. We note that for the semi-infinite tube \( L \) is undetermined, but from \( C P L/Q_0 T = 1 \) and with \( T = 1/\omega \) it follows that \( L = c_0/\omega \) is an appropriate characteristic length for this problem. Finally, we introduce the dimensionless nonlinear advection and diffusion term in a natural sense as follows:
\[
\gamma_a = A_0 V^2 \gamma_a, \\
\tau_w = Q_0 \omega \hat{\tau}_w. \tag{4.7}
\]
Observe that the actual (perturbed) wave velocity of the elastic wall is given by
\[
c = c(z, t) = \sqrt{\frac{A(z, t)}{\rho C}} = \sqrt{\frac{A_0 + C(p(z, t) - p_0)}{\rho C}} = c_0 \sqrt{1 + \delta \hat{p}(z, t)}, \tag{4.8}
\]
where \( \delta = Q_0/(A_0 c_0) = V/c_0 \). Thus the system (4.2) and (4.3) can be written in dimensionless formulation as
\[
\frac{\partial \hat{p}}{\partial t} + \frac{\partial \hat{q}}{\partial z} = 0, \tag{4.9}
\]
\[
\frac{\partial \hat{q}}{\partial t} + \delta \frac{\partial \gamma_a}{\partial z} + (1 + \delta \hat{p}) \frac{\partial \hat{p}}{\partial z} + \hat{\tau}_w = 0. \tag{4.10}
\]
The transformation of the boundary and initial conditions to their dimensionless form is trivial. In the sequel we will omit the ‘hats’ for the dimensionless quantities.
4.2 Basic example

In this basic example we neglect both the advection an diffusion term ($\gamma_a = \tau_w = 0$). The global equations of motion, as stated in (4.9) and (4.10), then simplify to

$$\frac{\partial p}{\partial t} + \frac{\partial q}{\partial z} = 0, \quad (4.11)$$

$$\frac{\partial q}{\partial t} + (1 + \delta P) \frac{\partial p}{\partial z} = 0. \quad (4.12)$$

We assume these equations to hold for $z > 0$ (semi-infinite vessel) and $t > 0$, and we impose the following boundary and initial conditions:

$$p(z, 0) = q(z, 0) = 0 \quad \text{for } z > 0,$$

$$q(0, t) = q_i(t) \quad \text{for } t > 0,$$

$$p(z, t) \to 0 \quad \text{for } z \to \infty, \quad t > 0 \quad (4.13)$$

for a given inflow $q_i(t)$, with $q_i(0) = 0$. The system (4.11) - (4.12) consists of two 1-D, nonlinear, wave equations for $p$ and $q$. This system of equations can be solved using the method of characteristics. To this end, let us assume that $z = \gamma(t)$ and $t = \gamma(z)$ and so $p(z, t) = P(\gamma)$ and $q(z, t) = Q(\gamma)$. The system of equations now transforms into

$$\frac{dP}{d\gamma} \frac{d\gamma}{dt} + \frac{dQ}{d\gamma} \frac{d\gamma}{dz} = 0, \quad (4.14)$$

$$\frac{dQ}{d\gamma} \frac{d\gamma}{dt} + (1 + \delta P) \frac{dP}{d\gamma} \frac{d\gamma}{dz} = 0. \quad (4.15)$$

For this system to have a non-trivial solution, the determinant of the coefficient matrix has to be zero, i.e.

$$\left( \frac{d\gamma}{dt} \right)^2 - (1 + \delta P) \left( \frac{d\gamma}{dz} \right)^2 = 0, \quad (4.16)$$

or

$$c^2 = (1 + \delta P) \frac{\gamma_t}{\gamma_z}^2 \quad \Rightarrow \quad \frac{\gamma_t}{\gamma_z} = \pm c, \quad (4.17)$$

where we used the notation $\gamma_t$ and $\gamma_z$ for the derivative of $\gamma$ with respect to $t$ and to $z$, respectively. We can now define two characteristics, say $\Gamma_1$ and $\Gamma_2$, along which successively $\gamma_2$ (referring to $-c$) or $\gamma_1$ (referring to $+c$) are constant.

4.2.1 Solution for constant wave velocity

As a first step, we consider a further simplification in which we assume the compliance $C$ very small (i.e. the wall is very stiff), implying that then $A \approx A_0$, and consequently $\delta = 0$. In that case the system of two wave equations becomes linear,

$$\frac{\partial \gamma_1}{\partial t} - \frac{\partial \gamma_1}{\partial z} = 0 \Rightarrow \gamma_1 = t + z \quad \text{along } \Gamma_1, \quad (4.18)$$

$$\frac{\partial \gamma_2}{\partial t} + \frac{\partial \gamma_2}{\partial z} = 0 \Rightarrow \gamma_2 = t - z \quad \text{along } \Gamma_2; \quad (4.19)$$

see Figure 4.1 (a). This gives rise to the following expressions for $p(z, t)$ and $q(z, t)$:

$$p(z, t) = P_1(t + z) + P_2(t - z), \quad (4.20)$$

$$q(z, t) = Q_1(t + z) + Q_2(t - z). \quad (4.21)$$

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Substituting these expressions into (4.11), we get
\[ P_1'(t+z) + P_2'(t-z) + Q_1'(t+z) - Q_2'(t-z) = 0, \tag{4.22} \]
yielding
\[ Q_1(t+z) = P_1(t+z) + Q_{1,0}. \tag{4.23} \]
\[ Q_2(t-z) = P_2(t-z) + Q_{2,0}. \tag{4.24} \]
From the initial conditions (4.13), it follows immediately that \( Q_1(t+z) = P_1(t+z) = Q_{1,0} = 0 \), for all \( z > 0, t > 0 \), and that \( Q_2(t-z) = P_2(t-z) = Q_{2,0} = 0 \), for all \( z > t \). This means that there is only an ongoing wave (propagating in positive \( z \)-direction), because there is no reflection in a semi-infinite vessel, and that this wave is only non-zero in the region \( 0 < z < t \). The condition at infinite is then already satisfied, while the boundary condition at \( z = 0 \) yields \( P_2(t) = q_i(t) \), for all \( t > 0 \), implying that \( P_2(t-z) = q_i(t-z) \), for \( 0 < z < t \). Thus, we arrive at the following explicit solution for this dimensionless linear system:
\[ p(z,t) = q(z,t) = q_i(t-z) \, H(t-z), \tag{4.25} \]
for \( z > 0, t > 0 \), where \( H(\cdot) \) is the Heaviside function.

### 4.2.2 General case for variable wave velocity

We now return to the general case in which \( c = c(z,t) = \sqrt{1 + \delta^2} \) is still unknown. In the same way as for the case \( \delta = 0 \), we introduce a variable \( \gamma \) such that \( p(z,t) = P(\gamma) \) and \( q(z,t) = Q(\gamma) \). This results in defining the two characteristics:
\[ \Gamma_1 = \{ \gamma_1, \gamma_2 \in \mathbb{R} | \gamma_2(z,t) = \gamma_2 \text{ is constant} \}, \tag{4.26} \]
\[ \Gamma_2 = \{ \gamma_1, \gamma_2 \in \mathbb{R} | \gamma_1(z,t) = \gamma_1 \text{ is constant} \}, \tag{4.27} \]
where \( \gamma_1 \) describes waves traveling in negative \( z \)-direction and \( \gamma_2 \) waves traveling in positive \( z \)-direction. See Figure 4.1 (b), for a sketch of the characteristics. Again, \( \gamma_1 \) and \( \gamma_2 \) satisfy (4.18) and
(4.19). Since for a semi-infinite vessel no backward traveling waves exist, we infer that \( p = P(\gamma_2) \) and \( q = Q(\gamma_2) \). This means that along \( \Gamma_1 \), \( P(\gamma_2) \) and \( Q(\gamma_2) \) are both constant.

From (4.14) and (4.17) it follows that
\[
\frac{dP}{d\gamma_2} \frac{\partial \gamma_2}{\partial t} + \frac{dQ}{d\gamma_2} \frac{\partial \gamma_2}{\partial z} = \left[ -\frac{dP}{d\gamma_2} \hat{c}(\gamma_2) + \frac{dQ}{d\gamma_2} \right] \frac{\partial \gamma_2}{\partial z} = 0, \tag{4.28}
\]
implying that (when \( \gamma_2, z \neq 0 \))
\[
\frac{dQ}{d\gamma_2} = \hat{c}(\gamma_2) \frac{dP}{d\gamma_2}. \tag{4.29}
\]
We can integrate (4.29) to obtain
\[
Q(\gamma_2) = Q(0) + \int_0^{\gamma_2} \hat{c}(\gamma) \frac{dP}{d\gamma} \, d\gamma = Q(0) + \int_{P(0)}^{P(\gamma_2)} \sqrt{1 + \delta P} \, dP \tag{4.30}
\]
\[
= Q(0) + \frac{2}{3\delta} \left[ \left[ 1 + \delta P(\gamma_2) \right]^{\frac{3}{2}} - \left[ 1 + \delta P(0) \right]^{\frac{3}{2}} \right] = \frac{2}{3\delta} \left[ \left[ 1 + \delta P(\gamma_2) \right]^{\frac{3}{2}} - 1 \right], \tag{4.31}
\]
because \( P = Q = 0 \) for \( \gamma_2 = 0 \). Inverting this relation, we obtain
\[
P(\gamma_2) = \frac{1}{\delta} \left[ 1 + \frac{3\delta}{2} Q(\gamma_2) \right]^{\frac{2}{3}} - 1. \tag{4.32}
\]
Let us take a closer look at the specific characteristic \( \Gamma_1 \) that intersects the \( t \)-axis at \( t = t_0 \), for arbitrary \( t_0 > 0 \) (see Figure 4.1). This means that \( \gamma_2 = \gamma_2(0, t_0) = \hat{\gamma}_2(t_0) \). Because \( P \) and \( Q \) are constant along \( \Gamma_1 \), we know that also \( \hat{c}(\gamma_1) = \sqrt{1 + \delta P(\gamma_2)} \) is constant. This leads to
\[
0 = \frac{\partial \gamma_2}{\partial t} \, dt + \frac{\partial \gamma_2}{\partial z} \, dz = \frac{\partial \gamma_2}{\partial t} \left[ dt - \frac{dz}{c} \right], \quad \text{along } \Gamma_1, \tag{4.33}
\]
from which follows that
\[
t - \frac{z}{\hat{c}} = \text{constant} = t_0, \quad \text{along } \Gamma_1, \tag{4.34}
\]
where \( \hat{c} = \hat{c}(\gamma_2(0, t_0)) = \hat{c}(t_0) \). Finally, we still need to satisfy the initial condition \( q(0, t) = q_i(t) \). To this end, we consider the specific characteristic \( \Gamma_2 \) that intersects the \( t \)-axis in \( t = t_0 \). Along each \( \Gamma_2 \), \( q(z, t) = Q(\gamma_2) \), and hence in \( (z, t) = (0, t_0) \) we get
\[
Q(\hat{\gamma}_2(t_0)) = q(0, t_0) = q_i(t_0), \quad \text{for every } t_0 > 0, \tag{4.35}
\]
and then from (4.32)
\[
P(\hat{\gamma}_2(t_0)) = p(0, t_0) = \frac{1}{\delta} \left[ 1 + \frac{3\delta}{2} q_i(t_0) \right]^{\frac{2}{3}} - 1, \tag{4.36}
\]
and, moreover,
\[
\hat{c}(t_0) = \sqrt{1 + \delta p(0, t_0)} = \left( 1 + \frac{3}{2} \delta q_i(t_0) \right)^{\frac{1}{3}}. \tag{4.37}
\]
Substituting this expression into (4.34), we obtain
\[
t - \frac{z}{\left( 1 + \frac{3}{2} \delta q_i(t_0) \right)^{\frac{1}{3}}} = t_0, \tag{4.38}
\]
from which \( t_0 \) can be solved as \( t_0 = T_0(z, t) \). An explicit numerical solution is obtained with Matlab.

Summarizing, we note that the solutions for \( p \) and \( q \) are

\[
q(z, t) = q_1(T_0(z, t))H(t - z), \tag{4.39}
\]

\[
p(z, t) = \frac{1}{\delta} \left\{ \left[1 + \frac{3\delta}{2} q_1(T_0(z, t))\right]^2 - 1 \right\} H(t - z). \tag{4.40}
\]

We characterize the deformation of the vessel wall by its relative displacement \( \Delta R \), defined as

\[
\Delta R(z, t) = \frac{R(z, t) - R_a}{R_a} = \hat{R}(\hat{z}, \hat{t}) - 1. \tag{4.41}
\]

According to (4.1) the deformed area \( A(z, t) = \pi R^2(z, t) \) is a local and linear (in accordance with the linear elasticity theory assumed here) function of the pressure, i.e.

\[
A(z, t) = \pi R^2(z, t) = \pi R_a^2 + C(p(z, t) - p_0) = \pi R_a^2 \left(1 + \frac{CP}{\pi R_a^2} \hat{p}(\hat{z}, \hat{t})\right), \tag{4.42}
\]

or, with \( CP/\pi R_a^2 = Q_0/c_0 \pi R_a^2 = V/c_0 = \delta \),

\[
\hat{R}(\hat{z}, \hat{t}) = \sqrt{1 + \delta \hat{p}(\hat{z}, \hat{t})} \equiv 1 + \frac{1}{2} \delta \hat{p}(\hat{z}, \hat{t}), \tag{4.43}
\]

where the latter linearization is completely in consistency with the basic assumptions of linear elasticity theory. Hence, assuming for a moment that \( \hat{p} \) is strictly of \( O(1) \), we note that a necessary condition for linear elasticity is here \( \delta \ll 1 \).

Using the linear elasticity form of (4.43), we obtain from (4.41)

\[
\Delta \hat{R}(\hat{z}, \hat{t}) = \frac{1}{2} \delta \hat{p}(\hat{z}, \hat{t}), \tag{4.44}
\]

with \( \hat{p}(\hat{z}, \hat{t}) = p(z, t) \) as given by (4.40).

**NOTE:** In this section we have derived solutions for \( q(z, t) \) and \( p(z, t) \) valid for arbitrary values of \( \delta \). However, since we have adopted linear elasticity already before, it is in principle inconsistent to employ these results for arbitrary large \( \delta \). Since in linear elasticity all terms of \( O(\delta^2) \) are consistently neglected with respect to the terms of \( O(\delta) \), the same must be done with the results in this section.

Thus, consistently neglecting \( O(\delta^2) \)-terms, leads us to the following (linear in \( \delta \)) results for \( q(z, t) \) and \( p(z, t) \) from (4.39) and (4.40):

\[
q(z, t) = q_1(T_0(z, t))H(t - z), \tag{4.45}
\]

\[
p(z, t) = \left(q_1(T_0(z, t)) - \frac{1}{4} \delta q_1^2(t - z)\right) H(t - z), \tag{4.46}
\]

while the relation for \( T_0(z, t) \), (4.38), reduces to

\[
t - \left(1 - \frac{1}{2} \delta q_1(t_0)\right) z = t_0. \tag{4.47}
\]

This relation can also be found as a special case of the solution obtained in the forthcoming Section 4.4.
4.2.3 The shell equation

The local relation (4.1) relates the deflection of the vessel wall in a point \( z \) directly to the pressure in the same point \( z \). A direct consequence of this is that when the internal pressure in the vessel makes a sudden jump (e.g. at a valve), the deflection follows this jump and thus becomes discontinuous. Mechanically, this means that the vessel wall will rupture, which is not realistic. The reason for this anomaly is that the local formula does not account for the effect (the stiffness) of neighboring parts of the vessel wall near \( z \). A model that does account for this non-local effect, and thus yields a smooth curve for the deflection, is described by the shell equation for a thin-walled tube, which is derived in Appendix A and results in (4.48):

\[
\frac{\partial^2 U}{\partial z^2}(z, t) + 4\beta^4 U(z, t) = \frac{1}{D} p(z, t),
\]

where \( U(z, t) \) is the radial displacement, \( p(z, t) \approx p(z, t) \) the internal pressure, \( D = (Eh^3)/(12(1 - \nu^2)) \) the plate constant and \( \beta^4 = 3/(R_D h)^2 \). Note that here \( \nu \) is Poisson’s ratio, which we give from here on the fixed value \( \nu = 0.5 \), because we assume the vessel wall incompressible, so \( D = Eh^3/9 \). The complete derivation of the shell equation can be found in Appendix A. The solution of this equation can be split into a homogeneous part and a particular part. For the homogeneous solution, we solve the corresponding characteristic equation to find

\[
U_h(z, t) = c_1 e^{(1+i)\beta z} + c_2 e^{(1-i)\beta z} + c_3 e^{(l-1)i\beta z} + c_4 e^{-(1+i)\beta z},
\]

where \( c_1, \ldots, c_4 \) are still functions of \( t \). This homogeneous solution delivers us the fundamental system of eigenfunctions that we can use to obtain a particular solution by the method of variation of constants. We choose the integration bounds in such a way that the particular solution \( U_p(z, t) \) goes to zero for \( z \to \infty \), provided that also \( p(z, t) \) goes to zero for \( z \to \infty \). We choose the total solution \( U(z, t) \) equal to this particular solution (for reasons that we will explain soon hereafter) and thus obtain the non-local result

\[
U(z, t) = \frac{1}{8\beta^3 D} \left[ \int_z^\infty (\cos(\beta(\xi - z)) + \sin(\beta(\xi - z))) e^{-\beta(z-\xi)} p(\xi, t) \, d\xi + \int_0^z (\cos(\beta(\xi - z)) - \sin(\beta(\xi - z))) e^{-\beta(z-\xi)} p(\xi, t) \, d\xi \right].
\]

We call this a non-local result, because the displacement \( U \) in the point \( z \) depends here on the pressure in all points of the wall near \( z \). If \( p(z, t) \) has a finite support, i.e. if we can write \( p(z, t) = p(z, t) H(l - z) \), for arbitrary \( l > 0 \) (possibly even with \( l = l(t) \)), with \( H(\cdot) \) the Heaviside function, then (4.50) can be written as

\[
U(z, t) = \frac{1}{8\beta^3 D} \left[ \int_z^l (\cos(\beta(\xi - z)) + \sin(\beta(\xi - z))) e^{-\beta(z-\xi)} p(\xi, t) \, d\xi + \int_0^z (\cos(\beta(\xi - z)) - \sin(\beta(\xi - z))) e^{-\beta(z-\xi)} p(\xi, t) \, d\xi \right].
\]

for \( 0 < z < l \), and

\[
U(z, t) = \frac{1}{8\beta^3 D} \int_0^l (\cos(\beta(\xi - z)) - \sin(\beta(\xi - z))) e^{-\beta(z-\xi)} p(\xi, t) \, d\xi,
\]

for \( z > l \).

Finally, we note that for the thin-walled arteries we consider here the parameter \( \beta \) is very large, which, more precisely, means that \( L\beta \gg 1 \) (where \( L \) is the characteristic length for the range of
For very large $\beta$ ($L\beta \to \infty$), $\beta \exp(-\beta \zeta)$ behaves like the Dirac delta function $\delta(\zeta)$, for $\zeta \geq 0$. This means that then (4.50) reduces to

$$U(z, t) = \frac{1}{8\beta^4 D} \left[ \int_{-\infty}^{\zeta} (\cos(\beta(\zeta - z)) + \sin(\beta(\zeta - z))) \delta(\zeta - z) p(\zeta, t) d\zeta + \int_{0}^{\zeta} (\cos(\beta(\zeta - z)) - \sin(\beta(\zeta - z))) \delta(z - \zeta) p(\zeta, t) d\zeta \right] = \frac{p(z, t)}{4\beta^4 D}.$$  \hspace{0.5cm} (4.53)

Hence, we conclude that for large $\beta$ the particular solution (4.53) tends to the same expression as (4.48) in case $1/4\beta^4 \to 0$ (i.e. after the neglect of $\left(\partial^4 U/\partial z^4\right)/4\beta^4$), namely

$$U(z, t) = \frac{p(z, t) \cdot 3R^2_a}{4Eh} = 3R^2_a a^{2/3} p(z, t).$$  \hspace{0.5cm} (4.54)

which at its turn completely corresponds to the simplified (affine) relation between $A(z, t)$ and $p(z, t)$ according to (4.1) as we shall show:

$$A(z, t) = \pi(R_a + U(z, t))^2 = \pi R^2_a + 2\pi R_a U(z, t) + ... = \pi R^2_a + Cp(z, t).$$  \hspace{0.5cm} (4.55)

or

$$U(z, t) = \frac{C}{2\pi R_a} p(z, t) = \frac{2R^2_a}{4Eh} p(z, t),$$  \hspace{0.5cm} (4.56)

with $C$ according to (A.19), or in dimensionless notation,

$$\hat{U}(\hat{z}, \hat{t}) = \frac{U(z, t)}{R_a} = \frac{1}{2} \delta(\hat{z}, \hat{t}),$$  \hspace{0.5cm} (4.57)

in accordance with (4.44).

### 4.3 Inclusion of radial diffusion

In this section we investigate the influence of the radial diffusion term $\tau_w$ as introduced in Section 3.4. This term represents the effect of the shear stress at the wall on the flow. To focus on the effect of radial diffusion purely, we neglect both the advection term ($\partial U/\partial z = 0$) and the compliance distribution in the coefficient of $\partial p/\partial z$ ($\delta = 0$). Moreover, we assume the diffusion term $\tau_w$ to be a linear combination of the flow rate $q$ and the pressure gradient $\partial p/\partial z$, as derived by Bessemets et al. in [4]. So

$$\tau_w(z, t) = \tau_1 q(z, t) + \tau_2 \frac{\partial p}{\partial z}(z, t),$$  \hspace{0.5cm} (4.58)

with $\tau_{1,2} \in \mathbb{R}$. We first step back to the dimensional equations of motion and substitute (4.58) for $\tau_w$. As posed before, we leave out the advection term and the compliance contribution. We thus arrive at (note that the first-order effect of the deformation of the wall is incorporated in the coefficient $C$ being greater than zero)

$$C \frac{\partial p}{\partial t} + \frac{\partial q}{\partial z} = 0,$$  \hspace{0.5cm} (4.59)

$$\frac{\partial q}{\partial t} + \frac{\lambda_0}{\rho} \frac{\partial p}{\partial z} + \tau_1 q + \tau_2 \frac{\partial p}{\partial z} = 0.$$  \hspace{0.5cm} (4.60)

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Introducing the scaled problem parameters as in (4.5), these equations of motion transform into
\[
\frac{C L^* P}{T Q_0} \frac{\partial p}{\partial t} + \frac{\partial q}{\partial z} = 0,
\]
(4.61)
\[
\frac{\partial q}{\partial t} + \frac{(A_0 + \tau_2 \rho) T P^*}{Q_0 \rho L^*} \frac{\partial p}{\partial z} + \tau_1 T q = 0.
\]
(4.62)
Note that we added a * to \( P \) and \( L \) to denote that these parameters will (slightly) differ from the expressions found in Section 4.1 due to the effect of \( \tau_2 \). If we choose
\[
P = \frac{Q_1}{C c^*_0}, \quad L = \frac{c^*_0}{\omega}, \quad \tilde{\tau} = T \tau_1,
\]
(4.63)
where \( c^*_0 = \sqrt{(A_0 + \tau_2 \rho)/\rho} \), the modified wave velocity, we see that the equations of motion transform into
\[
\frac{\partial p}{\partial t} + \frac{\partial q}{\partial z} = 0,
\]
(4.64)
\[
\frac{\partial q}{\partial t} + \frac{\partial p}{\partial z} + \tilde{\tau} q = 0,
\]
(4.65)
with boundary and initial conditions
\[
p(z, 0) = q(z, 0) = 0 \quad \text{for } z > 0,
\]
\[
q(0, t) = q_i(t) \quad \text{for } t > 0,
\]
\[
p(z, t) \to 0 \quad \text{for } z \to \infty, t > 0
\]
(4.66)
We try to solve the system (4.64) - (4.65) by using Laplace transformation. To this end, we introduce the Laplace transforms of \( p(z, t) \) and \( q(z, t) \) as follows:
\[
\mathcal{P}(z; s) = \int_0^\infty p(z, t) e^{-st} dt,
\]
(4.67)
\[
\mathcal{Q}(z; s) = \int_0^\infty q(z, t) e^{-st} dt.
\]
(4.68)
If we apply the Laplace transform to the system (4.64) and (4.65) and the boundary conditions (4.66), with the above definitions of the transforms of \( p(z, t) \) and \( q(z, t) \), we are left with
\[
s \mathcal{P} + \frac{d\mathcal{Q}}{dz} = 0,
\]
(4.69)
\[
\frac{d\mathcal{P}}{dz} + (\tilde{\tau} + s) \mathcal{Q} = 0,
\]
(4.70)
\[
\mathcal{P}(z; s) \to 0 (z \to \infty),
\]
(4.71)
\[
\mathcal{Q}(0; s) = \int_0^\infty q_i(t)e^{-st} dt = Q_i(s).
\]
(4.72)
The solution of this system of ordinary differential equations can easily be found by eliminating \( \mathcal{P}(z; s) \) from the second equation using the first one, which results in a differential equation in terms of \( \mathcal{Q}(z; s) \) only. The solutions are given by
\[
\mathcal{Q}(z; s) = Q_i(s)e^{-\sigma(s)z}, \quad \sigma(s) = \sqrt{(\tilde{\tau} + s)s}, \quad \text{Re}(\sigma) \geq 0,
\]
(4.73)
\[
\mathcal{P}(z; s) = \frac{\sigma(s)}{s} Q_i(s)e^{-\sigma(s)z}.
\]
(4.74)
Taking the inverse Laplace transform of the solution obtained above, we find for \((0 \leq z \leq t)\) that
\[
q(z, t) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} q(s) e^{st} ds = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} Q_i(s) e^{-s\tau} e^{st} ds
\]
\[
= \int_0^t q_i(t - x) f(z, x) dx,
\]
where
\[
f(z, t) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} e^{-s\tau} e^{st} ds,
\]
the inverse Laplace transform of \(\exp(-s\tau)z\). We used two properties of the inverse Laplace transform, stating that if in general \(F(s) \leftrightarrow f(t)\) and \(g(s) \leftrightarrow g(t)\), it holds that
\[
F(s)/s \leftrightarrow (H * f)(t),
\]
\[
F(s)g(s) \leftrightarrow (f * g)(t),
\]
where \(H(\cdot)\) is the Heaviside function.

