

Practical algorithms for linear Boolean-width

Citation for published version (APA):

ten Brinke, C. B., van Houten, F. J. P., & Bodlaender, H. L. (2015). Practical algorithms for linear Boolean-width. *arXiv*, (1509.07687v1).

Document status and date:

Published: 25/09/2015

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

Practical Algorithms for Linear Boolean-width*

Chiel B. ten Brinke¹, Frank J. P. van Houten¹, and Hans L. Bodlaender¹

¹ Department of Computer Science, Utrecht University
 PO Box 80.089, 3508 TB Utrecht, The Netherlands
 CtenBrinke@gmail.com, Frankv@nhouten.com, H.L.Bodlaender@uu.nl

Abstract

In this paper, we give a number of new exact algorithms and heuristics to compute linear boolean decompositions, and experimentally evaluate these algorithms. The experimental evaluation shows that significant improvements can be made with respect to running time without increasing the width of the generated decompositions. We also evaluated dynamic programming algorithms on linear boolean decompositions for several vertex subset problems. This evaluation shows that such algorithms are often much faster (up to several orders of magnitude) compared to theoretical worst case bounds.

1998 ACM Subject Classification G.2.2 [Discrete Mathematics]: Graph Theory — Graph algorithms; F.2.2 [Analysis of Algorithms and Problem Complexity]: Nonnumerical Algorithms and Problems — Computations on discrete structures

Keywords and phrases graph decomposition, boolean-width, heuristics, exact algorithms, vertex subset problems

Digital Object Identifier 10.4230/LIPIcs.xxx.yyy.p

1 Introduction

Boolean-width is a recently introduced graph parameter [2]. Similarly to treewidth and other parameters, it measures some structural complexity of a graph. Many NP-hard problems on graphs become easy if some graph parameter is small. We need a derived structure which captures the necessary information of a graph in order to exploit such a small parameter. In the case of boolean-width, this is a binary partition tree, referred to as the decomposition tree. However, computing an optimal decomposition tree is usually a hard problem in itself. A common approach to bypass this problem is to use heuristics to compute decompositions with a low boolean-width.

Algorithms for computing boolean decompositions have been studied before in [17, 10, 12, 7], but in this paper we study the specific case of linear boolean decompositions, which are considered in [1, 10, 12]. Linear decompositions are easier to compute and the theoretical running time of algorithms for solving practical problems is lower on linear decompositions than on tree shaped ones. For instance, vertex subset problems can be solved in $O^*(nec^3)$ due to a dynamic programming algorithm by Bui-Xuan et al. [3], but this can be improved to $O^*(nec^2)$ for linear decompositions. Here, nec is the number of d -neighborhood equivalence classes, i.e., the maximum size of the dynamic programming table.

* The research of the third author was partially funded by the Networks programme, funded by the Dutch Ministry of Education, Culture and Science through the Netherlands Organisation for Scientific Research.



We first give an exact algorithm for computing optimal linear boolean decompositions, improving upon existing algorithms, and subsequently investigate several new heuristics through experiments, improving upon the work by Sharmin [12, Chapter 8]. We then study the practical relevance of these algorithms in a set of experiments by solving an instance of a vertex subset problem, investigating the number of equivalence classes compared to the theoretical worst case bounds.

2 Preliminaries

A graph $G = (V, E)$ of size n is a pair consisting of a set of n vertices V and a set of edges E . The *neighborhood* of a vertex $v \in V$ is denoted by $N(v)$. For a subset $A \subseteq V$ we denote the neighborhood by $N(A) = \bigcup_{v \in A} N(v)$. In this paper we only consider simple, undirected graphs and assume we are given a total ordering on the vertices of a graph G . For a subset $A \subseteq V$ we denote the *complement* by $\bar{A} = V \setminus A$. A partition (A, \bar{A}) of V is called a *cut* of the graph. Each cut (A, \bar{A}) of G induces a bipartite subgraph $G[A, \bar{A}]$.

The *neighborhood across a cut* (A, \bar{A}) for a subset $X \subseteq A$ is defined as $N(X) \cap \bar{A}$.

► **Definition 1** (Unions of neighborhoods). Let $G = (V, E)$ be a graph and $A \subseteq V$. We define the set of *unions of neighborhoods across a cut* (A, \bar{A}) as

$$\mathcal{UN}(A) = \{N(X) \cap \bar{A} \mid X \subseteq A\}.$$

The number of unions of neighborhoods is symmetric for a cut (A, \bar{A}) , i.e., $|\mathcal{UN}(A)| = |\mathcal{UN}(\bar{A})|$ [8, Theorem 1.2.3]. Furthermore, for any cut (A, \bar{A}) of a graph G it holds that $|\mathcal{UN}(A)| = \#\mathcal{MLS}(G[A, \bar{A}])$, where $\#\mathcal{MLS}(G)$ is the number of maximal independent sets in G [17, Theorem 3.5.5].

► **Definition 2** (Decomposition tree). A *decomposition tree* of a graph $G = (V, E)$ is a pair (T, δ) , where T is a full binary tree and δ is a bijection between the leaves of T and vertices of V . If a is a node and L are its leaves, we write $\delta(a) = \bigcup_{l \in L} \delta(l)$. So, for the root node r of T it holds that $\delta(r) = V$. Furthermore, if nodes a and b are children of a node w , then $(\delta(a), \delta(b))$ is a partition of $\delta(w)$.

In this paper we consider a special type of decompositions, namely *linear decompositions*.

► **Definition 3** (Linear decomposition). A *linear decomposition*, or *caterpillar decomposition*, is a decomposition tree (T, δ) where T is a full binary tree and for which each internal node of T has at least one leaf as a child. We can define such a linear decomposition through a linear ordering $\pi = \pi_1, \dots, \pi_n$ of the vertices of G by letting δ map the i -th leaf of T to π_i .

► **Definition 4** (Boolean-width). Let $G = (V, E)$ be a graph and $A \subseteq V$. The *boolean dimension* of A is a function $\text{bool-dim} : 2^V \rightarrow \mathbb{R}$.

$$\text{bool-dim}(A) = \log_2 |\mathcal{UN}(A)|.$$

Let (T, δ) be a decomposition of a graph G . We define the *boolean-width* of (T, δ) as the maximum boolean dimension over all cuts induced by nodes of (T, δ) .

$$\text{boolw}(T, \delta) = \max_{w \in T} \text{bool-dim}(\delta(w))$$

The boolean-width of G is defined as the minimum boolean-width over all possible full decompositions of G , while the *linear boolean-width* of a graph $G = (V(G), E(G))$ of size n is defined as the the minimum boolean-width over all linear decompositions of G .

$$\text{boolw}(G) = \min_{(T, \delta) \text{ of } G} \text{boolw}(T, \delta)$$

$$\text{lboolw}(G) = \min_{\text{linear } (T, \delta) \text{ of } G} \text{boolw}(T, \delta)$$

It is known that for any graph G it holds that $\text{boolw}(G) \leq \text{treewidth}(G) + 1$ [17, Theorem 4.2.8]. The linear variant of treewidth is called *pathwidth* [11], or *pw* for short.

► **Theorem 5** (Appendix A.1). *For any graph G it holds that $\text{lboolw}(G) \leq \text{pw}(G) + 1$.*

The algorithms in this paper make extensive use of sets and set operations, which can be implemented efficiently by using bitsets. By using a mapping from vertices to bitsets that represent the neighborhood of a vertex we can store the adjacency matrix of a graph efficiently. We assume that bitset operations take $O(n)$ time and need $O(n)$ space, even though in practice this may come closer to $O(1)$. If one assumes that these requirements are constant, several time and space bounds in this paper improve by a factor n .

In this paper we assume that the graph G is connected, since if the graph consists of multiple connected components we can simply compute a linear decomposition for each connected component, after which we glue them together, in any arbitrary order.

3 Exact Algorithms

We can characterize the problem of finding an optimal linear decomposition by the following recurrence relation, in which P is a function mapping a subset of vertices A to the linear boolean-width of the induced subgraph $G[A, \bar{A}]$.

$$P(\{v\}) = |\mathcal{UN}(\{v\})| = \begin{cases} 1 & \text{if } N(v) = \emptyset \\ 2 & \text{if } N(v) \neq \emptyset \end{cases} \quad (1)$$

$$P(A) = \min_{v \in A} \{\max\{|\mathcal{UN}(A)|, P(A \setminus \{v\})\}\}$$

The boolean-width of the graph G is now given by $\log_2(P(V))$. Adaptation of existing techniques lead to the following algorithms for linear boolean-width, upon we hereafter improve:

- With dynamic programming a running time of $O(2.7284^n)$ is achieved. (See Theorem 19, Appendix A.2)
- With adaptation of the exact algorithm for boolean-width by Vatshelle [17], a running time of $O(n^3 \cdot 2^{n+\text{lboolw}(G)})$ is achieved. (See Theorem 20, Appendix A.2)

3.1 Improving the running time

We present a faster and easier way to precompute for all cuts $A \subseteq V$ the value $|\mathcal{UN}(A)|$, which results in a new algorithm displayed in Algorithm 2. In the following it is important that the \mathcal{UN} sets are implemented as hashmaps, which will only save distinct neighborhoods.

Algorithm 1 Compute $\mathcal{UN}(X \cup \{v\})$ given $\mathcal{UN}(X)$.

```

1: procedure INCREMENT-UN( $G, X, \mathcal{UN}_X, v$ )
2:    $\mathcal{U} \leftarrow \emptyset$ 
3:   for  $S \in \mathcal{UN}_X$  do
4:      $\mathcal{U} \leftarrow \mathcal{U} \cup \{S \setminus \{v\}\}$ 
5:      $\mathcal{U} \leftarrow \mathcal{U} \cup \{(S \setminus \{v\}) \cup (N(v) \cap (\bar{X} \setminus \{v\}))\}$ 
6:   return  $\mathcal{U}$ 

```

► **Lemma 6** (Appendix A.3). *The procedure Increment-UN is correct and runs in $O(n \cdot |\mathcal{UN}_X|)$ time using $O(n \cdot |\mathcal{UN}_X|)$ space.*

Algorithm 2 Return $\text{lboolw}(G)$, if it is smaller than $\log K$, otherwise return ∞ .

```

1: procedure INCREMENTAL-UN-EXACT( $G, K$ )
2:    $T_{\mathcal{UN}}(\emptyset) \leftarrow 0$ 
3:   COMPUTE-COUNT-UN( $G, K, T_{\mathcal{UN}}, \emptyset, \{\emptyset\}$ )
4:
5:    $P(X) \leftarrow \infty$ , for all  $X \subseteq V$ 
6:    $P(\emptyset) \leftarrow 0$ 
7:
8:   for  $i \leftarrow 0, \dots, |V| - 1$  do
9:     for  $X \subseteq V$  of size  $i$  do
10:      for  $v \in V \setminus X$  do
11:         $Y \leftarrow X \cup \{v\}$ 
12:        if  $P(X) \leq K$  then
13:           $P(Y) \leftarrow \min(P(Y), \max(T_{\mathcal{UN}}(Y), P(X)))$ 
14:
15:   return  $\log_2(P(V))$ 
16:
17: procedure COMPUTE-COUNT-UN( $G, K, T_{\mathcal{UN}}, X, \mathcal{UN}_X$ )
18:   for  $v \in V \setminus X$  do
19:      $Y \leftarrow X \cup \{v\}$ 
20:     if  $T_{\mathcal{UN}}(Y)$  is not defined then
21:        $\mathcal{UN}_Y \leftarrow \text{INCREMENT-UN}(G, X, \mathcal{UN}_X, v)$ 
22:        $T_{\mathcal{UN}}(Y) \leftarrow |\mathcal{UN}_Y|$ 
23:       if  $T_{\mathcal{UN}}(Y) \leq K$  then
24:         COMPUTE-COUNT-UN( $G, K, T_{\mathcal{UN}}, Y, \mathcal{UN}_Y$ )

```

► **Theorem 7** (Appendix A.4). *Given a graph G , Algorithm 2 can be used to compute $\text{lboolw}(G)$ in $O(n \cdot 2^{n+\text{lboolw}(G)})$ time using $O(n \cdot 2^n)$ space.*

This new algorithm improves upon the time in Theorem 20 by a factor n^2 , while the space requirements stay the same. Since the tightest known upperbound for linear boolean-width is $\frac{n}{2} - \frac{n}{143} + O(1)$ [10], this algorithm can be slower than dynamic programming, since $O(2^{n+\frac{n}{2}-\frac{n}{143}+O(1)}) = O(2.8148^{n+O(1)}) \supseteq O(2.7284^n)$, but this is very unlikely to happen in practice.

