Strategic fleet planning for city logistics

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**A R T I C L E   I N F O**

Article history:
Received 26 January 2016
Revised 5 October 2016
Accepted 6 October 2016
Available online 9 November 2016

Keywords:
Fleet management
City logistics
Area partitioning
Dynamic programming

**A B S T R A C T**

We study the strategic problem of a logistics service provider managing a (possibly heterogeneous) fleet of vehicles to serve a city in the presence of access restrictions. We model the problem as an area partitioning problem in which a rectangular service area has to be divided into sectors, each served by a single vehicle. The length of the routes, which depends on the dimension of the sectors and on customer density in the area, is calculated using a continuous approximation. The aim is to partition the area and to determine the type of vehicles to use in order to minimize the sum of ownership or leasing, transportation and labor costs. We formulate the problem as a mixed integer linear problem and as a dynamic program. We develop efficient algorithms to obtain an optimal solution and present some structural properties regarding the optimal partition of the service area and the set of vehicle types to use. We also derive some interesting insights, namely we show that in some cases traffic restrictions may actually increase the number of vehicles on the streets, and we study the benefits of operating a heterogeneous fleet of vehicles.

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**1. Introduction**

Urban areas are characterized by a high concentration of small commercial activities which generally results in a very high number of vehicles movements, often uncoordinated and performed with less-than-truckload shipments. The concept of city logistics (Ehmke, 2012) recognizes the need for efficient and environmentally-conscious urban transportation policies that can improve the efficiency of transportation systems as well as reduce energy consumption and vehicles emissions. One example is the introduction of electric vehicles into logistics fleets (Roumboutsos et al., 2014). Because of their high densities and relatively short distances, cities are particularly well-suited to the early adoption of alternative types of mobility ((European Commission, 2013), such as electric vehicles.

In order to provide incentive for investment in new low energy consumption vehicles and encourage their use, several cities have enacted regulations regarding urban freight transport. These often take the form of access limitations to certain types of vehicles at certain times of the day, depending on vehicle characteristics such as dimension, type of energy consumed, engine type, etc. (Muñuzuri et al., 2005). For example, in the Dutch city of ’s-Hertogenbosch, green and silent trucks are allowed to enter the city center at any time, whereas other commercial freight vehicles are admitted only between 7:00
and 12:00, and between 18:00 and 20:00. Italian cities such as Rome, Milan, Bologna and Florence, now restrict the access of diesel vehicles to the city center at certain times of the day (e.g. from 7:00 to 20:00 in Bologna).

Faced with increasingly restrictive access regulations and with the need to reduce costs, energy use and greenhouse gas emissions, logistics service providers are looking for ways to better manage their vehicles fleet. Stewart (2012) reports that about half of the organizations he surveyed would be willing to pay a 10% premium ownership cost for an electric vehicle, due to fuel savings, CO₂ reduction and "green branding" effect.

While there exists a rich body of literature on the fleet composition problem at the operational level, e.g. Golden et al. (1984) and Koç et al. (2014), relatively little has been done at the strategic level. One of the first publications of the fleet composition problem is due to Kirby (1959) who considers a homogeneous fleet. Loxton and Lin (2011) study a multi-period heterogeneous fleet dimensioning problem in which the cost function is the sum of fixed, variable and hiring costs. They assume that the number of vehicles of a given type required in a certain period is known. Loxton et al. (2012) investigate a stochastic version of the problem, in which the future vehicle requirements follow a given probability distribution. Both studies present a solution methodology based on dynamic programming and the golden section rule. Jabali et al. (2012) develop a continuous approximation model for the heterogeneous fleet composition problem and provide a mixed integer non-linear formulation along with upper and lower bounding procedures. Their study is the first to incorporate operational aspects, such as vehicles routes, within a strategic decision model. Finally, Nourinejad and Roorda (2016) extend the study of Jabali et al. (2012) for radial networks.

In this paper, we consider the strategic problem of determining an optimal fleet composition for a logistics service provider making deliveries to an urban area in the presence of access restrictions for certain types of vehicles. These restrictions take the form of a maximum length of time that each vehicle type can spend in the service area. The problem is to determine the number and types of vehicles to use in order to minimize the sum of ownership or leasing, transportation and labor costs without exceeding the transportation capacity of the vehicles and the maximum time allowed within the service area. Specifically, we consider a rectangular urban area which is partitioned into contiguous rectangular blocks, each served by a single vehicle. We use a continuous approximation model (Daganzo, 1984a; 1984b; 1987a; 1987b) to calculate the distances traveled. This approach is particularly useful in tackling strategic decision problems since it smooths out the minor dynamic and stochastic variations in the input parameters, as discussed in Francis and Smilowitz (2006) and in Jabali et al. (2012).

Our paper contributes to the literature in several ways. First, we model the strategic problem of managing a heterogeneous vehicle fleet to serve an urban area in the presence of access restrictions. Second, we propose an efficient dynamic programming (DP)-based algorithm to calculate an optimal solution for the case of two types of vehicles (e.g., electric and diesel), and we use a mixed integer linear programming (MILP) formulation for more general settings. Third, we establish structural properties of the optimal partitioning of the service area, such as where to use each type of vehicle. Finally, we show how the optimal fleet composition changes depending on the vehicle parameters, and we discuss the impact of city access restrictions on fleet composition.

The remainder of this paper is organized as follows. In Section 2, we describe the problem and we provide a MILP formulation. In Section 3, we present our DP formulation and derive analytical results: we first consider the single-strip-single-vehicle-type case, then the single-strip-multiple-vehicle-types case, and finally the multiple-strip-multiple-vehicle-type case. In Section 4, we report some numerical results on the impact of city access restrictions and on the benefits of using a heterogeneous fleet of vehicles and compare the performance of our MILP and DP formulations. All proofs are provided in the Appendix.

2. Model

In this section we first describe the problem setting. Subsequently we present the routing strategy and the partitioning policy. Finally, we introduce the MILP formulation.

2.1. Problem setting

The continuous approximation literature proposes two main ways of approximating the shape of service area: a rectangular shape as in Daganzo (1984b); Gaboune et al. (1994); Huang et al. (2013); Newell and Daganzo (1986b) and Ouyang (2007), and a ring-radial shape as in Newell and Daganzo (1986a) and Jabali et al. (2012). In the first case the depot is located outside the area, while in the second one the depot is located in the center. In practice the shape of the zone has no major impact on the quality of the approximation (Daganzo, 1984a; Eilon et al., 1971). Motivated by the widespread policy of operating an urban consolidation center at the entrance of a city (Quak and De Koster, 2006), we model the service area as a rectangular shape and we assume that the depot is located south of the area at a distance φ from the midpoint of its bottom edge (Fig. 1). We refer to the closest edge of the rectangle as the ‘bottom’ edge and the furthest edge as the ‘top’ edge. We also use the term ‘width’ to refer to the size of horizontal edges and ‘length’ to the size of vertical edges, even if the vertical distance may be smaller than the horizontal distance. Because we focus on an urban area, we use the $L_1$ (Manhattan) norm to calculate distances. A number $e$ of customers are located in this area, and are distributed according to a density function $\delta(x)$, where $x$ is a point within the area. As in Daganzo (2005) and Huang et al. (2013), we assume that the density function $\delta(x)$ does not vary significantly within the area and therefore, without any significant loss of accuracy,
it is approximated by a continuous function \( \delta \approx e^{\frac{\text{WL}}{W}} \). We also assume that the delivery quantity to each customer is equal to 1.

Different vehicle types can be used to perform the deliveries within the service area, e.g., electric and diesel vehicles. These differ in their transportation capacity (i.e., the maximum load they can carry), as well as in their usage cost, which is made up of two components: a fixed cost (if the vehicles are purchased, this cost is the depreciation on the purchase amount; if they are leased, this cost is the rental price paid per vehicle per shift), and a variable cost, which is proportional to the distance traveled. We also consider a limit on the duration of the vehicle routes as in Jabali et al. (2012) and Langevin and Soumis (1989). However, in our setting the time limit is only applied to the time spent in the service area and is allowed to differ across vehicle types, as is the case with the city center traffic restrictions discussed in the introduction (if a vehicle type is not subject to any access restrictions, this value is set equal to the length of a driver’s shift, minus the time required to drive back and forth between the depot and the entrance to the service area). We assume that the vehicle types with the larger transportation capacity have larger variable costs and stronger restrictions, since in practice, cities tend to impose stricter access restrictions on larger delivery vehicles which are also more expensive to operate. We do not make any assumption on how the fixed costs compare.

Our problem consist of partitioning the service area into contiguous rectangles called service sectors, each served by a single vehicle without exceeding the capacity and maximum within-area route duration constraints. As observed by Huang et al. (2013), this way of distributing the workload among vehicles is useful in practical settings since it allows the drivers to be responsible for a particular area. The objective is the minimization of the total travel cost which is the sum of the fixed vehicle cost, the variable vehicle cost and the labor cost (driver wages). We describe how these costs are calculated in the following sections.

2.2. Partitioning policy

We assume that the rectangular service area is partitioned into \( s \) strips having the same width equal to \( W/s \). Each strip is then divided into a number of sectors. As depicted in Fig. 2, we assume that the strips are numbered from left to right. Similarly, the sectors in each strip are numbered from bottom to top.

Let \( y_{ij}^{j} \) denote the length of the \( j \)th strip in the \( i \)th strip when the service area is partitioned into \( s \) strips. Let \( \phi_i^j \) denote the distance from the depot to the middle point of the bottom edge of strip \( i \). This value is \( \phi_i^j = \phi + \frac{|i+1-2i|}{2} \), where \( \phi \) is the vertical distance between the depot and the middle point of the bottom edge of the service area, and \( \frac{|i+1-2i|}{2} \) is the horizontal distance from this point to the middle point of the bottom edge of strip \( i \). The vehicle serving the \( j \)th sector in strip \( i \) must first drive through sectors 1 to \( j-1 \) in order to reach its sector. Therefore, the total transit distance from the depot to the bottom edge of the \( j \)th sector is \( \mu_{ij}^j = \phi_i^j + \sum_{i=1}^{j-1} y_{ij}^{j} \), where the first term is the distance between the depot and the bottom edge of strip \( i \), and the second one is the distance between the bottom edge of strip \( i \) and the bottom edge of sector \( j \).

The total distance \( y_{ij}^{j} \) traveled by a vehicle to serve customers in the \( j \)th sector in strip \( i \) when there are \( s \) strips is calculated using a continuous approximation model. In Section 2.3 we provide a description of the routing strategy and of the continuous approximation model.

2.3. Routing strategy

We use a continuous approximation model known as the dual strip strategy or half-width routing strategy (Daganzo, 1987a), to calculate the total distance traveled by a vehicle in a service sector. Suppose the area is partitioned into \( s \) strips of equal width, so that each strip and each sector has width \( W/s \). In particular, the \( j \)th service sector in strip \( i \) has width \( W/s \). 

\[ y_{ij}^{j} = \frac{W}{s} \]  

Fig. 1. Urban service area.
Fig. 2. (a) Service area partitioned into strips and sectors. (b) Distance $\varphi$ between the depot and the middle point of the bottom edge of the service area. Distance $\psi^k$ between the depot and the middle point of the bottom edge of the 1st strip when there are 4 strips. (c) Distance $\mu^k_{ij}$ between the depot and the middle point of the bottom edge of the 3rd sector in strip 1 when there are 4 strips.

