A robust control analysis for a steer-by-wire vehicle with uncertainty on the tyre forces

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A robust control analysis for a steer-by-wire vehicle with uncertainty on the tyre forces

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ABSTRACT
This paper proposes guidelines for controller design for steer-by-wire vehicles. The proposed controller removes the velocity dependency of the vehicle dynamics and accounts for the nonlinear characteristics of the lateral tyre force. Using an $H_{\infty}$ approach, bounds on the closed loop sensitivity are presented, which results in guidelines for the controller design. Using the designed controller, simulations using a multi-body model show that the yaw dynamics of the vehicle are stabilised for extreme manoeuvres.

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KEYWORDS
Steer-by-wire; vehicle dynamics; yaw-rate control; robust stability; velocity decoupling

1. Introduction
‘The average driver rates themselves as better than average’ is an often heard sentence when discussing driving skills. Be that as it may, but the average driver usually fails to maintain the vehicle in a stable and controlled state when surprised by unexpected vehicle behaviour. For one, it is difficult for the driver to recognise the coefficient of friction between the tyres and road and to identify the vehicle’s limits. Secondly, once the stability margin is surpassed, drivers are often surprised and react in a wrong way.[1]

Over the last decade major steps have been taken to improve vehicle safety by means of brake operated safety systems such as anti-lock braking system (ABS) and electronic stability control (ESC). The first system greatly increases vehicle controllability in emergency braking situations by preventing wheel lock. Since a locked wheel can only generate a force opposite to its direction of travel it is useless in an evasive maneuver as no lateral forces can be developed. By controlling the brake pressure ABS tries to maintain wheel rotation, thereby allowing the tyre to also generate a lateral force.[2] The ESC system influences the yaw and lateral dynamics of the vehicle by individually braking wheels, thereby creating a corrective yaw moment.

More recent developments in the field of vehicle control are four wheel steering [3] and active front steer.[4] Four wheel steering is mostly installed to enhance vehicle maneuverability at low velocities by applying a rear wheel steering angle opposite to the front wheel steering angle. At higher velocities the rear wheels are steered in the same direction as the front wheels to increase yaw and side-slip stability. Active front steer can be found on
modern BMWs and is capable of influencing the road wheel angle by means of actuating a planetary gear set in the steering column, thereby creating an additional steering angle at the front wheels. Its application is two-fold, with on one hand allowing for a variable steering ratio, depending on forward velocity and on the other hand influencing the vehicle yaw and side-slip stability.

A system that has not been explored as extensively as the aforementioned systems is steer-by-wire. Compared to the active front steer there is no physical connection between the steering wheel and front wheels anymore, allowing for greater flexibility. On the one hand the steering wheel torque can be chosen freely, this is discussed extensively in [5]. Besides this, the vehicles yaw dynamics can be influenced easily with steer-by-wire, not only in limit handling situations, but also for normal driving. For instance [6] uses a steer-by-wire system to change the vehicles front cornering stiffness as observed from the steering wheel. A full state feedback controller is introduced to modify the handling dynamics. With a modified Chevrolet Corvette it is shown that the observed cornering stiffness can be reduced in practice.

Besides the control of the vehicle handling dynamics using steer-by-wire, a challenging subject is the control of the steering wheel and local control of the steering rack. The reaction torque on the steering wheel as a function of changing ratio is discussed in [8] for example. In this paper a control method is suggested to cancel the unnatural feel resulting from changing the steering ratio. Hardware-in-the-loop tests confirm the effectiveness of the proposed control method. A steer-by-wire system with an unknown disturbance is discussed in [9]. To compensate for the unknown disturbance, a sliding mode controller is designed. The controller design is verified on a lab setup, which shows that the front wheels follow the commanded steering wheel angle in practice.

An extensive review of 68 papers on yaw-rate and side slip control is given in [10]. One of the main conclusions of this paper is that there is still a significant research effort required to address the subjective performance of systems. Furthermore, more research is needed to develop schemes that integrate systems to achieve high level performance objectives. A clear function of active steering in addition to current day ESC systems is found in the sense that it can attenuate the yaw response of the vehicle whilst for instance braking on a split-μ surface. The works of Ackermann [11] and Ackermann and Sienel [12] are also discussed, the author of [10] is critical about this work as he argues that the system’s performance is questionable in limit handling situations. Another critical aspect is the lack of hardware specifications for the tests that were performed.

A controller to stabilise the vehicle in high side-slip conditions is developed in [13]. A phase plot of the vehicle side slip angle and yaw-rate is used to identify the vehicle behaviour. It is found that three equilibria exist, one stable for straight line driving and two unstable equilibria corresponding to high side slip maneuvers. To control these unstable equilibria a state feedback controller is proposed. With measurements on a test vehicle it is shown that the controller works properly, even when the vehicle state is disturbed momentarily.

An often used addition to steer-by-wire of the front wheels is rear wheel steering. With this addition the yaw-rate and lateral velocity of the vehicle can be controlled independently of each other. For the front and rear wheel steering angles, two PI controllers are used with the control gains expressed as a function of the desired eigenvalues. Using the $H_\infty$ norms of the transfer functions, an optimisation is performed to determine the
gains. Vehicle side slip angle tracking up to 1 Hz and yaw-rate tracking up to 10 Hz is achieved. As an added benefit the phase lag between the yaw-rate and lateral acceleration is reduced.

