MASTER

Ghost counts of gas turbine meters

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Abstract

Turbine meters are often used to measure volume flow through pipes. The company responsible for natural gas transportation in the Netherlands (Gasunie) has recently observed what is called "ghost counts". Ghost counts are induced by strong acoustic pulsations inside the pipe systems, which results in rotation of the turbine in the absence of a mean flow.

In this research, the ghost counts have been investigated experimentally in two different setups built at Gasunie. In those setups we were able to control carefully the acoustic oscillations. The oscillations imposed are approximately harmonic. We develop three theoretical models in order to calculate the aerodynamic torque on the turbine. All the models consider the blades as flat plates, and quasi-steady behavior is assumed globally or locally by the models. One model is essentially empirical and the others use two dimensional potential flow theory for an inviscid incompressible fluid. In this potential flow model the force driving the rotation appears to be due to the singular behavior of the flow at the sharp edges of the blades. This results either into a suction force called "edge force" or a vortex shedding for which a simplified model is proposed. The theoretical results are compared with the experiments.

The approximate critical static friction torque of the rotor above which ghost counts starts is measured by two different methods. The acoustic velocities calculated to overcome the critical static torque differs substantially from what is expected, varying from 10% until 220%, depending on the apparatus and the Strouhal number. The Strouhal number is a measure for the ratio of the blade thickness and the fluid particle displacement in the acoustical standing wave. We make predictions of the ghost counts under field conditions. The values are consistent with previous field measurements.

The turbine steady rotation appears to be proportional to the acoustic velocity until a certain limit of the ratio between the blades velocity and the acoustic velocity. Above this critical ratio the flow pattern seems to change, and the rotor velocity starts decreasing for increasing acoustic velocity amplitude.

The influence of the geometry of the rotor blades on the aerodynamic force is discussed. A thick trailing edge reduces the sensitivity of the rotor at high oscillation frequencies (high Strouhal number). Since ghost counts are due to the asymmetrical blades geometry, it might be interesting to investigate wether a symmetrical design can be used.
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Chapter 1

Introduction

De Nederlandse Gasunie N.V. (Gasunie) is the company responsible for the transportation of natural gas in the Netherlands. Every year about 10^{10} m^3 of gas is transported for export or distribution to various customers [1]. This implies that volume flow measurements, usually made with turbine meters, is one of the main problems encountered by Gasunie. Gas turbine meters are rotors that can rotate freely with very low mechanical friction. The rotation speed of the turbine is an accurate measure of the volume flow rate if the flow is properly conditioned. It should be a stationary and fully developed flow. Special care is taken to avoid swirl in the flow, which would induce systematic errors [2].

Our work is concerned with measurement errors due to the unsteadiness of the flow associated with acoustic waves in the pipe system. Vast literature is found on measurement errors of gas turbine meters due to acoustic oscillation effects [3, 4, 5, 6, 7]. Studies of pulsation background with a mean flow over a wing is done by Szumowski [8]. In these studies the acoustical flow has a small amplitude compared with the main flow. Therefore, linearized theory around the steady flow is an efficient approach. We consider conditions for which this approximation fails. In particular, we will focus on the phenomenon called "ghost counts", rotation of the turbine meter in the absence of a mean flow through the pipe. These ghost counts are due only to acoustic oscillations. The oscillations for ghost counts to occur should be of high amplitude. This is only reached when there is an acoustic resonance in the pipe system. Since large turbine meters are excessively expensive the main pipe is usually divided into a row (or street) of narrower pipes in which smaller turbine meters can be replaced. When the volume flow is small, usually around summer, some of the pipes in the street are closed in order to increase the dynamic range of the measurements (fig. 1.1).

Figure 1.1: Summer configuration of measuring street in a natural gas transport system.

This is done by closing valves placed downstream of each meter. This results in a row of closed side branches, which can give strong acoustical resonances and display the ghost counts [9]. The
resonant acoustic field can be generated either by pulsations from compressors or flow instabilities of the grazing flow along a closed side branch [10, 11].

Ghost counts would not be a problem if there were perfect valves and junctions without any possibility of leak. This is however not the case. One has therefore either to be able to detect the leak or the ghost counts. The current project intends to understand the relationship between the acoustical pulsations and the turbine rotation. Besides industrial applications, the project has interesting academic value. A relative simple description of the complex phenomenon can be obtained. The problem is related to the flight insects [12, 13].

Obviously the turbine meter will not be affected by only uniform changes of the pressure in the fluid. It is the acoustical displacement of the fluid particles which induces the rotation of the turbine. The mechanical friction provides the threshold for the velocity fluctuations, below which the turbine will not rotate. In the absence of main flow, the acoustical velocity oscillation is symmetric around the zero velocity. For sake of simplicity we consider purely harmonic oscillations. The "ghost counts" are induced by an asymmetry of the profile of the turbine winglets. This can be seen in the photograph of the actual rotor and the scheme of a supposed flow pattern around the blades in fig. 1.2.

![Figure 1.2: Rotor photograph and flow pattern induced by the asymmetry of the blades.](image)

In order to understand the phenomenon Gasunie has build two setups consisting of a closed pipe in which a turbine gas meter is placed (fig.1.3). The acoustic field in the pipe is driven by a loudspeaker. A small setup with pipes diameter of 10cm operates at atmospheric pressure, while a large setup with pipes diameter of 30cm can operate with pressure up to 8bar. This allows to verify sealing rules and to change the ratio of aero-dynamic forces to mechanical friction forces.

In this report the results of the measurements are compared to two flow models. In both models the blades geometry is simplified to a flat plate. The first flow model attempts to predict the aerodynamic force on the rotor at rest (below the rotation threshold). This model supposes quasi-steady behavior close to one edge and calculating the edge force at this edge. A second model tries to understand the rotor behavior during rotation, by assuming a quasi-steady flow behavior. The first model used is similar to the models used in the literature of insect flight.

Next chapter gives some theoretical background important for the development of this research. First section (2.1) studies the general dynamics of the rotor, while the second section (2.2) discusses over the fluid dynamics theory. Chapter 3 builds both flow models, with section 3.2 defining the conditions and approximations of their applications. Chapter 4 gives characteristics and possibilities of both apparatus and the measurements equipments. The dynamic friction and the static friction are determined experimentally in subsections 4.3.2 and 4.3.3. Information over the data acquisition and data analysis are also given. At last, in chapter 5, the theoretical and the experimental results are presented and analyzed. Measurements of the starting conditions are in section 5.1, while the rotating
At industry fields:

At laboratory:

Figure 1.3: Laboratory simulation of field ghost counts at Gasunie.

behavior measurements are given in section 5.2. The most important conclusions are given in the last section (5.3).
Chapter 2

Theory

The theory that follows may be found in many sources, with the main material extracted from [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

2.1 Global description of the dynamics

We analyze the response of the rotor from a turbine gas meter to an oscillating flow. The present section will describe the general equation for the motion of the rotor, describing briefly the origin of each term in the equation, and showing how their values can be calculated.

2.1.1 Equation of motion

The general equation of motion, \( \sum T = d(I \cdot \Omega(t))/dt \), for the specific case of the rotor at the gas turbine meters becomes:

\[
T_{\text{aer}} + T_{\text{fr}} = I \frac{d\Omega(t)}{dt}
\]  

(2.1)

With \( T_{\text{aer}} \) the aerodynamic torque applied at the blades of the rotor, \( T_{\text{fr}} \) the total friction torque, \( I \) the rotor moment of inertia relative to its axis and \( \Omega(t) \) the angular velocity as a function of time.

Both torques are due to the motion the rotor relative to the fluid surrounding it. The origin of the aerodynamic force is explained in subsection 2.2.5 while a model for the prediction of its value is derived in chapter 3. The torque due to the friction has components of the friction with air and with the bearing. Formulas to calculate their values are derived in the next section. The moment of inertia of the rotor is known accurately, calculated by the gas turbine meter manufacturer. A rough calculation to check the values specified by the manufacturer is given in appendix A. The angular velocity as a function of the time has been measured at various conditions for both apparatus studied. The experiments are described in chapter 3.

From this apparently simple equation, which hides the complexity of the rotor motion, the values of the forces calculated from the aerodynamic forces model and from the friction theory may be compared with the experimental data acquired.

2.1.2 Friction

To increase durability and precision of the gas turbine meter, the bearing is filled with a thin layer of lubricant oil, separating partially the rotor from the shaft.

The static friction torque of the rotor \( (T_{\text{st}}) \) will be very low, with the dominant component being the friction with the shaft \( (T_{\text{ost}}) \). The kinematic friction torque \( (T_{\text{kn}}) \) can be divided between the torque from the friction at the bearing and with air on the rest of the rotor. At the bearing there is friction with oil \( (T_{\text{oll}}) \) and friction with the shaft \( (T_{\text{osn}}) \). With air we choose to decompose the friction in two parts, only with the blades \( (T_{\text{a}}) \) and with the rest of the rotor \( (T_{\text{air}}) \). The components for the static \( (\omega=0) \) and kinematic friction \( (\omega > 0) \) torques becomes:

\[
T_{\text{st}} = T_{\text{ost}}
\]

(2.2)
The friction from air on the rotor excluding the blades \((T_{\text{air}})\) and from the oil \((T_{\text{oil}})\) are explained by the shear stress along the surface of contact with the fluids. For a specific surface, the stress is a vector expressed in differential form as:

\[
\tau = \frac{d\vec{F}}{dS} \quad (2.4)
\]

With "\(d\vec{F}\)" the force acting on the surface element "\(dS\)" the element of area on the surface. We distinguish the normal shear stress "\(\tau_n\)" which we will relate to the pressure "\(p\)" and the shear stress "\(\tau_s\)" which is related to the tangential component of the viscous forces.

When the fluid is in contact with a solid surface, it must have the same velocities as the surface. This fact is known as a no-slip condition. The no-slip condition is not obvious, but consistent with experiments. The fluid layer in contact with the surface, due to viscosity, interacts with the layers further from the surface which have different velocities. Viscosity is a measure for the rate of momentum transfer between such layers. This causes the skin friction (integral of the shear stress over the surface) on the rotor.

If the velocity \((\bar{U})\) is directed along the surface and varies only in the direction perpendicular to the surface, and if the tangential shear stress \((\tau_s)\) is proportional to the velocity gradient, then the fluid is called Newtonian fluid:

\[
\tau_s = \mu \frac{dU(y)}{dy} \quad (2.5)
\]

With "\(\mu\)" the dynamic viscosity, "\(U(y)\)" the tangential velocity of the fluid particle and "\(y\)" the perpendicular distance of the fluid particle from the solid surface.

The air friction on the rotor blades \((T_a)\) is part of the model for the aerodynamic torque, described and predicted in chapter 3. In order to estimate the skin friction on the blades, suppose an average radius from the axis of \(R_{av}\), with an average fluid velocity of \(U_{av}\) and the total surface area of the blades as \(A_S\). The friction torque is estimated in appendix B and has the approximate value of:

\[
T = \sqrt{\frac{\mu P}{L}} S R_{av} U_{av}^{3/2} \quad (2.6)
\]

The contribution of the air friction torque on the lateral surface of a rotor of inner radius \(r_1\) and outer radius \(r_2\) is estimated to have the order of magnitude of \((\text{appendix B})\):

\[
T = \sqrt{\frac{\pi \mu_{\text{air}} P_{\text{air}} \Omega^2}{t}} (r_2^4 - r_1^4) \quad (2.7)
\]

With "\(\rho_{\text{air}}\)" the air density, "\(\mu_{\text{air}}\)" the air viscosity and "\(t\)" the time the rotor is turning with constant angular velocity \(\Omega\). The contribution of the friction.

To calculate the torque due to the oil in the bearings of the rotor, consider the oil velocity always tangential with magnitude depending only on the distance from the rotor axis. Let "\(a\)" be the radius of the shaft, "\(b\)" and "\(H\)" be respectively the radius and the width of the rotor at the bearings, "\(\mu_{\text{oil}}\)" the viscosity of the oil, "\(\Omega\)" the angular velocity of the rotor. The friction torque from the oil in the bearing \((T_{\text{oil}})\) is calculated in appendix B and is equal to:

\[
T_{\text{oil}} = \frac{4\pi \mu_{\text{oil}} H \Omega a^2 b^2}{(b^2 - a^2)} \quad (2.8)
\]

The torque from the contact of the rotor with the shaft \((T_{\text{kn}})\) can be approximated to a constant friction in the case of the kinematic friction. For the static friction it will have same magnitude but opposite direction as the resultant torque acting on the rotor \((T_R)\). Fig.2.1 shows such assumption:

The critical static torque value \((T_{\text{ost}})_{\text{max}}\) is the required torque to begin rotation. Let the friction coefficient between the two surfaces be "\(\kappa\)", "\(N\" the force normal to the surface, and "\(r_0\)" the radial distance from the axis at which the friction force is acting. The friction torque acting on the rotor will be:

\[
T_{\text{ost}} = \kappa_{\text{st}} N r_0 \quad (2.9)
\]

\[
T_{\text{kn}} = \kappa_{\text{kn}} N r_0 \quad (2.10)
\]

\(^{1}\)No direct experiment exists to check the no-slip condition, although its excellent agreement of the theory to predict experiments is such that the condition is considered true.
2.2 Some fluid dynamics

This section will explain various important definitions and concepts of fluid dynamics. The first subsection is dedicated to an introduction of the basic equations of fluid dynamics. In subsection 2.2.2, dynamic similarity is explained. Nondimensional numbers that were used during the research are defined. The section of "boundary layer and flow separation" introduces these two essential concepts. Potential flow theory, used to develop the force models in chapter 3, is explained briefly in subsection 2.2.4. At last, the forces acting on the blades due to the flow are explained qualitatively (aerodynamic forces).

2.2.1 Basic equations

Fluid dynamics considers the fluid as continuum, which we can divide in infinitesimal volume elements $dV$. An element of volume $dV$ is considered to be small compared to macroscopic length scales in the problem, but large if compared with the mean free path of the molecules (or other relevant molecular length scales). This implies that quantities such as velocity ($\vec{v}$), density ($\rho$) or pressure ($p$), are smooth functions of space and time coordinates ($\vec{r}, t$). From the conservation theorems of mass and momentum, four fundamental equations can be derived.

From the mass conservation, we have that the rate of mass change in a fixed volume should be equal to the flux of mass through the bounding surface of this volume. This results in what is called continuity equation, which in differential form is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad (2.11)$$

The scalar product can be expanded, and defining the total derivative as $D()/Dt = \partial()/\partial t + \vec{v} \cdot \nabla()$, eqn.2.11 becomes:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0 \quad (2.12)$$

If the flow is incompressible, which means that density variations are negligible, then the continuity equation reduces to:

$$\nabla \cdot \vec{v} = 0 \quad (2.13)$$

The momentum equation states that the sum of the forces applied to a solid particle is equal to the mass time the acceleration of this particle. Let $\vec{f}$ be the body force per unit volume and $\vec{S}$ the surface force per unit volume, the momentum equation for a fluid particle $dV$ in vector notation becomes:

$$\rho \frac{D\vec{v}}{Dt} = \vec{f} + \vec{S} \quad (2.14)$$
For incompressible flows ($\nabla \cdot \vec{v} = 0$), neglecting body forces, the momentum equation becomes:

$$\frac{D\vec{v}}{Dt} = -\nabla p + \mu \nabla^2 \vec{v}$$  \hspace{1cm} (2.15)

Where the first term of the surface forces ($-\nabla p$) is a result of the action of the pressure on the surface. The second term ($\mu \nabla^2 \vec{v}$) is due to the viscous forces along the surface. Some flows, far from boundaries, can be considered as inviscid, where the effect of viscosity is negligible. The momentum equation reduces in what is called Euler’s equations of motion (neglecting body forces):

$$\frac{D\vec{v}}{Dt} = -\nabla p$$  \hspace{1cm} (2.16)

The vorticity $\vec{\omega}$ is defined as $\vec{\omega} = \nabla \times \vec{v}$. If the vorticity is zero ($\nabla \times \vec{v} = 0$) over a certain region, the flow is called irrotational in this region. The circulation $\Gamma$ is defined as:

$$\Gamma = \oint_c \vec{v} \cdot ds = \int_S \vec{\omega} \cdot d\vec{s}$$  \hspace{1cm} (2.17)

Where the closed contour $c$ is spanned by the surface $S$ and $d\vec{s}$ is a vector normal to an element of area $ds$ of the surface $S$. If the flow is irrotational within the surface $S$, it is clear that the circulation is also zero.