Fig. 3. Delivery tour of a vehicle serving the $j$th service sector in strip 1 in a service area divided into 4 strips.

and length $y^t_{ij}$, and the total number of customers to visit in this sector is $(y^t_{ij} \delta W)/s$. According to this strategy, the sector is divided into two halves along its width. The vehicle enters from the middle point of the bottom edge then makes a single round trip within the sector visiting all customers without backtracking, and finally exits the sector from the point of entry (Fig. 3). According to the half-width routing strategy, the total distance traveled by the vehicle, denoted $\gamma^t_{ij}$, can be broken down between the transit distance (i.e., between the depot and the bottom edge of the sector) and the distance traveled within the sector, and is approximated by:

$$\gamma^t_{ij} = 2\mu^t_{ij} + 2y^t_{ij} + \frac{y^t_{ij} \delta W^2}{6s^2} = 2\left(\varphi + \frac{|s + 1 - 2ij|}{s} \frac{W}{2} + \sum_{l=1}^{j} y^t_{il}\right) + \frac{y^t_{ij} \delta W^2}{6s^2}. \quad (1)$$

In this equation, $2\mu^t_{ij}$ is the transit distance, $2y^t_{ij}$ is the approximate vertical distance traveled by the vehicle within the sector, and $y^t_{ij} \delta W^2/(6s^2)$ is the approximate horizontal distance traveled within the sector (see Daganzo, 1987a for more details). Note that the formula for the approximate horizontal distance comes from the fact that the average horizontal distance between two consecutive points in a strip is equal to one third of the width of the strip.

2.4. MILP formulation

We now show how to formulate the problem as a MILP. Let $K$ be the number of possible vehicle types. For every type $k \in \{1, \ldots, K\}$, we denote by $Q_k$ the vehicle capacity, by $f_k$ the vehicle fixed cost, and by $o_k$ its variable cost. Let $T_k$ be the
maximum within-area route duration for a vehicle of type $k$. The travel speed is $v$, which is assumed to be constant and identical for all vehicle types (and is a realistic assumption in the context of a congested city center). Let $h$ be the service time at the customer locations (i.e., the time to unload the goods), and let $d$ denote the driver wage (in $\text{\$/time unit}$). The service area can be partitioned into a maximum of $\overline{s}$ strips, and $m_i$ is the maximum number of sectors in which the $i$th strip can be partitioned when there are $s$ strips in total. We show how to calculate these values in Section 3 (Lemma 3.2) and in Appendix C.

The decision variables are as follows:

- $x_s$: binary variable equal to 1 if and only if the area is partitioned into $s$ strips;
- $x_{ij}^k$: binary variable equal to 1 if and only if the area is partitioned into $s$ strips, and sector $j \in \{1, \ldots, m_i^k\}$ in strip $i \in \{1, \ldots, \overline{s}\}$ is served by a vehicle of type $k \in \{1, \ldots, K\}$;
- $l_{ij}^k$: length of sector $j$ in strip $i$ served by a vehicle of type $k$ when the area is partitioned into $s$ strips;
- $u_{ij}^k$: total distance traveled by a vehicle of type $k$ to serve sector $j$ in strip $i$ when the area is partitioned into $s$ strips;
- $w_{ij}^k$: total distance traveled by vehicle of type $k$ to reach the beginning of sector $j$ in strip $i$ when the area is partitioned into $s$ strips.

The value of the last three variables is positive if the area is partitioned into $s$ strips and sector $j$ in strip $i$ is served by vehicle $k$, otherwise it is 0.

The formulation is

$$\text{Minimize} \sum_{s=1}^{\overline{s}} \sum_{i=1}^{\overline{s}} \sum_{k=1}^{m_i} f_k x_{ij}^k + \sum_{s=1}^{\overline{s}} \sum_{i=1}^{\overline{s}} \sum_{j=1}^{m_i} o_k u_{ij}^k + \sum_{s=1}^{\overline{s}} \sum_{i=1}^{\overline{s}} \sum_{j=1}^{m_i} \sum_{k=1}^{K} d_i u_{ij}^k + dhLW$$

subject to

$$\sum_{s=1}^{\overline{s}} x_s = 1$$

$$\sum_{k=1}^{K} \sum_{i=1}^{\overline{s}} \sum_{j=1}^{m_i} x_{ij}^k \leq x_s \quad s = 1, \ldots, \overline{s}, i = 1, \ldots, s, j = 1, \ldots, m_i^k$$

$$\delta v_{ij}^k \frac{W}{s} \leq Q_k x_{ij}^k \quad s = 1, \ldots, \overline{s}, i = 1, \ldots, s, j = 1, \ldots, m_i^k, k = 1, \ldots, K$$

$$\sum_{j=1}^{\overline{s}} \sum_{k=1}^{m_i} v_{ij}^k = Lx_s \quad s = 1, \ldots, \overline{s}, i = 1, \ldots, s$$

$$w_{ij}^k \geq \varphi_{ij} + \sum_{l=1}^{j-1} \sum_{k=1}^{m_i^k} v_{lj}^k - M(1 - x_{ij}^k) \quad s = 1, \ldots, \overline{s}, i = 1, \ldots, s, j = 1, \ldots, m_i^k, k = 1, \ldots, K$$

$$u_{ij}^k = 2(w_{ij}^k + v_{ij}^k) + v_{ij}^k \delta W^2 / (6\delta^2) \quad s = 1, \ldots, \overline{s}, i = 1, \ldots, s, j = 1, \ldots, m_i^k, k = 1, \ldots, K$$

$$u_{ij}^k / v + v_{ij}^k \delta Wh \leq T_k + \frac{2\varphi}{v} \quad s = 1, \ldots, \overline{s}, i = 1, \ldots, s, j = 1, \ldots, m_i^k, k = 1, \ldots, K$$

$$x_s \in \{0, 1\} \quad s = 1, \ldots, \overline{s}$$

$$x_{ij}^k \in \{0, 1\} \quad s = 1, \ldots, \overline{s}, i = 1, \ldots, s, j = 1, \ldots, m_i^k, k = 1, \ldots, K$$

$$v_{ij}^k \geq 0 \quad s = 1, \ldots, \overline{s}, i = 1, \ldots, s, j = 1, \ldots, m_i^k, k = 1, \ldots, K$$

$$u_{ij}^k \geq 0 \quad s = 1, \ldots, \overline{s}, i = 1, \ldots, s, j = 1, \ldots, m_i^k, k = 1, \ldots, K$$
\[ w_{ij}^{ks} \geq 0 \quad s = 1, \ldots, 3, i = 1, \ldots, s, j = 1, \ldots, m_i^{ks}, k = 1, \ldots, K. \] (14)

The objective function is the sum of four terms: the vehicle fixed cost, the vehicle variable cost, the driver wage for the time spent traveling, and the driver wage for the time spent serving the customers (which is a constant). Constraint (3) guarantees that the area is partitioned into a positive number of strips. Constraints (4) ensure that each sector is served by at most one vehicle type. Constraints (5) imply that the capacity of the vehicle is not exceeded (recall that we assume a unit quantity delivery at each customer location so that the total quantity to be delivered in a sector is equal to the customer density \( \delta \) multiplied by the area of the sector). Constraints (6) guarantee that the sum of the lengths of the sectors in every strip is equal to \( L \). Constraints (7) compute the distance between the depot and the beginning of sector \( j \) in strip \( i \) (to speed up the calculations, \( M \) can be replaced by \( \varphi + L \)). Constraints (8) calculate the total distance traveled by a vehicle to service sector \( j \) in strip \( i \). Constraints (9) ensure that the total time required to service sector \( j \) in strip \( i \) does not exceed the maximum within-area route duration. Note that since, the \( 2\varphi \) distance is traveled outside of the service area it does not count toward the maximum within-area route duration and therefore the corresponding travel time \( 2\varphi/v \) added to \( T_k \) in the right-hand side of (9). Finally the latest five constraints define the domains of the variables. As shown in Section 4 this MILP may be slow to generate a solution. In the remainder of this paper we study some analytical properties of the problem, which will be used as a basis for developing a fast exact solution procedure.

3. Analytical results

The notation used in the paper is presented in Table A.2. Without loss of generality in the rest of the paper we assume that the driver wage \( d = 0 \).

3.1. Single strip, single vehicle type

Here we consider a special case of our problem in which there is only one strip, i.e., \( s = 1 \) and only one vehicle type \( (K = 1) \) with fixed cost \( f \), variable cost \( o \), capacity \( Q \), and maximum within-area tour duration \( T \). Let \( W \) denote the width of the strip and let \( L \) be its length. Note that, for the problem to be feasible, we assume that \( T > 2L/v \), since otherwise reaching the top of the strip would take more than \( T \) time units, leaving no time to serve the customers. The problem is to determine the number and therefore the length of the sectors in the strip. We formulate this problem as a DP. Let \( g(y; l) \) be the cost of serving a sector of length \( y \) with a top edge at a distance of \( l \) from the bottom of the strip (and therefore a bottom edge at a distance of \( l - y \) from the bottom of the strip), where \( l \in [0, L] \). From (1), we have \( g(y; l) = f + o(2(\varphi + l) + yW^2\delta/6) \). Let \( V(l) \) be the value function, which is the total cost of serving the customers located at a vertical distance less than \( l + \varphi \) from the depot, or equivalently, at a vertical distance of \( l \) from the bottom of the strip. Our goal is to calculate \( V(L) \). The DP recursion is

\[
V(l) = \min_{0 \leq y \leq \bar{y}(l)} \left\{ g(y; l) + V(l - y) \quad \text{if} \quad 0 < l \leq L \right\} \quad \text{if} \quad l \leq 0, \tag{15}
\]

where \( \bar{y}(l) \) is the maximum length for a sector with a top edge at a distance of \( l \) from the bottom of the strip:

\[
\bar{y}(l) = \min \left\{ l, \frac{Q}{\delta W}, \frac{6(Tv - 2l)}{W^2\delta + 6WWh/}\right\}. \tag{16}
\]

The first term of the right-hand side comes from the fact that the length of the sector cannot exceed the remaining uncovered portion of the strip, the second term comes from the vehicle capacity constraint \( \delta Wy \leq Q \), and the last term comes from the maximum within-area route duration constraint which, using (1), is equivalent to \( 2L/v + yW^2/6 + yWgh \leq T \). Since we assume that \( T > 2L/v \), we have \( \bar{y}(l) \geq 0 \) for \( l \in [0, L] \). Fig. 4 provides a graphical representation of \( \bar{y}(l) \). We see that that for small values of \( l \), \( \bar{y}(l) \) is determined by \( l \) (i.e., the 45% degree line), for intermediate values it is determined
by the vehicle capacity constraint (i.e., the horizontal line) which does not vary with $l$ and finally, for high values of $l$, it is determined by the maximum within-area route duration constraint (i.e., the decreasing line), which becomes stricter as $l$ increases because the vehicles spend more time in transit on the way to their service sector and therefore have less time remaining for making the deliveries.

Our first result states that it is always optimal to set the length of a sector equal to its maximum possible value given the capacity and within-area duration constraints. Let $y_j$ be the length of the $j$th sector and $l_j = \sum_{i=1}^{j-1} y_i$ denote the distance from the top of sector $j$ to the bottom of the strip.

Proposition 3.1. Consider an optimal solution with $m$ sectors. It must be such that the length of each sector is equal to its maximum, i.e., $y_j = \Upsilon(l_j)$ for $j = 1, \ldots, m$.

The intuition behind Proposition 3.1 is that we need to make the sectors as long as possible, that is, as long as permitted by the capacity of the vehicle and the maximum within-area route duration constraints. The only sector for which these constraints may not be biding is the one closest to the depot: the vehicle assigned to that sector just covers the leftover part of the strip. In other words, it is optimal to make the shortest sector be the one closest to the depot because all vehicles need to drive through the bottom sectors on their way to their service sector, and therefore this keeps the distance to the start of each sector as low as possible.

