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An elasto-viscoplastic constitutive model for polymers at finite strains: Formulation and computational aspects

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1. Introduction

The use of polymeric materials is steadily increasing due to their ability to fulfil requirements for a large number of applications ranging from automotive, medical and electronic sectors. Their mechanical behaviour is usually highly nonlinear and it is extremely important to understand how their mechanical performance is affected by the molecular structure, the processing conditions and the geometry of the micro constituents. Over the last decades, a considerable effort has been made by the academic community to develop constitutive models that are able to describe the deformation behaviour of polymeric based materials.

Pioneering work to determine the behaviour of polymers dates back to 1930s. Eyring [1] proposed a molecular theory for the yield stress of amorphous polymers, considering the yield behaviour as a thermally activated process. Temperature and strain rate effects are accounted for in the theory. In 1940, Mooney [2] proposed a strain energy function for rubber elastic materials. Later, Haward and Thackray [3] developed a one dimensional constitutive model for glassy polymers. This work can be considered as one of the first constitutive models proposed for predicting the deformation behaviour of glassy polymers. According to it, the post yield behaviour of glassy polymers includes two different phases: firstly, a rate dependent plastic flow modelled by an Eyring dashpot, and secondly, a rate independent contribution of the entanglement modelled by a Langevin spring. The three dimensional version of Haward and Tackray model was proposed by Boyce et al. [4]. An alternative constitutive approach is based on the generalized compressible Leonov model. The first compressible version of the Leonov model [16] was proposed by Baaijens [5] to predict flow-induced residual stresses in injection moulded products. Later, the model was derived within a thermodynamically consistent framework by Tervoort et al. [6]. The rate of plastic strain was constitutively described by the stress-activated Eyring flow. This model was later extended by Timmermans [14] and Govaert et al. [7] to capture the typical characteristics of the post-yield behaviour of glassy polymers, namely the phenomenon of intrinsic strain softening and strain hardening. The generalized compressible Leonov model is currently known as the Eindhoven Glassy Polymer (EGP) model. Over the last decades, a wide range of constitutive models incorporating linear and non-linear visco-elastic and visco-plastic material behaviour have been developed to capture different aspects of the behaviour of polymers including molecular orientation, strain rate effects, failure, among others. For a review on finite element simulation of polymers, the reader is referred to [8,9].

The finite strain elasto-viscoplastic constitutive model developed is this work is inspired in the single mode EGP model [10], which has been extensively used by a large number of authors (e.g. see [7,11–15]) in the prediction of the deformation behaviour of polymeric materials. Therefore, the inelastic material behaviour at the constitutive level is modelled with the same rheological framework by Tervoort et al. [6]. The rate of plastic strain was constitutively described by the stress-activated Eyring flow. This model was later extended by Timmermans [14] and Govaert et al. [7] to capture the typical characteristics of the post-yield behaviour of glassy polymers, namely the phenomenon of intrinsic strain softening and strain hardening. The generalized compressible Leonov model is currently known as the Eindhoven Glassy Polymer (EGP) model. Over the last decades, a wide range of constitutive models incorporating linear and non-linear visco-elastic and visco-plastic material behaviour have been developed to capture different aspects of the behaviour of polymers including molecular orientation, strain rate effects, failure, among others. For a review on finite element simulation of polymers, the reader is referred to [8,9].
of the deformation gradient into elastic and inelastic contribution is assumed and logarithmic stretches are employed as strain measures.

Procedures for integration of the constitutive equations of a material model, usually defined by a set of evolution equations, in the context of finite element simulations have been thoroughly investigated by a large number of authors [17–22]. This numerical integration is carried out locally at each quadrature point in typical finite element implementations. This process has a strain driven structure where the stresses and updated internal variables, which characterize the inelastic response of the material, are pursued for a given strain increment and the previous values of the internal variables. The consistent linearization of the resulting discrete equations is crucial for the successful solution of the global boundary value problem with the Newton–Raphson scheme due to its asymptotic quadratic rate of convergence. The use of operator split techniques, which result in the classical elastic predictor/plastic corrector format of the time-discrete evolution problem, is widely accepted and has become standard nowadays.

The numerical integration algorithm, developed in this work, follows the procedure introduced by Eterovic and Bathe [23] where large strain kinematics are separated from the stress integration. The small strain numerical integration by means of a return mapping-type scheme requires, in the three-dimensional stress state, the solution of seven coupled non-linear equations. This large number of equations on the stress update algorithm, which is solved at each Gauss point of the finite element mesh, makes the simulation with this model rather expensive. Therefore, in this contribution, we show that a constitutive integration algorithm can be derived where the return mapping procedure, under any stress state, is reduced to the solution of only one single non-linear equation. Consequently, the computational burden of the model is significantly reduced.

The presentation is structured as follows. In Section 2, the hyperelastic based elasto-viscoplastic model is formulated. Section 3 describes in detail the algorithm for numerical integration of the model. For convenience, the closed form of the associated consistent tangent operator is presented. The performance of the model and the efficiency of the integration algorithm are assessed through a representative set of numerical examples in Section 4. Based on the obtained results, we present some concluding remarks in Section 5. Two appendices are included to describe some algebraic operations that were removed from the body text for the readers’ convenience. This should be useful for those interested in the implementation of the algorithm within an implicit finite element environment.

2. Hyperelastic-based multiplicative model

In this section, the hyperelastic-based finite strain inelastic constitutive framework adopted in this work, formulated in the spatial configuration, is presented. The main aspects of this approach, which will be conveniently particularized for the developed model, are: the multiplicative split of the deformation gradient, the use of a logarithmic strain measure, the existence of a free energy potential from which the hyperelastic law is derived, the existence of a dissipation potential from which the plastic flow rule is obtained and the additive decomposition of the total stress into driving and hardening stresses.

The mechanical model, which is schematically represented in Fig. 1, consists of two elements connected in parallel. The first element is composed by a linear spring that characterizes the elastic behaviour and a dashpot that characterizes the rate dependent yield behaviour and the non-linear viscoplastic material response. The second element is a spring that represents the strain hardening response.

Fig. 1. Rheological representation of the model in one dimension.

2.1. Multiplicative kinematics

The main assumption underlying this approach is the multiplicative decomposition of the deformation gradient. Based on this assumption, the deformation gradient, \( F \), is multiplicatively composed of the elastic deformation gradient, \( F^e \), and the plastic deformation gradient, \( F^p \) [17].

\[
F = F^e F^p. \tag{1}
\]

The elastic deformation gradient, \( F^e \), is acting on the elastic spring and the plastic deformation gradient, \( F^p \), is acting on the Eyring dashpot (viscoplastic element). The spring element, parallel to the elastic spring and Eyring dashpot, is affected by the total deformation gradient, \( F \). This decomposition suggests a local unstressed intermediate configuration obtained by elastic unloading from the final deformed configuration. Polar decomposition of the elastic and plastic deformation gradient results in:

\[
F^e = R^e U^e = V^e R^e, \tag{2}
\]

\[
F^p = R^p U^p = V^p R^p, \tag{3}
\]

where \( R^e, U^e \) and \( V^e \) are the elastic rotation tensor, elastic right stretch tensor and elastic left stretch tensor, respectively. The terms in relation (3) are the corresponding plastic terms. The velocity gradient is given by:

\[
\dot{L} = \dot{F} F^{-1}. \tag{4}
\]