If the flow is irrotational the velocity can be written as the gradient of a scalar potential ($\vec{v} = \nabla \phi$, subsection 2.2.4). Evaluating eqn.2.16 for irrotational flows gives the most simple forms of Bernoulli’s equation for steady (eqn.2.18) and unsteady (eqn.2.19) motion of fluid of constant density (neglecting body forces):

$$\text{const.} = \frac{v^2}{2} + \frac{p}{\rho}$$  \hspace{1cm} (2.18)

$$\text{const.} = \frac{v^2}{2} + \frac{p}{\rho} + \frac{\partial \phi}{\partial t}$$  \hspace{1cm} (2.19)

Where $v = |\vec{v}|$.

**Incompressibility assumption**

In general, no fluid is incompressible, although, under certain conditions, the density variations may be considered as negligible.

From the steady Bernoulli’s equation one sees that the pressure fluctuation has the order of magnitude $dp = \rho v^2$. Substituting this equation in the definition of the speed of sound ($c^2 = dp/d\rho$) one gets:

$$\frac{dp}{\rho} \approx \left(\frac{v}{c}\right)^2$$  \hspace{1cm} (2.20)

Thus, for steady flows, the assumption of incompressibility ($dp/\rho \ll 1$) requires that the square of the flow velocity should be small compared with the square of the sound velocity ($v^2 \ll c^2$).

For unsteady flows another condition arises. Suppose a significant change in velocity $\vec{v}$ that occurred during a time $t_0$ and within a distance $l$. From Euler’s equation neglecting body forces (eq.2.16) one can see that the corresponding change in pressure will have the order of magnitude of $dp = \rho v l / t_0$. Substituting this result into the definition of the speed of sound gives:

$$\frac{dp}{\rho} \approx \left(\frac{v}{c}\right) \left(\frac{l}{ct_0}\right)$$  \hspace{1cm} (2.21)

Therefore, besides the condition of steady flows ($v/c \ll 1$), we should have "$l/ct_0 \ll 1$". This means that the time of propagation of the pressure signal over the distance "$l$" is negligible compared with the time in which significant changes of pressure occur. If the acoustic oscillations have a period "$T$", then "$\lambda = cT$" ("$\lambda$" the wavelength) and the condition of incompressibility becomes $l/\lambda \ll 1$.

As it will be seen in subsection 2.2.2, the nondimensional numbers $v/c$ and $l/\lambda$ are called respectively Mach number ($M$) and Helmholtz number ($He$). Reasonable values to consider the flow as incompressible are "$M^2$" and "$He$" less then "$0.1$".
CHAPTER 2. THEORY

Flow around blades

The values of the Mach number were small for the experiments made. The Helmholtz number, relative to the blades length, were also small, so that the flow around the blades is considered as locally incompressible. This way, the flow around the blades is described by eqn.2.13, eqn.2.16 and a fifth equation:

$$\rho = \text{const.}$$  \hspace{1cm} (2.22)

Acoustic flow

Let an acoustic wave propagate inside a pipe. We assume a large wavelength compared with the pipe diameter. Such wave may be considered as a plane wave. The direction of the wave propagation is chosen to be \( "x" \). If the acoustic pressure deviations \( (p) \) are much smaller then the atmospheric pressure, one should have \[16, 15\]:

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = 0$$  \hspace{1cm} (2.23)

Where \( "c" \) is the wave propagation velocity. Let \( "\lambda" \) be the wavelength. For harmonic waves the acoustic field frequency \( (f) \) is related to the wavelength by:

$$f = \frac{c}{\lambda}$$  \hspace{1cm} (2.24)

For such a harmonic wave the acoustical pressure can be written in terms of a wave with amplitude \( "p^+" \) travelling in the positive \( x \) direction and a wave with amplitude \( "p^-" \) travelling in the negative \( x \) direction:

$$p = \text{Re} \left[ p^+ e^{i(\omega t-kx)} + p^- e^{i(\omega t+kx)} \right]$$  \hspace{1cm} (2.25)

Where \( \omega = 2\pi f \) and \( k = 2\pi/\lambda \). The corresponding velocity field \( "u" \) can be deduced from the linearized momentum equation \( \rho \partial u/\partial t = -\partial p/\partial x \):

$$u = \text{Re} \left[ \frac{p^+ e^{i(\omega t-kx)} - p^- e^{i(\omega t+kx)}}{\rho c} \right]$$  \hspace{1cm} (2.26)

At a closed pipe termination the flow velocity vanishes. If we assume such a condition at \( x = 0 \) we deduce that \( p^+ \) and \( p^- \) are equal, so that we have a standing wave:

$$p = 2p^+ \cos(\omega t) \cos(kx)$$  \hspace{1cm} (2.27)

$$u = -\frac{2p^+}{\rho c} \sin(\omega t) \sin(kx)$$  \hspace{1cm} (2.28)

In a pipe segment close a both ends, the acoustic field may be in resonance with standing waves inside the pipe. This happens for critical frequencies such that the pipe length fits an entire number of half wavelengths. Fig.2.2 shows a diagram of the maximum acoustic velocity for the first three modes of resonance.

In our discussion we assume that the wave propagation effects can be locally neglected in the flow around the rotor. The acoustic field in the pipe acts as a boundary condition for the flow around the rotor. The volume flow at the rotor is deduced from a measurement of the pressure \( p(0) = 2p^+ \) at the end wall (see appendix I and section 4.5).

2.2.2 Dynamic similarity

Two different setups will be in dynamic similarity when they have similar geometries and similar flows. Similar geometrie means that both apparatus have different size but their geometries are identical to a linear scaling factor.

Suppose both setups studied as geometrically similar, as it is will be seen approximately true in section 4.1. When effects such as gravity are neglected, there will be four independent parameters for
2.2 Theory

The Reynolds number \( Re \), the Strouhal number \( Sr \), the acoustical amplitude \( A \) and the Helmholtz number \( He \):

\[
Re = \frac{\mu U_{ac}d}{\rho} = \frac{U_{ac}d}{\nu} \quad (2.29)
\]

\[
Sr = \frac{fd}{U_{ac}} \quad (2.30)
\]

\[
A = \frac{U_{ac}}{U_0} \quad (2.31)
\]

\[
He = \frac{fd}{c} \quad (2.32)
\]

With \( \mu \) the dynamic viscosity, \( \nu = \mu/\rho \) the kinematic viscosity, \( U_{ac} \) a characteristic acoustical flow velocity, \( U_0 \) the mean flow velocity, \( d \) some characteristic length of both setup and \( f \) the frequency of the acoustic flow.

All the experiments were made without mean flow, or infinite acoustical amplitude. Others dimensionless parameters, such as Mach number \( M \), drag coefficient \( CD \) and lift coefficient \( CL \), are functions of these four dimensionless parameters:

\[
M = \frac{U_{ac}}{c} \quad (2.33)
\]

\[
CD = \frac{D}{\frac{1}{2} \rho U^2} \quad (2.34)
\]

\[
CL = \frac{L}{\frac{1}{2} \rho U^2} \quad (2.35)
\]

With \( D \) and \( L \) as the drag and lift forces respectively, which will be explained in subsection 2.2.5. Hence if \( Re \), \( Sr \) and \( He \) are the same in both flows, all the other dimensionless quantities will also be the same for both flows.

Each nondimensional number is important for specific features of the flow, for example, the Reynolds number (eqn.2.29) is a measure of the inertial forces relative to the viscous forces. It can be used to predict the transition from a laminar to a turbulent flow. It also gives information over the thickness of the viscous boundary layer, as explained in the next subsection. The Mach number (eqn.2.33) is a measure of the compressibility of the flow (subsection 2.2.1).

2.2.3 Boundary layers and flow separation

The concept of boundary layer theory was first introduced by Prandtl in 1904, which started to unify the mathematical description of inviscid and viscous flow. Inviscid flows ignores viscous effects and had been intensively studied, although this model could not predict any friction by assumption. On
the other hand, viscous fluids, due to the complexity of the equations of motion, could not be treated analytically (except in few special cases). Practical problems were treated empirically. Boundary layer theory filled partially the gap between inviscid and viscous fluids treatment, assuming with great success that the viscosity effects are concentrated next to the surface within a thin layer, called the viscous boundary layer. This simplifies considerably the equations of motion, as will be shown in the end of this section.

In order to explain briefly the concept of boundary layer, suppose a solid body of length "L" immersed in a uniform flow of constant velocity "U". The fluid should satisfy the no-slip condition at the solid wall (subsection 2.1.2) and, due to the fluid viscosity, the momentum is diffused from the surface into the flow. At high Reynolds number a high velocity gradient, or a high vorticity \( \omega = \Delta \times \vec{v} \), will be observed close to the solid surface. This vorticity is swept downstream by the flow. The formed high vorticity layer will be called boundary layer when it remains relatively thin, meaning that the layer thickness (\( \delta \)) compared with, for example, the solid body length "L" is very small (\( \delta \ll L \)). The flow within the boundary layer may be laminar or turbulent, depending on many parameters. The most important are the Reynolds number, the pressure distribution of the outer flow, the roughness of the wall and the level of disturbance of the outer flow [18].

It is known experimentally that the length scale for vorticity diffusion is approximately \( \sqrt{\nu t_0} \), with "\( t_0 \)" the time of flight. After a time around \( L/U \) the fluid particles will be swept away from the solid surfaces, limiting the size of the vorticity layer. For a laminar flow along a flat plate, as shown in fig.2.3, the vorticity layer thickness will have the order of magnitude of:

\[
\delta = \sqrt{\frac{\nu L}{U}}
\]  

Substituting \( L/\delta \) in eqn.2.36 becomes:

\[
\frac{L}{\delta} = \sqrt{\frac{UL}{\nu}} = \sqrt{Re_L}
\]

Therefore, for large Reynolds number (\( Re_L \gg 1 \)), the viscous layer will be thin enough to be considered as a boundary layer (\( \delta \ll L \)). The Reynolds number is a nondimensional measure of the flow speed and the body scale relative to the viscous diffusion effect.

The Navier-Stokes equations of motion (eqn.2.15) can be simplified considerably in a boundary layer flow. Considering a two dimensional flow with constant density and viscosity, neglecting body forces and the limit for large Reynolds number, results in what is called Prandtl’s boundary layer equations:

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}, \text{"x" component} \\
\frac{\partial p}{\partial y} &= 0, \text{"y" component} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \text{continuity equation}
\end{align*}
\]
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Where "x" is the coordinate along the wall while "y" is normal to the wall.

Outside the boundary layer, the flow is considered inviscid. It is governed by Euler's equation (eqn.2.16). If the flow can be considered as incompressible, then potential flow theory (subsection 2.2.4) can be used to describe the motion.

Under certain conditions the fluid of the boundary layer can no longer remain close to the wall, resulting in what is called flow separation. Fig.2.4 shows such phenomenon.

Figure 2.4: Scheme of flow separation from a solid surface at a stagnation point "S".

The flow of the boundary layer tends to decelerate when, for some reason, the pressure downstream of the flow is larger \((dp/dx > 0)\) then upstream. This is called as adverse pressure gradient (APG). The low velocity fluid particles of the boundary layer may not be able get over this pressure gradient. The flow may then stop and separate from the wall. The point that it separates is called stagnation point, or separation point \((S)\). The stagnation point is the encounter of the downstream flow and a reverse upstream flow, with \(\partial u/\partial y = 0\) at "S". The shear stress \((\tau = \mu \partial u/\partial y)\) will then have opposite directions to each side of the flow relative to the separation point. The exact position of the separation point, as well as complet description of the effect, is in general very difficult. This due to the high non-linearity of the phenomenon.

Flow with sharp edges, such as the trailing edge of the blades, usually separates exactly at this sharp edge. This due to the very high velocities required for the flow to follow the very low radius of curvature, and overcome the pressure gradient around the sharp edge. By imposing such a condition of flow separation at sharp edges, viscosity effects may be introduced (locally) within the framework of a frictionless theory. This is called Kutta condition. We discuss further flow separation around the rotor's blades in subsection 2.2.5.

2.2.4 Potential flow theory

In regions of the flow where it can be considered as inviscid and incompressible, the flow motion may be described by means of potential flow theory. Such theory may describe analytically various problems. First one obtains the velocity field corresponding to the imposed boundary conditions. This is a purely kinematic approach in which forces are not considered. Using the Bernoulli's equations for steady (eqn.2.18) and unsteady (eqn.2.18) flows, pressure forces may be calculated from the velocity field. Friction forces may not be predicted by this theory, due to the original assumption of an inviscid fluid. Introducing the Kutta condition of flow separation at sharp edges, one can however predict some effects of viscosity on the flow field.

Suppose the fluid once with zero vorticity, for example at rest. The fluid will remain irrotational if it satisfies the conditions of Kelvin's irrotational theorem, which says that an inviscid, barotropic flow with conservative body forces, the circulation around a closed curve moving with the fluid remains constant with time. Considering the actual fluid as inviscid, incompressible and only with gravity as body forces, it will satisfy Kelvin's theorem and remain irrotational. If the flow can be described in two dimensions, two new functions shall then be defined, characterizing a potential flow.

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The irrotational condition \((\nabla \times \mathbf{v} = 0)\) implies that there must be a function \(\phi\) such that \(\mathbf{v} = \nabla \phi\). Such a function is called the velocity potential, and in two dimension, with "\(u\)" and "\(v\)" the velocities in the x and y directions respectively, becomes:

\[
u = \frac{\partial \phi}{\partial x}
\]  
(2.41)
Another function, \( \psi \), can be defined as to satisfy the continuity equation of incompressible flow in two dimensions \( (\partial u/\partial x + \partial v/\partial y = 0) \):

\[
\begin{align*}
\psi &= \frac{\partial \phi}{\partial y} \\
\frac{\partial \psi}{\partial x} &= -\frac{\partial \psi}{\partial y}
\end{align*}
\]  

From this it can be seen that this lines of constant "\( \psi \)" fulfill the definition of streamlines.\(^2\)

The functions \( \phi \) and \( \psi \) are functions only of the fluid velocity, being reasonable to think them as smooth functions in space, having continuous partial derivatives. In the complex plane this corresponds to analytical functions. We define a function \( w(z) \), called complex potential, as:

\[
w(z) = \phi + i\psi
\]

Where \( z = x + iy \). From complex function theory it can be proved that \( w(z) \) will actually be an analytic function of \( z \). One sees that \( \phi \) and \( \psi \) defined by eqn.2.41 to eqn.2.44 satisfy Cauchy-Riemann equations (see also appendix C):

\[
\begin{align*}
\frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y} \\
\frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x}
\end{align*}
\]

From the Cauchy-Riemann equation and the fact that for an analytic function \( f \) \( \partial^2 f/\partial x \partial y = \partial^2 f/\partial y \partial x \), one sees that "\( \phi \)" and "\( \psi \)" are harmonic functions and the important equations follows:

\[
\begin{align*}
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} &= 0 \\
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} &= 0
\end{align*}
\]

These are the two dimension Laplace equations for the streamlines and for the velocity potential. One of the most important characteristics of Laplace equations are the uniqueness of the solutions, given the boundaries and the singular points where the functions are not analytic. A simple proof for this uniqueness for a fluid motion confined in a specific volume is given in appendix D.

