Renormalization group analysis for porous-media-like equations

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Renormalization group analysis for porous-media-like equations

Bachelor thesis
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List of Acronyms

PDE  Partial Differential Equation
PM  Porous Media
RG  Renormalization Group
Abstract

In paper [1], Braga et al. introduced a Renormalization Group method to analyse the long-time asymptotics of solutions of a certain class of partial differential equations. The approach has been illustrated by the verification of a conjecture about the long-time behavior of solutions to a certain class of nonlinear diffusion equations with periodic coefficients. Their numerical approach provides a detailed picture of the asymptotics of the solutions of the partial differential equation. To understand this technique, we studied the paper of Braga and collaborators [1] and have implemented their numerical scheme in MATLAB. Next, we have applied the Renormalization Group analysis technique to a modified porous media equation. With an adapted version of the numerical scheme, we have formulated a conjecture on the asymptotics of the solutions of a particular class of porous-media-like equations.

Key words. Partial differential equations, self-similar solutions, asymptotic behavior, Renormalization group, numerical scheme, fixed point, non-linear diffusion equation, porous-media equation, Fourier transformation

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1 Introduction

1.1 Motivation of this research

In the paper of Braga et al. [1] a Renormalization Group (RG) method is introduced to investigate the asymptotic behavior of solutions of certain class of Partial Differential Equations (PDEs). The approach is introduced for PDEs in one space dimension. The technique can be generalized to more dimensions, but we do not discuss it here. Braga and collaborators [1] illustrate the technique for a non-linear diffusion equation with periodic coefficients. They point out a relation between homogenization theory and the RG method with these periodic coefficients, see also [10]. We assume that the PDEs to be treated here admit similarity solutions for large time.

1.2 Background

We now briefly discuss the current situation of the Renormalization Group (RG) analysis of PDEs. Goldenfeld et. al [5] developed in 1992 a perturbative RG method for PDEs. They presented also a numerical scheme to give insight into the long time asymptotic behavior of solutions of PDEs. The numerical scheme was used for PDEs with self-similarity solutions and PDEs with travelling wave solutions. Bricmont, Kupianen and Lin (1994) introduced in [3] a non-perturbative RG method for PDEs. They have illustrated their method by applying the technique to several examples the of nonlinear parabolic equations. Also, they discuss many applications of the RG method. In 1995 Bricmont and Kupianen [2] presented a general method for studying long time asymptotics of nonlinear parabolic partial differential equations. Their method works for systems of coupled PDEs. Later, in 2003, Braga et al. [1] presented an efficient numerical approach based on the RG method for computation of self-similarity dynamics. A. G. Braga, F. Furtado and V. Isaia presented a systematic numerical procedure for the computation of asymptotically self-similar dynamics of physical systems whose evolution is modeled by PDEs.

1.3 Aim of the thesis

In this bachelor thesis we follow in detail the approach of Braga and collaborators [1]. We implement their numerical scheme. With this numerical scheme at hand, we investigate the asymptotic behavior of solutions of certain classes of PDEs. Our task is here twofold: On one hand, we wish to re-discover all their numerical results, while on the other hand we aim to apply RG analysis to a special class of porous-media-like equations. We have chosen for a porous-media-like equation, because it is a generalization of the diffusion equation.

For our Porous Media (PM)-like equation we conjecture the following. If \( u(x,t) \) is a...
solution to
\[
\begin{aligned}
\left\{ \begin{array}{l}
    u_t = [u \partial_x^2 u + (\partial_x u)^2] + \lambda F(u, \partial_x u, \partial_x^2 u), \quad x \in \mathbb{R}, \ t > 1. \\
    u(x, 1) = f(x), \quad x \in \mathbb{R}.
\end{array} \right.
\end{aligned}
\]
where \( F(u, v, w) \) is an analytic function of the parameters \( u, v, w \) around \( u = v = w = 0 \):
\[
F(u, v, w) = u^a v^b w^c + \sum_{\{i>a,j>b,k>c\}} c_{ijk} u^i v^j w^k,
\]
and \( a, b, c \in \mathbb{N} \) satisfy
\[
a + 2b + 3c - 4 > 0,
\]
\( f \in \mathcal{L}^1(\mathbb{R}), f \geq 0 \), then there is a constant \( A \geq 0 \), which depends on \( f \) and \( F \) such that
\[
t^\alpha u(\sqrt{tx}, t) \to A \left( C - kx^2 \right)_+ \quad \text{as} \quad t \to \infty
\]
with \( \alpha = 1/3, k = 1/6 \) and \( C \) depends on \( f \).

1.4 Organization of this thesis

This thesis is organized in the following manner. In Section 2 we present the conjecture of Braga et al. [1] which is about the asymptotic behavior of solutions of a diffusion equation. We explain the RG analysis technique to motivate this conjecture. In Section 3 we present the numerical procedure of Braga and collaborators [1] for the diffusion equation. We also discuss the implementation in detail here. Next, in Section 4 we present our numerical results to indicate the correctness of the conjecture. In Section 5 we present our conjecture and the RG analysis technique to motivate the conjecture. In Section 6 we present a modified version of the numerical procedure of Braga which applies for our porous-media-like equation. Next, in Section 7 we present our numerical results to support our conjecture. Finally, we conclude in Section 8. We have added in Section 10 our MATLAB code for the implementation of the numerical procedure of Section 3 and the numerical procedure of Section 6.
2 Renormalization Group analysis for the diffusion

We consider the following partial differential equation

\[
\begin{aligned}
\begin{cases}
    u_t &= (1 + \mu g(x))\partial_x^2 u + \lambda F(u, \partial_x u, \partial_x^2 u), & x \in \mathbb{R}, t > 1. \\
    u(x, 1) &= f(x), & x \in \mathbb{R}.
\end{cases}
\end{aligned}
\] (5)