Substitution of Eq. (1) in Eq. (4) results in:

\[
L = L^e + L^p F^e (F^e)^{-1}, \tag{5}
\]

where the elastic velocity gradient, \( L^e \), and plastic velocity gradient, \( L^p \), are defined by:

\[
L^e = F^e (F^e)^{-1}, \tag{6}
\]

\[
L^p = F^p (F^e)^{-1}. \tag{7}
\]

Based on the plastic velocity gradient, the rate of plastic deformation (plastic stretching tensor), \( D^p \), and plastic spin tensor, \( W^p \), are defined by:

\[
D^p = \text{sym}(\dot{L}^p) = \frac{1}{2} [ \dot{L}^p + (\dot{L}^p)^T ], \tag{8}
\]

\[
W^p = \text{skew}([\dot{L}^p]) = \frac{1}{2} [ \dot{L}^p - (\dot{L}^p)^T ], \tag{9}
\]

where the superscript \( T \) denotes the transpose of the tensor. The plastic spin tensor in this work is assumed to be null, \( W^p = 0 \), which is compatible with the hypothesis of plastic isotropy employed in...
the formulation. It is worth mentioning that there are different choices regarding the plastic spin tensor and this issue is addressed in detail in [14,24]. According to these references, choosing different plastic spin tensors does not remarkably affect the overall stress–strain curves. Nevertheless, if plastic anisotropy should be considered, an appropriate constitutive definition for the plastic spin tensor should be required.

2.2. Logarithmic strain measure

In the finite strain model, presented in this contribution, the logarithmic (or natural) Eulerian (or spatial) strain is adopted to measure elastic deformations. The use of logarithmic strain is motivated not only by its physical meaning, but also by the fact that using this strain measure provides remarkably simplified stress integration algorithm of the finite strain model in a way that the infinitesimal elastic predictor/return mapping algorithms can be naturally extended to finite strains. This simplification will be detailed in the following section. The Eulerian logarithmic elastic strain tensor is defined by:

\[ \varepsilon^e = \ln \mathbf{F} = \frac{1}{2} \ln \mathbf{B}; \]  

(10)

where \( \ln(\cdot) \) denotes the tensorial logarithm of (\( \cdot \)) and, \( \mathbf{B} \), is the left elastic Cauchy–Green deformation tensor given by:

\[ \mathbf{B}^e \equiv \mathbf{F}^T \mathbf{F} = \left[ \mathbf{V}^e \right]^2. \]  

(11)

2.3. Free energy potential

Following the formalism of thermodynamics with internal variables, in order to know the local state of the material, we should have knowledge of the state variables of the material. The state variables are categorized into two groups: observable variables, which are the total strain and temperature, and internal variables describing the current state of the material. It should be emphasized that in this study, constant temperature is considered and as a result, observable variables are reduced to the strain. Assuming the existence of a free energy potential, \( \psi \), with the following form:

\[ \psi = \psi(\varepsilon^e, \mathbf{A}), \]  

(12)

where \( \varepsilon^e \) is elastic logarithmic strain and \( \mathbf{A} \) is a set of internal variables, the following constitutive law is obtained:

\[ \mathbf{F} = \mathbf{F}(\psi(\varepsilon^e, \mathbf{A})), \]  

(13)

where \( \mathbf{F} \) is the Kirchhoff stress tensor and \( \mathbf{F} \) is the reference density. We know that the following relation holds:

\[ \tau = \mathbf{F} \sigma. \]  

(14)

where \( \sigma \) is the Cauchy stress tensor and \( \mathbf{F} \) is the determinant of the deformation gradient, \( \mathbf{F} = \text{det} \left( \mathbf{F} \right) \). Different free energy potentials, available in the literature and used for different kind of materials, can be used to derive the constitutive law for the stress as a function of strain. In this work, the so-called Hencky strain energy function (logarithmic strain-based hyperelasticity law), which is generally accepted for a wide range of applications, will be used. The Hencky strain energy function, in terms of principal stretches, is given by:

\[ \overline{\psi}(\lambda_1, \lambda_2, \lambda_3) = G \left[ \ln \lambda_1^2 + \ln \lambda_2^2 + \ln \lambda_3^2 \right] \]

\[ + \frac{1}{2} \left( K - \frac{2}{3} G \right) \left[ \ln \left( \lambda_1^2 \lambda_2^2 \lambda_3^2 \right) \right]^2, \]  

(15)

where \( \lambda_1, \lambda_2, \lambda_3 \) are the principal stretches in principal directions; \( K \) is the bulk modulus and \( G \) is the shear modulus of the material. Using the strain energy function introduced in relation (15) and also Eq. (13) results in the following relation between Kirchhoff stress and Eulerian logarithmic strain:

\[ \tau = D^e : \varepsilon^e, \]  

(16)

where \( D^e \) denotes the fourth order isotropic constant elastic tensor:

\[ D^e \equiv 2 G I_3 + \left( K - \frac{2}{3} G \right) (\mathbf{I} \otimes \mathbf{I}). \]  

(17)

The symbol \( I_3 \) represents the fourth order symmetric identity tensor and \( \mathbf{I} \) is the second order identity tensor. The tensors \( I_3 \) and \( \mathbf{I} \), in component form, can be expressed as:

\[ I_{ij} = \delta_{ij}, \]  

(18)

\[ I_k = \delta_k, \]  

(19)

where \( \delta \) is the Kronecker delta.

2.4. Dissipation potential

The flow rule of the model is characterized by the Eyring equation. The Eyring flow model in one dimension is described by [10]:

\[ \dot{\eta} = \frac{1}{A} \sinh \left( \frac{\tau}{\tau_0} \right), \]  

(20)

where \( \tau \) is the shear stress; \( \dot{\eta} \) is the plastic rate of shear; \( A \) and \( \tau_0 \) (a characteristic stress) are material constants at constant temperature:

\[ A = A_0 \exp \left( \frac{\Delta H}{RT} \right), \]  

(21)

\[ \tau_0 = \frac{RT}{V}. \]  

(22)

where \( \Delta H \) and \( V \) are the activation energy and the shear activation volume, respectively. The scalar \( A_0 \) is a constant or pre-exponential factor involving the fundamental vibration energy; \( R \) is the universal gas constant and \( T \) is the absolute temperature. The plastic rate of shear was explicitly given as a function of shear stress and Eyring properties. Performing an inversion in Eq. (20) results in:

\[ \dot{\eta} = \frac{\tau}{\tau_0 \sinh \left( \frac{\tau}{\tau_0} \right)}. \]  

(23)

From Eq. (23), a viscosity function can be defined:

\[ \eta(\dot{\eta}) = \tau_0 \sinh \left( \frac{\dot{\eta}}{\tau_0} \right). \]  

(24)

Using relation (24), the one dimensional Eyring flow equation, Eq. (23), can be rewritten as follows:

\[ \dot{\eta} = \frac{\tau}{\eta(\dot{\eta})}. \]  

(25)

The flow model in three dimensions can be described by the following relation for the equivalent rate of strain:

\[ \dot{\varepsilon} = \frac{1}{2} \text{tr} \left( \mathbf{s} \right) \]  

(26)

where \( \dot{\varepsilon} \) is an equivalent stress defined by:

\[ \tau_0 = \sqrt{\frac{1}{2} \text{tr} \left( \mathbf{s} \right) \mathbf{s} \cdot \mathbf{s}}. \]  

(27)

where \( \mathbf{s} \) is the deviatoric part of the stress tensor:

\[ \mathbf{s} \equiv \mathbf{I} - \mathbf{I} : \mathbf{\varepsilon}. \]  

(28)
The fourth order deviatoric identity tensor, $D_s$, is given by:

$$D_s = I - \frac{1}{3}(I \otimes I).$$  \hspace{1cm} (29)