A combination of steer-by-wire and longitudinal control is presented in [15], where an extremum seeking controller is introduced to control the longitudinal tyre forces. For steer-by-wire an linear quadratic regulator (LQR) controller based on yaw-rate and vehicle slip angle is designed. The desired yaw-rate is set by a saturated first-order transfer function, whilst the reference vehicle slip angle is set to zero. Simulations show that for both an emergency braking situation as well as braking on a μ-split surface the controlled car manages to follow the reference behaviour, whereas the open loop vehicle spins. Thus, a clear benefit of steer-by-wire over yaw-rate control with the brakes only is shown.

A nonlinear controller for an electric vehicle with four independent motors and steer-by-wire is developed in [16]. The controller is set up such that it considers all the tyre forces and moments as acting on the centre of gravity of the vehicle. Input–output linearisation is performed such that a linear controller can be implemented. The downside of this approach is that the full vehicle state has to be known. Control allocation is based on a pseudo-inverse of the input matrix, that describes the influence of each actuator on the centre of gravity. By means of an unscented Kalman filter the vehicle states are estimated. Simulations with a single sine steering input as well as with μ-split braking show that the controlled vehicle exhibits more stable behaviour and requires less input from the driver.

Common to the aforementioned control schemes is that either a linear controller is used on what is in essence a highly nonlinear system, or a nonlinear controller is used which requires detailed knowledge of the vehicle dynamics. Both methods show to be successful in the range of operation for which the controller is designed, but the behaviour for other conditions is not discussed. In [17] this problem is recognised. As a solution not the steering wheel angle, but the front tyre force is considered as input. Model predictive control is then used to determine the required control input. For each time step of the MPC controller, it is assumed that the input is kept constant over the horizon of 150 ms. During this 150 ms interval it is assumed by the author that the states can be calculated using a linear single-track model.

The goal of this paper is the control of the vehicle yaw dynamics using a steer-by-wire system. It is assumed that the local control of the motion of the steering rack and steering wheel is handled by suitable controllers. The design of these local controllers is beyond the scope of this paper, so the focus is on the control of the vehicle yaw dynamics. The aim is to provide guidelines on how to design the controller, such that, given an operating range, the stability of the yaw dynamics of the controlled vehicle can be assured. In these guidelines, forward velocity dependency of the vehicle dynamics will be included. This problem is recognised in [14] for example, but not implemented in the accompanying control structure. The forward velocity has a large influence on the vehicle dynamics, and therefore on the closed loop behaviour when including the vehicle in a control loop. One of the objectives in this paper is therefore to account for the dependency of the yaw dynamics on forward velocity. Besides the dependency of forward velocity, the tyres behave nonlinear in limit handling situations, which could influence the yaw stability of the vehicle. This nonlinear tyre behaviour was dealt with in [17] using model predictive control, which computationally expensive. In this paper the problem of the nonlinear tyre behaviour is recognised, it is therefore required that the controller accounts for possible variations in
the tyre behaviour. A robust control analysis is used to design a controller that is stable for these variations of the tyre behaviour.

The outline of this paper is as follows. In Section 2 the control objective is introduced, followed by a description of the vehicle model in Section 3. The influence of the tyre non-linearities is also explored in this section. Design of the controller is discussed in Section 4. The performance of the controller is evaluated in Section 5 using a multi-body model of the test vehicle. Finally, conclusions are presented in Section 6.

2. Control objective

The yaw-stability of a vehicle is of utmost importance. Besides this, the vehicle side slip angle is often seen as a limiting factor in vehicle handling.\cite{13,18} However, since this state is impossible to measure with affordable sensors, only the yaw-rate of the vehicle will be considered as input for the controller.

Stable yaw dynamics implies that when a bounded steering input is provided, the vehicle will exhibit a bounded yaw-rate. It should furthermore be robust against disturbances that excite the yaw dynamics. Ideally, the vehicle exactly follows the behaviour as expected by the driver, and thus exhibits a small yaw-rate error with respect to the reference behaviour. If an error does occur, it should ideally be suppressed faster than a human solving the problem.

The control lay-out that is used throughout this paper is shown in Figure 1. Starting from the left we see the steering wheel angle, $\delta_{sw}$, and forward velocity, $v_x$, which are inputs set by the driver. In a conventional vehicle these two inputs directly influence the vehicle dynamics. Here, however, the steering wheel angle and forward velocity are used as inputs for a reference model that provides a desired reference yaw-rate, $r_{zref}$. This reference model can be any physically feasible model. Note that this reference model can be designed such that it again has forward velocity dependent dynamics, such that the drive experiences the behaviour of a normal vehicle. The detailed design of this reference model is beyond the scope of this paper though. The reference yaw-rate, $r_{zref}$, is the input for the closed loop which contains the controller and the vehicle. Both the vehicle and the controller will be discussed in more detail in the upcoming sections. In summary, the objective for the closed loop system are formulated as

1. A small yaw-rate error $e = r_{zref} - r_z$ overshoot.
2. Stability, all closed loop poles should thus have a negative real part.
3. Fast convergence of the error dynamics. Ideally the poles are placed such that the response of the closed loop is beyond human capabilities, that is, a bandwidth beyond approximately 4 Hz,\cite{19} performing better than the best driver.
4. Closed loop dynamics that are independent of speed.

![Figure 1](image-url). Feedback control layout for yaw-rate control.
Note that all these objectives refer to the closed loop behaviour, that is, from $r_{z_{\text{ref}}}$ to $r_z$. This does not imply that the behaviour as seen by the driver, which includes the reference model, has to fulfill all these criteria.