In the special case where $T \geq 12L + Q(W + 6h)/(6\nu)$, we can provide a closed-form expression for the optimal solution: $m = \lceil 6W/\delta \rceil$, $y_j = Q/\delta W$ for $j = 2, \ldots, m$ and $y_1 = L - (m - 1)Q/\delta W$; in this case, the constraint on the maximum within-area route duration is so loose that the length of sectors $2$ to $m$ is determined by the maximum vehicle capacity $Q$ and the length of the first sector is set equal to the remaining length to cover.

In Appendix B we propose a recursive algorithm to calculate the optimal partition of the strip into sectors based on Proposition 3.1. We now derive some monotonicity properties for the optimal number of vehicles.

Lemma 3.1. The optimal number of sectors $m^*$ is independent of $f$ and $o$, is non-decreasing in the length and width of the strip ($L$ and $W$), the customer density ($\delta$), and non-increasing in the maximum within-area route duration ($T$) and the vehicle capacity ($Q$).

The optimal solution always minimizes the total number of sectors and hence, the total number of vehicles used. For this reason, when there is a single vehicle type, the fixed and variable vehicle cost parameters $f$ and $o$ are not relevant. The other relationships described in Lemma 3.1 are as expected.

3.2. Single strip, multiple vehicle types

In this section, we keep the assumption that there is a single strip of width $W$, but we allow the firm to choose between vehicles of $K$ different types. We label the vehicle types so that $Q_1 \leq \cdots \leq Q_K$, $o_1 \leq \cdots \leq o_K$ and $T_1 \geq \cdots \geq T_K$. For the problem to be feasible we assume that $T_1 > 2L/\nu$, that is, it is possible to serve the entire strip with vehicles of type 1. However we do not require that $T_k > 2L/\nu$ for $k = 2, \ldots, K$, which means that some vehicle types may not be able to serve customers in the top portions of the strip because they would not be able to reach them before within their maximum within-area route duration constraint. Let $t_j \in \{1, \ldots, K\}$ denote the type of the vehicle serving the $j$th sector, so that $f_{t_j}$, $o_{t_j}$, $Q_{t_j}$ and $T_{t_j}$ respectively denote the fixed cost, the variable cost, the capacity, and the maximum tour duration of the vehicle used to serve the $j$th sector.

As in the previous section, we formulate the problem as a DP. Let $g_k(y; l)$ denote the cost of serving a sector of length $y$, having a top edge at a distance of $l$ from the bottom of the strip, with a vehicle of type $k$. We have $g_k(y; l) = f_k + o_k(2(\psi + l) + yW^2\delta/6)$. The value function $V(l)$ is defined as in the previous section and the DP recursion is

$$V(l) = \begin{cases} \min_{k=1, \ldots, K} \{ \min_{0 \leq y \leq \Upsilon(l)} \{ g_k(y; l) + V(l - y) \} \} & \text{if } 0 \leq l \leq L \\ 0 & \text{if } l \leq 0 \end{cases}$$

(17)

where, for every vehicle type $k \in \{1, \ldots, K\}$, the maximum length for a sector ending at a distance $l$ from the bottom of the strip is

$$\Upsilon_k(l) = \min \left\{ l, Q_k/(\delta W), 6T_k\nu - 2l/(\delta W^2 + 6\nu\delta W) \right\}.$$

If for some $l$ and $k$, $\Upsilon_k(l)$ is negative, then vehicles of type $k$ cannot be used to feasibly serve the sector located at a distance $l$ from the bottom of the strip. As before, our goal is to calculate $V(L)$. We first analyze the structure of the optimal solution.

Proposition 3.2. An optimal solution contains at most one sector with a length shorter than its maximum value, i.e., we have $y_{j-1} = \Upsilon_{t_j}(l_{j-1})$ for all sectors $i = 1, \ldots, m$, except possibly for one of them. If such a sector exists, say $j$, i.e., $y_j < \Upsilon_{t_j}(l_j)$, then the following properties must hold: (i) $j > 1$, (ii) $o_{t_i} > o_{t_j}(12 + \delta W^2)/\delta W^2$ for $i = 1, \ldots, j - 1$. (iii) $y_i = \Upsilon_{t_i}(l_i) = Q_{t_i}/(W\delta)$ for $i = 1, \ldots, j - 1$, (iv) $Q_{t_i}/(W\delta) \leq y_j$ for $i = 1, \ldots, j - 1$.

Proposition 3.2 gives four necessary conditions for the optimal solution to include a sector with a length less than its maximum possible value: if such a sector exists, it cannot be the closest one to the depot (condition (i)), it must be preceded
only by sectors served by vehicles with a lower variable cost (condition (ii)) which operate at full truckload (condition (iii)) and the quantity delivered by the vehicle in this sector must exceed the quantity delivered by all the vehicles in the previous sectors (condition (iv)). If at least one of these conditions are not met, then it is optimal to set the length of each sector equal to its maximum possible value, i.e., \( y_i = \frac{p_i l_i}{2} \) for \( i = 1, \ldots, m \). In particular, this is the case if the variable costs of the vehicles are not too different, in the sense that they satisfy \( o_k < o_1 \frac{12 V W^2}{AW^2} \) (so that condition (ii) cannot be satisfied).

The following lemma provides an upper bound on the optimal number of sectors. We use this value to calculate the \( m_i^* \) values in our MILP formulation from Section 2.4.

**Lemma 3.2.** The optimal number of sectors in which the strip is partitioned is bounded by \( \bar{m} \). This is the optimal number of sectors in which the strip would be partitioned if it was served solely by vehicles of type 1 with a maximum within-area route duration. The latter value depends on the distance from the bottom of the strip \( l \) and is equal to

\[
T(l) = \begin{cases} 
T_K & l \in (0, T_Kv/2] \cr 
T_{K-1} & l \in ((T_Kv)/2, (T_{K-1}v)/2] \cr \vdots 
T_1 & l \in ((T_2v)/2, (T_1v)/2]. 
\end{cases}
\]

Note that \( \bar{m} \) can be easily calculated by Algorithm 1 in Appendix B except that \( T(l) \) is used instead of \( T \) in the expression for \( \bar{y}(l) \).

In what follows, we focus on the special case of \( K = 2 \). As before we assume \( o_1 \leq o_2, Q_1 \leq Q_2 \) and \( T_1 \geq T_2 \) so that, for example, one can think of the vehicles of type 1 as electric vehicles and vehicles of type 2 as diesel vehicles.

From Proposition 3.2 it follows that when \( o_1 \leq o_2 \leq o_1 \frac{12 V W^2}{AW^2} \), condition (ii) cannot be satisfied. Therefore each sector has a length equal to its maximum and the value function reduces to

\[
V(l) = \begin{cases} 
\min \left\{ g_1(\bar{y}_1(l); l) + V(l - \bar{y}_1(l)), g_2(\bar{y}_2(l); l) + V(l - \bar{y}_2(l)) \right\} & \text{if } 0 \leq l \leq L \\
0 & \text{if } l \leq 0, 
\end{cases}
\]

i.e., it is sufficient to compare only two values at each step of the DP.

In contrast, when \( o_2 > o_1 \frac{12 V W^2}{AW^2} \), there can be at most one sector served by a vehicle of type 2 with length lower than its maximum possible value. Nevertheless, in this case we can exploit the solution properties given in Proposition 3.2 to simplify the DP resolution, so that at most three values need to be compared at each step:

\[
V(l) = \begin{cases} 
\min \left\{ g_1(\bar{y}_1(l); l) + V(l - \bar{y}_1(l)), g_2(\bar{y}_2(l); l) + V(l - \bar{y}_2(l)), TC(l) \right\} & \text{if } l \leq 0 \\
\min \left\{ g_1(\bar{y}_1(l); l) + V(l - \bar{y}_1(l)), g_2(\bar{y}_2(l); l) + V(l - \bar{y}_2(l)) \right\} & \text{if } 0 < l \leq \min \left\{ 6(T_2v - 2\delta) - Q_1(W + 6hv)/12, L \right\} \\
\text{otherwise}, 
\end{cases}
\]

where \( TC(l) \) is the cost of serving the sector ending at distance \( l \) from the bottom of the strip by a vehicle of type 2 operating at less than full capacity, and all the previous sectors use vehicles of type 1 at full capacity. The formula used to calculate this value is derived in Appendix E. Example 3.1 shows a case where the optimal solution includes a sector with length lower than its maximum possible value.

**Example 3.1.** Let \( L = 46, W = 8, \delta = 0.2, \phi = 0, f_1 = 223, f_2 = 75, o_1 = 8, o_2 = 25.5, Q_1 = 6, Q_2 = 32, T_1 = T_2 = \infty, v = 30 \).

As depicted in Fig. 5 the optimal solution for this example is to partition the strip into four sectors. Sectors 1 and 2 are served by vehicles of type 1 operating at full truckload, and their lengths are equal to the maximum possible value, i.e., \( y_1 = y_2 = Q_2/(W\delta) = 3.75 \). Sector 3 is served by a vehicle of type 2 operating at less-than-full truckload and its length is less than the maximum possible value, i.e., \( y_3 = 18.5 < Q_2/(W\delta) = 20 \). Finally, sector 4 is served by a vehicle of type 2 operating at full truckload and its length is equal to the maximum possible value, i.e., \( y_4 = 20 = Q_2/(W\delta) \).

We now analyze the properties of the optimal solution. To this end, we first provide an example to illustrate the tradeoffs between the two types of vehicle.

**Example 3.2.** Consider two alternative solutions for the partitioning of a strip of length \( L \). In solution A there exists a sector of length \( x \) served by a vehicle of type 2 at the top of the strip and a sector of length \( L - x \) served by a vehicle of type 1 at bottom of the strip, such that \( x \geq L/2 \). In solution B the order of the two sectors is reversed: the sector of length \( L - x \) served by a vehicle of type 1 is at the top of the strip, and the sector of length \( x \) served by a vehicle of type 2 is at the bottom of the strip.

We compare the two solutions in terms of the total distance traveled by the two vehicles. From (1), we can see that the total vertical and traversal distances within the strip are equal in both solutions. The difference lies in the total transit distance. In solution A, the vehicle of type 1 starts the bottom of the strip immediately, while the vehicle of type 2 needs to drive \( L - x \) units before reaching its service sector. In contrast, in solution B, the vehicle of type 1 needs to drive \( x \) units before reaching its service sector, and the vehicle of type 2 starts service immediately. Since the total fixed cost is identical in both solutions and \( L - x < x \), the tradeoff between total transit distance and variable costs determines which solution is optimal: if \( (L - x)a_2 < x^* a_1 \), or equivalently \( (L - x)/x < o_1/o_2 \), then solution A is preferred, otherwise solution B is.
In the absence of maximum within-area route duration constraints, we are able to prove some structure of the optimal solution.

**Lemma 3.3.** If there are no within-area route duration constraints, i.e., \(T_1 = T_2 = \infty\), then there exists an optimal solution which has one of the following three structures: (i) All sectors are served by the same vehicle type (see Fig. 6a and b); (ii) The sectors which are closest to the depot are served by one vehicle type and the remaining ones are served by the other type (see Fig. 6c and d); (iii) The first and last sectors are served by the same vehicle type, while the sectors in the middle are served by the other one (see Fig. 6e and f).

In some rare cases there could be multiple optimal solutions, some of which may not have one of the three structures stated in this lemma (this may happen in particular if \(o_1/o_2 = Q_1/Q_2\)). However, in these cases, we can guarantee that there is always at least one optimal solution which does. **Lemma 3.3** implies that in the absence of maximum within-area route duration constraints, there can be at most two switches in the vehicle types which are used to serve the sectors when going from the bottom to the top of the strip. In contrast, in the presence of maximum within-area route duration constraints, the (unique) optimal solution may be such that the vehicle types used to serve the sectors change several times from top to bottom, as shown in the following example area.