It is important to mention that relations (27) and (26) for the equivalent stress, $\tau^{eq}$, and equivalent rate of strain, $\dot{\tau}^{eq}$, are defined such that in a pure shear stress state, they reduce to the one dimensional shear stress, $\tau$, and the plastic rate of shear, $\dot{\tau}$, respectively. Although the magnitude of the plastic strain for one-dimensional (1-D) and three-dimensional (3-D) cases is obtained from relations (20) and (26), in order to fully characterize the flow behaviour of the material in a generic 3-D case, the flow vector, $N$, should be defined. As mentioned in the beginning of this section, the existence of a dissipation potential, $\Psi$, from which flow vector would be determined is postulated. The dissipation potential is defined as a convex scalar value function, non-negative and zero valued at the origin with the following form:

$$\Psi = \frac{1}{2}\mathbf{N} : \mathbf{s} \cdot \mathbf{s}.$$  \hspace{1cm} (30)

As a result, the flow vector is obtained:

$$\mathbf{N} = \frac{\partial \Psi}{\partial \mathbf{t}} = -\frac{1}{2}\mathbf{N} : \mathbf{N},$$  \hspace{1cm} (31)

where $|\mathbf{s}|$ is the norm of $\mathbf{s}$ defined by:

$$|\mathbf{s}| = \sqrt{s_1^2 + s_2^2 + s_3^2}.$$  \hspace{1cm} (32)

The plastic flow rule for the model presented here, is given by:

$$\dot{\mathbf{d}}^p = \dot{\tau}^{eq} \frac{\partial \Psi}{\partial \mathbf{t}} = \dot{\tau}^{eq} \mathbf{N},$$  \hspace{1cm} (33)

where $\Psi$ is the dissipation potential, $\mathbf{N}$ is the flow vector and $\dot{\mathbf{d}}^p$ is the spatial plastic stretching tensor:

$$\dot{\mathbf{d}}^p = \mathbf{R}^T \mathbf{D}^p \mathbf{R}^T.$$  \hspace{1cm} (34)

The spatial plastic stretching tensor, $\mathbf{d}^p$, is the plastic stretching tensor, $\dot{\mathbf{d}}^p$, rotated to the current (spatial) configuration by the elastic rotation, $\mathbf{R}$. Combining relations (33) and (31), the multi-dimensional plastic flow rule of the model is obtained as:

$$\dot{\mathbf{d}}^p = \dot{\tau}^{eq} \frac{1}{2}\mathbf{N} : \frac{\mathbf{s}}{|\mathbf{s}|}.$$  \hspace{1cm} (35)

Using relation (26) and also Eq. (27), the plastic flow rule can be expressed in another form:

$$\dot{\mathbf{d}}^p = \frac{1}{A} \sinh \left( \frac{\tau^{eq}}{\tau_0} \right) \frac{\mathbf{s}}{2\tau^{eq}}.$$  \hspace{1cm} (36)

or equivalently,

$$\dot{\mathbf{d}}^p = \frac{\mathbf{s}}{2A \left[ \sinh \left( \frac{\tau^{eq}}{\tau_0} \right) \right]}.$$  \hspace{1cm} (37)

which can be represented as:

$$\dot{\mathbf{d}}^p = \frac{\mathbf{s}}{2\eta(\tau^{eq})}.$$  \hspace{1cm} (38)

where the viscosity function, $\eta(\tau^{eq})$, is given by:

$$\eta(\tau^{eq}) = A \frac{\tau^{eq}}{\sinh \left( \frac{\tau^{eq}}{\tau_0} \right)}.$$  \hspace{1cm} (39)

Eq. (38) is the extension of the one dimensional non-Newtonian fluid relationship, Eq. (25), to the multi-dimensional case. Hence, it can be stated that, the plastic flow rule of this model is characterized by the generalized Eyring equation. The rate of equivalent plastic shear can be expressed in terms of the multi-dimensional plastic flow rule:

$$\dot{\gamma}^{eq} = \sqrt{2\mathbf{d}^p : \mathbf{d}^p}.$$  \hspace{1cm} (40)

Although the parameter $A$ was already introduced in Eq. (21), in order to account for the pressure and softening behaviours of the material, this parameter can be generalized to the following relation \cite{7,14}:

$$A = A_0 \exp \left[ \frac{\Delta H}{RT} + \frac{\mu P}{\tau_0} - D \right].$$  \hspace{1cm} (41)

The symbol $\mu$ represents a pressure coefficient related to the shear activation volume, $V'$, and the pressure activation volume, $\Omega$. According to:

$$\mu = \frac{\Omega}{V'},$$  \hspace{1cm} (42)

In relation (41), $P$ is the total hydrostatic pressure:

$$P = p_0 + p,$$  \hspace{1cm} (43)

where the scalar $p$ is the hydrostatic pressure defined by:

$$p = -\frac{1}{3} \mathbf{t} tr(\mathbf{t}).$$  \hspace{1cm} (44)

and the scalar $p_0$ is the superimposed hydrostatic pressure of the analysis. The scalar $D$ represents a softening parameter. The post-yield behaviour of the majority of thermoplastics and thermosetting polymers is governed initially by strain softening and then by strain hardening. After the yield point, the true stress decreases with increasing deformation and this phenomenon is called softening. The physical justification for the softening behaviour is not yet perfectly understood but it appears that it is related to the physical aging process \cite{7}. For glassy polymers, the strain softening behaviour promotes plastic localization \cite{7}. Hasan et al. \cite{25} proposed a phenomenological law for the evolution of the softening parameter, $D$:

$$D = h \left( 1 - \frac{D}{D_m} \right)^{\tau_0},$$  \hspace{1cm} (45)

where $D_m$ is the saturation value of the softening parameter with the initial condition $D = 0$, the scalar quantity $h$ influences the softening slope and $\tau_0$ is an equivalent plastic strain rate defined by \cite{7}:

$$\tau_0 = \sqrt{\mathbf{d}^p : \mathbf{d}^p}.$$  \hspace{1cm} (46)

Using relations (27), (39), (40), (41) and (45) together with some algebraic manipulations, provided in Appendix A, results in the following viscosity function:

$$\eta = A_0 \exp \left[ \frac{\Delta H}{RT} - \frac{\mu P}{\tau_0} - D - D_m \exp \left( -\frac{h \sqrt{3\tau_0}}{\sqrt{2D_m}} \right) \right] \frac{\tau^{eq}}{\sinh \left( \frac{\tau^{eq}}{\tau_0} \right)}.$$  \hspace{1cm} (47)

where $\tau^{eq}$ is the accumulated plastic strain. It should be emphasized that we do not use any explicit yield function, i.e. at any time of the deformation and at any point of the structure, the total strain is assumed to be additively composed of elastic strain (or reversible strain) and inelastic strain (or non-instantaneously reversible or non-reversible strain) as given by

$$\mathbf{e} = \mathbf{e}^0 + \mathbf{e}^p.$$  \hspace{1cm} (48)

In order to facilitate the notation, the later is referred to by superscript $p$ standing for “plastic”. In fact, this decomposition of strain is consistent with the physical phenomena happening during the deformation. Bending and stretching of strong chain covalent bonds and also small displacement of adjacent molecules,
which is resisted by the presence of van der Waals and hydrogen secondary bonds, contribute to the elastic deformation. At small deformations (less than 5%), viscoplastic strain should be considered in terms of a cooperative movement of molecular chain segments. Chain entanglements have remarkable contributions in resistance of polymers to visco-plastic flow [26].