3. Vehicle model

For design and analysis of the yaw-rate controller, the well known single track model is used.[20] In this model only the planar vehicle dynamics are considered, furthermore, it is assumed that the longitudinal velocity, $v_x$, varies slowly compared to the other dynamics. The longitudinal dynamics are therefore neglected. The lateral and yaw motion of the vehicle’s centre of gravity are the only degrees of freedom under consideration, the equations of motion for these degrees of freedom can be written as

$$
\dot{x} = \begin{bmatrix}
\dot{v}_y \\
\dot{r}_z
\end{bmatrix} = \begin{bmatrix}
\dot{v}_y \\
\dot{r}_z
\end{bmatrix} = Ax + B\delta_{\text{rack}}
$$

(1)

$$
y = r_z = Cx
$$

(2)

$$
A = \begin{bmatrix}
\frac{C_{\alpha 1} + C_{\alpha 2}}{aC_{\alpha 1} - bC_{\alpha 2}} \\
\frac{m v_x}{I_{zz} v_x} \\
\frac{m v_x}{a^2 C_{\alpha 1} + b^2 C_{\alpha 2}} \\
\frac{v_x}{I_{zz} v_x}
\end{bmatrix},
B = \begin{bmatrix}
\frac{C_{\alpha 1}}{m} \\
\frac{C_{\alpha 1} a - C_{\alpha 2} b}{a^2 C_{\alpha 1} + b^2 C_{\alpha 2}}
\end{bmatrix},
C = [0 \ 1]
$$

(3)

Here, $r_z$ represents the vehicle yaw-rate and $v_y$ the lateral velocity. The vehicle mass and inertia are represented by $m$ and $I_{zz}$, respectively. The parameters $a$ and $b$ represent the distance between the front and rear tyre to the centre of gravity, respectively (see Figure 2). The input to the model is the rack angle $\delta_{\text{rack}}$, which is introduced by the steering rack to the front wheels. Finally, the front cornering stiffness is $C_{\alpha 1}$ and rear cornering stiffness $C_{\alpha 2}$. The single track vehicle model, though simple, does represent the relevant dynamics of the vehicle. By including nonlinear tyre behaviour, a good representation of the lateral and yaw dynamics of the vehicle is achieved. A model with more degrees of freedom would overcomplicate the following analysis and would remove the mathematical insight that is still present with the presented two degrees-of-freedom model.

Typically, to achieve high lateral accelerations, large tyre side slip angles are required. As a result, the tyres exhibit nonlinear behaviour as is shown in Figure 3. Obviously, this has an influence on the vehicle dynamics. To incorporate this behaviour in the model, the local gradient of the lateral force, $K_{\alpha}$ is introduced. This variable can be interpreted as the

Figure 2. Single-track vehicle model.
Lateral force, $F_y$ versus side slip angle $\alpha$ with the cornering stiffness indicated by $C_\alpha$ and the local gradient of the lateral force $K_\alpha$.

Figure 3. Lateral force, $F_y$ versus side slip angle $\alpha$ with the cornering stiffness indicated by $C_\alpha$ and the local gradient of the lateral force $K_\alpha$.

cornering stiffness of the tyre when linearising around a tyre side slip operating point. The vehicle dynamics thus become

$$ \dot{x}_p = \begin{bmatrix} \dot{v}_y \\ \dot{r}_z \end{bmatrix} = A_p x_p + B_p \delta_{\text{rack}}, $$

$$ y_p = r_z = C_p x_p, $$

$$ A_p = - \begin{bmatrix} aK_{\alpha1} + bK_{\alpha2} & aK_{\alpha1} - bK_{\alpha2} \\ \frac{mv_x}{aK_{\alpha1} - bK_{\alpha2}} & \frac{mv_x}{a^2K_{\alpha1} + b^2K_{\alpha2}} \end{bmatrix} + v_x $$

$$ B_p = \begin{bmatrix} \frac{m}{aK_{\alpha1}} \\ \frac{I_{zz}}{I_{zz}} \end{bmatrix}, \quad C_p = [0 \ 1] $$

with $x_p$ the perturbed state vector, $y_p$ the perturbed output state and $A_p$, $B_p$ and $C_p$ the perturbed state space matrices. The local gradient of the lateral force is assumed to be within the range

$$ K_{\alpha1} \in [K_{\alpha1}^-, K_{\alpha1}^+] \quad \text{with} \quad K_{\alpha1}^{\text{nom}} = \frac{K_{\alpha1}^- + K_{\alpha1}^+}{2}, $$

$$ K_{\alpha2} \in [K_{\alpha2}^-, K_{\alpha2}^+] \quad \text{with} \quad K_{\alpha2}^{\text{nom}} = \frac{K_{\alpha2}^- + K_{\alpha2}^+}{2}. $$