**Example 3.3.** Let \(W = 10, L = 60.5, \varphi = 0, \delta = 0.3, v = 30, f_1 = 150, f_2 = 90, o_1 = 5, o_2 = 25, Q_1 = 6, Q_2 = 70, T_1 \geq 6, T_2 = 6\). As displayed in Fig. 7 the optimal solution has 14 sectors: sectors 1–7,9,11–14 are served by vehicles of type 1 operating at full truckload, i.e., \(y_i = Q_1/(W\delta) = 2\) for \(i = 1, \ldots, 7, 9, 11, \ldots, 14\) and sectors 8 and 10 are served by vehicles of type 2 operating at less than full truckload, because the length of their sector is determined by the maximum within-area route duration constraint, i.e., \(y_8 = 6(T_2v - 2l_8)/(\delta W^2 + 6hv\delta W) = 21.5\) and \(y_{10} = 6(T_2v - 2l_{10})/(\delta W^2 + 6hv\delta W) = 15\).

The following lemma provides sufficient conditions under which the optimal solution uses only one type of vehicle.
Lemma 3.4. (a) If \( f_1 \leq f_2 \) and \( Q_1 > \frac{6(T_f \varphi)_W}{3W^2 + 6W + 12} \), then it is optimal to use only vehicles of type 1. (b) If both vehicle types have the same trip duration maximum, i.e. \( T_1 = T_2 \) and \( f_1 \geq f_2 + (o_2 - o_1)(2(L + \varphi) + (WQ_2)/6) \), then it is optimal to use only vehicles of type 2.

In Lemma 3.4(a), the within-area route duration constraint for vehicles of type 2 is so strict that these vehicles are not able to carry more load than the vehicles of type 1, hence, they lose their advantage and are never used. The condition in Lemma 3.4(b) implies that the increase in fixed cost when switching from a type 2 to a type 1 vehicle more than offsets the maximum possible decrease in variable cost which comes with this switch, and therefore type 1 vehicles are never used.

In practice, a mixed fleet may bring extra logistical and maintenance complexity, so that some logistics service providers may prefer to have only one type of vehicle.

3.3. Multiple strips, multiple vehicle types

In this section we consider the case of multiple strips, that is, the rectangular area is partitioned into \( s \) strips of equal width, i.e. \( W/s \). Let \( V_{W/s,s}(L) \) denote the optimal cost of serving customers in a strip of width \( W/s \) and length \( L \), when the middle of the bottom edge the strip is located at a distance of \( \varphi \) from the depot. This value can be calculated using recursion (17) (in the special case of \( K = 2 \), one can use the simplified DP formulations (18) or (19), depending on whether \( o_2 \) is lower or greater than \( o_1(V_{W/s,s}/W)^{1/2} \)).

Let \( C(s) \) denote the optimal cost when the area is partitioned into \( s \) strips; we have \( C(s) = \sum_{i=1}^{s} V_{W/s,s_i}(L) \), where \( s_i \) is given by (1). The following lemma establishes the symmetry of the optimal solution around the middle strip(s).

**Lemma 3.5.** The optimal solution for a given number of strips is always symmetric around the middle strip(s), i.e., \( y^*_i = y^*_{s-s+i-1, j} \) and \( t^*_i = t^*_{s-s+i, j} \) for \( i = 1, \ldots, \lceil s/2 \rceil \), where \( s^* \) is the optimal number of strips.

From Lemma 3.5 we can write \( C(s) = 2 \sum_{i=1}^{\lceil s/2 \rceil} V_{W/s,s_i}(L) \) for \( s \) even and \( C(s) = 2 \sum_{i=1}^{\lfloor s/2 \rfloor} V_{W/s,s_i}(L) + V_{W/s,s_{\lceil s/2 \rceil}}(L) \) for \( s \) odd. The minimum cost for the area partitioning problem is \( C^* = \min_{s=1, \ldots, \bar{s}} C(s) \). Because \( C(s) \) is not necessarily monotone in \( s \), as shown in Fig. 8, the optimal value of \( s \) can only be obtained by comparison of \( C(1), \ldots, C(\bar{s}) \), where \( \bar{s} \) is an upper bound on the number of strips, which we show how to calculate in Appendix C.

We now provide an example and illustrate the optimal solution graphically.

**Example 3.4.** Let \( W = 50 \), \( L = 30 \), \( \varphi = 0 \), \( \delta = 0.5 \), \( v = 30 \), \( f_1 = 80 \), \( f_2 = 50 \), \( o_1 = 5 \), \( o_2 = 10 \), \( Q_1 = 5 \), \( Q_2 = 40 \), \( T_1 = 8 \), \( T_2 = 3 \). Fig. 9 represents the optimal partitioning of the area: there are 14 strips, with the central ones (5–10) being partitioned into 2 sectors and the other ones being partitioned into 3 sectors. We see that strip partitioning is symmetric around the middle strip as discussed in Lemma 3.5. In the central strips (5–10), it is optimal to use only vehicles of type 2 so as to take advantage of their larger capacity. As one moves away from the center, the length of the top sectors becomes smaller.
because the vehicles take more time traveling horizontally to reach the bottom of their service strip and therefore, they have less time left to serve the customers before hitting the maximum within-area tour duration. Eventually, in strips 1–4 and 10–14 vehicles of type 1 are used to serve the customers at the top of the strips because these vehicles have looser time access restrictions.

4. Numerical analysis

The purpose of our numerical study is threefold. First, we study the impact of city access restrictions on the optimal fleet composition. Second, we study the optimal fleet composition; in particular, we quantify the benefits of having a heterogeneous versus a homogeneous fleet of vehicles. Third, we compare our MILP formulation from Section 2.4 and DP formulation from Section 3.2 in terms of computational time.

4.1. Impact of city access restrictions

We first investigate the impact of imposing maximum within-area route duration on the optimal fleet composition. We focus on the single-strip case with two vehicles, i.e., \( K = 2 \) such that \( Q_1 \leq Q_2 \) and \( o_1 \leq o_2 \) and refer to type 1 as the electric vehicles and to type 2 as the diesel vehicles. Fig. 10 depicts how the optimal total number of electric and diesel vehicles varies with \( T_2 \), for different values of \( f_1, f_2, o_1, o_2, Q_1 \) and \( Q_2 \). Other input parameters are set as follows: \( L = 30, W = 5, \varphi = 3, \delta = 0.5, \nu = 30 \). After extensive testing, we have found that the three cases depicted in Fig. 10 are representative of the possible patterns that can be observed.

In Fig. 10a both vehicle types have the same capacity and variable cost but the electric type has a higher fixed cost. When \( T_2 \) is very large, meaning that the access restrictions on diesel vehicles are very loose, it is optimal to use only diesel vehicles. As \( T_2 \) becomes smaller, the diesel vehicles are replaced by electric vehicles on a one-to-one basis and the total number of vehicles serving the area remains the same.
(a) $f_1 = 100, f_2 = 50$, $o_1 = o_2 = 5$, $Q_1 = Q_2 = 5$, $T_1 = 8$.

(b) $f_1 = f_2 = 100$, $o_1 = o_2 = 5$, $Q_1 = 6$, $Q_2 = 16$, $T_1 = 8$.

(c) $f_1 = 100, f_2 = 20$, $o_1 = o_2 = 5$, $Q_1 = 5$, $Q_2 = 50$, $T_1 = 8$.

Fig. 10. Optimal number of vehicles as a function of $T_2$.

In Fig. 10b both types of vehicle have the same fixed and variable costs but the diesel type has a larger capacity. When $T_2$ is large, it is optimal to use only diesel vehicles to exploit the economies of scale from transporting larger quantities. As $T_2$ becomes smaller the diesel vehicles are progressively replaced by electric ones but the ratio is not one-to-one as the vehicle types have different capacities, so that the total number of vehicles actually increases. Therefore, in this case, restricting the time period during which diesel vehicles are allowed to enter the city center contributes to decreasing the number of diesel vehicles in the fleet but it increases the total number of vehicles serving the area.

In Fig. 10c both types of vehicle have the same variable cost but the diesel type has a larger capacity and a lower fixed cost. When $T_2$ is large it is optimal to use only diesel vehicles since they are cheaper to buy or lease than the electric ones and they yield economies of scale. As $T_2$ decreases, it is first optimal to increase the number of diesel vehicles because more vehicles are needed to serve the area. As $T_2$ decreases further, it becomes optimal to add electric vehicles. Finally for very small values of $T_2$, one should replace the diesel vehicles by electric vehicles. Therefore, in this case, the number of diesel vehicles is not monotone in $T_2$ and the total number of vehicles serving the area becomes larger as $T_2$ decreases.

These examples illustrate that city access restrictions, in the form of a maximum within-area route duration constraint, may have unintended consequences. It is possible that, if cities restrict city center access to large diesel vehicles, they may in fact trigger an increase in the total number of diesel vehicles, along with the total number of vehicles serving the area.

4.2. Optimal fleet composition

We next investigate how the optimal fleet composition, i.e., the number of vehicles of each type, changes with the area and vehicle parameters. As in the previous section we assume there are two vehicle types with $Q_1 \leq Q_2$ and $o_1 \leq o_2$.

Fig. 11 represents how the percentage of electric vehicles in the optimal solution to the area partitioning problem varies with the ratios $o_1/o_2$ and $Q_1/Q_2$. The pictures are based on the calculation of the optimal solution for 10,000 discrete points, when both ratios vary between 0 and 1 in increments of 0.01. A value of 100% corresponds to an all-electric vehicle fleet, while a value of 0% corresponds to an all-diesel vehicle fleet. Any intermediate value indicates that it is optimal to use a heterogeneous fleet of vehicles. Fig. 11a is the base scenario where the parameters are set as follows: $W = 10, L = 50, \varphi = 3, \delta = 0.5, v = 30 h = 0, f_1 = f_2 = 0, o_2 = 20, Q_2 = 50, T_1 = T_2 = \infty$. We use the same parameters for Figs. 11b–d except for one or two variables: $f_1 = 130$ in Fig. 11b, $T_2 = 3$ in Fig. 11c and $L = 50, \delta = 0.3$ in Fig. 11d. Based on extensive numerical experiments, we have established that these four cases are indeed representative of the tradeoffs involved. In each of the four graphs of Fig. 11, we also identify the instance yielding the maximum cost savings resulting from using a heterogeneous fleet of vehicles versus a homogeneous one. So for example, the maximum cost savings is 3.3% in Fig. 11a.

The four graphs of Fig. 11 all exhibit the same basic pattern: if electric vehicles have a small capacity and a high variable cost, i.e. $Q_1/Q_2 \to 0$ and $o_1/o_2 \to 1$, then it is preferable to use a homogeneous fleet of diesel vehicles (i.e., the proportion of electric vehicles used is 0%). On the other hand, if electric vehicles have a large capacity and a low variable cost, i.e. $Q_1/Q_2 \to 1$ and $o_1/o_2 \to 0$, then it is preferable to use a homogeneous fleet of electric vehicles (i.e., the proportion of electric
Fig. 11. Percentage usage of electric vehicles. The numbers in bold correspond to the maximum percentage cost savings yielded by using a heterogeneous fleet versus a homogeneous one.

(a) Case 1: $W = 10, L = 30, f_1 = f_2 = 0, T_1 = T_2 = \infty, \delta = 0.5$.

(b) Case 2: $W = 10, L = 30, f_1 = 130, f_2 = 0, T_1 = T_2 = \infty, \delta = 0.5$.

(c) Case 3: $W = 10, L = 30, f_1 = f_2 = 0, T_1 = \infty, T_2 = 2.8, \delta = 0.5$.

(d) Case 4: $W = 10, L = 50, f_1 = f_2 = 0, T_1 = T_2 = \infty, \delta = 0.3$.

vehicles used is 100%). The cases where a heterogeneous fleet of vehicles is optimal all lie on the frontier between these two extreme regions. This suggests that the benefits of having a heterogeneous fleet are the greatest when there is not too much difference between the cost of operating an all-electric vehicle fleet and the cost of operating an all-diesel vehicle fleet. In the base case of Fig. 11a, 60% of the instances have an all-electric optimal fleet, 37% have an all-diesel optimal fleet and the remaining 3% have a heterogeneous fleet.