It remains to determine \(f(z, t)\). This can be done by some simple mathematical manipulation and using a table for inverse Laplace transforms (see [7]). This leads to
\[
f(z, t) = -\frac{\partial}{\partial z} \left\{ \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{e^{-s\tau\sqrt{(\frac{z}{2})^2 - \frac{c^2}{4}}}}{\sqrt{(\frac{z}{2})^2 - \frac{c^2}{4}}} e^{st} ds \right\}
\]
\[
= -e^{-\frac{3}{2}t} \frac{\partial}{\partial z} \left\{ \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{e^{-s\tau\sqrt{\frac{c^2}{4} - \frac{z^2}{4}}}}{\sqrt{\frac{c^2}{4} - \frac{z^2}{4}}} e^{st} ds \right\}
\]
\[
= -e^{-2\frac{c}{2}t} \frac{\partial}{\partial z} \left[ I_0 \left( 2\frac{c}{2} \sqrt{\frac{c^2}{4} - \frac{z^2}{4}} \right) H(t - z) \right]
\]
\[
= e^{-2\frac{c}{2}t} \left[ \delta(t - z) + 2\frac{c}{2} \frac{I_1 \left( 2\frac{c}{2} \sqrt{\frac{c^2}{4} - \frac{z^2}{4}} \right)}{\sqrt{\frac{c^2}{4} - \frac{z^2}{4}}} H(t - z) \right],
\]
where we used the substitutions \(s = p - \frac{c}{2}\) and consequently \(c = c + \frac{c}{2}\) in the first line and \(c = \frac{c}{4}\) in the second line. Combining (4.75) and (4.79), we finally arrive at
\[
q(z, t) = \int_0^t q_i(t - x) e^{-2\frac{c}{2}x} \left[ \delta(x - z) + 2\frac{c}{2} \frac{I_1 \left( 2\frac{c}{2} \sqrt{\frac{c^2}{4} - \frac{z^2}{4}} \right)}{\sqrt{\frac{c^2}{4} - \frac{z^2}{4}}} H(x - z) \right] dx
\]
\[
= q_i(t - z)e^{-2\frac{c}{2}z} + 2\frac{c}{2} \int_0^{t-z} \frac{I_1 \left( 2\frac{c}{2} \sqrt{\frac{c^2}{4} - \frac{z^2}{4}} \right)}{\sqrt{\frac{c^2}{4} - \frac{z^2}{4}}} e^{-2\frac{c}{2}(t-x)} q_i(s) ds,
\]
for \(0 \leq z \leq t\).

To solve the equation for \(p(z, t)\) we use the resulting solution of \(q(z, t)\) in (4.80) and we introduce the auxiliary function \(\mathbb{B}(z, t)\) as
\[
\mathbb{B}(z, t) = \int_z^t I_0 \left( 2\frac{c}{2} \sqrt{\frac{c^2}{4} - \frac{z^2}{4}} \right) q_i(t - x)e^{-2\frac{c}{2}x} dx H(t - z),
\]
where
\[
q_i(t) = \int_0^t q_i(\xi) d\xi.
\]
With this definition of the function $B(z, t)$ we rewrite the expression for $q(z, t)$ in (4.80) to

$$q(z, t) = -\frac{\partial}{\partial z} \int_{z}^{t} I_{0} \left(2\hat{\tau} \sqrt{x^2 - z^2}\right) q(t - x) e^{-2\hat{\tau} x} \, dx = -\frac{\partial^{2}B}{\partial z \partial t}(z, t).$$ (4.83)

Using the original equation given in (4.65), we see that

$$\frac{\partial p}{\partial z} = -\left(\frac{\partial q}{\partial t} + 4\hat{\tau} q\right) = \left(\frac{\partial}{\partial t} + 4\hat{\tau}\right) \frac{\partial B}{\partial t} = \frac{\partial}{\partial z} \left(\frac{\partial}{\partial t} + 4\hat{\tau}\right) \frac{\partial B}{\partial t}.$$ (4.84)

resulting in

$$p(z, t) = \left(\frac{\partial}{\partial t} + 4\hat{\tau}\right) I_{0} \left(2\hat{\tau} \sqrt{x^2 - z^2}\right) q(t - x) e^{-2\hat{\tau} x} \, dx$$

$$= \int_{z}^{t} I_{0} \left(2\hat{\tau} \sqrt{x^2 - z^2}\right) \left(\dot{q}_{t}(t - x) + 4\hat{\tau} q_{t}(t - x)\right) e^{-2\hat{\tau} x} \, dx$$

$$= \int_{0}^{t - z} I_{0} \left(2\hat{\tau} \sqrt{(t - s)^2 - z^2}\right) \left(\dot{q}_{s}(s) + 4\hat{\tau} q_{s}(s)\right) e^{-2\hat{\tau} (t - s)} \, ds,$$ (4.85)

for $0 \leq z \leq t$.

**NOTE:** For

$$A(x, z) = I_{0} \left(2\hat{\tau} \sqrt{x^2 - z^2}\right)$$ (4.86)

it holds that

$$\frac{\partial^{2}A}{\partial x^{2}} - \frac{\partial^{2}A}{\partial z^{2}} = 4\hat{\tau}^{2} A.$$ (4.87)

With this relation one easily checks that the obtained solutions for $p$ and $q$ also satisfy (4.64).

So, summarizing we see that for $0 \leq z \leq t$

$$q(z, t) = q_{t}(t - z) e^{-\hat{\tau} z} + 2\hat{\tau} z \int_{0}^{t - z} I_{0} \frac{\left(2\hat{\tau} \sqrt{(t - s)^2 - z^2}\right)}{\sqrt{(t - s)^2 - z^2}} e^{-2\hat{\tau} (t - s)} q_{s}(s) \, ds,$$ (4.88)

$$p(z, t) = \int_{0}^{t - z} I_{0} \frac{\left(2\hat{\tau} \sqrt{(t - s)^2 - z^2}\right)}{\sqrt{(t - s)^2 - z^2}} \left(\dot{q}_{s}(s) + 4\hat{\tau} q_{s}(s)\right) e^{-2\hat{\tau} (t - s)} ds.$$ (4.89)

The corresponding asymptotic expansions with respect to $\hat{\tau}$, for $0 < \hat{\tau} \ll 1$, are given by:

$$q(z, t) = q_{t}(t - z) - 2\hat{\tau} z q_{t}(t - z) + 2\hat{\tau}^{2} \left(z^{2} q_{t}(t - z) + z \int_{0}^{t - z} q_{s}(s) \, ds\right) + O\left(\hat{\tau}^{3}\right),$$ (4.90)

$$p(z, t) = q_{t}(t - z) + 2\hat{\tau} \int_{0}^{t - z} \left(2q_{s}(s) - (t - s) \dot{q}_{s}(s)\right) ds +$$

$$\hat{\tau}^{2} \int_{0}^{t - z} \left(3(t - s)^2 - z^2\right) \dot{q}_{s}(s) - 8(t - s) q(s) \, ds + O\left(\hat{\tau}^{3}\right).$$ (4.91)

An alternative derivation of $p(z, t)$ and $q(z, t)$, using a more elegant fundamental mathematical approach, can be found in Appendix B.
We conclude this section by deriving an expression for our dimensionless coefficient \( \hat{\tau} \). To this end, we follow Bessems [1], who defines the dimensional radial diffusion term as

\[
\tau_w(z, t) = \frac{2 \pi R_a}{\rho} \left( \frac{2 \eta}{(1 - \zeta_c)} R_a A_0 q(z, t) + \frac{R_a (\zeta_c - 1)}{4} \frac{\partial p}{\partial \zeta}(z, t) \right),
\]

where \( \zeta_c \) is the square of the dimensionless core-diameter and defined as

\[
\zeta_c = \max \left[ 0, \left( 1 - \frac{\sqrt{2}}{\alpha} \right)^2 \right],
\]

with \( \alpha = r_a \sqrt{\omega/\nu} \), the Womersley number according to (3.20). All other quantities involved are as introduced before. So we see that

\[
\tau_1 = \frac{4 \nu}{R_a^2 (1 - \zeta_c)} \quad \text{and} \quad \tau_2 = \frac{A_0 (\zeta_c - 1)}{2 \rho}. \tag{4.94}
\]

If we define \( \vartheta = \sqrt{(1 + \zeta_c)}/2 \), we derive with the definition of \( \tau_2 \) that \( \zeta_c \approx c_0 \vartheta \) and so \( P^* = P/\vartheta \) and \( L^* = L \vartheta \). Here, \( c_0, P \) and \( L \) are the originally defined parameters from Section 4.1. Finally, we arrive for \( \hat{\tau} \) at

\[
\hat{\tau} = \frac{\tau_1}{4 \omega} = \frac{\nu}{R_a^2 (1 - \zeta_c)} = \frac{1}{\alpha^2 (1 - \zeta_c)}. \tag{4.95}
\]

We conclude by noting that the formula for \( \zeta_c \), (4.93), according to Bessems et al. [1], only holds for relatively large \( \alpha \), implying that \( \zeta_c \) is close to one. Hence, also the parameter \( \vartheta \approx 1 \) (in fact \( \vartheta = 1 + \mathcal{O}(\alpha^{-1}) \)).

### 4.4 Inclusion of advection

In this section we will investigate the contribution of the advection term to the flow. We start from the dimensionless equation of motion given in (4.9) and (4.10). To specifically investigate the effect of the advection term in (4.10), we neglect the radial diffusion term (\( \tau_w = 0 \)). Moreover, following Bessems et al. [3], we assume the nonlinear advection to be of the form

\[
\gamma_a(z, t) = \gamma_{a_1} q^2(z, t) + \gamma_{a_2} q(z, t) \frac{\partial p}{\partial \zeta}(z, t) + \gamma_{a_3} \left( \frac{\partial p}{\partial \zeta}(z, t) \right)^2, \tag{4.96}
\]

where \( \gamma_{a_1,2,3} \) are real positive constants. Once more, we follow the choice of Bessems et al. by putting \( \gamma_{a_2} = \gamma_{a_3} \approx 0 \) yielding \( \gamma_a(z, t) = \gamma_{a_1} q^2(z, t) \). As we will see, this choice for \( \gamma_a(z, t) \) allows us to also incorporate the compliance contribution in the equations of motion.

We thus obtain

\[
\frac{\partial p}{\partial t} + \frac{\partial q}{\partial z} = 0, \tag{4.97}
\]

\[
\frac{\partial q}{\partial t} + 2 \gamma_{a_1} q \frac{\partial q}{\partial z} + (1 + \delta p) \frac{\partial p}{\partial \zeta} = 0, \tag{4.98}
\]

with boundary and initial conditions

\[
p(z, 0) = q(z, 0) = 0 \quad \text{for} \quad z > 0, \]

\[
q(0, t) = q_i(t) \quad \text{for} \quad t > 0, \tag{4.99}
\]

\[
p(z, t) \to 0 \quad \text{for} \quad z \to \infty, t > 0
\]
We can solve this system of equations in a completely analogous way as done for the system in Section 4.2.2 but we assume $\delta^2 \to 0$ and solve the system up to the first order in $\delta$. So, again we introduce a variable $\gamma$ in such a way that $p(z,t) = P(\gamma)$ and $q(z,t) = Q(\gamma)$ and we define the two characteristics:

$$\Gamma_1 = \{ \gamma_1, \gamma_2 \in \mathbb{R} | \gamma_1(z,t) = \gamma_2 \text{ is constant} \},$$  \hfill (4.100)

$$\Gamma_2 = \{ \gamma_1, \gamma_2 \in \mathbb{R} | \gamma_1(z,t) = \gamma_2 \text{ is constant} \}. $$  \hfill (4.101)

where $\gamma_1$ describes waves traveling in negative $z$-direction and $\gamma_2$ waves traveling in positive $z$-direction. Substitution of $p(z,t) = P(\gamma)$ and $q(z,t) = Q(\gamma)$ in (4.97) and (4.98) gives (in matrix notation)

$$
\begin{pmatrix}
\frac{\partial \gamma_1}{\partial t} & \frac{\partial \gamma_1}{\partial z} \\
(1 + \delta P) \frac{\partial \gamma_1}{\partial t} & \frac{\partial \gamma_1}{\partial z} + 2\delta \gamma_1 Q \frac{\partial \gamma_1}{\partial z}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial P}{\partial \gamma} \\
\frac{\partial Q}{\partial \gamma}
\end{pmatrix} = 0.
$$

leading to the solvability condition

$$\gamma_1^2 + 2\delta \gamma_1 Q \gamma_1 - (1 + \delta P) \gamma_1^2 = 0.$$  \hfill (4.103)

As solution for this condition we find

$$\frac{\gamma_1}{\gamma_2} = \pm \sqrt{\delta^2 \gamma_1^2 Q^2 + 1 + \delta P - \delta \gamma_1 Q},$$

where the $+$ sign and the $-$ sign refer to a solution along $\Gamma_1$ and $\Gamma_2$, respectively. Along $\Gamma_2$ (so $\gamma_1$ is constant) we see that (4.102), after substitution of (4.104) leads to

$$\frac{dP}{\gamma_2} \frac{\partial \gamma_2}{\partial t} + \frac{dQ}{\gamma_2} \frac{\partial \gamma_2}{\partial z} = \frac{dP}{\gamma_2} \left( \sqrt{\delta^2 \gamma_1^2 Q^2 + 1 + \delta P + \delta \gamma_1 Q} \right) \frac{\partial \gamma_2}{\partial z} = 0.$$  \hfill (4.105)

implying that (when $\gamma_{2,z} \neq 0$, with $O(\delta^2) \to 0$)

$$\frac{dQ}{d\gamma_2} = \left( 1 + \frac{1}{2} \delta P + \delta \gamma_1 Q \right) \frac{dP}{\gamma_2}.$$  \hfill (4.106)

Considering $Q(\gamma_2) = Q(P)$, we obtain an ordinary differential equation for $\gamma_2$

$$\frac{dQ}{dP} - \delta \gamma_1 Q = 1 + \frac{1}{2} \delta P,$$  \hfill (4.107)

$$Q(0) = 0.$$  \hfill (4.108)

having solution (again up to $O(\delta)$)

$$Q(P) = \frac{1 + 2\gamma_1}{2\gamma_1^2 \delta} \left( e^{\delta \gamma_1 P} - 1 \right) - \frac{1}{2\gamma_1} P$$

$$= \frac{1 + 2\gamma_1}{2\gamma_1^2 \delta} \left( 1 + \delta \gamma_1 P + \frac{1}{2} \delta^2 \gamma_1^2 P^2 - 1 \right) - \frac{1}{2\gamma_1} P$$

$$= P + \frac{1}{4} (1 + 2\gamma_1) \delta P,$$  \hfill (4.109)

or equivalently as a function of $\gamma_2$

$$P(\gamma_2) + \frac{1}{4} (1 + 2\gamma_1) \delta P(\gamma_2) - Q(\gamma_2) = 0.$$  \hfill (4.110)
Since we want to solve the system 4.102 up to order \(O(\delta)\), we assume \(P(\gamma_2) = P_0(\gamma_2) + \delta P_1(\gamma_2)\) and together with 4.110 we find that

\[
P_0(\gamma_2) = Q(\gamma_2), \tag{4.111}
\]

\[
P_1(\gamma_2) = -\frac{1}{4}(1 + 2\gamma_1)Q^2(\gamma_2), \tag{4.112}
\]

and so we obtain an inverted relation of 4.109

\[
P(\gamma_2) = Q(\gamma_2) - \frac{1}{4}\delta(1 + 2\gamma_1)Q^2(\gamma_2). \tag{4.113}
\]

Along \(\Gamma_1\) we know that \(P\) and \(Q\) are both constant, which means that 4.104 along \(\Gamma_1\) (+sign) is constant. We take a specific characteristic on \(\Gamma_1\) that intersects the \(t\)-axis at \(t = t_0\), for arbitrary \(t_0 > 0\). This means that \(\gamma_2 = \gamma_2(0, t_0) = \hat{\gamma}_2(t_0)\) is constant. This leads to

\[
0 = d\gamma_2 = \frac{\partial \gamma_2}{\partial t} dt + \frac{\partial \gamma_2}{\partial z} dz = \frac{\partial \gamma_2}{\partial t} \left[ dt - \frac{dz}{\sqrt{\delta^2 \gamma_1^2 Q^2 + 1 + \delta P + \delta \gamma_1 Q}} \right], \tag{4.114}
\]

from which follows that

\[
t = \frac{z}{\sqrt{\delta^2 \gamma_1^2 Q^2 + 1 + \delta P + \delta \gamma_1 Q}} = t_0, \tag{4.115}
\]

along \(\Gamma_1\). Finally, we consider the specific characteristic on \(\Gamma_2\) that intersects the \(t\)-axis in \(t = t_0\). In the point \((z, t) = (0, t_0)\) we have \(\gamma_2 = \hat{\gamma}_2(t_0)\) and so we get

\[
Q(\hat{\gamma}_2(t_0)) = q(0, t_0) = q_1(t_0), \tag{4.116}
\]

and consequently

\[
P(\hat{\gamma}_2(t_0)) = q_1(t_0) - \frac{1}{4}\delta(1 + 2\gamma_1)q_1^2(t_0). \tag{4.117}
\]

for every \(t_0 > 0\). Moreover, we see that

\[
t_0 = T_0(z, t) = t - \frac{z}{\sqrt{\delta^2 \gamma_1^2 Q^2(\hat{\gamma}_2(t_0)) + 1 + \delta P(\hat{\gamma}_2(t_0)) + \delta \gamma_1 Q(\hat{\gamma}_2(t_0))}} = t - \frac{z}{\sqrt{\delta^2 \gamma_1^2 q_1^2(t_0) + 1 + \delta \left(q_1(t_0) - \frac{1}{4}\delta(1 + 2\gamma_1)q_1^2(t_0)\right) + \delta \gamma_1 q_1(t_0)}}. \tag{4.118}
\]

Summarizing, we note that the solutions for \(p\) and \(q\) are

\[
q(z, t) = q_1(T_0(z, t))H(t - z), \tag{4.119}
\]

\[
p(z, t) = \left(q_1(T_0(z, t)) - \frac{1}{4}\delta(1 + 2\gamma_1)q_1^2(T_0(z, t))\right)H(t - z). \tag{4.120}
\]

with \(T_0(z, t)\) the solution of 4.118; this solution is obtained numerically with for example Matlab.

Note: If we put \(\gamma_1\) equal to zero in the obtained solutions 4.119 and 4.120, we see that the result is in accordance with the linearized solutions of Section 4.2.2 (equations 4.45 and 4.46).

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To conclude this section, we take a closer look at the constant $\gamma_0$. Bessems et al. (3) introduces $\gamma_0(z, t)$ (dimensional) as (with $\gamma_2 = \gamma_3 = 0$)

$$\gamma_0(z, t) = \frac{\delta_1}{A_0} q^2(z, t),$$  \hspace{1cm} (4.121)

where

$$\delta_1(\zeta_c) = \frac{2 - 2\zeta_c (1 - \ln(\zeta_c))}{(1 - \zeta_c)^2},$$  \hspace{1cm} (4.122)

and were $\zeta_c$ as defined in the previous section. From this expression and with use of (4.121) we can easily determine our dimensionless coefficient $\gamma_{a1}$

$$\gamma_{a1} = A_0 V^2 \hat{\gamma}_a = A_0 V^2 \gamma_{a1} \hat{q}^2 = \frac{\delta_1}{A_0} q^2 = \frac{\delta_1 Q_0^2}{A_0} \hat{q}^2,$$  \hspace{1cm} (4.123)

leading to

$$\gamma_{a1} = \frac{\delta_1 Q_0^2}{A_0^3 V^2} = \delta_1,$$  \hspace{1cm} (4.124)

4.5 Preliminary conclusions from this chapter

In this chapter we have found closed-form solutions for the global, one dimensional, inlet-flow problem for a semi-infinite deformable vessel. In order to specifically investigate the effects of advection and diffusion, we have solved the following three partial problems

1. **No advection, no diffusion**: investigates especially the effect of the deformation of the vessel wall.
2. **Diffusion, no advection**: investigates the effect of the shear wall stress on the flow.
3. **Advection, no diffusion**: investigates the effect of the nonlinear advection terms.

To enable the calculations, we borrowed explicit expressions for the terms mentioned above from Bessems et al. [3]. In the three-dimensional analysis, to be presented in the next chapter, we hope to find a justification for these choices or alternative expressions that can also be used in a more or less analogous way to obtain closed-form solutions. In Chapter 5 we give explicit numerical presentations of the results of this, and next, chapter, which can serve to quantify the different effects considered here. We hope that this will lead to definitive conclusions regarding the classification of the orders of magnitudes of the effects.
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Chapter 5

The local equations of motion

In this chapter, we consider the same inlet-flow as in the preceding chapter, but now in a local, i.e. three-dimensional, formulation. We solve the equations of motions up to leading order by imposing asymptotic expansions for the axial velocity, the flow-rate and the pressure function in terms of the small parameter \( \delta \), the dimensionless compliance of the wall. This leads to a zeroth order and a first-order set of equations of motions and enables us to derive zeroth-order expressions for the flow characteristics; especially the shear force and the nonlinear advection contribution. We start from a simplified situation in which the vessel is undeformable and that serves as a preparation step in solving the equations of motion for a deformable vessel. Again, we are looking for as much as possible analytical solutions. To this end, we impose a certain approximation to the axial velocity. This approximation also affects all other quantities and therefore we need to discuss the accuracy of the approximation we make. As for the results of Chapter 4, the results of this chapter will graphically be presented in Chapter 6.

5.1 Problem set-up

In this chapter, we investigate the inlet flow in a three-dimensional semi-infinite thin-walled vessel of initial (undeformed) radius \( R_a \). As in the previous chapter we use a cylindrical coordinate system and assume the flow to be rotationally symmetric. This means that the velocity \( \mathbf{v} = v_r(r, z, t) \mathbf{e}_r + v_z(r, z, t) \mathbf{e}_z \) and the pressure \( p = p(r, z, t) \) for \( 0 < r < R_a \) and \( 0 < z < \infty \).

The general equations describing the inlet flow problem are (see (3.29)-(3.31))

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r v_r \right) + \frac{\partial v_z}{\partial z} = 0, \tag{5.1}
\]

\[
\frac{\partial p}{\partial r} = 0 \quad \Rightarrow \quad p = p(z, t), \tag{5.2}
\]

\[
\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{v}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right). \tag{5.3}
\]

Moreover, for the problem we consider here, the initial and boundary conditions are

\[
\mathbf{v}(r, z, 0) = p(z, 0) = 0, \tag{5.4}
\]
and
\[ v_r(R, z, t) = \frac{\partial R}{\partial t}(z, t), \quad v_z(R, z, t) = 0, \]
(5.5)
at \( r = R = R(z, t) \), the deformed inner radius of the vessel, respectively. At the inlet \( z = 0 \), we only describe the total influx \( q(0, t) = q_i(t) \), but we choose the distribution in accordance with the obtained solution form; we refer to this as the relaxed problem. Finally, we require the solution \((v_r, v_z, p - p_0)\) to go to zero for \( z \to \infty \).