We assume \( \mu, \lambda \) to be real parameters and \( \mu \) s.t. \((1 + \mu g(x)) > 0\) for all \( x \in \mathbb{R} \). Here is \( g(x) \) a smooth and periodic function with zero mean (for example \( g(x) = \sin(x) \)). The function \( f(x) \) is smooth (i.e. \( f \in C^\infty(\mathbb{R}) \)) and rapidly decaying function (i.e. \( f \in L^1(\mathbb{R}) \)).

\[
F(u, v, w) \text{ is an analytic function of the parameters } u, v, w \text{ around } u = v = w = 0:
\]

\[
F(u, v, w) = u^a v^b w^c + \sum_{\{i>a, j>b, k>c\}} c_{ijk} u^i v^j w^k,
\] (6)

where \( a, b, c \in \mathbb{N} \). Let \( T \) be the period of the function \( g \), and let us denote with \( H(g, \mu) \) the harmonic mean of \( 1 + \mu g(x) \) via

\[
H(g, \mu) := \left[ \frac{1}{T} \int_0^T \frac{1}{1 + \mu g(x)} dx \right]^{-1}.
\] (7)

As mentioned in Section 1, we only consider PDEs for which the solutions are self-similar. We show now that a self-similar solution exists for \( \lambda = \mu = 0 \). From the numerics it will become clear that the solution of the general situation behaves self-similarly. A self-similar solution is a solution of the form

\[
u(x, t) \sim t^{-\alpha} \phi(xt^{-\beta}) = t^{-\alpha} \phi(\eta), \quad \text{with } \eta = xt^{-\beta}.
\] (8)

To check the self-similarity of the linear heat equation, we first calculate \( u_t, \partial_x u, \partial_x^2 u \):

\[
\begin{aligned}
u_t &= -\alpha t^{-\alpha-1} \phi(\eta) - t^{-\alpha} \phi'(\eta) xt^{-\beta-1}\beta = t^{-\alpha-1} (-\alpha \phi(\eta) - \beta \phi'(\eta) \eta), \\
\partial_x u &= t^{-\alpha-\beta} \phi'(\eta), \\
\partial_x^2 u &= t^{-\alpha-2\beta} \phi''(\eta).
\end{aligned}
\] (9)

Next, we plug those formulas in linear heat equation and find

\[
t^{-\alpha-1} [-\alpha \phi(\eta) - \beta \phi'(\eta) \eta] = t^{-\alpha-2\beta} \phi''(\eta).
\] (10)

We obtain a self-similar solution, if we choose \(-\alpha - 1 = -\alpha - 2\beta \Leftrightarrow \beta = 1/2\).
2.1 Conjecture by Braga et al. [1]

We present in this Subsection the Conjecture:

**Conjecture.** Let \( u(x,t) \) be the solution of (5). Given that the lowest order exponents \( a, b, c \) satisfy the inequality

\[
a + 2b + 3c - 3 > 0 \quad (11)
\]

and \( f \in \mathcal{L}^1(\mathbb{R}) \), then there is a constant \( A \), which depends on \( f, g, \mu, \lambda \) and \( F \) such that

\[
t^\alpha u(\sqrt{t}x,t) \to \frac{A}{\sqrt{4\pi \sigma}} e^{-\frac{x^2}{4\sigma}} \quad \text{as} \quad t \to \infty \quad (12)
\]

with \( \alpha = 1/2 \) and \( \sigma = H(g, \mu) \).

2.2 The verification of the conjecture for the linear heat equation

Let us verify the conjecture for the heat equation

\[
\begin{cases}
  u_t = \partial_x^2 u, & x \in \mathbb{R}, t > 0. \\
  u(x,0) = f(x) \in \mathcal{L}^1(\mathbb{R}), & x \in \mathbb{R}.
\end{cases} \quad (13)
\]

The unique solution of the Cauchy problem of (13) can be found as follows. First use the separation of variables technique to find the fundamental solution. The fundamental solution \( \Phi : \mathbb{R} \times (0, \infty) \to \mathbb{R} \) is defined by

\[
\Phi(x,t) := \frac{1}{\sqrt{4\pi t}} \exp \left(-\frac{x^2}{4t}\right). \quad (14)
\]

Next, apply the convolution operation to \( \Phi(x,t) \) and \( u(x,0) = f(x) \), we arrive then at the solution

\[
u(x,t) = (\Phi * f)(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp \left(-\frac{(x-y)^2}{4t}\right) f(y) dy, \quad (15)
\]

see [2] for details. Since (13) is a linear equation, we find that an linear combination of solutions of (13) is again a solution. We can now check the conjecture:

\[
t^{1/2} u(\sqrt{t}x,t) = t^{1/2} \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp \left(-\frac{(\sqrt{t}x-y)^2}{4t}\right) f(y) dy
\]

\[
= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp \left(-\frac{x^2}{4} + 2\frac{xy}{4\sqrt{t}} - \frac{y^2}{4t}\right) f(y) dy. \quad (16)
\]

Since

\[
\exp \left(-\frac{(\sqrt{t}x-y)^2}{4t}\right) f(y) \leq f(y) \quad \text{for all} \quad y \in \mathbb{R}, \quad (18)
\]
\[
\exp \left( -\frac{x^2}{4} + 2\frac{xy}{4\sqrt{t}} - \frac{y^2}{4t} \right) \to \exp \left( -\frac{x^2}{4} \right) \quad \text{as } t \to \infty \tag{19}
\]

and \( f \in \mathcal{L}^1(\mathbb{R}) \) we apply Lebesgue’s dominated convergence theorem [9, p. 26-27]. As \( t \to \infty \), we now obtain

\[
t^{1/2}u(\sqrt{t}x, t) \to \frac{\exp \left( -\frac{x^2}{4} \right)}{\sqrt{4\pi}} \int_{\mathbb{R}} f(y)dy. \tag{20}
\]