In Fig. 11b, the fixed cost of the electric vehicle is increased to 130, which leads to a decrease in the number of electric vehicles used by moving the heterogeneous fleet region down, closer to the 45-degree line. Compared to the previous case, the proportion of instances for which it is optimal to have an all-electric fleet decreases to 50%, while the proportion of instances for which it is optimal to have an all-diesel fleet increases to 49%. Also, the proportion of instances for which it is optimal to have a heterogeneous fleet reduces to 1%.

In Fig. 11c, the maximum within-area route duration of the diesel vehicles is set equal to 2.8, but there are still no time restrictions on the usage of the electric vehicles. This leads to a decrease in the proportion of diesel vehicles (to 10%) and to an increase in the proportion of electric vehicles (to 70%). Also note that the proportion of solutions with a heterogeneous fleet increases to 20% compared to the base case. When $Q_1/Q_2$ becomes sufficiently large ($Q_1/Q_2 \geq 0.776$), it is optimal to use only electric vehicles, even if $a_1 = a_2$. This is because the maximum length of the sector that can be served by an electric vehicle is longer than the maximum sector length that can be served by a diesel vehicle due to the time restrictions
of the latter. Therefore, in this situation, the diesel vehicle provides no advantage compared with the electric vehicle since it has to make shorter trips and is more expensive.

Finally, Fig. 11d depicts the results obtained when the same number of customers are distributed over a larger area, i.e., the length of the area is increased to \( L = 50 \) and the density is reduced from \( \delta = 0.5 \) to \( \delta = 0.3 \), so that the total number of customers remains constant at 150. This change leads to a decrease in the proportion of solutions for which only diesel vehicles are used (to 32%) and to an increase in the proportion of solutions for which a heterogeneous fleet is optimal (8%). This is because the area is longer and the travel distances are longer, so that it becomes optimal to use more electric vehicles which are cheaper to operate per unit of distance.

In our numerical experiments, we also calculated the percentage cost savings resulting from the use of a heterogeneous fleet of vehicles compared to a homogeneous one. While we found some rare cases where this value was very large (more than 50%), the average cost saving we obtained using the data from Fig. 11 was only about 0.1%, with a maximum value of about 10%. This suggests that the benefits of operating a heterogeneous vehicle fleet are generally small so that in practice, they may not offset the added complexity of managing more than one vehicle type.

4.3. MILP versus DP

In this section, we solve 144 instances to compare the computational performance of our DP algorithm with an MILP solver. Both methods were coded in a C++ environment and we used the CPLEX V12.5.1 concert libraries to solve the MILP. The computational tests were performed on a 2.7 GHz Intel Core i5 processor, with 8 GB RAM, operating on MacBook Pro. We used \( K = 2 \), \( L \in \{5, 10, 20, 50\} \), \( W \in \{5, 10, 20, 30\} \), \( \phi = 0 \), \( \delta = 0.5 \), \( \nu = 30 \), \( f_1 = 10 \), \( f_2 = 10 \), \( o_1 = 2 \), \( o_2 \in \{2, 4, 6\} \), \( Q_1 = 5 \), \( Q_2 = \{10, 15, 20\} \), \( T_1 = 24 \), \( T_2 = \{6, 12\} \) and selected 144 problem instances out of the possible parameter combinations. Table 1 reports the average computational time to solve the instances. The value \( \text{Opt.} \) indicates the percentage of instances solved to optimality within a time limit of one hour.

While CPLEX could solve only 60% of the instances to optimality within an average time of 492.5 s, the DP algorithm could solve all instances within less than one second.

5. Conclusions

In this paper, we study the strategic problem of a logistics service provider managing a fleet of vehicles making deliveries in a city under vehicle access restrictions. We represent the city as a rectangular service area divided into sectors, each served by a single vehicle. The length of the routes is calculated by means of a continuous approximation. The objective is the minimization of the total cost of fleet ownership or leasing, the fuel cost and labor cost. We formulate the problem as a mixed integer partitioning problem and as a dynamic program. We provide an efficient method to compute an optimal solution by exploiting some key structural properties of the problem; this method is shown to be much faster than solving the MILP by CPLEX on a large number of problem instances. We also show that the optimal solution may exhibit a complex structure, with vehicles of different types serving adjacent sectors, and some vehicles running at less-than-full-truckload capacity or returning to the depot before the end of the within-area route duration limit. Numerically, we find that this type of city access restrictions may, in some cases, be counterproductive because it may lead to an increase in the number of diesel vehicles and an increase in the total number of vehicles used in the service area. We also found that on average, operating a heterogeneous fleet only leads to a small decrease in cost compared to operating a homogeneous fleet, which may not be outweighed by the increase in logistical complexity resulting from operating several vehicle types.

Acknowledgments

The work was partly supported by the Dutch Institute for Advanced Logistics under the project 4C4D and by the Canadian Natural Sciences and Engineering Research Council under grant 2015-06189. This support is gratefully acknowledged. The authors also thank Carlos F. Daganzo for his valuable insights on the dual strip strategy. Thanks are due to the editor and to a referee for their valuable comments.

Appendix A. Notations

The following table summarizes the notations used in the paper.
Table A.2
Summary of the notations.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_k$</td>
<td>fixed cost for vehicle of type $k$</td>
</tr>
<tr>
<td>$c_k$</td>
<td>variable cost for vehicle of type $k$</td>
</tr>
<tr>
<td>$Q_k$</td>
<td>capacity of vehicle of type $k$</td>
</tr>
<tr>
<td>$T_k$</td>
<td>maximum within-area tour duration for a vehicle of type $k$</td>
</tr>
<tr>
<td>$v$</td>
<td>vehicle speed</td>
</tr>
<tr>
<td>$d$</td>
<td>driver wage</td>
</tr>
<tr>
<td>$h$</td>
<td>time to serve a single customer</td>
</tr>
<tr>
<td>$K$</td>
<td>number of vehicle types</td>
</tr>
<tr>
<td>$L$</td>
<td>length of service area</td>
</tr>
<tr>
<td>$W$</td>
<td>width of service area</td>
</tr>
<tr>
<td>$\delta$</td>
<td>customer density</td>
</tr>
<tr>
<td>$\tau$</td>
<td>maximum number of strips</td>
</tr>
<tr>
<td>$\psi$</td>
<td>distance between the depot and the middle point of the bottom edge of the service area</td>
</tr>
<tr>
<td>$\psi_i^j$</td>
<td>distance between the depot and the middle point of the bottom edge of the $i$th strip when the area is partitioned into $s$ strips</td>
</tr>
<tr>
<td>$\mu_{ij}$</td>
<td>distance between the depot and the middle point of the bottom edge of the $j$th sector in strip $i$ when the area is partitioned into $s$ strips</td>
</tr>
<tr>
<td>$y_j$</td>
<td>length of sector $j$ when there is only one strip</td>
</tr>
<tr>
<td>$y_j^s$</td>
<td>length of sector $j$ in strip $i$ when there are $s$ strips</td>
</tr>
<tr>
<td>$y^s_{ij}$</td>
<td>total distance traveled by a vehicle to serve sector $j$ in strip $i$ when the area is partitioned into $s$ strips</td>
</tr>
<tr>
<td>$w^k_{ij}$</td>
<td>total distance traveled by a vehicle of type $k$ to serve sector $j$ in strip $i$ when the area is partitioned into $s$ strips</td>
</tr>
<tr>
<td>$w_{ij}^s$</td>
<td>total distance traveled by a vehicle of type $k$ to reach the bottom of sector $j$ in strip $i$ when the area is partitioned into $s$ strips</td>
</tr>
<tr>
<td>$v_{ij}^s$</td>
<td>length of sector $j$ in strip $i$ served by a vehicle of type $k$ when the area is partitioned into $s$ strips</td>
</tr>
<tr>
<td>$l_i$</td>
<td>distance from the top of sector $j$ to the bottom of the strip when there is only one strip</td>
</tr>
<tr>
<td>$\bar{y}(l)$</td>
<td>maximum length for a sector with a top edge at a distance of $l$ from the bottom of the strip when there is only one strip served by a single vehicle type</td>
</tr>
<tr>
<td>$\bar{y}_1(l)$</td>
<td>maximum length for a sector with a top edge at a distance of $l$ from the bottom of the strip when there is only one strip served by a vehicle of type $k$</td>
</tr>
<tr>
<td>$\bar{m}$</td>
<td>upper bound on the optimal number of sectors in a strip when there is only one strip</td>
</tr>
<tr>
<td>$\bar{m}_i$</td>
<td>upper bound on the optimal number of sectors in strip $i$ when the area is partitioned into $s$ strips</td>
</tr>
<tr>
<td>$\bar{u}_j$</td>
<td>vehicle type used in sector $j$ when there is only one strip</td>
</tr>
<tr>
<td>$\bar{u}^i$</td>
<td>vehicle type used in sector $j$ of strip $i$ when the area is partitioned into $s$ strips</td>
</tr>
</tbody>
</table>

Appendix B. Recursive algorithm

In this section we propose a recursive algorithm to calculate the optimal partition of a strip into sectors with one vehicle type.

Appendix C. Solution properties and formula for calculating $s$

In this section of the Appendix, we prove the existence of an upper bound on the number of strips and we provide a formulation for calculating it. We first provide a Lemma which is useful in obtaining this result.

Algorithm 1 Optimal partition of strip into sectors with one vehicle type.

Step 0: Set $l = L$ and $j = 1$.
Step 1: $y_j = \bar{y}(l)$.

if $\bar{y}(l) > l$ then
    set $l = L - \bar{y}(l)$ and $j = j + 1$ then repeat Step 1. Else Stop.
end if

Step 2: $m = j$. Renumber the sectors: $y_j := y_{m-j+1}$ for $j = 1, \ldots, m$.

Lemma C.1. The total distance traveled by all the vehicles serving an area, where each strip is served by a single vehicle, is increasing in $s$ for $s \geq \lfloor W^2\delta/12 \rfloor$. 

Proof. Let \( s \geq \lceil \sqrt{W^2 \delta / 12} \rceil \), the difference in total distance traveled between solutions with \( s + 1 \) and \( s \) strips, denoted \( \Delta_{s,s+1} \), is equal to

\[
\Delta_{s,s+1} = \sum_{i=1}^{s+1} \left( 2(\varphi_i^{s+1} + L) + \frac{L \delta W^2}{6 s^2} \right) - \sum_{i=1}^{s} \left( 2(\varphi_i^s + L) + \frac{L \delta W^2}{6 (s+1)^2} \right)
\]

(\(C.1\))

\[
= 2 \left( \sum_{i=1}^{s+1} \frac{\varphi_i^{s+1} - \varphi_i^s}{s} \right) - 2L - \frac{L \delta W^2}{6 s (s+1)}.
\]

(\(C.2\))

Given the definition of \( \varphi_i^s \), it is easy to check that \( \sum_{i=1}^{s+1} \varphi_i^{s+1} - \sum_{i=1}^{s} \varphi_i^s \) is equal to \( \frac{W^2}{4 \pi^2} + \varphi \) if \( s \) is even and to \( \frac{W^2}{4 \pi^2} + \varphi \) if \( s \) is odd, which is positive in both cases. Therefore \( \Delta_{s,s+1} \geq 2L - \frac{L \delta W^2}{6 (s+1)} \geq 2L - \frac{L \delta W^2}{6 s^2} \), which is positive because \( s + 1 > s \geq \lceil \sqrt{W^2 \delta / 12} \rceil \). \( \square \)

The next lemma provides a formulation for the upper bound computation.