This system will be supplemented by the equations of the global formulation, which will further on be used to obtain the local solution (we note here that \( C > 0 \) for a deformable wall, but \( C = 0 \) for a rigid wall). For these equations we refer to (4.2) and (4.3). These equations hold for \( z > 0 \), \( t > 0 \), and read
\[
C \frac{\partial p}{\partial t} + \frac{\partial q}{\partial z} = 0, \quad (5.6)
\]
\[
\frac{\partial q}{\partial t} + \frac{\partial q_a}{\partial z} + A_0 \left( 1 + \frac{C}{A_0} \rho \right) \frac{\partial p}{\partial z} + \tau_w = 0, \quad (5.7)
\]
with
\[
q(z, t) = 2\pi \int_0^{R(z, t)} r v_z(r, z, t) \, dr,
\]
\[
q_a(z, t) = 2\pi \int_0^{R(z, t)} r v_z^2(r, z, t) \, dr,
\]
(5.8)
\[
\tau_w(z, t) = -2\pi \nu R(z, t) \frac{\partial v_z}{\partial r} \bigg|_{R(z, t)},
\]
\[
R(z, t) = R_a \left( 1 + \frac{C}{2\pi R_a^2} p(z, t) \right),
\]
and
\[
p(z, 0) = q(z, 0) = 0 \quad \text{for} \quad z > 0,
\]
\[
R(z, 0) = R_a \quad \text{for} \quad z > 0,
\]
\[
q(0, t) = q_i(t) \quad \text{for} \quad t > 0; \ \text{prescribed}
\]
\[
p(z, t) \to 0 \quad \text{for} \quad z \to \infty, \ t > 0.
\]
(5.9)

First, we make these equations dimensionless, using the same dimensionless quantities as introduced in Chapters [3] and [3]. To keep up the readability, we repeat these definitions here:
\[
t := \omega^{-1} t, \quad z := L \hat{z} = \varepsilon R_a \hat{z}, \quad r := R_a \hat{r}, \quad R := R_a \hat{R},
\]
\[
v_r := \varepsilon V \hat{v}_r, \quad v_z := V \hat{v}_z, \quad p := P \hat{p}, \quad q := Q_0 \hat{q},
\]
(5.10)
where \( \omega = 2\pi \) and
\[
L := \frac{c_0}{\omega}, \quad \varepsilon := \frac{R_a}{L} \ll 1, \quad V := \frac{Q_0}{\pi R_a^2}, \quad Q_0 := \max_{t>0} q_i(t), \quad P := \rho c_0 V,
\]
(5.11)
and, moreover, we introduce the (small) parameter \( \delta \) by \((\delta \text{ must be small, as it represents the dimensionless compliance, and because we applied linear elasticity theory to calculate the deformations of the wall; in linear elasticity theory, } \mathcal{O}(\delta^2)\text{-terms are consistently neglected):}
\]
\[
\delta = \frac{V}{c_0} \ll 1.
\]
(5.12)

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Secondly, we eliminate \( R \) through
\[
R(z, t) = \sqrt{1 + \delta p(z, t)} = 1 + \frac{1}{2} \delta p(z, t) + O(\delta^2),
\] (5.13)
and \( v_r \) by means of
\[
v_r (r, z, t) = -\frac{1}{r} \int_0^r x \frac{\partial v_z}{\partial z} (x, z, t) \, dx,
\] (5.14)
which follows by integrating (5.1), using the regularity of \( v_r \) in \( r = 0 \). With this relation, the boundary condition (5.9), is trivially satisfied, as we can show as follows:
\[
v_r (R, z, t) = -\frac{1}{R(z, t)} \int_0^{R(z, t)} x \frac{\partial v_z}{\partial z} (x, z, t) \, dx
\]
\[
= -\frac{1}{R(z, t)} \left\{ \frac{\partial}{\partial z} \int_0^{R(z, t)} x v_z (x, z, t) \, dx - R(z, t) v_z (R(z, t), z, t) \frac{\partial R}{\partial z} \right\}
\]
\[
= -\frac{1}{2R(z, t)} \frac{\partial q}{\partial z} (z, t) = -\frac{1}{2R(z, t)} \frac{\partial p}{\partial t} (z, t) = \frac{1}{\delta} \frac{\partial R}{\partial t} (z, t),
\] (5.15)
according to (5.13).

Finally, we introduce the advection function \( \Gamma_a \) as
\[
\Gamma_a (r, z, t) = v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z}
\]
\[
= -\frac{1}{r} \int_0^r x \frac{\partial v_z}{\partial z} (x, z, t) \, dx \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z}
\]
\[
= -\frac{1}{r} \frac{\partial}{\partial r} \left\{ v_z \int_0^r x \frac{\partial v_z}{\partial z} (x, z, t) \, dx \right\} + \frac{1}{r} v_z r \frac{\partial v_z}{\partial z} + v_z \frac{\partial v_z}{\partial z}
\]
\[
= -\frac{1}{r} \frac{\partial}{\partial r} \left\{ v_z \int_0^r x \frac{\partial v_z}{\partial z} (x, z, t) \, dx \right\} + v_z \frac{\partial v_z}{\partial z}.
\] (5.16)

With all this we arrive at the following system of equations for the unknowns \( v_z (r, z, t), p(z, t) \) and \( q(z, t) \), all in dimensionless form
\[
\frac{\partial p}{\partial t} + \frac{\partial q}{\partial z} = 0,
\] (5.17)
\[
\frac{\partial q}{\partial t} + \delta \frac{\partial \gamma_a}{\partial z} + (1 + \delta p) \frac{\partial p}{\partial z} + \tau_w = 0,
\] (5.18)
\[
\frac{\partial v_z}{\partial t} + \delta \Gamma_a = -\frac{\hat{\nu}}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) = -\frac{\partial p}{\partial z},
\] (5.19)
with
\[
q(z, t) = 2 \int_0^{R(z, t)} r v_z (r, z, t) \, dr,
\]
\[
\gamma_a = \gamma_a (z, t) = 2 \int_0^{R(z, t)} r v_z^2 \, dr,
\] (5.20)
\[
\tau_w (z, t) = -2 \hat{\nu} R(z, t) \frac{\partial v_z}{\partial r} \bigg|_{R(z, t)},
\]
\[
R(z, t) = 1 + \frac{1}{2} \delta p(z, t).
\]
where \( \hat{v} = v/(\omega R^2) \) and

\[
\begin{align*}
p(z, 0) &= q(z, 0) = v_z(r, z, 0) = 0 & \text{for } z > 0, \\
v_z(R, z, t) &= 0 & \text{for } z > 0, t > 0 \\
q(0, t) &= q_i(t) & \text{for } t > 0, \text{ prescribed} \\
p(z, t) &\to 0 & \text{for } z \to \infty, t > 0.
\end{align*}
\]

Since the constitutive equation for \( R(z, t) \) is only accurate up to \( O(\delta) \) (i.e. \( O(\delta^2) \to 0 \)), it seems consistent to neglect all \( O(\delta^2) \)-terms in the above system. To this end, we expand all variables asymptotically as follows:

\[
\begin{align*}
v_z(r, z, t; \delta) &= v_{z,0}(r, z, t) + \delta v_{z,1}(r, z, t) + O(\delta^2), \\
p(z, t; \delta) &= p_0(z, t) + \delta p_1(z, t) + O(\delta^2), \\
q(z, t; \delta) &= q_0(z, t) + \delta q_1(z, t) + O(\delta^2).
\end{align*}
\]

We substitute these relations into (5.17) - (5.21), develop them in powers of \( \delta \), split up the system in one of \( O(\delta^0) \) and one of \( O(\delta^1) \) and neglect all terms of \( O(\delta^2) \). This finally amounts to the following systems of equations.

1. Zeroth-order system:

\[
\begin{align*}
\frac{\partial p_0}{\partial t} + \frac{\partial q_0}{\partial z} &= 0, \\
\frac{\partial q_0}{\partial t} + \frac{\partial p_0}{\partial z} + \tau_{w,0} &= 0, \\
\frac{\partial v_{z,0}}{\partial t} &= \frac{\hat{v}}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_{z,0}}{\partial r} \right) = -\frac{\partial p_0}{\partial z}, \\
q_0(z, t) &= 2 \int_0^1 r v_{z,0}(r, z, t) \, dr, \\
\tau_{w,0}(z, t) &= -2\hat{v} \frac{\partial v_{z,0}}{\partial r} \bigg|_{r=1}, \\
p_0(z, 0) &= q_0(z, 0) = v_{z,0}(r, z, 0) = 0 & \text{for } z > 0, \\
v_{z,0}(1, z, t) &= 0 & \text{for } z > 0, t > 0 \\
q_0(0, t) &= q_i(t) & \text{for } t > 0, \text{ prescribed} \\
p_0(z, t) &\to 0 & \text{for } z \to \infty, t > 0.
\end{align*}
\]

2. First-order system:

\[
\begin{align*}
\frac{\partial p_1}{\partial t} + \frac{\partial q_1}{\partial z} &= 0, \\
\frac{\partial q_1}{\partial t} + \frac{\partial q_{a,0}}{\partial z} + \frac{\partial p_1}{\partial z} + p_0 \frac{\partial p_0}{\partial z} + \tau_{w,1} &= 0, \\
\frac{\partial v_{z,1}}{\partial t} + \Gamma_{a,0} &= \frac{\hat{v}}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_{z,1}}{\partial r} \right) = -\frac{\partial p_1}{\partial z},
\end{align*}
\]

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with

\[ q_1(z, t) = 2 \int_0^1 rv_{z,1}(r, z, t) \, dr, \]

\[ \gamma_{a,0} = \gamma_{a,0}(z, t) = 2 \int_0^1 r v_{z,0}^2 \, dr, \]

\[ \tau_{w,1}(z, t) = -2\nu \left\{ \frac{\partial v_{z,1}}{\partial r} + \frac{1}{2} p_0(z, t) \left( \frac{\partial v_{z,0}}{\partial r} + \frac{\partial^2 v_{z,0}}{\partial r^2} \right) \right\} \bigg|_{r=1}, \]

\[ \Gamma_{a,0} = \frac{\partial^2 v_{z,0}}{\partial z^2} - v_{z,0} \frac{\partial v_{z,0}}{\partial z}, \]

and

\[ p_1(z, 0) = q_1(z, 0) = v_{z,1}(r, z, 0) = 0 \quad \text{for } z > 0, \]

\[ v_{z,1}(1, z, t) = \frac{1}{4\nu} p_0(z, t) \tau_{w,0}(z, t) \quad \text{for } z > 0, \quad t > 0, \]

\[ q_1(0, t) = 0 \quad \text{for } t > 0, \]

\[ p_1(z, t) \to 0 \quad \text{for } z \to \infty, \quad t > 0. \]

### 5.2 Relaxed solution for inlet flow in a rigid vessel

In this section, we consider the problem of a sudden (from \( t = 0 \) on) inlet flow in a semi-infinite \((z > 0)\) rigid vessel, of radius \( R (\equiv R_a)\). The fluid is initially (for \( t < 0 \)) in rest. We prescribe the total inlet flow \( q_i(t) \) at \( z = 0 \), but not its exact distribution. Instead, we consider the relaxed solution, which means that we assume that the local (a function of \( r \)) inflow boundary condition is such that the resulting flow in the tube is independent of \( z \). We consider this, very simplified, problem here to get already some grip on the more complicated problem to be solved in the next section and in the hope that some aspects of the solution derived here can also be used in the next section.

For a rigid vessel, the compliance \( C = 0 \), and then (5.6) yields

\[ \frac{\partial q}{\partial z} = 0 \quad \Rightarrow \quad q = q(t) = q_i(t), \tag{5.35} \]

in accordance with (5.9). As a consequence, also the axial velocity \( v_z \) is independent of \( z \) (by the way, the radial velocity \( v_r = 0 \), here). Finally, the pressure can be written as

\[ p = p(z, t) = p_i(t) - \Pi(t)z, \tag{5.36} \]

where the inlet pressure \( p_i \) is further irrelevant. With all this, and with \( v_z = v_z(r, t) := w(r, t) \) and \( \delta = 0 \), (5.18) and (5.19) yield

\[ \frac{dq}{dt} + \tau_w = -\frac{\partial p}{\partial z} = \Pi(t), \tag{5.37} \]

and

\[ \frac{\partial w}{\partial t} - \frac{\dot{\nu}}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) = \frac{dq}{dt} + \tau_w, \tag{5.38} \]

where

\[ \tau_w = -2\nu \frac{\partial w}{\partial r}(1, t). \tag{5.39} \]
Equation (5.38) can be solved by Laplace transforms with respect to the time \( t \), defined by:

\[
\mathcal{W}(r; s) = \mathcal{L}\{w(r, t), t, s\} = \int_0^\infty w(r, t)e^{-st}\, dt,
\]

(5.40)

\[
\mathcal{Q}(s) = \mathcal{L}\{q(t), t, s\} = \int_0^\infty q(t)e^{-st}\, dt,
\]

(5.41)

\[
\mathcal{T}(s) = \mathcal{L}\{\tau_w(t), t, s\} = \int_0^\infty \tau_w(t)e^{-st}\, dt.
\]

(5.42)

By taking the Laplace transform of (5.38), we find an ordinary differential equation in terms of the radius \( r \), for \( 0 \leq r < 1 \),

\[
\hat{v} \frac{d^2 \mathcal{W}}{dr^2} + \hat{v} \frac{d \mathcal{W}}{dr} - s \mathcal{W} = -\mathcal{T} - s \mathcal{Q}.
\]

(5.43)

The solution of this differential equation can be split into a homogenous part and a particular part. By putting the right-hand side of the differential equation equal to zero, we see that the solution of the homogenous part reads

\[
\mathcal{W}_{\text{hom}}(r; s) = c_1 I_0 \left( r \sqrt{\frac{z}{\nu}} \right) + c_2 K_0 \left( r \sqrt{\frac{z}{\nu}} \right).
\]

(5.44)

Here, \( I_0(\cdot) \) is the modified Bessel-function of the first kind of order 0, \( K_0(\cdot) \) the modified Bessel-function of the second kind of order 0, and \( c_1, c_2 \) constants in \( \mathbb{R} \). By observing that both \( \mathcal{Q} \) and \( \mathcal{T} \) are independent of \( r \), the particular solution of the differential equation is simply given by

\[
\mathcal{W}_{\text{par}}(r; s) = \mathcal{Q}(s) + \frac{\mathcal{T}(s)}{s}.
\]

(5.45)

The condition that the solution should be bounded for \( r = 0 \), leads to \( c_2 = 0 \). From the boundary condition \( w(1, t) = 0 \), and so \( \mathcal{W}(1; s) = 0 \), we see that

\[
c_1 = -\frac{1}{I_0 \left( \sqrt{\frac{z}{\nu}} \right)} \left( \mathcal{Q}(s) + \frac{\mathcal{T}(s)}{s} \right) .
\]

(5.46)

As the total solution for the Laplace transform of the radial velocity \( w(r, t) \) we thus find

\[
\mathcal{W}(r; s) = \left( \mathcal{Q}(s) + \frac{\mathcal{T}(s)}{s} \right) \left( 1 - \frac{I_0 \left( r \sqrt{\frac{z}{\nu}} \right)}{I_0 \left( \sqrt{\frac{z}{\nu}} \right)} \right).
\]

(5.47)

The definition of \( \tau_w \) gives

\[
\mathcal{T}(s) = -2\hat{v} \frac{d \mathcal{W}}{dr}(1, s) = 2\hat{v} \left( \sqrt{\frac{z}{\nu}} \right) \frac{I_1 \left( \sqrt{\frac{z}{\nu}} \right)}{I_0 \left( \sqrt{\frac{z}{\nu}} \right)} \left( \mathcal{Q}(s) + \frac{\mathcal{T}(s)}{s} \right) .
\]

(5.48)

and so (with \( I_0(z) - (2/z)I_1(z) = I_2(z) \))

\[
\mathcal{T}(s) = 2\hat{v} \left( \sqrt{\frac{z}{\nu}} \right) \frac{I_1 \left( \sqrt{\frac{z}{\nu}} \right)}{I_2 \left( \sqrt{\frac{z}{\nu}} \right)} \mathcal{Q}(s) .
\]

(5.49)

Finally, the Laplace transform \( \mathcal{W} \) can be written as

\[
\mathcal{W}(r; s) = \frac{1}{s} \frac{I_0 \left( \sqrt{\frac{z}{\nu}} \right) - I_0 \left( r \sqrt{\frac{z}{\nu}} \right)}{I_2 \left( \sqrt{\frac{z}{\nu}} \right)} s \mathcal{Q}(s) = s \mathcal{Q}(s) \Lambda(r; s) ,
\]

(5.50)
with
\[
\Lambda(r; s) = \frac{1}{s} \frac{I_0\left(\sqrt{\frac{s}{\nu}}\right) - I_0\left(\sqrt{\frac{r}{\nu}}\right)}{I_2\left(\sqrt{\frac{s}{\nu}}\right)}.
\] (5.51)

Taking the inverse Laplace transform with respect to \(s\), and using the convolution theorem for Laplace transforms, we find the solution \(w(r, t)\) of (5.38) as
\[
w(r, t) = \mathcal{L}^{-1}\left\{s \mathcal{Q}(s) \Lambda(r; s), s, t\right\} = (\lambda * \dot{q}_i)(r, t)
= \int_0^t \lambda(r, t - \tau) \dot{q}_i(\tau) \, d\tau,
\] (5.52)
where we have used that \(q(t) = q_i(t)\), and where the function \(\lambda(r, t)\) is the inverse Laplace transform of \(\Lambda(r; s)\), i.e.
\[
\lambda(r, t) = \mathcal{L}^{-1}\{\Lambda(r, s), s, t\} = \frac{1}{2\pi i} \lim_{z \to \infty} \int_{c-i\infty}^{c+i\infty} \Lambda(r, s) e^{zt} \, ds.
\] (5.53)

The above integral can be computed by determining the residues of the function \(\Lambda(r, s) e^{zt}\) and by applying the l'Hôpital rule. This is done in the following way:

1. The function \(\Lambda(r; s)\) has a singularity for \(s = 0\). For \(s \to 0\) a Laurent series for \(\Lambda\) can be computed. This leads to
\[
\Lambda(r; s) = \frac{2(1 - r^2)}{s} + \mathcal{O}(1).
\] (5.54)
The residue of \(\Lambda(r; s)\) in \(s = 0\) is given by the coefficient of \(s^{-1}\), i.e.
\[
\text{Res}_{s=0} \Lambda(r; s) = 2 \left(1 - r^2\right).
\] (5.55)

2. Other singularities of \(\Lambda(r; s)\) can be found by computing the zeros of the function
\[
N(s) = I_2\left(\sqrt{\frac{s}{\nu}}\right).
\] (5.56)
This function does not have any singularities in the positive real plane, except for \(s = 0\). To investigate the singularities in the negative real plane, denoted with \(s_k\) \((k = 1, 2, \ldots)\), we introduce the variable \(\sigma\), according to \(\sqrt{s/\nu} = \tau \sigma\). This results in
\[
N(s) = I_2\left(i\sigma\right) = -J_2\left(\sigma\right),
\] (5.57)
and so
\[
\Lambda(r; s) = \frac{1}{2\nu \sigma^2} \frac{J_0(\sigma) - J_0(\rho \sigma)}{J_2(\sigma)} = \tilde{\Lambda}(r; \sigma).
\] (5.58)
The zeros \(\sigma_k\) of \(\tilde{N}(\sigma)\), corresponding to \(s_k\), are taken on the positive real axis and can be computed by numerical methods. Using that
\[
N'(s) = \frac{1}{2\nu} \sqrt{\frac{\nu}{s}} \left( I_1\left(\sqrt{\frac{s}{\nu}}\right) - 2 \sqrt{\frac{\nu}{s}} I_2\left(\sqrt{\frac{s}{\nu}}\right)\right)
= \frac{1}{2\nu \sigma} \left( J_1(\sigma) - \frac{2}{\sigma} J_2(\sigma)\right),
\] (5.59)
or, with \(J_2(\sigma_k) = 0\),
\[
N'(s_k) = \frac{1}{2\nu} \frac{J_1(\sigma_k)}{\sigma_k} = \frac{1}{4\nu} J_0(\sigma_k),
\] (5.60)
where we used the recurrence relation \((2\alpha/x)J_\alpha(x) = J_{\alpha-1}(x) + J_{\alpha+1}(x)\) in the final step. We find with use of the \(t\)Hôpital rule that

\[
\text{Res}_{\nu = \nu_k} \Lambda(r; s) = -\frac{4}{\sigma_k^2} \frac{J_0(\sigma_k) - J_0(r\sigma_k)}{J_0(\sigma_k)},
\]

(5.61)

\textbf{NOTE:} There are no complex valued zeros of \(N(s)\).

Altogether this gives for \(\lambda(r, t)\)

\[
\lambda(r, t) = \sum_{k=0}^{\infty} \text{Res}_{\nu = \nu_k} \left( \Lambda(r, s)e^{\nu t} \right)
= 2 \left( 1 - r^2 \right) - \sum_{k=1}^{\infty} \frac{4}{\sigma_k^2} \frac{J_0(\sigma_k) - J_0(r\sigma_k)}{J_0(\sigma_k)} e^{-\nu \sigma_k^2 t}.
\]

(5.62)

To conclude we find for the axial velocity \(w(r, t)\) the expression

\[
w(r, t) = \int_0^t \lambda(r, t - \tau) \dot{q}_i(\tau) d\tau
= 2 \left( 1 - r^2 \right) \int_0^t \dot{q}_i(\tau) d\tau - \int_0^t \left\{ \sum_{k=1}^{\infty} \frac{4}{\sigma_k^2} \frac{J_0(\sigma_k) - J_0(r\sigma_k)}{J_0(\sigma_k)} \left\{ \int_0^t e^{-\nu \sigma_k^2 (t - \tau)} \dot{q}_i(\tau) d\tau \right\} \right\} d\tau
= 2 \left( 1 - r^2 \right) q_i(t) - \sum_{k=1}^{\infty} \frac{4}{\sigma_k^2} \frac{J_0(\sigma_k) - J_0(r\sigma_k)}{J_0(\sigma_k)} \left\{ \int_0^t e^{-\nu \sigma_k^2 (t - \tau)} \dot{q}_i(\tau) d\tau \right\}.
\]

(5.63)

For later purposes and to keep up the readability of this result, we here introduce the following short-hand notations (for \(k = 1, 2, \ldots\)):

\[
\mathcal{J}_k(r) = \frac{4}{\sigma_k^2} \frac{J_0(\sigma_k) - J_0(r\sigma_k)}{J_0(\sigma_k)},
\]

(5.64)

\[
\Phi_k(t) = \int_0^t e^{-\nu \sigma_k^2 \tau} \dot{q}_i(t - \tau) d\tau.
\]

(5.65)

Note that we changed the integration variable in the definition of \(\Phi_k(t)\) with respect to (5.63) in order to make the time derivative of the sum of \(\Phi_k(t)\) converge. Moreover, note here that for a given, and not too complicated, expression for \(q_i(t)\), these integrals can be calculated analytically. And so we finally arrive at

\[
w(r, t) = 2 \left( 1 - r^2 \right) q_i(t) - \sum_{k=1}^{\infty} \mathcal{J}_k(r) \Phi_k(t).
\]

(5.66)

### 5.3 Zeroth order solution for the axial velocity

Our next step will be to find a zeroth order solution for a sudden inlet flow in a semi-infinite, flexible (so \(R = R(z, t) = 1 + \delta/2p(z, t)\)) vessel, as described by the system under item 1. on page 52. One should realize that the flexibility of the vessel wall is already incorporated by the inclusion of the first term in (5.6) or (5.68). For the sake of clarity, we will omit the 0 index in the sequel.
problem is described by the system

\[ \frac{\partial w}{\partial t} - \frac{\hat{v}}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) = \Pi(z, t) := - \frac{\partial p}{\partial z}, \quad (5.67) \]

\[ \frac{\partial p}{\partial t} + \frac{\partial q}{\partial z} = 0, \quad (5.68) \]

\[ \frac{\partial q}{\partial t} + \frac{\partial p}{\partial z} + \tau w = 0, \quad (5.69) \]

with boundary and initial conditions

\[ p(z, 0) = q(z, 0) = v_z(r, z, 0) = 0 \quad \text{for } z > 0, \]

\[ v_z(1, z, t) = 0 \quad \text{for } z > 0, t > 0, \]

\[ q(0, t) = q_i(t) \quad \text{for } t > 0, q_i(t) \text{ is prescribed}, \]

\[ p(z, t) \rightarrow 0 \quad \text{for } z \rightarrow \infty, t > 0. \]

Note that

\[ \Pi(z, t) = - \frac{\partial p}{\partial z} = \frac{\partial q}{\partial t} + \tau w. \quad (5.71) \]

We suggest three possible methods to solve the above system of equations by expressing the solution in terms of \( \Pi(z, t) \) or \( q(z, t) \):

1. The first option is to use the technique of separation of variables by introducing a Fourier-Bessel series (in analogy with the standard Fourier serie) for the axial velocity. This leads to the general form

   \[ w(r, z, t) = 2 \sum_{k=1}^{\infty} \frac{J_0(\lambda_k r)}{\lambda_k J_1(\lambda_k)} \Omega_k(z, t), \quad (5.72) \]

   where

   \[ \Omega_k(z, t) = \int_z^t \Pi(z, \tau) e^{-i \lambda_k^2 \tau} H(t - \tau) \, d\tau, \quad (5.73) \]

   and where \( \lambda_k (k = 1, 2, \ldots) \) are the consecutive zeros of the zeroth-order Bessel function of the first kind \( J_0(\cdot) \).

2. The second option is to follow the method used in the previous section, but with the prescribed \( q_i(t) \) now replaced by the unknown \( q(z, t) \). The result is (for \( 0 < z < t \))

   \[ w(r, z, t) = 2 \left( 1 - r^2 \right) q(z, t) - \sum_{k=1}^{\infty} \mathcal{F}_k(r) \left\{ \int_0^t e^{-i \lambda_k^2 \tau} q(z, \tau) \, d\tau \right\}. \quad (5.74) \]

3. The third option is again to use separation of variables, but this time by expressing the axial velocity as a finite sum of Zernike polynomials. We find (again for \( 0 < z < t \))

   \[ w(r, z, t) = \left( 1 - Z_{2(K+1)}(r) \right) q(z, t) + \sum_{k=1}^{K} w_k(z, t) \left( Z_{2k}(r) - Z_{2(K+1)}(r) \right), \quad (5.75) \]

   with \( q(z, t) \) and \( w_k(z, t) \) still unknown.