Hence, we indeed find that the large time behavior of \( t^{1/2}u(x\sqrt{t}, t) \) is

\[
A \frac{e^{-\frac{x^2}{4\pi}}}{\sqrt{4\pi\sigma}} \quad \text{with } \mu = \lambda = 0, A = \int_{\mathbb{R}} f(y)dy \quad \text{and } \sigma = H(g, 0) = 1.
\]

### 2.3 Renormalization Group analysis for the linear heat equation

To introduce the basic concepts of Renormalization Group analysis, we start off with the linear heat equation (13). It will appear that it is easier to start at time \( t = 1 \) instead of at \( t = 0 \). Since change this is just a linear transformation, the long time behavior of the solution will not be affected. We define a sequence \( \{u_n\}_{n=0}^{\infty} \) as

\[
u_0 := u, \quad u_n(x, t) := L^nu_{n-1}(L^nx, L^nt) \quad n \in \mathbb{N}, n \geq 1. \tag{21}\]

with \( L \in \mathbb{R}, L > 1 \) a fixed constant. Further in this report, it will become clear clear that the choice of \( \kappa \) is not important. We choose \( \kappa \) to be \( \frac{1}{2} \). So, for \( u_1(x, t) \) we get

\[
u_1(x, t) = L^{1/2}u_0(L^{1/2}x, L^{1/2}t) = L^{1/2}u(L^{1/2}x, L^{1/2}t). \tag{22}\]

Taking now \( x' := L^{1/2}x \) and \( t' := L^{1/2}t \), we find

\[
u_2(x, t) = L^{1/2}u_1(L^{1/2}x, L^{1/2}t) = L^{1/2}u_1(x', t') = L^{1/2}L^{1/2}u(L^{1/2}x', L^{1/2}t') = Lu(Lx, L^2t). \tag{23}\]

The induction argument gives

\[
u_n(x, t) := L^{n/2}u(L^{n/2}x, L^nt) \quad \text{for all } n \in \mathbb{N}. \tag{24}\]

If \( u(x, t) \) is a solution of (13) then \( u_n(x, t) \) is also a solution of it. This can be verified by a direct calculation. First, we compute \( \partial_x^2 u_n(x, t) \):

\[
\partial_x^2 u_n(x, t) = \partial_x^2 \left( L^{n/2}u(L^{n/2}x, L^nt) \right) = L^{3n/2}\partial_x^2 u(L^{n/2}x, L^nt). \tag{25}\]

Next, we calculate \( \partial_t u_n(x, t) \) and use (25):

\[
\partial_t u_n(x, t) = \partial_t \left( L^{n/2}u(L^{n/2}x, L^nt) \right) = L^{3n/2}u(L^{n/2}x, L^nt) = \partial_x^2 u_n(x, t). \tag{26}\]
We define the sequence \( \{ f_n \}_{n=0}^{\infty} \) as follows
\[
f_n(x) := L^{n/2}u(L^{n/2}x, L^n) \quad \text{for all } n \in \mathbb{N}.
\] (27)

Note that \( f_0(x) = u(x,1) \) and \( f_1(x) = L^{1/2}u(L^{1/2}x, L) \). We define the Renormalization transformation \( R_{L,n} : L^1(\mathbb{R}) \to L^1(\mathbb{R}) \) as follows
\[
R_{L,n}(f_n(x)) := L^{1/2}u_n(L^{1/2}x, L^n) \quad \forall n \in \mathbb{N}.
\] (28)

We can also write
\[
R_{L,n}(f_n(x)) = R_{L,n}(L^{n/2}u(L^{n/2}x, L^n)) = L^{(n+1)/2}u(L^{(n+1)/2}x, L^{n+1}).
\] (29)

Equation (29) indicates that \( R_{L,n} \) can be constructed as
\[
R_{L,n}(f_n(x)) = f_{n+1}(x), \quad \text{for all } n \in \mathbb{N}.
\] (30)

Since \( L > 1 \) is a fixed constant, we can 'renormalize' it by \( L^n \). We find then
\[
R_{L^n,0}(f_0(x)) = R_{L^n,0}(u(x,1)) = L^{n/2}u(L^{n/2}x, L^n) = f_n(x).
\] (31)

Also note that, with \( R^n_L := R_{L,n-1} \circ \ldots \circ R_{L,0} \), we get
\[
R^n_L(f_0(x)) = R^n_L(f_1(x)) = \ldots = R^n_{L,n-1}(f_{n-1}(x)) = f_n(x).
\] (32)

These equations hold for each \( f \). By combining (30) and (31) we arrive at
\[
R_{L^n,0} = R^n_L.
\] (33)

This means that \( R_L \) has the semigroup property. Since we have defined \( R_L \) for \( L \in \mathbb{R}, L > 1 \), there is no identity and no inverse Renormalization transformation. Notice that for the choice \( t := L^n, A = \hat{f}(0) \) in the limiting procedure (12) we get
\[
t^n u(\sqrt{t}x, t) = L^{n/2}u(L^{n/2}x, L^n),
\] (34)

for \( \alpha = 1/2 \). Studying the long time asymptotics of solutions to (13) is equivalent to studying fixed points (and their basin of attraction) of the Renormalization transformation \( R_L \). A function \( g : \mathbb{R} \to \mathbb{R} \) is a fixed point of \( R_L \) if and only if \( R_L(g) = g \). The semigroup property allows us to investigate the limiting procedure (12) for the choice \( t := L^n \), by iterating the Renormalization transformation \( R_L \). We say that a function \( h : \mathbb{R} \to \mathbb{R} \) is in the basin of attraction of a fixed point \( g \) if and only if
\[
\lim_{n \to \infty} R^n_L(h(x)) = g(x) \quad \forall x \in \mathbb{R}.
\] (35)