2.5. Additive decomposition of total stress

The total stress, $\tau^\text{total}$, in the model, is additively composed, in a parallel assemblage, of driving stress and hardening stress:

$$\tau^\text{total} = \tau^\text{driving} + \tau^\text{hardening}. \quad (49)$$

In the final phase of the deformation behaviour, which typically occurs at large strains, softening reaches its saturation value and the true stress starts to increase with increasing strain. This phase is known as hardening. Two different kinds of hardening could be considered: isotropic hardening and kinematic hardening. Physically, isotropic hardening could be associated with increasing the covalent bonds, with the entanglements density and with the change of the conformation of molecular links inside the amorphous phase. Kinematic hardening could be physically related to the existence of structural defects in the amorphous phase such as the existence of free radicals or chain scission points [26]. In this study, only isotropic hardening is considered.

The hardening behaviour of glassy polymers is commonly modelled as a generalized rubber elastic spring with finite extensibility. The so-called three-chain and eight-chain models proposed by Arruda and Boyce [27] and also the full chain model proposed by Wu and van der Giessen [28] are typically used to model the hardening behaviour. Another approach for modelling the hardening behaviour was introduced by Havard [29] which consists in the application of network models, employing Gaussian chain statistics (which results in a neo-Hookean strain hardening response), to uniaxial experimental stress–strain curves. Nevertheless, for glassy polymers, which show a strong strain softening behaviour, the application of the Gaussian model to strain hardening is not straightforward and mechanical preconditioning is required [7].

The hardening stress in this model is characterized with the following relation:

$$\tau^\text{hardening} = H \varepsilon_d. \quad (50)$$

where $H$ is the hardening modulus (one of the material properties) and $\varepsilon_d$ is the deviatoric part of the total strain:

$$\varepsilon_d = I_2 : \varepsilon. \quad (51)$$

Govaert et al. [7] have defined the hardening stress using the deviatoric isochoric left Cauchy–Green deformation tensor, which results from the generalization of the neo-Hookean relation between stresses and strains to three dimensions. In this contribution, the total deviatoric strain, relation (51), is employed at small strains and extended to finite strains with the use of logarithmic strains.

3. The integration algorithm

Operator split algorithms are widely used for numerical integration of constitutive equations in the context of elastoplasticity and elasto-viscoplasticity [20,30]. Numerical implementation of constitutive models into finite element codes basically requires the appropriate derivation of the state update procedure for the specific model and the computation of the consistent tangent operator. Here, we shall focus on the particularization of the fully implicit elastic predictor/return mapping method to the finite strain model introduced in the previous section. It must be emphasized that the derivation of the state update procedure and consistent tangent operator will be performed at the small strain format and then will be extended to the finite strain counterparts. The finite strain extension employed here preserves the most important properties of the small strain formulation [17]. In particular, volume preserving plastic deformations, finite plastic incompressibility and associativity and maximum plastic dissipation at large strains [17]. Due to its suitable features, this kind of extension has been widely used by different authors [31–33].

3.1. The state update procedure

Let us consider a typical time interval $[t_n, t_{n+1}]$. Assuming that the incremental displacement, $\Delta \varepsilon$, is known, we can update the deformation gradient. The incremental deformation gradient, $F_n$, is obtained by [17]:

$$F_n = I + \nabla \varepsilon(\Delta u). \quad (52)$$

where $\nabla \varepsilon(\Delta u)$ is the gradient of the incremental displacement. With the incremental deformation gradient, we can update the deformation gradient at time step $t_{n+1}$:

$$F_{n+1} = F_n F_n^{-1}. \quad (53)$$

Now, it is the turn to evaluate the elastic trial state. Assuming that the total elastic strain, $\varepsilon^e_{n+1}$, at $t_n$ is known, the elastic left Cauchy–Green deformation tensor is given by:

$$B^e_{n+1} = \exp[2 \varepsilon^e_{n+1}]. \quad (54)$$

The elastic trial left Cauchy–Green deformation tensor at $t_{n+1}$ is obtained by:

$$B^e_{n+1} = F_n B^n_{n+1}(F_n)^T. \quad (55)$$

The elastic trial strain, which is in fact the driving parameter in the computational implementation of the model under study, at $t_{n+1}$ is computed by:

$$\varepsilon^e_{n+1} = \ln[B^e_{n+1}] = {1 \over 2} \ln[B^e_{n+1}]. \quad (56)$$

It must be noted that so far, everything is done at the kinematical level i.e. it is completely independent of the material model. Therefore, at this point, we should update stress and the state variables based on the constitutive relations of the model. The set of variables $\{\sigma_n, \varepsilon^e_{n+1}, \varepsilon^p_{n+1}, \tau^p_n\}$ is known at time $t_n$ and the main problem is to determine the same set $\{\sigma_{n+1}, \varepsilon^e_{n+1}, \varepsilon^p_{n+1}, \tau^p_{n+1}\}$ at time $t_{n+1}$.

3.1.1. Return mapping

We know that total strain is additively composed of elastic strain and plastic strain:

$$\varepsilon^p_{n+1} = \varepsilon^p_{n+1} + \varepsilon^p_{n+1}. \quad (57)$$

As mentioned before, we will establish the integration algorithm of the model within the context of infinitesimal theory and the finite strain framework adopted preserves the format of both the state update algorithm and consistent tangent operator. Hence, we shall adopt the small strain counterpart of the spatial plastic stretching tensor, relation (38), to work with:

$$\dot{\varepsilon}^{p} = {\textbf{s}} \over 2\eta, \quad (58)$$

which is the rate of plastic strain within the infinitesimal formulation. Integrating relation (58) during time step $[t_n, t_{n+1}]$ leads to the following:

$$\varepsilon^p_{n+1} = \varepsilon^p_n + \Delta t \over 2\eta \dot{\varepsilon}^{p}_{n+1}. \quad (59)$$

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The subscript \( d \) in relation (59) stands for the deviatoric part of the plastic strain and from relation (58), it is obvious that the plastic strain is totally deviatoric and there is no volumetric plastic strain. The rate of accumulated plastic strain is defined by:

\[
\dot{\varepsilon}_{p}^{d} = \frac{2}{3} \dot{\varepsilon}^d : \varepsilon^d. 
\]  
(60)

Integrating Eq. (60) over the time step \([t_n, t_n+1]\) and using relation (58) results in:

\[
\varepsilon_{p}^{d} = \frac{\sqrt{3} \Delta t}{\eta_n} \varepsilon_{p}^{d, n+1},
\]  
(61)

where the effective stress at \( t_{n+1} \) is given by:

\[
\varepsilon_{p}^{d, n+1} = \frac{1}{2} S_{n+1} : S_{n+1},
\]  
(62)

and we can express the total strain split, relation (57), on the deviatoric space as:

\[
\varepsilon_{d}^{n+1} = \varepsilon_{d}^{n+1} + \varepsilon_{p}^{d, n+1}.
\]  
(63)

The deviatoric strain at \( t_{n+1} \), \( \varepsilon_{d}^{n+1} \), can be obtained from the determination of the plastic strain, \( \varepsilon_{d}^{n+1} \), and accumulated plastic strain, \( \varepsilon_{p}^{d, n+1} \). This means that for a generic 3D problem, we would have to solve a system of seven coupled equations: six for plastic strain tensor and one for the accumulated plastic strain. In the following, it will be shown how to manipulate the relations such that the integration algorithm turns out to be significantly more efficient. The stress deviator, \( S_{n+1} \), is given by:

\[
S_{n+1} = 2 G (\varepsilon_{d}^{n+1} - \varepsilon_{p}^{d, n+1}).
\]  
(64)