Remark that in all three options there still is a lot of work to be done, since we need to express the solution in terms of the initial inlet flow function \( q_i(t) \) instead of \( q(z, t) \). In this section we opt for the first solution method to work out in all detail.
We start by introducing a Fourier-Bessel series for \( w(r, z, t) \), i.e.

\[
w(r, z, t) = \sum_{k=1}^{\infty} w_k(z, t) J_0(\lambda_k r),
\]

where \( w_k(z, t) \) are Fourier-Bessel coefficients and \( \lambda_k \) the consecutive roots of \( J_0(\cdot) \). If we substitute (5.76) into (5.67) and rewrite the right-hand side of that same equation by means of

\[
1 = \sum_{k=1}^{\infty} \frac{2 J_0(\lambda_k r)}{\lambda_k J_1(\lambda_k)},
\]

we arrive at

\[
\sum_{k=1}^{\infty} \left[ \frac{\partial w_k}{\partial t} J_0(\lambda_k r) + \hat{\nu} \lambda_k^2 J_0(\lambda_k r) w_k(z, t) \right] = \sum_{k=1}^{\infty} \Pi(z, t) \frac{2 J_0(\lambda_k r)}{\lambda_k J_1(\lambda_k)},
\]

where we used that

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{d J_0(\lambda_k r)}{dr} \right) = -\lambda_k^2 J_0(\lambda_k r).
\]

Since (5.78) should hold for every \( k \), we see that, after dividing by \( J_0(\lambda_k r) \), we are left with

\[
\frac{\partial w_k}{\partial t} + \hat{\nu} \lambda_k^2 w_k = \frac{2 \Pi}{\lambda_k J_1(\lambda_k)}, \quad \text{for all } k = 1, 2, \ldots
\]

This equation has as general solution

\[
w_k(z, t) = e^{-\hat{\nu} \lambda_k^2 t} \left[ c_k(z) + \frac{2}{\lambda_k J_1(\lambda_k)} \int_0^t \Pi(z, \tau) e^{\hat{\nu} \lambda_k^2 \tau} d\tau \right].
\]

Since \( w(z, r, t) = 0 \) for \( t = 0 \) (see (5.70)), we conclude that \( c_k = 0 \) and thus

\[
w(r, z, t) = 2 \sum_{k=1}^{\infty} \frac{J_0(\lambda_k r)}{\lambda_k J_1(\lambda_k)} \Omega_k(z, t),
\]

with \( \Omega_k(z, t) \) as defined in (5.73). This solution is indeed the same as the one suggested in (5.82).

Our next goal is to determine \( \Pi(z, t) \), and consequently \( \Omega_k(z, t) \), in terms of the inlet flow function \( q_i(t) \). First of all we derive with (5.82) and the definitions for \( q(z, t) \) and \( r_w(z, t) \) in Section 5.1

\[
q(z, t) = 2 \int_0^1 r w(r, z, t) dr = 4 \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} \Omega_k(z, t),
\]

\[
\tau_w(z, t) = -\hat{\nu} \frac{\partial w}{\partial r}(1, z, t) = 4 \hat{\nu} \sum_{k=1}^{\infty} \Omega_k(z, t).
\]

With these expressions and (5.68), we find a relation for \( \Pi(z, t) \):

\[
\frac{\partial \Pi}{\partial t} = \frac{\partial^2 q}{\partial z^2} = 4 \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} \frac{\partial^2 \Omega_k}{\partial z^2}.
\]

**NOTE:** The equation (5.69) is automatically full-filled since

\[
\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} = \frac{1}{4}.
\]

A derivation of the latter equality, as well as a derivation of (5.77), can be found in Appendix C.
We will solve (5.85) by means of Laplace transforms. For this, we change the coordinate system from \((z, t)\) to \((\eta, \xi)\), with \(\eta = z\) and \(\xi = t - z\), such that

\[
\Pi(z, t) = \tilde{\Pi}(z, t - z) = \tilde{\Pi}(\eta, \xi).
\] (5.87)

This gives

\[
\begin{align*}
\Omega_k(z, t) &= \int_z^t \Pi(z, \tau) e^{-i\lambda_k^2(t-\tau)} d\tau \\
&= \int_0^{\xi+\eta} \Pi(\eta, \tau) e^{-i\lambda_k^2(\xi+\eta-\tau)} d\tau \\
&= \int_0^\xi \Pi(\eta, y+\eta) e^{-i\lambda_k^2(\xi-y)} dy \\
&= \int_0^\xi \tilde{\Pi}(\eta, y) e^{-i\lambda_k^2(\xi-y)} dy = \tilde{\Omega}_k(\eta, \xi).
\end{align*}
\] (5.88)

Rewriting (5.85) in terms of \(\tilde{\Omega}_k\) and \(\tilde{\Pi}_k\), we obtain

\[
\partial \tilde{\Pi} \partial \xi = 4 \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} \left( \frac{\partial^2}{\partial \eta^2} - 2 \frac{\partial^2}{\partial \eta \partial \xi} + \frac{\partial^2}{\partial \xi^2} \right) \tilde{\Omega}_k.
\] (5.89)

We will solve (5.89) for \(\tilde{\Omega}_k\) and at the end we go back to \(\Pi\). We define the Laplace transform of \(\tilde{\Omega}_k(\eta, \xi)\) by

\[
P(\eta; s) = L\left( \tilde{\Pi}(\eta, \xi); \xi, s \right) = \int_0^\infty \tilde{\Pi}(\eta, \xi) e^{-s\xi} d\xi.
\] (5.90)

Applying the Laplace transform (with respect to \(\xi\)) to both sides of (5.89) and using the above definition, we arrive at

\[
sP(\eta; s) - \tilde{\Pi}(\eta, 0) = 4 \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} \left( \frac{\partial^2}{\partial \eta^2} \Theta_k(\eta; s) - 2 \frac{\partial}{\partial \eta} \left( s \Theta_k(\eta; s) - \tilde{\Omega}_k(\eta, 0) \right) + s^2 \Theta_k(\eta; s) - s \tilde{\Omega}_k(\eta, 0) - \frac{\partial \tilde{\Omega}_k}{\partial \xi}(\eta, 0) \right),
\] (5.91)

where

\[
\Theta_k(\eta; s) = L\left( \tilde{\Omega}_k(\eta, \xi), \xi, s \right) = \frac{P(\eta; s)}{s + i\lambda_k^2}.
\] (5.92)

From (5.88) we immediately see that \(\tilde{\Omega}_k(\eta, 0) = 0\). From this same identity we can derive that \(\partial_\xi \tilde{\Omega}_k(\eta, 0) = \tilde{\Pi}(\eta, 0)\), which means that these two terms cancel out. This reduces (5.91) to

\[
\frac{\partial^2}{\partial \eta^2} \Theta_k(\eta; s) - 2s \frac{\partial}{\partial \eta} \Theta_k(\eta; s) + \left( s^2 - \rho^2(s) \right) \Theta_k(\eta; s) = 0,
\] (5.93)

with

\[
\rho(s) = \sqrt{\frac{s}{\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} \frac{I_0\left( \sqrt{s} \right)}{I_1\left( \sqrt{s} \right)}}}.
\] (5.94)

see Appendix C for the derivation of the last step in (5.94). The general solution of this equation, meeting the condition \(\Theta \rightarrow 0\) for \(\eta \rightarrow \infty\), is

\[
\Theta_k(\eta; s) = \Theta_0(s) e^{-\rho(s)\eta},
\] (5.95)

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where the function \( \mathcal{P}_0(s) \) is still to be determined. To this end, we propose a boundary condition to \( \Pi(\eta; s) \). We know that \( \Pi_0(\xi) := \Pi(0, \xi) := \Pi(0, t) = \Pi_0(t) \). With \( (5.73) \) and \( (5.83) \) we find

\[
q_i(t) = \lim_{z \to 0} q(z, t) = \frac{4}{\lambda_k^2} \int_0^t \hat{P}_i(\tau) e^{-i \lambda_k^2 \tau} \, d\tau.
\]

The Laplace transform of \( (6.1) \) yields

\[
\mathcal{Q}_i(s) = \mathcal{P}_0(s) \sum_{k=1}^{\infty} \frac{4}{\lambda_k^2 (s + i \lambda_k^2)} = \frac{s}{\rho^2(s)} \mathcal{P}_0(s),
\]

with \( \mathcal{Q}_i(s) \) the Laplace transform of \( q_i(t) \). So, we conclude that

\[
\mathcal{P}(\eta; s) = \frac{\rho^2(s)}{s} \mathcal{Q}_i(s) e^{i \eta e^{s(\rho(s)\eta)}}.
\]

Moreover, we find that

\[
\mathcal{W}(r; \eta; s) = \mathcal{L}(\tilde{w}(r, \eta, \xi); \xi, s) = 2 \sum_{k=1}^{\infty} \frac{I_0(\lambda_k \tau)}{\lambda_k J_1(\lambda_k)} \mathcal{L}(\hat{Q}_i(\eta, \xi); \xi, s)
\]

\[
= \mathcal{P}(\eta, s) \sum_{k=1}^{\infty} \frac{4}{\lambda_k (s + i \lambda_k^2)} J_1(\lambda_k)
\]

\[
= \rho^2(s) \frac{I_0(\sqrt{s}) - I_0(\sqrt{\pi^2})}{s} = \frac{s}{s I_0\left(\sqrt{s}\right)} \mathcal{Q}_i(s) e^{i \eta e^{s(\rho(s)\eta)}}.
\]

the derivation of the last step of \( (5.99) \) can be found in Appendix C. With \( (5.94) \) we derive

\[
\frac{s}{s I_0\left(\sqrt{s}\right)} \frac{I_0(\sqrt{s}) - I_0(\sqrt{\pi^2})}{s} = s \Lambda(r; s),
\]

with \( \Lambda(\cdot, \cdot) \) as defined in \( (5.51) \). Substitution of \( (5.100) \) in \( (5.99) \) yields

\[
\mathcal{W}(r; \eta; s) = \Lambda(r; s) s \mathcal{Q}_i(s) e^{i \eta e^{s(\rho(s)\eta)}}
\]

(5.101)

Inverse transformation of \( (5.101) \) results in (analogous to the procedure in \( (5.51) - (5.66) \) in the preceding section)

\[
\tilde{w}(r, \eta, \xi) = \mathcal{L}^{-1}\left[ \Lambda(r; s) s \mathcal{Q}_i(s) e^{i \eta e^{s(\rho(s)\eta)}} \right]
\]

\[
= \int_0^\xi \mathcal{L}^{-1}\left[ \Lambda(r; s) s \mathcal{Q}_i(s) \right] (\xi - x) \, dx
\]

\[
= \int_0^\xi g(r, x) f(\eta, \xi + \eta - x) \, dx = \int_0^{t-z} g(r, x) f(z, t - x) \, dx
\]

\[
= \int_0^t g(r, t - x) f(z, x) \, dx = w(r, z, t)
\]

(5.102)

where

\[
g(r, t) = \mathcal{L}^{-1}\left[ \Lambda(r; s) s \mathcal{Q}_i(s) \right] (x) = \int_0^t \lambda(r, t - \tau) \tilde{q}_i(\tau) \, d\tau
\]

\[
= 2 \left( 1 - r^2 \right) q_i(t) - \sum_{k=1}^{\infty} \lambda_k(\tau) \Phi_k(t),
\]

(5.104)
which is identical to (5.66), where \( \lambda(\cdot, \cdot) \) the inverse transformation of \( \Lambda(\cdot, \cdot) \) (as in (5.62)), and

\[
f(z, t) = \mathcal{L}^{-1}\left\{ e^{-\rho(s)z}; s, t \right\}.
\]  

(5.105)

The evaluation of the latter inverse Laplace transform can be found in Appendix B.

5.4 An approximative zeroth order solution

In the previous section we derived an exact solution for the zeroth order approximation of the axial velocity. This solution is given by

\[
w(r, z, t) = \int_z^t g(r, t - x) f(z, x) \, dx,
\]  

(5.106)

where

\[
g(r, t) = 2 \left( 1 - r^2 \right) q_1(t) - \sum_{k=1}^{\infty} \Phi_k(r) \lambda_k(t).
\]  

(5.107)

and

\[
f(z, t) = \mathcal{L}^{-1}\left\{ e^{-\rho(s)z}; s, t \right\}.
\]  

(5.108)

It turns out that the obtained solution for the inverse Laplace transform \( f(z, t) \), however exact, is rather complex, and in the form of an infinite series containing integrals that can only be calculated numerically. Of course, we can substitute this series-representation for \( f(z, t) \), truncated after a finite number of terms, into (5.103) and compute \( w(r, z, t) \) for any given \( q_1(t) \).

However, for the sake of simplicity, we opt here for an alternative approach in which we use an approximation for \( f(z, t) \), which seems us useful for practical purposes and certainly for small values of \( \hat{\nu} \), when the boundary layer at the wall is most steep. For larger values of \( \hat{\nu} \) the flow profile starts to resemble more and more the classical, quadratic, Poiseuille profile. An extra advantage inherent to the approximation for \( f(z, t) \) that we shall derive below is that it enables us, as we will show further on, to compare this solution with the one of Bessems et al., [10], that was used in the preceding chapter. We proceed with the derivation of this approximation.

Observe that, if \( \mathcal{F}(z, s) = e^{-\rho(s)z} \), \( \mathcal{F} \) satisfies

\[
\frac{d^2 \mathcal{F}}{dz^2} - \rho^2(s) \mathcal{F} = 0, \quad \mathcal{F}(0, s) = 1,
\]  

(5.109)

where

\[
\rho^2(s) = \frac{s}{4 \sum_{k=1}^{\infty} \frac{1}{\lambda_k(t)^2 + s}}.
\]  

(5.110)

Substituting (5.110) into (5.109) and using (5.86), leads us to

\[
\sum_{k=1}^{\infty} \frac{1}{\lambda_k(t)^2} \left\{ \frac{1}{\hat{\nu} \lambda_k^2} - \frac{\lambda_k^2}{s} \frac{d^2 \mathcal{F}}{dz^2} - s \mathcal{F} \right\} = 0.
\]  

(5.111)

For the first approximation, we only take into account the contribution for \( k = 1 \). So \( \mathcal{F} \approx \mathcal{F}^{(1)} \), where \( \mathcal{F}^{(1)} \) is the solution of

\[
\frac{d^2 \mathcal{F}^{(1)}}{dz^2} - \left( \hat{\nu} \lambda_1^2 + s \right) \mathcal{F}^{(1)} = 0, \quad \mathcal{F}^{(1)}(0, s) = 1.
\]  

(5.112)
So
\[ F^{(1)}(z; s) = e^{-\sqrt{s} t + \hat{\nu} \lambda^2_1 z}}, \quad (5.113) \]
and, analogous to \(4.79\), but with \(\hat{\nu} \to \hat{\nu} \lambda^2_1/4\),
\[ f^{(1)}(z, t) = \mathcal{L}^{-1} \left\{ F^{(1)}(z; s); s, t \right\} \]
\[ = e^{\frac{-t}{2} \hat{\nu} \lambda^2_1/4} \left[ \delta(t - z) + \frac{1}{2} \hat{\nu} \lambda^2_1 z \frac{I_1 \left( \frac{1}{2} \hat{\nu} \lambda^2_1 \sqrt{t^2 - z^2} \right)}{\sqrt{t^2 - z^2}} H(t - z) \right]. \quad (5.114) \]
Notice that for \(\hat{\nu} \downarrow 0\) the exact solution for \(f(z, t)\) and the approximated one \(f^{(1)}(z, t)\) coincide, and thus, it seems reasonable to assume that \(f^{(1)}(z, t)\) is a good approximation for \(f(z, t)\) for smaller values of \(\hat{\nu}\) (say for \(\hat{\nu} \lambda^2_1 \ll 1\)).

### 5.4.1 Derivation of the approximative flow characteristics

We now substitute the approximation \(f^{(1)}(z, t)\), \(\text{[5.114]}\), for \(f(z, t)\) into \(\text{[5.103]}\), but we keep the full solution for \(g(r, t)\). Again, to keep the final result readable, we introduce the number \(\alpha_1 = \hat{\nu} \lambda^2_1/2\) and following short-hand notation:
\[ \mathcal{A}(z, t) = I_1 \left( \frac{\alpha_1 \sqrt{t^2 - z^2}}{\sqrt{t^2 - z^2}} \right). \quad (5.115) \]
With these abbreviations, \(\text{[5.103]}\) results in (with \(w(r, z, t) \approx w^{(1)}(r, z, t)\)), and for \(0 < z < t\)
\[ w^{(1)}(r, z, t) = \int_z^t g(r, t - x) f^{(1)}(z, x) \, dx \]
\[ = e^{-\alpha_1 z} \left\{ 2 \left( 1 - t^2 \right) q_i(t - z) - \sum_{k=1}^{\infty} \mathcal{J}_k(r) \Phi_k(t - z) \right\} \]
\[ + 2 \alpha_1 z \left( 1 - t^2 \right) \int_z^t \mathcal{A}(z, x) q_i(t - x) e^{-\alpha_1 x} \, dx \]
\[ - \alpha_1 z \sum_{k=1}^{\infty} \mathcal{J}_k(r) \left\{ \int_z^t \mathcal{A}(z, x) \Phi_k(t - x) e^{-\alpha_1 x} \, dx \right\} \quad (5.116) \]
If we define the functions \(\mathcal{H}(z, t)\) and \(\mathcal{H}_k(z, t)\) (for \(k = 1, 2, \ldots\)) as
\[ \mathcal{H}(z, t) = \int_z^t \mathcal{A}(z, x) q_i(t - x) e^{-\alpha_1 x} \, dx, \quad (5.117) \]
\[ \mathcal{H}_k(z, t) = \int_z^t \mathcal{A}(z, x) \Phi_k(t - x) e^{-\alpha_1 x} \, dx, \quad (5.118) \]
then the final result for \(w^{(1)}(r, z, t)\) reads
\[ w^{(1)}(r, z, t) = e^{-\alpha_1 z} \left\{ 2 \left( 1 - r^2 \right) q_i(t - z) - \sum_{k=1}^{\infty} \mathcal{J}_k(r) \Phi_k(t - z) \right\} + \]
\[ \alpha_1 z \left\{ 2 \left( 1 - r^2 \right) \mathcal{H}(z, t) - \sum_{k=1}^{\infty} \mathcal{J}_k(r) \mathcal{H}_k(z, t) \right\}. \quad (5.119) \]
The goal is to compare the solutions for the flow characteristics in the global equations of motion with the ones in the local case. With the approximative expression \( w^{(1)}(r, z, t) \) for the zeroth order axial flow velocity, we are able to also derive zeroth order expressions for \( q(z, t) \) and \( p(z, t) \), but especially expressions for \( \tau_w(z, t) \) and \( \gamma_a(z, t) \). Using the definitions given in (5.28) leads us, for \( 0 < z < t \), to

\[
q(z, t) = e^{-\alpha_1 z} q_i(t - z) + \alpha_1 z H(z, t),
\]

\[
\tau_w(z, t) = 8 \hat{v} \left( e^{-\alpha_1 z} q_i(t - z) + \alpha_1 z H(z, t) \right) + 4 \hat{v} \sum_{k=1}^{\infty} \left( e^{-\alpha_1 z} \Phi_k(t - z) + \alpha_1 z \mathcal{H}_k(z, t) \right)
\]

Remark the resemblance between the solution for \( q(z, t) \) with the one we found in (4.88) in Section 4.3. Since we make an approximation for \( f(z, t) \) to satisfy (5.68) or (5.69). From the observation that \( q(z, t) \) resembles the solution from Chapter 4, it seems reasonable to opt for the first possibility and so we choose to take also for \( p(z, t) \) the same comparable form as the solution for \( p(z, t) \) in Section 4.3. So we find

\[
p(z, t) = \int_{z}^{t} \int_{0}^{r} \left( \alpha_1 \sqrt{x^2 - z^2} \right) \left( \hat{q}_i(t - x) + 2 \alpha_1 q_i(t - x) \right) e^{-\alpha_1 x} \, dx.
\]

Finally, we can use definition (5.33) to derive a zeroth order expression for \( \gamma_a(z, t) \). Note that this definition is depending on the zeroth order expression for the axial velocity but is part of the first order system (due to the \( \delta \) coefficient of \( \partial \gamma_a / \partial z \) in (5.18)). Since

\[
\left( w^{(1)}(r, z, t) \right)^2 = \left( 2 \left( 1 - r^2 \right) \left( e^{-\alpha_1 z} q_i(t - z) + \alpha_1 z H(z, t) \right) \right)^2
\]

\[
- \sum_{k=1}^{\infty} \mathcal{J}_k(r) \left( e^{-\alpha_1 z} \Phi_k(t - z) + \alpha_1 z \mathcal{H}_k(z, t) \right)^2
\]

\[
= 4 \left( 1 - r^2 \right)^2 q^2(z, t) + \left( \sum_{k=1}^{\infty} \mathcal{J}_k(r) \left( e^{-\alpha_1 z} \Phi_k(t - z) + \alpha_1 z \mathcal{H}_k(z, t) \right) \right)^2
\]

\[
- \sum_{k=1}^{\infty} 4 \left( 1 - r^2 \right) \mathcal{J}_k(r) q(z, t) \left( e^{-\alpha_1 z} \Phi_k(t - z) + \alpha_1 z \mathcal{H}_k(z, t) \right),
\]

we find that

\[
\gamma_a(z, t) = \frac{4}{3} q^2(z, t) + \sum_{k=1}^{\infty} \frac{4}{\sigma_k^2} \left( e^{-\alpha_1 z} \Phi_k(t - z) + \alpha_1 z \mathcal{H}_k(z, t) \right)^2
\]

\[
- q(z, t) \sum_{k=1}^{\infty} \frac{8}{\sigma_k^2} \left( e^{-\alpha_1 z} \Phi_k(t - z) + \alpha_1 z \mathcal{H}_k(z, t) \right),
\]

where we used the orthogonality property for Bessel functions for the quadratic term of \( \mathcal{J}_k(r) \), such that

\[
\int_{0}^{1} r \mathcal{J}_k(r) \mathcal{J}_l(r) \, dr = \begin{cases} 
0 & \text{for } k \neq l, \\
\frac{4}{\sigma_k^2} & \text{for } k = l.
\end{cases}
\]
5.4.2 The accuracy of the approximative solution

By introducing the approximation \( f^{(1)}(z, t) \) for \( f(z, t) \) we agreed on also introducing an error with respect to the exact solution. In this subsection we try to estimate the difference. One of our primary goals was to solve all systems analytically as far as possible, which supports the choice for \( f^{(1)}(z, t) \). However, there are numerical methods available for calculating a numerical approximation of the exact inverse Laplace transforms. For a first impression, we compare our approximation \( f^{(1)}(z, t) \) with a numerical approximation. We opt to use the built-in function `invlap.m` in Matlab, which is based on [2].

Since \( f^{(1)}(z, t) \) contains a delta function, we want to avoid the generation of a delta function in calculating the numerical solution. To this end, we do not compare \( f^{(1)}(z, t) \) with \( f(z, t) \), but \( h^{(1)}(z, t) = -\partial_z f^{(1)}(z, t) \) with \( h(z, t) = -\partial_z f(z, t) \). The resulting plots, for \( 0 < t < 2\pi \), \( z = 1 \) and a typical value for \( \hat{\nu}(=0.03) \), are shown in Figure 5.1. We observe that around \( t = z \) there is a difference between the behavior of both functions, but in general we may conclude that the difference of both solutions is small compared to the \( \mathcal{O}(1) \) behavior of the flow characteristics.