We can check that the fundamental solution of (15) is indeed a fixed point of \( R_L \). We find for \( R_{L,0}(\Phi(x,t)) \) now
\[
R_{L,0}(\Phi(x,t)) = R_{L,0}\left( \frac{1}{\sqrt{4\pi t}} \exp\left( -\frac{x^2}{4t} \right) \right)
= \frac{L^{1/2}}{\sqrt{4\pi t}} \exp\left( -\frac{(L^{1/2}x)^2}{4Lt} \right)
= \frac{1}{\sqrt{4\pi t}} \exp\left( -\frac{x^2}{4t} \right) = \Phi(x,t)
\] (36)
We use Fourier transformations to investigate the basin of attraction. Let \( h : \mathbb{R} \to \mathbb{R} \) be a function with \( h \in L^1(\mathbb{R}) \). We choose to use the following Fourier transformation \( \mathcal{F}[h] : \mathbb{R} \to \mathbb{C} \) of \( h(x) \):

\[
\mathcal{F}[h](\omega) := \int_{\mathbb{R}} h(x)e^{-i\omega x}dx,
\]

with inverse Fourier transform is

\[
h(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}[h](\omega)e^{i\omega x}d\omega.
\]

Bricmont, Kupianen and Lin [3] and then Bricmont and Kupainen [2] found that following relation holds

\[
\mathcal{F}[\mathcal{R}_{L,0}f_0](\omega) = e^{-\omega^2(1-L^{-1})} \mathcal{F}[f_0](L^{-1/2}\omega).
\]

L.T. Rolla [4] shortly sketches the proof. Since we require a similar approach in Section 5.2, we discuss the proof in detail. Essentially, we need \( \mathcal{F}[\Phi(x,L-1)](\omega L^{-1/2}) \). In the computation of this Fourier Transform, by definition (37), we use the transformation of variables technique. For \( \mathcal{F}[\Phi(x,L-1)](\omega L^{-1/2}) \) we get now

\[
\varpi := x + 2i\omega \frac{L-1}{\sqrt{L}}
\]

\[
\zeta := \frac{\varpi}{2\sqrt{L-1}}
\]

\[
\mathcal{F}[\Phi(x,L-1)](\omega L^{-1/2}) = \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi} (L-1)} \exp\left(-\frac{x^2}{4(L-1)} - xi\omega L^{-1/2}\right)dx
\]

\[
= \exp(-\omega^2(1-L^{-1})) \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi} (L-1)} \exp\left(-\frac{x^2}{4(L-1)}\right)dx
\]

\[
= \exp(-\omega^2(1-L^{-1})) \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi} (L-1)} \exp\left(-\frac{x^2}{4(L-1)}\right)dx
\]

\[
= \exp(-\omega^2(1-L^{-1})) \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi} (L-1)} \exp\left(-\zeta^2\right)d\zeta
\]

\[
= \exp(-\omega^2(1-L^{-1})).
\]

Next, we use the definition of \( \mathcal{R}_{L,0}f_0 \) and we get

\[
\mathcal{F}[\mathcal{R}_{L,0}f_0](\omega) = \mathcal{F}[L^{1/2}u \left(L^{1/2}x, L\right)](\omega).
\]

The scaling property of the Fourier transform [9, p. 179] gives us

\[
\mathcal{F}[\mathcal{R}_{L,0}f_0](\omega) = \mathcal{F}[L^{1/2}u(x,L)](\omega L^{-1/2})L^{-1/2}
\]

\[
= \mathcal{F}[u(x,L)](\omega L^{-1/2}).
\]
Since we can write the solution of equation (13) as a convolution of the fundamental solution and the initial data, we arrive at

\[ \mathcal{F} [ \mathcal{R}_{L,0} f_0 ] (\omega) = \mathcal{F} [ \phi (x, L - 1) * f_0(x) ] (\omega L^{-1/2}). \] (43)

Next, we use the property of the Fourier transform which states that the Fourier transform of a convolution is the product of the Fourier transforms [9, p. 179]. This yields

\[ \mathcal{F} [ \mathcal{R}_{L,0} f_0 ] (\omega) = \mathcal{F} [ \phi (x, L - 1) ] (\omega L^{-1/2}) \mathcal{F} [ f_0(x) ] (\omega L^{-1/2}) \] (44)

Combining this result with equation (40), we arrive at

\[ \mathcal{F} [ \mathcal{R}_{L,0} f_0 ] (\omega) = \exp (-\omega^2 (1 - L^{-1})) \mathcal{F} [ f_0 ] (\omega L^{-1/2}), \] (45)

which proves the claim.