Substitution of Eq. (59) into Eq. (66) gives:

\[
S_{n+1} = 2 G \left( \varepsilon_{d}^{n+1} - \varepsilon_{p}^{d, n+1} - \Delta t \frac{\varepsilon_{p}^{d, n+1}}{\eta_n} \right).
\]  
(65)

Rearranging relation (67) results in:

\[
\varepsilon_{p}^{d, n+1} = \frac{2 G}{1 + (\Delta t G/\eta_n)} (\varepsilon_{d}^{n+1} - \Delta t e_d).
\]  
(68)

The deviatoric strain at \( t_{n+1} \), is equal to the deviatoric strain at \( t_n \) plus the incremental deviatoric strain:

\[
\varepsilon_{d}^{n+1} = \varepsilon_{d}^{n} + (\Delta t) e_d.
\]  
(69)

Substituting relation (69) in relation (68) yields to:

\[
\varepsilon_{p}^{d, n+1} = \frac{2 G}{1 + (\Delta t G/\eta_n)} (\varepsilon_{d}^{n} + \Delta t e_d - \varepsilon_{p}^{d, n}).
\]  
(70)
\[ \frac{dR(n_{n+1})}{dn_{n+1}} = 1 - \left[ K_1(n_{n+1}) \left( \frac{C_2(n_{n+1})}{C_3(n_{n+1})} \right) \right] + C_1 \left( \frac{C_2(n_{n+1})K_2(n_{n+1}) - C_2(n_{n+1})K_3(n_{n+1})}{C_1(n_{n+1})} \right), \]  
\tag{83}

where

\[ K_1(n_{n+1}) = \frac{dC_1}{dn_{n+1}} = C_1 \exp \left( -\frac{b \sqrt{3} C_4}{\sqrt{2} D_{in}} \right) \frac{h \Delta t |\mathbf{\sigma}^{n+1}|}{2(n_{n+1} + \Delta G)^2}, \]  
\tag{84}

\[ K_2(n_{n+1}) = \frac{dC_2}{dn_{n+1}} = \sqrt{\frac{1}{2} \left( \frac{\Delta G |\mathbf{\sigma}^{n+1}|}{\tau_0(n_{n+1} + \Delta G)} \right)} \left( \cosh \frac{C_2}{\tau_0} \right), \]  
\tag{85}

\[ K_3(n_{n+1}) = \frac{dC_3}{dn_{n+1}} = \sqrt{\frac{1}{2} \left( \frac{\Delta G |\mathbf{\sigma}^{n+1}|}{\tau_0(n_{n+1} + \Delta G)} \right) \left( \cosh \frac{C_3}{\tau_0} \right)}. \]  
\tag{86}

Having computed the aforementioned factors, we can apply the Newton–Raphson method to solve the residual equation in an iterative fashion:

\[ \eta_{n+1}^{k+1} = \eta_{n+1}^{k} - R(n_{n+1}) \left[ \frac{dR}{dn_{n+1}} \right]^{-1}, \]  
\tag{87}

where the superscript \((k)\) and \((k-1)\) stand for two consecutive Newton–Raphson iterations. Once the iterations on the viscosity functions, \(R(n_{n+1})\), converge, we can update all the other variables through:

\[ s_{n+1}^{e} = \eta_{n+1}^{k+1} + \Delta G \mathbf{\sigma}^{n+1}, \]  
\tag{88}

\[ \mathbf{E}_{n+1}^{e} = \frac{\eta_{n+1}^{k+1} + \Delta G}{2 G} \mathbf{\sigma}^{n+1}, \]  
\tag{89}

\[ \mathbf{E}_{n+1}^{p} = \mathbf{E}_{n+1}^{e} + \frac{\Delta t}{2(n_{n+1} + \Delta G)} \mathbf{\sigma}^{n+1}, \]  
\tag{90}

\[ \mathbf{E}_{n+1}^{r} = \mathbf{E}_{n+1}^{p} + \frac{\Delta t}{3(n_{n+1} + \Delta G)} \sqrt{\frac{3}{2}} |\mathbf{\sigma}^{n+1}|, \]  
\tag{91}

The discretized form of the hardening stress, relation (50), is given by:

\[ \tau_{hardening} = H \mathbf{E}_{n+1}^{p}. \]  
\tag{92}

It is worth emphasizing that relation (73) is the key relation to condense the system of equations to only one scalar equation. Substitution of relation (73) in relations (75) and (77) results in expressions for the effective stress and accumulated plastic strain which are functions of viscosity. By inserting these functions of viscosity in relation (63), the system of equations reduces to one single equation.

### 3.2. Consistent tangent operator

In order to complete the numerical treatment of the model within an implicit quasi-static integration scheme, we need to obtain the consistent tangent operator. The global tangent stiffness matrix is assembled using the tangent operators, which are derived by consistently linearizing the integration scheme. The spatial tangent modulus is given by [17]:

\[ a_{ij} = \frac{1}{2} \mathbf{\theta} : \mathbf{L} : \mathbf{B}^{\text{ref}} - \mathbf{a}_{ij} \delta_{ij}, \]  
\tag{93}

where \(\delta\) is the Kronecker delta, \(\mathbf{D}\) is the small strain, either elastic or viscoplastic, consistent tangent operator. The fourth order tensor \(\mathbf{L}\) is defined by:

\[ \mathbf{L} = \frac{\partial \ln \mathbf{B}^{\text{ref}}}{\partial \mathbf{B}^{\text{ref}}}, \]  
\tag{94}

and the fourth order tensor \(\mathbf{B}\) is defined by the cartesian components:

\[ \mathbf{B}_{ijkl} = \delta_{ik} \left( \mathbf{B}_{j\cdot l}^{\text{ref}} \right)_{,j} + \delta_{il} \left( \mathbf{B}_{k\cdot j}^{\text{ref}} \right)_{,l}. \]  
\tag{95}

It should be noted here that the only material related term in relation (93) is \(\mathbf{D}\) and the otheer contributions of the spatial tangent module are independent of the material model, i.e. all the components taking part in relation (93), other than \(\mathbf{D}\), are purely related to the kinematical level. As a matter of fact, the perfect separation of the finite strain kinematics and the material related contributions to the integration algorithm, previously observed, is also possible through the assembly of the spatial tangent module, as shown in relations (93)–(95).