Lemma C.2. The optimal solution has a number of strips \( s^* \) which is less or equal to \( \bar{s} = \max \left\{ \max_{k=1,\ldots,K} s_k, \lceil \sqrt{W^2 \delta / 12} \rceil \right\} \), where for \( k = 1, \ldots, K \),

\[
s_k = \max \left\{ \left[ \frac{WL\delta}{Q_k} \right], \frac{6W_k (1 - \delta h L v) + \sqrt{(6W_k \delta h L v - 6W_k)^2 - 4\delta L W^2 (6 (2L - T_k v) + 6W_k)}}{12(W_k + 2L - T_k v)} \right\}
\]

(\(C.3\))

and \( W_k = \min \{ W, T_k v - 2L - \epsilon \} \) where \( \epsilon \) is a small positive number.

Proof. For a given vehicle type \( k \), \( W_k/2 \) measures the maximum horizontal distance from the middle point of the top edge of the service area that can be reached by a vehicle of type \( k \). Vehicles such that \( W_k = W \) can reach the upper left corner of the rectangle, but vehicles such that \( W_k < W \) cannot be used to serve the top left and right corners of the service area because they are unable to reach these customers within their route duration limit \( T_k \).

The variable \( s_k \) measures the minimum number of strips needed so that vehicles of type \( k \) can feasibly use one vehicle per strip to serve the rectangle of height \( L \) and width \( W_k \) (with \( W_k/2 \) on each side of the depot). It is the smallest value satisfying \( \frac{W_k \delta}{s_k} \geq Q_k \) and \( 2L \frac{s_k - 1}{s_k} W_k + L + \frac{W_k^2 \delta^2}{6 s_k^2} + \frac{W_k \delta h L v}{s_k} \leq T_k v \). Solving the first inequality as an equation gives \( s_k = \frac{WL\delta}{Q_k} \), which is the first term (\(C.3\)) for the second we obtain a quadratic equation in \( s_k \):

\[
6s_k^2[T_k v - 2L - W] + 6W s_k (1 - \delta h L v) - W^2 L \delta = 0.
\]

Since \( T_k v - 2L - W_k = \epsilon > 0 \), this equation has only one positive root, which is the second term in (\(C.3\)). As a result, for all values of \( s > s_k \), it is always feasible to serve all strips within a distance \( W_k/2 \) of the depot with only one vehicle of type \( k \).

The rest of the proof is by contradiction. Suppose there exists an optimal solution \( S' \) for which the area is divided into \( s' \) strips such that \( s' > \bar{s} \). In this solution, there must be only one vehicle per strip since all vehicles types used in a strip can feasibly serve it entirely, therefore it is optimal to use the vehicle that can serve the entire strip at the lowest cost. Let \( n'_i \) denote the number of strips served by vehicles of type \( i \) in solution \( S' \). Consider an alternative solution \( S'' \) with \( s'' = \bar{s} \), where each strip is served by a single vehicle. Let \( n''_i \) denote the number of strips served by vehicles of type \( i \) in solution \( S'' \). We set \( n'_1 = \min\{n'_1, \bar{s}\}, n'_2 = \min\{n'_2, \bar{s} - n'_1\}, \ldots, n'_K = \min\{n'_K, \bar{s} - \sum_{i=1}^{K-1} n'_i\} \). Let \( k \in \{1, \ldots, K\} \) be the largest value such that \( n'_k > 0 \). We have \( n''_i = n'_i \) for \( i = 1, \ldots, k - 1 \). \( n''_k \leq n'_k \) and \( n''_i = 0 < n'_i \) for \( i = k + 1, \ldots, K \). Let \( D'_i \) and \( D''_i \) be the total distances traveled by all the vehicles of type \( i \in \{1, \ldots, K\} \) in solutions \( S' \) and \( S'' \), respectively. We know that \( \sum_{i=1}^{K} D'_i \geq \sum_{i=1}^{K} D''_i \) since, by Lemma C.1, the total distance travelled increases in the number of strips as \( s \geq \lceil \sqrt{W^2 \delta / 12} \rceil \). Also, \( D''_i \geq D'_i \) for \( i = 1, \ldots, k - 1 \) since the number of strips served by vehicles of type \( i \) in \( S'' \) is the same as for \( S \), but these strips are wider. We have
Proof. F.1. Appendix hence where 1 where \( \hat{s} \) > 1. We \( \sum_{i=1}^k D_i' \geq \sum_{i=1}^k D_i'' \) and \( D_i' < D_i'' \) for \( i = 0, \ldots, k-1 \). Since \( TC(S'') < TC(S') \), we have a contradiction. \( \square \)

We observed that we can obtain a tighter upper bound on the value of \( s' \) using a more complex algorithm. We used this alternate bound in our numerical study since it contributes to reducing the computational time. However, as this does not affect the main results of the paper, we decided to present a simpler formulation which calculates a looser upper bound.

Appendix D. \( TC(l) \) Formulation

In this section of the Appendix, we present the formulation for calculating \( TC(l) \) from Section 3.2 of the paper.

Appendix E. Formulation

Formally, we have

\[
TC(l) = \min_{k=\lceil l_W \gamma_2(l)/Q_1 \rceil} \ldots \min_{k=\lceil l_W \gamma_2(l)/Q_1 \rceil} \left\{ \sum_{i=1}^k g_i \left( \frac{Q_1}{W\delta} \cdot i \frac{Q_1}{W\delta} \right) + g_2 \left( l - i \frac{Q_1}{W\delta}, l \right) \right\}
\]

\[
= \min_{k=\lceil l_W \gamma_2(l)/Q_1 \rceil} \ldots \min_{k=\lceil l_W \gamma_2(l)/Q_1 \rceil} \left\{ k f_1 + o_1 \left( 2 \sum_{i=0}^{k-1} \frac{Q_1}{W\delta} (k - i) + 4k + \frac{Q_1 W}{6} k \right) + f_2 + o_2 \left( 2(l + \varphi) + \frac{l - k Q_1 W}{6} \right) \right\}
\]

where \( TC(l) \) is set equal to \( \infty \) if \( l W\delta \gamma_2(l)/Q_1 > l W\delta Q_1 - 1 \). The minimum and maximum values for \( k \) come from the observation that, by Proposition 3.2(iv), the sector must have a length \( y \) such that \( Q_1 (\delta W) = y_1 (l) \gamma_2 (l) < y \gamma_2 (l) \gamma_1 (l) \gamma_1 (l) < Q_2 (W\delta) \). Hence \( k \) must be such that \( Q_1 (\delta W) < l - k Q_1 (\delta W) < y \gamma_2 (l) \).

Appendix F. Proofs

F.1. Proof of Proposition 3.1

Proof. The proof is by contradiction. Let \( \hat{s} \) be the lowest index such that \( y_{\hat{s}} \neq y(l_{\hat{s}}) \) in the optimal solution. Since \( y(l) \) is the maximum value satisfying both capacity and maximum within-area tour duration constraints, we must have \( y_{\hat{s}} < y(l_{\hat{s}}) \). Also \( \hat{s} > 1 \) since it must be \( y_1 = l_1 = y(l_1) \), as otherwise \( \sum_{j=1}^{m} y_j \neq L \). Consider an alternate solution with the same number of sectors and same sector length for all sectors, except sectors \( \hat{s} \) and \( \hat{s} - 1 \), such that the length of sector \( \hat{s} \) is increased by \( \epsilon \) and the length of sector \( \hat{s} - 1 \) is decreased by \( \epsilon \), where \( \epsilon \) is a small positive value. The difference in total cost between the optimal and the alternate solutions is

\[
g(y_{\hat{s}}, l_\hat{s}) + g(y_{\hat{s} - 1}, l_\hat{s} - y_{\hat{s}}) - g(y_{\hat{s}} + \epsilon, l_\hat{s}) - g(y_{\hat{s} - 1} - \epsilon, l_\hat{s} - y_{\hat{s}} - \epsilon)
\]

\[
= o \left( 2l_\hat{s} + y_{\hat{s}} \frac{W^2 \delta}{6} \right) + o \left( 2(l_\hat{s} - y_{\hat{s}}) + y_{\hat{s} - 1} \frac{W^2 \delta}{6} \right) - o \left( 2l_\hat{s} + (y_{\hat{s}} + \epsilon) \frac{W^2 \delta}{6} \right) - o \left( 2(l_\hat{s} - y_{\hat{s}} - \epsilon) + (y_{\hat{s} - 1} - \epsilon) \frac{W^2 \delta}{6} \right)
\]

\[
= 2\epsilon \delta < 0,
\]

which is a contradiction. \( \square \)
F.2. Proof of Lemma 3.1

**Proof.** First we show that \(m^*\) is non-decreasing in \(L\). Consider two strips with respective lengths \(L'\) and \(L''\) such that \(L' < L''\). For both strips, we use Algorithm 1 to obtain the optimal number of sectors. Let \(l'\) and \(l''\) be the variable used in this algorithm when the length of the strip is \(L'\) and \(L''\) respectively. In the first iteration, we have \(l' = L' < l'' = L''\). There are three cases: (i) if \(\bar{y}(l'') = l''\), then \(\bar{y}(l'') = l'\) and the number of sector is the same for both strips; (ii) if \(\bar{y}(l'') < l''\) and \(\bar{y}(l') = l'\), then the method stops for \(l'\) but not for \(l''\), which means that there is at least one more sector with \(l''\); (iii) if \(\bar{y}(l'') < l''\) and \(\bar{y}(l') < l'\), then the algorithm continues for both strips. Also in this case, we must have \(\bar{y}(l'') \geq \bar{y}(l')\) because \(\bar{y}(l)\) is non-increasing in \(l\) for values of \(l\) such that \(\bar{y}(l) < l\). As a result the next iteration starts with \(l'' := l'' - \bar{y}(l'')\) and \(l' := l' - \bar{y}(l')\), which is a similar starting point. We can therefore repeat the same argument. Since there is no case in which the strip with the greater length stops the recursive method before the strip with the shorter length does, the result must be true.

The fact that \(m^*\) is non-decreasing in \(\delta\) and \(W\) and non-increasing in \(T\) and \(Q\) follows directly from the fact that \(\bar{y}(l)\) is non-increasing in \(\delta\) and \(W\) and non-decreasing in \(T\) and \(Q\), as can be seen from (16). Given that \(\bar{y}(l)\) does not depend on \(f\) and \(o\), it follows that \(m^*\) is independent of these two cost parameters. □

F.3. Proof of Proposition 3.2

**Proof.** Property (i) holds because the length of the first sector is always equal to its maximum since \(y_1 = l_1 = \bar{y}_1(l_1)\). Property (ii) can be proven by contradiction. Suppose there exists a sector \(k < j\) such that \(o_k \leq o_k(12 + 6W^2)/(6W^2)\). Consider an alternate solution with the same number of sectors \(m\) and the same vehicle types used in each sector, but with sector lengths \(y'_1, \ldots, y'_m\) such that \(y'_i = y_i\) for \(i \neq k\), \(y'_j = y_j + \epsilon\) and \(y'_k = y_k - \epsilon\), with \(\epsilon \in (0, \bar{y}_k(l_k) - y_j)\). In other words only sectors \(j\) and \(k\) are different in the two solutions. This alternate solution is feasible since \(\bar{y}_j(l_j)\) is non-increasing in \(l_j\) for \(i > 1\). The difference in total cost between the optimal and the alternate solution is given by

\[
\sum_{i=k}^{j} g_i \left( y_i, L - \sum_{l=1}^{m} y_l \right) - \sum_{i=k}^{j} g_i \left( y'_i, L - \sum_{l=1}^{m} y'_l \right) = f_i + o_i \left[ 2(l_i + \varphi) + \frac{y_i W^2 \delta}{6} \right] + f_j + o_j \left[ 2(l_j + \varphi) + \frac{y_j W^2 \delta}{6} \right] + \sum_{i=k+1}^{j} o_i \left[ 2(l_i + \varphi) + \frac{y_i W^2 \delta}{6} \right] - o_k \left[ 2(l_k + \varphi) + \frac{y_k W^2 \delta}{6} \right] - o_l \left[ 2(l_l + \varphi) + \frac{y_l W^2 \delta}{6} \right] = \frac{\epsilon W^2 \delta}{6} + o_k \left( 2\epsilon + \frac{W^2 \delta}{6} \right) + \sum_{i=k+1}^{j} o_i \left[ 2(l_i + \varphi) + \frac{2(y_j - \epsilon) W^2 \delta}{6} \right] - \sum_{i=k+1}^{j} o_i \left[ 2(l_i + \varphi) - \frac{2(y_k - \epsilon) W^2 \delta}{6} \right] > 0.
\]

The last term is positive since \(o_k \leq o_k(12 + 6W^2)/(6W^2)\) by the contradiction assumption; hence, we have a contradiction. The proof of property (iii) consists of two parts: (a) we show that it is never optimal to have a sector \(k < j\) such that \(\bar{y}_k(l_k) = \frac{6(T_k - 2L_k)}{W^2 + 6W \ln \Theta}\); (b) we show that it is never optimal to have a sector \(k < j\) such that \(\bar{y}_k(l_k) < Q_k/(W \delta)\).