We have seen in equation (36) that \( \Phi(x,1) \) is a fixed point of \( \mathcal{R}_L \). It also holds that \( \mathcal{F} [ \Phi(x,1) ] \) is a fixed point of \( \mathcal{F} [ \mathcal{R}_{L,0} ] \). To check this, we first compute \( \mathcal{F} [ \Phi(x,1) ] \). In this computation we use the transformation of variables \( \varpi := x + 2i\omega \) and \( \varsigma := \varpi/2 \). For \( \mathcal{F} [ \Phi(x,1) ] \) we get now

\[ \mathcal{F} [ \Phi ] (\omega) = \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi}} \exp \left( -\frac{x^2}{4} - x\omega \right) dx \]
\[ = \exp (-\omega^2) \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi}} \exp \left( -\frac{1}{4} (x + 2i\omega)^2 \right) dx \]
\[ = \exp (-\omega^2) \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi}} \exp \left( -\frac{\varpi^2}{4} \right) d\varpi \]
\[ = \exp (-\omega^2) \int_{\mathbb{R}} \frac{1}{\sqrt{\pi}} \exp \left( -\varsigma^2 \right) d\varsigma \]
\[ = \exp (-\omega^2). \] (46)

Combining (45) and (46), we find that \( \mathcal{F} [ \Phi(x,1) ] \) is a fixed point of \( \mathcal{F} [ \mathcal{R}_{L,0} ] \):

\[ \mathcal{F} [ \mathcal{R}_{L,0} \Phi(x,1) ] (\omega) = \exp (-\omega^2 (1 - L^{-1})) \mathcal{F} [ \Phi ] (L^{-1/2} \omega) \]
\[ = \exp (-\omega^2 (1 - L^{-1})) \exp \left( -\left( L^{-1/2} \omega \right)^2 \right) \]
\[ = \exp (-\omega^2) \]
\[ = \mathcal{F} [ \Phi(x,1) ] (\omega). \] (47)

It also holds that \( \mathcal{F} [ \Phi(x,1) ] \) is the only fixed point of \( \mathcal{F} [ \mathcal{R}_{L,0} ] \), this can be verified by the following nice trick. We compute the derivative of the fixed point to obtain an ODE. Let \( \psi \in \mathcal{L}^1(\mathbb{R}) \) be a fixed point of \( \mathcal{F} [ \mathcal{R}_{L,0} ] \), then we have

\[ \mathcal{F} [ \psi ] (\omega) = e^{-\omega^2(L^{-1})} \mathcal{F} [ \psi ] (L^{-1/2} \omega). \] (48)
Now, we compute the derivative of $F[\psi]$ using the relation (48).
\[
\frac{d}{d\omega} F[\psi](\omega) = \lim_{L \to 1} \frac{F[\psi](L^{1/2}\omega) - F[\psi](\omega)}{L^{1/2}\omega - \omega} \\
= \lim_{L \to 1} \frac{F[\psi](\omega)(e^{-\omega^2(L-1)} - 1)}{\omega(L^{1/2} - 1)} \\
= \lim_{L \to 1} \frac{F[\psi](\omega)(-\omega^2(L - 1) + O((L - 1)^2))}{\omega(L^{1/2} - 1)} \\
= \lim_{L \to 1} \frac{F[\psi](\omega)(-\omega(L^{1/2} + 1) + O(L - 1))}{\omega(L^{1/2} - 1)} \\
= -2\omega \cdot F[\psi](\omega) .
\] (49)

We obtain an ordinary differential equation for $F[\psi]$:
\[
\frac{d}{d\omega} F[\psi](\omega) = -2\omega \cdot F[\psi](\omega).
\] (50)

The solution of (50) is
\[
F[\psi](\omega) = C \cdot \exp(-\omega^2),
\] (51)
for some $C \in \mathbb{R}$. So we see that the only fixed points of $R_L$ are multiples of $F[\Phi](\omega)$. Let $f$ be a function in the basin of attraction of $\Phi$, i.e.
\[
R_L^n f = R_L^n f \to A\Phi, \quad A \in \mathbb{R}.
\] (52)

Next, we determine the functions in the basin of attraction of $\Phi$. Let $\tau$ be a function in the basin of attraction of $\Phi$ and let the Fourier transform of $\tau$ be $F[\tau]$:
\[
F[\tau](\omega) := \int_{\mathbb{R}} \tau(x)e^{-i\omega x}dx.
\] (53)

We expand in the integral the Taylor series of $e^{-i\omega x}$ around $\omega = 0$ and arrive at
\[
F[\tau](\omega) := \int_{\mathbb{R}} \tau(x)(1 - i\omega x + O(\omega^2)) dx \\
= F[\tau](0) + c\omega + O(\omega^2),
\] (54)

with $c := -\int_{\mathbb{R}} i\tau(x)dx$. We compute the Renormalization transformation of $F[\tau](\omega)$ we use equation (45), this yields
\[
F[R_L\tau](\omega) = e^{-\omega^2(L^{-1})}F[\tau](L^{-1/2}\omega) \\
= e^{-\omega^2(L^{-1})}F[\tau](0) + c\omega L^{-1/2} + O(\omega^2 L^{-1}).
\] (56)

As $L \to \infty$ we obtain
\[
F[R_L\tau](\omega) \to e^{-\omega^2}F[\tau](0).
\] (57)

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We define the Banach space $\mathcal{B}$ as (see [4, p. 27], [2, p. 4])

$$\mathcal{B} := \{ \pi(x) \in \mathcal{L}^2(\mathbb{R}) \mid \mathcal{F}[\pi] \in \mathcal{C}^1(\mathbb{R}), \|\pi\|_\mathcal{B} < \infty \} \quad (59)$$

with norm

$$\|\pi\|_\mathcal{B} := \sup_{\omega \in \mathbb{R}} \{(1 + \omega^4) (\|\mathcal{F}[\pi](\omega)\| + \|\mathcal{F}[\pi'](\omega)\|)\}.$$ 

(60)

For every $\tau \in \mathcal{B}$ we find that $\tau$ is in the basin of attraction of $\Phi$. This follows from a Banach Fixed Point argument, for the proof see L. T. Rolla ([4, p. 27-29]).