Another important result that follows from the definition of complex potential (eqn.2.45), velocity potential (eqn.2.41 and eqn.2.42) and streamlines (eqn.2.43 and eqn.2.44):

\[
\begin{align*}
dw &= \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy \\
dw &= \left( \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} \right) dx + \left( \frac{\partial \phi}{\partial y} + i \frac{\partial \psi}{\partial y} \right) dy \\
dw &= (u - iv) dx + i(u - iv) dy = (u - iv)(dx + idy) \\
\frac{dw}{dz} &= u - iv
\end{align*}
\]

For steady flow, using steady Bernoulli’s equation and the complex potential, an important theorem can be derived, called Blasius theorem (appendix G). It states that the force per unit length in the

\(^2\)Streamlines are lines in a two dimensional flow which for all positions of this line the velocity are tangential, \( dy/dx = v/u \). Consider \( \psi \) any analytic function in the complex plane:

\[
d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy
\]

If \( d\psi = 0 \) and using eqn.2.43 and eqn.2.44 is clear that \( \psi \) satisfies the definition of streamlines \( (dy/dx = v/u) \).
direction parallel to the \((x,y)\) plane on a solid body of contour "c", immersed in a steady flow of complex potential defined by \(w(z)\), will be (see fig.G.1):

\[
F_x - iF_y = \frac{i\rho}{2} \oint_C \left( \frac{dw(z)}{dz} \right)^2 dz
\]  

(2.50)

**Conformal mapping**

Complex function theory combined with conformal mapping is very useful in fluid dynamics. By means of a conformal mapping "\(z = z(\zeta)\)" complicated flow patterns in the \(z\) plane, defined by a complex potential "\(w(z)\)", can be transformed into simpler flows in the \(\zeta\) plane, defined by new complex potentials "\(W(\zeta) = w(z(\zeta))\)". Some basic properties of such methods are given in appendix H.

A very useful conformal transformation in airfoil theory is called Zhukovski transformation, also explained in appendix H. It can transform the complex potential around a circle of radius "\(A\)" in the \(\zeta\)-plane, into different complex potentials around different kinds of airfoils in the \(z\)-plane:

\[
z = \zeta + \frac{A^2}{\zeta}
\]  

(2.51)

The inverse transformation can do the opposite, simplifying the description of the flow around the airfoil transforming it to the flow around a circle:

\[
\zeta = \frac{z + \sqrt{z^2 - 4A^2}}{2}
\]  

(2.52)

From this, one gets:

\[
\frac{d\zeta}{dz} = \frac{1}{2} + \frac{2z}{\sqrt{z^2 - 4A^2}}
\]  

(2.53)

And the velocity around the airfoil can be found:

\[
\frac{dw(z)}{dz} = \frac{dW(\zeta)}{d\zeta} \frac{d\zeta}{dz}
\]  

(2.54)

**Examples of potential flows**

The velocity potential "\(\phi\)" and the stream function "\(\psi\)" are linear functions, and can be superposed to define new potentials and stream functions. From this fact we shall build the complex potential of the flow around a flat plate in fig.2.5 from the superpositions of simpler flows described next.

![Figure 2.5](image)

Figure 2.5: (a)- Flow around a flat plate of length \(4A\). (b)- The equivalent flow around a cylinder of radius \(A\) obtained by means of the Zhukhovski transformation.

Suppose a uniform parallel flow making an angle "\(\alpha\)" with the \(x\)-axis, and velocity magnitude of "\(U = \sqrt{u^2 + v^2}\)". The complex potential will be (appendix E):

\[
W(\zeta) = U\zeta e^{-i\alpha}
\]  

(2.55)
A simple mathematical description of a vortex flow can be done considering it as a point vortex, and the complex complex potential will be (appendix E):

\[ W(\zeta) = -\frac{i\Gamma}{2\pi} \ln(\zeta - \zeta_0) \]  

(2.56)

With "\( \zeta_0 \)" the position of the center of the vortex and "\( \Gamma \)" a circulation (positive for rotation in the counter clockwise direction). Clearly, if the vortex is at the origin then:

\[ W(\zeta) = -\frac{i\Gamma}{2\pi} \ln\zeta \]  

(2.57)

From the circle theorem (appendix F), if a cylinder of radius "\( A \)" is immersed in a flow defined by a complex potential \( f(\zeta) \), the resulting flow will satisfy the following complex potential:

\[ W(\zeta) = f(\zeta) + \overline{f}\left(\frac{A^2}{\zeta}\right) \]  

(2.58)

Where \( \overline{f}(z) \) indicates the complex conjugate of the function \( f(z) \), which is obtained by taking the complex conjugate \( \overline{f}(\overline{z}) \) of \( f(z) \) and replacing \( z \) by \( \overline{z} \) (see appendix F).

A circulation around a cylinder can be expressed as a vortex located at the center of the cylinder. Consequently, if the same cylinder with circulation \( \Gamma \) is immersed in an uniform parallel flow with its center at the origin of the coordinate system, the complex potential becomes of the flow similar to fig.2.5b will be:

\[ W(\zeta) = U\zeta e^{-i\alpha} + \frac{UA^2e^{i\alpha}}{\zeta} - i\frac{\Gamma}{2\pi} \ln\zeta \]  

(2.59)

The circulation around the cylinder (\( \Gamma \)) can be tuned in order to satisfy the Kutta condition at the sharp trailing edge. Tangential flow separation at a sharp edge in the physical plane corresponds to a finite value of \( "dw/dz" \) at this point. Due to the singularity of \( "d\zeta/dz" \), this implies that \( dW/d\zeta \) should vanish: \( dW/d\zeta = 0 \) at \( \zeta = A \). Substituting in eqn.2.59 gives:

\[ \Gamma = -4\pi AU \sin(\alpha) \]  

(2.60)

The description for the flow \( (w(\zeta) = W(\zeta(\zeta))) \) around the flat plate (fig.2.5) can then be done substituting the Zhukhovski transformation \( (\zeta = \zeta(\zeta), \text{eqn.}2.52) \) in the complex potential \( (W(\zeta)) \) of the flow around the cylinder (eqn.2.59). Using Blasius theorem (eqn.2.50) the force on the flat plate may then be found. The flow obtained above corresponds to a steady flow around a wing when there is no flow separation at the upstream (leading) edge, and there is flow separation at the downstream (trailing) edge. This approximation should be valid at very low Strouhal numbers.

For further reference we can also consider the flow induced by a parallel flow and a vortex placed outside the cylinder. If a vortex "\( \Gamma \)" is at a position \( \zeta_0 \) from the center of the cylinder, using the circle theorem the complex potential becomes:

\[ W(\zeta) = U\zeta e^{-i\alpha} + \frac{UA^2e^{i\alpha}}{\zeta} - i\frac{\Gamma}{2\pi} \ln\zeta - i\frac{\Gamma}{2\pi} \ln(\zeta - \zeta_0) + i\frac{\Gamma}{2\pi} \ln\left(\frac{A^2}{\zeta} - \zeta_0\right) \]  

(2.61)

Such potential could be used for a description of the unsteady flow around the blades of the rotor at high Strouhal number. Appendix K describes the evolution of the vortex shed while close to a sharp edge. Similar, but more rigorous description is done by Cortelesi [26]. For further positions, numerical integration should be used in order to describe the vortex evolution. Flow visualization and a description of the unsteady vortex evolution is given for a setup simulating a insect wing [12], [13]. From this complex potential, the flow around the flat plate may be described more accurately, with the force calculated from the integration of the unsteady Bernoulli’s equation (eqn.2.19) over the surface.
2.2.5 Aerodynamic forces

The resultant force on the blades will have two causes, the skin friction and the integral of the pressure along surface of the blades. For practical reasons, the resultant force is decomposed in two components, one perpendicular and other parallel to the undisturbed flow relative to the body. Airfoils are designed to maximize the perpendicular force, called lift force \( L \), and minimize the parallel force, called drag force \( D \). The lift "L" keeps the airplane in the air, being a force perpendicular to the motion and does not perform work. The drag "D" retards the plane and is the cause of the flight expenses. A simple expression for the lift force for irrotational, incompressible, two dimensional steady flows can be found using Blasius theorem (appendix G). Let \( \rho \) be the fluid density, \( U \) the fluid velocity and \( \Gamma \) the circulation, the lift force is found from Kutta-Zhukovski theorem (appendix G):

\[
L = -\rho U \Gamma
\]

(2.62)

Usually the lift and drag forces are expressed in terms of a nondimensional quantity, introduced in subsection 2.2.2 as the lift coefficient \( C_L \) and the drag coefficient \( C_D \):

\[
|\bar{L}| = C_L \frac{1}{2} \rho U^2
\]

(2.63)

\[
|\bar{D}| = C_D \frac{1}{2} \rho U^2
\]

(2.64)

Typical measurements for the lift coefficient of an airfoil as a function of the angle of attack is shown in fig.2.6. Above a critical angle of attack \( \alpha_0 \), when the lift coefficient begins to decay with the angle of attack, the wing is said to have stalled \( (\alpha > \alpha_0) \). The stall happens due to the flow separation on the upper part of the airfoil, similar to what is shown in fig.2.7c. This would decrease the pressure difference between the upper and lower surfaces, main contribution for the lift force. On the other hand, the drag force keeps increasing rapidly for higher angles of incidence. The straight dotted line in the figure expresses the lift coefficient if, for some reason, the flow would continue not separating even when \( \alpha > \alpha_0 \). This way the potential flow theory could still be used.

Figure 2.6: Typical measurements of the lift coefficients.

Fig.2.7 shows a sketch of the actual blades of the rotor, immersed in a flow of uniform velocity \( U \) and angle of incidence \( \alpha \). Figures a, b and c shows the different perturbations of the flow with the blades immersed. The blades have a rounded leading edge and a sharp trailing edge, with the resultant aerodynamic force represented by "\( \vec{F} \)".

Fig.2.7a shows an ideal inviscid flow, without satisfying the no slip condition over all surface. The flow turns around both edges of the blades and continues downstream without generation of any circulation \( \Gamma \). Such an ideal flow will have no steady resultant force applied at the blades, as follows from Blasius theorem (eqn.2.50) and the absence of friction forces. In fig.2.7b the flow separates only at the sharp edge. The average velocity on the upper part will be higher than on the lower part of the blade. As follows from Bernoulli’s equation (eqn.2.19) this results in a considerable force due to the pressure difference. In fig.2.7c, the flow separates on the sharp edge and on the upper side of the
blade, resulting in a large wake downstream the blade, and the consequent resultant force shown in the figure.

The separation on the top of the blade surface depends mostly on the Reynolds number and on the angle of incidence. In some conditions, the flow can separate from the surface, reattaching in position downstream the flow. Such phenomenon creates a "separation bubble", changing considerably the flow pattern (fig.2.8).

Figure 2.8: Flow separation and later reattachment on the blades surface (separation bubble).

The actual flow around the blades is very complex, and we will use simplified models. The model to calculate the resultant force, described next chapter, considers the blades as a flat plate, which can also exhibit the profiles shown in fig.2.7. Such assumption simplifies considerably the mathematical equations involved and gives reasonable results.
Chapter 3

Models

3.1 Basic flow models

3.1.1 Steady potential flow model

A first basic model for the blades is obtained by considering the blades of the rotor as flat plates, with flow separation imposed only at the trailing edge (Kutta condition). The complex potential for the flow around a cylinder is described in subsection 2.2.4 (eqn.2.59). For a flow velocity \( U \), a cylinder of radius \( A \) with its center coinciding with the origin of the \( \zeta \)-plane, eqn.2.59 becomes:

\[
w(\zeta) = U\zeta e^{-i\alpha} - \frac{i\Gamma}{2\pi} \ln \zeta + \frac{UA^2 e^{i\alpha}}{\zeta}
\]  

(3.1)

And therefore:

\[
\frac{dw(\zeta)}{d\zeta} = Ue^{-i\alpha} - \frac{i\Gamma}{2\pi \zeta} - \frac{UA^2 e^{i\alpha}}{\zeta^2}
\]  

(3.2)

Using the Zhukhovski transformation (eqn.2.52), the current flow may be mapped into the flow around a flat plate, as shown in fig.2.5. The circulation may be calculated as in eqn.2.60, and gives \( \Gamma = -4\pi AU \sin \alpha \). The force on the plate due to a steady flow can be found using Blasius theorem (appendix G), which is the integration of the pressure along the flat plate surface. The evaluation can be done explicitly or using the general Kutta-Zhukhovski lift theorem (appendix G), which states:

\[
L = -\rho U T
\]  

(3.3)

With "L" the the lift force per unit length of the blades in the direction parallel to the flow plane \((x,y)\) and perpendicular to the undisturbed flow. For this specific case we find using eqn.2.60:

\[
L = 4\rho U^2 \pi \sin \alpha
\]  

(3.4)

Since the undisturbed flow makes an angle \( \alpha \) with the x-axis (fig.3.1), is clear that the force "\( F_x \)" in the direction of the plate will be:

\[
F_x = -L \sin \theta = -4\pi \rho AU^2 \sin^2 \alpha
\]  

(3.5)

While the force "\( F_y \)" normal to the plate will be:

\[
F_y = L \cos \theta = 4\pi \rho AU^2 \sin \alpha \cos \alpha
\]  

(3.6)

The force on the plate is calculated in a two dimensional theory per unit height, and if the plate has length "\( T \)" \((T = 4A)\) and height "\( H \)"; then one side of the surface area will be the length times the height \((S = TH)\). The force becomes:

\[
F_x = -S\pi \rho U^2 \sin^2 \alpha
\]  

(3.7)

\[
F_y = S\pi \rho U^2 \sin \alpha \cos \alpha
\]  

(3.8)
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There will be a moment acting on the flat plate, since the pressure distribution is not symmetrical relative to the center. This torque which is normal to the rotation of axis of the rotor will have no consequence on the motion, therefore no attention has been given to it.

**Edge force with net circulation around the flat plate**

It is clear that the resultant force will have a component parallel to the flat plate surface, $F_x$, and another component perpendicular, $F_y$. Since the resultant force is only due to the pressure distribution, there must be a force at either edges of the flat plate. Physically, this edge force can be explained by the infinite velocity necessary for the flow to turn around the leading edge, resulting in an infinitely negative pressure (eqn.2.19). Although the thickness of the plate tends to zero, a finite force will act at the leading edge, directed outwards and parallel to the plate. To calculate this force, consider a very small circular contour $\epsilon$ around the edge on the flat plate (fig.3.2).

**Figure 3.2:** Force on the leading edge with kutta condition imposed on trailing edge. The domain of radius "$\epsilon$" used to calculate the edge force is indicated.