With this range of the local gradient of the lateral force, the nonlinear behaviour of the lateral tyre force as shown in Figure 3 is accounted for. This validates the approach of approximating the vehicle dynamics with the model that was presented in Equations (4)–(6). The local gradient of the lateral force, $K_{\alpha1}^{\text{nom}}$ and $K_{\alpha2}^{\text{nom}}$, are chosen as the average between the minimum value, $K_{\alpha1}^-$ and $K_{\alpha2}^-$, and the maximum value, $K_{\alpha1}^+$ and $K_{\alpha2}^+$, respectively. If $K_{\alpha1}^-$ goes to zero, steering cannot influence the vehicle dynamics anymore. It is assumed that the mass, inertia and distances to the center of gravity are constant and that the tyres do not saturate fully. In real-life the mass, inertia and center of gravity distances could be estimated using for example stroke sensors and a known vertical stiffness of the suspension. The vehicle parameters are summarised in Table 1. The transfer function the steering
Table 1. Single track model vehicle parameters, based on a BMW 318i.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>1734</td>
<td>kg</td>
</tr>
<tr>
<td>$a$</td>
<td>1.422</td>
<td>m</td>
</tr>
<tr>
<td>$b$</td>
<td>1.303</td>
<td>m</td>
</tr>
<tr>
<td>$I_{zz}$</td>
<td>1900</td>
<td>kgm$^2$</td>
</tr>
<tr>
<td>$K_{α1}$</td>
<td>52,000</td>
<td>N/rad</td>
</tr>
<tr>
<td>$K_{α1}^+$</td>
<td>148,000</td>
<td>N/rad</td>
</tr>
<tr>
<td>$K_{α2}$</td>
<td>64,000</td>
<td>N/rad</td>
</tr>
<tr>
<td>$K_{α2}^+$</td>
<td>189,000</td>
<td>N/rad</td>
</tr>
</tbody>
</table>

rack angle, $δ_{rack}$ to the yaw-rate, $r_z$, with constant $v_x$ can be written as

$$G_p(s) = C_p(sI - A_p)^{-1}B_p = \frac{K_{α1}v_x((a + b)K_{α2} + amv_xs)}{(a + b)^2K_{α1}K_{α2} + m(bK_{α2} - aK_{α1})v_x^2 + ((K_{α1} + K_{α2})I_{zz} + m(a^2K_{α1} + b^2K_{α2}))v_xs + I_{zz}v_x^2s^2}$$

(9)

with $s$ being the Laplace variable.

### 3.1. Vehicle yaw-rate response

The vehicle yaw-rate response is influenced by the magnitude of the front and rear local gradient of the lateral force, $K_{αi}$. An example of the transfer functions for combinations of the minimum and maximum values of $K_{αi}$ is shown in Figure 4. From this figure it can be seen that the steady-state yaw-rate response

$$k_{δ→r_z} = \frac{(a + b)K_{α1}K_{α2}v_x}{(a + b)^2K_{α1}K_{α2} + m(bK_{α2} - aK_{α1})mv_x^2}$$

(10)

is influenced significantly by both the front and rear $K_{αi}$. Besides this dependency, Equation (10) also shows a dependency on forward velocity. This forward velocity dependency is also visible in the eigenfrequency, $f_e$ and dimensionless damping ratio, $β_e$

$$f_e = \frac{1}{2\pi} \frac{(a + b)^2K_{α1}K_{α2} + (bK_{α2} - aK_{α1})mv_x^2}{I_{zz}mv_x^2},$$

(11)

$$β_e = \frac{K_{α1}(I_{zz} + a^2m) + K_{α2}(I_{zz} + b^2m)}{2\sqrt{I_{zz}m((a + b)^2K_{α1}K_{α2} + (bK_{α2} - aK_{α1})mv_x^2)}},$$

(12)

For a typical road vehicle, this eigenfrequency is in the order of 1 Hz. At higher frequencies the yaw-rate response is only influenced by the front local gradient of the front tyre lateral force

$$G_p(s)|_{s→∞} = \frac{K_{α1}a}{I_{zz}s}.$$  

(13)

Figure 4 also shows that for $K_{α1}^+$ and $K_{α2}^-$ the phase behaves significantly different from the other cases. This is caused by the vehicle being oversteered and beyond its critical forward
velocity, which results in a right half plane pole and thus an unstable vehicle. This critical velocity can be derived by setting the denominator of Equation (10) to zero. In this particular case the front gradient of the lateral force is high ($K_{\alpha 1} = 148,000$ N/rad) and the rear one is low ($K_{\alpha 2} = 64,000$ N/rad), the vehicle is significantly oversteered and has a critical velocity of only 17.9 m/s.

4. Controller design

In a steer-by-wire vehicle there is no mechanical connection between the steering wheel angle $\delta_{sw}$ and rack motion, represented by the rack angle $\delta_{rack}$. Controllers that manage both the rack position and feedback torque on the steering wheel are required. In this paper only the control of the steering rack is considered, for more details on the steering wheel side see for example.[5,7–9] For control of the yaw dynamics of the vehicle, a feedback controller is used as is shown in Figure 1, here the vehicle is represented by Equation (9). It is assumed that the reference model can be any physically feasible model that produces a reference yaw-rate, which only depends on forward velocity, $v_x$, and driver steering wheel input, $\delta_{sw}$. Obviously, stable reference behaviour is desirable. Furthermore, it is assumed that the forward velocity, $v_x$, varies slowly in comparison with the yaw-rate, $r_z$.  