4 Heuristics

4.1 Generic form of the heuristics

The goal when using a heuristic is to find a linear ordering of the vertices in a graph in such a way that the decomposition that corresponds to this ordering will be of low boolean-width. A basic strategy to accomplish this is to start the ordering with some vertex and then by some selection criteria append a new vertex to the ordering that has not been appended yet. This strategy is used in heuristics introduced by Sharmin [12, Chapter 8], and a similar approach is shown in Algorithm 3.

Algorithm 3 Greedily generate an ordering based on the score function and the given starting vertex.

```

1: procedure GENERATEVERTEXORDERING( $G, ScoreFunction, init$ )
2:    $Decomposition \leftarrow (init)$ 
3:    $Left \leftarrow \{init\}$ 
4:    $Right \leftarrow V \setminus \{init\}$ 
5:   while  $Right \neq \emptyset$  do
6:      $Candidates \leftarrow$  set returned by candidate set strategy
7:     if there exists  $v \in Candidates$  belonging to a trivial case then
8:        $chosen \leftarrow v$ 
9:     else
10:       $chosen \leftarrow \underset{v \in Candidates}{\operatorname{argmin}} (ScoreFunction(G, Left, Right, v))$ 
11:       $Decomposition \leftarrow Decomposition \cdot \{chosen\}$ 
12:       $Left \leftarrow Left \cup \{chosen\}$ 
13:       $Right \leftarrow Right \setminus \{chosen\}$ 
14: return  $Decomposition$ 

```

At any point in the algorithm we denote the set of all vertices contained in the ordering by $Left$, and the remaining vertices by $Right$. While $Right$ is not empty, we choose a vertex from a candidate set $Candidates \subseteq Right$, based on a set of trivial cases, and, if no trivial case applies, by making a local greedy choice using a score function that indicates the quality of the current state $Left, Right$.

4.1.1 Selecting the initial vertex

Selecting a good initial vertex can be of great influence on the quality of the decomposition. Sharmin proposes to use a double breadth first search (BFS) in order to select the initial vertex. This is done by initiating a BFS, starting at an arbitrary vertex, after which a vertex of the last level of the BFS is selected. This process is then repeated by using the found vertex as a starting point for the second BFS. However, the fact that an arbitrary vertex is used for the first BFS already influences the boolean-width of the computed decomposition. During our experiments we noticed that performing a single BFS sometimes gave better results. But since we will see in Chapter 5 that applications are a lot more expensive in terms of running time, it is wise to use all possible starting vertices when trying to find a good decomposition.

4.1.2 Pruning

Starting from multiple initial vertices allows us to do some pruning. If we notice during the algorithm that the score of the decomposition that is being constructed exceeds the score of the best decomposition found so far, we can stop immediately and move to the next initial vertex. For this reason, it is wise to start with the most promising initial vertices (e.g. obtained by the double BFS method), and after that try all other initial vertices.

4.1.3 Candidates

The most straightforward choice for the set $Candidates$ is to take $Right$ entirely. However, we may do unnecessary work here, since vertices that are more than 2 steps away from any

vertex in $Left$ cannot decrease the size of \mathcal{UN} . This means that they should never be chosen by a greedy score function, which means that we can skip them right away. By this reasoning, the set of $Candidates$ can be reduced to $N^2(Left) \cap Right = N(Left \cup N(Left)) \cap Right$. Especially for larger sparse graphs, this can significantly decrease the running time.

4.1.4 Trivial cases

A vertex is chosen to be the next vertex in the ordering if it can be guaranteed that it is an optimal choice by means of a trivial case. Lemma 8 generalizes results by Sharmin [12], since the two trivial cases given by her are subcases of our lemma, namely $X = \emptyset$ and $X = \{u\}$ for all $u \in Left$. Note that we can add a wide range of trivial cases by varying X , such as $X = Left$ and $\forall u, w \in Left : X = \{u, w\}$, but this will increase the complexity of the algorithm.

► **Lemma 8** (Appendix A.5). *Let $X \subseteq Left$. If $\exists v \in Right$ such that $N(v) \cap Right = N(X) \cap Right$, then choosing v will not change the boolean-width of the resulting decomposition.*

4.1.5 Relative Neighborhood Heuristic

For a cut $(Left, Right)$ and a vertex v define

$$Internal(v) = (N(v) \cap N(Left)) \cap Right$$

$$External(v) = (N(v) \setminus N(Left)) \cap Right$$

In the original formulation by Sharmin [12] $\frac{|External(v)|}{|Internal(v)|}$ is used as a score function. However, if we use $\frac{|External(v)|}{|Internal(v)| + |External(v)|} = \frac{|External(v)|}{|N(v) \cap Right|}$ we get the same ordering by Lemma 9, without having an edge case for dividing by zero. Furthermore, in contrast to Sharmin's proposal of checking for each vertex $w \in N(v)$ if $w \in N(Left) \cap Right$ or not, we can compute these sets directly by performing set operations. We will refer to this heuristic by RELATIVENEIGHBORHOOD.

► **Lemma 9** (Appendix A.6). *The mapping $\frac{a}{b} \mapsto \frac{a}{a+b}$ is order preserving.*

Two variations on this heuristic can be obtained through the score functions $\frac{|External(v)|}{|N(v)|}$ and $1 - \frac{|Internal(v)|}{|N(v)|}$, which work slightly better for sparse random graphs and extremely well for dense random graphs respectively. We will refer to these two variations by RELATIVENEIGHBORHOOD₂ and RELATIVENEIGHBORHOOD₃.

One can easily see that the running time of these three algorithms is $O(n^3)$ and the required space amounts to $O(n)$. Notice however that this algorithm only gives us a decomposition. If we need to know the corresponding boolean-width we need to compute it afterwards, for instance by iteratively applying INCREMENT-UN on the vertices in the decomposition, and taking the maximum value. This would require an additional $O(n^2 \cdot 2^k)$ time and $O(n \cdot 2^k)$ space, where k is the boolean-width of the decomposition.

4.1.6 Least Cut Value Heuristic

The LEASTCUTVALUE heuristic by Sharmin [12] greedily selects the next vertex $v \in Right$ that will have the smallest boolean dimension across the cut $(Left \cup \{v\}, Right \setminus \{v\})$. This vertex is obtained by constructing the bipartite graph $BG = G[Left \cup \{v\}, Right \setminus \{v\}]$ for each $v \in Right$, and counting the number of maximal independent sets of BG using the CCM_{IS} [9] algorithm on BG , with the time of CCM_{IS} being exponential in n .

4.1.7 Incremental Unions of Neighborhoods Heuristic

Generating a bipartite graph and then calculating the number of maximal independent sets is a computational expensive approach. A different way to compute the boolean dimension of each cut is by reusing the neighborhoods from the previous cut, similarly to INCREMENTAL-UN-EXACT. We present a new algorithm, called the INCREMENTAL-UN-HEURISTIC, in Algorithm 4. A useful property of this algorithm is that the running time is output sensitive. It follows that if a decomposition is not found within reasonable time, then the decomposition that would have been generated is not useful for practical algorithms.

Algorithm 4 Greedy heuristic that incrementally keeps track of the Unions of Neighborhoods.

```

1: procedure INCREMENTAL-UN-HEURISTIC( $G, init$ )
2:    $Decomposition \leftarrow (init)$ 
3:    $Left, Right \leftarrow \{init\}, V \setminus \{init\}$ 
4:    $\mathcal{UN}_{Left} \leftarrow \{\emptyset, N(init) \cap Right\}$ 
5:   while  $Right \neq \emptyset$  do
6:      $Candidates \leftarrow$  set returned by candidate set strategy
7:     if there exists  $v \in Candidates$  belonging to a trivial case then
8:        $chosen \leftarrow v$ 
9:        $\mathcal{UN}_{chosen} \leftarrow$  INCREMENT-UN( $G, Left, \mathcal{UN}_{Left}, v$ )
10:    else
11:      for all  $v \in Candidates$  do
12:         $\mathcal{UN}_v \leftarrow$  INCREMENT-UN( $G, Left, \mathcal{UN}_{Left}, v$ )
13:        if  $chosen$  is undefined or  $|\mathcal{UN}_v| < |\mathcal{UN}_{chosen}|$  then
14:           $chosen \leftarrow v$ 
15:           $\mathcal{UN}_{chosen} \leftarrow \mathcal{UN}_v$ 
16:         $Decomposition \leftarrow Decomposition \cdot chosen$ 
17:         $Left \leftarrow Left \cup \{chosen\}$ 
18:         $Right \leftarrow Right \setminus \{chosen\}$ 
19:         $\mathcal{UN}_{Left} \leftarrow \mathcal{UN}_{chosen}$ 
20:    return  $Decomposition$ 

```

► **Theorem 10** (Appendix A.7). *The INCREMENTAL-UN-HEURISTIC procedure runs in $O(n^3 \cdot 2^k)$ time using $O(n \cdot 2^k)$ space, where k is the boolean-width of the resulting linear decomposition.*

4.1.8 Unsuccessful ideas

- First Improvement — Preliminary experiments pointed out that it not only gave worse results in terms of boolean-width, but it also increased the time needed to compute a decomposition, which can be explained by the output sensitivity of the INCREMENTAL-UN-HEURISTIC. In other words, even though the best improvement strategy takes more time to determine the next vertex for a single iteration, it is worthwhile to put effort in finding a good cut, as it also decreases the time for future cuts.
- Lookaheads — This technique does not only look at the change of \mathcal{UN} resulting from choosing a candidate v , but also recursively considers the changes of the algorithm after v has been chosen, up to a fixed depth. With each level of depth added, the time complexity increases with a factor n , but experiments turned out that the benefits were only marginal.

- Minimal Neighborhood Cover — This heuristic tries to minimize the number of neighborhoods in *Left* that are needed to cover the neighborhood of the vertex to be chosen.
- Max Cardinality Search — This heuristics selects vertices in such an order that at each step the vertex with most neighbors in *Left* is chosen. In practice this heuristic performed similar to other already known polynomial heuristics.

5 Vertex subset problems

Boolean decompositions can be used to efficiently solve a class of vertex subset problems called (σ, ρ) vertex subset problems, which were introduced by Telle [13]. This class of problems consists of finding a (σ, ρ) -set of maximum or minimum cardinality and contains well known problems such as the maximum independent set, the minimum dominating set and the maximum induced matching problem. The running time of the algorithm for solving these problems is $O(n^4 \cdot nec_d(T, \delta)^3)$ [3], where $nec_d(T, \delta)$ is the number of equivalence classes of a problem specific equivalence relation, which can be bounded in terms of boolean-width. In Section 6 we investigate how close the value of $nec_d(T, \delta)$ comes to any of the theoretical bounds.