The exact linearization of the constitutive relations together with some algebraic manipulations, described in detail in

### Table 1

<table>
<thead>
<tr>
<th>Material properties for the Leonov model.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>E (MPa)</strong></td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>PET</td>
</tr>
<tr>
<td>PS</td>
</tr>
<tr>
<td>PC</td>
</tr>
<tr>
<td>PA-6</td>
</tr>
</tbody>
</table>

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Appendix B, result in the following closed form relation for the viscoplastic tangent operator.

\[
D = \left( \frac{2G_{n+1}}{\eta_{n+1} + \Delta G} + H \right) \left[ I - \frac{1}{3} (I \otimes I) \right] + K (I \otimes I)
+ \left( \frac{\Delta G}{\eta_{n+1} + \Delta G} \right) \left[ F_1 (1 - F_2 - F_3) \right] (s_{n+1} \otimes s_{n+1})
- \left[ \frac{\Delta G K C_1 \mu (1 + C_2)}{\eta_{n+1} (\eta_{n+1} + \Delta G) C_3 \tau_0 X} \right] (s_{n+1} \otimes I),
\]

(96)

where

\[
F_1 = \frac{\sqrt{2G_{n+1}}}{{C_1} \eta_{n+1} + \Delta G} |S_{n+1}^{ap}|, \quad (97)
\]

\[
F_2 = \frac{C_2}{\eta_{n+1}} \exp \left( -\frac{h \sqrt{3C_4}}{Q} \right), \quad (98)
\]

\[
F_3 = \frac{C_2}{\eta_{n+1}} \cosh \left( \frac{C_2}{\tau_0} \right), \quad (99)
\]

\[
X = 1 - K_1 \left( \frac{C_2}{C_3} \right) + C_1 \left( \frac{K_2 C_1 - C_2 K_3}{C_3} \right), \quad (100)
\]

4. Numerical examples

In this section, the ability of the model to capture the typical behaviour of polymeric based materials under different scenarios will be illustrated. In addition, the performance and accuracy of the numerical implementation will be, with some numerical examples, evaluated. The material properties, for the materials used in this section, are provided in Table 1. The materials used are PET copolyester 9921W, Polystyrene-N5000, Lexan 101\textsuperscript{R}, which is a commercial grade of Polycarbonate, and Nylon-6. These materials are referred to as PET, PS, PC and PA-6 in Table 1 and in what follows, respectively. It should be noted that we do not aim to com-

Table 2

<table>
<thead>
<tr>
<th>Iteration number</th>
<th>Relative residual norm (%) at increment 150</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PA-6</td>
</tr>
<tr>
<td>1</td>
<td>0.854515</td>
</tr>
<tr>
<td>2</td>
<td>0.291957</td>
</tr>
<tr>
<td>3</td>
<td>0.347145</td>
</tr>
</tbody>
</table>

Fig. 3. Comparison of cylinder upsetting simulations under strain rate of \( \zeta = -0.001 \,(1/s) \); solid line: present study, solid circles: Van Melick et al. [34].

Fig. 4. The number of required iterations versus increment number for the cylinder upsetting simulations. (a) PS, (b) PC, (c) PA-6 and (d) PET.

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pare directly the results obtained from numerical examples with experimental results available in the literature. This is due to the fact that we are using the properties obtained and calibrated by other researchers. Therefore, since the model presented in this contribution is not exactly what the references of this work employed, the determination and calibration of the material properties should be done independently in order to make a critical comparison to experimental results. The material properties for PET, PS, PC and PA-6 are taken from [15,34,14], respectively. Since, the deformation behaviour of polymers is dependent on the strain rate, Section 4.1 presents uniaxial compression tests under different strain rates. Section 4.2 describes a plane strain compression tests and compares the results with a compression test of the same material under different stress state. Due to the importance of temperature and its significant impact on the mechanical behaviour of polymers, Section 4.3 provides cylinder upsetting simulations on PS under three different temperatures and compares the evolution of the stress–strain curves. Hydrostatic pressure also affects the stress–strain behaviour of polymers and thus, some cylinder upsetting simulations under different superimposed hydrostatic pressures are given in Section 4.4. To demonstrate the efficiency and robustness of the derived and implemented algorithm, two more
2D problems, namely necking of a cylindrical bar and compression of a notched bar are given in Sections 4.5 and 4.6, respectively. Finally, Section 4.7 presents a 3D tensile simulation of a dumbbell shape specimen.

### 4.1. Cylinder upsetting-strain rate effect

In this example, uniaxial compression tests are performed on cylinders using all aforementioned materials and a comparison is made between stress–strain curves. In order to keep the strain rate constant during loading, the following relation is used for the displacement applied [35]:

\[
u(t) = l_0(\exp(\dot{\varepsilon}t) - 1),
\]

where \(u(t)\) is the total displacement, \(\dot{\varepsilon}\) is the strain rate, \(l_0\) is the initial length of the specimen and \(t\) is the total time. The simulations are performed under a strain rate of \(\dot{\varepsilon} = -0.001\) (1/s). The time interval between load steps is \(\Delta t = 5\) s. In addition, the simulations are done assuming room temperature, \(T = 20\) °C, and atmospheric superimposed hydrostatic pressure, \(p_0 = 0.1E+06\) Pa. The height of the specimen and also the diameter is considered to be equal to 6 mm. The test is approximated as a 2D axisymmetric problem.

Due to symmetry, a rectangular with 6 mm height and 3 mm width is spatially discretized with 72 eight noded quadrilateral elements. The total displacement is \(u = 3.8\) mm and it is applied in 200 increments. In Fig. 2, the stress–strain curves obtained for the four materials are provided. A comparison is conducted between the obtained results, in this study, and the numerical results presented by Van Melick et al. [34] for PS. Fig. 3 depicts the numerical stress–strain curves for PS under strain rate of \(\dot{\varepsilon} = -0.001\) (1/s). As depicted in Fig. 3, there is a reasonable agreement between the numerical simulations of this study and numerical results obtained by other authors. One of the most important features of the numerical implementation is the convergence rate of the algorithm. Consequently, it is necessary to investigate the convergence of the solution. In Table 2, the convergence of the problem at the global level in increment 150 is shown by providing the values of relative residual norm for each material. In order to explore the convergence rate of the algorithm, Fig. 4 shows the number of required iterations versus the increment number for the cylinder upsetting simulations. In order to observe the effect of the strain rate on the deformation behaviour of polymers, the cylinder upsetting problem is analysed under three different strain rates, \(\dot{\varepsilon} = -0.0001\) (1/s), \(\dot{\varepsilon} = -0.0005\) (1/s) and \(\dot{\varepsilon} = -0.001\) (1/s), on PS. Fig. 5 depicts stress–strain curves of PS cylinder upsetting under different strain rates. As expected, it can be seen that by increasing the strain rate, despite almost the same deformation pattern for all strain rates, the level of stress increases. It should be emphasized that the strain

![Fig. 10](image1.png)

**Fig. 10.** The contour plots of displacement field, accumulated plastic strain and effective stress of the cylinder under tensile deformation; (a) displacement field [mm]; (b) accumulated plastic strain; (c) effective stress [MPa].

![Fig. 11](image2.png)

**Fig. 11.** Contour plots of the accumulated plastic strain of necking simulation, (a) finer mesh and (b) coarser mesh.
rates are kept relatively low since increasing strain rate causes thermo-mechanical effects [35], which are not included in the model.

4.2. Compression of a cube

In this example, cube compression simulations, modelled as plane strain compression, are performed at room temperature, atmospheric superimposed hydrostatic pressure and strain rate $\varepsilon = -0.001 (1/s)$ on PA-6. In Fig. 6, both cylinder upsetting (axisymmetric) and cube compression (plane strain) results are shown. According to Arruda and Boyce [27], chains are uniaxially oriented in plane strain compression; in contrast, under uniaxial compression chains have planar orientation. Since under axisymmetric compression, chains have additional paths for deformation, larger strains are allowed under lower stresses whereas under plane strain compression, stress increases at higher speed due to the same paths of the chain deformations.

4.3. Cylinder upsetting-temperature effect

To analyse how temperature affects the stress-strain relation in polymers, an axisymmetric compression test has been simulated, using PS material properties, at three different temperatures, $T = 20 \degree C, 40 \degree C, 60 \degree C$ and under plane strain conditions. The superimposed hydrostatic pressure is assumed to be $p_0 = 0.1$ MPa and the strain rate is $-\dot{\varepsilon} = 0.001 (1/s)$. It is important to remark that the properties of many polymeric materials are highly dependent on temperature. Therefore, a set of relations taken from [34] are used to update the value of some properties at temperatures other than room temperature:

$$E(T) = E_0 \left( -0.002696T + 1.79 \right),$$
$$D_\infty(T) = D_\infty(0) \left( -0.0127T + 4.516 \right),$$
$$H(T) = H_0 \left( -0.01334T + 4.91 \right),$$

In relations (102), the scalars $E_0, D_\infty(0)$ and $H_0$ are the reference values for the Young modulus, saturation value of the softening parameter. 