An other way to investigate the accuracy of the approximation, is to see in how far the approximative flow characteristics are in agreement with the equations of motion (5.67)-(5.69). In the sequel of this section we omit \(^{(1)}\), since it is clear that we are dealing with the approximative solutions. From (5.116) we deduce that

\[
\frac{\partial w}{\partial t} = e^{-\alpha_1 z} \left\{ 2 \left(1 - r^2\right) \dot{q}_i(t - z) - \sum_{k=1}^{\infty} \mathcal{H}_k(r) \dot{\Phi}_k(t - z) \right\} + \\
\alpha_1 z \left\{ 2 \left(1 - r^2\right) \frac{\partial \mathcal{H}}{\partial t} - \sum_{k=1}^{\infty} \mathcal{H}_k(r) \frac{\partial \mathcal{H}_k}{\partial t} \right\} \\
= \frac{3q}{\partial t} + 4\delta \sum_{k=1}^{\infty} \frac{J_0 (\sigma_k) - J_0 (\sigma_k r)}{J_0 (\sigma_k)} \left( e^{-\alpha_1 z \Phi(t - z) + \alpha_1 z I_k(z, t)} \right),
\]  

\( (5.126) \)

where in the last step we used that, according to (5.65),

\[
\dot{\Phi}_k(t) = \dot{q}_i(t) - \hat{\nu} \sigma_k^2 \Phi_k(t),
\]  

\( (5.127) \)
and that for $0 \leq r < 1$

$$\sum_{k=1}^{\infty} g_k(r) = 1 - 2r^2. \quad (5.128)$$

Moreover

$$-\frac{\hat{v}}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) = 8\hat{v} \left( e^{-\alpha_1 z} q_i(t - z) + \alpha_1 z \mathcal{H}(z, t) \right) +$$

$$\frac{\hat{v}}{r} \sum_{k=1}^{\infty} 4J_1(\sigma_k r) \left( e^{-\alpha_1 z} \Phi_k(t - z) + \alpha_1 z \mathcal{H}_k(z, t) \right) +$$

$$\hat{v} \sum_{k=1}^{\infty} \frac{2(J_0(\sigma_k r) - J_2(\sigma_k r))}{\sigma_k J_0(\sigma_k)} \left( e^{-\alpha_1 z} \Phi_k(t - z) + \alpha_1 z \mathcal{H}_k(z, t) \right)$$

$$= 8\hat{v} q(z, t) + 4\hat{v} \sum_{k=1}^{\infty} \frac{J_0(\sigma_k r) - J_2(\sigma_k r)}{J_0(\sigma_k)} \left( e^{-\alpha_1 z} \Phi_k(t - z) + \alpha_1 z \mathcal{H}_k(z, t) \right). \quad (5.129)$$

which, taken together, yields

$$\frac{\partial w}{\partial t} - \frac{\hat{v}}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) = \frac{\partial q}{\partial t} + 8\hat{v} q + 4\hat{v} \sum_{k=1}^{\infty} \left( e^{-\alpha_1 z} \Phi_k(t - z) + \alpha_1 z \mathcal{H}_k(z, t) \right). \quad (5.130)$$

We immediately conclude that indeed

$$\frac{\partial w}{\partial t} - \frac{\hat{v}}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) = \frac{\partial q}{\partial t} + \tau_w. \quad (5.131)$$

From Chapter 4 we know that for the approximative $q(z, t)$ from (5.120) and $p(z, t)$ from (5.122), (5.68) is already satisfied. So, as a criterion for the correctness of our approximation, we will use (5.71). From expression (5.122) for $p(z, t)$ we see that

$$\Pi(z, t) = -\frac{\partial p}{\partial z} = (\hat{q}_i(t - z) + 2\alpha_1 q_i(t - z)) e^{-\alpha_1 z} +$$

$$\alpha_1 z \int_{z}^{t} A(z, x) (\hat{q}_i(t - x) + 2\alpha_1 q_i(t - x)) e^{-\alpha_1 x} \, dx$$

$$= \frac{\partial q}{\partial t}(z, t) + 2\alpha_1 q(z, t). \quad (5.132)$$

So, we might conclude that the approximation is permitted whenever the difference of $\Pi(z, t)$ and $\partial q/\partial t + \tau_w$ is small, i.e.

$$\left( 8\hat{v} - 2\alpha_1 \right) q(z, t) + 4\hat{v} \sum_{k=1}^{\infty} \left( e^{-\alpha_1 z} \Phi_k(t - z) + \alpha_1 z \mathcal{H}_k(z, t) \right) \approx 0. \quad (5.133)$$

which means

$$\max_{z,t} \left| \left( 8\hat{v} - 2\alpha_1 \right) q(z, t) + 4\hat{v} \sum_{k=1}^{\infty} \left( e^{-\alpha_1 z} \Phi_k(t - z) + \alpha_1 z \mathcal{H}_k(z, t) \right) \right| \ll \max_{z,t} |\Pi(z, t)|. \quad (5.134)$$

We will discuss this result in the next chapter, when we choose an explicit function for $q_i(t)$.
5.5 Preliminary conclusions from this chapter

In this chapter we found an analytical zeroth order (in $\delta$) solution for the axial velocity as part of the local 3-D equations of motions describing the flow through a semi-infinite, deformable vessel. As part of the solution tactic, we first solved the same problem for a rigid (undeformable) vessel with constant internal radius. The solution for the axial velocity depends on a rather complicated inverse Laplace transform, which for reasons of simplification, has been approximated analytically. This approximated solution for the axial velocity enabled us to also find zeroth order approximated solutions for the pressure distribution, the volumetric flow rate, the radial diffusion term and the advection term.

In the next chapter we hope to compare the resulting expressions for the approximated flow quantities from this chapter, based on the local 3-D axial velocity, with the solutions from the global 1-D equations of motion in the previous chapter. Of special interest will be comparison of the radial diffusion term and the advection term, which in the global case both are based on Bessems et al. [10]. We hope to establish a classification, based on these comparisons, for the flow characteristics for a set of representative human arteries.
Chapter 6

Results and discussion

In Chapter 4 we derived solutions for the flow rate and pressure distribution from the global equations of motion. The expressions for the radial diffusion and the advection contribution are based on Bessems [10]. In Chapter 5 we derived a zeroth order approximation for the axial velocity from the local equations of motion, which gave rise to expressions for the flow rate and pressure distribution according to this zeroth-order approximation. More importantly, we were also able to derive the radial diffusion and the advection contribution due to this expression for the axial velocity. In this chapter we will plot the results from the previous two chapters for a given expression for the inlet-flow function. We compare the results from the plots and discuss the differences between the solutions from the global equations of motion and the solutions from the local ones.

6.1 Results

An overview of all the expressions derived in the previous two chapters are recapitulated in Table 6.3. We added the subscripts \( c, v, d, a \) to distinguish between solutions for constant wave velocity, variable wave velocity, radial diffusion and advection, respectively, for the global equation of motion and the subscript \( l \) the solutions from the local equations of motion.

All solutions derived are expressed in terms of the inlet-flow function \( q(0, t) = q_i(t) \). In order to be able to plot the solutions and compare them we need an expression for the scaled inlet-flow function \( q_i(t) \). We choose for

\[
q_i(t) = \frac{2t(\pi - t)}{(1 + t^2)(\sqrt{1 + \pi^2} - 1)} H(t) H(\pi - t),
\]

which is zero for \( t < 0 \) and \( t > \pi \) and has a maximum value of one. A plot of (6.1) can be found in Figure 6.1. Besides the inlet-flow function, we also introduced a lot of geometrical and physical parameters, inclusive the dimensionless numbers, for describing the flow through the vessel. All these quantities are calculated for a set of seven typical representative human arteries, denoted with vessel (A)-(H) (leaving out vessel (G), since it is very much the same as vessel (F)), based on data from [10] and [1]. See Table 6.2, page 73 for all details. The choice of arteries follows the choice of Bessems [1] and the position of the arteries is schematically presented in Figure 6.2. In the analysis in Chapter 4 and 5, the dimensionless compliance \( \delta \) plays an important role. Therefore, we choose to base our results on three vessels, which correspond to three typical values of this number \( \delta \); namely vessel A \( (\delta = 0.12) \), vessel B \( (\delta = 0.24) \) and vessel D \( (\delta = 0.04) \).
Figure 6.1: Plot of the scaled inlet-flow function as defined in equation (6.1)

Table 6.1: Average percentages for the differences of the solutions for $q_{c,v,d,a}(z,t)$ with respect to $q_{l}(z,t)$, $p_{c,v,d,a}(z,t)$ with respect to $p_{l}(z,t)$ and for the global solutions for $\tau_w$ and $\gamma_a$ with respect to local ones.

<table>
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<th>$\delta$</th>
<th>$q_c$</th>
<th>$q_t$</th>
<th>$q_d$</th>
<th>$q_a$</th>
<th>$p_c$</th>
<th>$p_t$</th>
<th>$p_d$</th>
<th>$p_a$</th>
<th>$\tau_w$</th>
<th>$\gamma_a$</th>
</tr>
</thead>
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<td>A</td>
<td>0.12</td>
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<td>+3%</td>
<td>-11%</td>
<td>+1%</td>
<td>+2%</td>
<td>+1%</td>
<td>-4%</td>
<td>-9%</td>
<td>+10%</td>
<td>+30%</td>
</tr>
<tr>
<td>B</td>
<td>0.24</td>
<td>+10%</td>
<td>+10%</td>
<td>-17%</td>
<td>+9%</td>
<td>+7%</td>
<td>+1%</td>
<td>-4%</td>
<td>-19%</td>
<td>+5%</td>
<td>+18%</td>
</tr>
<tr>
<td>D</td>
<td>0.04</td>
<td>+43%</td>
<td>+43%</td>
<td>-13%</td>
<td>+43%</td>
<td>+26%</td>
<td>+5%</td>
<td>+21%</td>
<td>+6%</td>
<td>+93%</td>
<td></td>
</tr>
</tbody>
</table>

NOTE: In Table 6.2, $L_v$ is the actual length of the vessel (for this we took the sum of the values from [10] per artery), and $L = c_0/\omega$ is the characteristic length for the wave propagation in a semi-infinite vessel, as it is used in our model. The parameter $\varepsilon = R_a/L$ is defined on $L$, rather than on $L_v$, and is for all vessels considered less than 5%, which justifies the neglect of $O(\varepsilon^2)$-terms in (3.24) and (3.25), Chapter 3. However, we note that although $L_v < L$ always, even if we would define $\varepsilon$ on $L_v$, then still $\varepsilon \leq 5\%$, except for case $\Lambda$, for which $\varepsilon \approx 36\%$. In the latter case the aforementioned approximation becomes critical (albeit not in our semi-infinite model).

In Figure 6.3 on page 75 we plotted the axial velocity as a function of $r$ for certain time $t$ and a few values of $z$. In this same Figure, but in the other column, we plotted $\Pi(z,t) = -\partial p/\partial z$ and the error, as derived in (5.134). In Figure 6.4–6.7, on pages 76–79, the solutions for the flow rate $q(z,t)$, the pressure distribution $p(z,t)$, the shear force $\tau_w(z,t)$ at the wall and the advection distribution $\gamma_a(z,t)$ are plotted; in column (a) as a function of $t$ for two values of $z$, and in column (b) as a function of $z$ for two values of $t$. Except for the plots of the error in Figure 6.3, all plotted quantities are dimensional, whereas the variables $r$, $z$ and $t$ are dimensionless.

In order to get some inside into the difference between the global solutions and the local solutions, we calculated the ratio of the maximum of a global solution and the maximum of the corresponding local solution and took the average of the results; in all cases there are four results: two from column (a) and two from column (b) in each of the plots. The results from these calculations can be found in Table 6.1.

68 Mathematical modeling of blood flow through a deformable thin-walled vessel
6.2 Comments on and discussion of the results

In this section we comment on the results, as presented in the previous section and we draw conclusions based on the observations we make. Here, we follow the order of the figures on pages 76/79.

1. The axial velocity \( w(r, z, t) \) is plotted in Figure 6.3, column (a), as a function of \( r \) for \( t = \pi \) and \( z = \{0, 0.1, 0.25, 0.5, 0.75\} \pi \). For small values of \( z \) (with respect to time \( t \)), we see that the axial velocity is negative for \( r \rightarrow 1 \), which means that in a small boundary layer near the vessel wall, fluid is flowing backward for small values of the axial coordinate. This effect is mathematically caused by the contribution of the \( z \) term in the expression of \( w \) (see Table 6.3). For small values of \( \hat{v} \), i.e. vessel (A) and (B), the axial velocity profile near by the flow front, i.e. for \( z \) such that \( t-z \ll t \), resembles that of a plug flow. This is caused by the effect of the combination of the \( \exp(-\alpha_1 t) q_i \) and \( z \) term. For vessel (D), the larger value of \( \hat{v} \), a more quadratic Poiseuille-like profile establishes for \( z \) near to \( t \). Finally, we notice that the larger \( \delta \) gets, the larger the value of the axial velocity at the center of the vessel, i.e. in \( r = 0 \), becomes. This is caused by the elasticity of the vessel wall, which is higher for larger values of \( \delta \) and leads to a larger cross-sectional area and therefore to a higher flow rate and a higher velocity.

2. In Section 5.4.2, we discussed the accuracy of our solution due to the approximation we made for the inverse Laplace transform \( f(z, t) \). In Figure 6.3 column (b), we plotted \( \Pi(t, z) \) as a function of \( t \), where we choose \( \Pi = -\partial p/\partial z \), and the error defined by (5.134) which is the difference of \( \Pi = -\partial p/\partial z \) and \( \Pi = \partial q/\partial t + \tau_w \). In all three different cases for \( \delta \) we conclude that the error is much smaller (less than 15%) than \( \Pi(z, t) \), which means that the approximation we made for \( f(z, t) \) is suitable. We leave a detailed discussion about the behavior of \( \Pi(z, t) \) out of consideration, since we do not need it to draw the final conclusions.

3. The solutions for the volumetric flow rate \( q(z, t) \) are shown graphically in Figure 6.4, where in column (a) we plotted \( q(z, t) \) as a function of \( t \) for two values of \( z \), whereas in column (b) \( q(z, t) \) is plotted as a function of \( z \) for two values of \( t \). The same has been done for the pressure distribution \( p(z, t) \) in Figure 6.5. Here, we combine the discussion of the plots for \( q \) and \( p \), since much of the observations hold for both cases.

First of all we note, as a trivial observation, that for \( z = 0 \) (see column (a)), all solutions for \( q(z, t) \) are exactly the same, which is in agreement with the solutions we obtain if we substitute \( z = 0 \) in the solutions for \( q(z, t) \) in Table 6.3.

The solutions with constant wave velocity, i.e. \( q_c \) and \( p_c \) (red), perform like stationary waves, which means that the wave profiles (which are equal to \( q_i(t) \)) do not change in time and place and travel with constant wave speed \( c_0 \). For both vessel (A) and (B) we conclude from the plots and from Table 6.1 that the solutions show great resemblance (in both the profile and the amplitude) with the solutions from the local equations of motion (green). For vessel (D) (i.e. for small \( \delta \)), we see that the amplitudes differ a lot and so for this case the solution with constant wave velocity does not satisfy and we definitely need to include the contribution of the radial diffusion; notice that for this case \( \hat{v} \) is relatively large, and hence, induces a large viscous resistance.

**NOTE:** For solutions with constant wave velocity, we have \( \delta = 0 \) in (4.11). However, we should realize that this does not mean that the vessel is rigid. The deformation of the vessel is still incorporated in \( C \partial p/\partial t \).

For the case in which the wave velocity is variable, so for the solutions \( q_v \) and \( p_v \) (orange), the waves travel with wave speed \( c \), the wave phase increases with increasing \( \delta \) and the amplitudes of the waves change (attenuation). So, here no qualitative differences but only...
quantitative ones are observed. An exception has to be made for the wave front at $z = t$. Here we see that the derivatives of $q$ and $p$ for the larger values of $t$, $z$ and $\delta$ at the wave front $z = t$ are infinite. This singular behavior can be explained from the fact that the characteristics are not parallel to each other as for $\delta = 0$, but intersect with the front characteristic $z = t$ (the characteristic of $t = 0$), for $\delta > 0$, while $q$ and $p$ are still nonzero. The result is a shockwave. The irregularity at the wave front is caused by the fact that we neglected all $O(\epsilon^2)$-terms in the derivation of the global equations of motion. If we did incorporate these terms, then the axial diffusion, i.e. $\partial^2 v_z/\partial z^2$, would still have been present. This term is small in general, however at the wave front $z = t$ it is significant and gives rise to a boundary layer in which the discontinuity we observe now, would have been spread out to a smooth progression in both $q$ and $p$. Again, in the same way as for the solutions for constant wave velocity, we state here that the solutions are of the same order, however different in shape, of the solution obtained in Chapter 5.

The solutions in which the radial diffusion term is included, i.e. $q_d$ and $p_d$ (blue), perform like a traveling wave which is almost completely in phase with the solution from the local equations of motion and has the same profile. This can be expected since the expressions for both $q_e$ and $p_e$ are the same as the ones found in Chapter 5, apart from a small difference in the characteristic parameters. In all plots we observe that the amplitude is smaller than the amplitude for local solutions, but the difference is, also regarding the results in Table 6.1, within an acceptable range for $q_d$ (approximately 15%) and very good for $p_d$ (ranging from -4% to +5%). Finally, we should notice that the solutions for $p_d$ are still larger than zero for $\pi < t < 2\pi$ in column (a) and for $0 < z < \pi$ in column (b). This effect is due to the contribution of $q_i$ in the solution $p_d$ (and also $p_l$) at $t = \pi$ or $z = \pi$, respectively, where the value of $q_i$ is positive, not zero.

For $q_d$ and $p_d$ (cyan), i.e. the solutions where we incorporated the advection term, we observe that the results show a triangular-like profile with fast decreasing amplitude for the larger values of $\delta$. Just as for the solutions with the variable wave velocity, remarkable discontinuities at $z = t$ are present. Here, the problem lies in the fact that we neglected higher order $\delta$-terms, i.e. terms of $O(\delta^2)$, in (4.106), (4.110), but also in the definition of $\delta_0$ in (4.118). Due to this assumption, the solution performs a phase difference which gets higher whenever $\delta$ is larger. We may conclude, also regarding the results in Table 6.1, that the solutions obtained for the inclusion of the advection term, even for the small value of $\delta$, are highly unreliable. Note that in the other cases we were able to find an exact solution, but here we had to neglect higher order terms of $\delta$ to find a solution. We would have found an exact solution when we would have neglected the compliance contribution in (4.98). However, we can not do this since we are then leaving out a term which is of the same order as the effect we are investigating, namely $\partial^2 v_z/\partial z^2$.

4. In Figure 6.6 the resulting plots for $r_w$ according to the solutions in Chapter 4 (blue) and 5 (red) can be found as a function of time $t$ for two values of $z$ in column (a) and as a function of $z$ for two values of $t$ in column (b). In column (a) we see, that for both solutions, the value of $r_w$ is positive in the beginning but gets negative after some time, whereas in column (b) the same effects is visible but in opposite order. Furthermore, we observe that the solution from the global equations of motion shows a discontinuity at $z = t$, which is due to the fact that $\partial p/\partial z$ is discontinuous at $z = t$. The solution from the local equations of motion does not perform this discontinuity.

However, here we observe another important effect. In column (a) for example, we see that the solution at $z = 0$ is not equal to zero for $t > \pi$, but gradually tends to zero for $t > \pi$. This effect is caused again, just as in the discussion for $p_d$, by $q_i$ and finds its origin in the $\sum \exp(\alpha_i z) \Phi_i(t - z)$ term. Physically this effects seems plausible, when we imagine that the shear force at the wall fades away gradually after the passing of a single pulse.
Furthermore, we notice from the plots and from Table \[6.1\] that, maybe except for a small environment near $z = t$, both the solutions are in agreement with each other in a qualitative but also quantitative way, i.e. in both the amplitude and the shape of the solution. The influence of $\delta$, or in this context more suitable, the influence of $\tilde{v}$ only effects the magnitude in the way that the smaller $\tilde{v}$ gets, the bigger the amplitude of $\tau_w$ (initially) becomes.

5. The solutions for $\gamma_a$ are plotted in Figure \[6.7\] again as a function of $t$ for two values of $z$ in column (a) and as a function of $z$ for two values of $t$ in column (b). The solution for $\gamma_a$ from the global equations of motion (blue) shows the same effects as we described for the solution $q_a$ and $p_a$, i.e. the discontinuity at $z = t$ due to the neglect of the $O(\delta^2)$-terms. We observe an almost triangular-like behavior and a larger fluctuation in the amplitude for all three values of $\delta$. This in contrast with the red curves, the solutions for $\gamma_a$ from the local equations of motion, which show a nice smooth traveling wave. We might conclude that, also from Table \[6.1\] the global solution has not the desired effects and therefore it is unusable to compare and to draw conclusions from it. One final remark has to be made on the solution of $\gamma_a$ from the local equations of motion if we compare it with the solution of $\tau_w$ from the local equations of motion. We see for vessel (A) and (B) that both quantities are almost of the same order, whereas for vessel (D), the small value of $\delta$, the value of $\gamma_a$ is a factor 10 smaller than the value of $\tau_w$ which implies that for the smaller vessels the radial diffusion is much more important than advection, which is very logical and as can be expected.

6. As already mentioned before, most of the work done is based on Bessems \[1\]. So finally, it seems reasonable to compare the results obtained in Figures \[6.4\]-\[6.7\] with the results in \[1\], especially Figures \[6.4\]-\[6.6\], where plots of $q$, $p$ and $\tau_w$ can be found for the same vessels (A), (B) and (D) as discussed in this report. Unfortunately, no plots of $\gamma_a$ are available in \[1\]. However, since we already concluded that our results for $\gamma_a$ do not meet our expectations, we don’t feel the urge to to compare them with other investigations, such as the ones in Bessems \[1\].

In we compare the plots for $q$, we see, although we only regarded a single pulse whereas Bessems in \[1\] regards a periodic pulse, that both figures show great resemblance. Of course, this is expected, since we based our values for $Q_0$ on these pictures. The difference in shape and in time scale is due to the choice we made for the inlet flow function $q_t$, which is a very simplify representation of the real inlet flow. For the graphs of the pressure distribution $p(z, t)$ we see that the shape of the graphs show the same behavior. However, the maximum values of $p(z, t)$ in Figure \[6.5\] and Figure \[6.4\] of \[1\] differ about a factor five. One of the reasons could be that we regard a semi-infinite vessel. This means that in our case no reflection of waves will occur. If we would have taken a finite vessel in which reflection occurs, then demping and strengthening will take place which will increase the maximum value for $p(z, t)$.

If we want to compare the results of Bessems in \[1\] for $\tau_w$ with with our results, we need to realize that there is a small difference in the definition of $\tau_w$. Except that there is a difference in sign between both definitions, we also see that our definition of $\tau_w$ compares with $(2\pi R_a/\rho)\tau_{w,B}$, where $\tau_{w,B}$ is the definition Bessems uses in \[1\]. Since $(2\pi R_a/\rho) = O(10^{-5})$, at least for the three vessel (A), (B) and (D) we discuss here, we need to incorporate a factor $10^{-5}$ for the plots of $\tau_w$ in \[1\]. If we flip the results in Figure 6.6 of \[1\] around the $x$-axis, which is due to the difference in sign of both definitions of $\tau_w$, we see that the solutions behave alike, even change of sign is present in both result after some time $t$. We also observe the same peak for $\tau_w$ around $z = t$. Even the amplitudes, when taking into account $2\pi 10^{-5}$, are of the same order of magnitude for vessel (A) and (B). The difference for vessel (D) is approximately a factor 2. All together we may conclude that the results for $\tau_w$ are very accurate.
Figure 6.2: Representation of the human arterial system. Figure taken from [1], page 90

Mathematical modeling of blood flow through a deformable thin-walled vessel
<table>
<thead>
<tr>
<th>TYPE</th>
<th>(A) Aorta ascendens</th>
<th>(B) Aorta thoracalis</th>
<th>(C) Aorta abdominalis</th>
<th>(D) A. subclavia</th>
<th>(E) A. iliaca externa</th>
<th>(F)-(G) A. femoralis</th>
<th>(H) A. tibialis anterior</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_v$</td>
<td>[m]</td>
<td>0.04</td>
<td>0.16</td>
<td>0.16</td>
<td>0.10</td>
<td>0.08</td>
<td>0.32</td>
</tr>
<tr>
<td>$R_a$</td>
<td>[$10^{-2}$m]</td>
<td>1.46</td>
<td>0.77</td>
<td>0.58</td>
<td>0.41</td>
<td>0.29</td>
<td>0.25</td>
</tr>
<tr>
<td>$h$</td>
<td>[$10^{-3}$m]</td>
<td>1.63</td>
<td>0.99</td>
<td>0.81</td>
<td>0.67</td>
<td>0.55</td>
<td>0.51</td>
</tr>
<tr>
<td>$E$</td>
<td>[$10^6$Pa]</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>$Q_0$</td>
<td>[$10^{-6}$m$^3$/s]</td>
<td>420</td>
<td>260</td>
<td>70</td>
<td>14</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>$V = Q_0/(\pi R_a^2)$</td>
<td>[m/s]</td>
<td>0.63</td>
<td>1.40</td>
<td>0.66</td>
<td>0.27</td>
<td>0.30</td>
<td>0.36</td>
</tr>
<tr>
<td>$C = (3\pi R_a^3)/(2Eh)$</td>
<td>[$10^{-9}$m$^2$/s/$kg$]</td>
<td>22.49</td>
<td>5.43</td>
<td>2.84</td>
<td>1.21</td>
<td>0.52</td>
<td>0.36</td>
</tr>
<tr>
<td>$c_0 = \sqrt{(\pi R_a^2)/(\rho C)}$</td>
<td>[m/s]</td>
<td>5.32</td>
<td>5.71</td>
<td>5.96</td>
<td>6.44</td>
<td>6.94</td>
<td>7.20</td>
</tr>
<tr>
<td>$L = c_0/\omega$</td>
<td>[m]</td>
<td>0.85</td>
<td>0.91</td>
<td>0.95</td>
<td>1.03</td>
<td>1.10</td>
<td>1.15</td>
</tr>
<tr>
<td>$\varepsilon = R_a/L$</td>
<td>[10$^{-2}$]</td>
<td>1.72</td>
<td>0.85</td>
<td>0.61</td>
<td>0.40</td>
<td>0.26</td>
<td>0.22</td>
</tr>
<tr>
<td>$\delta = V/c_0$</td>
<td>[-]</td>
<td>0.12</td>
<td>0.24</td>
<td>0.11</td>
<td>0.04</td>
<td>0.04</td>
<td>0.05</td>
</tr>
<tr>
<td>$\alpha = R_a\sqrt{\omega/\nu}$</td>
<td>[-]</td>
<td>19.56</td>
<td>10.31</td>
<td>7.77</td>
<td>5.49</td>
<td>3.89</td>
<td>3.35</td>
</tr>
<tr>
<td>$Re = (R_a V)/\nu$</td>
<td>[-]</td>
<td>2616</td>
<td>3070</td>
<td>1097</td>
<td>311</td>
<td>251</td>
<td>255</td>
</tr>
<tr>
<td>$Sr = \alpha^2/(\nu Re)$</td>
<td>[-]</td>
<td>8.49</td>
<td>4.09</td>
<td>8.99</td>
<td>24.30</td>
<td>22.92</td>
<td>20.10</td>
</tr>
<tr>
<td>$P = (Q_0 T)/(LC)$</td>
<td>[Pa]</td>
<td>3507</td>
<td>8375</td>
<td>4142</td>
<td>1793</td>
<td>2207</td>
<td>2694</td>
</tr>
<tr>
<td>$D = (Eh^3)/9$</td>
<td>[$10^{-5}$Nm]</td>
<td>19.25</td>
<td>4.31</td>
<td>2.36</td>
<td>1.34</td>
<td>0.74</td>
<td>0.59</td>
</tr>
<tr>
<td>$\beta = \sqrt{3/(R_a h)^2}$</td>
<td>[m$^{-1}$]</td>
<td>270</td>
<td>477</td>
<td>607</td>
<td>794</td>
<td>1042</td>
<td>1166</td>
</tr>
<tr>
<td>$\zeta_c = (1 - \sqrt{2/\alpha})^2$</td>
<td>[-]</td>
<td>0.86</td>
<td>0.74</td>
<td>0.67</td>
<td>0.55</td>
<td>0.40</td>
<td>0.33</td>
</tr>
<tr>
<td>$\hat{t} = 1/(\alpha^2(1 - \zeta_c))$</td>
<td>[-]</td>
<td>0.02</td>
<td>0.04</td>
<td>0.05</td>
<td>0.07</td>
<td>0.11</td>
<td>0.13</td>
</tr>
<tr>
<td>$\delta_1$ (see (4.122))</td>
<td>[-]</td>
<td>1.05</td>
<td>1.10</td>
<td>1.13</td>
<td>1.20</td>
<td>1.29</td>
<td>1.35</td>
</tr>
<tr>
<td>$\hat{v} = v/(\alpha R_a^2)$</td>
<td>[$10^{-1}$]</td>
<td>0.02</td>
<td>0.09</td>
<td>0.17</td>
<td>0.33</td>
<td>0.66</td>
<td>0.89</td>
</tr>
<tr>
<td>$\bar{\theta} = \sqrt{\frac{1}{4}(1 + \zeta_c)}$</td>
<td>[-]</td>
<td>0.96</td>
<td>0.93</td>
<td>0.91</td>
<td>0.88</td>
<td>0.84</td>
<td>0.82</td>
</tr>
</tbody>
</table>