5.1 Definitions

► **Definition 11** ((σ, ρ) -set). Let $G = (V, E)$ be a graph. Let σ and ρ be finite or co-finite subsets of \mathbb{N} . A subset $X \subseteq V$ is called a (σ, ρ) -set if the following holds

$$\forall v \in V : |N(v) \cap X| \in \begin{cases} \sigma & \text{if } v \in X, \\ \rho & \text{if } v \in V \setminus X. \end{cases}$$

In order to confirm if a set X is a (σ, ρ) -set we have to count the number of neighbors each vertex $v \in V$ has in X , where it suffices to count up until a certain number of neighbors. As an example, when we want to confirm if a set X is an independent set, which is equivalent to checking if X is a $(\{0\}, \mathbb{N})$ -set, it is irrelevant if a vertex v has more than one neighbor in X . We capture this property in the function $d : 2^{\mathbb{N}} \rightarrow \mathbb{N}$, which is defined as follows:

► **Definition 12** (d-function). Let $d(\mathbb{N}) = 0$. For every finite or co-finite set $\mu \subseteq \mathbb{N}$, let $d(\mu) = 1 + \min(\max_{x \in \mathbb{N}} x : x \in \mu, \max_{x \in \mathbb{N}} x : x \notin \mu)$. Let $d(\sigma, \rho) = \max(d(\sigma), d(\rho))$.

► **Definition 13** (d-neighborhood). Let $G = (V, E)$ be a graph. Let $A \subseteq V$ and $X \subseteq A$. The *d-neighborhood* of X with respect to A , denoted by $N_A^d(X)$, is a multiset of vertices from \overline{A} , where a vertex $v \in \overline{A}$ occurs $\min(d, |N(v) \cap X|)$ times in $N_A^d(X)$. A d-neighborhood can be represented as a vector of length $|\overline{A}|$ over $\{0, 1, \dots, d\}$.

► **Definition 14** (d-neighborhood equivalence). Let $G = (V, E)$ be a graph and $A \subseteq V$. Two subsets $X, Y \subseteq A$ are said to be *d-neighborhood equivalent* with respect to A , denoted by $X \equiv_A^d Y$, if it holds that $\forall v \in \overline{A} : \min(d, |X \cap N(v)|) = \min(d, |Y \cap N(v)|)$. The number of equivalence classes of a cut (A, \overline{A}) is denoted by $nec(\equiv_A^d)$. The number of equivalence classes $nec_d(T, \delta)$ of a decomposition (T, δ) is defined as $\max(nec(\equiv_A^d), nec(\equiv_{\overline{A}}^d))$ over all cuts (A, \overline{A}) of (T, δ) .

Note that $N_A^1(X) = N(X) \cap \overline{A}$. It can then be observed that $|\mathcal{UN}(A)| = nec(\equiv_A^1)$ [17, Theorem 3.5.5] Also note that $X \equiv_A^d Y$ if and only if $N_A^d(X) = N_A^d(Y)$.

5.2 Bounds on the number of equivalence classes

We present a brief overview of the most relevant bounds that are currently known, for which we make use of a *twin class partition* of a graph.

► **Definition 15** (Twin class partition). Let $G = (V, E)$ be a graph of size n and let $A \subseteq V$. The *twin class partition* of A is a partition of A such that $\forall x, y \in A$, x and y are in the same partition class if and only if $N(x) \cap \bar{A} = N(y) \cap \bar{A}$. The number of partition classes of A is denoted by $ntc(A)$ and it holds that $ntc(A) \leq \min(n, 2^{\text{bool-dim}(A)})$ [2].

For all bounds listed below, let $G = (V, E)$ be a graph of size n and let d be a non-negative integer. Let (A, \bar{A}) be a cut induced by any node of a decomposition (T, δ) of G , and let $k = \text{bool-dim}(A) = \text{nec}(\equiv_A^1)$.

► **Lemma 16.** [3, Lemma 5] $\text{nec}(\equiv_A^d) \leq 2^{d \cdot k^2}$.

► **Lemma 17.** [17, Lemma 5.2.2] $\text{nec}(\equiv_A^d) \leq (d + 1)^{\min(ntc(A), ntc(\bar{A}))}$.

► **Lemma 18** (Appendix A.8). $\text{nec}(\equiv_A^d) \leq ntc(A)^{d \cdot k}$.

By Lemma 16 we conclude that we can solve (σ, ρ) problems in $O^*(8^{dk^2})$. This shows that applications are more computationally expensive than using heuristics to find a decomposition.

6 Experiments

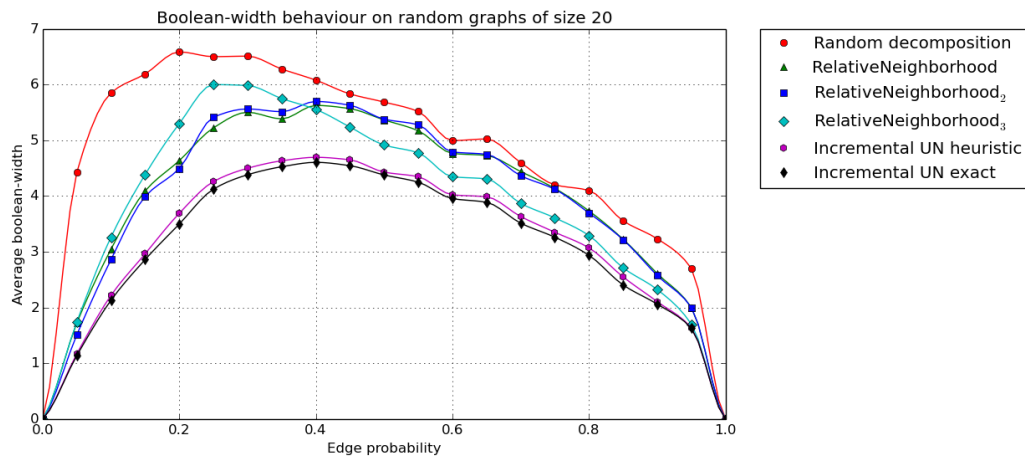
The experiments in this section are performed on a 64-bit Windows 7 computer, with a 3.40 GHz Intel Core i5-4670 CPU and 8GB of RAM. We implemented the algorithms using the C# programming language and compiled our programs using the *csc* compiler that comes with Visual Studio 12.0.¹

6.1 Comparing Heuristics on random graphs

We will look at the performance of heuristics on randomly generated graphs, for which we used the Erdős-Rényi-model [5] to generate a fixed set of random graphs with varying edge probabilities. By using the same set of graphs for each heuristic, we rule out the possibility that one heuristic can get a slightly easier set of graphs than another. In these experiments we start a heuristic once for each possible initial vertex, so n times in total. For the RELATIVENEIGHBORHOOD heuristic we select the best decomposition based upon the sum of the score function for all cuts, since computing all actual linear boolean-width values would take $O(n^3 \cdot 2^k)$ time, thereby removing the purpose of this polynomial time heuristic. For the set *Candidates* we take $N^2(\text{Left}) \cap \text{Right}$, which avoids that we exclude certain optimal solutions, as opposed to Sharmin [12], who restricted this set to $N(\text{Left}) \cap \text{Right}$. However, this does not affect the results significantly.

We let the edge probability vary between 0.05 and 0.95 with steps of size 0.05. For each edge probability value, we generated 20 random graphs. The result per edge probability is taken to be the average boolean-width over these 20 graphs, which are shown in Figure 1. It can be observed that the INCREMENTAL-UN-HEURISTIC procedure performs near optimal. Furthermore we see that the RELATIVENEIGHBORHOOD variants perform somewhere in between the optimal value and the value of random decompositions.

¹ Source code of our implementations can be found on <https://github.com/Chiel92/boolean-width> and <https://github.com/FrankvH/BooleanWidth>



■ **Figure 1** Performance of different heuristics on random generated graphs consisting of 20 vertices, with varying edge probabilities, in terms of linear boolean-width.

6.2 Comparing heuristics on real-world graphs

In order to get an idea of how the INCREMENTAL-UN-HEURISTIC compares to existing heuristics we compare them by both the boolean-width of the generated decomposition and the time needed for computation. We cannot compare the heuristics to the optimal solution, because computing an exact decomposition is not feasible on these graphs. The graphs that were used come from *Treewidthlib* [14], a collection of graphs that are used to benchmark algorithms using treewidth and related graph problems.

We ran the three different heuristics mentioned in Section 4 with *Candidates = Right* and with an additional two variations on the INCREMENTAL-UN-HEURISTIC (IUN) by varying the set of start vertices. The first variation, named 2-IUN, has two start vertices which are obtained through a single and double BFS respectively. The n-IUN heuristic uses all possible start vertices. For all other heuristics we obtained the start vertex through performing a double BFS. In Table 1 and 2 we present the results of our experiments.

■ **Table 1** Linear boolean-width of the decompositions returned by different heuristics.

Graph	$ V $	Edge Density	Relative	LeastCut	IUN	2-IUN	n-IUN
barley	48	0.11	5.70	5.91	5.91	4.70	4.58
pigs-pp	48	0.12	10.35	7.13	7.13	7.13	6.64
david	87	0.11	9.38	6.27	6.27	6.27	5.86
celar04-pp	114	0.08	11.67	7.27	7.27	7.27	7.27
lbbk-pp	127	0.18	16.81	9.98	9.98	9.53	9.53
miles1500	128	0.64	8.17	5.58	5.58	5.58	5.29
celar10-pp	133	0.07	10.32	11.95	11.95	7.64	6.91
munin2-pp	167	0.03	15.17	9.61	9.61	9.61	7.61
mulsol.i.5	186	0.23	7.55	5.29	5.29	5.29	3.58
zeroin.i.2	211	0.16	7.92	4.46	4.46	4.46	3.81
boblo	221	0.01	19.00	4.32	4.32	4.32	4.00
fpsol2.i-pp	233	0.40	5.58	6.07	6.07	5.78	4.81
munin4-wpp	271	0.02	13.04	9.27	9.27	9.27	7.61

■ **Table 2** Time in seconds of the heuristics used to find linear boolean decompositions.

Graph	$ V $	Edge Density	Relative	LeastCut	IUN	2-IUN	n-IUN
barley	48	0.11	< 0.01	0.18	0.01	0.02	0.16
pigs-pp	48	0.12	< 0.01	0.76	0.02	0.04	0.52
david	87	0.11	0.02	3.15	0.04	0.06	1.62
celar04-pp	114	0.08	0.04	5.73	0.14	0.23	9.85
1bkb-pp	127	0.18	0.06	198.05	1.14	4.18	107.32
miles1500	128	0.64	0.06	44.57	0.10	0.14	7.05
celar10-pp	133	0.07	0.06	8.93	1.96	4.72	18.43
munin2-pp	167	0.03	0.11	3.81	0.80	3.37	30.21
mulsol.i.5	186	0.23	0.09	37.88	0.13	0.27	8.80
zeroin.i.2	211	0.16	0.06	18.70	0.09	0.11	5.85
boblo	221	0.01	0.29	3.39	0.28	0.56	46.22
fpsol2.i-pp	233	0.40	0.18	189.11	0.36	0.74	56.63
munin4-wpp	271	0.02	0.61	57.87	1.98	6.66	367.37

It is expected that the IUN heuristic and LEASTCUTVALUE heuristic give the same linear boolean-width, since both these heuristics greedily select the vertex that minimizes the boolean dimension. The RELATIVENEIGHBORHOOD heuristic performs worse than all other heuristics in nearly all cases. While the difference might not seem very large, note that algorithms parameterized by boolean-width are exponential in the width of a decomposition. The 2-IUN heuristic outperforms IUN in three cases while n-IUN gives a better decomposition in 11 out of 13 cases, which shows that a good initial vertex is of great influence on the width of the decomposition.