Figure 4. Yaw-rate response as a function of steering input for a range of values of the local gradient of the lateral force for $v_x = 30$ m/s.
4.1. Disturbance model

The prime objective is to design a stabilising controller for the plant as given by Equation (12). A suitable control method for this is $H_\infty$-control, which allows for the inclusion of uncertain parameters in the control structure. To model these uncertainties, a nominal model, $G_n(s)$, is defined, where $K_{a1} = K_{a1}\text{nom}$ and $K_{a2} = K_{a2}\text{nom}$. Furthermore, in Section 3.1 it is shown that, as a function of the uncertain parameters, a pole of the plant could move to the right half plane. A suitable method of representing this pole moving to the right half plane is the inverse multiplicative uncertainty,[21] where the perturbed plant is defined as

$$G_p(s) = G_n(s)(1 + w_{II}(s)\Delta_{II})^{-1}. \quad (14)$$

Here $w_{II}(s)$ is a weighting filter designed such that the uncertainty $|\Delta_{II}(j\omega)| \leq 1$ for all $\omega$. To visualise this uncertainty, $w_{II}(s)\Delta_{II}(j\omega)$ creates a disc shaped region with radius $l_{II}(\omega)$ around the nominal plant, $G_n(s)$, in the Nyquist plot. The radius of this disc, $l_{II}(\omega)$, describes the magnitude of uncertainty in the plant and can be calculated as [21]

$$l_{II}(\omega) = \max_{G_p \in \Pi} \left| \frac{G_n(j\omega) - G_p(j\omega)}{G_p(j\omega)} \right| \quad (15)$$

with $\Pi$ all possible plants within the set of plants defined by the parameter variation of (7), (8). The objective is now to find a weight $w_{II}(j\omega)$ such that

$$|w_{II}(j\omega)| \geq l_{II}(\omega), \forall \omega. \quad (16)$$

This radius of the disc shaped region, $l_{II}(\omega)$, can be calculated using standard MATLAB tools such as wcgain. For the parameters as listed in Table 1 this is shown in Figure 5. This figure shows that with increasing forward velocity, the uncertainty at low frequencies increases. At high frequencies the uncertainty becomes independent of forward velocity. Both these observations will be proven analytically in Section 4.2.

Robust stability for the controlled system can be guaranteed if

$$|1 + L(s)(1 + w_{II}(s)\Delta_{II})^{-1}| > 0, \quad \forall \omega, \quad (17)$$

which can be rewritten as

$$|1 + L(s)| - |w_{II}(s)| > 0, \quad \forall \omega, \quad (18)$$

$$\left| \frac{w_{II}(s)}{1 + L(s)} \right| < 1, \quad \forall \omega, \quad (19)$$

which is derived from the Nyquist stability criterion. The distance to the point $-1$ in the Nyquist plot is given by $|1 + L(s)|$ with $L(s) = C(s)G_n(s)$ and $C(s)$ any stabilising controller. The uncertainty weighting filter $w_{II}(s)$ potentially brings the perturbed plant closer...
Finally, this can be rewritten to the stability condition

$$|S(s)| < \frac{1}{|w_{II}(s)|}, \quad \forall \omega$$

(20)

with $S(s)$ the sensitivity function which is defined as

$$S(s) = \frac{1}{1 + L(s)} = \frac{1}{1 + C(s)G_n(s)}.$$  

(21)

It can be shown that this condition is sufficient to guarantee stability for each $\Delta_I$, even if it is a time varying operator, see [22].

### 4.2. Analytical uncertainty bounds

Figure 4 shows that depending on the the choice of the local gradient of the lateral force the vehicle yaw response greatly deviates from the nominal model. In this section analytical bounds will be presented that capture this deviation such that the uncertainty weighting, $w_{II}(s)$, as presented in the previous section can be derived analytically.

The steady state yaw-rate gain, $k_{\dot{\delta} \to r_z}$, was previously expressed in Equation (10), for the nominal plant $G_n(s)$, with $K_{\alpha 1} = K_{\alpha 1\text{nom}}$ and $K_{\alpha 2} = K_{\alpha 2\text{nom}}$, this becomes

$$G_n(j\omega)\big|_{s \to 0} = k_{\dot{\delta} \to r_z} = \frac{(a + b)K_{\alpha 1\text{nom}}K_{\alpha 2\text{nom}}v_x}{(a + b)^2K_{\alpha 1\text{nom}}K_{\alpha 2\text{nom}} + (bK_{\alpha 2\text{nom}} - aK_{\alpha 1\text{nom}})mv_x^2}.$$  

(22)
The uncertainty radius at \( \omega = 0 \) thus becomes

\[
li_{0} = li_{f}(\omega)_{\omega \to 0} = \max_{G_{p} \in \Pi} \frac{|k_{n,\delta \to rz} - k_{\delta \to rz}|}{k_{n,\delta \to rz}} = \frac{mv_{x}^{2}(aK^{+}K^{+} - K^{-}K^{-}) + bK^{-}K^{+}(K^{-} - K^{+})}{K^{+}K^{-}(mv_{x}^{2} - aK^{-}K^{-} + K^{-}K^{+})}. (23)
\]

The maximum radius at \( \omega = 0 \) for \( v_{x} \to \infty \) is defined as

\[
\lim_{v_{x} \to \infty} li_{f}(\omega)_{\omega \to 0} = a(K^{-}K^{-}K^{-}K^{+} - K^{-}K^{-}K^{+}K^{+}) + b(K^{-}K^{-}K^{+}K^{+} - K^{-}K^{-}K^{+}K^{+}) - aK^{-}K^{-}K^{+}K^{+} + bK^{-}K^{-}K^{+}K^{+}. (24)
\]

This is the upperbound of the uncertainty.

