Research paper

Detecting precursors of localization by strain-field analysis

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Localization in the deformation field, even though initiated locally at the microscopic scale, leads upon increased deformation to fracture at the macroscopic scale, thereby violating the separation of length scales. Localization and damage can be accounted for in macroscopic modeling by appropriate enrichments at that level, however doing so requires (i) detecting the onset of localization prior to its actual occurrence and (ii) quantifying the kinematical characteristics of the localization band. This paper serves that goal. A key methodology is developed to analyze the evolution of strain- and displacement-fields during deformation. A key ingredient in this analysis is the use of the Minkowski functionals (also known as intrinsic volumes, varemass integrals, or curvature integrals) from integral geometry, to detect emerging patterns in thresholded strain- and displacement-fields. Doing so, the onset of localization in the microstructure is detected as the emergence of a correlated and narrow pattern of high strains, prior to the actual loss of material stability. Furthermore, the developed localization band is characterized in terms of a weak displacement discontinuity, incorporating the width and direction of the band. The developed methodology uses kinematical fields only, and is therefore applicable to both numerical and experimental deformation-field data. For illustration purposes, numerical data from a finite-element simulation of a deformed voided microstructure is used, without any loss of generality.

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1. Introduction

Multiscale methods provide an essential contribution to bridging scales in mechanics of materials and material science. Nowadays, they are widely applied in designing, engineering and processing of advanced materials, e.g. composite materials, advanced alloys, biological materials. In recent years, various multiscale methods have been proposed to tackle scale bridging, e.g. asymptotic and homogenization methods (Cailletaud et al., 2003), heterogeneous multiscale methods (E et al., 2007; Chen, 2009), variational multiscale methods (Hughes et al., 1998), computational homogenization (Kouznetsova et al., 2001; Smit et al., 1998; Miehe et al., 1999; Geers et al., 2010; Schröder and Hackl, 2014; Oller, 2014) and experimental studies (Efthathiou et al., 2010). However, several classes of engineering problems represent cases in which scales strongly interact. In other words, there is no clear separation of length scales, and hence a clear distinction of phenomena into fine- or coarse-scale features is cumbersome, if not even impossible. A prominent example in this respect is material damage. Initiation and growth of damage is typically associated with the emergence of narrow regions with localizing strains at the microscale (Coenen et al., 2012b; Nguyen et al., 2011). While the progressive degradation of a material starts at the micro-scale, it gradually propagates to the macro-scale until material stability is lost, resulting in overall failure, i.e. fracture. As damage spans a wide range of length scales, it intrinsically violates scale separation and compromises the applicability of existing multiscale methods. In turn, this implies that, depending on the progression of the damage, the material model that is used to study the macroscopic behavior needs to be enriched. To that end, it is mandatory to devise a methodology that detects the necessity for an enrichment of the macroscopic model prior to the actual loss of stability. Therefore, an appropriate characterization of pattern formation in the microscale features is required in order to identify a precursor of localization.

Several studies have investigated strain localization by applying prototypical modeling, numerical simulations and experimental investigations. Various experimental techniques allow to investigate deformation and strain localization of materials at various length scales, including non-contact optical and interferometric methods (Cloud, 1998), atomic force microscopy (Tanaka et al., 2007; Man et al., 2002), transmission electron microscopy (TEM) (Saito et al., 2005), scanning electron microscope (SEM) (Crostack et al., 2001; Sutton et al., 2007b; Tanaka et al., 2011) and digital image correla-
tion (DIC) (Pan et al., 2009; Marty et al., 2015; Arikawa et al., 2011; Kammers and Daly, 2013; Sutton et al., 2007a). Several advanced numerical methods are dedicated to micro-mechanical modeling that incorporates damage (Uthaiansuk et al., 2009; Tekoglu and Paridoen, 2010; Legarth and Niordson, 2010; Ghosh et al., 2009; Kim and Lee, 2010) for which a continuous-discontinuous homogenization scheme is required to capture localization (Coenen et al., 2011, 2012b; Nguyen et al., 2012, 2011; Bosco et al., 2014; T. Belytschko, 2010; Ji et al., 2015; Paul and Kumar, 2012). Note that identifying material instabilities may be involved, see e.g. Benallal and Comi (1996); Szabó (2000); Benallal et al. (2010); Altmeyer et al. (2013). Here, the onset of localization will be captured naturally on the basis of the observed kinematics.

Localization occurs as a rapid collective growth and concentration of the microfluctuations in the strain field, culminating into a narrow high-gradient region in the sample, typically accompanying strain softening and thereby inducing material instability. Localization generally manifests itself as an irregular band of intense strains crossing the microstructure and provoking the emergence of a pronounced displacement variation. The latter is typically described by a weak discontinuity (Bosco et al., 2014; Liu, 2015). A weak discontinuity divides the micro-scale into regions of small and large displacements, respectively, separated by a smooth but pronounced transition. At the macro-scale, this may also be captured as a weak discontinuity or a discrete jump in the displacement field. Both cases require that the kinematical characteristics of the strong or weak discontinuity are properly quantified and embedded into the (thereby extended) macroscopic description to account for the localization, see e.g. Vernerey et al. (2007, 2008) and Wang and Lee (2010).

There exists a strong demand for a technique, applicable both to numerical and experimental strain- and displacement-fields, that efficiently analyzes micro-fluctuations in order to detect localization patterns, prior to the loss of material stability, thereby signaling when an enrichment of the macroscopic model (Coenen et al., 2012b) becomes necessary. To predefine the onset of localization in a deforming microstructure, digital image analysis is applied to investigate developing patterns in the micro-fluctuations in the deformation field. The latter makes use of a set of sequential snapshots of strains or displacements in the deforming microsample. These snapshots may be obtained from an experimental analysis, e.g. using DIC, or by numerical modeling at the micro-scale. The resulting micro-fluctuation field is decomposed into its stochastic (i.e. uncorrelated) part and spatially correlated part, respectively. Strain localization typically entails evolving correlated patterns. The kinematical characteristics of these patterns need to be qualified and quantified, necessitating a systematic analysis of the overall morphology and topology of the patterns.