Looking at the times displayed in Table 2 for computing each decomposition we see that the RELATIVENEIGHBORHOOD heuristic is significantly faster. This is to be expected because of the $O(n^3)$ time, compared to the exponential time for all other heuristics. The interesting comparison that we can make is the difference between the IUN heuristic and LEASTCUTVALUE heuristic. While both of these heuristics give the same decomposition, IUN is significantly faster. Additionally, even 2-IUN and n-IUN are often faster than the LEASTCUTVALUE heuristic.

In Table 8 (Appendix B.2) we show linear boolean-width upperbounds that are obtained through using the IUN heuristic with all starting vertices and *candidates = Right*. We compare this with the best known tree-width and boolean-width values. Examining these results, it seems that linear boolean-width seem to be more useful in practice than boolean-width heuristics. However, one should note that on certain graph classes, for instance graphs which look like trees, boolean-width is a lot lower than linear boolean-width.

6.3 Vertex subset experiments

We have used the linear decompositions given by the n-IUN heuristic to compute the size of the maximum induced matching (MIM) in a selection of graphs, of which the results are presented in Table 3. The maximum induced matching problem is defined as finding the largest $(\{1\}, \mathbb{N})$ set, with $d(\{1\}, \mathbb{N}) = 2$. The choice for the MIM problem is arbitrary, any vertex subset problem with $d = 2$ will have the same number of equivalence classes and therefore they all require the same time when computing a solution. We present the computed value of $nec_d(T, \delta)$, together with theoretical upperbounds. For $d = 2$ a tight

upperbound in terms of boolean-width is not known. Note that we take the logarithm of each value, since we find this value easier to interpret and compare to other graph parameters. We let $UB_1 = 2^{d \cdot \text{boolw}^2}$, $UB_2 = (d+1)^{\min ntc}$ and $UB_3 = ntc^{d \cdot \text{boolw}}$, with $ntc = \max_{w \in T} ntc(\delta(w))$ and $\min ntc = \max_{w \in T} \min(ntc(\delta(w)), ntc(\overline{\delta(w)}))$.

The column *MIM* displays the size of the MIM in the graph, while the time column indicates the time needed to compute this set. Missing values for *nec* and MIM are caused by a lack of internal memory, because of the $O^*(nec_d(T, \delta)^2)$ space requirement. One can immediately see that there is a large gap between the upperbound for nec_2 in terms of boolean-width and nec_2 itself. Another interesting observation we can make by looking at the graphs zeroin.i.2 and boblo, is that a lower boolean-width does not imply a lower nec_2 . We even encountered this for decompositions of the same graph: for the graph barley we observed $\text{boolw}(T, \delta) = 4.58$ and $\text{boolw}(T', \delta') = 4.81$, while $\log_2(nec_2(T, \delta)) = 7.00$ and $\log_2(nec_2(T', \delta')) = 6.75$. This suggests that this upperbound does not justify minimizing nec_2 through boolean-width in practice.

■ **Table 3** Results of using the algorithm by Bui-Xuan et al. [3] for solving (σ, ρ) problems on graphs, using decompositions obtained through the n-IUN heuristic.

Graph	boolw	$\log_2(nec)$	$\log_2(UB_1)$	$\log_2(UB_2)$	$\log_2(UB_3)$	<i>MIM</i>	Time (s)
barley	4.58	7.00	42.04	12.68	27.51	22	3
pigs-pp	6.64	10.31	88.28	19.02	49.17	22	1147
david	5.86	9.37	68.63	22.19	44.61	34	919
celar04-pp	7.27	11.15	105.61	28.53	65.74	-	-
1bkb-pp	9.53	-	181.47	52.30	98.49	-	-
miles1500	5.29	9.30	55.87	34.87	49.69	8	4038
celar10-pp	6.91	10.34	95.41	25.36	59.70	50	10179
munin2-pp	7.61	11.82	115.97	19.02	54.60	-	-
mulsol.i.5	3.58	6.11	25.70	14.26	24.80	46	22
zeroin.i.2	3.81	6.58	28.99	20.60	28.18	30	59
boblo	4.00	6.17	32.00	9.51	20.68	148	41
fpsol2.i-pp	4.81	8.07	46.22	22.19	36.61	46	934
munin4-wpp	7.61	12.13	115.97	19.02	57.98	-	-

7 Conclusion

We have presented a new heuristic and a new exact algorithm for finding linear boolean decompositions. The heuristic has a running time that is several orders of magnitude lower than the previous best heuristic and finds a decomposition in output sensitive time. This means that if a decomposition is not found within reasonable time, then the decomposition that would have been generated is not useful for practical algorithms. Running the new heuristic once for every possible starting vertex results in significantly better decompositions compared to existing heuristics.

We have seen that if $\text{lboolw}(T, \delta) < \text{lboolw}(T', \delta')$, then there is no guarantee that $nec(T, \delta) < nec(T', \delta')$. While in general it holds that minimizing boolean-width results in a low value of number of equivalence classes, we think that it can be worthwhile to focus on minimizing the nec_d instead of the boolean-width when solving vertex subset problems. However, the number of equivalence classes is not symmetric, i.e., for a cut (A, \overline{A})

$nec_d(A) \neq nec_d(\overline{A})$, which makes it harder to develop fast heuristics that focus on minimizing nec_d since we need to keep track of both the equivalence classes of A and \overline{A} .

Further research can be done in order to obtain even better heuristics and better upperbounds on both the linear boolean-width and boolean-width on graphs. For instance, combining properties of the INCREMENTAL-UN-HEURISTIC and the RELATIVE-NEIGHBORHOOD heuristic might lead to better decompositions, as they make use of complementary features of a graph. Another approach for obtaining good decompositions could be a branch and bound algorithm that makes us of trivial cases that are used in the heuristics. To decrease the time needed by the heuristics one can investigate reduction rules for linear boolean-width. While most reduction rules introduced by Sharmin [12] for boolean-width do not hold for linear boolean-width, they can still be used on a graph after which we can use our heuristic on the reduced graph. Although the resulting decomposition after reinserting the reduced vertices will not be linear, the asymptotic running time for applications does not increase [15]. Another topic of research is to compare the performance of vertex subset algorithms parameterized by boolean-width to algorithms parameterized by treewidth [16].

References

- 1 R. Belmonte and M. Vatshelle. Graph classes with structured neighborhoods and algorithmic applications. *Theoretical Computer Science*, 511(0):54 – 65, 2013. Exact and Parameterized Computation.
- 2 B.-M. Bui-Xuan, J. A. Telle, and M. Vatshelle. Boolean-width of graphs. In *IWPEC 2009*, volume 5917 of *LNCS*, pages 61–74. Springer, 2009.
- 3 B.-M. Bui-Xuan, J. A. Telle, and M. Vatshelle. Fast dynamic programming for locally checkable vertex subset and vertex partitioning problems. *Theoretical Computer Science*, 511:66–76, 2013.
- 4 V. M.F. Dias, C. M.H. de Figueiredo, and J. L. Szwarcfiter. On the generation of bicliques of a graph. *Discrete Applied Mathematics*, 155(14):1826 – 1832, 2007.
- 5 P. Erdős and A. Rényi. On random graphs. *Publicationes Mathematicae 6: 290–297*, 1959.
- 6 S. Gaspers, D. Kratsch, and M. Liedloff. On independent sets and bicliques in graphs. In *Proc. WG 2008*, volume 5344 of *LNCS*, pages 171–182. Springer, 2008.
- 7 E. M. Hvidevold, S. Sharmin, J. A. Telle, and M. Vatshelle. Finding good decompositions for dynamic programming on dense graphs. In *IWPEC 2012*, volume 7112 of *LNCS*, pages 219–231. Springer, 2012.
- 8 K. H. Kim. *Boolean matrix theory and its applications (Monographs and textbooks in pure and applied mathematics)*. Marcel Dekker, 1982.
- 9 F. Manne and S. Sharmin. Efficient counting of maximal independent sets in sparse graphs. In *Experimental Algorithms*, volume 7933 of *LNCS*, pages 103–114. Springer, 2013.
- 10 Y. Rabinovich, J. A. Telle, and M. Vatshelle. Upper bounds on boolean-width with applications to exact algorithms. In *IWPEC 2013*, volume 8246 of *LNCS*, pages 308–320. Springer, 2013.
- 11 N. Robertson and P. D. Seymour. Graph minors. I. Excluding a forest. *Journal of Combinatorial Theory, Series B*, 35(1):39 – 61, 1983.
- 12 S. Sharmin. *Practical Aspects of the Graph Parameter Boolean-width*. PhD thesis, University of Bergen, Norway, 2014.
- 13 J. A. Telle. Complexity of domination-type problems in graphs. *Nordic Journal of Computing*, 1(1):157–171, 1994.
- 14 Treewidthlib. <http://www.staff.science.uu.nl/~bodla101/treewidthlib/>. A benchmark for algorithms for treewidth and related graph problems.

14 Practical Algorithms for Linear Boolean-width

- 15 F. J. P. van Houten. Experimental research and algorithmic improvements involving the graph parameter boolean-width. Master's thesis, Utrecht University, The Netherlands, 2015.
- 16 J. M. M. van Rooij, H. L. Bodlaender, and P. Rossmanith. Dynamic programming on tree decompositions using generalised fast subset convolution. In *Algorithms - ESA 2009*, volume 5757 of *LNCS*, pages 566–577. Springer, 2009.
- 17 M. Vatshelle. *New width parameters of graphs*. PhD thesis, University of Bergen, Norway, 2012.

A

 Omitted proofs

A.1 Proof of Theorem 5

► **Claim.** For any graph G it holds that $\text{lboolw}(G) \leq \text{pw}(G) + 1$.

Proof. We give a method of construction that gives us a linear boolean decomposition of a graph G from a path decomposition of G . Recall that a linear boolean decomposition can be defined through a linear ordering $\pi = \pi_1, \dots, \pi_n$ of V . The idea is that given a path decomposition X_1, \dots, X_n we select vertices one by one from a subset X_i and append them to the linear ordering π , after which we move on to X_{i+1} . For shorthand notation we denote

$$\chi_i = \bigcup_{j=1}^i X_j.$$

Let $S_i = \{u \mid u \in \chi_i : N(u) \cap \overline{\chi_i} \neq \emptyset\}$. For each $u \in S_i$ it holds that $\exists j > i \exists w \in X_j$ for which $\{u, w\} \in E$. By definition of a path decomposition we know that there is a subset X_j with $u, w \in X_j$, and since all subsets containing a certain vertex are subsequent in the path decomposition, it follows that $u \in X_i$ and $u \in X_{i+1}$, implying that $S_i \subseteq X_i$ and $S_i \subseteq X_{i+1}$. By definition, the unions of neighborhoods of χ_i can only consist of neighborhoods of subsets of S_i , thus it follows that $|\mathcal{UN}(\chi_i)| = 2^{\text{bool-dim}(\chi_i)} \leq 2^{|S_i|} \leq 2^{|X_i|} \leq 2^{\text{pw}(G)+1}$. What remains to be shown is that while appending vertices one by one from a subset X_{i+1} , the number of unions of neighborhoods will not exceed $2^{|X_{i+1}|}$ at any point. For each vertex $v \in X_{i+1}$ there are two possibilities. If $v \in S_i$, then appending v to the linear ordering will not increase the boolean dimension, since v 's neighborhood was already an element of the unions of neighborhoods constructed so far. If $v \notin S_i$, then it is possible that v will contribute a new neighborhood to the unions of neighborhoods, which will cause factor 2 increase in the worst case. There are at most $|X_{i+1} \setminus S_i|$ such vertices, and because $S_i \subseteq X_{i+1}$, it follows that $|X_{i+1} \setminus S_i| = |X_{i+1}| - |S_i|$. We conclude that at any point during construction it holds that

$$|\mathcal{UN}(\chi_{i+1})| = 2^{\text{bool-dim}(\chi_{i+1})} \leq 2^{|S_i|} \cdot 2^{|X_{i+1}| - |S_i|} = 2^{|X_{i+1}|} \leq 2^{\text{pw}(G)+1}$$

◀

A.2 Adaptation of existing exact algorithms

Straightforward dynamic programming leads to the following result.