Table 6.2: Table of constants for a set of different arteries. The labels (A)-(H) correspond to the labels in Figure 6.2. The values for $L_v$, $R_a$, $h$ and $E$ are taken from [4] and the values for $Q_0$ are estimated from [1], page 95, Figure 6.5. For the values of $L_v$, the length of the vessel, we took the sum of the values in [4] per artery and for the values of $R_a$ and $h$ we took the arithmetic mean of the values in [4] per artery. The values for the material parameters of the fluid are $\rho = 1.05 \cdot 10^3$kg/$m^3$ and $\nu = 3.5 \cdot 10^{-6}$m$^2$/s, while the radial frequency $\omega = 2\pi$. Finally, we note that we have used the value 0.5 for the Poisson ratio of the wall, assuming that the material of the wall is incompressible.
**OVERVIEW SOLUTIONS**

**GLOBAL EQUATIONS OF MOTION**

**Constant wave velocity**

(4.25): \( q_z(z, t) = q_i(t - z) \ H(t - z) \)

(4.25): \( p_z(z, t) = q_i(t - z) \ H(t - z) \)

**Variable wave velocity**

(4.39): \( q_z(z, t) = q_i(T_0(z, t)) \ H(t - z) \)

(4.40): \( p_z(z, t) = \left\{ \frac{1}{\pi} \left[ 1 + \frac{1}{2} q_i(T_0(z, t)) \right] \right\}^2 H(t - z) \)

with \( t_0 \) solution of \( t - \left( \frac{1}{1 + \frac{1}{2} q_i(t_0)} \right) \right\) = \( t_0, (4.38) \)

**Radial diffusion**

(4.88): \( q_d(z, t) = \left[ e^{-2i \tau \ s} q_i(t - z) + 2i \tau \int_0^{t - z} \ H(t - \cdots) \right] e^{-2i \tau \ s} q_i(s) \ ds \]

(4.89): \( p_d(z, t) = \int_0^{t - z} \ H(t - \cdots) \left( \frac{4 \tau \ s H(t - \cdots)}{z} \right) e^{-2i \tau \ s} \ ds \]

where \( \tau_w = \tau q_d(z, t) + \frac{1}{2} \frac{\partial p_d}{\partial z}, (4.58) \)

**Advection**

(4.119): \( q_a(z, t) = q_i(T_0(z, t)) \ H(t - z) \)

(4.120): \( p_a(z, t) = \left( q_i(T_0(z, t)) - \frac{1}{2} \delta(t - \frac{1}{2} \gamma a) q_1^2(T_0(z, t)) \right) \ H(t - z) \)

where \( \gamma a(z, t) = \gamma a q_1^2(z, t), (4.96) \)

**LOCAL EQUATIONS OF MOTION**

(5.119): \( w(r, z, t) = 2 \left( 1 - r^2 \right) \left[ e^{-\alpha z q_i(t - z)} + \alpha z \mathcal{H}(z, t) \right] - \sum_{k=1}^{\infty} \mathcal{J}_k(r) \left[ e^{-\alpha z \Phi_k(t - z)} + \alpha z \mathcal{H}_k(z, t) \right] \)

(5.120): \( q(z, t) = \left( e^{-\alpha z q_i(t - z)} + \alpha z \mathcal{H}(z, t) \right) \]

(5.122): \( p(z, t) = \int_0^t \ H(t - \cdots) \left( \hat{q}_i(t - x) + 2q_1 q_i(t - x) \right) e^{-\alpha z} \ dx \ H(t - z) \)

(5.121): \( \tau_w(z, t) = 8 \hat{\nu} \left( e^{-\alpha z q_i(t - z)} + \alpha z \mathcal{H}(z, t) \right) + 4 \hat{\nu} \sum_{k=1}^{\infty} \left( e^{-\alpha z \Phi_k(t - z)} + \alpha z \mathcal{H}_k(z, t) \right) \)

(5.124): \( \gamma a(z, t) = \frac{4}{\hat{\nu}} q_2(z, t) + \sum_{k=1}^{\infty} \frac{1}{\alpha k} \left( e^{-\alpha z \Phi_k(t - z)} + \alpha z \mathcal{H}_k(z, t) \right)^2 \)

\( q(z, t) \sum_{k=1}^{\infty} \frac{1}{\alpha k} \left( e^{-\alpha z \Phi_k(t - z)} + \alpha z \mathcal{H}_k(z, t) \right)^2 \)

where \( \mathcal{H}(z, t), \mathcal{H}_k(z, t), \Phi_k(t) \), and \( \mathcal{J}_k(r) \), \( \mathcal{H}_k(z) \), as defined in Chapter [5].

Table 6.3: An overview of all solutions obtained in Chapter [4] and Chapter [5]. Notice the subscripts we added for the solutions of the global equations of motion in order to mark the differences in the plots of this chapter.
Figure 6.3: Column (a) Plots of axial velocity $w(\hat{r}, \hat{z}, \hat{t})$ for vessels A, B and D as a function of $\hat{r}$ for $\hat{t} = \pi$ and $\hat{z} = \{0, 0.1, 0.25, 0.5, 0.75\} \pi$. Column (b) Plots of $\Pi(\hat{\xi}, \hat{t}) = -\hat{\partial}p/\hat{\partial}z$ and the error, defined in (5.134), for vessels A, B and D as a function of $\hat{t}$ for $\hat{z} = 0$ (solid line) and $\hat{z} = \pi$ (dashed line). All quantities and variables are dimensionless (in contrast with all other plots)!
Figure 6.4: Plots of $q_c(\hat{z}, \hat{t})$, $q_v(\hat{z}, \hat{t})$, $q_d(\hat{z}, \hat{t})$, $q_a(\hat{z}, \hat{t})$ and $q_l(\hat{z}, \hat{t})$ for vessels A, B and D and for column (a) as a function of $\hat{t}$ for a set $\hat{z} = 0$ (solid line) and $\hat{z} = \pi$ (dashed line); and for column (b) as a function of $\hat{z}$ for a set $\hat{t} = \pi$ (solid line) and $\hat{t} = 2\pi$ (dashed line). The plotted quantities are dimensional, based on the quantities of Table 6.2 whereas the variables $\hat{t}$ and $\hat{z}$ are dimensionless.
Vessel A ($\delta = 0.12$)

Vessel B ($\delta = 0.24$)

Vessel D ($\delta = 0.04$)

Figure 6.5: Plots of $p_c(\hat{z}, \hat{t})$, $p_d(\hat{z}, \hat{t})$, $p_d(\hat{z}, \hat{t})$, $p_d(\hat{z}, \hat{t})$ and $p_l(\hat{z}, \hat{t})$ for vessels A, B and D and for column (a) as a function of $\hat{t}$ for a set $\hat{z} = 0$ (solid line) and $\hat{z} = \pi$ (dashed line); and for column (b) as a function of $\hat{z}$ for a set $\hat{t} = \pi$ (solid line) and $\hat{t} = 2\pi$ (dashed line). The plotted quantities are dimensional, based on the quantities of Table 6.2 whereas the variables $\hat{t}$ and $\hat{z}$ are dimensionless.

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Figure 6.6: Plots of $\tau_w(\hat{z}, \hat{t})$ in the global case and the local case for vessels A, B and D and for column (a) as a function of $\hat{t}$ for a set $\hat{z} = 0$ (solid line) and $\hat{z} = \pi$ (dashed line); and for column (b) as a function of $\hat{z}$ for a set $\hat{t} = \pi$ (solid line) and $\hat{t} = 2\pi$ (dashed line). The plotted quantities are dimensional, whereas the variables $\hat{t}$ and $\hat{z}$ are dimensionless.
Figure 6.7: Plots of $\gamma_a(\hat{z}, \hat{t})$ in the global case and the local case for vessels A, B and D and for column (a) as a function of $\hat{t}$ for a set $\hat{z} = 0$ (solid line) and $\hat{z} = \pi$ (dashed line); and for column (b) as a function of $\hat{z}$ for a set $\hat{t} = \pi$ (solid line) and $\hat{t} = 2\pi$ (dashed line). The plotted quantities are dimensional, whereas the variables $\hat{t}$ and $\hat{z}$ are dimensionless.
6.3 Preliminary conclusions from this chapter

In this chapter we presented the results from the solutions from Chapter 4 and Chapter 5, i.e., a set of plots of the flow rate $q$, the pressure distribution $p$, the shear force at the wall $\tau_w$ and the advection distribution $\gamma_a$. We distinguished between the results from the global equations of motion and the local equations of motion. Next to these plots, we generated a table in which the average percentages of the differences from the solutions of the global equations of motion with respect to the zeroth order approximative solutions from the local equations of motion are listed. Also, a list of parameters for a representative set of arteries and based on real measurements is included and is used to generate the results.

We performed a detailed analysis and discussion of the results we gathered. For each of the flow characteristics described above, we performed a qualitative and quantitative investigation in order to compare the solutions from the global equations of motion with the local ones. At the end, as an important subject of discussion, we also compared our results with the results obtained by Bessems in [1], since we based most of this research on this work. As a special remark, we note here that all our scaled variables, and also their derivatives as far as we could check this, remained of $O(1)$; this confirms the correctness of the scaling we introduced in Chapter 3.

In the next, and final chapter we come to the conclusions of this report. We shortly summarize the steps taken to come to the results we obtained in this chapter and try to collect all important issues from the discussion in this chapter, to come to the final conclusion and give answers to the questions posed in the introduction in Chapter 1. Finally, we will do some recommendations for further investigation.
Chapter 7

Conclusion and recommendation

In Chapter 6 we presented numerical results from the expressions derived in the foregoing two chapters, we also discussed these results in full detail and we drew some temporary conclusions. In this final chapter we combine the results from Chapter 4 and 5 with the results and conclusions from the previous chapter and we finally conclude whether we can or can not meet the goals and questions we posed in the problem description in Section 1.2. To this end, we start with summarizing the intermediate steps we followed in this report, followed by an enumeration of the most important results and conclusions. And the end of this chapter we finish with a few suggestions for further investigation of the problem.

7.1 Recapitulation

In this project we modeled the flow of blood through a semi-infinite deformable vessel, with special interest for the order of magnitude of the radial diffusion term and the nonlinear advection term. We assumed the blood to be an incompressible Newtonian fluid and we modeled the vessel as a linearly elastic, thin-walled vessel. From the continuity equation and the Navier-Stokes equations, we derived, based on a dimensional analysis, a set of three-dimensional time-dependent local equations of motion. The global, one-dimensional, equations of motion are obtained by integrating the local ones over the cross-sectional area.

In solving the global equations of motion we distinguished three different cases. First we neglected both the diffusion and advection term in the set of equations. We solved the resulting system using the method of characteristics, where we first considered the case of a constant wave velocity with corresponding parallel characteristics and secondly the case of a variable wave velocity with intersecting characteristics. For the second case, we incorporated the radial diffusion term but neglected the advection term in the equations of motion. This system is solved using Laplace transforms with respect to the time variable \( t \). Finally, we solved the system in which we neglected the radial diffusion term but now incorporated the advection term. As for the first case, we here also used the method of characteristics. However, in this case we had to neglect higher order terms of the dimensionless compliance parameter, i.e. terms of \( \mathcal{O}(\delta^2) \), in order to find an analytical solution up to leading order.

The local equations are solved up to the zeroth order of \( \delta \), by assuming asymptotic expansions for the axial velocity, the pressure distribution and the flow rate in positive powers of \( \delta \). First we solved, as an intermediate step, the zeroth order system for the axial velocity in case of a rigid
vessel, i.e. a vessel with constant radius. Secondly, we solved the zeroth order system for the general case, using the solution for the rigid vessel. In both cases we used Laplace transforms as an important part of the solution tactic. The obtained solution for the axial velocity in the general case contained an inverse Laplace transform, which was difficult to determine analytically. This is why we introduced an approximation for the inverse Laplace transform, resulting in an approximative zeroth order solution for the axial velocity. From this solution and the definitions for the flow rate and the shear stress at the wall, we were able to find a zeroth order approximation for the flow rate and the radial diffusion term, and, by comparison with the results for the global equations of motion, an expression for the pressure distribution. Even for the advection term we succeeded to find an approximative solution, however not as a part of the zeroth order system, but as a part of the first order system.

Finally, we chose an expression for the initial flow rate. From this choice, we plotted numerical results obtained for the flow characteristics for both the global and the local equations of motion and compared the solutions. We added a table in which we listed the average percentage maximal difference between the solutions from the global equations of motion and zeroth order approximative ones from the local equations of motion. We did this in order to get inside into the relevances of the effects described. Based on the plots and the table, we performed an extensive analysis and discussion on the effects observed.

7.2 Main achievements

In this project we wanted to investigate the order of relevance of the radial diffusion and the advection term in the equations of motion. Based on the results of Chapter 6 and the discussion performed in this same chapter, we come to the final conclusions in this section by summarizing the main achievements obtained.

We saw that the approximation for the inverse Laplace transform, as introduced in Chapter 5, is justified since the resulting error caused by the approximation is of negligible order. Based on this approximation, solutions for the flow characteristics were derived from the local equations of motion and compared with the solution for the flow characteristics from global ones. Since the dimensionless quantities and their derivatives are of $O(1)$, as we observed from the plots in Chapter 6, we conclude that the scaling parameters, as introduced in Section 4.1, are correct. The solutions for the flow rate $q$ and the pressure distribution $p$ without radial diffusion and advection do not differ significantly from the approximative zeroth order solution for the larger values of the dimensionless compliance parameter $\delta$. For the small values of $\delta$, the inclusion of the radial diffusion term becomes more important due to a growing contribution of the viscous resistance. The neglect of the higher order terms of $\delta$ in the case where radial diffusion is excluded and nonlinear advection is included, showed large variations in the solutions, mainly differences in phase and amplitude, due to the neglect of higher order terms of $\delta$. Therefore, we conclude that our solutions for $q$ and $p$ in this specific case are unreliable. In general, only quantitative differences are observed and no qualitative ones; in all cases we observe a traveling wave solution with decreasing amplitude. One other important observation was the presence of a discontinuous behavior at the wave front $z = t$, caused by the absence of the axial diffusion term as a result of the dimensional analysis. Finally, we noticed a difference of a factor ten with respect to reference [10], Bessems et al., regarding the maximal value of the pressure distribution. This last difference needs further investigation.

The plots for the shear stress $\tau_w$ at the wall for both the global an local equations of motion showed great resemblance in both shape and amplitude for all three representative values of $\delta$. In all cases we observed two remarkable effects. First of all we saw a discontinuity at the wave front for the global solution, due to the discontinuity of the spatial derivative of the pressure
distribution at the wave front. Secondly, we observed a change of sign for $\tau_w$ after some time $t$ in both the global and local solutions. For the two largest values of $\delta$, our results coincide with the results found in literature. For the smallest value of $\delta$, there is a difference of a factor two which, according to our perspectives, is not too disastrous. As for the solutions for the flow rate and the pressure distribution in the case of inclusion of nonlinear advection, we also observe the described unreliability in the plots for $\gamma_a$ in the global solutions. Therefore, it is hard to compare the solutions for the global equations of motion with the local ones. However, we were able to compare the local solution for $\gamma_a$ with the local solution for $\tau_w$. For the two larger values of $\delta$ both quantities are of the same order, whereas for the small value of $\delta$, $\gamma_a$ is a factor ten smaller then $\tau_w$.

Summarizing, we conclude that the solutions obtained for the flow rate $q$, the pressure distribution $p$ and the wall shear stress $\tau_w$ are accurate within an acceptable range. Especially for small values of the dimensionless compliance parameter $\delta$, i.e. $\delta < 0.1$, the wall shear stress, as based on the definition of Bessems in [1], plays an important role. For the nonlinear advection term, although we strongly expect that the influence is minimal in all cases for $\delta < 0.1$, it is not possible to definitely draw this conclusion from the results obtained in this project.

7.3 Recommendations

We may state that the physical behavior of blood flow through a vessel has been widely studied in all kinds of vessels by means of numerical solution methods. However, we choose to approach the problem in an, as far as possible, analytical way. In this context there are still some aspects which need some more detailed investigation.

As already discussed intensively in Chapter 6, the results obtained for the inclusion of the advection distribution are far from accurate, and thus incomplete. Here, it might for example be useful to incorporate higher order terms in $\delta$ (at least all terms up to $\mathcal{O}(\delta^2)$). Another option would be, although perhaps not completely physically relevant, to investigate what the effect will be of leaving out the compliance contribution, i.e. the $\delta p$ term, in the global equations of motion. Here, still some more work has to be done in order to compare it with the results of Chapter 5.

Secondly, we observed that there is a difference of a factor ten in the results for the pressure distribution as derived by Bessems in [11] and ours. We suggest that it is possibly worthwhile to drop the assumption of a semi-infinite vessel and replace it by a finite vessel. The effect will be a reflection at the end of the vessel, which will increase the maximum pressure.

Except from the lack of results for the advection contribution and the suggestion to investigate the effect of a finite vessel instead of a semi-infinite vessel, there are some more interesting ways to extend the model and investigate, again as far as possible in an analytical sense, the effect of the extension on the flow characteristics. The possibilities we could think of are for example: (i) a more detailed investigation of the effects of the boundary layer by incorporating the axial diffusion term $\partial^2 p / \partial z^2$; (ii) add extra source terms to the global equations of motion which describe for example phenomena like leakage, aneurysm or branching; (iii) use non-linear elasticity theory to describe the elastic behavior of the vessel wall, the latter is specifically relevant for $\delta > 0.1$. Remark that this would change the constitutive equation for describing the relation between the cross-sectional area and the (derivatives of the) pressure distribution.
Mathematical modeling of blood flow through a deformable thin-walled vessel
Bibliography


Appendix A

Linear elasticity theory for a thin-walled vessel

Consider a linearly elastic thin-walled vessel under internal pressure \( p = p(z, t) \). Due to changes in this pressure the cross-sectional area of the vessel changes from \( A_0 \) at \( p = p_0 \) to \( A = A(z, t) \). The area \( A \) is assumed to be an affine function of the pressure change \( p - p_0 \), i.e.

\[
A(z, t) = A_0 + C(p(z, t) - p_0),
\]

(A.1)

This appendix provides a derivation of this formula. Also we come up with an expression for the compliance \( C \) in terms of the physical parameters of the vessel wall. Moreover, we will derive a non-local equation for the radial displacement of the vessel wall, commonly referred to as the shell equation.

A.1 Basic equations

The deformations \( \varepsilon_{ij} \) and the stresses \( \sigma_{ij} \) of the vessel wall are related according to Hooke’s law (and its inverse)

\[
\sigma_{ij} = \frac{E}{1 + \nu} \left( \varepsilon_{ij} + \frac{\nu}{1 - 2\nu} \delta_{ij} \varepsilon_{kk} \right),
\]

(A.2)

\[
\varepsilon_{ij} = \frac{1 + \nu}{E} \left( \sigma_{ij} - \frac{\nu}{1 + \nu} \delta_{ij} \sigma_{kk} \right),
\]

(A.3)

for given Young’s modulus \( E \) and Poisson ratio \( \nu \). Let \( R_d \) be the undeformed mean radius of the vessel and \( R(z, t) = R_d + u_r \) the deformed one, where \( u_r = u(r, z, t) \) is the displacement of the (thin) wall. The axial displacement is \( u_z = w(r, z, t) \) and the tangential displacement \( u_\theta = 0 \) (rotational symmetry). The deformations are given by

\[
\varepsilon_{rr} = \frac{\partial u}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u}{r},
\]

\[
\varepsilon_{zz} = \frac{\partial w}{\partial z}, \quad \varepsilon_{rz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right).
\]

(A.4)
Hooke’s law, written out in components, reads

\[
\begin{align*}
\sigma_{rr} &= \frac{E}{(1-2\nu)(1+\nu)}((1-\nu)\varepsilon_{rr} + \nu(\varepsilon_{\theta\theta} + \varepsilon_{zz})), \\
\sigma_{\theta\theta} &= \frac{E}{(1-2\nu)(1+\nu)}((1-\nu)\varepsilon_{\theta\theta} + \nu(\varepsilon_{rr} + \varepsilon_{zz})), \\
\sigma_{zz} &= \frac{E}{(1-2\nu)(1+\nu)}((1-\nu)\varepsilon_{zz} + \nu(\varepsilon_{rr} + \varepsilon_{\theta\theta})), \\
\sigma_{r\theta} &= \frac{E}{(1+\nu)}\varepsilon_{r\theta}.
\end{align*}
\]

The equations of equilibrium are

\[
\begin{align*}
\frac{1}{r} \frac{\partial}{\partial r}(r \sigma_{rr}) + \frac{\partial \sigma_{r\theta}}{\partial z} - \frac{1}{r} \sigma_{\theta\theta} &= 0, \\
\frac{1}{r} \frac{\partial}{\partial r}(r \sigma_{r\theta}) + \frac{\partial \sigma_{zz}}{\partial z} &= 0.
\end{align*}
\]

The boundary conditions at the inner side: \( r = R^- = R_a - h/2 \) and the outer side: \( r = R^+ = R_a + h/2 \) of the vessel wall of thickness \( h \), with \( h/R_a \ll 1 \), are given by

\[
\begin{align*}
r &= R^- : & \sigma_{rr} &= -p(z,t), \quad \sigma_{r\theta} = 0, \\
r &= R^+ : & \sigma_{rr} &= -p_0, \quad \sigma_{r\theta} = 0.
\end{align*}
\]

The vessel can be either in a state of plane strain (\( \varepsilon_{zz} = 0 \)) or in a state of plane stress (\( \sigma_{zz} = 0 \)). Since arteries are commonly prestretched with a fixed axial stretch, we opt here for a state of plane strain.