The goals of this study are (i) to detect the precursors of localization and (ii) to quantify the kinematical parameters required to enrich the macroscopic description in the presence of localization, based on the analysis of micro-fluctuations in the strain- and displacement-fields. Analysis of complex spatially fluctuating structures is a relevant subject in statistical physics, for which several methods and tools have been proposed. The present article employs the so-called Minkowski functionals (Hadwiger, 1957; Santaló, 1976; Schneider, 1993; Weil, 1983; Munkres, 2000; Ohser and Müller, 2000), also known as intrinsic volumes (quermass integrals, curvature integrals), to analyze the evolving patterns in the micro-fluctuation field. Minkowski functionals represent a set of morphological descriptors describing the geometry of objects using global integral quantities, in contrast to differential-geometric tools that provide local information. Analysis by Minkowski functionals finds wide application in physics, soft matter science, and medicine (Mecke and Stoyan, 2000; Petri et al., 2013; Hüttner, 2003; Arns et al., 2010; Li et al., 2012).

The paper is organized as follows. Section 2 reviews the Minkowski functionals as a mathematical tool for analyzing the morphology of digital images. A complete procedure for analyzing the micro-fluctuation field in order to identify correlated patterns leading to localization is presented in Section 3. Section 4 is dedicated to the kinematic enrichment in the macro-scale model in order to account for the key characteristics of the localization band. Finally, conclusions are presented in Section 5.

2. Analyzing digital images: Minkowski functionals

The evolution of any scalar field, e.g. the equivalent total strain field, can be represented by a set of digital snapshots at discrete time instants. Digital images of the scalar field can be obtained by either numerical simulations or experimental techniques, thereby making the proposed approach applicable to both types of analysis. The evolution of the patterns in the scalar field of interest is captured and described by Minkowski functionals.

Minkowski functionals are mathematically represented as integral measurements of shape (Hadwiger, 1957; Santaló, 1976; Schneider, 1993; Weil, 1983; Munkres, 2000; Ohser and Müller, 2000). In d-dimensional space \( R^d \), there are \( d+1 \) scalar Minkowski functionals \( W_i (\nu) \) \((i = 0 .. d)\) that describe a domain \( \Omega \) with regular boundary \( \partial \Omega \). The present study is restricted to \( R^2 \) space, i.e. to the analysis of 2D images. The three scalar Minkowski functionals are then represented as follows.

\[
W_0 (\Omega) = \int_{\Omega} d \Omega. \quad W_1 (\Omega) = \frac{1}{2} \int_{\partial \Omega} d \nu. \quad W_2 (\Omega) = \frac{1}{2} \int_{\Omega} d k r.
\]

(1)

with \( d \Omega \) an infinitesimal area element, \( d \nu \) a one-dimensional line element on the boundary \( \partial \Omega \), and \( k \) the local principle curvature of the boundary. These three Minkowski functionals \( W_i \) are related to the area \( W_0 \), boundary length \( W_1 \), and Euler characteristic \( W_2 \) that describes the topology in terms of the connectivity of the structures in the domain \( \Omega \subset R^2 \).

While the above definitions (1), particularly the ones for \( W_1 \) and \( W_2 \), refer to a differentiable smooth boundary, alternative formulations have been developed (Mantz et al., 2008; Michielsen and De Raedt, 2001) that are applicable to the case of pixelized (2D) and voxelized data (3D). As a consequence, Minkowski functionals are widely applied to analyze texture of gray-scale 8-bit digital images. Gray-scale images are presented by a set of pixels with intensity \( I_0 \) in the range \( I_0 \in [0, 255] \). Intensities \( I_0 = 0 \) and \( I_0 = 255 \) correspond to the black and white color, respectively. A threshold \( I_0 = [0, 255] \) divides the image into foreground and background patterns: If the intensity of the pixel \( I_0 > I_0 \), then it is re-assigned to the maximum \( I_0 = 255 \), otherwise it is re-assigned to \( I_0 = 0 \). Doing so, a black-and-white binary image is obtained, which amounts to defining the pixel-equivalent of \( \Omega \) in (1). The black and white patterns in the binary image are referred to as background and foreground patterns, respectively. The Minkowski functionals \( W_i \) for gray-scale digital images are calculated on the thresholded variants, and are therefore functions of the threshold value \( I_0 \). \( W_i = W_i (I_0) \). Quantifying the area, boundary length and connectivity (Euler characteristic) of the black structures for each value of \( I_0 \), increasing the threshold \( I_0 \), results in an increase of the background area \( W_0 \). Corresponding changes in the boundary length are reflected by \( W_1 \). The Euler characteristic, represented by the third functional \( W_2 \), is defined as the difference between the number of disconnected background components (black structures) minus the number of foreground (white) inclusions in the background domain. Analysis of all three functionals through the whole range of thresholds \( I_0 \) will be employed for analyzing the micro-fluctuations in the deformation field, and thereby ultimately detecting the precursors for localization. With reference to analyzing
3. Deforming microstructures: identification of localizing patterns

The overarching goal of this paper is to develop a scheme for detecting precursors of localization by strain-field analysis, as summarized schematically in Fig. 1, irrespective of whether the strain field is obtained experimentally or numerically. Therefore, both experimental as well as numerical data could be used to develop and illustrate the precursor-detection scheme. In what follows, strain fields from a numerical finite-element-method (FEM) simulation are used.

3.1. Micro-scale problem definition: micro-sample with voids

In the present study, a set of digital snapshots of the deforming microstructure is generated numerically in order to analyze a strain field at the micro-scale revealing emerging patterns. For this purpose, the progressive deformation of a two-dimensional microstructural volume element (MVE) is used. An example of a typical MVE used for the analysis is shown in Fig. 2.