► **Theorem 19.** *A linear boolean decomposition of minimum boolean-width can be computed in $O(2.7284^n)$ time using $O(n \cdot 2^n)$ space.*

Proof. As a preprocessing step we compute for all cuts $A \subseteq V$ the values $|\mathcal{UN}(A)|$ by computing $\#\mathcal{MLS}(G[A, \overline{A}])$. Computing $\#\mathcal{MLS}$ for any graph can be done in $O(1.3642^n)$ time [6]. Doing this for all A takes $O(2.7284^n)$ time.

We solve recurrence relation (1) in a bottom-up fashion. For each iteration, the minimum of $|A|$ numbers has to be taken. Suppose $|A| = k$, then this takes $O(k)$ time for each iteration. When solving the recurrence relation, $|A|$ goes from 1 to n . Since there are $\binom{n}{k}$ subsets of size k , it takes $\sum_{k=1}^n \binom{n}{k} k = O(n \cdot 2^{n-1}) = O(n \cdot 2^n)$ time to compute all values for lboolw .

Because the preprocessing step of computing bool-dim is the bottleneck, the total time is $O(2.7284^n)$. The space requirements amount to $O(n \cdot 2^n)$, since bool-dim and lboolw contain at most 2^n entries of integers of at most n bits. ◀

The currently fastest known exact algorithm for boolean-width runs in $O^*(2^{n+K})$ [17], where K is a known upperbound for the boolean-width of the current graph. By performing a

binary search on K , we can achieve an output sensitive asymptotic running time. Theorem 20 is a direct adaptation to linear boolean-width.

► **Theorem 20.** *A linear boolean decomposition of minimum boolean-width for a graph G can be computed in $O(n^3 \cdot 2^{n+\text{lboolw}(G)})$ time using $O(n \cdot 2^n)$ space.*

Proof. As a preprocessing step we compute for all cuts $A \subseteq V$ the values $|\mathcal{UN}(A)|$, using a polynomial time delay algorithm, which lists maximal independent sets in $G[A, \bar{A}]$ with at most $O(n^3)$ time in between two results [4]. We can use the upperbound K as a limit for this algorithm, such that computing $\max(|\mathcal{UN}(A)|, K)$ takes at most $O(n^3 \cdot K)$ time.

Now consider relation (1). This can be solved in $O(n \cdot 2^n)$ time by the same reasoning as in Theorem 19. This results in a total running time of $O(n^3 \cdot 2^{n+\text{lboolw}(G)})$ by binary search on K . The space requirements amount to $O(n \cdot 2^n)$, since the tables `bool-dim` and `lboolw` contain at most 2^n entries of integers of at most n bits. ◀

A.3 Proof of Lemma 6

► **Claim.** The procedure `Increment-UN` is correct and runs in $O(n \cdot |\mathcal{UN}_X|)$ time using $O(n \cdot |\mathcal{UN}_X|)$ space.

Proof. For proof by induction, assume that all unions of neighborhoods for the cut (X, \bar{X}) saved inside the set \mathcal{UN}_X are computed correctly. For each neighborhood in \mathcal{UN}_X we only perform two actions to obtain new neighborhoods. The first action is removing v , since v cannot be in any neighborhood of $X \cup \{v\}$. The second operation is adding $N(v)$ to an existing neighborhood, which also results in a valid new neighborhood across the cut. It is clear that if a neighborhood is added to \mathcal{U} , then it is a valid neighborhood across the cut $(X \cup \{v\}, \bar{X} \setminus \{v\})$. We now show that all valid neighborhoods of the cut $(X \cup \{v\}, \bar{X} \setminus \{v\})$ are contained in \mathcal{U} . Assume for contradiction that S is a valid neighborhood not contained in \mathcal{U} . By definition, there is a set R for which $N(R) \cap (\bar{X} \setminus \{v\}) = S$. If $v \notin R$, then $N(R) \cap \bar{X} \in \mathcal{UN}_X$, meaning that we add $N(R) \cap (\bar{X} \setminus \{v\})$ to \mathcal{U} , contradicting our assumption. If $v \in R$, then $N(R \setminus \{v\}) \cap \bar{X} \in \mathcal{UN}_X$. During the algorithm we construct $(N(R \setminus \{v\}) \cup N(v)) \cap (\bar{X} \setminus \{v\})$, which is equal to $N(R) \cap (\bar{X} \setminus \{v\})$. This means that $N(R) \cap (\bar{X} \setminus \{v\})$ is added to \mathcal{U} , also contradicting our assumption. It follows that a neighborhood is contained in the set \mathcal{U} if and only if it is a valid neighborhood across the cut $(X \cup \{v\}, \bar{X} \setminus \{v\})$.

The time is determined by the number of sets S saved in \mathcal{UN}_X . The number of unions of neighborhoods that we iterate over does not exceed $|\mathcal{UN}_X|$. The set operations that are performed for each S take at most $O(n)$ time. This results in the total time for this algorithm to be $O(n \cdot |\mathcal{UN}_X|)$. The space requirements amount to $O(n \cdot |\mathcal{UN}_X|)$, for storing \mathcal{U} which contains at most $O(|\mathcal{UN}_X|)$ sets of size at most $O(n)$. ◀

A.4 Proof of Theorem 7

► **Claim.** Given a graph G , Algorithm 2 can be used to compute `lboolw`(G) in $O(n \cdot 2^{n+\text{lboolw}(G)})$ time using $O(n \cdot 2^n)$ space.

Proof. Iteratively double K in Algorithm 2, starting with $K = 1$, until it returns a number that is not ∞ . By Lemma 21 this will take $O(\sum_{\log K=1}^{\text{lboolw}(G)} n \cdot 2^{n+\log K}) = O(n \cdot 2^{n+\text{lboolw}(G)+1}) = O(n \cdot 2^{n+\text{lboolw}(G)})$ and take $O(n \cdot 2^n)$ space. ◀

► **Lemma 21.** *Given a graph $G = (V, E)$ of size n and an integer K , Algorithm 2 computes the linear boolean width, if it is at most $\log K$, in $O(n \cdot K \cdot 2^n)$ time using $O(n \cdot 2^n)$ space.*

Proof. Consider the first part of procedure INCREMENTAL-UN-EXACT, where the call to the procedure COMPUTE-COUNT-UN is made. It may not be immediately clear that $T_{\mathcal{UN}}$ is always computed when necessary, since there may be X such that $T_{\mathcal{UN}}(X)$ is not computed, while $T_{\mathcal{UN}}(X) \leq K$. Suppose that $X \subseteq V$ of size i occurs in an optimal decomposition and $T_{\mathcal{UN}}(X)$ has not been computed. Since we are dealing with linear decompositions, there exists an ordering v_1, \dots, v_i of X such that for all $1 \leq j \leq i$, the set $X_j = \bigcup_{0 \leq j' \leq j} v_{j'}$ also occurs in the optimal decomposition. Obviously this implies that $T_{\mathcal{UN}}(X_j) \leq K$ for all j . But this means that for all these X_j the if-statement on line 23 evaluates to true. But that means that $T_{\mathcal{UN}}(X)$ must be computed, contradiction. Thus we conclude that $T_{\mathcal{UN}}$ is computed correctly throughout the algorithm. The second part of procedure INCREMENTAL-UN-EXACT simply solves the recurrence in a bottom-up dynamic programming fashion. Finally, the procedure INCREMENT-UN is correct by Lemma 6.

We now analyze the running time. Consider the procedure COMPUTE-COUNT-UN. We observe that the procedure can only be called once for each $X \subseteq V$, because as soon as the call is made, $T_{\mathcal{UN}}(X)$ will be defined and line 20 prevents further calls with equal X . At every call the for-loop has to make at most n iterations, thus we obtain $O(n \cdot 2^n)$ iterations in total. If line 20 evaluates false, the body of the for-loop takes constant time. If line 20 evaluates true, the call to INCREMENT-UN takes $O(n \cdot 2^K)$ time (by Lemma 6), as $|\mathcal{UN}_X| \leq K$ (otherwise by line 23 the call to COMPUTE-COUNT-UN would not have been made). Because line 20 only returns true at most $O(2^n)$ times, the time of COMPUTE-COUNT-UN amounts to $O(n \cdot 2^{n+K})$. Consider the rest of the code in INCREMENTAL-UN-EXACT. The three outer for-loops account for $n \cdot 2^n$ executions of the inner code block, which take $O(1)$ time, resulting in $O(n \cdot 2^n)$ time in total. Thus, in total the time amounts $O(n \cdot 2^{n+K})$.

For the space requirements, we observe that the tables $T_{\mathcal{UN}}$ and S are of size at most 2^n storing numbers of n bits. Moreover, the recursion of COMPUTE-COUNT-UN can be at most n deep, so only n unions of neighborhoods have to be stored, which are at most of size $n \cdot 2^K$. Since $O(n \cdot 2^K) \subseteq O(n \cdot 2^{n/2}) \subsetneq O(n \cdot 2^n)$, the total space requirements amount to $O(n \cdot 2^n)$. ◀

A.5 Proof of Lemma 8

► **Claim.** Let $X \subseteq \text{Left}$. If $\exists v \in \text{Right}$ such that $N(v) \cap \text{Right} = N(X) \cap \text{Right}$, then choosing v will not change the boolean-width of the resulting decomposition.

Proof. The choice for v will not change the unions of neighborhoods in any way, which means that $\mathcal{UN}(\text{Left}) = \mathcal{UN}(\text{Left} \cup \{v\})$. Thus, for any vertex in $\text{Right} \setminus \{v\}$ it will hold that it will interact in the exact same way with $\mathcal{UN}(\text{Left})$ as it would with $\mathcal{UN}(\text{Left} \cup \{v\})$, resulting in the boolean dimension of the computed ordering being the same. ◀

A.6 Proof of Lemma 9

► **Claim.** The mapping $\frac{a}{b} \mapsto \frac{a}{a+b}$ is order preserving.

Proof. Suppose $\frac{a}{b} \leq \frac{c}{d}$. Then $ad - bc \leq 0$. Now we see that

$$\frac{a}{a+b} - \frac{c}{c+d} = \frac{a(c+d) - c(a+b)}{(c+d)(a+b)} = \frac{ac + ad - ac - bc}{(c+d)(a+b)} = \frac{ad - bc}{(c+d)(a+b)} \leq 0$$

Thus $\frac{a}{a+b} \leq \frac{c}{c+d}$. ◀

A.7 Proof of Theorem 10

► **Claim.** The INCREMENTAL-UN-HEURISTIC procedure runs in $O(n^3 \cdot 2^k)$ time using $O(n \cdot 2^k)$ space, where k is the boolean-width of the resulting linear decomposition.