The leading edge position is $z_0 = -2A$, which corresponds to $\zeta_0 = -A$ on the $\zeta$-plane. From eqn.3.1, the velocity around the contour in the $\zeta$-plane, with $\zeta_0 \to -A$, becomes:

$$ \frac{dw(\zeta)}{d\zeta} = -4U \sin \alpha $$  \hspace{1cm} (3.9)

Using the Zhukhovski transformation (eqn.2.52) close to the edge, $z_0 \to -2A$, the value of $d\zeta/dz$ will be very large, very close to:

$$ \left[ \frac{d\zeta}{dz} \right]_{\text{edge}} = \frac{1}{2} + \frac{z}{2\sqrt{(z-2A)(z+2A)}} = \frac{i\sqrt{A}}{2\sqrt{z+2A}} $$  \hspace{1cm} (3.10)

The value of $dw(z)/dz$ close to the leading edge of the plate will then be:

$$ \left[ \frac{dw(z)}{dz} \right]_{\text{edge}} = \left[ \frac{dw(\zeta)}{d\zeta} \right]_{\text{edge}} \left[ \frac{d\zeta}{dz} \right]_{\text{edge}} = \frac{2U\sqrt{A}}{\sqrt{z+2A}} \sin \alpha $$  \hspace{1cm} (3.11)
The force on the edge can be found evaluating Blasius theorem around the closed contour $\gamma$, such that eqn.3.11 is valid. This gives:

$$F_{x0} - iF_{y0} = 2iAU^2 \rho \sin^2 \alpha \lim_{\epsilon \to 0} \oint_{\gamma} \left( \frac{1}{z - (-2A)} \right) dz$$

(3.12)

Since $z_0 = -2A$ is the only singularity, from Cauchy integral theorem:

$$\oint_{\gamma} \left( \frac{f(z)}{z - z_0} \right) dz = f(z_0)2\pi i$$

(3.13)

And from eqn.3.12, yields:

$$F_{x0} = -4\pi \rho AU^2 \sin^2 \alpha$$

(3.14)

As expected, this corresponds exactly to eqn.3.7. Therefore, the force parallel to the plate is only due to the force on the edge ($F_x = F_{x0}$), while the pressure distribution along the horizontal surface results into the normal force $F_y$ (eqn.3.8).

Due to the Kutta condition at the downstream edge ($z = 2A$), the velocity is not singular, so that the edge force at that side vanishes.

**Edge force without net circulation around the flat plate**

For later use, we consider in a similar way as described above the case of an ideal flow without separation at neither edges. The edge force is calculated for the potential flow around a flat plate without circulation. This is the case where there is no flow separation at all (fig.3.3).

![Figure 3.3: Force on leading and trailing edges for ideal inviscid flow. The domains of radius $\gamma$ used to calculate the edge forces are indicated.](image)

The mapped complex potential using Zhukhovski transformation, as in fig.2.5 without circulation, will be (eqn.3.1):

$$w'(\zeta) = UA^2 e^{i\alpha} + \frac{U A^2 e^{i\alpha}}{\zeta}$$

(3.15)

Close to the leading edge, $\zeta \to -A$, the velocity in $\zeta$-plane becomes:

$$\frac{dw'(\zeta)}{d\zeta} \bigg|_{\text{edge}} = -2U i \sin \alpha$$

(3.16)

Which gives:

$$\frac{dw'(z)}{dz} \bigg|_{\text{edge}} = \frac{dw'(\zeta)}{d\zeta} \bigg|_{\text{edge}} \frac{dz}{d\zeta} = \frac{U \sqrt{A} \sin \alpha}{\sqrt{z + 2A}}$$

(3.17)

Performing the integral, similar to eqn.3.12, gives:

$$F_{x0} = -\pi \rho AU^2 \sin^2 \alpha$$

(3.18)

The edge force at $z_0 = 2A$ may be calculated in a similar way. This gives that the edge force at one edge is compensated by an equal and opposite force on the opposite edge. Comparing with eqn.3.14 one sees that the magnitude of this force is exactly a factor four lower then the edge force on the leading edge of a blade with the flow separation at the downstream edge.
3.1.2 Empirical steady flow model

Fig. 3.4 shows the flow separating at both edges of a flat plate. A complicated flow pattern appears downstream the plate, making potential flow theory inapplicable.

An empirical formula to calculate the resultant force for this case shall be used. It is given by [24], which for angles of incidence greater than 12° gives:

\[
\vec{F}_n = \frac{\rho U^2 S}{2} C_n \vec{F}_n, \text{ with }
\]

\[
C_n = \frac{\sin(\alpha)}{0.283 + 0.222 \sin(\alpha)}, \text{ and }
\]

\[
(3.19) \\
(3.20) \\
(3.21)
\]

With "S" the surface area of one side of the plate (length times height), "U" is the undisturbed velocity of the flow and "\(\alpha\)" is the angle of incidence of the flow on the flat plate. By definition, "\(\vec{F}_n = (0, 1, 0)\)" is a unit vector perpendicular to the plate (y direction).

3.1.3 Quasi-steady approximation of edge forces

In some circumstances an unsteady flow may be considered as an infinite sequence of steady flow, which is called quasi-steady state approximation. This way, the description of steady flows may be used on unsteady flows but simply considering a time dependent velocity. This is is valid when the local flow acceleration \(\frac{\partial u}{\partial t}\) is negligible compared with the convective acceleration \(u \frac{\partial u}{\partial x}\):

\[
\frac{\partial u}{\partial t} \ll u \frac{\partial u}{\partial x}
\]

If the flow does not separate close at an edge, the velocity becomes singular at the edge \((u \to \infty)\). "\(\partial u/\partial x\)" is infinitely large because both "\(u \to \infty\)" and "\(dx \to 0\)". Then the local acceleration will indeed be negligible compared with the convective acceleration. This way, the quasi-steady approximations will be valid close to this edge. The edge forces calculated for steady flows in the previous sections will be the same as the edge forces in an unsteady flow with the same complex potential "\(w(z)\)" (subsection 2.2.4).

3.1.4 Local flow separation

For some conditions, the flow might separate from the surface and attach again in a different position. Around an airfoil, such phenomenon might occur close to the rounded edge, or close to the sharp edge. The local separation on the rounded edge was shown in fig. 2.8. A discussion of the second situation: unsteady flow separation at the sharp edge, will be done here. We limit ourselves to high Strouhal numbers so that the vortex formed by flow separation remains close to the edge.

In fig.3.5 the local separation for two directions of the flow is represented for a flat plate ((a) and (b)), and for a blade ((c) and (d)). On the upper part of fig.3.5a and (b), the local separation of the flow around the edge is emphasized, while for the lower part, a general view of the flow is shown. We define here the leading and the trailing edge by means of their position relative to the regular steady...
flow direction. This way, the leading edge corresponds to the rounded edge of the blades and the trailing edge corresponds to the sharp edge of the blades.

For the flat plate, in the case of fig.3.5a, we shall assume the local flow separation only around the trailing edge. As a consequence, there will be a finite velocity at the sharp edges, and consequently no edge force. Around the leading edge we suppose the flow able to follow the surface, resulting in infinite velocity and the consequent edge force (eqn.3.18). As discussed in the previous paragraph this edge force can be calculated from a quasi-steady approximation.

![Figure 3.5: Local flow separation at sharp edges for a flat plate ((a) and (b)) and for a blade ((c) and (d)).](image)

The flow separation generates a vortex close to the sharp edge. This vortex is assumed to have the only effect of removing the pressure singularity and the consequent edge force where it is shed. The flow separation also implies that the boundary layer vorticity is injected into the main flow. We will assume this vorticity of small magnitude and confined in a small region close to the edge. Due to these assumptions, there will be no significant change of the global circulation for the flow around the flat plate (around the blades).

For regions not so close to the trailing edge, the complex potential is given by eqn.3.15 and the (leading) edge force can be calculated from eqn.3.18. If the flow has its direction indicated by fig.3.5b, the same arguments might be used, and the Kutta condition is imposed once again at the trailing edge. The expression for the edge force around the leading edge will also be given by eqn.3.18, and the resultant force originated from the edge force with the flow in both directions may be calculated.

Let us now consider the case of the flat plate placed in a parallel flow with an harmonic time dependence \( U = U_0 \cos(2\pi ft) \). The flow will alternate between the two situations (a) and (b) of fig.3.5. The perpendicular forces to the surfaces are opposite in direction for the two directions of the flow \( F_{y-} \text{ and } F_{y+} \). If the small perturbation around the edge of the potential flow is neglected (a symmetric flow is imposed), then from this simple model, the average over an oscillation period of the perpendicular force will be equal to zero. Consequently, the resultant force on the airfoils may be calculated simply from the calculations of the edge forces.

Fig.3.5 (c) and (d) shows the phenomenon represented on the actual blades used for the experiments. The real thickness and the sharp edge angle of the blades results in clear differences between the flow around the different profiles. The flow around the rounded edge and the equivalent sharp edge of the flat plate are roughly similar. This indicates that the edge force calculated for the flat plate should also be roughly similar with the blades force in the x-direction. On the other side, around the sharp edge and the equivalent sharp edge of the flat plate, there will be a clear difference. The edge force vanishes in the case of the flat plate, while for the actual blades the edge force in the x-direction at the sharp edge is certainly not zero. Due to the slower average flow velocity in fig.3.5d the force \( F_{0z+} \) will be larger then the force when the flow is inverted \( F_{0z-} \).
3.2 Application

3.2.1 Starting conditions

Suppose the flow as described in subsection 3.1.4, fig.3.5. The flow is assumed irrotational everywhere, except for regions close to one of the edges. The resultant average force will have only components due to the edge force, since the mean perpendicular forces to each half of the acoustic oscillation period are equal but of opposite direction. Let the angle of incidence be $\pi/4$, which is the actual angle of incidence of both rotor blades when it is at rest. Then the edge force (eqn.3.18) assuming the quasi-steady approximations becomes:

$$F_{x0} = \frac{\pi \rho S}{4} \sin^2 \left(\frac{\pi}{4}\right) U^2$$

(3.23)

Consider an average radius $(r_{av})$ at which the force is applied, and assume the acoustic velocity oscillations defined by $U = U_0 \sin(2\pi t/T_{ac})$, where $U_0$ is the maximum velocity amplitude and $T_{ac}$ is the acoustic oscillation period. The average torque for one acoustic period $(\bar{T}_{ac})$ on the direction of the rotation will be simply:

$$\bar{T}_{x0} = \frac{\pi \rho r_{av} S}{8} \sin^3 \left(\frac{\pi}{4}\right) U_0^2$$

(3.24)

3.2.2 Rotation behavior

Let a group of plates surround a rotor, similar to the actual turbine meter with the blades. The angle of incidence $(\alpha)$ changes with the acoustic velocity and rotation of the rotor (see fig.3.6).

$$\alpha \approx \Theta - \Lambda(t)$$

(3.25)

$$\Lambda(t) = \arctan \left( \frac{U_{rav}}{u(t)} \right)$$

(3.26)

Symmetric acoustic flow perturbation

Suppose the acoustic flow around the flat plate as represented in fig.3.7. The flow separates from both edges of the flat plate and for both directions.

The force on the plate may be calculated from the empirical formula shown in subsection 3.1.2 (eqn.3.19 to eqn.3.20).
Suppose the acoustic velocity to half the acoustic oscillation period equal to a constant \("u_0"\), while for the other half of the acoustic period equal to \("-u_0"\). From the equations in subsection 3.1.2, the rotor equation of motion (eqn.2.1) becomes:

\[
0 = \frac{r_{av}\rho S}{4} (C_{n1} - C_{n2}) (|\Omega r_{av}|^2 + |u_0|^2) + T_{fr}
\]

\[
C_{n1/n2} = \frac{\sin(\Theta + \Lambda)}{0.283 + 0.222 \sin(\Theta + \Lambda)}, \text{ and}
\]

\[
\Lambda = \arctan\left(\frac{\Omega r_{av}}{u_0}\right)
\]

Suppose that no friction acts on the rotor, \(T_{fr} = 0\), then eqn.3.27 reduces to:

\[
\sin(\Theta - \Lambda) - \sin(\Theta + \Lambda) = 0
\]

The solutions for the equation are \(\Theta = \pi/2, \Theta = 3\pi/2, \Lambda = 0\) or \(\Lambda = \pi\). The conditions of \(\Theta\) represents the plate at the vertical position, while the conditions of \(\Lambda = 0\) or \(\Lambda = \pi\) represents the state when the rotor is at rest, \(\Omega = 0\) (\(\Lambda = \arctan(\Omega r_{av}/u)\)). With the plate at the vertical position the model cannot predict any force to accelerate the rotor, since the resulting force is always perpendicular to the plate. When the rotor is at rest, the incidence of the acoustic flow is symmetrical to the plates, resulting on forces of equal magnitude but opposite directions and consequently no mean acceleration. In fig.3.8, the resulting force coefficient for this modelling \((C_{n1} - C_{n2})\) is shown as a function of \(\Omega r_{av}/u_0 = \tan(\Lambda)\), for the specific case of \(\Theta = \pi/4\) (equal to the angle of actual rotor blades). It is clear that once the blades are moving the model predict a resulting force that retards the rotor, bringing it to rest. This model certainly cannot explain the ghost counts, which the rotor is accelerated from rest and kept at steady rotation only by the acoustic field.

Figure 3.8: Resultant force coefficient for symmetric flow perturbation.
Asymmetric acoustic flow perturbation

Consider a second situation for a symmetric oscillating flow around the rotor. We now consider an asymmetric perturbation of the flow around the flat plates corresponding to the rotor blades (fig.3.9). In one direction the acoustic flow separates only at the trailing edge of the flat plate, which behaves as an airfoil (fig.3.9a). On the opposite direction, the flow separates on both edges (fig.3.9b).

![Figure 3.9: Asymmetric flow perturbation.](image)

For the first case the force on the plate may be calculated from eqn.3.7 and eqn.3.8, derived in subsection 3.1.1. The second case we may use the empirical formula shown in subsection 3.1.2. Suppose, once again, the rotor with steady rotation and the acoustic flow oscillations with velocity equal to "u_0" and "-u_0" for each half acoustic period. The rotor equation of motion (eqn.2.1) becomes:

\[
0 = \frac{\tau_{av}}{4} (C_L \cos(\Lambda) - C_n \cos(\Theta)) (|\Omega r_{av}|^2 + |u_0|^2) + T_f
\]

(3.31)

\[C_n = \frac{\sin(\Theta + \Lambda)}{0.283 + 0.222 \sin(\Theta + \Lambda)}, \text{ and} \]

(3.32)

\[C_L = 2\pi \sin(\Theta - \Lambda) \]

(3.33)

\[\Lambda = \arctan \left( \frac{\Omega r_{av}}{u_0} \right) \]

(3.34)

If the friction is neglected then \(C_L \cos(\Lambda) - C_n \cos(\Theta) = 0\). Fig.3.10 show a graphic for such coefficient as a function of \(\Omega r_{av}/u_0 = \tan(\Lambda)\), with \(\Theta\) equal to \(\pi/4\). It is seen that if the rotor is at rest and the acoustic flow is imposed, a positive resultant force acts on the rotor. It will accelerate the rotor until \(V_{bl}/u_0\) is approximately 0.58, equivalent to \(\Lambda = 30^\circ\). After this point, according to the theory, there will be a retarding force.

![Figure 3.10: Resultant force coefficient for asymmetric flow perturbation.](image)
If the friction is not neglected, then from eqn.3.31 one gets:

\[
\begin{align*}
u_0 &= \sqrt{\frac{-4T_f}{r_{nu}\cdot A_s(\tan^2(\Lambda) + 1)f(\Lambda)}} \\
f(\Lambda) &= 2\pi \sin(\Theta - \Lambda) \cos(\Lambda) - \frac{\sin(\Theta + \Lambda)}{0.283 + 0.222 \sin(\Theta + \Lambda)}
\end{align*}
\] (3.35) (3.36)

The results for such a model are given in subsection 5.2.3 (fig.5.18), where they are compared with the experimental data. The ratio of the blade velocity with the acoustic velocity varies from zero until the frictionless limit \((V_{bl}/u_0 = 0.58)\). This because the friction becomes negligible compared to the high aerodynamic forces at high acoustic velocities.
Chapter 4

Experimental setup

4.1 Setup

To perform the experiments two distinct setups have been used, which were both designed and built by Jan Mulder (Gasunie-Research, Groningen). Previous measurements of the small setup are presented in a report of Mulder [27]. The larger setup was first investigated during this research. Some important features of both setup is shown in the table below (fig.4.1) while a scheme of both setup is shown in fig.4.2 (small setup) and fig.4.3 (large setup). Appendix J shows photographs of both setups.