The initial MVE is a square of size 1 mm × 1 mm and contains 70 randomly distributed circular voids with a constant radius \( r_v = 0.03 \) mm at a volume fraction of 20%. Note that the actual size of the microstructure only matters if a model with an intrinsic length scale is used. Voids do neither overlap with each other nor cross the boundary of the MVE, and the minimum distance between the faces of the voids and the boundary is fixed to \( d_v = 0.3 \) \( r_v \). Emphasis is here put on the analysis of the patterns in the strain field to establish the methodology rather than on the particular MVE, i.e. the considered MVE need not be representa-
tive. Voids are not crossing the boundary simply for convenience of the implementation. Obviously, this may affect the MVE response, but it does not alter the methodology for the analysis of the strain field proposed in this paper.

The matrix material consists of an isotropic von Mises elastoplastic material with linear hardening, with an elastic stiffness $E = 200$ GPa and Poisson’s ratio $\nu = 0.3$. The plastic properties are characterized by the initial yield stress $\sigma_y = 250$ MPa and hardening modulus $H = 60$ MPa. To investigate the occurrence of patterns that may lead to localization in the microstructure, uniaxial tension of the MVE with the following boundary conditions is applied. The left boundary of the MVE is fixed in both directions, a uniform static displacement is applied to the right boundary, whereas the top/bottom boundaries are traction free. Note that these boundary conditions are not the ones used for conventional homogenization, since these would restrict localization through their inappropriate nature, e.g. periodic boundary conditions. Other more complex boundary conditions can be used instead (Coenen et al., 2012c). Pixel-formatted strain fields are derived by constructing digital images from the numerical simulations of the deformed MVE. The snapshots of the equivalent total strain are captured after each discrete step of deformation and analyzed for localizing patterns using the Minkowski functionals. The equivalent total strain $\varepsilon_{\text{eq}} = \sqrt{(2/3)\varepsilon : \varepsilon}$ is derived from the nodal data.

3.2. Strain inside the voids

MVEs with voids represent a heterogeneous material with infinitely soft inclusions. Due to the voided structure, digital images of the strain distribution in the MVE are undefined inside the voids. This hampers a straightforward implementation of the Minkowski-functional analysis of the strain-field images. To overcome this artifact, the strain field is approximated inside the voids by interpolating the nodal data on the boundaries of the voids. Particularly, for each void, the average strain over its entire boundary is calculated, and then boundary zones are identified with higher-than-average strain. All these zones are then paired with each other, and the strain is linearly approximated between them. A comparative analysis with the MVEs using soft filler particles instead of voids has shown that this interpolation procedure is physically meaningful for both localizing and non-localizing cases. Therefore, the details of the procedure for interpolating the strain in the voids is not of particular relevance for the further analysis. Since the numerical MVE calculations merely serve the purpose of generating strain- and displacement-fields, which could equally well be taken from experiments, it is the presence of localization as such on which we focus for this prototypical system.

Fig. 3 shows the resulting continuous images of the equivalent total strain $\varepsilon_{\text{eq}}$ obtained by numerical analysis of a deforming voided sample. The maximum/minimum value of the equivalent total strain is related to the white/black color, respectively, and the gray colors refer to intermediate strain-values. For the respective states of deformation, $\varepsilon_{\text{eq}}$ covers the range $[\varepsilon_{\text{min}}, \varepsilon_{\text{max}}]$, represented in each subfigure of Fig. 3 by gray-levels between black ($\varepsilon_{\text{min}}$) and white ($\varepsilon_{\text{max}}$), respectively; specifically: (a) $\varepsilon_{\text{eq}} \in [0.0, 0.13] \times 10^{-2}$, (b) $\varepsilon_{\text{eq}} \in [0.0, 0.56] \times 10^{-2}$, (c) $\varepsilon_{\text{eq}} \in [0.0, 3.8] \times 10^{-2}$, (d) $\varepsilon_{\text{eq}} = 0.2 \times 10^{-2}$. All images are shown in the undeformed geometry.

3.3. Persistent patterns in the strain field

The analysis of patterns in the strain field derived from the digital images is performed in terms of the Minkowski functionals $W_i$. At each discrete step of deformation $i$, the Minkowski functionals $W_i (\nu = 0, 1.2)$ are calculated for all possible values of the threshold strain $\varepsilon_{\text{th}} \in [\varepsilon_{\text{min}}, \varepsilon_{\text{max}}]$, with $\varepsilon_{\text{min}}$ and $\varepsilon_{\text{max}}$ the minimum and maximum values of strain in the MVE at that state of deformation $i$. As explained earlier, each value of threshold strain $\varepsilon_{\text{th}}$ relates to the corresponding intensity threshold in the interval $[0.255]$ in the gray-scale snapshots. As a result, one obtains at every state of deformation $i$ the Minkowski functionals as functions of the threshold strain $\varepsilon_{\text{th}} \in [\varepsilon_{\text{min}}, \varepsilon_{\text{max}}]$ to characterize the background patterns (area, boundaries and connectivity of the structures).

The primary goal of the present analysis is to systematically identify localizing patterns. Localization develops as a local correlated growth of a band-like pattern. The third Minkowski functional $W_2$, i.e. the connectivity of the background structure with respect to the foreground, is most sensitive to changes in the pattern morphology, and it is therefore used next to study the onset of localization.

An important ingredient in this analysis is the so-called persistent patterns. A persistent pattern can be defined by a large contrast in gray-scale values, i.e. in $\varepsilon_{\text{eq}}$-values, where the transition between low- and high-value regions have only a small spatial extent, i.e. relatively sharp transitions. This implies that when choosing various thresholds $\varepsilon_{\text{th}}$ between the high and low values, the corresponding resulting black-white structures vary only marginally. In turn, this means that, for that range of $\varepsilon_{\text{th}}$-values, the corresponding Minkowski functionals are only marginally dependent on $\varepsilon_{\text{th}}$. As a result, a persistent pattern is recognized by the connectivity $W_2(\varepsilon_{\text{th}})$ being constant (horizontal plateau) over an extended $\varepsilon_{\text{th}}$-range, within a certain tolerance. These persistent patterns could therefore also be called good-contrast patterns. If there is a horizontal plateau over a limited $\varepsilon_{\text{th}}$-range only, the pattern is not persistent. In order to study the onset of localization in terms of persistent patterns, only the longest horizontal plateau in $W_2(\varepsilon_{\text{th}})$ is considered in what follows, as a function of the global strain $\varepsilon_R$. For what follows it is pointed out that the horizontal plateau is defined for all global strains, albeit the length of the horizontal plateau is relatively short at small deformations.