Proof. The time is determined by the number of sets saved in \mathcal{UN}_{Left} . The worst case consisting of $Candidates = Right$ will result in at most n iterations and calls to INCREMENT-UN. This call takes $O(n \cdot 2^{|\mathcal{UN}_{Left}|})$ time by Lemma 6. By definition $|\mathcal{UN}_{Left}|$ never exceeds 2^k , where k is the boolean-width of the resulting decomposition. Because we need to make n greedy choices to process the entire graph, we conclude that the total time for this algorithm is $O(n^3 \cdot 2^k)$. For the space requirements we observe that all structures in the algorithm require $O(n)$ space, except for the unions of neighborhoods. Since there are only stored two of them at any time and they require at most $O(n \cdot 2^k)$ space, the total space requirements amount to $O(n \cdot 2^k)$. ◀

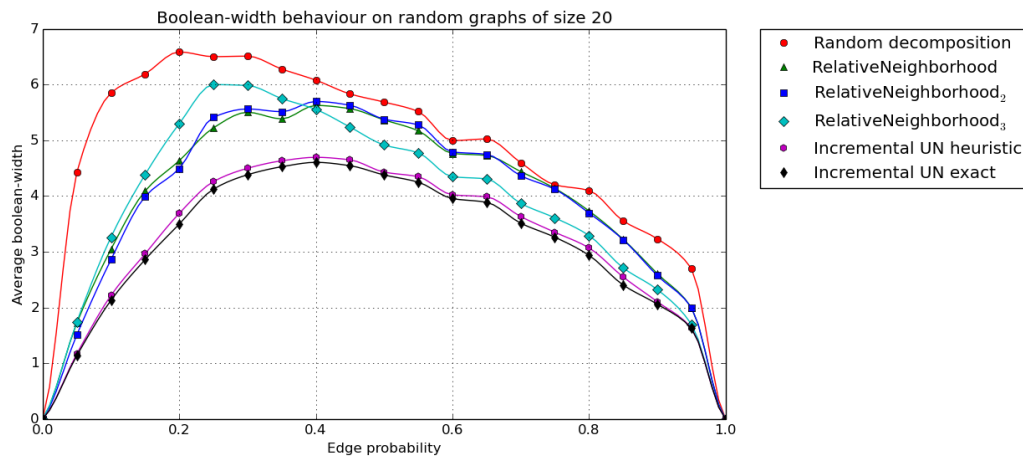
A.8 Proof of Lemma 18

► **Claim.** $nec(\equiv_A^d) \leq ntc(A)^{d \cdot k}$.

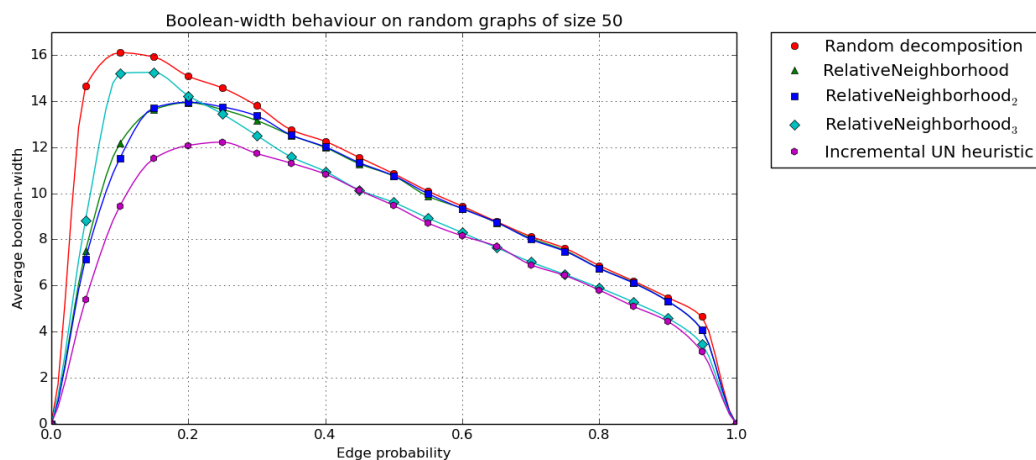
Proof. We make use of a graph parameter called *maximum induced matching-width* [1]. Let $mim(A)$ denote the maximum matching-width of A . It has been shown that for a graph G and for any subset $A \subseteq V$ it holds that $mim(A) \leq \text{bool-dim}(A)$ [17, Theorem 4.2.10]. From [17, Lemma 5.2.3] we know that $nec(\equiv_A^d) \leq ntc(A)^{d \cdot mim(A)}$, thus $nec(\equiv_A^d) \leq ntc(A)^{d \cdot k}$. ◀

B Figures and Tables

B.1 Figures



■ **Figure 2** Performance of different heuristics on random generated graphs consisting of 20 vertices, with varying edge probabilities, in terms of linear boolean-width.



■ **Figure 3** Performance of different heuristics on random generated graphs consisting of 50 vertices, with varying edge probabilities. Because of feasibility limitations, the INCREMENTAL-UN-EXACT algorithm is only used for the in Figure 3. While the optimal values are now unknown, it is clear that INCREMENTAL-UN-HEURISTIC outperforms all other heuristics. Interestingly enough, RELATIVE-NEIGHBORHOOD₃ peers with INCREMENTAL-UN-HEURISTIC as soon as the edge probability exceeds 0.4. Moreover, RELATIVE-NEIGHBORHOOD and RELATIVE-NEIGHBORHOOD₂ do not perform better than a random decomposition generator after the edge probability exceeds 0.4. We also observe that the highest boolean-width values are reached when the edge probability is around 0.1–0.2, indicating that the size of the graphs has an influence on the edge-probability-boolean-width-curve. Also note that it seems that dense random graphs have lower linear boolean-width than sparse graphs. Therefore it may be profitable to use RELATIVE-NEIGHBORHOOD₃ when dense graphs are encountered.

B.2 Tables

■ **Table 4** Linear boolean-width of the decompositions returned by the heuristics described in Section 4, with *Candidates = Right*. For 2-IUN we use two start vertices; one is obtained through a single BFS search, while the other is obtained through a double BFS search. The n-IUN heuristic uses all n start vertices, and all other heuristics use start vertices obtained through performing a double BFS.

Graph	$ V $	Edge Density	Relative	LeastCut	IUN	2-IUN	n-IUN
alarm	37	0.10	3.32	3.00	3.00	3.00	3.00
barley	48	0.11	5.70	5.91	5.91	4.70	4.58
pigs-pp	48	0.12	10.35	7.13	7.13	7.13	6.64
BN_100	58	0.17	15.84	11.56	11.56	10.86	10.86
eil76	76	0.08	8.86	8.33	8.33	8.33	8.33
david	87	0.11	9.38	6.27	6.27	6.27	5.86
ljhg	101	0.17	12.86	8.67	8.67	8.49	8.41
laac	104	0.25	20.29	12.40	12.40	12.40	12.33
celar04-pp	114	0.08	11.67	7.27	7.27	7.27	7.27
1a62	122	0.21	18.92	11.68	11.68	11.28	11.14
1bkb-pp	127	0.18	16.81	9.98	9.98	9.53	9.53
1dd3	128	0.17	16.61	9.98	9.98	9.90	9.90
miles1500	128	0.64	8.17	5.58	5.58	5.58	5.29
miles250	128	0.05	7.95	7.13	7.13	5.39	4.58
celar10-pp	133	0.07	10.32	11.95	11.95	7.64	6.91
anna	138	0.05	12.65	8.67	8.67	8.51	7.94
pr152	152	0.04	12.69	11.19	11.19	10.36	8.29
munin2-pp	167	0.03	15.17	9.61	9.61	9.61	7.61
mulsol.i.5	186	0.23	7.55	5.29	5.29	5.29	3.58
zeroin.i.2	211	0.16	7.92	4.46	4.46	4.46	3.81
boblo	221	0.01	19.00	4.32	4.32	4.32	4.00
fpsol2.i-pp	233	0.40	5.58	6.07	6.07	5.78	4.81
munin4-wpp	271	0.02	13.04	9.27	9.27	9.27	7.61

■ **Table 5** Time in seconds of the heuristics used to find the linear boolean decompositions of which the boolean-width is displayed in Table 4.

Graph	$ V $	Edge Density	Relative	LeastCut	IUN	2-IUN	n-IUN
alarm	37	0.10	< 0.01	0.02	< 0.01	< 0.01	0.06
barley	48	0.11	< 0.01	0.18	0.01	0.02	0.16
pigs-pp	48	0.12	< 0.01	0.76	0.02	0.04	0.52
BN_100	58	0.17	< 0.01	25.10	0.41	1.24	17.17
eil76	76	0.08	0.02	5.00	0.13	0.29	8.35
david	87	0.11	0.02	3.15	0.04	0.06	1.62
ljhg	101	0.17	0.03	24.46	0.21	0.48	14.75
laac	104	0.25	0.04	754.54	5.66	11.81	375.31
celar04-pp	114	0.08	0.04	5.73	0.14	0.23	9.85

Continued on next page

Table 5 – Continued from previous page

Graph	$ V $	Edge Density	Relative	LeastCut	IUN	2-IUN	n-IUN
1a62	122	0.21	0.06	585.95	3.10	11.57	376.26
1bkb-pp	127	0.18	0.06	198.05	1.14	4.18	107.32
1dd3	128	0.17	0.07	117.21	0.92	2.74	91.19
miles1500	128	0.64	0.06	44.57	0.10	0.14	7.05
miles250	128	0.05	0.02	0.56	0.05	0.10	1.24
celar10-pp	133	0.07	0.06	8.93	1.96	4.72	18.43
anna	138	0.05	0.06	20.81	0.22	0.57	19.95
pr152	152	0.04	0.10	50.74	1.76	5.66	120.06
munin2-pp	167	0.03	0.11	3.81	0.80	3.37	30.21
mulsol.i.5	186	0.23	0.09	37.88	0.13	0.27	8.80
zeroin.i.2	211	0.16	0.06	18.70	0.09	0.11	5.85
boblo	221	0.01	0.29	3.39	0.28	0.56	46.22
fpsol2.i-pp	233	0.40	0.18	189.11	0.36	0.74	56.63
munin4-wpp	271	0.02	0.61	57.87	1.98	6.66	367.37

■ **Table 6** Results of using the algorithm by Bui-Xuan et al. [3] for solving (σ, ρ) problems on graphs, using decompositions obtained using the IUN heuristic using all starting vertices. The columns UB indicate theoretical upperbounds on the number of equivalence classes, with $UB_1 = 2^{d \cdot \text{boolw}^2}$, $UB_2 = (d+1)^{\min ntc}$ and $UB_3 = ntc^{d \cdot \text{boolw}}$, with $ntc = \max_{w \in T} ntc(\delta(w))$ and $\min_{w \in T} ntc = \max_{w \in T} \min(ntc(\delta(w)), ntc(\overline{\delta(w)}))$.