<table>
<thead>
<tr>
<th>Set up</th>
<th>Pipe Diameter</th>
<th>Length</th>
<th>Width</th>
<th>Material</th>
<th>Gas meter Outer Diameter</th>
<th>Inner Diameter</th>
<th>Loudspeaker Displacement Limit</th>
<th>Diameter</th>
<th>Power limit</th>
<th>Static Pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td>0.1m</td>
<td>1m</td>
<td>1.8mm</td>
<td>Wood</td>
<td>0.108m</td>
<td>0.07m</td>
<td>0.3m</td>
<td>13mm</td>
<td>0.1m</td>
<td>50W</td>
</tr>
<tr>
<td>Large</td>
<td>0.31m</td>
<td>2m/6m</td>
<td>11mm</td>
<td>Steel</td>
<td>0.3m</td>
<td>0.2m</td>
<td>0.9m</td>
<td>40mm</td>
<td>0.27m</td>
<td>500W</td>
</tr>
</tbody>
</table>

Figure 4.1: Table of setup features

4.1.1 Small setup

In the small setup, the gas turbine meter (Instromet-G250) is placed between two plastic pipes, PVC of diameter d=100mm and length L=1.8mm. A loudspeaker (Visaton-w100s) is fixed at the end of one pipe, while a closing wooden plate is fixed at the opposite end of the other pipe. Four orifices (13mm diameter) were made in order to install the dynamical pressure transducers. Their positions were distributed along both pipes, so the calculations of the acoustic velocity from the acoustic pressure (section 4.5) could be checked. One of the transducers was at the center of the closing wooden plate, opposite to the speaker, while the others were along the pipes, at distances from the speaker of 0.03m, 1.33m and 1.8m. The measurements were made at atmospheric pressure and the junctions between the pipes and the gas turbine meter were not perfectly sealed.

4.1.2 Large setup

Three important differences between the large setup and the small apparatus should be emphasized: firstly the larger dimension of the larger setup. Secondly, the possibility to change the mean static pressure within the larger setup, changing consequently the air density. At last, both the position of the loudspeaker and piston could be changed, changing also the resonance frequency of the large setup. These differences allow studies of the influence of the gas density and increase the range of the acoustic field frequency and amplitude for which we observe the phenomena of ghost counts.

A scheme of the large setup is shown in fig.4.3. The gas turbine meter is connected between two pipes, with equal diameter of 300mm and lengths of 2m and 6m. The end of each pipe is sealed with a closing plate. Both the pipes and the closing plates are steel made. All the connections are sealed, being able to support the 7bar air pressure difference that is used in the experiments. The closing
plates have two holes, one for inserting a shaft and another for placing either the transducer or the high-pressure air source hose. The shafts are used to move the loudspeaker (Peerless xls-10), placed at the longer pipe, and the piston, placed at the shorter pipe. This makes possible to change their position and consequently modify the resonance frequency. Although, most of the measurements were made without the piston, with a transducer placed at the closed end. This is due to three reasons: The larger pipe length without the piston, the advantage to eliminate a possible leak between the piston and the pipe that could disturb the acoustic field and, at last, a possibility to place the dynamical pressure transducer at the closing steel plate. Three holes are made to fix the pressure source hoses, two at the closing lids, and one close to the gas meter, so that the pressure can rise more uniformly inside the pipe. Nine holes were made for placing the dynamical pressure transducers along the pipes, from the closing lid of the smaller pipe at a distance of 0m, 0.51m, 0.84m, 1.17m, 1.63m, 2.95m, 3.41m, 3.74m, and 4.07m.

4.2 Measurements equipments

The acquisition of the signals was all made by means of a twelve input channel data acquisition device (LMS Roadrunner compact).

A harmonic voltage signal is applied at the loudspeaker in the following way. A signal generator is connected to a power amplifier (Bruel&Kjaer-2706) from which the signal is applied at the loudspeaker. The amplitude and the frequency of the signal could be controlled independently, being also recorded by the LMS roadrunner device.

The measurements of the acoustic pressure was done with piezo-electric transducers (Kistler-7031), installed at different positions of the pipe. The transducers were connected to a set of charge amplifiers (Brue&Kjaer-2635), which were then connected to the data analyzer. The acoustic pressure signal registered were approximately harmonic, being considered as such for the calculations.

The blades frequency is measured using a standard measuring equipment made by Instromet. The equipment is fixed at the gas turbine meter and each time one blade passes through a certain point the device creates a voltage pulse that is recorded by the data analyzer. The blade passage frequency
recorded is then converted to rotor rotation frequency (\(\Omega\)) by means of a simple calculation with the data analyzer.

The data analysis required was made with the Matlab program and with software provided by LMS Skalar instruments (Cada-PC v3.87.18). The procedure for the different data handling are described at each experiment individually, in the following sections.

### 4.3 Rotor features

Special attention should be given to the rotor, which is the main object of the research. Detailed description of the geometry of the rotor (including the blades) as well as the description for the measurements of static and dynamic friction will be given.

#### 4.3.1 Geometry

A photograph of a rotor similar to the one used in the small setup is shown in fig.4.4. The material of the rotor is aluminium. The diameter of the rotor is such that a very small gap is left between the blades top surface and the gas meter.

![Small rotor photograph.](image)

**Figure 4.4:** Small rotor photograph.

![Blades scheme of both apparatus.](image)

**Figure 4.5:** Blades scheme of both apparatus.

<table>
<thead>
<tr>
<th>Gas meter type</th>
<th>Diameter size</th>
<th>Length size</th>
<th>Number of blades</th>
<th>Blades height ((h))</th>
<th>Blades thickness (d)</th>
<th>Blades bottom length (Lb)</th>
<th>Blades top length (Lt)</th>
<th>Moment of inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>G250</td>
<td>0.1m</td>
<td>0.3m</td>
<td>16</td>
<td>14.5mm</td>
<td>1.6mm</td>
<td>28.9mm</td>
<td>34.0mm</td>
<td>1.35·10^{-4} kgm²</td>
</tr>
<tr>
<td>G2500</td>
<td>0.3m</td>
<td>0.9m</td>
<td>24</td>
<td>41.5mm</td>
<td>3.0mm</td>
<td>42.1mm</td>
<td>45.9mm</td>
<td>1.07·10^{-4} kgm²</td>
</tr>
</tbody>
</table>

**Figure 4.6:** Table of rotor and blades features

The blades around rotor are represented in fig.4.5, with its dimensions for both setups given on the following table (fig.4.6). The angle that the blade makes with the rotor axis varies some degrees
from the bottom of the blades to the top, explaining the difference of the bottom length and the top length. For the both apparatus this angle varies from 40° to 49°. The gas turbine meters, as well as the rotors, are produced by Instromet, and the main features are presented at the table on fig.4.6. The exact values for the rotors moment of inertias are given by Instromet and a rough calculation of a 15cm rotor is given at the appendix A.

4.3.2 Dynamic friction

In order to measure the torque on the rotor, the blades passing frequency is recorded as a function of time. A simple calculation with the data analyzer device transforms the blades frequency into the rotor angular frequency ($\Omega$). From the acquisition of the time evolution of the angular frequency, the resultant torque on the rotor may be found using the relation $T(\Omega) = I d\Omega/dt$ (eqn.2.1).

The rotor is accelerated by means of the acoustic field. After the rotor has reached a steady velocity, the loudspeaker is turned off and the angular frequency ($\Omega$) decay with time is registered. When there is no acoustic field, there will be only a torque on the rotor due to the friction (eqn.2.3). In order to find the friction torque, the measured points of the angular frequency as a function of time are fitted, assuming a fourth order dependence with time, $\Omega = A_0 t^4 + A_1 t^3 + A_2 t^2 + A_3 t + A_4$. For the range of the angular frequencies studied, such assumption gives the necessary precision. This due to the very low residuals of the fitted curves, and higher order fitting would (almost) not change the results. After the curve is fitted, the coefficients are found and the torque is calculated from $T = I(4A_0 t^3 + 3A_1 t^2 + 2A_2 t + A_3)$.

Some results for the small and the large rotors are given in fig.4.7 and fig.4.8, with the torque plotted respective to the blades average velocity (velocities at the middle of the blades). For the small setup the graphic shows two different measurements at the same conditions, while the measurements for the large setup are at different absolute pressure.

![Figure 4.7: Small apparatus torque decay without acoustic field.](image)

In fig.4.8 is clear that for low rotor velocities, the measurements shows that the dynamic torque have approximately a linear dependency with blades velocity. For higher values of the blades velocity, the torque looses the linear dependency and some higher order dependency prevails. This is explained by the theory presented in subsection 2.1.2, where four dependencies of the friction relative to the rotor velocity are identified: A square dependency (due to aerodynamic pressure forces, eqn.3.7 and eqn.3.8), a dependency of $\Omega^{3/2}$ (due to the skin friction with air, eqn.2.7), a linear dependency (due to the friction with oil at the bearing, eqn.2.8) and a constant dependency (due to the friction with the shaft, eqn.2.10). Therefore, at low velocities, when higher orders terms than one are neglected,
CHAPTER 4. EXPERIMENTAL SETUP

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Torque to four different pressures

Figure 4.8: Large apparatus torque decay without acoustic field.

the dominant contribution of the friction is with the shaft and with the oil at the bearing. In the limit of the blades velocity equal to zero, all the dynamic torques in fig.4.7 and fig.4.8 should have the same value, equal to the constant friction with the shaft. There are probably three causes for this inconsistence. One is the high variations of the friction when the rotor is at very low velocities. Fig.4.9 shows a typical measurement of the large apparatus just before stopping, where the variations are clear. Another problem is due to the fitting procedure, which larger ranges of the angular frequency fitted gives higher residuals, so higher inaccuracy of the fitted curve. And at last, the angular frequency is acquired from discrete values, at each time one of the blades pass through a certain position. The last value registered of the average blades velocity is very close to zero, but not exactly zero. In fig.4.9, as well as all the measurements, this slight difference was subtracted from the original data.

Figure 4.9: Variations of average blades velocity decay at very low speed.

In fig.4.10 the torques for a fixed velocity are plotted versus the air densities. The extrapolation
to when the air density is equal to zero would give the friction values without the contribution of air, only from the bearing. As seen in subsection 2.1.2 (eqn.2.3), this friction should be equal to the sum of the friction with the shaft and with the oil. Larger values of the rotor velocity indicates slightly larger values of the rotor friction with the shaft. This indicates the contribution of the oil shear stress with the shaft, which is proportional to the rotor velocity. Therefore, the main contribution of friction from the bearing should come from the constant dynamic friction of the rotor with the shaft, $T_{\text{dyn}} = \mu_{\text{dyn}} N r$ (eqn.2.10). The value for the dynamic friction ($T_{\text{dyn}}$) is approximately $6 \cdot 10^{-6} \text{Nm}$ for the small apparatus and $0.9 \cdot 10^{-4} \text{Nm}$ for the large apparatus.

![Dynamic torque extrapolation for fixed velocity](image)

Figure 4.10: Friction torque dependence on air density relative to different blades velocity: $\Omega r = 0.18 \text{m/s}$ (o); $\Omega r = 0.12 \text{m/s}$ (+); $\Omega r = 0.08 \text{m/s}$ (−)

### 4.3.3 Static friction

Fig.4.11 shows the large turbine meter, with simple equipments in an attempt to measure the static friction. The weight of a small piece of tape is measured. This tape is fixed at a known radius on the rotor ($r$). The tape will induce a torque on the rotor, which can be calculated from the measurements of the weight and position of the tape. The critical value of the torque for which the rotor begins rotating indicates the maximum of the static friction of the torque of the rotor ($T_{\text{ost max}}$). According to the angle that the radial direction of the tape makes with the horizontal plane ($\theta$), the torque applied at the rotor can easily be changed, $T = Pr \cos \theta$ ("$T$" the torque at the rotor, "$P$" the weight of the tape and "$r$" the average radius which the torque is applied). This makes it easier to find ","($T_{\text{ost max}}$). To hold and release the rotor we use a photograph shooter, which enables a fast release of the rotor without significant disturbance.

The critical static torque ($T_{\text{ost max}}$) measured was $5.6 \cdot 10^{-6} \text{Nm}$ for the small apparatus and $1.0 \cdot 10^{-4} \text{Nm}$ for the large apparatus. Comparing with the constant portion of the dynamic friction ($T_{\text{dyn}}$), the values of the static friction ($T_{\text{ost}}$) gives slightly lower values for the small apparatus and slightly higher values for the large apparatus. Inaccuracy of the experiments is probably due mainly to the difficulty to release the rotor without disturbing it, giving a lower value than what it should be. For the small apparatus the results are clearly not accurate enough, since the static friction values are expected to be higher than the dynamic friction. The experimental procedure does not have the necessary precision to distinguish the difference between the static and dynamic friction with the shaft, but confirms the order of magnitude found from the dynamic measurements.
4.4 Resonance modes

In order to observe the resonance modes of both setups, a constant sinusoidal voltage signal is applied at the loudspeaker. The frequency is scanned linearly between a wide range of 1Hz up to 360Hz to the small setup, and from 1Hz up to 200Hz to the larger setup. The change is made continuously at a time rate of 1Hz per 20s. Fig.4.12 and fig.4.13 shows the acquired pressure for the transducer at the closed end (maximum of pressure) as a function of the frequency scanned. Fig.2.2 shows a diagram of the maximum acoustic velocity for the first three modes of resonance.

For both setups it is only possible to rotate the rotor at the first and third resonance peak shown in fig.4.12 and fig.4.13. The fifth mode, even though it gives a high acoustic velocity at the blades position, is not able to induce the ghost counts for the available power at the loudspeaker. The rotor does not rotate at resonance frequency corresponding to entire multiples of the wavelength inside the
4. EXPERIMENTAL SETUP

Acoustic pressure at cicsec end transducer.

Figure 4.13: Resonance modes of large setup.

The acoustic velocity at the center of the setup, where the gas meter is located, has a value close to zero.

The quality factor is defined here as equal to the ratio of the resonance peak frequency to the resonance bandwidth between the $p'/p'_{\text{max}} = 1/√2$ points around the peak. The approximate value of the quality factor for the first and third peaks are: 6 and 25 for the small apparatus, and 15 and 40 for the large apparatus at 8bar.

4.5 Acoustic velocity calculations

The measurements of the acoustic pressure had offset and broadband noise, which would depend on the transducer and the charge amplifier used. Since the acoustic field frequency is known, a band pass filtering is done using the data analyzer. Tests of the final data shows that it has been cleaned to a satisfactory level (less then 1% of noise level). The acoustic velocity is then found from the acoustic pressure measured at one transducer, with the expressions for the calculations at appendix 1. One example of the acoustic velocity calculated from various pressure transducers, at various absolute pressure, is shown in fig. 4.14. The variations of the measurements of the acoustic pressure at the resonance frequency are plotted in fig.4.15. From this result, and others not shown, is seen that the deviation for each calculation from one transducers is small. This validates the calculation procedure as well as the data acquisition and analysis. For all the following results, the acoustic velocity is calculated from the transducer placed at the closed end.
Figure 4.14: Example of acoustic velocity at the rotor calculated from various pressure transducers.

Figure 4.15: Overview of acoustic velocity measurements indicating the scatter between results for different transducers.
Chapter 5

Results

5.1 Starting conditions

At a certain value of the acoustic velocity, without mean flow, the rotor begins to rotate. The measurement of this acoustic velocity value required is done the following way: The voltage signal applied at the loudspeaker is kept at a constant frequency \( f' \), while the amplitude is increased very slowly. This can be done either manually or using a special function of the signal generator. At the moment which the rotor begins rotating, the amplitude of the acoustic velocity is determined from the acoustic pressure (sec 4.5). Fig. 5.1 shows some measurements for the small apparatus, with the first mode of resonance on the left, and the third mode on the right.

![Graph showing acoustic velocity vs. frequency for different modes](image)

Figure 5.1: Critical amplitude of the acoustic velocity at which the ghost count starts for different frequencies for the first and third mode of resonance.