Fig. 4 shows the third Minkowski functional $W_3$ as a function of the strain threshold, $W_3 = W_3(\varepsilon_{\text{th}})$. The corresponding images of the strain distribution are depicted in Fig. 3. At the stage of elas-
tic deformation, the patterns in the strain field are mainly uncorrelated and predefined by the initial heterogeneity (void distribution) of the microstructure, see Fig. 3(a). As observed in Fig. 4(a), the functional $W_2$ is quite sensitive to changes of the threshold $\varepsilon_{th}$. The fact that the horizontal plateau, depicted by the horizontal dashed line in each subfigure of Fig. 4, is relatively short implies that there is no persistent pattern in Fig. 4(a). Further deformation of the micro-sample and evolution of the patterns in the strain field cause the emergence of a larger and more pronounced horizontal plateau on the $W_2$-curve, specifically in Fig. 4(c) and (d). Continuous growth of the horizontal plateau indicates the emergence of a persistent pattern.

The definition of the horizontal plateau admits minor fluctuations of the $W_2$-curve. The plateau is therefore defined within a certain vertical tolerance tol$_{W_2}$. In the analysis performed here, a tolerance tol$_{W_2}$ was chosen in the range tol$_{W_2} \in [1, 3]$, the specific choice eventually showing a minor influence on the final results. Accounting for the finite tolerance tol$_{W_2}$ diminishes the influence of minor changes in the persistent pattern, e.g. minor loss of connectivity or collapse of separated tiny foreground structures with increase of $\varepsilon_{th}$.

The range of $\varepsilon_{th}$ to be considered for defining the horizontal plateau is limited by the vertical dashed line in Fig. 4, which has the following origin. For large values of the threshold $\varepsilon_{th}$, the corresponding patterns present a set of a few small isolated islands, e.g. high-stain regions nearby and inside the voids that persist in the foreground for almost the entire range of $\varepsilon_{th}$. Their small size is confirmed by the first and second Minkowski functionals, see Fig. 5(a) and (b), respectively. Note that the horizontal plateau in the tail of $W_2(\varepsilon_{th})$ occurs during all stages of deformation, and is not representative for any localization pattern, but rather for isolated islands of high strain. To identify actual localization, a cutoff in the tail of the $W_2(\varepsilon_{th})$-curve is therefore required. Since the interest in this study is on localization, the focus is on persistent coherent patterns that cross the entire MVE. Therefore, a foreground pattern that represents a localization band must have at least a minimum characteristic length $L_{\text{min}}$, as measured by the Minkowski functional $W_1$. The minimal contour-length of a localization band inside an MVE of 1 mm $\times$ 1 mm is 2 mm, because the band has got two sides. Since, the Minkowski functional $W_1$ contains also the perimeter of the MVE as explained at the end of Section 2, this leads to $L_{\text{min}} = 6$ mm. All patterns with a boundary length smaller than $L_{\text{min}}$ are to be disregarded. In other words, threshold values $\varepsilon_{th}$ for which $W_1(\varepsilon_{th}) < L_{\text{min}}$ are excluded from the horizontal-plateau analysis, which is marked in Figs. 4 and 5 for $W_1(\varepsilon_{th})$ with the dashed vertical line at the critical value $\varepsilon_{th, L_{\text{min}}}$, where $W_1(\varepsilon_{th, L_{\text{min}}}) = L_{\text{min}}$.

In summary, in this Section 3.3, the field of equivalent total strain $\varepsilon_{eq}$ for a given global strain $\varepsilon_g$ has been analyzed with the goal to extract the most prominent features of the strain fluctua-
tions. This has been achieved by applying a variable threshold to the gray-scale $\varepsilon_{eq}$-images to obtain a series of black-white images. For the latter, the Minkowski functional $W_2$, i.e. the connectivity, is examined as a function of the threshold value, $W_2(\varepsilon_{th})$. The existence of an extended plateau of $W_2(\varepsilon_{th})$ indicates that the corresponding pattern has good (strain-) contrast, which is therefore called a persistent pattern, at that respective global strain $\varepsilon_g$. As a quantitative criterion for the existence of a persistent pattern, one can require that the width of the plateau must be larger than a certain fraction of the admissible threshold-range $[0, \varepsilon_{th,\text{min}}]$. A natural criterion for a pattern to be considered as persistent would be to require that the width of the plateau must cover at least 20% of the range of admissible threshold values.

3.4. Evolution of the pattern and occurrence of correlated structures

A horizontal plateau in the $W_2$-curve, at a given global strain $\varepsilon_g$, indicates a characteristic feature of the strain field; if the horizontal plateau is sufficiently long that feature is called a persistent pattern, as discussed in Section 3.3. In general, increasing the deformation of the micro-sample (i.e., increasing the global strain $\varepsilon_g$) influences the strain field and might thus cause significant changes in the horizontal plateau, with properties described by the corresponding values of the three Minkowski functionals, $W_0^p$, $W_1^p$, $W_2^p$. The value $W_0^p$ of the horizontal plateau is the connectivity, i.e. the difference between the number of disconnected background components (black structures) minus the number of foreground (white) inclusions in the background domain, as discussed in Section 2.

During elastic deformation and stable material behavior, patterns persist in a rather narrow strain range, see Fig. 4(a), i.e. there are only weak-contrast features. Progressive evolution of the micro-fluctuation field with increasing deformation results in a marked change of the pattern. This change is reflected by a rapid increase in the length of the horizontal plateau, see Fig. 4(d), which becomes a persistent pattern. Furthermore, increasing deformation also results in a significant increase of the plateau value $W_2^p$ (decrease of $|W_2^p|$), see Fig. 6.