Graph	boolw	$\log_2(nec)$	$\log_2(UB_1)$	$\log_2(UB_2)$	$\log_2(UB_3)$	MIM	Time (s)
alarm	3.00	4.32	18.00	7.92	13.93	18	< 1
barley	4.58	7.00	42.04	12.68	27.51	22	3
pigs-pp	6.64	10.31	88.28	19.02	49.17	22	1147
BN_100	10.86	-	235.93	36.45	105.53	-	-
eil76	8.33	12.63	138.81	22.19	65.10	-	-
david	5.86	9.37	68.63	22.19	44.61	34	919
ljhg	8.41	13.53	141.58	41.21	81.75	-	-
laac	12.33	-	304.08	72.91	141.25	-	-
celar04-pp	7.27	11.15	105.61	28.53	65.74	-	-
1a62	11.14	-	248.09	60.23	121.61	-	-
1bkb-pp	9.53	-	181.47	52.30	98.49	-	-
1dd3	9.90	-	196.11	52.30	103.17	-	-
miles1500	5.29	9.30	55.87	34.87	49.69	8	4038
miles250	4.58	7.24	42.04	15.85	31.72	52	37
celar10-pp	6.91	10.34	95.41	25.36	59.70	50	10179
anna	7.94	11.94	125.98	33.28	75.48	-	-
pr152	8.29	12.76	137.45	22.19	63.13	-	-
munin2-pp	7.61	11.82	115.97	19.02	54.60	-	-
mulsol.i.5	3.58	6.11	25.70	14.26	24.80	46	22
zeroin.i.2	3.81	6.58	28.99	20.60	28.18	30	59
boblo	4.00	6.17	32.00	9.51	20.68	148	41

Continued on next page

Table 6 – Continued from previous page

Graph	boolw	$\log_2(nec)$	$\log_2(UB_1)$	$\log_2(UB_2)$	$\log_2(UB_3)$	<i>MIM</i>	Time (s)
fpsol2.i-pp	4.81	8.07	46.22	22.19	36.61	46	934
munin4-wpp	7.61	12.13	115.97	19.02	57.98	-	-

■ **Table 7** Width of linear boolean decompositions found with the IUN heuristic using the start vertices returned by performing a double BFS, and with $candidates = N^2(Left) \cap Right$ in order to decrease the computation time. The values of the two others heuristics are taken from [12]. Missing entries are caused by a lack of internal memory which is caused by the $O(n \cdot 2^k)$ space requirement, with k being the linear boolean-width of the computed decomposition. The last column indicates the time of the IUN heuristic.

Graph	$ V $	Edge Density	LeastUncommon	Relative	IUN	Time (s)
link-pp	308	0.02	34.81	28.68	17.44	610.09
diabetes-wpp	332	0.01	8.58	18.58	5.32	1.53
link-wpp	339	0.02	35.00	29.03	16.79	374.04
celar10	340	0.02	20.81	15.00	10.17	1.83
celar11	340	0.02	19.54	14.70	10.80	1.88
rd400	400	0.01	34.73	21.32	17.01	1,007.03
diabetes	413	0.01	29.32	19.32	-	-
fpsol2.i.3	425	0.10	15.87	8.92	7.67	2.11
pigs	441	0.01	24.04	18.00	12.39	20.08
celar08	458	0.02	24.95	15.00	10.17	2.12
d493	493	0.01	20.29	48.10	16.73	708.57
homer	561	0.01	36.22	28.49	-	-
rat575	575	0.01	16.48	37.23	-	-
u724	724	0.01	18.72	50.09	-	-
inithx.i.1	864	0.05	11.98	7.22	6.81	7.31
munin2	1003	< 0.01	31.25	12.13	11.91	61.17
vm1084	1084	< 0.01	15.21	48.95	-	-
BN_24	1819	< 0.01	4.91	2.32	2.58	610.72
BN_25	1819	< 0.01	4.64	2.32	2.58	601.41
BN_23	2425	< 0.01	8.48	3.17	2.58	1,808.29
BN_26	3025	< 0.01	6.98	2.32	3.58	4,532.83

■ **Table 8** Linear boolean-width upperbounds that are obtained through using the IUN heuristic with all starting vertices and $candidates = Right$. The tw column gives an upperbound on the treewidth, while the bw column gives an upperbound on the boolean-width, which values are taken from [12]. Cursive graph names marked with an asterisk indicate the graphs for which, in theory, the linear boolean decomposition will give a higher bound on the running time than the boolean decomposition, i.e., graphs for which $2^{2bw} > 2^{3bw}$. From this it seems that linear boolean-width seem to be more useful in practice than boolean-width heuristics. However, one should note that on certain graph classes, for instance graphs which look like trees, boolean-width is a lot lower than linear boolean-width.

Graph	$ V $	Edge Density	tw	bw	lbw	lbw/bw
celar06-pp-003	4	0.5	2	1	1	1.00

Continued on next page

Table 8 – *Continued from previous page*

Graph	$ V $	Edge Density	tw	bw	lbw	lbw/bw
<i>diabetes-pp-001*</i>	6	0.8	4	1	1.58	1.58
<i>munin3-pp-001*</i>	7	0.81	5	1	1.58	1.58
<i>munin3-pp-002*</i>	7	0.81	5	1	1.58	1.58
celar06-pp-000	8	0.43	3	1	1	1.00
diabetes-pp-002	8	0.61	4	2.32	2.32	1.00
mainuk-pp	9	0.78	6	1.58	1.58	1.00
rl5934-pp-001	10	0.44	4	2.81	3.17	1.13
f3795-pp-001	10	0.44	4	2.81	3	1.07
f3795-pp-003	10	0.44	4	2.81	3	1.07
f3795-pp-002	10	0.44	4	2.81	3.17	1.13
pathfinder-pp-001	11	0.58	5	2.58	3.32	1.29
myciel3	11	0.36	5	3	3.46	1.15
pcb3038-pp-001	11	0.4	5	3	2.81	0.94
f3795-pp-004	11	0.42	4	3	3.46	1.15
pathfinder-pp	12	0.65	6	2.58	2.81	1.09
celar11-pp-002	13	0.59	7	2.81	3.17	1.13
celar04-pp-001-000	15	0.74	9	1.58	2	1.27
weeduk	15	0.47	7	1.58	1.58	1.00
fungiuk	15	0.34	4	2	1.58	0.79
pcb3038-pp-002	15	0.3	5	3	2.81	0.94
mildew-wpp	15	0.3	4	2.58	3.32	1.29
celar04-pp-001	16	0.78	10	1.58	2	1.27
celar06-pp	16	0.84	11	1.58	1.58	1.00
celar10-pp-001	16	0.51	8	3	3.46	1.15
celar09-pp-001	16	0.51	8	3	3.17	1.06
celar08-pp-002	16	0.51	8	3	3.32	1.11
celar07-pp-002	16	0.45	7	3	3.32	1.11
barley-pp-001	16	0.42	7	3.32	3.32	1.00
celar11-pp-004	16	0.36	6	3.17	3.58	1.13
munin2-pp-005	16	0.3	5	3	3.58	1.19
munin2-pp-006	16	0.3	5	3	3.58	1.19
munin2-pp-003	16	0.3	5	3.17	3.7	1.17
munin2-pp-004	16	0.3	5	3.17	3.7	1.17
munin2-pp-007	17	0.35	7	3.46	3.58	1.03
munin2-pp-011	17	0.35	7	3.46	3.58	1.03
munin2-pp-010	17	0.35	7	3.46	3.81	1.10
munin2-pp-008	17	0.35	7	3.46	3.58	1.03
munin2-pp-009	18	0.31	6	3.46	3.81	1.10
munin2-pp-012	18	0.31	6	3.46	3.81	1.10
celar01-pp-002	19	0.65	10	2	2.32	1.16
celar02-pp	19	0.67	10	2	2	1.00
celar05-pp-001	19	0.66	11	2	2.32	1.16
celar11-pp-001	19	0.65	10	2	2.32	1.16
f3795-pp-005	19	0.22	4	3.32	3.58	1.08
water-pp-001	21	0.45	9	3.81	4.09	1.07
anna-pp	22	0.64	12	3.46	3.81	1.10

Continued on next page

Table 8 – Continued from previous page

Graph	$ V $	Edge Density	tw	bw	lbw	lbw/bw
water-pp	22	0.42	9	4.17	4.32	1.04
water-wpp	22	0.42	9	4.17	4.32	1.04
munin4-pp-001	23	0.26	8	3.58	4	1.12
munin4-pp-002	23	0.26	8	3.58	4	1.12
myciel4	23	0.28	10	5	5.49	1.10
BN_29	24	0.18	5	2	2.32	1.16
BN_28	24	0.18	5	2	2.32	1.16
queen5_5	25	0.53	18	5.29	5.67	1.07
barley-pp	26	0.24	7	3.7	3.46	0.94
fl3795-pp-006	26	0.16	5	3.81	4.17	1.09
david-pp	29	0.47	13	4.09	4.32	1.06
barley-wpp	29	0.2	7	3.81	3.58	0.94
pcb3038-pp-003	29	0.12	5	4.32	4.75	1.10
celar02-wpp	30	0.33	10	2.81	2.58	0.92
water	32	0.25	9	4.39	4.75	1.08
BN_16-pp-015	34	0.28	11	3.58	4.39	1.23
celar06-wpp	34	0.28	11	3	3.17	1.06
BN_16-pp-014	34	0.28	11	3.81	4.86	1.28
1bx7-pp	34	0.31	11	4.7	4.39	0.93
mildew	35	0.13	4	3	3.32	1.11
queen6_6	36	0.46	25	7.65	8.08	1.06
alarm	37	0.1	4	2.58	3	1.16
celar03-pp-001	38	0.34	14	5.81	6.11	1.05
<i>munin4-pp-003*</i>	38	0.16	8	3.58	5.39	1.51
munin4-pp-004	38	0.16	8	4.17	5.39	1.29
celar08-pp-001	39	0.38	16	5.09	5.21	1.02
oesoca	39	0.09	3	2.32	3	1.29
1bx7	41	0.24	11	4.91	4.75	0.97
oesoca42	42	0.08	3	2.32	3.17	1.37
celar07-pp-001	45	0.32	16	5.46	5.86	1.07
celar01-pp-001	47	0.25	15	5.88	6.36	1.08
celar05-pp-002	47	0.25	15	6.07	5.83	0.96
myciel5	47	0.22	19	8.12	6.49	0.80
1ubq-pp	47	0.16	12	5.95	8.79	1.48
pigs-pp-001	47	0.12	9	5.95	7.07	1.19
1brf-pp	48	0.36	22	7.01	7.25	1.03
1rb9	48	0.37	22	6.77	7.17	1.06
celar11-pp-003	48	0.23	15	5.73	4.58	0.80
<i>mainuk*</i>	48	0.18	7	3.58	6.49	1.81
barley	48	0.11	7	4	3.7	0.93
pigs-pp	48	0.12	9	5.7	6.64	1.16
1brf	49	0.35	22	7.01	7.3	1.04
queen7_7	49	0.4	35	10.36	10.97	1.06
1kth-pp	51	0.33	20	7.06	5.86	0.83
li07-pp	51	0.28	15	5.55	7.18	1.29
eil51.tsp	51	0.11	9	5.78	5.78	1.00