![Fig. 12. Geometry and mesh of the notched bar compression example.](image)

![Fig. 13. Accumulated plastic strain for the compression simulation on the notched round bar.](image)

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Equilibrium convergence table for necking of a cylindrical bar and compression of a notched round bar.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iteration number</td>
<td>Relative residual norm (%)</td>
</tr>
<tr>
<td></td>
<td>Necking of a cylindrical bar</td>
</tr>
<tr>
<td></td>
<td>Increment (45)</td>
</tr>
<tr>
<td>1</td>
<td>10.2200</td>
</tr>
<tr>
<td>2</td>
<td>0.755383</td>
</tr>
<tr>
<td>3</td>
<td>0.464948E-02</td>
</tr>
<tr>
<td>4</td>
<td>0.872957E-05</td>
</tr>
<tr>
<td>5</td>
<td>0.280901E-07</td>
</tr>
</tbody>
</table>

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eter and hardening modules obtained at room temperature, $T = 20^\circ C$, and at strain rate of $\dot{\varepsilon} = 0.001$ (1/s). In Fig. 7, it is shown that if the analysing temperature increases, the level of stress would considerably decrease.

4.4. Cylinder upsetting-effect of superimposed hydrostatic pressure

In order to show the influence of superimposed hydrostatic pressure on the deformation behaviour of polymers, axisymmetric compression tests, employing PET material properties, have been numerically simulated under two different superimposed hydrostatic pressures, $p_0 = 0.1$ MPa and 300 MPa. A strain rate of $\dot{\varepsilon} = -0.001$ and room temperature of, $T = 20^\circ C$, are assumed for the test. In Fig. 8, it is clearly depicted that by increasing the superimposed hydrostatic pressure, the stress level will be raised.

4.5. Necking of a cylindrical bar

In this example, the necking of a cylindrical bar assuming PC material properties, room temperature and atmospheric superimposed hydrostatic pressure is presented. The geometry and mesh of the example are given in Fig. 9. As can be seen in Fig. 9, the problem is modelled as an axisymmetric tensile test and the specimen is spatially discretized with 1800 eight noded quadrilateral elements with reduced four integration Gauss points. The total number of degrees of freedom, after enforcing the (symmetric) boundary conditions, is equal to 11,308. The total displacement of $u = 2$ mm is applied within 100 increments and the displacement rate is $\dot{u} = 0.5$ mm/min. In order to trigger necking, a very small reduction in the area, of almost 1%, is applied on the right edge of the specimen. In Fig. 10, the accumulated plastic strain of...
the deformed cylinder at the end of deformation together with the
effective stress and displacement contour plots are shown. The
necking simulation is performed using a desktop with an Intel(R)
Core(TM) i7-3770 CPU with 3.40 GHz speed and 16 GB of installed
ram. The whole simulation took 127 s. To illustrate objectivity, the
necking simulation is also performed with a coarser finite element
discretization. The sample is spatially discretized with 525 eight
nodded quadrilateral elements with reduced integration and the
mesh has 1676 Cartesian nodes. The contour plots of the accumu-
lated plastic strain using both spatial discretizations are given in
Fig. 11. By simply comparing Fig. 11a and b, it could be concluded
that there is a good agreement (for both distribution of the accu-
mulated plastic strain and also the values) between the two differ-
ent spatial discretizations. This is due to the fact that the model
includes rate effects on the viscoplastic formulation which implicit-
ly introduces a characteristic length through the viscosity
function.

4.6. Compression of a notched round bar

The next example is a notched bar under uniaxial compression.
The geometry and mesh of the problem is shown in Fig. 12. The
specimen is spatially discretized with 350 eight noded quadri-
lateral elements with reduced four integration Gauss points. The
specimen is assumed to be made of PS and is subjected to a total
displacement of 2 mm in 100 increments under ambient con-
dition and room temperature. The deformation speed is
\( u = 0.6 \text{ mm/min} \). In Fig. 13, the contour plot of accumulated plastic
strain at the end of deformation is shown. In order to show the
numerical robustness of derived and implemented algorithm, the
equilibrium convergence of the later boundary value problems,
necking of a cylindrical bar and compression of a notched round
bar, are presented in Table 3. The simulation with the desktop, pre-
viously mentioned, took 45 s to finish. Besides, to check also the
convergence rate of the problem throughout the simulations, the
increments number with the corresponding
number of required iterations for convergence for both aforemen-
tioned simulations i.e. necking of the cylindrical bar and also com-
pression of the notched bar.

4.7. Tensile test on a dumbbell shape specimen

The final example presented in this contribution is a dumbbell
shape specimen subjected to uniaxial tensile loading. The geometry
and spatially discretized specimen are shown in Fig. 15. PA-6
material properties are considered for this example. The specimen
is discretized with 1100 eight noded elements with eight integra-
tion Gauss points. Total displacement of \( u = 15 \text{ mm} \) is applied to
the specimen in 200 increments under room temperature and
hydrostatic superimposed hydrostatic pressure. The load speed is
considered \( u = 4.5 \text{ mm/min} \). In Fig. 16, the force–displacement
curve is shown. The contour plot of the effective stress on the
deformed configuration together with the mesh on the initial con-
figuration are shown in Fig. 17.

5. Conclusions

An elasto-viscoplastic model based on the single mode EGP
model was presented. A fast robust one equation implementation
for the return mapping process of the numerical integration algo-
rithm of the model was provided. The ability of the model, with
the integration algorithm presented, to capture the effect of strain rate, hydrostatic pressure and temperature on the deformation
behaviour of polymers is shown through some numerical examples.
The efficiency of the integration algorithm was investigated by
showing the global convergence rates for different numerical
examples. In summary and in view of the aforementioned issues,
it can be concluded that the constitutive relations of the elasto-
viscoplastic model has been successfully implemented through
finite element with only one equation for the return mapping of
the state update procedure. The ability to capture reasonable
results, efficiency and robustness of the numerical implementation
were validated through some examples.

Since the model presented is only able to predict the deformation
behaviour of homogeneous polymers, an improved model could also be developed for porous polymers. To do so, the flow
potential of the presented model should be modified to account
for porosity. Another extension could be developing an elasto-
viscoplastic model to characterize the deformation behaviour of
polymers under different loading conditions.

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cial Humano/Fundo Social Europeu).

Appendix A

In this appendix, the algebraic manipulations to derive relation
(47) for the viscosity are provided.