For both measurements we see considerable deviations between results at the different frequencies. Such variations are also seen if two consecutive measurements are made at the same frequency. Probable reasons for the deviations are the varying static friction torque from the bearings. The oil and the solid surfaces at the bearings are not perfectly homogeneous. This causes different critical static friction \( (T_{ast})_{max} \), depending on the initial position of the rotor [14]. We cannot exclude the possibility of changes in flow configuration from one experiment to the other. Another source of errors is the acquisition or the acoustic velocities calculations. This might give systematic errors but cannot explain the large observed scatter. The standard deviation for the measurements of the first mode and the third mode are roughly similar, approximately 20% of the mean value.

Similar measurements as shown in fig. 5.1 were made in the large apparatus at different absolute pressures. Four different pressure were measured, 1bar, 2bar, 4bar and 8bar. Fig. 5.2 shows an overview of the measurements of the average of the amplitude of the acoustic velocity required to begin rotation. The results are plotted together for the small and the large apparatus as a function of
the air density \( \rho_{\text{air}} \). The average of the frequencies measured in each mode is taken, and the third mode of resonance has average frequency about three times the first mode. We see that in order to begin rotation the third mode requires an average acoustic velocity amplitude about two times the first. Therefore, in order to reach the same force to begin rotation at higher frequency, one should rise the velocity amplitude. Due to the large deviations of the measurements, the probable differences of the acoustic velocities amplitude within the same resonance mode could not be identified.

![Graph](image-url)

**Figure 5.2:** Average amplitude of the acoustic velocity at which the ghost count starts.

In fig.5.3, the Reynolds number based on the blades length \( \text{Re} = uL/\nu \) is plotted against the air densities, with the velocity \( u \) equal to the average acoustic velocities shown in fig.5.2 \( (V_{ac}) \). We see that higher air densities results in lower significance of the viscous forces relative to the inertial forces. It's also straightforward that the first mode of resonance displays higher significance of the viscous force relative to the third mode. For Reynolds number lower then \( 10^5 \) we expect a laminar flow in the boundary layer [18].

In fig.5.4 the Strouhal number based on the blades thickness \( \text{Sr} = fL/d \) to begin rotation is plotted against the air densities. The Strouhal number, in this case, is a measure of the particles displacement \( u/2\pi f \) relative to the blades thickness \( d \). We see from the graph that increasing the air densities, at the same mode of resonance, increases the Strouhal number to begin rotation. This due to the fact that for the same particles displacement, higher air densities implies higher forces on the blades (eqn.3.3 and eqn.3.19). Thus, less particles displacement (higher Strouhal number) are required to begin rotation.

At fixed air densities, the third mode of resonance displays higher Strouhal number relative to the first mode. While the acoustic field frequency increases about three times, the velocity amplitude increases about two times. This results in an acoustic displacement of about one third less, or Strouhal number of one and a half times more.

The torque to begin rotation is calculated from eqn.3.24, which has the form of \( k\rho u^2 \), with \( k \) a constant, \( \rho \) the air density and \( u \) the flow velocity. The results of the calculations are given in fig.5.5 and fig.5.6 for respectively the first and third mode of resonance. The torques are calculated from the maximum average acoustic velocity in fig.5.2. The calculated values are expected to be equal to the critical static torque given in subsection 4.3.3. For the small apparatus, the torque calculated gives a value of about two times for the first mode of resonance and five times for the third mode. The large apparatus, for the first mode, gives a value of about 20% lower than expected. For the measurement when the air density is 9.6kg/m³, it gives about 40% less than expected. For the third mode of resonance, the calculations gives values approximately 4.5 times. Analysis of these results
CHAPTER 5. RESULTS

5.1.1 Analysis

From the measurements in fig.5.2 the torques on the rotor are calculated and plotted in fig.5.7 and fig.5.8 for both apparatus and both modes of resonance. The end of each line represents the average acoustic velocity amplitude measured and the torque calculated to begin rotation. The critical static torque expected should be around \((T_{\text{ost}})_{\text{max}} = 6 \times 10^{-6} Nm\) for the small apparatus and \((T_{\text{ost}})_{\text{max}} = 10^{-4} Nm\) for the large (subsections 4.3.2 and 4.3.3). The values calculated for the torque and the measured values of the critical static friction should be equal. In fig.5.8, the torque on the rotor is

Figure 5.3: Average Reynolds \((Re = uL/\nu)\) number at which the ghost count starts.

Figure 5.4: Average Strouhal number \((Sr = fd/\nu)\) at which the ghost count starts.

and prediction at field condition are given next subsection.
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Rotor starting conditions - 1st mode

Figure 5.5: Ratio between the calculated critical torque \( T_{cal} \) and critical static torque \( (T_{ost})_{max} \), for the first mode of resonance.

Rotor starting conditions - 3rd mode

Figure 5.6: Ratio between critical torque calculated and critical static torque given for the third mode of resonance.

also calculated for field conditions of natural gas transportation, with static pressure as "\( p_{st} = 60 \text{bar} \)" and the natural gas density as "\( \rho_{gas} = 40 \text{kg/m}^3 \)". The pipe and the rotor geometries are assumed to be similar to the large apparatus. The acoustic velocity amplitude simulated to reach the critical static torque is found to be approximately \( U_{ac} = 0.06 \text{m/s} \), which gives an acoustic pressure amplitude of \( p = \rho_0 c_0 U_{ac}^2 = 10^3 \text{Pa} \) (\( c_0 \approx 400 \text{m/s} \)). Measurements on field conditions shows ghost counts with acoustic pressure amplitude of \( 4 \cdot 10^3 \) [9]. Hence our model can explain the occurrence of ghost counts.

The model in subsection 3.1.1 consider the blades as a flat plate, which ignores the effect of the real
CHAPTER 5. RESULTS

Figure 5.7: Torque calculated by edge force for small apparatus ("o", "+") and maximum static friction torque \((T_{ost})_{max}\).

Figure 5.8: Torque calculated by edge force for large apparatus ("o", "+") and maximum static friction torque \((T_{ost})_{max}\).

blades thickness. The higher frequency of the third mode, and consequent higher Strouhal number, results in the velocity required to begin rotation about two times for the third mode than for the first mode. The model will then give approximately four times the value for the torque applied at the rotor for the third mode (eqn.3.18).
CHAPTER 5. RESULTS

5.2 Rotation behavior

A different kind of measurements studies the steady rotation of the rotor due to the acoustic field. The experiments are performed similar to section 4.4, where the loudspeaker voltage signal is kept at constant amplitude while the frequency is varied. The frequency is swept at a time rate not lower than 1 Hz per 5 min, such that the steady rotation is approached. The signals of the acoustic pressure, the blades velocity and the applied voltage at the loudspeaker are recorded.

5.2.1 Small setup

Fig. 5.9 and fig. 5.10 shows the amplitude of the acoustic velocity at the rotor position and the average blades velocity \(V_{bl} = \Omega r_{av}\), with \(\Omega\) the angular velocity and \(r_{av}\) the average blades radius). The harmonic voltage signal applied to the loudspeaker at the measurement of fig. 5.9 is of 4V peak to peak, while in fig. 5.10 it is of 8V. It is clear that the blades velocity follows the acoustic velocity.

In fig. 5.9 we see that the blades velocity is quite unsteady at low blades velocities. This irregularity of the blades velocity may be due to variations of the friction in the bearings.

![Graph showing rotating behavior for the small apparatus turbine at the first mode of resonance with voltage applied on speaker of 4V](image)

In fig. 5.10 it can be seen some wiggles of the blades velocity with variations of about 2%, while the acoustic field shows no similar oscillation as the blades velocity. The period of the wiggles is dependent on the blades velocity, being larger for smaller blades velocities. For low enough blades velocities the wiggles cannot be seen. The measurement shown in fig. 5.10 has the period of the wiggles of about 780s, equivalent to about 470 complet rotations of the rotor. Such small effect could also be seen at constant acoustic field frequency. In view of the roughness of our research, no serious attempt was done in order to understand nor to predict the effect.

We note that a lower amplitude of the acoustic field corresponds to a larger value of the Strouhal number \((Sr = f d/V_{ac} \approx 0.21)\). The experiments on the starting conditions and the results in fig. 5.9 in comparison with fig. 5.10 seem to confirm a significant Strouhal number effect of the resultant torque.

In fig. 5.11 the blades velocity and the acoustic velocity at the rotor are acquired for the third mode of resonance in the same way as for the first mode. The response of the blades velocity to the acoustic amplitude is less efficient for the third mode then for the first mode, as will be seen more clearly in fig. 5.12. By less efficient we mean that the ratio of the blades velocity relative to the acoustic velocity is lower.

Fig. 5.12 shows the ratio of the blades velocity and the acoustic velocity for the measurements shown in fig. 5.10 and fig. 5.11. The voltage signal applied at the loudspeaker in both experiments are
CHAPTER 5. RESULTS

Figure 5.10: Rotating behavior for the small apparatus turbine at the first mode of resonance with voltage applied on speaker of 8V ($Sr = f_d/V_{ac} \approx 0.11$)

Figure 5.11: Rotating behavior for the small apparatus turbine at the third mode of resonance with voltage applied on speaker of 8V ($Sr = f_d/V_{ac} \approx 0.38$)

the same, and the acoustic velocity amplitude differs about 10%. The first mode presents the ratio about two times the third mode, making evident the less "efficiency" of the acoustic field at higher Strouhal numbers.

The measurements on fig.5.13 are done in a similar way of the measurements for the starting conditions. The frequency of the voltage signal applied on the loudspeaker in kept constant, while the amplitude is changed until close to the limit of the loudspeaker possibilities. The amplification was done using the signal generator, which could only amplify from zero volts until some voltage limit within the maximum time range of 1000s. In order to accompany the steady rotation, the amplification of the acoustic velocity should be as slow as possible. Therefore, the restriction mentioned would result in a fast amplification and the registered blades velocity were not in steady rotation with the acoustic velocities. The points shown in the graph were registered for different measurements, in which the
time waited is supposed large enough to have a steady rotation. This explains the higher position of the points recorded comparing with the curve to each respective acoustic velocity.

An unique effect during the steady rotation can be noticed in the measurements of fig.5.13. The blades velocity starts to decrease even though the acoustic field amplitude is increasing. The effect could only be observed for the small apparatus at the first mode of resonance.

The effect is seen to happen in a certain limit of the ratio between the blades velocity and the acoustic velocity. This ratio appears to be constant for the experiments performed. Only two of them are shown in fig.5.13. After this limit, the accelerating force decays for higher acoustic flow velocities. The accelerating force is mainly due to the lift force in one direction of the flow. We expect a change in the flow pattern after this limit, and there will be a curve of the steady rotor velocity similar to the lift coefficient in fig.2.6.

Figure 5.12: Ratio of acoustic velocity and blades velocity. First ($Sr \approx 0.11$) and third ($Sr \approx 0.38$) mode of resonance.
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5.2.2 Large setup

The acquisition of the steady rotation of the large setup was done for various absolute pressures. The measurements are done identically as the small setup, described in the beginning of the current section. The results for the acquisition of the acoustic velocity is shown in fig.5.14 and fig.5.15, while the blades velocities are plotted in fig.5.16 and fig.5.17.

The acoustic velocities are calculated from the acoustic pressures (section 4.5). It is clearly seen that the acoustic velocity gives lower amplitude at higher air densities. A shift of the resonance frequency with changes of the absolute pressure is seen. We do not have an explanation for this effect.
Similar to the small setup, we can see in fig.5.16 and fig.5.17 that the blades response follows the acoustic velocity. The third mode of resonance also appears to be less efficient (lower blades velocity relative to the acoustic field) relative to higher Strouhal numbers.

5.2.3 Analysis

Fig.5.18 shows the application of the model for the steady rotation developed in subsection 3.2.2 (eqn.3.30 and eqn.3.35). The straight line \( \frac{V_{bl}}{V_{ac}} = 0.58 \) represents the ratio of the steady rotation predicted by the model if there was no friction on the rotor (eqn.3.30). The other curved lines are
CHAPTER 5. RESULTS

Figure 5.17: Large apparatus blades velocity for third mode of resonance.

done using eqn.3.35, with the values for the dynamic friction considered constant and equal to what is found in subsection 4.3.2. Different measurements of the steady rotations at both setups are also plotted.

Figure 5.18: Experimental ("x";"o";"*";"+") and theoretical ("--" ; ".." ; "---") ratio of blades velocity and acoustic velocity for steady rotation.

The ideal model gives large discrepancies, being closer to the experiments for larger pressures and for lower Strouhal numbers. The fitted curve made by the measurements of the first mode to the small apparatus is similar to the model, but for the big shift.
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5.3 Conclusions

From section 5.1 we see that the critical acoustic velocity amplitude above at which the ghost counts begin presents a scatter of approximately 20%. We suppose these variations to be due to two reasons, the fluctuations of the critical static friction \( (T_{\text{ast}})_{\text{max}} \) and instability of the flow configuration from one experiment to another.

For the rotor at rest, the torque is calculated from the edge force model of the flat plates (subsection 3.2.1). Section 5.1 shows the values found in order to begin rotation. The expected value to begin rotation is the critical static torque \( (T_{\text{ast}})_{\text{max}} \), which is measured for both apparatus in subsection 4.3.3 and its order of magnitude is confirmed in subsection 4.3.2. For the first mode of resonance the observed critical torque agrees within a factor of two with the theory for the small rotor. For the large rotor the theory predicts the experimental result within 20%. For the third mode of resonance, the discrepancies are around 4.5 times of what is expected for both setups.

The minimum acoustic pressure amplitude to begin ghost counts was calculated for field conditions in subsection 5.1.1. The calculations were made for a 30cm turbine meter under field conditions \( (p_{\text{st}})_{\text{max}} = 60 \text{bar} \) and \( p_{\text{gas}} = 40 \text{kg/m}^3 \). The minimum value of the acoustic pressure amplitude is found to be \( p_0 \approx 10^3 \). Previous measurements at field conditions observed ghost counts at \( p_0 \approx 4 \cdot 10^3 \) \cite{9}. Hence, our model explains the occurrence of those ghost counts.

Both experiments for the starting conditions (section 5.1) and for the rotating behavior (section 5.2) indicate that higher acoustic frequencies imply higher acoustic amplitudes in order to have the same resultant torque at the rotor. This inverse dependence of the frequency of the acoustic field to generate aerodynamic force was not simulated at the flat plate model. The force on the blades due to the acoustic oscillations seems to be smaller than the force calculated for a flat plate. At the same air density, higher Strouhal number implies a larger attenuation. This is expected to be related to the effect on the blades thickness that is not included in our model. Our results indicate that the thick trailing edge of the blades in the rotor reduces the ghost counts, and in this respect is a good design.

In subsection 5.2.1 it can be seen that, in the absence of mean flow, the blades velocity appears to be proportional to the acoustic velocity amplitude up to a certain limit, after which the dependence is not linear. This limit corresponds to the acoustic velocity amplitude two times the blades velocity. A change in the flow pattern seems to occur after this limit. The effect was not simulated by the models.

The quasi-steady models applied in subsection 3.2.2 do not consider harmonic acoustic oscillations and unsteady effects are ignored. The model gives substantial differences from the experimental results. Higher pressures and lower Strouhal numbers gives less discrepancy with the experimental results. The model shows inadequate for small rotor velocities.

A straightforward solution to cancel ghost counts is a symmetrical design of the blades. Although it would be efficient in this respect, we expect that symmetrical blades would give less accuracy of the steady flow measurements due to two reasons. Mainly because the actual blades geometry (similar to airfoils) gives a high lift force, responsible to accelerate the rotor. This way, the rotation of the rotor changes faster to variations of the flow velocity. Another reason is that a rounded trailing edge would give unstable flow separation. Then, slight changes of the edge geometry or of the surface roughness might give considerable changes of the rotor response. Even though we see clear prejudice in changing the blades geometry, it might be worthwhile further investigation in this aspect.