### A.2 Derivation of the compliance \( C \)

In relation (A.1), the change of the area \( A \) in the point \( z \) is directly related to the change in the pressure in the point same \( z \). This implies that the reactions of the neighbouring points are not accounted for; this is why we call this a local relation. In other words, the vessel is either considered as a free ring or a tube under uniform pressure. The latter approximation is allowed if the changes in \( p \) in axial direction have a typical length scale much larger than the thickness \( h \) (or better, as we will see further on, than \( \sqrt{R_a h} \)). This is, in general, true for our problems. Under these conditions, Laplace’s formula holds, expressing the radial equilibrium of a circular segment of the wall (see Figure A.1), stating that

\[
pr_a d\theta - 2\sigma_{\theta\theta} \sin \left( \frac{d\theta}{2} \right) h = 0,
\]

or, for \( d\theta \to 0 \),

\[
\sigma_{\theta\theta} = \frac{R_a}{h} (p - p_0).
\]

Since \( |\sigma_{rr}| \sim (p - p_0) \) and \( R_a/h \gg 1 \), we see that \( |\sigma_{rr}| \ll |\sigma_{\theta\theta}| \), and therefore we neglect \( \sigma_{rr} \) in the sequel. With the deformed radius written as \( R = R_a + u \), and with the use of \( u/R_a \ll 1 \), we obtain for the deformed area \( A \)

\[
A = \pi R^2 = \pi (R_a + u)^2 = \pi R_a^2 + 2\pi R_a u + \pi u^2 \approx A_0 + 2\pi R_a u.
\]
From the inverse Hooke’s law (A.3), with $\sigma_{rr} \approx 0$, we obtain for the state of plane strain,

$$\varepsilon_{zz} = \frac{1}{E} (\sigma_{zz} - \nu \sigma_{\theta\theta}) = 0,$$

(A.16)

that $\sigma_{zz} = \nu \sigma_{\theta\theta}$. Furthermore, we know that $u = R_a \varepsilon_{\theta\theta}$. These two results are substituted into equation (A.15). This leads to

$$A \approx A_0 + 2\pi R_a u = A_0 + 2\pi R_a^2 \varepsilon_{\theta\theta} = A_0 + 2\pi R_a^2 \frac{1}{E} (\sigma_{\theta\theta} - \nu \sigma_{zz})$$

$$= A_0 + 2\pi R_a^2 \frac{1}{E} (\sigma_{\theta\theta} - \nu^2 \sigma_{\theta\theta}) = A_0 + \frac{2\pi R_a^2 (1 - \nu^2)}{E} \sigma_{\theta\theta}$$

$$= A_0 + \frac{2\pi (1 - \nu^2) R_a^3}{E h} (p - p_0),$$

(A.17)

with use of Laplace’s formula (A.14). This yields for the compliance $C$ in (A.1)

$$C = \frac{2\pi (1 - \nu^2) R_a^3}{E h}.$$

(A.18)

Assuming, as is common practice, that the elastic material of the artery wall is incompressible, we take $\nu = 0.5$, to obtain

$$C = \frac{3\pi R_a^3}{2E h}.$$

(A.19)

### A.3 The shell equation

In this section we will derive a non-local equation which relates the radial displacement $u$ in the point $z$ to the pressure $p$ in all neighboring points of $z$, thus accounting for the stiffness of the near part of the wall. In contrast to the local relation (A.1), this results in a continuous radial displacement $u$ even if the pressure is discontinuous. To this end we start with calculating the
global forces and moments. These global forces and moments can be obtained by integrating the local equations of equilibrium over the wall of the vessel. The result will be a set of equations that only depends on the spatial coordinate $z$ and the time $t$.

1. Integration of (A.9) yields

$$
\int_{R^-}^{R^+} \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r \sigma_{rr}) + \frac{\partial \sigma_{rz}}{\partial z} - \frac{1}{r} \sigma_{\theta\theta} \right\} r \, dr = r \sigma_{rr} \bigg|_{R^-}^{R^+} + \frac{\partial}{\partial z} \int_{R^-}^{R^+} r \sigma_{rz} \, dr - \int_{R^-}^{R^+} \sigma_{\theta\theta} \, dr = -R^+ p_0 + R^- p(z,t) + R_a \frac{\partial Q_z}{\partial z} - N_\theta = 0,
$$

or

$$
\frac{\partial Q_z}{\partial z} - \frac{1}{R_a} N_\theta = -\bar{p}(z,t), \quad \text{(A.20)}
$$

where

$$
\bar{p}(z,t) = \frac{R^-}{R_a} p(z,t) - \frac{R^+}{R_a} p_0, \quad \text{(A.21)}
$$

$$
Q_z(z,t) = \frac{1}{R_a} \int_{R^-}^{R^+} r \sigma_{rz}(r,z,t) \, dr, \quad \text{(A.22)}
$$

$$
N_\theta(z,t) = \int_{R^-}^{R^+} \sigma_{\theta\theta}(r,z,t) \, dr. \quad \text{(A.23)}
$$

Here, $Q_z$ is the shear force and $N_\theta$ the normal force in azimuthal direction per unit of length ("hoop force") (N/m).

2. Integration of (A.10) gives

$$
\int_{R^-}^{R^+} \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r \sigma_{rz}) + \frac{\partial \sigma_{zz}}{\partial z} \right\} r \, dr = r \sigma_{rz} \bigg|_{R^-}^{R^+} + \frac{\partial}{\partial z} \int_{R^-}^{R^+} r \sigma_{zz} \, dr = R_a \frac{\partial N_z}{\partial z} = 0,
$$

or

$$
\frac{\partial N_z}{\partial z} = 0, \quad \text{(A.24)}
$$

where

$$
N_z(z,t) = \frac{1}{R_a} \int_{R^-}^{R^+} r \sigma_{zz}(r,z,t) \, dr. \quad \text{(A.25)}
$$

Here, $N_z$ is the normal force in axial direction per unit of length (N/m).

3. Integration of (A.10) multiplied by $(r - R_a)$ yields

$$
\int_{R^-}^{R^+} (r - R_a) \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r \sigma_{rz}) + \frac{\partial \sigma_{zz}}{\partial z} \right\} r \, dr = \int_{R^-}^{R^+} \left\{ \frac{\partial}{\partial r} ((r - R_a) r \sigma_{rz}) - r \sigma_{rz} + r(r - R_a) \frac{\partial \sigma_{zz}}{\partial z} \right\} \, dr =
$$

$$
(r - R_a) r \sigma_{rz} \bigg|_{R^-}^{R^+} + \frac{\partial}{\partial z} \int_{R^-}^{R^+} r(r - R_a) \sigma_{zz} \, dr - \int_{R^-}^{R^+} r \sigma_{rz} \, dr = R_a \frac{\partial M_z}{\partial z} - R_a Q_z = 0,
$$

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or

$$\frac{\partial M_z}{\partial z} - Q_z = 0,$$  \hspace{1cm} (A.26)

where

$$M_z(z, t) = \frac{1}{R_a} \int_{R_a}^{R^+} r(r - R_a) \sigma_{zz}(r, z, t) \, dr.$$  \hspace{1cm} (A.27)

Here, $M_z$ is the bending moment per unit of length (N/m).

Note that these moments and forces are exact integrals of the local equations of equilibrium. No approximation is made with respect to $h/R_a$. So, these global equations of equilibrium also hold for thick-walled vessels.

Eliminating $Q_z$ from (A.26) in favor of $N_0$ by means of (A.20), we arrive at

$$\frac{\partial^2 M_z}{\partial z^2} - \frac{1}{R_a} N_0 = -\rho(z, t)$$  \hspace{1cm} (A.28)

In order to come to the shell equation, we assume asymptotic expansions for the displacements $u(r, z, t)$ and $w(r, z, t)$ in terms of positive powers of the small parameter $h/R_a$. Since we know that $(r - R_a)/R_a = \mathcal{O}(h/R_a)$, we arrive at

$$u(r, z, t) = U_0(z, t) + \frac{r - R_a}{R_a} U_1(z, t) + \ldots,$$  \hspace{1cm} (A.29)

$$w(r, z, t) = W_0(z, t) + \frac{r - R_a}{R_a} W_1(z, t) + \ldots,$$  \hspace{1cm} (A.30)

where the ... stand for terms which are small in second order of $h/R_a$ with respect to the leading order terms. As said before, in section A.1, we have opted here for a state of plane strain. However, since the bending of the wall under a $z$-dependent pressure $p(z, t)$ is a problem that essentially depends on $z$, it is in principle not right to call this a plane strain problem. What we mean here is that due to the prescribed and fixed axial pre-stretch, there is no extra axial stretch in the mean over the wall. Hence, although there can, and will, be an axial displacement of the wall (due to the bending of the wall), but averaged over the thickness of the wall the resulting mean displacement will be zero. The direct consequence of this assumption is that in (A.30) the leading term $W_0(z, t)$ must be taken zero.

We can now express the deformations in (A.4) in terms of $U_i$ and $W_j$ ($i = 0, 1$) by substitution of the asymptotic expansions. This leads to (neglecting the $\mathcal{O}((h/R_a)^2)$ terms)

$$\varepsilon_{\theta \theta} = \frac{u}{r} = \frac{u}{R_a} \frac{1}{1 + (r - R_a)/R_a} = \frac{u}{R_a} \left(1 - \frac{r - R_a}{R_a}\right),$$

$$= \frac{1}{R_a} \left(U_0 + \frac{r - R_a}{R_a} U_1\right) \left(1 - \frac{r - R_a}{R_a}\right) = \frac{U_0}{R_a} + \frac{1}{R_a} \frac{r - R_a}{R_a} (U_1 - U_0),$$  \hspace{1cm} (A.31)

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = \frac{r - R_a}{R_a} \frac{\partial W_1}{\partial z},$$  \hspace{1cm} (A.32)

$$\varepsilon_{r z} = \frac{1}{2} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = \frac{1}{2} \left(\frac{\partial U_0}{\partial z} + \frac{W_1}{R_a}\right) + \frac{1}{2} \frac{r - R_a}{R_a} \left(\frac{\partial U_1}{\partial z} + \frac{2 W_2}{R_a}\right).$$  \hspace{1cm} (A.33)

In accordance with the Kirchhoff hypothesis, which states that normal cross-sections remain normal after bending, there is no shear in the $rz$-plane. This means that $\varepsilon_{r z} = 0$ and so

$$2\varepsilon_{r z} = \frac{\partial U_0}{\partial z} + \frac{W_1}{R_a} + \mathcal{O}(d) = 0,$$  \hspace{1cm} (A.34)
which leads to

\[ W_1(z, t) = -R_a \frac{\partial U_0}{\partial z}(z, t). \]  

(A.35)

As in the previous section (where it was shown that \(|\sigma_{rr}| = \mathcal{O}(h/2R_a)|\sigma_{\theta\theta}|,\) on basis of Laplace’s formula) we choose \(\sigma_{rr} = 0\), resulting in

\[ \sigma_{rr} = (1 - \nu)\varepsilon_{rr} + \nu(\varepsilon_{\theta\theta} + \varepsilon_{zz}) = (1 - \nu) \frac{\partial u}{\partial r} + \nu \frac{r}{R_a} \frac{\partial w}{\partial z} \]

\[ = (1 - \nu) \frac{U_1}{R_a} + \nu \frac{U_0}{R_a} = 0, \]  

(A.36)

and so

\[ U_1(z, t) = -\frac{\nu}{1 - \nu} U_0(z, t). \]  

(A.37)

Moreover, since \(\sigma_{rr} = 0\), we also find by substitution that (A.6) and (A.7) simplify to

\[ \sigma_{\theta\theta} = \frac{E}{1 - \nu^2} (\varepsilon_{\theta\theta} + \nu \varepsilon_{zz}), \]  

(A.38)

\[ \sigma_{zz} = \frac{E}{1 - \nu^2} (\nu \varepsilon_{\theta\theta} + \varepsilon_{zz}). \]  

(A.39)

Using the above precalculations we are now able to express \(\sigma_{\theta\theta}\) and \(\sigma_{zz}\) in terms of the leading-order displacement \(U_0(z, t)\) and so we arrive at

\[ \sigma_{\theta\theta} = \frac{E}{1 - \nu^2} (\varepsilon_{\theta\theta} + \nu \varepsilon_{zz}) \]

\[ = \frac{E}{1 - \nu^2} \left( \frac{U_0}{R_a} + \frac{1}{R_a} \frac{r - R_a}{R_a} \left( U_1 - U_0 + \frac{r - R_a}{R_a} \frac{\partial W_1}{\partial z} \right) \right) \]

\[ = \frac{E}{1 - \nu^2} \left( \frac{U_0}{R_a} - \frac{r - R_a}{R_a} \left( \frac{1}{R_a} \frac{U_1 - U_0}{R_a} + \frac{r}{R_a} \frac{\partial W_1}{\partial z} \right) \right) \]

\[ = \frac{E}{1 - \nu^2} \left( \frac{U_0}{R_a} - \frac{r - R_a}{R_a} \left\{ \frac{1}{1 - \nu} \frac{U_0}{R_a} + \nu R_a \frac{\partial^2 U_0}{\partial z^2} \right\} \right), \]  

(A.40)

\[ \sigma_{zz} = \frac{E}{1 - \nu^2} (\nu \varepsilon_{\theta\theta} + \varepsilon_{zz}) \]

\[ = \frac{E}{1 - \nu^2} \left( \nu \left( \frac{U_0}{R_a} + \frac{1}{R_a} \frac{r - R_a}{R_a} \left( U_1 - U_0 \right) + \frac{r - R_a}{R_a} \frac{\partial W_1}{\partial z} \right) \right) \]

\[ = \frac{E}{1 - \nu^2} \left( \nu \frac{U_0}{R_a} + \frac{r - R_a}{R_a} \left( \frac{\nu}{R_a} \left( U_1 - U_0 \right) + \frac{\partial W_1}{\partial z} \right) \right) \]

\[ = \frac{E}{1 - \nu^2} \left( \nu \frac{U_0}{R_a} - \frac{r - R_a}{R_a} \left\{ \nu \frac{U_0}{1 - \nu} + R_a \frac{\partial^2 U_0}{\partial z^2} \right\} \right). \]

(A.41)

Note that the expressions for \(\sigma_{\theta\theta}\) and \(\sigma_{zz}\) are only correct up to an order of \((h/2R_a)^2\). With (A.40)-(A.41), we calculate the global forces and moments given at the beginning of this section by
simple integration. This leads to

\[ N_0(z, t) = \int_{R^-}^{R^+} \sigma_{\theta \theta} \, dr = \int_{R^-}^{R^+} \frac{1}{1 - \nu} \frac{U_0}{R_a} \frac{\partial^2 U_0}{\partial z^2} \, dr \]

\[ = \frac{E h}{1 - \nu^2} \frac{U_0(z, t)}{R_a}, \]

(A.42)

\[ M_z(z, t) = \frac{1}{R_a} \int_{R^-}^{R^+} r(r - R_a) \sigma_{zz} \, dr \]

\[ = - \frac{E h^3}{12 (1 - \nu^2)} \left( \frac{v^2}{1 - v} \frac{U_0}{R_a^2} + \frac{\partial^2 U_0}{\partial z^2} (z, t) \right). \]

(A.43)

We note here that the term \( \frac{U_0}{R_a^2} \) is small (i.e. \( \Theta(h/R_a) \)) compared to \( \frac{\partial^2 U_0}{\partial z^2} \). To prove this, we need a result that will be derived a few lines further on, namely that the characteristic axial length for the radial displacements due to bending is \( \sqrt{h R_a} \). This implies that

\[ \frac{\partial^2 U_0}{\partial z^2} \sim \frac{U_0}{h R_a} = \frac{R_a U_0}{h R_a^2} \gg \frac{U_0}{R_a^2}, \]

(A.44)

and thus we may approximate (A.43) as

\[ M_z(z, t) = - D \frac{\partial^2 U_0}{\partial z^2} (z, t), \]

(A.45)

with

\[ D = \frac{E h^3}{12 (1 - \nu^2)}, \]

(A.46)

the plate constant. Substituting (A.45) and (A.42) into (A.28), we arrive at

\[ - D \frac{\partial^4 U_0}{\partial z^4} (z, t) - \frac{E h}{1 - \nu^2} \frac{U_0(z, t)}{R_a^2} = - \tilde{p}(z, t) \]

(A.47)

Introducing the parameter \( \beta^4 = 3/(R_a h)^2 \) we obtain the final version of the shell equation as

\[ \frac{\partial^4 U_0}{\partial z^4} + 4 \beta^4 U_0 = \frac{1}{D} \tilde{p}(z, t). \]

(A.48)

This shows that the characteristic length in \( z \)-direction is directly related to \( 1/\beta \sim \sqrt{R_a h} \).
Mathematical modeling of blood flow through a deformable thin-walled vessel
## Appendix B

### Inverse Laplace transformation

In Chapter 4, as well as in Chapter 5, we used the concept of Laplace transformation to find a solution of the global and local equations of motions, respectively. For a function \( f(t), t > 0 \), the Laplace transform \( F(s), s \in \mathbb{C} \), and its inverse are defined by

\[
F(s) = \mathcal{L}\{f(t); t, s\} := \int_{0}^{\infty} f(t)e^{-st} \, dt, \tag{B.1}
\]

\[
f(t) = \mathcal{L}^{-1}\{F(s); s, t\} := \lim_{T \to \infty} \frac{1}{2\pi \mathbf{i}} \int_{c-iT}^{c+iT} F(s)e^{st} \, ds. \tag{B.2}
\]

In this appendix we derive expressions for the inverse Laplace transform in two cases from the above mentioned chapters, where we apply contour-integration to come to a result.

#### B.1 Inverse Laplace transform (1)

In this section we calculate by hand the inverse Laplace transform, used in equation 4.79 in section 4.3, defined by

\[
G(z, t) = \lim_{T \to \infty} \frac{1}{2\pi \mathbf{i}} \int_{c-iT}^{c+iT} \frac{e^{-\sigma(s)z}}{\sigma(s)} e^{st} \, ds, \tag{B.3}
\]

for \( 0 < z < t \) with \( \sigma(s) = \sqrt{(\tilde{\tau} + s)s} \). We define a contour \( \mathcal{C} \) as

\[
\mathcal{C} = \bigcup_{i=0}^{9} \mathcal{C}_i, \tag{B.4}
\]

where \( \mathcal{C}_i (i = 0, 1, \ldots, 9) \) as defined in Figure B.1. For this contour holds

\[
\int_{\mathcal{C}} \frac{e^{-\sigma(s)z}}{\sigma(s)} e^{st} \, ds = 0, \tag{B.5}
\]

since there are no poles within the contour \( \mathcal{C} \). This means that

\[
G(z, t) = \lim_{T \to \infty} \frac{1}{2\pi \mathbf{i}} \int_{\mathcal{C}_0} \frac{e^{-\sigma(s)z}}{\sigma(s)} e^{st} \, ds = -\lim_{T \to \infty} \frac{1}{2\pi \mathbf{i}} \sum_{i=1}^{9} \int_{\mathcal{C}_i} \frac{e^{-\sigma(s)z}}{\sigma(s)} e^{st} \, ds \tag{B.6}
\]

We will now calculate all integrals on the righthand-side of (B.6) to determine the inverse Laplace transform. As we will see, only the integrals over \( \mathcal{C}_4 \) and \( \mathcal{C}_6 \) will contribute.
1. Along $C_1$ we know that $s = T e^{i\phi} + c$, where $T \gg 1$ and $\phi \in [\frac{\pi}{2}, \pi]$. This gives

$$\sigma(s) = T e^{i\phi} \sqrt{1 + (\tilde{\tau} + 2c) e^{-i\phi} T^{-1} + c (\tilde{\tau} + c) e^{-2i\phi} T^{-2}}, \quad (B.7)$$

such that

$$G_1(z, t) = \lim_{T \to \infty} \frac{e^{ic}}{2\pi} \int_{\frac{\pi}{2}}^\pi e^{-Te^{i\phi}} \sqrt{1 + (\tilde{\tau} + 2c) e^{-i\phi} T^{-1} + c (\tilde{\tau} + c) e^{-2i\phi} T^{-2}} e^{iT e^{i\phi} t} d\phi$$

$$= \lim_{T \to \infty} \frac{e^{ic}}{2\pi} \int_{\frac{\pi}{2}}^\pi e^{i(t-z)T} \cos(\phi) e^{i(t-z)T} \sin(\phi) d\phi = 0. \quad (B.8)$$

since $(t-z)T > 0$ and $\cos(\phi) < 0$ for $\frac{\pi}{2} \leq \phi \leq \pi$. With the same reasoning (now with $\phi \in [-\pi, -\frac{\pi}{2}]$) we see that $G_9(z, t) = 0$.

2. Along $C_2$ we see that $s \in [c - T, -\tilde{\tau}]$, which means that $\sigma(s) = \sqrt{(\tilde{\tau} + s)s} \in \mathbb{R}^+$. The same holds for $C_8$, but we remark that the integration is in opposite direction. So we find that

$$G_2(z, t) + G_8(z, t) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{-i}^{-T} e^{-\sigma(s) z} \sigma(s) e^{it} ds + \lim_{T \to \infty} \frac{1}{2\pi i} \int_{-i}^{-T} e^{-\sigma(s) z} \sigma(s) e^{it} ds = 0. \quad (B.9)$$

3. Along $C_3$ we know that $s = r e^{i\psi} - \tilde{\tau}$, where $0 < r \ll 1$ and $\psi \in [0, \pi]$. This gives

$$\sigma(s) = e^{\frac{1}{2}i\psi} \sqrt{r} e^{i\psi} - \tilde{\tau}. \quad (B.10)$$
such that

\[ G_3(z, t) = \lim_{r \to 0} \frac{e^{-it}}{2\pi r} \int_0^\pi e^{\frac{t}{r} \sqrt{r^2 + \frac{z^2}{r^2} - 1}} e^{r e^{it}} r e^{it} \, d\varphi \]

\[ = \lim_{r \to 0} \frac{e^{-it}}{2\pi} \int_0^\pi e^{\frac{t}{r} \sqrt{r^2 + \frac{z^2}{r^2} - 1}} e^{r e^{it}} e^{\frac{t}{r} \sqrt{r^2} \sqrt{r^2} \sqrt{r^2} \varphi} \, d\varphi = 0. \]  

(B.11)

Consequently also \( G_5(z, t) \) and \( G_2(z, t) \) are both equal to zero.

4. Along \( C_4 \) we have that \( s \in [-\tilde{t}, 0] \), such that \( \sigma(s) = \sqrt{(\tilde{t} + s)s} = i\sqrt{-(\tilde{t} + s)s} \in \mathbb{C} \). This gives

\[ G_4(z, t) = \lim_{r \to \infty} \frac{1}{2\pi r} \int_{-\tilde{t}}^0 e^{\frac{t}{r} \sqrt{-(\tilde{t} + s)s}} e^{r t} \, ds \]

\[ = -\frac{1}{2\pi} \int_{-\tilde{t}}^0 e^{\frac{t}{r} \sqrt{-(\tilde{t} + s)s}} e^{r t} \, ds \]  

(B.12)

5. Finally, along \( C_6 \) we have that \( s \in [-\tilde{t}, 0] \), such that \( \sigma(s) = \sqrt{(\tilde{t} + s)s} = -i\sqrt{-(\tilde{t} + s)s} \in \mathbb{C} \). This gives

\[ G_6(z, t) = \lim_{r \to \infty} \frac{1}{2\pi r} \int_{0}^{-\tilde{t}} e^{\frac{t}{r} \sqrt{-(\tilde{t} + s)s}} e^{r t} \, ds \]

\[ = -\frac{1}{2\pi} \int_{0}^{-\tilde{t}} e^{\frac{t}{r} \sqrt{-(\tilde{t} + s)s}} e^{r t} \, ds \]  

(B.13)

These integrals still seem difficult to evaluate. However, if we change the coordinate system, things get a little easier. To this end, put \( t = Z \cos(X) \) and \( z = iZ \sin(X) \) (such that \( t^2 - z^2 = Z^2 \)) and change the integration variable \( s = \frac{t}{\tilde{t}} (\sin(\varphi) - 1) \). This gives

\[ G_4(z, t) = -\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{\frac{t}{\tilde{t}} \cos(\varphi)} Z \sin(X) \, e^{\frac{t}{\tilde{t}} Z \sin(\varphi) - 1} \, d\varphi \]

\[ = -\frac{1}{2\pi} e^{-\frac{t}{\tilde{t}} Z \cos(X)} \int_{-\pi/2}^{\pi/2} e^{\frac{t}{\tilde{t}} Z \sin(\varphi + X)} \, d\varphi \]

\[ = -\frac{1}{2} e^{-\frac{t}{\tilde{t}} Z \cos(X)} I_0 \left( \frac{\tilde{t}}{2} Z \right). \]  

(B.14)

This last result can for example be obtained by Mathematica, or using the general integral representation for Bessel functions;

\[ I_n(x) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-i(x \varphi - x \sin(\varphi))} \, d\varphi, \]  

(B.15)

which for example can be found in [11]. In the same way we find that

\[ G_6(z, t) = -\frac{1}{2\pi} e^{-\frac{t}{\tilde{t}} Z \cos(X)} \int_{-\pi/2}^{\pi/2} e^{\frac{t}{\tilde{t}} Z \sin(\varphi - X)} \, d\varphi \]

\[ = -\frac{1}{2} e^{-\frac{t}{\tilde{t}} Z \cos(X)} I_0 \left( \frac{\tilde{t}}{2} Z \right) = G_4(z, t). \]  

(B.16)
NOTE: Since we have taken the contour left from $c$, the above derivation holds for $0 < z < t$. For $z > t$, the contour has to be taken right from $c$ and since there are no poles or square root singularities within this right half-side of the $s$-plane, the result would be trivially $G \equiv 0$.

So, we conclude that

$$G(z, t) = -G_4(z, t) - G_6(z, t) = e^{-\frac{\bar{t}}{2}Z \cos(X)} I_0 \left( \frac{\bar{t}}{2}Z \right)$$

B.17

which is equal to the result in Section 4.3 with $\bar{\tau} = 4 \hat{\tau}$.