More quantitatively, the effect of the increasing deformation can be described as follows. During elastic deformation $\varepsilon_g \leq 0.9 \times 10^{-3}$, uncorrelated foreground structures mainly represent regions located at the boundary and inside of the voids. Increasing deformation $\varepsilon_g > 0.9 \times 10^{-3}$ induces a rapid increase of $W_2^p$, indicating a significant reduction of the uncorrelated components, i.e. the formation of a connected pattern. Between $\varepsilon_g = 0.9 \times 10^{-3}$ and $\varepsilon_g = 1.6 \times 10^{-3}$, the persistent pattern develops and stabilizes, and for a global strain $\varepsilon_g \geq 1.6 \times 10^{-3}$ one observes only minor further changes in connectivity. At increasing deformation, $W_2^p$ gradually tends to zero, corresponding to a single pattern.

Fig. 7(a) shows the range of the threshold values marking the boundaries of the horizontal plateau, $\varepsilon_{th}^p$, with increasing deformation of the MVE. The beginning/end of the vertical segment (at each $\varepsilon_g$) conforms to the strain at the beginning/end of the respective horizontal plateau. The width of the $\varepsilon_{th}^p$-range at each state of deformation, $\Delta \varepsilon_{th}^p$, is shown in Fig. 7(b). One observes that when a correlated pattern is developing (see Fig. 6), the width of the horizontal plateau increases, which implies that the correlated pattern is persistent, with increasingly good contrast.

The plateau characteristics, defined by the third functional $W_2$, indicate a global change of the initial pattern and the formation of a correlated persistent structure. In addition, the correlated growth of the patterns in the strain field causes a noticeable change of the characteristic area and boundary length revealed by the first and second functional at increasing deformation, $\varepsilon_g$, see Fig. 8. The quantities $W_0^p$ and $W_1^p$ in Fig. 8 are defined as the area and boundary length of the horizontal-plateau pattern scaled by the area and side-length of the MVE, respectively. Specifically, at each global strain $\varepsilon_g$, the lower/upper end of the vertical segment in
the figure corresponds to the area and boundary length of the pattern at the beginning/end of the horizontal plateau (as defined based on $W_2$). The symbols • represent the mean values of the area and boundary length, respectively. As the global strain reaches a value $\varepsilon_g \gtrsim 0.9 \times 10^{-3}$, a global change of the pattern, previously indicated by changes in the level $W_2^p$ and the plateau-width $\Delta \varepsilon_{th}^p$, is accompanied by a marked drop of the area and boundary length. A minor increase of the area and boundary length in the range $\varepsilon_g = 0.9 \times 10^{-3} - 1.6 \times 10^{-3}$ indicates progressive development of the (localization) pattern. For larger global strains, the pattern characteristics remain stable.

The Minkowski functionals $W_n$ applied to the strain images identify persistent patterns in the strain field and allow to track their evolution. A rapid transition in the level $W_2^p$ accompanied by a pronounced increase of the plateau-width $\Delta \varepsilon_{th}^p$ and a significant reduction of the characteristic area $W_0^p$ and boundary length $W_1^p$ of the pattern are the key characteristics revealed. Such behavior points to the local growth of the strain field and the formation of correlated patterns that potentially can lead to localization.

To clarify whether the observed behavior of the strain fluctuations predefines the formation of localizing patterns and the onset of localization, the above results are reconsidered in the context of the homogenized stress-strain curve, see Fig. 9. The actual transition from stable to unstable material behavior occurs at the peak in the overall stress-strain curve of the MVE, namely at $\varepsilon_g = 1.6 \times 10^{-3}$. The occurrence of long-range correlated patterns,
identified in Fig. 6 and denoted in Fig. 9 by □ at $\varepsilon_g = 0.9 \times 10^{-3}$, precedes the onset of localization in the strain field. The development of the correlated patterns towards a stable and persistent pattern continues up to $\varepsilon_g = 1.6 \times 10^{-3}$, see Fig. 6 and Fig. 8. This point corresponds to the peak of the homogenized stress-deformation curve (■) and indicates the onset of localization and unstable behavior of the material.

In summary, in this Section 3.4, it has been shown under what circumstances a persistent pattern represents a correlated, connected pattern. To that end, the plateau value of connectivity for the corresponding plateau, $W^p_0$, has been examined, leading to the observation that this quantity makes a rather pronounced transition towards a connected structure, as the global strain $\varepsilon_g$ is increased. That transition is accompanied by drastic drops in the plateau values $W^p_0$ and $W^p_1$, and by a significant increase in the plateau width $\Delta^p_{\text{th}}$, indicating an increasingly good contrast. Finally, it has been noticed that these observed transitions upon increasing $\varepsilon_g$ occur prior to the actual onset of mechanical instability.

3.5. Distinguishing localization from correlated patterns

Up to this point, it has been made clear that the proposed approach reveals global changes in the patterns of the strain field, which can be identified prior to the onset of the mechanical instability. However, the existence of correlated patterns in the strain field may not guarantee factual evolution towards localization. The distinction between correlated localizing and non-localizing patterns is discussed in this section.

To render the proposed approach more quantitative and to prove its ability to trace the occurrence of localizing patterns, the horizontal plateaus are investigated in both localizing and non-localizing cases. Using the same problem definition and MVE as discussed in Section 3.1, a non-localizing (stable) behavior of the material is achieved by increasing the hardening modulus to $H = 6000$ GPa. The corresponding global stress-strain curve is shown in Fig. 10.