Further research should be done mainly to improve of the model of the aerodynamic force. The real blades geometry should be taken into account. Unsteady effects should be calculated with the pressure integrated over the actual blades surface. Experimentally, the small setup can easily be reproduced, and its length can be increased significantly without much expense. This would enable studies at much lower Strouhal numbers, similar to field conditions.
Appendix A

Estimation of rotor moment of inertia

Figure A.1: Definition of rotor geometric features.

The moment of inertia of the rotor is done considering four different regions: I, II, III and the blades (fig.A.1). The general formula to calculate the moment of inertia is:

\[ I = \int_V \rho r^2 dV \] (A.1)

With \( \rho \) the density of the rotor material, and \( r \) the position of the volume element \( dV \). Evaluating the integral for the different regions one get:

\[ I = M_1(R_2^2 - R_1^2) + M_2(R_3^2 - R_2^2) + M_3(R_4^2 - R_3^2) + \frac{N \rho d_n L M H^3}{3} \] (A.2)

With \( M_n = \rho \pi d_n (R_{n+1}^2 - R_n^2)/2 \) and \( d_n \) the thickness of the rotor at that region. The last term express the approximate moment of inertia of the blades. Since we did not have access to the actual rotors inside the gas turbine meter, we calculate the approximate values for a rotor of 7.5cm radius similar to the ones used in both setups:

\[ \rho_{at} = 2700 \text{kg/m}^3 \] (A.3)
\[ R_1 = 0.9 \cdot 10^{-2} \text{m} \] (A.4)
The approximate moment of inertia may be found from eqn A.2. For this rotor gives:

\[ I_{\text{15cm}} = 1.7 \cdot 10^{-4} \text{kg m}^2 \]  

Therefore, the value given by the company (fig. 4.6) may be correct, and will be considered as veracious.
Appendix B

Estimation of friction forces

Horizontal flow along a flat plate

The boundary layer thickness ($\delta$) is estimated in subsection 2.2.3, which is found to have its the order of magnitude equal to (eqn.2.36):

$$\delta = \sqrt{\frac{\nu L}{U}}$$

(B.1)

With $U$ the fluid undisturbed velocity, $L$ the length travelled along the flat plate and $\nu$ the kinematic viscosity. Assuming a simple plane Couette flow, which the velocity increases linearly from zero ($y = 0$) until $U$ ($y = \delta$), yields:

$$u = \frac{y U}{\delta}$$

(B.2)

The shear stress at a position $L$ of the flat plate will be:

$$\tau_s = \mu \frac{du}{dy} = \frac{\mu U}{\delta} = \sqrt{\frac{\mu \rho}{L} U^{3/2}}$$

(B.3)

Considering the velocity only changing perpendicular to the flat plate ($y$ direction). The friction force on an area of the flat plate is the integration of the shear stress along this area. Therefore, a flat plate rotating with average radius from the axis of "$R_{av}$", with an average fluid velocity of "$U_{av}$" and total surface area "$S$", it will have a friction torque of:

$$T = \sqrt{\frac{\mu \rho}{L} S R_{av} U_{av}^{3/2}}$$

(B.4)

Rotating rotor in a fluid at rest

Consider a rotor rotating with constant angular velocity "$\Omega$". Let the internal radius be $r_1$ and external radius $r_2$, with the width being negligible. Suppose the fluid turning with the rotor between the surface and the position where it is at rest ($z = \delta$), such that the shear stress is only due to change of velocity in $z$ direction. Such assumption is a rough approximation and it's not coherent with real flows. Although, to our system the rotor turns in a closed pipe of equal diameter and the fluid is considered as incompressible, such that, to our degree of accuracy, the assumption is valid. The shear stress "$\tau_s$" at a distance "$r$" from the axis, using the results for the flat plate (eqn.B.3), is estimated to be:

$$\tau_s(r) = \sqrt{\frac{\mu \rho}{L(r)} U(r)^{3/2}}$$

(B.5)

With $U(r)$ the undisturbed fluid velocity relative to the rotor at a distance $r$ from the rotor center, and $L(r)$ the distance travelled by the fluid along the rotor surface after a time $t$, $L(r) = \Omega rt$ ("$\Omega$" constant angular velocity). The value of the friction on both rotor lateral surfaces will be:

$$F = 2 \int_{r_1}^{r_2} \tau_s(r)dr = \sqrt{\frac{\mu \rho \Omega^3}{t}} (r_2^2 - r_1^2)$$

(B.6)
APPENDIX B. ESTIMATION OF FRICTION FORCES

And the approximate torque simply:

\[ T = \frac{\pi \mu \rho \Omega^3}{t} (r_2^4 - r_1^4) \]  

**B.7**

Two coaxial cylinders

Let the inner cylinder of radius \( a \) rotate with constant angular velocity \( \Omega_a \), and the outer cylinder of radius \( b \) rotate with constant angular velocity \( \Omega_b \). Suppose the flow velocity always tangential with constant angular velocity \( \Omega \), and assume its magnitude dependent only on the distance \( r \) from the cylinder axis. The coordinates of a fluid particle will be:

\[ x = r \cos(\Omega t) \]
\[ y = r \sin(\Omega t) \]

Then, the velocity \((v_x, v_y)\) at the positions \((x,y)\) will be:

\[ v_x = -\Omega r \sin(\Omega t) = -\Omega y \]
\[ v_y = \Omega r \cos(\Omega t) = \Omega x \]

The shear stress (eqn.2.5), for this case becomes:

\[ \tau_{xy} = \mu \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) = \mu \left( x \frac{\partial w}{\partial x} - y \frac{\partial w}{\partial y} \right) \]  

**B.8**

At \( y = 0 \), results:

\[ \left[ \tau_{xy} \right]_{y=0} = \mu r \frac{d\Omega}{dr} \]  

**B.9**

The shear stress will be the same around the rings of radius \( r \). The torque, \( T \), on a cylinder with length \( H \) and surface \( 2\pi r H \), will be:

\[ T = 2\pi \mu H r^3 \frac{d\Omega}{dr} \]  

**B.10**

Since the flow is steady, the torque cannot depend on \( r \), and \( d\Omega/dr \) must be proportional to \( r^{-3} \). Then, \( \Omega = -A/r^2 + B \), with "A" and "B" some constant. From the boundary conditions of \( \Omega = \Omega_a \) at \( r = a \) and \( \Omega = \Omega_b \) at \( r = b \) the values of \( A \) and \( B \) can be found and eqn.B.10 becomes:

\[ T = \frac{4\pi \mu H a^2 b^2}{b^2 - a^2} (\Omega_b - \Omega_a) \]  

**B.11**
Appendix C

Cauchy-Riemann equations

Let \( w(z) = \phi(x,y) + i\psi(x,y) \), with \( \phi(x,y) \) and \( \psi(x,y) \) analytic functions of the complex plane. Simplifying the notation for \( w(z) = \phi(x,y) + i\psi(x,y) = w = \phi + i\psi \), results:

\[
\begin{align*}
\frac{\partial w}{\partial x} &= \frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} \\
\frac{\partial w}{\partial y} &= \frac{dw}{dz} = \frac{\partial \phi}{\partial y} + i \frac{\partial \psi}{\partial y}
\end{align*}
\]

Comparing the equations one gets:

\[
\frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = -i \left( \frac{\partial \phi}{\partial y} + i \frac{\partial \psi}{\partial y} \right)
\]

The real and imaginary terms must be equal, resulting in the Cauchy-Riemann equations:

\[
\begin{align*}
\frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y} \\
\frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x}
\end{align*}
\]

For \( \phi(x,y) = \text{constant} \) and \( \psi(x,y) = \text{constant} \) we get:

\[
\begin{align*}
\frac{d\phi}{dx} &= \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} \bigg|_\phi = 0 \\
\frac{d\psi}{dx} &= \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} \bigg|_\psi = 0
\end{align*}
\]

Using Cauchy-Riemann equations proves that all constants lines of \( \phi(x,y) \) and \( \psi(x,y) \) intersect at right angles:

\[
\frac{dy}{dx} \bigg|_\phi \cdot \frac{dy}{dx} \bigg|_\psi = -1
\]
Appendix D

Uniqueness of potential fluid motion

The fluid motion of a potential flow will be proven to have unique solution if specified the boundaries and the singularities of the motion. Consider the fluid inside a volume "V", with boundary surface S on which the normal velocity \( (\vec{v} \cdot \vec{n}) \) is given. The general solution satisfies:

\[
\nabla^2 \phi = 0, \text{ inside } S \tag{D.1}
\]
\[
\frac{\partial \phi}{\partial n} = f(\vec{r}), \text{ on } S \tag{D.2}
\]

Suppose two distinct solutions, \( \phi_1 \) and \( \phi_2 \), and define \( \phi_3 = \phi_1 - \phi_2 \), then:

\[
\nabla^2 \phi_3 = 0, \text{ inside } S \tag{D.3}
\]
\[
\frac{\partial \phi_3}{\partial n} = 0, \text{ on } S \tag{D.4}
\]

The kinetic energy associated with \( \phi_3 \) will be:

\[
Kn = \frac{1}{2} \rho \int_V (\nabla \phi_3)^2 dV \tag{D.5}
\]

Using the relation \((\nabla \phi_3)^2 = \nabla \cdot (\phi_3 \nabla \phi_3) - \phi_3 \nabla^2 \phi_3 = \nabla \cdot (\phi_3 \nabla \phi_3)\) and the divergence theorem, then eqn. D.1 becomes:

\[
Kn = \frac{1}{2} \rho \int_V \phi_3 \frac{\partial \phi_3}{\partial n} dS \tag{D.6}
\]

Which from eqn. D.4 results that \( Kn = 0 \). Since \( Kn \) is an integral of only positive values, \( \Delta \phi_3 \) must be zero inside \( S \), and consequently \( \phi_1 \) and \( \phi_2 \) must be equal.
Appendix E

Complex potential of some simple flows

-Uniform flow
Suppose a uniform parallel flow of velocity magnitude \( U \) and making an angle \( \alpha \) with the x-axis (fig.E.1). The velocity components \((u,v)\) in the \((x,y)\) direction of the flow will be:

\[
\begin{align*}
    u &= U \cos(\alpha) \\
    v &= U \sin(\alpha)
\end{align*}
\]

Figure E.1: Parallel uniform flow

Then from potential flow theory:

\[
\frac{dw}{dz} = u - iv = U e^{-i\alpha}
\]

Which gives:

\[
w(z) = U z e^{-i\alpha}
\]  \hspace{1cm} (E.1)

-Vortex flow
Let a point vortex be at the origin of a coordinate system, and have positive circulation \( \Gamma \) (fig.E.2). The velocity components \((u,v)\) in the complex plane \((x,y)\) can be described as:

\[
\begin{align*}
    u &= -\frac{\Gamma \sin \theta}{2\pi r} \\
    v &= \frac{\Gamma \cos \theta}{2\pi r}
\end{align*}
\]  \hspace{1cm} (E.2)

(E.3)
Which yields:

\[
\frac{dw}{dz} = u - iv = -i \frac{\Gamma}{2\pi z}
\]

And the complex potential of the point vortex flow can be found:

\[
w(z) = -i \frac{\Gamma}{2\pi} \ln z
\]  \hspace{1cm} \text{(E.4)}

If the point vortex is at a position \(z_0\) with circulation also equal to \(\Gamma\) a translation of the coordinate system can be done to coincide the point vortex center with the origin of the new coordinate system \(z' = z - z_0\), and then:

\[
w(z') = -i \frac{\Gamma}{2\pi} \ln(z - z_0)
\]  \hspace{1cm} \text{(E.5)}
Appendix F

Circle theorem

Let $f(z) = h(z) + ig(z)$ be a function of the complex plane with $z = x + iy$. A new function, $\overline{f}(z)$, can be defined such that:

$$\overline{f}(z) = h(z) - ig(z) \quad \text{(F.1)}$$

Which yields:

$$f(z) + \overline{f}(\overline{z}) = h(z) + h(\overline{z}) + i [g(z) - g(\overline{z})] = \text{real number} \quad \text{(F.2)}$$

**Theorem:**

Let $f(z)$, $z = x + iy$, be a complex potential of a two dimension, irrotational and incompressible flow. Suppose $f(z)$ an analytic function for every points on the complex plane, with possible singularities only when $|z| > A$, "A" a positive constant. If a circle of radius "A" is introduced in the flow with center at $(x, y) = (0, 0)$, the new complex potential becomes:

$$w(z) = f(z) + \overline{f} \left( \frac{A^2}{z} \right) \quad \text{(F.3)}$$

**Proof:** The possible singularity on the region of $|z| > A$ will still be the same, and no new singularity will be added, since $\overline{f}(z_0) < A$, for all $|z_0| > A$.

The circle boundaries should become a constant streamline ($\Psi_{|z|=A} = \text{constant}$) and when $z = Ae^{i\alpha}$ then $\overline{f}(A^2/z) = \overline{f}(\overline{z})$. Then from eqn. F.2:

$$w(z)_{z=Ae^{i\alpha}} = f(z) + \overline{f}(\overline{z}) = \text{real number}$$

The streamlines along the circle boundary will then have constant zero streamline value, and the new complex potential function will then satisfy all the boundaries conditions with the circle introduced, characterizing the new complex potential.
Appendix G

Blasius force and Kutta-Zhukovskii theorems

Blasius force theorem:

Let a solid body boundaries be defined by a closed contour "c". If a potential flow is described by \( w(z) \), the force per unit length applied on the body will be:

\[
F_x - iF_y = \frac{i\rho}{2} \oint_c \left( \frac{dw(z)}{dz} \right)^2 dz
\]

\[(G.1)\]

![Figure G.1: Scheme for Blasius theorem demonstration.](image)

**Proof:** The force at an element \( ds \) of the contour "c" will be (fig.G.1, with \( \theta=0 \)):

\[
dF_x - idF_y = Pds(-\sin\theta - i\cos\theta) = -iPds e^{-i\theta}
\]

With \( P \) the perpendicular force per unit area on the surface. Using steady Bernoulli’s equation, which in a simplistic form is \( \frac{p}{\rho} + \frac{1}{2}U^2 = K_0 \), \( K_0 \) a constant, results:

\[
dF_x - idF_y = ids \left( \frac{\rho U^2}{2} + K_0 \right) e^{-i\theta}
\]

\[(G.2)\]
Where $U$ is the velocity along the contour "c" and $K_0$ is the sum of the constants velocity and pressure of the undisturbed flow. The solid boundary contour "c" is along a streamline, with $U = u + iv = |U|e^{i\theta}$, then $dw/dz = u - iv = Ue^{-i\theta}$, which results:

$$U^2 = \left(\frac{dw}{dz}\right)^2 e^{2i\theta}$$

Substituting $U^2$ and $ds = de^{i\theta}$ in eqn.G.2, yields:

$$dF_x - idF_y = \left[\frac{i\rho}{2} \left(\frac{dw}{dz}\right)^2 + K_0e^{-2i\theta}\right] dz$$

Performing the integral along a closed contour, it’s clear that the constant integrand does not contribute for the force, resulting in the Blasius theorem (eqn.G.1).