At higher frequencies the yaw-rate response reduces to

\[
G(j\omega)_{s \to \infty} = \frac{aK^{-}mv_{x}^{2}s}{I_{zz}mv_{x}^{2}s^{2}} = \frac{aK^{-}}{I_{zz}s}. (25)
\]

Substituting this into Equation (15) and rewriting results in

\[
l_{i|\infty} = li_{f}(\omega)_{\omega \to \infty} = -1 + \frac{K^{-}K^{-}}{K^{-}K^{-}}, (26)
\]

which is independent of forward velocity, \( v_{x} \), as Figure 5 already showed. Given the stability requirement of Equation (20), this poses a limit on the allowable ratio between the average and minimum local gradient of the lateral force, since \( S(\omega = \infty) = 1 \) per definition, that is,

\[
\frac{K^{-}K^{-}}{K^{-}K^{-}} < 2. (27)
\]

Obviously, this constraint has been accounted for in the parameter selection as was shown in Table 1.

With these uncertainty bounds defined analytically, an expression for the weight \( wi(s) \) can be found. Recall Figure 5, the shape of these lines can be approximated by a second-order filter of the form

\[
w_{f}(s) = li_{0} \frac{1}{(2\pi f_{w1})^{2}}s^{2} + \frac{\beta_{w1}}{\pi f_{w1}}s + 1 \frac{1}{(2\pi f_{w2})^{2}}s^{2} + \frac{\beta_{w2}}{\pi f_{w2}}s + 1 (28)
\]

with \( f_{w1} \) and \( f_{w2} \) crossover frequencies and \( \beta_{w1}, \beta_{w2} \) damping terms. For this second-order filter it holds that

\[
\left(\frac{f_{w2}}{f_{w1}}\right)^{2} = \frac{l_{i|\infty}}{li_{0}}. (29)
\]
Using this, together with Equations (23) and (26), the magnitude of $f_{w1}$ and $f_{w2}$ can be derived as

$$f_{w1} = f_w + f_w \Delta = f_w + \frac{1 - \sqrt{\frac{h_{\infty}}{l_{\infty}}}}{1 + \sqrt{\frac{h_{\infty}}{l_{\infty}}}} f_w, \quad (30)$$

$$f_{w2} = f_w - f_w \Delta = f_w - \frac{1 - \sqrt{\frac{h_{\infty}}{l_{\infty}}}}{1 + \sqrt{\frac{h_{\infty}}{l_{\infty}}}} f_w. \quad (31)$$

The parameters $f_w$, $\beta_{w1}$ and $\beta_{w2}$ are chosen such that up to $v_x = 40 \text{ m/s}$ the weighting filter, $w_{ii}(s)$, is an upper bound for the disturbance radius $l_{ii}(\omega)$, a summary of the parameters is given in Table 2.

### 4.3. Performance controller

One of the control objectives is to make the nominal closed loop behaviour independent of the forward velocity. Under the assumption that the nominal vehicle model is known, a controller can be designed that cancels these nominal dynamics. Recall that the steady state gain, eigenfrequency, dimensionless damping were provided in Equations (10)–(12), respectively. With the zero defined as

$$s = -\frac{(a + b)K_{w2}}{amv_x} \quad (32)$$

the controller is defined as

$$C_1(s) = \frac{1}{k_n} \frac{amv_x}{(a+b)K_{w2}\text{nom}} s + 1 \frac{1}{(2\pi f_c)^2 s^2 + \frac{\beta_c}{\pi f_c} s + 1} \frac{1}{(2\pi f_c)^2 s^2 + \frac{\beta_c}{\pi f_c} s + 1}. \quad (33)$$

Note the addition of the second-order dynamics. These are included to achieve high frequency roll-off. The open loop transfer function from $r_{z\text{ref}} - r_z$ to $r_z$ becomes

$$L_1(s) = C_1(s)G_n(s) = \frac{1}{(2\pi f_c)^2 s^2 + \frac{\beta_c}{\pi f_c} s + 1}, \quad (34)$$

where the eigen frequency $f_c$ and dimensionless damping $\beta_c$ can be chosen freely to achieve the desired open loop transfer function and closed loop sensitivity. The sensitivity function of the nominal controlled plant $S_n(s)$ is shown in Figure 6. Here, $f_c$ and $\beta_c$ have been chosen as 10 Hz and 2, respectively, such that the peak in the sensitivity is smaller than $1/w_{ii}(s)$.
Figure 6. Sensitivity of the nominal plant with the $v_x$ cancelling controller, $S_n(s) = 1/(1 + L_1(s))$ and the uncertainty bounds, $w_U(s)$ as a function of forward velocity. Here, $f_c = 10$ Hz and $\beta_c = 2$.

Figure 6 shows that for high frequencies the sensitivity is smaller than the inverse of the uncertainty bound. However, for low frequencies, it surpasses these bounds. This shows the need for a controller that stabilises the plant for all uncertainties and thus suffices Equation (20). It is easy to seen that introducing an integrator in the control structure decreases the sensitivity. Besides that, for stability, the bandwidth should be chosen well beyond the frequency of the right half plane pole, which is at approximately 0.7 Hz at $v_x = 40$ m/s. The objective is therefore to place bandwidth at approximately 12 Hz. The proposed controller thus becomes

$$C_2(s) = k_p \frac{s + \frac{5}{2\pi f_L}}{s + \frac{1}{2\pi f_L}} \frac{s + \frac{1}{2\pi f_L}}{s + \frac{1}{2\pi f_L}}.$$  \hspace{1cm} (35)$$