### 2.4 Renormalization of the nonlinear diffusion equation with periodic coefficients

In this Section we shortly derive the renormalized version of the PDE (5) as stated by A. G. Braga et al. ([1]) in their paper. We want to apply the presented numerical procedure (Section 3) to the problem (5). Therefore, we need the renormalized version of the PDE in step 1 of the numerical procedure. This can simply be verified by a direct calculation.

We assume that $F$ is of the form $F(u, \partial_x u, \partial_x^2 u) = u^a (\partial_x u)^b (\partial_x^2 u)^c$. Let $u = u(x, t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$. Let $L > 1$ and sequences of positive scaling exponents $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be fixed. Now we define a sequence of rescaled functions inductively by

$$u_0 = u, \quad u_n(x, t) = L^{\alpha_n} u_{n-1}(L^{\beta_n} x, L^n t) \quad \text{for } n \geq 1. \quad (61)$$

By iteration we now obtain

$$u_n(x, t) = L^{\alpha_n + \ldots + \alpha_1} u_0(L^{\beta_n + \ldots + \beta_1} x, L^n t). \quad (62)$$

If the original function $u$ is a global solution to equation (5), then the sequence $u_n$ satisfies the scaled problem:

$$\begin{cases} 
\partial_t u_n &= \chi_n [1 + \mu g(\omega_n x)] \partial_x^2 u_n + \lambda_n (u_n)^a (\partial_x u_n)^b (\partial_x^2 u_n)^c \\
\partial_x u_n &= f_n(x), \quad x \in \mathbb{R},
\end{cases} \quad (63)$$

with $\chi_n = L^{a - 2(\beta_n + \ldots + \beta_1)}$, $\omega_n = L^{\beta_n + \ldots + \beta_1}$, $\lambda_n = \lambda L^{n - (b + 2c)(\beta_n + \ldots + \beta_1) + (1 - a - b - c)(\alpha_n + \ldots + \alpha_1)}$. The initial data $f_n$ is

$$f_n(x) = L^{\alpha_n + \ldots + \alpha_1} u(L^{\beta_n + \ldots + \beta_1} x, L^n) = \xi_n u(\omega_n x, L^n). \quad (64)$$

with $\xi_n = L^{\alpha_n + \ldots + \alpha_1}$. As mentioned, this can be verified by a direct calculation. We first calculate $\partial_x u_n$ and $\partial_x^2 u_n$:

$$\partial_x u_n = \omega_n \xi_n u_x(\omega_n x, L^n t), \quad (65)$$

$$\partial_x^2 u_n = \omega_n^2 \xi_n \partial_x^2 u(\omega_n x, L^n t). \quad (66)$$

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We now rearrange the terms and can express $\partial_x u, \partial_x^2 u$ respectively in $\partial_x u_n, \partial_x^2 u_n$:

$$\begin{align*}
\partial_x u(\omega_n x, L^nt) &= \xi_n^{-1} \omega_n u_n(x, t), \\
\partial_x^2 u(\omega_n x, L^nt) &= \xi_n^{-1} \omega_n^{-2} \partial_x^2 u_n(x, t).
\end{align*}$$

(67)

(68)

We now compute $\partial_t u_n$ using (P) and (67):

$$\begin{align*}
\partial_t u_n &= L^n \xi_n u_t(\omega_n x, L^nt) \\
&= L^n \xi_n \left[ (1 + \mu g(\omega_n x)) \partial_x^2 u(\omega_n x, L^nt) + \lambda u^a(\omega_n x, L^nt) (\partial_x u)^b(\omega_n x, L^nt) (\partial_x^2 u)^c(\omega_n x, L^nt) \right] \\
&= L^n \xi_n \left[ (1 + \mu g(\omega_n x)) \xi_n^{-1} \omega_n^{-2} \partial_x^2 u_n(x, t) + \lambda u_n^a(x, t) \xi_n^b \omega_n^{-b} (\partial_x u_n)^b(x, t) \right] \\
&\quad \cdot \xi_n^{e} \omega_n^{-2e} (\partial_x^2 u_n)^c(x, t)(\omega_n x, L^nt) \\
&= \chi_n [1 + \mu g(\omega_n x)] \partial_x^2 u_n + \lambda_n (u_n)^a (\partial_x u_n)^b (\partial_x^2 u_n)^c
\end{align*}$$

(69)

If we choose $\beta_n = 1/2$ for all $n \in \mathbb{N}$, which yields $\chi_n = 1$, then the 'diffusion' part of equation (90) is invariant. Notice that this choice of $\beta_n$ corresponds to the requirement of the self-similarity behavior of the solution. We also choose $\alpha_n$ to be $1/2$ for all $n \in \mathbb{N}$. By this choice, the sequence $u_n$ as defined in (61) becomes the same as the sequence $u_n$ for the linear heat equation, as defined in (21). $u_n$ satisfies now

$$\begin{cases}
\partial_t u_n &= [1 + \mu g(L^{n/2} x)] \partial_x^2 u_n \\
&\quad + \lambda L^{n(3-(a+2b+3c)/2)} u_n^a (\partial_x u_n)^b (\partial_x^2 u_n)^c \\
u_n(x, 1) &= f_n(x), \quad x \in \mathbb{R}.
\end{cases}$$

(70)

Since we required $a, b, c$ to satisfy equation (11), we expect that the effect of the perturbation function to be negligible for large $n$. 

---

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3 Numerical procedure for the diffusion equation (5)

In this Section the numerical procedure (of A. G. Braga et al. ([1])) for the integration of equation (5) is now presented. The procedure constructs the sequence \( f_n \) as defined in equation (27). From this \( f_n \) we can easily find the solution of (5) at time \( t = L^n \). First, we let \( f_0 = f \), where \( f \) is the initial condition at \( t = 1 \). For \( m = 0, \ldots, n \) we have the following three steps:

1. Evolve \( f_m \) forward from \( t = 1 \) to \( t = L \) using a discretization method for the PDE (13), this yields \( u_m(x, L) \). (The discretization method will be discussed further in this paper.)