### B.2 Inverse Laplace transform (2)

In this section, we derive an expression for $f(z, t)$, defined in (5.105) as

$$f(z, t) = L^{-1} \left\{ e^{-\rho(s)z} \right\} = -\frac{d}{dz} L^{-1} \left\{ \frac{e^{-\rho(s)z}}{\rho(s)} \right\} =: -\frac{d}{dz} h(z, t),$$

B.18

where

$$\rho(s) = \sqrt{\frac{I_0 \left( \sqrt{\frac{z}{\bar{r}}} \right)}{I_2 \left( \sqrt{\frac{z}{\bar{r}}} \right)}}.$$

B.19

We proceed with the evaluation of the inverse transform $h(z, t)$; notice the analogy with the previous section. The inverse Laplace transform $g(z, t)$ is defined by

$$h(z, t) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+ iT} e^{-\rho(s)z} \frac{e^{s t}}{\rho(s)} \frac{ds}{2\pi i}.$$

B.20

Remark the resemblance with the inverse Laplace transform calculated in the previous section. We will evaluate the inverse transformation (B.20) by contour integration; see Figure B.2 for the sketch of the contour. The function $\rho(s)$ has roots for $s = -\hat{\nu} \lambda_k^2$ and vertical asymptotes for $s = -\hat{\nu} \sigma_k^2$, where $\lambda_k$ are the positive roots of $J_0$ and $\sigma_k$ are the positive roots of $J_2$ ($k = 1, 2, \ldots$) As in the previous section, the part of the contour at infinity, i.e. $C_I$ and $C_{III}$, and the circles around the square root singularities $s = -\hat{\nu} \lambda_k^2$ do not contribute to $g(z, t)$.

The only contribution comes from the subsets $\delta_{\pm k}$ of $C_{\pm k}$ ($k = 0, 1, 2, \ldots$), where

$$\delta_{\pm k} = \left\{ s \mid -\hat{\nu} \lambda_{k+1}^2 < s < -\hat{\nu} \sigma_k^2 \right\},$$

B.21

with $\sigma_0 = 0$ and where the imaginary parts of $\rho(s)$ have different sign just above or below the negative real axis, say

$$\rho(s) = \mp \hat{\tau} \hat{\rho}(s) + \mathcal{O}(\epsilon),$$

B.22

for $s \in \delta_{\pm k} \pm \epsilon t$, $0 < \epsilon \ll 1$, where $\hat{\rho}(s)$ is a positive real function of $s$ for $s \in \delta_{\pm k}$ given by

$$\hat{\rho}(s) = \sqrt{-s^2 I_0 \left( \sqrt{\frac{s}{\bar{r}}} \right) I_2 \left( \sqrt{\frac{s}{\bar{r}}} \right)}.$$

B.23
Figure B.2: Definition sketch of the contour for calculating the inverse Laplace transform (B.20).
The contribution of all these parts $S_{\pm k}$ to $g(z, t)$ is

$$\lim_{T \to \infty} \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left\{ \int_{-\nu^2_{n+1}}^{-\nu^2_n} e^{i\rho(s)z} - i \frac{\rho(s)}{\dot{\rho}(s)} e^{it} ds + \int_{-\nu^2_n}^{-\nu^2_{n+1}} e^{i\rho(s)z} - i \frac{\rho(s)}{\dot{\rho}(s)} e^{it} ds \right\} =$$

$$\frac{1}{2\pi} \sum_{n=0}^{\infty} \int_{-\nu^2_{n+1}}^{-\nu^2_n} \left\{ \frac{\rho(s)}{\dot{\rho}(s)} + \frac{e^{-i\rho(s)z}}{\dot{\rho}(s)} \right\} e^{it} ds =$$

$$\frac{1}{\pi} \sum_{n=0}^{\infty} \int_{-\nu^2_{n+1}}^{-\nu^2_n} \cos \left( \frac{\rho(s)z}{\dot{\rho}(s)} \right) e^{it} ds,$$  \hspace{1cm} (B.24)

and so we obtain

$$h(z, t) = -\frac{1}{\pi} \sum_{n=0}^{\infty} \int_{-\nu^2_{n+1}}^{-\nu^2_n} \frac{\cos \left( \frac{\rho(s)z}{\dot{\rho}(s)} \right) e^{it}}{\dot{\rho}(s)} ds H(t - z)$$  \hspace{1cm} (B.25)

Again, as in the calculation of (B.3), because of the absence of poles and/or square root singularities in the right half of the $s$-plane, the solution for $z > t$ is equal to zero. Therefore, since the solution is valid for $0 < z < t$, we added $H(t - z)$ in (B.25).

Finally we see that

$$f(z, t) = -\frac{\partial}{\partial z} h(z, t) = \delta(t - z) - \frac{1}{\pi} \sum_{n=0}^{\infty} \int_{-\nu^2_{n+1}}^{-\nu^2_n} \sin \left( \frac{\rho(s)z}{\dot{\rho}(s)} \right) e^{it} ds H(t - z).$$  \hspace{1cm} (B.26)
Appendix C

Fourier-Bessel series

In Section 5.3 we derived an expression for the zeroth order axial flow velocity starting from the assumption that the axial flow velocity can be written as a Fourier-Bessel serie. In this appendix apply the concept of Fourier-Bessel series, which is strongly related to Fourier series for periodic functions, in two examples that are mentioned in the derivation of the axial flow velocity. The Fourier-Bessel series of \( f(r) \), which is defined on the interval \((0, a)\), is given by

\[
f(r) = \sum_{k=1}^{\infty} c_k J_n(\lambda_k r), \quad 0 < r < a,
\]

with Fourier-Bessel coefficients

\[
c_k = \frac{2}{a^2 J_1^2(\lambda_k a)} \int_0^a r f(r) J_n(\lambda_k r) \, dr,
\]

where \( n > -1 \) real and \( \lambda_k (k = 1, 2, 3, \ldots) \) the positive zeros of \( J_n(\lambda) = 0 \). See [11] for all details about Fourier-Bessel series and the derivation of the above definition.

C.1 Example (1)

In this first example we choose \( f(r) = 1, n = 0 \) and \( a = 1 \). Then we find for corresponding the Fourier-Bessel coefficients

\[
c_k = \frac{2}{J_1^2(\lambda_k)} \int_0^1 r J_0(\lambda_k r) \, dr = \frac{2}{\lambda_k^2 J_1^2(\lambda_k)} \int_0^{\lambda_k} z J_0(z) \, dz
\]

\[
= \frac{2}{\lambda_k^2 J_1^2(\lambda_k)} [z J_1(z)]_0^{\lambda_k} = \frac{2}{\lambda_k J_1(\lambda_k)},
\]

and so we see that for \( 0 < r < 1 \) we find

\[
1 = \sum_{k=1}^{\infty} \frac{2 J_0(\lambda_k r)}{\lambda_k J_1(\lambda_k)}.
\]
If we multiply both sides of (C.4) by \( r \) and consecutively integrate the result with respect to \( r \), we obtain

\[
\int_0^1 \sum_{k=1}^{\infty} \frac{2r J_0 (\lambda_k r)}{\lambda_k J_1 (\lambda_k)} \, dr = \sum_{k=1}^{\infty} \frac{2}{\lambda_k J_1 (\lambda_k)} \int_0^1 r J_0 (\lambda_k r) \, dr = \sum_{k=1}^{\infty} \frac{2}{\lambda_k^2 J_1 (\lambda_k)} [J_1 (\lambda_k r)]_0^1
\]

\[
= \sum_{k=1}^{\infty} \frac{2}{\lambda_k^2} = \int_0^1 r \, dr = \frac{1}{2},
\]

and so we find that

\[
\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} = \frac{1}{4}.
\]

\[\text{(C.5)}\]

\[\text{(C.6)}\]

### C.2 Example (2)

In this second example we choose \( n = 0 \) and \( a = 1 \), just as in the previous example. Let \( f(r; s) \) be given by

\[
f(r; s) = \frac{I_0 \left( \sqrt{\frac{r}{s}} \right) - I_0 \left( \sqrt{\frac{r}{s}} \right)}{I_0 \left( \sqrt{\frac{r}{s}} \right)}. \quad \text{(C.7)}
\]

The Fourier-Bessel coefficients for \( f(r; s) \) are given by

\[
c_k = \frac{2}{J_1^2 (\lambda_k)} \int_0^1 r \left( \frac{I_0 \left( \sqrt{\frac{r}{s}} \right) - I_0 \left( \sqrt{\frac{r}{s}} \right)}{I_0 \left( \sqrt{\frac{r}{s}} \right)} \right) J_0 (\lambda_k r) \, dr = \frac{2}{J_1^2 (\lambda_k)} \frac{s J_1 (\lambda_k)}{\lambda_k (s + \tilde{\nu} \lambda_k^2)}, \quad \text{(C.8)}
\]

and so we find for \( 0 < r < 1 \) that

\[
\frac{I_0 \left( \sqrt{\frac{r}{s}} \right) - I_0 \left( \sqrt{\frac{r}{s}} \right)}{I_0 \left( \sqrt{\frac{r}{s}} \right)} = \sum_{k=1}^{\infty} \frac{2s}{\lambda_k (s + \tilde{\nu} \lambda_k^2)} J_0 (\lambda_k r). \quad \text{(C.9)}
\]

Again, by multiplication of both sides of (C.9) by \( r \) and integration of the result with respect to \( r \), we find

\[
\int_0^1 r \frac{J_0 \left( \sqrt{\frac{r}{s}} \right) - J_0 \left( \sqrt{\frac{r}{s}} \right)}{J_0 \left( \sqrt{\frac{r}{s}} \right)} \, dr = \frac{I_2 \left( \sqrt{\frac{r}{s}} \right)}{2I_0 \left( \sqrt{\frac{r}{s}} \right)} =
\]

\[
\sum_{k=1}^{\infty} \frac{2s}{\lambda_k^2 (s + \tilde{\nu} \lambda_k^2)} J_1 (\lambda_k) \int_0^1 r J_0 (\lambda_k r) \, dr = \sum_{k=1}^{\infty} \frac{2s}{\lambda_k^2 (s + \tilde{\nu} \lambda_k^2)}. \quad \text{(C.10)}
\]

And so we find that

\[
\sum_{k=1}^{\infty} \frac{4}{\lambda_k^2 (s + \tilde{\nu} \lambda_k^2)} = \frac{I_2 \left( \sqrt{\frac{r}{s}} \right)}{s I_0 \left( \sqrt{\frac{r}{s}} \right)}. \quad \text{(C.11)}
\]
Appendix D

Global equations of motion with radial diffusion

In Section 4.3 we derived a solution for the volumetric flow rate \( q(z, t) \) from the global equations of motion, in which we incorporated the radial diffusion term \( \tau_w \) and neglected the advection term \( \partial_z \gamma_a \). This solution is obtained by introducing Laplace transforms for the flow rate and the pressure distribution. In this appendix we opt for an alternative method to derive the expression for the flow rate. From a change of the coordinate system, we arrive at a linear case of Cauchy’s problem. In solving the resulting problem we follow the method executed in [8], which uses contour integration and Green’s divergence theory.

Once more, we recall that the global equations of motion with inclusion of the radial diffusion term are given by

\[
\begin{align*}
\frac{\partial p}{\partial t} + \frac{\partial q}{\partial z} &= 0, \quad (D.1) \\
\frac{\partial q}{\partial t} + \frac{\partial p}{\partial z} + \tilde{\tau} q &= 0, \quad (D.2)
\end{align*}
\]

In Section 4.2 we saw that for the case \( \delta = 0 \), the characteristic parameters are given by \( \eta = t + z \) and \( \xi = t - z \). In this perspective, we assume \( q(z, t) = Q(\eta, \xi) \) and \( p(z, t) = P(\eta, \xi) \). Substitution in (4.64) and (4.65) gives

\[
\begin{align*}
\frac{\partial P}{\partial \eta} + \frac{\partial P}{\partial \xi} + \frac{\partial Q}{\partial \eta} - \frac{\partial Q}{\partial \xi} &= 0, \quad (D.3) \\
\frac{\partial Q}{\partial \eta} + \frac{\partial Q}{\partial \xi} + \frac{\partial P}{\partial \eta} - \frac{\partial P}{\partial \xi} + \tilde{\tau} Q &= 0. \quad (D.4)
\end{align*}
\]

Successively adding and subtracting (D.3) and (D.4), we obtain

\[
\begin{align*}
2 \frac{\partial P}{\partial \eta} &= -2 \frac{\partial Q}{\partial \eta} - \tilde{\tau} Q, \quad (D.5) \\
2 \frac{\partial P}{\partial \xi} &= 2 \frac{\partial Q}{\partial \xi} + \tilde{\tau} Q. \quad (D.6)
\end{align*}
\]

Elimination of \( P \) by cross-differentiation with respect to \( \xi \) and \( \eta \) and subtraction of the resulting equations finally leads to (with \( \tilde{\tau} = \tilde{\tau}/4 \))

\[
\frac{\partial^2 Q}{\partial \eta \partial \xi} + \tilde{\tau} \frac{\partial Q}{\partial \eta} + \tilde{\tau} \frac{\partial Q}{\partial \xi} = 0, \quad (D.7)
\]
Finding the solution $Q(\eta, \xi)$ of (D.7) is a special case of the so-called Cauchy problem. In order to find this solution, we define the differential operator and its adjoint corresponding to (D.7) by

$$L[Q] = Q_{\eta \xi} + \hat{\tau} Q_{\eta} + \hat{\tau} Q_{\xi}, \quad (D.8)$$

$$M[\Phi] = \Phi_{\eta \xi} - \hat{\tau} \Phi_{\eta} - \hat{\tau} \Phi_{\xi}, \quad (D.9)$$

where $\Phi = \Phi(\eta, \xi)$ is a new unknown function. Simple substitution shows that

$$\Phi L[Q] - Q M[\Phi] = \left( \hat{\tau} \Phi Q + \frac{1}{2} \Phi Q_{\xi} - \frac{1}{2} \Phi_{\xi} Q \right)_{\eta} + \left( \hat{\tau} \Phi Q + \frac{1}{2} \Phi Q_{\eta} - \frac{1}{2} \Phi_{\eta} Q \right)_{\xi}. \quad (D.10)$$

By using Green’s divergence theorem, stating that

$$\iint_\mathcal{D} (U_x + V_y) \, dx \, dy = \int_\mathcal{C} (U \, dy - V \, dx), \quad (D.11)$$

where $\mathcal{D}$ is the region of integration and $\mathcal{C}$ the corresponding closed contour, as defined in Figure D.1, we see that

$$\iint_\mathcal{D} (\Phi L[Q] - Q M[\Phi]) \, dq \, d\xi = \int_\mathcal{C} \left[ \left( \hat{\tau} \Phi Q + \frac{1}{2} \Phi Q_{\xi} - \frac{1}{2} \Phi_{\xi} Q \right) \, d\xi - \left( \hat{\tau} \Phi Q + \frac{1}{2} \Phi Q_{\eta} - \frac{1}{2} \Phi_{\eta} Q \right) \, d\eta \right]. \quad (D.12)$$

We want to eliminate the derivatives of $Q$ from the contour integral. This can be done by integration by parts.

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1. Along $C_1$ we see that
\[
- \int_C^A (\dot{\tau} \Phi Q + \frac{1}{2} \Phi Q_{\eta} - \frac{1}{2} \Phi_{\eta} Q) \, d\eta = \\
- \int_C^A (\dot{\tau} \Phi Q - \frac{1}{2} \Phi_{\eta} Q) \, d\eta - \frac{1}{2} \int_C^A \Phi Q_{\eta} \, d\eta = \\
- \int_C^A (\dot{\tau} \Phi Q - \frac{1}{2} \Phi_{\eta} Q) \, d\eta - \frac{1}{2} \left( \Phi(A)Q(A) - \Phi(O)Q(O) - \int_C^A \Phi_{\eta} Q \, d\eta \right) = \\
- \int_C^A (\dot{\tau} \Phi - \Phi_{\eta}) \, Q \, d\eta - \frac{1}{2} \Phi(A)Q(A) + \frac{1}{2} \Phi(O)Q(O). \tag{D.14}
\]

2. Along $C_2$ we see that
\[
\int_A^B (\dot{\tau} \Phi Q + \frac{1}{2} \Phi Q_{\xi} - \frac{1}{2} \Phi_{\xi} Q) \, d\xi = \\
\int_A^B (\dot{\tau} \Phi Q - \frac{1}{2} \Phi_{\xi} Q) \, d\xi + \frac{1}{2} \int_A^B \Phi Q_{\xi} \, d\xi = \\
\int_A^B (\dot{\tau} \Phi Q - \frac{1}{2} \Phi_{\xi} Q) \, d\xi + \frac{1}{2} \left( \Phi(B)Q(B) - \Phi(A)Q(A) - \int_A^B \Phi_{\xi} Q \, d\xi \right) = \\
\int_A^B (\dot{\tau} \Phi - \Phi_{\xi}) \, Q \, d\xi + \frac{1}{2} \Phi(B)Q(B) - \frac{1}{2} \Phi(A)Q(A). \tag{D.15}
\]

3. Along $C_3$ we see that
\[
- \int_B^C (\dot{\tau} \Phi Q + \frac{1}{2} \Phi Q_{\eta} - \frac{1}{2} \Phi_{\eta} Q) \, d\eta = \\
- \int_B^C (\dot{\tau} \Phi Q - \frac{1}{2} \Phi_{\eta} Q) \, d\eta - \frac{1}{2} \int_B^C \Phi Q_{\eta} \, d\eta = \\
- \int_B^C (\dot{\tau} \Phi Q - \frac{1}{2} \Phi_{\eta} Q) \, d\eta - \frac{1}{2} \left( \Phi(C)Q(C) - \Phi(B)Q(B) - \int_B^C \Phi_{\eta} Q \, d\eta \right) = \\
- \int_B^C (\dot{\tau} \Phi - \Phi_{\eta}) \, Q \, d\eta - \frac{1}{2} \Phi(C)Q(C) + \frac{1}{2} \Phi(B)Q(B). \tag{D.16}
\]

4. Along $C_4$ we see that
\[
\int_C^O (\dot{\tau} \Phi Q + \frac{1}{2} \Phi Q_{\xi} - \frac{1}{2} \Phi_{\xi} Q) \bigg|_{\xi=\xi} \, d\xi - \int_C^O (\dot{\tau} \Phi Q + \frac{1}{2} \Phi Q_{\eta} - \frac{1}{2} \Phi_{\eta} Q) \bigg|_{\eta=\eta} \, d\eta = \\
\int_C^O \left( \frac{1}{2} \frac{\partial}{\partial \xi} (\Phi Q) - \Phi_{\xi} Q \right) \bigg|_{\eta=\xi} \, d\xi - \int_C^O \left( \frac{1}{2} \frac{\partial}{\partial \eta} (\Phi Q) - \Phi_{\eta} Q \right) \bigg|_{\xi=\eta} \, d\eta = \\
\int_C^O \Phi_{\eta} Q \bigg|_{\xi=\eta} \, d\eta - \int_C^O \Phi_{\xi} Q \bigg|_{\eta=\xi} \, d\xi + \frac{1}{2} \Phi(O)Q(O) - \frac{1}{2} \Phi(C)Q(C). \tag{D.17}
\]
So finally we are left with

\[
\iint_{\mathcal{D}} \left( \Phi [\mathcal{Q}] - \mathcal{M}[\Phi] \right) \, d\eta d\xi = \\
\Phi(Q, O) Q(O) - \Phi(C) Q(C) + \Phi(B) Q(B) - \Phi(A) Q(A) \\
- \int_{O}^{A} (\hat{\tau} \Phi - \Phi_{\eta}) \, Q \, d\eta + \int_{A}^{B} (\hat{\tau} \Phi - \Phi_{\xi}) \, Q \, d\xi \\
- \int_{B}^{C} (\hat{\tau} \Phi - \Phi_{\eta}) \, Q \, d\eta + \int_{C}^{O} \Phi_{\eta} Q \bigg|_{\xi = \eta} \, d\eta - \int_{C}^{O} \Phi_{\xi} Q \bigg|_{\eta = \xi} \, d\xi
\]  

(D.18)

Along \( C_{1} \) it holds that \( \eta = t - z = 0 \), which means that \( t = z \). From the conditions proposed on \( q(z, t) \) we see that \( Q(\eta, 0) = 0 \). So, the integral over the interval \( (O, A) \) and the terms \( \frac{1}{2} \Phi(Q, O) \) and \( \Phi(A) Q(A) \) disappear from \( (D.18) \). Moreover, along \( C_{4} \) we have that \( \eta = \xi \), which means that \( t + z = t - z \) or equivalently \( z = 0 \). Since \( q(0, t) = q_{i}(t) \), we can substitute \( Q(\cdot, \cdot) = q_{i}(\cdot) \) for the integral over the interval \( (C, O) \) and \( Q(C) \) by \( q_{i}(\xi_{0}) \). We conclude that

\[
\iint_{\mathcal{D}} \left( \Phi [\mathcal{Q}] - \mathcal{M}[\Phi] \right) \, d\eta d\xi = \\
\Phi(B) Q(B) - \Phi(C) q_{i}(\xi_{0}) + \int_{B}^{C} (\hat{\tau} \Phi - \Phi_{\xi}) \, Q \, d\xi \\
- \int_{B}^{C} (\hat{\tau} \Phi - \Phi_{\eta}) \, Q \, d\eta + \int_{C}^{O} \Phi_{\eta} Q \bigg|_{\xi = \eta} \, d\eta - \int_{C}^{O} \Phi_{\xi} Q \bigg|_{\eta = \xi} \, d\xi
\]  

(D.19)

The next goal is to determine the function \( \Phi(\eta, \xi) \). The first observation is that \( \mathcal{M}[\Phi] \) should be equal to \( 0 \) in order to eliminate \( Q \) from the left-hand side of \( (D.19) \). Since we want to express the solution \( Q(\eta, \xi) \) in the point \( B \) in terms of the initial data proposed on \( Q(\eta, \xi) \) on \( C_{1} \) and \( C_{4} \), we want the integrals over \( C_{2} \) and \( C_{3} \) to disappear from \( (D.19) \). This leads to two ordinary differential equations for two characteristic initial conditions on \( \Phi(\eta, \xi) \). If we on top of this demand that \( \Phi(B) = 1 \), this leads to

\[
\hat{\tau} \Phi - \Phi_{\xi} = 0, \quad \text{along } C_{2},
\]

(D.20)

\[
\hat{\tau} \Phi - \Phi_{\eta} = 0, \quad \text{along } C_{3},
\]

(D.21)

implying that

\[
\Phi(\eta_{0}, \xi) = e^{\hat{\tau}(\xi - \xi_{0})}, \quad \text{for } \xi \leq \xi_{0},
\]

(D.22)

\[
\Phi(\eta, \xi_{0}) = e^{\hat{\tau}(\eta - \eta_{0})}, \quad \text{for } \eta \leq \eta_{0}.
\]

(D.23)

These conditions are satisfied by assuming that

\[
\Phi(\eta, \xi) = \mathcal{A}(\eta, \xi; \eta_{0}, \xi_{0}) e^{\hat{\tau}(\eta - \eta_{0})} e^{\hat{\tau}(\xi - \xi_{0})}.
\]

(D.24)

Substitution of \( (D.24) \) in the adjoint equation \( \mathcal{M}[\Phi] = 0 \) leads to the telegraph equation

\[
\frac{\partial^{2} \mathcal{A}}{\partial \eta \partial \xi} - \hat{\tau}^{2} \mathcal{A} = 0,
\]

(D.25)

\[
\mathcal{A}(\eta_{0}, \xi; \eta_{0}, \xi_{0}) = 1, \quad \text{for } \xi \leq \xi_{0},
\]

(D.26)

\[
\mathcal{A}(\eta, \xi_{0}; \eta_{0}, \xi_{0}) = 1, \quad \text{for } \eta \leq \eta_{0}.
\]

(D.27)
where the boundary conditions are such that \((D.24)\) satisfies the characteristic initial conditions \((D.22)\) and \((D.23)\). The solution of \((D.25)\) is the so-called Riemann function, given by

\[
A(\eta, \xi; \eta_0, \xi_0) = I_0 \left( 2\hat{\tau} \sqrt{(\eta_0 - \eta)(\xi_0 - \xi)} \right),
\]

\((D.28)\)

where \(I_0(\cdot)\) represents the modified Bessel function of order zero.

Since we know \(A\), we are able to express \(Q\) in the arbitrary point \((\eta_0, \xi_0)\) in terms of the initial data proposed on \(q(z, t)\) by substitution of \(\Phi\) in \((D.19)\). This gives, for \(0 \leq \xi_0 \leq \eta_0\),

\[
Q(\eta_0, \xi_0) = e^{i(\eta_0 - \eta_0)} q_i(\xi_0) +
\hat{\tau}(\eta_0 - \xi_0) \int_0^{\xi_0} \frac{I_1 \left( 2\hat{\tau} \sqrt{(\eta_0 - s)(\xi_0 - s)} \right)}{\sqrt{(\eta_0 - s)(\xi_0 - s)}} e^{-i(\eta_0 + \hat{\xi}_0 - 2s)} q_i(s) \, ds.
\]

\((D.29)\)

From the \((\eta, \xi)\)-domain we can easily go back to the \((z, t)\)-domain. This gives for \(q(z, t)\):

\[
q(z, t) = \left[ e^{-2\hat{\tau}z} q_i(t - z) +
2\hat{\tau}z \int_0^{t - z} \frac{I_1 \left( 2\hat{\tau} \sqrt{(t - s)^2 - z^2} \right)}{\sqrt{(t - s)^2 - z^2}} e^{-2\hat{\tau}(t-s)} q_i(s) \, ds \right] H(t - z).
\]

\((D.30)\)

In a completely analogous way, we can derive for \(p(z, t)\).
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