This controller includes a proportional action, $k_p$, that adjusts the gain of the controller such that a bandwidth of 12 Hz is achieved. The integral action achieves good low frequent tracking and ensures that the sensitivity function goes to $-\infty$ for $\omega \to 0$. Since a pure integrator creates a 90° phase delay, its action is limited to 5 Hz, thereby preventing too much phase delay at the bandwidth. Finally, a lead filter is included to increase the phase margin and thus the stability. A wide frequency band lead filter with its centre frequency at 60 Hz is chosen such that enough phase lead is created over a wide frequency range. With the parameters summarised in Table 3 nominal stability is guaranteed. The peak value of
Table 3. Controller parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_c$</td>
<td>0.9</td>
<td>Hz</td>
</tr>
<tr>
<td>$\beta_c$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$k_p$</td>
<td>3.1487</td>
<td></td>
</tr>
<tr>
<td>$f_l$</td>
<td>5</td>
<td>Hz</td>
</tr>
<tr>
<td>$f_t$</td>
<td>60</td>
<td>Hz</td>
</tr>
</tbody>
</table>

The sensitivity

$$S_2(s) = \frac{1}{1 + C_1(s)C_2(s)G_n(s)}$$

is 1.0466, and the value of $1/w_{II}(s)$ at that frequency is 1.0833, thereby guaranteeing robust stability. The resulting sensitivity function is shown in Figure 7 which shows that the sensitivity is smaller than the inverse of the disturbance weight $w_{II}(s)$ for all velocities considered here. Besides this, the low values of the sensitivity results in good reference tracking, since obviously

$$S_2(s) = \frac{r_{zref} - r_z}{r_{zref}}.$$  (37)

Figure 7. Sensitivity of the controlled plant $S_2(s) = 1/(1 + C_1(s)C_2(s)G_n(s))$ with the uncertainty bounds, $w_{II}(s)$ shown for different forward velocities.
Figure 8. Sensitivity of the controlled plant $S_2(s) = 1/(1 + L(s))$ including all bounds of the disturbance.

4.4. Robust performance

Besides nominal performance and robust stability, robust performance is of importance for the closed loop control system. This is indicated by how well the perturbed system tracks the reference behaviour, $r_{\text{ref}}$. This again translates into the sensitivity function which is shown in Figure 8. This figure shows that, although the local gradient of the lateral stiffness does have influence in the tracking behaviour, at low frequencies there is still suitable tracking performance. It is also visible that for the oversteered vehicle with $K_{\alpha 1}^-$ and $K_{\alpha 2}^-$ the peak in the sensitivity is the largest.

5. Simulation results

In this section a multi-body model of a BMW 318i will be used to show the performance of the controller designed in the previous sections. The multi-body model includes nonlinear tyre behaviour, which implies that the local gradient of the lateral force decreases when a tyre experiences large side slip angles. As a reference the single-track model, (1)–(3), is used. The parameters of this reference model are chosen such that a neutral steered vehicle
is created, see Table 4 for a list of parameters. The parameters find their base in the test-vehicle parameters, however they have been altered to acquire a neutral steered vehicle. Note that this is a choice of the authors, any physically feasible model can be used here. The steering ratio, however, has been changed to achieve the same steady-state response at a lateral acceleration of approximately 8 m/s² as the conventional vehicle.

### 5.1. Multi-body model

A multi-body model offers the possibility to model a vehicle in great detail. Here, the software package MATLAB/Simulink/SimMechanics is used to model the vehicle. The chassis is modelled as a single rigid mass to which four suspensions are connected. Each suspension is modelled in detail with all physical links as they are present in the real car. The left and right suspension systems are connected by means of an anti-roll bar. Furthermore, the steering system dynamics are taken into account. Finally, the tyres are modelled with the TNO Delft-Tyre package which represents the forces and moments created by the tyre by means of the Magic Formula. For more details on this model the authors refer to [23].

Many tests have been done to validate this multi-body model. As an example a double-lane change is shown. The objective of this test is to complete a pre-defined trajectory with the highest possible entry velocity.[24] For the simulation model the forward velocity and steering input as measured during the experiments is used as input. The results of the measurement and simulation can be seen in Figure 9. Obviously, the steering angle in simulation and measurement is the same, as this is used as input. The longitudinal velocity of the model is matched to the model by the cruise control. Besides these two inputs, it can also be seen that the yaw-rate, lateral acceleration and roll-rate of the simulation model match the measurements quite well. Based on this, and various other simulations presented in [23], the conclusion can be drawn that the multi-body model represents the behaviour of the BMW for high lateral acceleration manoeuvres, where the tyres are in their nonlinear regime, accurately. It is therefore assumed that this multi-body model is a good representation of a real vehicle.

### 5.2. Sine with dwell

A double lane change, such as in the model verification, is a test often used to asses vehicle behaviour in limit handling situations.[24] However, it is a closed loop test which includes the driver in the loop. To avoid having to model the driver and its variability, an open loop sine with dwell test [25] is simulated. A constant forward velocity of $v_x = 30$ m/s is chosen

### Table 4. Reference single-track model vehicle parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>1734</td>
<td>kg</td>
</tr>
<tr>
<td>$a$</td>
<td>1.3625</td>
<td>m</td>
</tr>
<tr>
<td>$b$</td>
<td>1.3625</td>
<td>m</td>
</tr>
<tr>
<td>$I_{zz}$</td>
<td>1900</td>
<td>kgm²</td>
</tr>
<tr>
<td>$C_{\alpha 1}$</td>
<td>130,000</td>
<td>N/rad</td>
</tr>
<tr>
<td>$C_{\alpha 2}$</td>
<td>130,000</td>
<td>N/rad</td>
</tr>
<tr>
<td>$i_y$</td>
<td>30</td>
<td>rad/rad</td>
</tr>
</tbody>
</table>
and the steering wheel angle is first increased sinusoidally to $85^\circ$ and after $1.07$ s decreased to $-85^\circ$. It is then held at that steering wheel angle for $0.5$ s. Finally the steering wheel angle is set to $0^\circ$ again.