2. Set \( \beta_{m+1} = 1/2 \) and solve \( \alpha_{m+1} \) from the equation \( L^{\alpha_{m+1}} = u_m(0, L) \).

3. Set \( f_{m+1} = L^{\alpha_{m+1}} u_m(L^{\beta_{m+1}} x, L) \).

The solution of the PDE at time \( L^n \) can easily be found by a scaling of \( f_n \) [1]:

\[
    u(x, L^n) = A_n L^{-n \alpha_n} f_m(B_n L^{-n \beta_n} x),
\]

where

\[
    A_n := L^{n \alpha_n -(\alpha_n+\ldots+\alpha_1)} \quad \text{and} \quad B_n := L^{n \beta_n -(\beta_n+\ldots+\beta_1)}.
\]

If \( A_n \to A, B_n \to B, \alpha_n \to \alpha, \beta_n \to \beta \) and \( f_n \to \phi \) (as \( n \to \infty \)) then we expect that

\[
    L^{n \alpha} u(L^{n \beta} x, L^n) \rightarrow A \phi(B x),
\]

as \( n \to \infty \). This is exactly how we will proceed. In Section 4 we check those limiting procedures. However, let us first discuss the steps of the numerical method in more detail.

3.1 Numerical procedure for the diffusion equation (step 1)

In this Subsection we discuss the numerical method we use solve the PDE from \( t = 1 \) to \( t = L > 1 \). The discretization method we use is the explicit finite-difference (Euler) method. First, we divide the space \( \mathbb{R} \) from \(-K \) to \( K \) in parts of meshsize \( \Delta x \) for some large \( K \in \mathbb{R}_+ \). Also, we divide the time in parts of size \( \Delta t \). So we write \( x = j \Delta x \) and \( t = i \Delta t \). To find the discrete version of (13) we use Taylor series. We expand the Taylor series of \( u(x + \Delta x, t) \) and \( u(x - \Delta x, t) \):

\[
    u(x + \Delta x, t) = u(x, t) + \partial_x u(x, t) \Delta x + \frac{1}{2} \partial^2_x u(x, t)(\Delta x)^2 + \frac{1}{6} \partial^3_x u(x, t)(\Delta x)^3 + O(\Delta x)^4
\]

\[
    u(x - \Delta x, t) = u(x, t) - \partial_x u(x, t) \Delta x + \frac{1}{2} \partial^2_x u(x, t)(\Delta x)^2 - \frac{1}{6} \partial^3_x u(x, t)(\Delta x)^3 + O(\Delta x)^4.
\]

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For the approximation of $\partial^2_x u$ we use the central second differentiation and obtain inside the mesh

$$\partial^2_x u(x, t) = \frac{u(x - \Delta x, t) - 2u(x, t) + u(x + \Delta x, t)}{(\Delta x)^2} + O(\Delta x)^2. \quad (74)$$

We use the forward differentiation for $u_t$, so we expand the Taylor series of $u(x, t + \Delta t)$:

$$u(x, t + \Delta t) = u(x, t) + \partial_t u(x, t) \Delta t + O(\Delta t)^2$$

This yields for $\partial_t u(x, t)$:

$$\partial_t u(x, t) = \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} + O(\Delta t). \quad (76)$$

Next, we make an approximation by ignoring the terms $O(\Delta t)$, $O(\Delta x)^2$. We define $u^{i+1}_{j} := u(j\Delta x, i\Delta t)$, then the approximation of (13) yields in

$$u^{i+1}_{j} = u^i_{j} + \frac{\Delta t}{(\Delta x)^2} (u^i_{j-1} - 2u^i_{j} + u^i_{j+1}). \quad (77)$$

To check the stability of this approximation, we define $s := \frac{\Delta t}{(\Delta x)^2}$. Then we have

$$u^i_{j} = u^{i-1}_{j} + \frac{\Delta t}{(\Delta x)^2} (u^{i-1}_{j-1} - 2u^{i-1}_{j} + u^{i-1}_{j+1})$$

$$= (1 - 2s)u^{i-1}_{j} + s(u^{i-1}_{j-1} + u^{i-1}_{j+1})$$

$$= (1 - 2s)u^{i-1}_{j} + 2s \frac{u^{i-1}_{j-1} + u^{i-1}_{j+1}}{2}.$$

We see that we need $1 - 2s > 0$ for stability, which results in $s = \frac{\Delta t}{(\Delta x)^2} < \frac{1}{2}$. So for obtain a stable system $\Delta t$ should be small enough compared to $(\Delta x)^2$. This means that when we enlarge the number of parts in the space with a factor $p \in \mathbb{R}_+$, then we should also replace the time steps by $\frac{1}{p^2} \Delta t$.