The strain field for the non-localizing case is analyzed in terms of Minkowski functionals with identical tolerance parameters as for the localizing case. The characteristics of the horizontal plateau for the non-localizing case are compared with the localizing case considered before. Fig. 11(a) shows the transition of the level $W^p_0$ for both localizing (□) and non-localizing (△) cases, respectively. The marked transitions in $W^p_0$ for $\varepsilon_g > 0.9 \times 10^{-3}$ capture the development of correlated patterns in the strain field. However, in the non-localizing case the magnitude of the transition is significantly smaller than in the localizing case, and the number of disconnected components in the pattern remains considerably large. With increasing deformation of the micro-sample, the level of the plateau in the non-localizing case slowly decreases, indicating an increasing number of separated components within the pattern. This trend is opposite to the one observed for the localizing case, where the level tends towards zero, indicating a single connected pattern. The distinct evolution of the localizing and non-localizing patterns is confirmed by the different behavior of the plateau-width $\Delta^p_{\text{th}}$, shown in Fig. 11(b). The behavior is similar in the elastic regime only, where the plateau-width increases linearly with global strain. After the global strain reaches $\varepsilon_g = 0.9 \times 10^{-3}$, the curve of the plateau-width in the localizing case (□) shows a significantly faster growth compared to the non-localizing case (△). This manifests that in the localizing case there are relatively sharp transitions in strain, while in the non-localizing case the strain varies relatively mildly, spreading over the disconnected patterns.

To establish a quantitative criterion for distinguishing the localizing and non-localizing cases, the level $W^p_0$ and plateau-width $\Delta^p_{\text{th}}$ are normalized. Localization develops as an irregular band crossing the MVE domain. If the initial number of characteristic heterogeneities (i.e. voids) in the initial pattern equals $N$, a typical localization band in this two-dimensional context passes through

![Fig. 10. Global stress-strain curve in non-localizing case.](image)

![Fig. 11. Level $W^p_0$ of the horizontal plateau (a) and plateau-width $\Delta^p_{\text{th}}$ (b) for localizing (□) and non-localizing (△) cases.](image)
\(\sqrt{N}\) heterogeneities. While \(N\) may not always be available, it is closely related to the connectivity value of the horizontal plateau at small deformations, \(W_2^0(\varepsilon_g = 0)\). At the beginning of deformation, the mechanical response is still elastic, whereby the corresponding level \(W_2^0\) is proportional to the number \(N\) of characteristic heterogeneities in the MVE. Therefore, a meaningful indicator for the identification of a localization band is that \(W_2^0\) drops with increasing deformation to a value below \(\sqrt{W_2^0}(\varepsilon_g = 0)\). To emphasize this, the level \(W_2^0\) for the whole range of deformation is normalized by its initial value in the elastic regime, \(W_2^0(\varepsilon_g = 0)\). The normalized level of the plateau, \(R_{\varepsilon_0}^p\), is depicted in Fig. 12(a). If the normalized parameter \(R_{\varepsilon_0}^p\) drops below the defined threshold value \(1/\sqrt{W_2^0}(\varepsilon_g = 0)\), localization occurs. For the considered MVE, this threshold is about 0.14, as shown by the horizontal dashed line in Fig. 12(a). The localizing case drops clearly below the indicated threshold of 0.14, whereas the \(R_{\varepsilon_0}^p\) in the non-localizing case drops to approximately 0.55 only. Therefore, the normalized parameter \(R_{\varepsilon_0}^p\) together with its threshold value serve as the prime indicator to identify localization.

The occurrence of localization is confirmed by the normalized plateau-width \(R_{\Delta \varepsilon_m}^\varepsilon\), shown in Fig. 12(b). The plateau-width \(\Delta \varepsilon_m\) normalized by the global strain \(\varepsilon_g\), yielding \(R_{\Delta \varepsilon_m}^\varepsilon = \Delta \varepsilon_m / \varepsilon_g\). Localization reveals a rapid increase of \(R_{\Delta \varepsilon_m}^\varepsilon\) after the initial stages of deformation, i.e. for \(\varepsilon_g > 0.9 \times 10^{-3}\), to \(R_{\Delta \varepsilon_m}^\varepsilon > 1\). The non-localizing case, on the contrary, shows only a minor increase of \(R_{\Delta \varepsilon_m}^\varepsilon\), to \(R_{\Delta \varepsilon_m}^\varepsilon \approx 2\).

Note that the proposed approach does not rely on any prior knowledge of the deforming microstructure, e.g. the actual number of voids is not used. It enables an early identification of the onset of localization, accompanied by the formation of long-range correlated patterns in the strain field that satisfy the specific criteria discussed above.

In summary, in this Section 3.5, criteria have been developed to identify correlated patterns that are indicative of imminent localization, which are distinguished from correlated patterns in non-localizing cases. Particularly, \(R_{\varepsilon_0}^p < 1/\sqrt{W_2^0}(\varepsilon_g = 0)\) has been found to be an indicator for precursors of localization, as well as \(R_{\Delta \varepsilon_m}^\varepsilon > 1\), which signals strong strain concentration. With this, one has completed the development of a precursor-detection methodology, as described in Fig. 1(left).

### 4. Decomposition of the micro-fluctuation field: enriched kinematics characterizing a micro-to-macro transition

As discussed in Section 1, the occurrence of localization violates the separation of length scales. Therefore, the onset of localization at the micro-scale requires a kinematic enrichment of the macro-scale description of the material (Verner, 2007; 2008; Coenen et al., 2012b), which is discussed in this section. In the following, the main steps summarized in Fig. 1(right) for the characterization of the localizing band are addressed in detail.

#### 4.1. Micro-scale kinematics

In the previous section, persistent and long-range correlated patterns in the strain field indicating localization have been examined. Strain localization in the MVE typically occurs as an irregular complex band. As an exemplary situation, Fig. 13(a) depicts the equivalent total strain in the MVE at a global strain \(\varepsilon_g = 3.0 \times 10^{-3}\), reflecting the localization band. The corresponding magnitude of the displacement vector is shown in Fig. 13(b). The displacement-field reflects a weak discontinuity across the localization region, which divides the MVE in two domains with small (dark color) and large (light color) displacements, respectively.

The kinematics of the displacement discontinuity (band) can be characterized by a smooth step function \(H_m\) (Coenen et al., 2012b) shown in Fig. 13(c). The horizontal parts at the top/bottom of the curve correspond to the regions of small/large displacement, respectively, whereas the smooth transition in between represents the weak discontinuity with a displacement jump \(\Delta \overline{u}_b\) over the characteristic width \(\Delta w_b\). In the present analysis, \(\Delta w_b\) refers to the undeformed configuration. To complete the enrichment (often denoted by \(W_m\)) of the macroscopic continuum description, also the orientation of the band must be determined.