**Kutta-Zhukovski theorem:**

Suppose an uniform flow at infinity. Let this flow past a two dimensional body of boundary contour $C$, which presents a circulation $\Gamma$. Such body will have a lift force perpendicular to the undisturbed flow, 90 degrees opposite to the circulation, with magnitude:

$$|L| = |\rho U\Gamma|$$

(G.3)

![Figure G.2: Scheme for Kutta-Zhukovski theorem demonstration.](image)

**Proof:** Consider an inclined flow in the x-direction of angle $\alpha$ through a body of contour $C$ (fig.G.2). Let $w(z)$ be the complex potential with $dw/dz$ an analytic function everywhere. Using Laurent series ($f(z) = \sum_{n=0}^{\infty} a_n z^n$) and from the fact that the flow has uniform velocity at infinity then:

$$\frac{dw}{dz} = Ue^{-i\alpha} + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

(G.4)

Let a circular contour $C'$ surround the body of contour $C$, with no singularity between the two closed contours. From Cauchy integral theorem (eqn.3.13) it can be proved that the Blasius force theorem (eqn.G.1), proved for body boundaries, will have same value around the contour $C'$. The Blasius integral becomes:

$$F_x - iF_y = \frac{i\rho}{2} \oint_{C'} \left(Ue^{-i\alpha} + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots\right)^2 dz$$

(G.5)
Around the contour $C'$ we have that $z = R^n e^{i\theta}$ and $dz = iR e^{i\theta} d\theta$, so that the general form of each term on the integral becomes:

$$\frac{i\theta}{2} \oint_{0}^{2\pi} a'_n R^{1-n} e^{i\theta(1-n)} i d\theta, \text{ for } n \geq 0$$  \hspace{1cm} (G.6)

From this is clear that the integral vanishes for all values of $n$ except when $n = 1$. From simple multiplication of eqn.G.4 is easily seen that $a'_1 = 2a_1 U e^{-i\alpha}$, and eqn.G.5 becomes:

$$F_x - iF_y = 2\pi a_1 \rho U e^{-i\alpha}$$  \hspace{1cm} (G.7)

In order to find $a_1$, apply the same arguments above on eqn.G.4:

$$2\pi i a_1 = \oint_{C'} \frac{dw}{dz} dz = \oint_{C} \frac{dw}{dz} dz$$  \hspace{1cm} (G.8)

Since the integral is performed along a streamline, the integral of the velocity potential is zero and the integral of the streamline is the circulation. Then:

$$\oint_{C} \frac{dw}{dz} dz = \oint_{C} d\phi + i \oint_{C} d\psi = \Gamma$$  \hspace{1cm} (G.9)

Which gives the value of $a_1 = -i\Gamma/2\pi$ and substituting in eqn.G.7 results:

$$F_x - iF_y = -\rho U \Gamma e^{-i\alpha}$$  \hspace{1cm} (G.10)

Representing both the uniform flow and the the force on the $z$-plane, one sees that force is indeed perpendicular to the undisturbed flow, 90 degrees opposite to the circulation, and with magnitude $\rho U \Gamma$. 


Appendix H

Conformal mapping

Conformal mapping is of great applicability in potential flow theory. Various sources present such method with emphasis on fluid dynamics [19, 17, 22]. Its basic properties shall be explained here.

Let an analytic function of $z$, $F(z)$, transform a connected region (R) in the $z$-plane ($z = x + iy$), into another connected region (I), represented in the $\xi$-plane ($\zeta = \xi + i\eta$, in fig.H.1 $\text{Im} = \text{Im}(x) + i\text{Im}(y)$).

\[
\zeta = F(z) \tag{H.1}
\]

![Diagram](Image)

Figure H.1: General conformal transformation

Two conditions are necessary to have a conformal mapping: i) there are no two points in the connected region R that the values taken by $F(z)$ are the same, and, ii) the points in the connected region R taken by $F(z)$ completely fill the connected region I. When such conditions happen $F(z)$ will be an analytic function of $z$, with the inverse transformation, $F^{-1}(z)$, also analytic and conformal:

\[
z = F^{-1}(\zeta) \tag{H.2}
\]

A flow in region R, described by a complex potential $w(z) = \phi(x,y) + i\psi(x,y)$, can then be transformed to another flow described by the new complex potential $W(\zeta) = \Phi(\xi,\eta) + i\Psi(\xi,\eta)$. Constant lines of the velocity potentials and the streamlines will be mapped at constants lines at the transformed plane, so that the boundaries will be mapped into boundaries in the transformed plane.

An interesting property of such a transformation, which gives its name, is that it preserves the angle for infinitesimal curves. To prove such property let $z_1$, $z_2$, and $z_3$ represent three points in the $z$-plane, and $I_1$, $I_2$, and $I_3$ represent the respective transformed points in the $\zeta$-plane (fig.H.2).
Then:
\[
\frac{I_2 - I_1}{z_2 - z_1} = \frac{\mathcal{F}(z_2) - \mathcal{F}(z)}{z_2 - z_1} \quad (H.3)
\]
\[
\frac{I_3 - I_1}{z_3 - z_1} = \frac{\mathcal{F}(z_3) - \mathcal{F}(z)}{z_3 - z_1} \quad (H.4)
\]
Suppose \( \mathcal{F}(z) \) and \( \mathcal{F}(\zeta) \) as analytic functions of \( z \) and \( \zeta \) within the surrounding region of the points specified, and suppose \( z_2 - z_1 \to 0 \) and \( z_3 - z_1 \to 0 \), then:
\[
\frac{I_2 - I_1}{z_2 - z_1} = \frac{I_3 - I_1}{z_3 - z_1} = \mathcal{F}'(z) \quad (H.5)
\]
Then, taking the modulus and the argument of the infinitesimal segments, one gets:
\[
\frac{\text{arg}(\overline{z_1z_3}) - \text{arg}(\overline{z_1z_2})}{\overline{z_1z_2}} = \frac{\text{arg}(\overline{I_1I_3}) - \text{arg}(\overline{I_1I_2})}{\overline{I_1I_2}} \quad (H.6)
\]
\[
\frac{\overline{I_1I_2}}{\overline{z_1z_2}} = \frac{\overline{I_1I_3}}{\overline{z_1z_3}} \quad (H.7)
\]
From eqn.\( H.6 \) one sees the equality of the angles transformed to correspondent infinitesimal segments, and from eqn.\( H.7 \) the geometry similarity.

**-Zhukhovski transformation**

The Zhukhovski transformation is defined as:
\[
z = \zeta + \frac{A^2}{\zeta} \quad (H.8)
\]
Representing \( \zeta \) in polar coordinates, \( \zeta = r \cos \theta + ir \sin \theta \), equation (1) becomes:
\[
z = \left( r + \frac{A^2}{r} \right) \cos \theta + i \left( r - \frac{A^2}{r} \right) \sin \theta \quad (H.9)
\]
To visualize the transformation imagine concentric circles in the \( \zeta \)-plane with the center at the origin and radius \( R \) (\( |\zeta_0| = R \)), \( R > A \) (fig.\( H.3 \)). The transformation becomes:
\[
\zeta_0 = R \cos \theta + i R \sin \theta \quad (H.10)
\]
\[
z_0 = \left( R + \frac{A^2}{R} \right) \cos \theta + i \left( R - \frac{A^2}{R} \right) \sin \theta \quad (H.11)
\]
Is clear from equation (2) and (3) that the mapped region from \( \zeta \)-plane will elongate in the \( x \) direction, and shorten in the \( y \) direction, mapping circles in the \( \zeta \)-plane into ellipses in the \( z \)-plane (fig.\( H.3 \)).

Given that \( R > A \), it can be seen that each point in the \( \zeta \)-plane is mapped once, and only once, in the \( z \)-plane. It is also is clear that a specified connected region in the \( \zeta \)-plane will define a
connected region in the $z$-plane, satisfying the conditions for the conformal mapping. The inverse of the conformal transformation will also be valid, and shall be used often to transform the real airfoil into a circle in the imaginary plane:

$$
\zeta = \frac{z + \sqrt{z^2 - 4A^2}}{2} \quad \text{(H.12)}
$$

Which yields:

$$
\frac{d\zeta}{dz} = \frac{1}{2} + \frac{2z}{\sqrt{z^2 - 4A^2}} \quad \text{(H.13)}
$$

When $R = A$, the mapped region will be:

$$
z = 2A \cos \theta \quad \text{(H.14)}
$$

In principle, such a transformation is not conformal at this boundary, since two points within $0 < \theta \leq 2\pi$ gives same value in the transformed plane, contradicting the conditions for conformal mapping. We shall assume $R \to 0$, such that the transformation stills conformal and the ellipse thickness tends to zero, together with $\sin \theta$. For $0 < \theta < \pi$ the transformed points will then lie above the flat line, $\sin \theta > 0$ in eqn.H.11, while for $\pi < \theta < 2\pi$ the transformed points lies under the flat line ($\sin \theta < 0$). Fruitful discussion of such assumption is given by Lighthill [17].
Appendix I

Acoustic velocity calculation

The acoustic waves were considered as plane waves, dependent only on the x direction (pipe axis). This was possible due to the large acoustic wavelength compared with the pipe diameter, of at least the ten times larger. The plane wave equation, with the acoustic pressure deviations (p) much smaller then the atmospheric pressure, will be:

\[
\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = 0 \tag{I.1}
\]

Where "c" is the propagation velocity. For a harmonic plane wave, the equations for the acoustic pressure and velocity must satisfy:

\[
p = \rho_0 c^2 \left( P^+ e^{i(\omega t-kx)} + P^- e^{i(\omega t+kx)} \right) \tag{1.2}
\]

\[
u = \frac{1}{\rho_0 c} \left( P^+ e^{i(\omega t-kx)} - P^- e^{i(\omega t+kx)} \right) \tag{1.3}
\]

With "\rho_0" the propagation media density, "\omega" the the frequency in radians per second (\( \omega = 2\pi/T \), "T" the period) and "k" the wave number (\( k = \omega/c = 2\pi/\lambda \), "\lambda" the wavelength). The first and second terms of the equation represents respectively a wave propagating in the positive and in the negative "x" direction.

The acoustic velocity at the blades position will then be calculated from the acoustic pressure measured at one transducer. Consider the acoustic field for both setups represented in fig.1.1. Where

Figure I.1: Acoustic model scheme

"A" is the sectional area and "P^+" indicates an acoustic wave travelling to the right and "P^-" to the left. The origin of the (one dimension) coordinate system is imposed to be at one border of the gas turbine meter. Both eqn.1.2 and eqn.1.3 must be valid at any time, then let \( t=0 \). From the fact that the acoustic velocity should be zero at the closed end (\( u_2 = 0 \) at \( x = L \)), and from the acoustic pressure measured at one transducer (\( p_t \)) on the position \( x = x_t \), the following equation derives:

\[
p_t = P^+_2 e^{-ikx_t} + P^-_2 e^{ikx_t} \tag{I.4}
\]

\[
0 = P^+_2 e^{-ikx_t} + P^-_2 e^{ikx_t} \tag{I.5}
\]

And the coefficients \( P^+_2 \) and \( P^-_2 \) may be found:

\[
P^+_2 = \frac{P_t e^{ikx_t}}{1 + e^{2ik(x_t-L)}} \tag{I.6}
\]
APPENDIX I. ACOUSTIC VELOCITY CALCULATION

\[ P_2^- = \frac{P_1 e^{-ikx_t}}{1 + e^{2ik(L-x_t)}} \]  \hspace{1cm} (1.7)

The acoustic field over all the closed end side of the pipe (region 2) is then determined. Let \( A \) be the area at the pipes, with subscript \( 3 \) indicating the region inside the gas turbine meter. Let the flow be locally incompressible. This approximation should be reasonable due to the low Reynolds number and low Mach number of the experiments. Consequently, from the mass conservation, the following equation arises at the position between the pipe and the gas turbine meter:

\[ A_3 u_3 = A_2 u_2 \]  \hspace{1cm} (1.8)

We shall assume the velocity at the blades \( (u_b) \) equal to the velocity at the border of the gas turbine meter. Such assumption rely on the large wavelength compared with the gas meter length (not less then five times larger). The interior area of the gas turbine meter decreases of about 6%, due to the blades frontal area. To our modest degree of precision, such difference is neglected, and the turbine section area is considered as constant. The acoustic velocity at the blades position, from eqn.I.3 and eqn.I.6 to eqn.I.8, results:

\[ U_b = \frac{A_2}{A_3 \rho_0 c} \left( P_2^+ - P_2^- \right) \]  \hspace{1cm} (1.9)

When the transducer is at the closed end position, \( x_t = L \), eqn.I.9 reduces simply to:

\[ U_b = \frac{A_2 p_t}{A_3 \rho_0 c} \sin(kx_t) \]  \hspace{1cm} (1.10)
Appendix J

Apparatus photographs

Figure J.1: Small apparatus photograph.
Figure J.2: Large apparatus photograph.

Figure J.3: Large apparatus gas turbine meter photograph.
Appendix K

Point vortex motion close to the edge

A local transformation of the flow around the edge, considering the plate as infinite, can be done using the following mapping:

\[ \zeta = \sqrt{z} \]  \hspace{1cm} (K.1)

![Figure K.1: Vortex dynamics.](image)

The complex potential flow in the \( \zeta \)-plane can be easily derived (subsection 2.2.4), and using eqn. K.1 gives:

\[ w(z) = U_0(t)\sqrt{z} - \frac{i\Gamma}{2\pi} \left( \ln(\sqrt{z} - \sqrt{z_v}) - \ln(\sqrt{z} - \sqrt{z_v^2}) \right) \]  \hspace{1cm} (K.2)

In order to calculate the vortex velocity, the velocity induced by the point vortex singularity should be excluded:

\[ U^*(t) = \lim_{z \to z_v} \frac{d}{dz} \left[ w(z) + \frac{i\Gamma}{2\pi} \ln(z - z_v) \right] \]  \hspace{1cm} (K.3)

To perform the limit it is convenient to make \( z = z_v(1 + \epsilon) \) with \( \epsilon \to 0 \), and evaluating the limit of the derivative gives:

\[ U^*(t) = \frac{U_0(t)}{2\sqrt{z_v}} + \frac{i\Gamma}{2\pi} \left( \frac{1}{2(z_v - |z_v|)} + \frac{1}{4z_v} \right) \]  \hspace{1cm} (K.4)

The kutta condition at the edge implies zero velocity \( (dW(\zeta)/d\zeta = 0) \) at the origin, \( \zeta = 0 \). This gives the circulation value, which in terms of \( z \) is:

\[ \Gamma(t) = \frac{-\pi U_0(t)\sqrt{R}}{\sin \frac{\theta}{2}} \]  \hspace{1cm} (K.5)
With $z = Re^{i\theta}$ ($\theta = Q$ in fig.K.1). Substituting this value into eqn.K.5 yields:

$$U^*(t) = \frac{U_0(t)e^{\frac{z}{2}}}{2\sqrt{R}} + \frac{U_0(t)i}{4\sqrt{R}\sin\left(\frac{\theta}{2}\right)} \left( \frac{1}{e^{i\theta} - 1} + \frac{e^{i\theta}}{2} \right)$$  \hspace{1cm} (K.6)

Evaluating this expression, and spreading real and imaginary terms, gives the velocity components of the point vortex:

$$u_v = \frac{U_0(t)\cos\left(\frac{\theta}{2}\right)}{4\sqrt{R}} \left( 1 - \frac{1}{2\sin^2\left(\frac{\theta}{2}\right)} \right)$$  \hspace{1cm} (K.7)

$$v_v = \frac{U_0(t)\sin\left(\frac{\theta}{2}\right)}{4\sqrt{R}}$$  \hspace{1cm} (K.8)

Clearly $v_v/u_v = \tan(\theta)$, which evaluating gives $\theta = 3\pi/2$. Then one sees that $u_v = 0$, with the vortex along the $y$-axis and with velocity:

$$v_v = \frac{U_0(t)}{4\sqrt{R}}$$  \hspace{1cm} (K.9)

It worths emphasizing that the equations derived are valid only for the starting of the vortex, later on, when second and higher terms of $(z - z_v)$ cannot be neglected, such equations will not be valid. Numerical integration methods should be used to evaluate the vortex motion.
Bibliography


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