Part (a). The proof is by contradiction. Suppose there exists a sector \(k < j\) such that \(y_k \leq \bar{y}_k(l_k) = \frac{6(T_k - 2L_k)}{W^2 + 6W \ln \Theta}\). Let \(\alpha = \bar{y}_k(l_k) - y_k \geq 0\). Also let \(\epsilon = \bar{y}_k(l_k) - y_j\), which is strictly positive by definition of sector \(j\). By property (ii), \(o_k \neq o_j\), therefore, given the vehicle numbering, we also have \(T_k \geq T_j\) and \(Q_k \leq Q_j\). which implies that \(\bar{y}_k(l_k) = \frac{6(T_k - 2L_k)}{W^2 + 6W \ln \Theta} > \bar{y}_j(l_j) = \frac{6(T_j - 2L_j)}{W^2 + 6W \ln \Theta}\). Since \(y_j \leq \bar{y}_j(l_j)\), we have \(y_j \leq \bar{y}_k(l_k)\). There can be two cases: (1) \(\alpha \leq \epsilon\), (2) \(\alpha > \epsilon\). In case (1), consider an alternate solution \(S'\) with the same number of sectors, but with lengths \(y'_1, \ldots, y'_m\) such that \(y'_i = y_i\) for
i = 1, \ldots, m with i \neq k, j, y_j' = y_j + (e - \alpha) and y_k' = y_k - (e - \alpha). And vehicle types t_1', \ldots, t_m' such that t_i' = t_i for i = 1, \ldots, m and i \neq k, j, t_j' = t_k and t_k' = t_j. In other words sector j gets longer and k gets shorter and they switch their vehicle types. This solution is feasible since y_j' = \bar{y}_t(k_i) - \alpha < \bar{y}_t(l_j) and y_k' = \bar{y}_t(k_k) - \epsilon = \bar{y}_t(k_l) - \epsilon = \bar{y}_t(k_k) - \bar{y}_t(l_j)) + \bar{y}_t(l_j) - y_j = \bar{y}_t(k_l) - \bar{y}_t(l_j) + \bar{y}_t(l_j) - y_j) = \bar{y}_t(k_l) - (\bar{y}_t(k_k) - \bar{y}_t(l_j) + \bar{y}_t(l_j) - y_j) < \bar{y}_t(l_k) - \bar{y}_t(l_j) < \bar{y}_t(l_k) < \bar{y}_t(l_k) - \epsilon + \alpha).

The cost difference between the original and the alternate solution is

\[ g_{t_i}(y_i, l_i) + g_{t_k}(y_k, l_k) + \sum_{i=k+1}^{j-1} g_{t_i}(y_i, l_i) - g_{t_i}(y_i + (e - \alpha), l_i) \]

\[ = g_{t_i}(y_k - (e - \alpha), l_k - (e - \alpha)) - \sum_{i=k+1}^{j-1} g_{t_i}(y_i, l_i - (e - \alpha)) \]

\[ = \alpha_t [2(l_j + \varphi) + y_j k W^2 \delta] + \alpha_t [2(l_k + \varphi) + y_k k W^2 \delta] + \sum_{i=k+1}^{j-1} \alpha_t [2(l_i + \varphi) + y_i k W^2 \delta] \]

\[ = \alpha_t [2(l_j + \varphi) + \frac{y_j k W^2 \delta}{6}] + \alpha_t [2(l_k + \varphi) + \frac{y_k k W^2 \delta}{6}] + \sum_{i=k+1}^{j-1} \alpha_t [2(l_i + \varphi) + \frac{y_i k W^2 \delta}{6}] \]

which is positive because \( \epsilon > \alpha, \alpha_t > \alpha_t \), and \( l_j > l_k \).

In case (2), consider an alternative solution \( S' \) with the same number of sectors and the same sector lengths, but with vehicle types \( t_1', \ldots, t_m' \) such that \( t_i' = t_i \) for \( i = 1, \ldots, m \) and \( i \neq k, j \), \( t_j' = t_k \) and \( t_k' = t_j \). In other words, sectors \( k \) and \( j \) exchange their vehicle types. This solution is feasible for \( \alpha > \epsilon \) since \( y_j < \bar{y}_t(l_j) \leq \bar{y}_t(k_l) \) and \( y_k \leq \bar{y}_t(k_k) - \epsilon \leq \bar{y}_t(k_l) - \bar{y}_t(l_j)) = \bar{y}_t(k_l) - (\bar{y}_t(k_k) - \bar{y}_t(l_j)) = \bar{y}_t(l_k) \). The cost difference between the original and the alternate solution is

\[ g_{t_i}(y_i, l_i) + g_{t_k}(y_k, l_k) + \sum_{i=k+1}^{j-1} g_{t_i}(y_i, l_i) - g_{t_i}(y_i, l_i) - g_{t_i}(y_k, l_k) - \sum_{i=k+1}^{j-1} g_{t_i}(y_i, l_i) \]

\[ = \alpha_t [2(l_j + \varphi) + \frac{y_j k W^2 \delta}{6}] + \alpha_t [2(l_k + \varphi) + \frac{y_k k W^2 \delta}{6}] - \alpha_t [2(l_j + \varphi) + \frac{y_j l_k W^2 \delta}{6}] - \alpha_t [2(l_k + \varphi) + \frac{y_k l_k W^2 \delta}{6}] \]

\[ = (\alpha_t - \alpha_t) \left( 2(l_j - l_k) + \frac{W^2 \delta}{6} (y_j - y_k) \right) \]

\[ = (\alpha_t - \alpha_t) \left( 2(l_j - l_k) + \frac{W^2 \delta}{6} (\bar{y}_t(l_j) - \bar{y}_t(l_k)) + (\alpha - \epsilon) \right) \]

\[ \geq (\alpha_t - \alpha_t) \left( 2(l_j - l_k) + \frac{W^2 \delta}{6} \frac{12 (l_k - l_j)}{W^2 \delta + 6W l h \delta} \right) \]

\[ = (\alpha_t - \alpha_t) \left( 2(l_j - l_k) \left( 1 - \frac{W^2 \delta}{W^2 \delta + 6W l h \delta} \right) \right) . \]

which is positive since \( \alpha_t > \alpha_t \) and \( l_j > l_k \).

Part (b). The proof is by contradiction. Suppose there exists a sector \( k < j \) such that \( y_k < Q_{t_j} / (W \delta) \). Consider an alternate solution \( S' \) with the same number of sectors and vehicle types serving each sector, but with lengths \( y_1', \ldots, y_m' \) such that
\( y_j' = y_j \) for \( i = 1, \ldots, m \) with \( i \neq k, j, y_j' = y_j + \epsilon \) and \( y_k' = y_k - \epsilon \). Where \( \epsilon \) is a (positive or negative) value such that \( y_k' \geq 0, y_j' > 0 \) and \( y_j' < \bar{y}_k(l_j) \). In other words, we shift some of the lengths of sector \( j \) to sector \( k \) or the other way around and we keep the length of all other sectors unchanged. This alternate solution is feasible since \( \bar{y}_k(l_j) = \min(l_j, Q_k / \delta) \) for \( i = 2, \ldots, j - 1 \) by Part (a). The difference in cost between solution \( S \) and \( S' \) is \( \epsilon(2 \sum_{t=k}^{i-1} a_{t_k} - W^2 \delta) / 6(o_{t_k} - o_{t_j}) \). Depending on whether \( \epsilon \) is positive or negative, this value can be made positive, hence we have a contradiction. Note that property (iii) implies that there is at most one sector with length shorter than its maximum possible value.

The proof of property (iv) is by contradiction. Suppose there exists a sector \( k < j \) such that \( y_j < Q_k / \delta \). By Property (iii) we know that \( y_k = \bar{y}_k(l_k) = Q_k / \delta \) which implies \( y_j < y_k \). There can be two cases: (1) \( \bar{y}_k(l_j) = Q_k / \delta \) and (2) \( \bar{y}_k(l_j) = \frac{6(T_k - \epsilon - 2l_k)}{W^2 \delta + 6 W h v \delta} < Q_k / \delta \). In case (1), consider an alternate solution \( S' \) with the same number of sectors with lengths \( \bar{y}_y', \ldots, \bar{y}_y' \) such that \( y_j' = y_j \) for \( i = 1, \ldots, m \), with \( i \neq k \) and \( i \neq j \) and \( y_j' = y_j \). \( y_j' = y_k \) and vehicle types \( t_1', \ldots, t_m' \) such that \( t_i' = t_i \) for \( i = 1, \ldots, m \) and \( t_k' = t_j \) and \( t_k' = t_k \). In other words, the lengths and types of sectors \( k \) and \( j \) are switched. This solution is feasible since \( \bar{y}_k(l_i) = y_k \) and \( \bar{y}_k(l_k) \geq \bar{y}_k(l_j) > y_j \). The cost difference between two solutions is \( 2(o_{t_k} - o_{t_j})(l_j - l_k) + \sum_{i=k+1}^{j-1} o_{t_k}(y_k - y_j) \), which is positive since \( o_{t_k} > o_{t_j} \) by property (ii), \( j > l_k \) and \( y_k > y_j \). Hence, we have a contradiction. In case (2) the proof is similar to that of Property (iii) Part (a). By property (ii) we have that \( o_{t_k} < o_{t_j} \), therefore, given the vehicle type number, \( t_k < t_j \) and we also have \( T_k < T_j \) and \( Q_k < Q_j \). Let \( \epsilon = \bar{y}_k(l_j) - y_j \) which is strictly positive since \( y_j < \bar{y}_k(l_j) \) and \( T_k < T_j \) so that \( \bar{y}_k(l_k) > \bar{y}_k(l_j) \). Consider an alternate solution \( S' \) with the same number of sectors, but with lengths \( \bar{y}_y', \ldots, \bar{y}_y' \) such that \( y_j' = y_j \) for \( i = 1, \ldots, m \) with \( i \neq k \) and \( i \neq j \). \( y_j' = y_j + \epsilon = \bar{y}_k(l_j) \) and \( y_k' = y_k - \epsilon = \bar{y}_k(l_k) - \epsilon \). And vehicle types \( t_1', \ldots, t_m' \) such that \( t_i' = t_i \) for \( i = 1, \ldots, m \) and \( i \neq k \) and \( i \neq j \) and \( t_k' = t_k \) and \( t_k' = t_j \). In other words sector \( j \) gets longer and sector \( k \) gets shorter and they switch their vehicle types. We show that this alternate solution is feasible. First note that \( y_j' = \bar{y}_k(l_j) = \bar{y}_k(l_j) \). Second, if \( \bar{y}_k(l_k) = Q_k / \delta \), we define \( \bar{y}_k(l_k) = \frac{6(T_k - \epsilon - 2l_k)}{W^2 \delta + 6 W h v \delta} \). And we have \( y_k' = \bar{y}_k(l_k) - \epsilon < \bar{y}_k(l_k) - (\bar{y}_k(l_j) - y_j) = \bar{y}_k(l_k) - \bar{y}_k(l_j) - y_j \). We also have \( y_j' = \bar{y}_k(l_j) - y_j \). \( \bar{y}_k(l_j) - y_j < \bar{y}_k(l_k) - \epsilon = \bar{y}_k(l_k) - \epsilon \). The cost difference between the original and the alternate solution is