By substituting relation (41) in relation (39) we obtain:

\[
\eta = A_0 \exp \left[ \frac{\Delta H}{RT} + \frac{\mu P}{\tau_0} - D \right] \frac{\tau^n}{\sinh \left( \frac{\tau^n}{\tau_0} \right)}. \tag{A.1}
\]
The first term of the above relation, in more detail, is:

\[ d \mathbf{s}^{\text{strain}}_{n+1} = d(2G \varepsilon^{\text{true}}_{n+1}) = d(2G \varepsilon_{n+1}) = 2G \left[ I_1 - \frac{1}{3} (I \otimes I) \right] : d \varepsilon_{n+1}. \]  

(8.8)

In order to complete the first term in relation (8.5), we have to compute the following:

\[ d \left( \frac{\eta_{n+1}}{\eta_{n+1} + \Delta \eta} \right) = \frac{\Delta \eta \, d \eta_{n+1}}{(\eta_{n+1} + \Delta \eta)^2}. \]  

(8.9)

Consequently and based on relations (8.5) and (8.9), we need to compute the terms \( d \eta_{n+1} \) and \( d p_{n+1} \) in order to obtain \( d \tau_{n+1}^{\text{B.1}} \). Two relations are considered to compute the mentioned terms. In order to be consistent with the state update algorithm, the first relation is the residual equation, used in state update algorithm and introduced in relation (78). The second equation is the following:

\[ p_{n+1} + K I : \varepsilon_{n+1} = 0. \]  

(8.10)

Therefore, we can write the following system of two equations.

\[
\begin{align*}
R_1(\eta_{n+1}, p_{n+1}) &= \eta_{n+1} - (C_1 C_2/C_3) = 0, \\
R_2(\eta_{n+1}, p_{n+1}) &= p_{n+1} + K I : \varepsilon_{n+1} = 0.
\end{align*}
\]  

(8.11)

According to the above system of equations, we can write:

\[
\begin{align*}
\left[ \frac{\partial R_1}{\partial \eta_{n+1}} \right] &\left[ \frac{\partial R_1}{\partial p_{n+1}} \right] \left[ d \eta_{n+1} \right] = -\left( \frac{\partial R_1}{\partial \varepsilon_{n+1}} : d \varepsilon_{n+1} \right), \\
\left[ \frac{\partial R_2}{\partial \eta_{n+1}} \right] &\left[ \frac{\partial R_2}{\partial p_{n+1}} \right] \left[ d p_{n+1} \right] = -\left( \frac{\partial R_2}{\partial \varepsilon_{n+1}} : d \varepsilon_{n+1} \right).
\end{align*}
\]  

(8.12)

Now, it is required to compute the derivatives of the two equations in order to \( \eta_{n+1}, p_{n+1} \) and \( \varepsilon_{n+1} \). The first term, \( (\partial R_1/\partial \eta_{n+1}) \), is already computed and presented in relation (83). The derivatives of the equations in order to strain, \( \varepsilon_{n+1} \), are the following:

\[
\begin{align*}
- \frac{\partial R_1}{\partial \varepsilon_{n+1}} &= F_1 [1 - F_2 - F_3] (I_4 : \mathbf{s}^{\text{strain}}_{n+1}) - C_1 \mu K \frac{I}{C_0 C_3}, \\
&= F_1 [1 - F_2 - F_3] \mathbf{s}^{\text{strain}}_{n+1} - C_1 \mu K \frac{I}{C_0 C_3}, \tag{8.16}
\end{align*}
\]  

(8.16)

where the factors \( F_1, F_2 \) and \( F_3 \) are introduced in relations (B.18)–(B.20).

\[ F_1 = \frac{\sqrt{2} C_1 \eta_{n+1}}{C_3 (\eta_{n+1} + \Delta \eta) \mathbf{s}^{\text{strain}}_{n+1}}, \]  

(8.18)

\[ F_2 = C_2 h \Delta t \exp \left( -h \sqrt{2} C_1 \frac{\mathbf{s}^{\text{strain}}_{n+1}}{D_\infty} \right), \]  

(8.19)

\[ F_3 = \frac{C_2}{C_1 C_0} \cosh \left( \frac{C_2}{C_0} \right). \]  

(8.20)

Having determined the components of relation (B.12), we can write:

\[ d \mathbf{s}^{\text{strain}}_{n+1} = - \frac{\eta_{n+1}}{\eta_{n+1} + \Delta \eta} G \mathbf{s}^{\text{strain}}_{n+1} \]  

(8.6)

or equivalently, broken down to,

\[ d \mathbf{s}^{\text{strain}}_{n+1} = - \left( \frac{\eta_{n+1}}{\eta_{n+1} + \Delta \eta} G \right) \mathbf{s}^{\text{strain}}_{n+1} - \left( \frac{\eta_{n+1}}{\eta_{n+1} + \Delta \eta} G \right) \mathbf{s}^{\text{strain}}_{n+1}. \]  

(8.7)

Appendix B

Exact linearization of the state update relations is presented in order to show how to derive relation (96) for the plastic consistent tangent operator. The tangent operator is obtained as the derivative of stress in order to strain:

\[ D = \frac{d \mathbf{s}_{n+1}}{d \varepsilon_{n+1}}. \]  

(8.1)

According to Eq. (49), we have:

\[ d \varepsilon_{n+1} = d \varepsilon^{\text{driving}}_{n+1} + d \varepsilon^{\text{hardening}}_{n+1}. \]  

(8.2)

Thus, relation (8.1) is rewritten as follows:

\[ D = \frac{d \varepsilon^{\text{driving}}_{n+1} + d \varepsilon^{\text{hardening}}_{n+1}}{d \varepsilon_{n+1}}. \]  

(8.3)

First, we compute the first term in the above relation, \( d \varepsilon^{\text{driving}}_{n+1} / d \varepsilon_{n+1} \). The stress decomposition into deviatoric and hydrostatic contributions in discretized format is given by:

\[ \varepsilon^{\text{driving}}_{n+1} = \mathbf{s}_{n+1} - p_{n+1} I. \]  

(8.4)

Differentiation of the above relation gives:

\[ d \varepsilon^{\text{driving}}_{n+1} = d \mathbf{s}_{n+1} - dp_{n+1} I. \]  

(8.5)

The first term of the above relation, in more detail, is:

\[ d \mathbf{s}_{n+1} = \left( \frac{\eta_{n+1}}{\eta_{n+1} + \Delta \eta} G \right) \mathbf{s}^{\text{strain}}_{n+1} \]  

(8.6)

or equivalently, broken down to,

\[ d \mathbf{s}_{n+1} = \left( \frac{\eta_{n+1}}{\eta_{n+1} + \Delta \eta} G \right) \mathbf{s}^{\text{strain}}_{n+1} + \left( \frac{\eta_{n+1}}{\eta_{n+1} + \Delta \eta} G \right) \mathbf{s}^{\text{strain}}_{n+1}. \]  

(8.7)
\[
\begin{align*}
&\frac{d\eta_{n+1}}{d\rho_{n+1}} = \left[\frac{(\partial R_1/\partial\eta_{n+1})}{(\partial R_2/\partial\rho_{n+1})} \right]^{-1} \left[\frac{(-\partial R_1/\partial\rho_{n+1})}{(\partial R_2/\partial\eta_{n+1})} \right] \cdot \frac{d\rho_{n+1}}{d\eta_{n+1}}.
\end{align*}
\]

By computing the aforementioned invert matrix and performing some straightforward algebraic manipulations, we can achieve the following relations:

\[
\begin{align*}
&d\rho_{n+1} = -K_1 : d\eta_{n+1},
\end{align*}
\]

Using relations (8.3), (8.5), (8.7), (8.8), (8.22) and (23), the driving part of the consistent tangent operator is obtained. In order to complete the relation, we need to add the hardening contribution to the tangent operator, \(d\rho_{n+1} \rightarrow d\rho_{n+1}^{\text{hardening}}\). The hardening part of the tangent operator is given by:

\[
\begin{align*}
&\frac{d\rho_{n+1}^{\text{hardening}}}{d\rho_{n+1}} = \left[\frac{1}{X} \right] \cdot \frac{d\rho_{n+1}}{d\rho_{n+1}},
\end{align*}
\]

References