**5.3. LQR control**

As a comparative study, an LQR controller has been developed.[15] The system under consideration is described by Equatons (2)–(4). It is assumed that both states can be measured.
The $Q$ matrix, scaling the states in the LQR cost function, is chosen as

\[
Q = \begin{bmatrix}
0.001 & 0 \\
0 & 1
\end{bmatrix},
\]

such that the yaw-rate is favoured. Output weighting is chosen as $R = 0.001$, such that the focus lies on the yaw-rate control, without penalising the steering action. The controller that is synthesised places the poles at $-4.67$ and $-131.6$ for the vehicle side slip angle and yaw-rate, respectively.

The LQR controller behaviour is analysed with the multi-body model, the results are shown in Figure 10. First the uncontrolled baseline vehicle is shown, which clearly spins out. From this figure it furthermore becomes apparent that the LQR controller is not capable of stabilising the vehicle behaviour, showing large yaw oscillations. This is caused by the, linear, vehicle parameters for which the controller is designed, which deviate significantly

![Figure 10](image-url). Sine with dwell manoeuvre with yaw-rate, steering angle and lateral acceleration for the vehicle with and without control with a forward velocity of $v_x = 30$ m/s with LQR control.
from the multi-body model parameters. The high lateral accelerations cause the tyres to behave non-linearly, thus altering the vehicle parameters.

### 5.4. $H_{\infty}$ bounded controller

The results of the sine with dwell manoeuvre for the controller proposed in this paper are shown in Figure 11. This figure shows that the uncontrolled baseline vehicle follows the initial steering motion nicely. After steering to $-85^\circ$ the yaw-rate stays almost constant, indicating that the vehicle has become unstable and thus spins. This can also be seen in the lateral acceleration which remains at a constant level even after the steering angle has been decreased. The controlled vehicle tracks the reference yaw-rate well and remains stable all through the manoeuvre. As shown in Figure 11 the controller achieves this by reducing the steering angle, thereby effectively damping the yaw overshoot. In the final return to zero steering wheel angle, the controller applies a countersteering action to stabilise the yaw dynamics.

![Figure 11. Sine with dwell manoeuvre with yaw-rate, steering angle and lateral acceleration for the vehicle with and without control with a forward velocity of $v_x = 30$ m/s.](image-url)
5.5. Split-$\mu$ braking

A situation that catches normal drivers off guard is $\mu$-split braking. This occurs for example in icy conditions, when the centre of the road is free of ice and the shoulder hasn't yet thawed. When the brakes are applied, one side of the vehicle develops a larger longitudinal force than the other, resulting in a yaw moment around the centre of gravity. As a result the vehicle will move towards the grippier side of the road or spin. The $\mu$-split braking manoeuvre has been performed with the simulation model. For this case the friction coefficient of the tyres on one side of the vehicle has been halved. At $t = 0\text{s}$, with $v_x = 30 \text{ m/s}$ a braking action is the performed with $5 \text{ m/s}^2$ until $10 \text{ m/s}$ is reached. The steering wheel is assumed to fixed.

The result of this $\mu$-split braking manoeuvre can be seen in Figure 12. This figure shows that without control the vehicle will develop a yaw-rate and exhibit unstable behaviour. As a result of this, the front wheels will develop a small steering angle due to the finite stiffness in the steering system. The controlled system manages to keep the yaw-rate close to zero,

![Figure 12. Split-$\mu$ braking with forward velocity, yaw-rate, steering angle and lateral acceleration for the vehicle with and without control.](image-url)
thereby maintaining straight line braking. It does this by applying a steering wheel angle of approximately 26°.

6. Conclusions

In this paper analytical bounds are presented that aid in the design of controllers for steer-by-wire vehicles including nonlinear tyre behaviour. Using $H_\infty$ analysis with an inverse multiplicative uncertainty it is shown that for high frequencies the bound on robust stability is only defined by the ratio between the nominal gradient of the lateral force of the front tyre and the minimum gradient of the lateral force. At low frequencies this bound on robust stability is a function of forward velocity, however, by using an integrator in the controller, this bound is met automatically.

Furthermore, forward velocity decoupling is provided by means of an inverse model approach. This approach accounts for the nominal vehicle dynamics, thereby making the nominal closed loop behaviour independent of forward velocity.

Performance of the controller is verified using a multi-body model of the proposed test vehicle. A sine with dwell manoeuvre at high lateral acceleration is performed which destabilises the uncontrolled vehicle and LQR controller vehicle. The vehicle with the controller developed in this paper tracks a reference yaw-rate accurately and remains stable. Besides the lane change test, performance of the controller is shown in $\mu$-split braking, showing that the controller is capable of maintaining zero yaw-rate nearly the entire manoeuvre.

Disclosure statement

No potential conflict of interest was reported by the authors.

References