#### 4.2. Discontinuity-enrichment at the macro-scale

In the presence of strain localization, the displacement discontinuity at the micro-scale has to be accounted for at the macro-scale. To quantify the displacement jump \(\Delta \overline{u}_b\), the displacement field is split into its \(x\)- and \(y\)-components; corresponding snapshots of the \(u_x\) and \(u_y\)-fields at a global strain \(\varepsilon_g = 3.0 \times 10^{-3}\) are shown in Fig. 14(a) and (c), respectively. These fields are analyzed in terms of Minkowski functionals. Note that the displacement field inside voids is interpolated in a similar way as discussed in Section 3.1.
The first Minkowski functional $W_0$ (i.e. the area) is shown in Fig. 14(b) and (d) for the $x$- and $y$-components, respectively. The curves represent the area growth of the background pattern with increasing threshold $u_{x,y,\text{th}}$ in the range $[\min(u_{x,y}), \max(u_{x,y})]$ for $u_x$ and $u_y$. The $W_0$-curves show, after vanishingly small values, an initial rapid increase of the background area, which represents the region of low displacements (dark color). After the rapid growth, the curve levels off, reflecting a minor increase of the area over a relatively wide range of threshold values. This plateau-like behavior represents the transition between the low- and high-displacement regions in Fig. 14(a) and (c), and is depicted by the dashed horizontal line in Fig. 14(b) and (d). At a later stage, the $W_0$-curves start to increase rapidly again, eventually reaching their maximum. This represents the region of large displacement (light color).

The width of the plateau (dashed line) in Fig. 14(b) and (d) is determined as follows. The tangent is computed for the entire range of thresholds $u_{x,y,\text{th}}$. The dashed line corresponds to the $u_{x,y,\text{th}}$-range for which the slope of the tangent drops below 1% of
its overall average value. The width of the plateau determines the magnitude of the displacement jump \( \Delta u_{dp,b} \) of that respective displacement component, i.e., the difference between \( u_{dp,b} \) at the end and beginning of the dashed line. The complete displacement jump can then be described by the vector \( \Delta \mathbf{u} = (\Delta u_{x,b}, \Delta u_{y,b}) \), defining both the magnitude and the direction of the jump. Following Section 3.4, unstable material behavior (localization) initiates at a deformation \( \varepsilon_g = 1.6 \times 10^{-3} \). Fig. 15 shows the displacement jump through the band in x-direction, \( \Delta u_{x,b} \) (left), and y-direction, \( \Delta u_{y,b} \) (right), after the onset of localization, i.e., where a pronounced displacement jump exists.

The width \( \Delta W_0 \) of the displacement band is extracted by again using the first Minkowski functional \( W_0 \), but this time on the magnitude \( |\mathbf{u}| \) of the displacement rather than on its x- and y-components. Performing this analysis reveals a horizontal plateau similar to Fig. 14(b) and (d). The band area corresponds to the difference of \( W_0 \) at the beginning and end of the horizontal plateau. Fig. 16(a) depicts a binary image of the displacement band area at a global strain \( \varepsilon_g = 3.0 \times 10^{-3} \). It is evident from this example that the band may be strongly irregular and non-uniform. Local heterogeneities may induce a lot of variation in the width of the band as illustrated by Fig. 16(a), which may complicate an adequate quantification of the displacement-band width for the macroscopic enrichment. A characteristic measure for the width of the band can be obtained by applying so-called erosion (Richards, 2013; Haralick et al., 1987) to the binary image of the area. Erosion is a basic mathematical operation on a binary morphology. At each erosion step, the foreground pattern (white pixels) is eroded by subtracting a layer of one-pixel thickness at its boundary. By progressive erosion, the localization band (see e.g., Fig. 16(a)) will become increasingly fragmented. Therefore, performing successive erosion steps on a localization band will lead to a changing connectivity \( W_2 \). Note that a pixel has a specific (spatial) size, and hence an erosion step by a one-pixel layer corresponds to eroding the structure by a layer of a specific (spatial) thickness. This implies that one can determine the absolute value of connectivity \( |W_2| \) as a function of the thickness of the eroded layer, \( \Delta w \), as shown in Fig. 16(b). The number of pixels required to change the connectivity of the connected structure (first change in Fig. 16(b)) corresponds to the minimum width of the band. The first peak of the curve (marked by ▲ in Fig. 16(b)) indicates a maximum break-up of the structure and corresponds to the characteristic width of the displacement band, excluding inflated parts. The maximum width of the band is reached when the area of the foreground pattern in

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**Fig. 15.** Displacement jump across the band at increasing global deformation: magnitude of the jump in x-direction (a), and in y-direction (b).

**Fig. 16.** Area of the displacement band, (a); absolute value of the connectivity as a function of the thickness of the eroded layer applied to the displacement band shown in (a), (b).
$W_0$ becomes zero, i.e. when even the inflated parts in Fig. 16(a) are fully eroded.

Fig. 17 shows the minimum, maximum and characteristic widths of the displacement band, respectively, with increasing deformation of the MVE. The minimum width ($\omega$) of the band is constant over the entire deformation range after the onset of localization. It characterizes the narrowest regions of the band, with a rapid transition from small to large displacements. The characteristic width of the band ($\Delta$), $\Delta W_0$, coincides with the minimum width right after the onset of localization, followed by an insignificant growth at later stages of deformation. In contrast, the maximum width ($\epsilon$), corresponding to the inflated part in the displacement band shown in Fig. 16(a), shows a significant increase with increasing deformation, eventually converging to a constant value. For the macroscopic enrichment, the width required should be representative for the entire MVE and not only local (distorted) parts of the band. The characteristic width defined here seems most appropriate for this purpose.