Continued on next page

Table 8 – *Continued from previous page*

Graph	$ V $	Edge Density	tw	bw	lbw	lbw/bw
ligq-pp	52	0.37	23	6.74	7.45	1.11
1kth	52	0.32	20	7.04	6.87	0.98
lg6x	52	0.31	19	6.89	7.21	1.05
ligq	54	0.35	23	6.89	7.61	1.10
zeroin.i.1-pp	54	0.89	46	1.58	1.58	1.00
1e0b-pp	55	0.33	24	7.69	8.32	1.08
munin4-pp-006	55	0.11	8	4.32	5.17	1.20
munin4-pp-005	55	0.11	8	4.39	5.17	1.18
1j75	56	0.36	27	8.51	8.94	1.05
1k61-pp	56	0.37	26	8.02	8.37	1.04
lsem-pp	56	0.37	26	8.09	8.5	1.05
1bbz-pp	56	0.35	25	8.18	8.36	1.02
1bf4-pp	57	0.39	26	7.63	7.79	1.02
1cka	57	0.38	27	8.55	8.87	1.04
lsem	57	0.36	26	8.32	8.66	1.04
zeroin.i.2-pp	57	0.69	32	2.81	3.32	1.18
zeroin.i.3-pp	57	0.69	32	3	3.32	1.11
1bbz	57	0.34	25	8.3	8.36	1.01
loai-pp	57	0.32	22	7.94	8.28	1.04
1jo8	58	0.37	27	8.46	8.73	1.03
loai	58	0.32	22	7.87	8.15	1.04
celar01-pp-003	58	0.19	15	6.97	6.89	0.99
lg2b-pp	59	0.37	28	8.5	8.99	1.06
ligd-pp	59	0.36	25	7.66	7.9	1.03
1kq1-pp	59	0.35	27	8.63	8.94	1.04
1pwt-pp	59	0.38	29	8.85	9.24	1.04
1i07	59	0.23	15	5.52	5.93	1.07
1k61	60	0.33	26	8.32	8.81	1.06
1kq1	60	0.34	27	8.79	8.89	1.01
1ku3-pp	60	0.33	23	7.46	7.53	1.01
1e0b	60	0.29	24	8.13	8.42	1.04
knights8_8-pp	60	0.09	16	10.77	11.3	1.05
lgut-pp	61	0.33	22	7.19	7.54	1.05
1i2t	61	0.35	27	8.38	9.03	1.08
ligd	61	0.34	25	7.75	7.9	1.02
1pwt	61	0.36	29	8.81	9.27	1.05
1ku3	61	0.32	23	7.53	7.61	1.01
lg2b	62	0.34	28	8.72	9.05	1.04
1fr3-pp	62	0.32	21	7.16	7.29	1.02
celar04-pp-002	62	0.17	16	6.86	7.26	1.06
1bf4	63	0.34	26	7.9	8.09	1.02
1r69	63	0.35	30	9.12	9.51	1.04
munin1-pp-001	63	0.09	11	5.58	6.43	1.15
lgcq-pp	64	0.36	30	8.95	9.38	1.05
queen8_8	64	0.36	45	13.16	14.05	1.07
1a8o	64	0.27	25	9.11	9.3	1.02

Continued on next page

Table 8 – Continued from previous page

Graph	$ V $	Edge Density	tw	bw	lbw	lbw/bw
knights8_8	64	0.08	16	11.06	11.64	1.05
1fjl	65	0.29	26	7.9	8.49	1.07
1c9o	66	0.34	29	8.75	8.88	1.01
1hg7	66	0.33	29	8.81	9.13	1.04
1ezg	66	0.25	23	8.33	7	0.84
1en2-pp	66	0.21	17	7.46	8.54	1.14
munin1-pp	66	0.09	11	5.58	6.43	1.15
1c4q	67	0.34	31	9.45	9.71	1.03
1fse	67	0.33	27	8.58	8.75	1.02
1kw4	67	0.3	28	9.39	5.73	0.61
1gut	67	0.28	22	7.47	7.36	0.99
1fr3	67	0.28	21	7.29	7.47	1.02
1b67-pp	67	0.25	16	6.61	9.61	1.45
1gcq	68	0.33	30	9.36	9.65	1.03
1ail-pp	68	0.28	24	8.11	8.33	1.03
1d3b-pp	68	0.3	25	8.54	5.78	0.68
1b67	68	0.25	16	6.61	8.52	1.29
1c75	69	0.29	30	9.88	8.31	0.84
1ail	69	0.27	24	8.07	9.68	1.20
1d3b	69	0.29	25	8.44	8.53	1.01
1en2	69	0.2	17	7.24	7	0.97
1cc8	70	0.34	32	9.35	9.63	1.03
1dj7-pp	70	0.3	27	8.12	8.22	1.01
1i27-pp	70	0.3	27	8.67	8.82	1.02
1l9l	70	0.29	29	9.26	10	1.08
1ljo-pp	71	0.31	30	8.92	9.02	1.01
1dp7-pp	71	0.3	27	9.21	9.15	0.99
graph03-pp-001	71	0.11	20	12.53	12.24	0.98
1mgq-pp	72	0.31	28	8.98	9.08	1.01
1i27	73	0.28	27	8.78	9.06	1.03
multsol.i.1-pp	73	0.83	50	2.32	2.58	1.11
1dj7	73	0.28	27	9.66	8.22	0.85
1ldd	74	0.31	32	9.6	9.73	1.01
1ljo	74	0.29	30	8.88	9.06	1.02
1mgq	74	0.3	28	8.91	9.06	1.02
huck	74	0.11	10	2.81	3.32	1.18
1ubq	74	0.08	12	6.61	7.75	1.17
1ig5	75	0.29	33	10.45	10.64	1.02
1dp7	76	0.27	27	9.01	9.3	1.03
celar10-pp-002	76	0.15	16	7.25	6.58	0.91
celar08-pp-003	76	0.15	16	7.41	6.58	0.89
celar09-pp-002	76	0.15	16	7.46	6.58	0.88
1iqz	77	0.29	33	10	10.1	1.01
1qtn-pp	77	0.25	24	8.56	8.33	0.97
munin3-pp-003*	79	0.09	7	4.17	12.73	3.05
graph03-pp	79	0.1	20	12.99	5.61	0.43

Continued on next page

Table 8 – Continued from previous page

Graph	$ V $	Edge Density	tw	bw	lbw	lbw/bw
sudoku-elim1	80	0.28	45	9.47	12	1.27
<i>jean*</i>	80	0.08	9	3.91	6.54	1.67
celar05-pp	80	0.13	15	7.2	4.58	0.64
sudoku	81	0.25	45	9	12.7	1.41
celar03-pp	81	0.13	14	6.19	6.11	0.99
graph03-wpp	84	0.09	20	12.74	12.92	1.01
1fk5	85	0.23	31	10.76	10.1	0.94
1aba	85	0.25	29	10.13	10.81	1.07
graph01-pp-001	85	0.09	24	13.4	13.66	1.02
1ctj-pp	86	0.25	33	10.78	11.07	1.03
1ctj	87	0.25	33	10.74	11.04	1.03
1ptf	87	0.3	38	11.21	10.86	0.97
1qtn	87	0.21	24	9.15	8.97	0.98
david	87	0.11	13	5.32	5.86	1.10
graph05-pp-001	87	0.1	24	12.68	13.31	1.05
1awd	89	0.28	38	10.8	11.13	1.03
celar03-wpp	89	0.11	14	6.17	6.49	1.05
celar05-wpp	89	0.11	15	7.52	6.54	0.87
graph01-pp	89	0.08	24	14.62	13.96	0.95
munin1-wpp	90	0.05	11	7.23	7.58	1.05
1jhg-pp	91	0.19	25	8.34	8.41	1.01
graph05-pp	91	0.1	24	13.84	13.49	0.97
celar07-pp	92	0.12	16	6	6	1.00
a280.tsp-pp	92	0.06	14	8.23	7.38	0.90
<i>kroE100.tsp-pp*</i>	92	0.06	10	6.48	14.84	2.29
1g2r-pp	93	0.26	37	11.87	11.51	0.97
graph01-wpp	93	0.07	24	14.69	11.41	0.78
1czp	94	0.27	38	11.47	11.6	1.01
1g2r	94	0.25	37	12.17	14.19	1.17
graph05-wpp	94	0.09	24	14.38	13.18	0.92
1c5e	95	0.26	36	11.06	10.83	0.98
myciel6	95	0.17	35	13.4	7.86	0.59
homer-pp	95	0.17	31	14.61	13.88	0.95
kroA100.tsp-pp	95	0.06	10	7.61	6.58	0.86
celar11-pp	96	0.1	15	6.64	5.98	0.90
munin3-pp	96	0.07	7	4.32	5.86	1.36
celar07-wpp	97	0.01	16	6	7.17	1.20
<i>kroC100.tsp-pp*</i>	97	0.06	10	6.94	11.97	1.72
1plc	98	0.25	35	11.28	11.1	0.98
1lkk-pp	99	0.24	34	11	10.84	0.99
1d4t-pp	99	0.23	35	11.88	6.58	0.55
celar11-wpp	99	0.1	15	7.17	4.91	0.68
1i0v	100	0.24	41	12.21	12.47	1.02
celar02	100	0.06	10	3.32	4.91	1.48
<i>celar06*</i>	100	0.07	11	3.81	14.85	3.90
graph05	100	0.08	24	13.7	13.36	0.98

Continued on next page

Table 8 – Continued from previous page

Graph	$ V $	Edge Density	tw	bw	lbw	lbw/bw
graph01	100	0.07	24	14.61	14.21	0.97
graph03	100	0.07	20	13.29	8.41	0.63
1erv	101	0.25	41	12.26	12.44	1.01
1jhg	101	0.17	25	8.87	11.97	1.35
1iib-pp	102	0.27	40	11.98	11.76	0.98
1d4t	102	0.22	35	12.87	10.31	0.80
1iib	103	0.26	40	12.62	11.79	0.93
1b0n	103	0.19	32	10.81	11.17	1.03
1lkk	103	0.22	34	11.89	13.56	1.14
1aac	104	0.25	41	12.29	12.33	1.00
1bkf-pp	105	0.23	36	11.1	11.4	1.03
1bkf	106	0.23	36	11.69	11.44	0.98
1bkr	107	0.24	44	14.4	13.75	0.95
1rro	107	0.23	43	15.36	3.58	0.23
1f9m	109	0.23	45	14.27	13.56	0.95
<i>pathfinder*</i>	109	0.04	6	3.32	10.83	3.26
celar04-pp	110	0.09	16	7.29	7.27	1.00
1fs1	114	0.21	34	13.79	7.36	0.53
celar04-wpp	116	0.07	16	7.95	11.1	1.40
1gef-pp	117	0.22	43	12.93	13.35	1.03
1gef	119	0.21	43	13.6	13.35	0.98
mulsol.i.5-pp	119	0.36	31	3	3	1.00
1a62-pp	120	0.21	37	14.7	11.14	0.76
1a62	122	0.21	37	13.62	9.68	0.71
1dd3-pp	124	0.17	31	14.6	9.25	0.63
ch130.tsp-pp	125	0.05	12	8.67	9.53	1.10
1kb-pp	127	0.18	30	15.55	9.9	0.64
miles1500	128	0.64	77	4.86	5.29	1.09
1dd3	128	0.17	31	11.68	4.58	0.39
miles500	128	0.14	22	9.42	7.04	0.75
<i>miles250*</i>	128	0.05	9	4.95	9.61	1.94
1kb	131	0.17	30	14.53	6.91	0.48
celar10-pp	133	0.07	16	9.08	7.7	0.85
anna	138	0.04	12	6.67	7.25	1.09
celar09-wpp	142	0.06	16	8.49	7	0.82
celar01-pp	157	0.07	15	7.39	7	0.95
celar01-wpp	158	0.06	15	7.09	7.61	1.07
munin2-pp	167	0.03	7	5.49	6.91	1.26
mulsol.i.3	184	0.23	32	4.95	3.58	0.72
mulsol.i.4	185	0.23	32	4.81	3.58	0.74
mulsol.i.5	186	0.23	31	4.95	3.58	0.72
mulsol.i.2	188	0.22	32	4.81	3.58	0.74
celar08-wpp	190	0.05	16	9.64	11.48	1.19
mulsol.i.1	197	0.2	50	4	4.17	1.04
zeroin.i.3	206	0.17	32	5.39	3.81	0.71
zeroin.i.1	211	0.19	50	3.7	3.32	0.90

Continued on next page

Table 8 – *Continued from previous page*

Graph	$ V $	Edge Density	tw	bw	lbw	lbw/bw
zeroin.i.2	211	0.16	32	5.39	3.81	0.71
fpsol2.i.1-pp	233	0.4	66	4.91	4.81	0.98