\[
\begin{align*}
g_c(y_j, l_j) + g_t(y_k, l_k) + \sum_{i=k+1}^{j-1} g_c(y_i, l_i) - g_t(y_j + \epsilon, l_j) - g_t(y_k - \epsilon, l_k - \epsilon) - \sum_{i=k+1}^{j-1} g_c(y_i, l_i) - \epsilon) \\
= o_k \left[ 2(l_j + \varphi) + \frac{y_j W^2 \delta}{6} \right] + o_k \left[ 2(l_k + \varphi) + \frac{y_k W^2 \delta}{6} \right] + \sum_{i=k+1}^{j-1} o_k \left[ 2(l_i + \varphi) + \frac{y_i W^2 \delta}{6} \right]
- \left[ 2(l_k + \varphi - \epsilon) - \frac{y_k W^2 \delta}{6} \right] - \sum_{i=k+1}^{j-1} o_k \left[ 2(l_i + \varphi - \epsilon) - \frac{y_i W^2 \delta}{6} \right]
= 2o_k \epsilon + (o_{t_j} - o_{t_k}) \left[ 2(l_j - l_k) + \frac{W^2 \delta}{6} (y_j - y_k + \epsilon) \right] + \sum_{i=k+1}^{j-1} o_k \epsilon
= 2o_k \epsilon + (o_{t_j} - o_{t_k}) \left[ 2(l_j - l_k) + \frac{W^2 \delta}{6} (\bar{y}_k(l_j) - \bar{y}_k(l_k)) \right] + \sum_{i=k+1}^{j-1} o_k \epsilon
> 2o_k \epsilon + (o_{t_j} - o_{t_k}) \left[ 2(l_j - l_k) + \frac{W^2 \delta}{6} (12(l_k - l_j)) \right] + \sum_{i=k+1}^{j-1} o_k \epsilon
= 2o_k \epsilon + (o_{t_k} - o_{t_j}) \left[ 2(l_j - l_k) \left(1 - \frac{W^2 \delta}{W^2 \delta + 6 W h v \delta} \right) \right] + \sum_{i=k+1}^{j-1} o_k \epsilon
\end{align*}
\]

which is positive because \( \epsilon > 0, o_{t_j} < o_{t_k} \) and \( l_j > l_k \). \( \Box \)

**F.4. Proof of Lemma 3.2**

**Proof.** Let \( \bar{y}_i(l) = \min \left\{ l, o_{t_k} \frac{6(T_k - \epsilon - 2l_k)}{W^2 \delta + 6 W h v \delta} \right\} \). For all \( l \in [0, l] \), we have \( \bar{y}_i(l) \leq \bar{y}_k(l) \) for \( k = 1, \ldots, K \). The proof is by contradiction. Suppose there exists an optimal solution \( S' \) where \( m^* > \bar{m} \). This is only possible if there is a sector \( j \) such that
$y_j < \bar{y}^\text{min}(l_j)$. By Proposition 3.2 properties (i) and (iv), it must be $j > 1$ and $y_j > q_{j-1}/(\delta W)$. But this would imply that $\bar{y}^\text{min}(l_j) > q_{j-1}/(\delta W) \geq \tilde{y}_{j-1}(l_j)$, which is a contradiction. □

F.5. Proof of Lemma 3.3

**Proof.** Consider a solution $S$ with two non-consecutive sectors $k$ and $i$, such that $k + 1 < i$, which are served by the same type of vehicles, i.e., $t_k = t_i$, such that all the sectors in between are served by the other vehicle type, i.e. $t_j = 3 - t_k$ for $j = k + 1, \ldots, i - 1$. We show that for $S$ to be optimal it must be $k = 1$. This will prove that there cannot be other structures than (A), (B) or (C). In particular any structure with the vehicle types switching more than twice from the depot to the end of the area will not be possible, since it would violate this property.

The proof is by contradiction, i.e., suppose $k > 1$. There are two cases (i) $t_k = t_i = 1$ and (ii) $t_k = t_i = 2$.

In Case (i), consider an alternate solution $S'$ obtained by swapping sectors $k$ and $k + 1$, that is, setting $t'_k = 2$, $t'_{k+1} = 1$. $y_{k+1}' = y_k$ and $y'_k = y_{k+1}$ (and leaving all other sectors unchanged). This solution is feasible since, for $T_1 = T_2 = \infty$, $y_k = \tilde{y}_1(l_k) - \tilde{y}_1(l_{k+1})$. Also, $\tilde{y}_2(l_{k+1} - y_k) = \min\{q_2/(W \delta), l_{k+1} - y_k\}$. We know that $(i) y_{k+1} \leq \tilde{y}_2(l_{k+1}) \leq q_2/(W \delta)$. Moreover, since $k > 1$ it must be $y_k + y_{k+1} < l_k$ and therefore $(i) y_{k+1}' = y_k$ and $y'_k = y_{k+1}$. From (i) and (ii) it follows $y_{k+1} \leq \tilde{y}_2(l_{k+1} - y_k)$. The difference in costs between solution $S$ and $S'$ is $\Delta S = 2(o_2 y_k - o_1 y_{k+1})$, which must be negative if $S$ is optimal. Next, let $S''$ be an alternate solution obtained by swapping sectors $i - 1$ and $i$, that is, setting $t''_i = 2$, $t'_{i-1} = 1$. $y''_i = y_{i-1}$ and $y''_{i-1} = y_i$ (and leaving all other sectors unchanged). This solution is feasible since $y_{i-1} \leq \tilde{y}_2(l_{i-1}) \leq \tilde{y}_2(l_i)$ and $y_i = \tilde{y}_1(l_i) = \tilde{y}_1(l_i - y_i)$. The difference in costs between solution $S$ and $S''$ is $\Delta S'' = 2(o_1 y_{i-1} - o_2 y_i)$. From Proposition 3.2, if there is a sector with length less than its maximum, then it must be the first sector served by a vehicle of type 2 starting from the bottom of the strip, i.e., sector $k + 1$. Hence, we have $y_k = \tilde{y}_1(l_k) = \tilde{y}_2(l_k) = y$ and $y_i = \tilde{y}_1(l_i)$, and $y_{i-1} = \tilde{y}_2(l_{i-1})$. Moreover since $k + 1 > 1$ and $T_1 = T_2 = \infty$, we must have $\tilde{y}_1(l_k) = \tilde{y}_1(l_i) = q_1/(W \delta)$ and $\tilde{y}_2(l_{i-1}) = max(l_{i-1}, q_2/(W \delta)) \leq \tilde{y}_2(l_{i-1}) = max(l_{i-1}, q_2/(W \delta))$ by (16). This implies that $y_k = y_i$ and $y_{k+1} \leq y_{i-1}$. Therefore, $\Delta S'' = 2(o_1 y_{i-1} - o_2 y_i) \geq 2(o_1 y_{k+1} - o_2 y_k) = -\Delta S > 0$, which is a contradiction.

In case (ii), note that sector $k$ cannot be preceded by a vehicle of type 1, since otherwise we could be in case (i) as well. Therefore by Proposition 2, all sectors lengths have to be equal to their maximum value so that it is easy to show that $\Delta S'' = -\Delta S > 0$, which is a contradiction to $S$ being optimal. □

F.6. Proof of Lemma 3.4

**Proof.** We first prove (a). If $Q_1 > \frac{6(T_2 v) W}{\delta W^2 + 6 h v \delta W} + 12$ then for all $l \geq 0$, then we have

$$\bar{y}_2(l) = \begin{cases} l & \text{for } l \leq \frac{6 T_2 v}{\delta W^2 + 6 h v \delta W + 12} \\ \frac{6(T_2 v - 2l)}{\delta W^2 + 6 h v \delta W + 12} & \text{for } \frac{6 T_2 v}{\delta W^2 + 6 h v \delta W + 12} < l \leq \frac{6 T_2 v}{2} \end{cases}$$

and

$$\bar{y}_1(l) = \begin{cases} l & \text{for } l \leq \min\left\{ \frac{6 T_1 v}{\delta W^2 + 6 h v \delta W + 12}, \frac{6 Q_1}{\delta W^2 + 6 h v \delta W + 12} \right\} \\ \min\left\{ \frac{6 Q_1}{\delta W^2 + 6 h v \delta W + 12}, \frac{6(T_1 v)}{\delta W^2 + 6 h v \delta W + 12}, \frac{6 T_1 v - Q_1 (W + 6h v)}{12} \right\} & \text{for } \frac{6 Q_1}{\delta W^2 + 6 h v \delta W + 12} < l \leq \frac{T_1 v}{2} \end{cases}$$

The comparison of these two equations implies that $\bar{y}_1(l) \geq \bar{y}_2(l)$ for all $l < \frac{T_2 v}{2}$. Hence, for $l < \frac{T_2 v}{2}$, we can write the DP recursion as

$$V(l) = \min_{0 \leq y \leq \bar{y}_2(l)} \{ g_1(y, l), g_2(y, l) \} + V(l - y).$$

When $f_1 \leq f_2$, $g_2(y, l) \leq g_1(y, l)$ for all $l > 0$ and $y \leq \bar{y}_2(l)$, which means that it is optimal to select vehicle 1. For $l \in [T_2 v/2, t_1 v/2)$, only vehicles of type 1 are feasible. Therefore it is optimal to use vehicles of type 1 for all values of $l$ such that the problem is feasible.

We next prove (b). Consider the first iteration of the DP recursion

$$V(L) = \min_{0 \leq y \leq \bar{y}_2(L)} \{ g_1(y, L) + V(L - y) \}, \quad \min_{0 \leq y \leq \bar{y}_1(L)} \{ g_2(y, L) + V(L - y) \} \quad (E1)$$

Let $\hat{l}(y) = \frac{6(f_1 - f_2) - (o_2 - o_1)(w^2 \delta y + 12 \Phi)}{12(o_2 - o_1)}$ be the maximum value of $l$ such that, for a given $y$, and all $l \leq \hat{l}(y)$, $g_2(y, l) = f_2 + o_2(2l + \phi) + w^2 \delta y / 6 \leq g_1(y, l) = f_1 + o_1(2l + \phi) + w^2 \delta y / 6$. If $\hat{l}(Q_2/\delta W) > L$, which is equivalent to $f_1 \geq f_2 + (o_2 -
Proof. The result follows directly for the realization that for all \( s \), \( \varphi_i^s = \varphi_{s-i+1}^s \) for \( i = 1, \ldots, \left\lfloor \frac{s}{2} \right\rfloor \), therefore \( V_{W/S, \varphi_i^s} (L) = V_{W/S, \varphi_{s-i+1}^s} (L) \) for all \( W/s \) and \( L \). □

References

Daganzo, C.F., 1984a. The distance traveled to visit \( n \) points with a maximum of \( c \) stops per vehicle: an analytic model and an application. Transp. Sci. 18 (4), 331–350.