To complete the quantitative characterization of the displacement band, the orientation of the band is required for the enrichment of a macroscopic model. In general, due to the heterogeneities, the displacement band has a complex shape and crosses the MVE as a complex path. However, at the macroscopic level only the average orientation of the band matters. To quantify that band orientation, here only the points where the band crosses the boundary of the MVE are used. These boundary-crossing points are identified by analyzing binary images of the displacement band, see Fig. 16(a). The pixels on the boundary of the image are analyzed and two intervals filled with white pixels are determined. The centers positions of these two intervals, denoted by $\text{I}$ in Fig. 18, are taken as the boundary-crossing points of the band, which in turn allows to determine the band orientation.

The resulting quantified kinematical properties of the localization band, i.e. displacement jump, band width, and its orientation, can be used to extend the macro-scale description of the continuum, e.g. through an embedded or X-FEM based discontinuity (Coenen et al., 2012a; Bosco et al., 2015).

In summary, in this Section 4, the main features of the localization band have been quantified, to be used for an enrichment of the macro-scale description. For global strains $\varepsilon_g$ at which localization indeed occurs, the following analysis has been performed. On the one hand, variable thresholds have been applied to the grayscale images for the components of the displacement field, $(u_x, u_y)$, to arrive at two series of black-white images. The latter have been analyzed in terms of the Minkowski functional $W_0$. The width of the plateau in $W_0$ with respect to the corresponding threshold is used to quantify the displacement jump across the band, $\Delta W$. On the other hand, a variable threshold has been applied to the grayscale images for the magnitude of the displacement vector field, yielding a series of black-white images. Using the Minkowski functional $W_0$ combined with the erosion technique, the width $\Delta W$ of the band has been quantified. Finally, the orientation of the band has been determined in terms of the crossing points of the band with the boundary of the MVE. Hence, the macroscale characterization of the localization band, see Fig. 1(right), is completed.

5. Conclusions

This paper presents a novel methodology for identifying a reliable precursor of strain localization in a microstructural volume element that precedes mechanical instability, summarized in Fig. 1(left). Furthermore, the properties and evolution of the emerging localization band are quantitatively characterized, see Fig. 1(right). The proposed method is tailored to recognize the relevant patterns in digital images of the strain- and displacement-fields. The method is generally applicable to field images of both numerical and experimental origin.

The analysis of strain localization is based on image-analysis tools from integral geometry, namely the Minkowski functionals. The latter are computed for thresholded images of the equivalent strain to assess pattern formation and to detect the onset of localization. For every state of deformation, the resulting values of the Minkowski functionals represent geometrical characteristics (area, boundary length, connectivity) of the evolving strain patterns. The detection of localization precursors proceeds in several steps, as shown in this paper. First, the Minkowski functionals are used to detect good-contrast patterns that dominate the entire strain field, those patterns being called persistent patterns. For their identification, the gray-scale strain-field images are thresholded, and if the Minkowski functionals vary only marginally over an extended range of threshold-strains, the correspondingly thresholded strain-field image shows a persistent pattern. Secondly, it is examined whether the persistent pattern, which at the early stages of deformation consists of several separate high-strain regions, evolves to a correlated pattern that spans the entire sample, by interconnecting the separate high-strain regions. And thirdly, it is established under what conditions the detected correlated patterns are indeed indicative for a precursor to localization. By comparing localizing...
and non-localizing cases, a generalized criterion based on the characteristics of the persistent pattern has been developed to properly identify a quantitative precursor to localization and to describe the localization band itself.

The emerging localization pattern triggers a transition from stable to unstable material behavior. The localization develops as a complex irregular band. Strain softening in the band degrades the load-carrying capacity of the microstructure and introduces a weak displacement discontinuity along the band, throughout the entire sample. Since this violates the separation of scales, it is necessary to enrich the macro-scale kinematics by appropriate characteristics of the localization band to account for the damage adequately. The corresponding properties of the localization band are determined by using the Minkowski-functional analysis for thresholded displacement images. The displacement jump across as well as the width of the localization band are determined in this way. Finally, the enrichment of the macroscopic description is supplemented by determining the average orientation of the localization band.

The proposed methodology presents a flexible and computationally efficient numerical technique to analyze the transition from damage to localization in a deforming material. Using image processing of scalar kinematical fields, it naturally applies to both numerical and experimental data. It is only for illustrative purposes that digital images of the strain- and displacement-fields were generated by numerical means, by performing a finite element calculation of a deforming microstructural volume element (MVE) of a voided microstructure. Specifically, two systems with different characteristics of the matrix material have been studied, where one of the systems shows localization while the other one does not. These two systems are used to generate prototypical deformation fields, which could just as well have been taken from experiments, and they are sufficient for showing that we have indeed developed a methodology that can discern precursors for localization from other fluctuations. The methodology itself, by way of its development, is completely independent of the models used in the illustrative examples. It is also emphasized that the stress-strain curves of the two examples are only used to show whether the respective prototypical sample is localizing (or not), while the further details of the model are not essential. Clearly, other examples where the onset of localization may not coincide with the peak in the global load-displacement curve are not excluded, and by construction the method would work here as well. Note that the peak is not used for the method development, since the method detects the precursors of localization prior to the point of the actual instability. Using kinematical data only, the proposed method is independent from any particular constitutive model.

The considered fields (strain/displacement) are the most common kinematical fields used in numerical simulations or experimental measurements. While the strain field has been employed for the detection of precursors for localization, the displacement field has been used in order to characterize the localization band in the actual case of mechanical instability. Since the entire methodology should also be applicable to the analysis of experimental data, other fields than kinematic fields can not be used. Experimental DIC-methods typically determine displacement fields, from which strain fields are extracted. Stress fields are not available (since this requires the precise knowledge of all constitutive models and parameters). Moreover, stress fields do not allow to uniquely identify localization characteristics.

In general, pattern recognition based on Minkowski functionals may show a great potential for other patterns as well, besides localization. Macroscopic enriched continua are widely present in the literature, and often conglomerated in the term “micromorphic continua”. The physical basis for these enrichments is rooted in the fine-scale kinematics, for which correlated patterns are known to be a dominant characteristic. It is within this context this offers a clear future potential.

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