Similarly, the approximation of (63) results in

$$u^{i+1}_{j} = u^i_{j} + \frac{\Delta t}{(\Delta x)^2} [1 + \mu g(\omega_{ij})] (u^{i-1}_{j-1} - 2u^i_{j} + u^{i+1}_{j+1})$$

$$+ \Delta t \lambda_i (u^i_{j})^a \left( \frac{u^i_{j-1} - u^i_{j+1}}{\Delta x} \right)^b \left( \frac{u^i_{j-1} - 2u^i_{j} + u^i_{j+1}}{(\Delta x)^2} \right)^c,$$

with stability constraint ([1])

$$\left(1 + \max_x (|\mu g(x)|) \frac{\Delta t}{(\Delta x)^2} \right) < \frac{1}{2}. \quad (79)$$

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3.2 Numerical procedure for the diffusion equation (step 2)

In this subsection we discuss the details of step 2 of the numerical procedure. In this step we use the self-similarity behavior of the solution. Let $u$ be the solution of the PDE. Then we have for large time

$$u_m(x,t) = L^{\alpha m + \ldots + \alpha_1} u(L^{\beta m + \ldots + \beta_1} x, Lt) \approx L^{\alpha m + \ldots + \alpha_1} t^{-\alpha} \phi(0),$$

which results in

$$\frac{u_m(0,1)}{u_m(0,L)} \approx \frac{L^{\alpha m + \ldots + \alpha_1} 1^{-\alpha} \phi(0)}{L^{\alpha m + \ldots + \alpha_1} L^{-\alpha} \phi(0)} \approx L^{\alpha m + 1}.$$

3.3 Numerical procedure for the diffusion equation (step 3)

Next, we discuss the details of step 3 of the numerical procedure. In the first place, note that there happens mathematically actually nothing, i.e. we just define a function. However, numerically it is interesting. Before step 3 we have the set of data

$$\{u_m(x = j \Delta x, t = L)\},$$

and in step (3) we require the following set of data

$$\{L^{\alpha m + 1} u_m(x = j L^{\beta m + 1} \Delta x, t = L)\}.$$

This set of data can be obtained by using interpolation and extrapolation techniques. However, the interpolation and extrapolation can be avoided by using a nice rescaling trick found by L. Chen et al. [5]. Their idea is to rescale the mesh size $\Delta x$ instead of rescaling the discrete sites $j$. Let $(\Delta x)'$ be the new mesh size. Then we require n step (3) the following set of data

$$\{L^{\alpha m + 1} u_m(x = j L^{\beta m + 1} (\Delta x)', t = L)\}.$$

Now, the choice of the new mesh size is $(\Delta x)' := L^{-\beta m + 1} \Delta x$. With this choice the new mesh points arrive at the locations of the old mesh points:

$$\{L^{\alpha m + 1} u_m(x = j L^{\beta m + 1} (\Delta x)', t = L)\} = \{L^{\alpha m + 1} u_m(x = j (\Delta x), t = L)\}.$$

Thus, we define $f_{m+1}$ at the new mesh points as

$$\{f_{m+1}(x = j (\Delta x)', t = L) := L^{\alpha m + 1} u_m(x = j L^{\beta m + 1} (\Delta x)', t = L)\}.$$
Notice that the system size shrinks for each RG iteration. For the system to remain stable, the time step has to shrink also. We choose the new time step as $(\Delta t)' := L^{-2\beta_{m+1}} \Delta t$. This means that solving the PDE from time $t = 1$ to $t = L$ (step 1 of the numerical procedure) requires more iterations as $m$ increases. To avoid the need of boundary conditions, we add points with value zero to the system before we enlarge $m$ with one (see [2]). This works numerically because the initial data is almost zero on the boundary. The rescaling procedure is visualized in Figure 1. Thus, by using this rescaling technique we do not need any interpolation and extrapolation techniques.

Figure 1: Rescaling mesh size in step 3 of the numerical procedure.
4 Numerical results for the diffusion equation

In this Section we use the numerical procedure to indicate the correctness of the conjecture presented by A. G. Braga et al. ([1]). We want to recover their numerical results. As mentioned in equation (73), we expect the Conjecture to hold if the sequences (as defined in Section 3) \( \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{A_n\}_{n=0}^{\infty}, \{B_n\}_{n=0}^{\infty} \) and \( \{f_n\}_{n=0}^{\infty} \) converge. For the sequences \( \{\beta_n\}_{n=0}^{\infty} \) and \( \{B_n\}_{n=0}^{\infty} \) we have already obtained convergence since we choose \( \beta_n = \frac{1}{2} \) for all \( n \in \mathbb{N} \) in step 2 of the numerical procedure.

We use for the initial profile function 27 grid points on the interval \([-5, 5]\) and we choose the scaling factor \( L \) to be 1.021 (just as Braga and collaborators did). The number of grid points increases during the iteration process, as explained in Section 3. We choose the initial time step \( \Delta t \) to be 0.0007. For the numerical simulation for will use the following three initial profile functions:

\[
\begin{align*}
f_{0,1}(x) & := \exp \left( -\frac{x^2}{4} \right), \\
f_{0,2}(x) & := 1 - \left| \frac{x}{5} \right|, \\
f_{0,3}(x) & := \frac{1}{2} \left( 1 - \sin \left( \frac{\pi x}{5} \right) \right).
\end{align*}
\] (87)

We visualize the initial profile functions in Figure 2. To indicate the correctness of the

![Figure 2: Plot of initial profiles f_{0,1} (red continuous line), f_{0,2} (blue dotted line), f_{0,3} (black dash-dotted line).](image)

Conjecture of A. G. Braga and collaborators numerically, we first investigate the linear
heat equation (i.e. $\lambda = \mu = 0$). Next, we add the periodic function to the diffusion part. After that, the influence of the parameter $\mu$ is checked. Subsequently, we choose an other periodic function to check the conjecture for. Finally, we add the perturbation function to the PDE. In all simulations, we obtain the convergence of the profile functions $f_m$ to (a multiple of) equation (14).

4.1 Numerical results for the linear heat equation

In this Subsection we verify the correctness of the Conjecture for $\lambda = \mu = 0$. For each of the initial profile functions we iterate the numerical procedure for $m = 0, 1, \ldots, 300$. We show in Figure 3 $\alpha_m$ ($m = 0, 1, \ldots, 300$) for the linear heat equation ($\mu = \lambda = 0$). Indeed, we see that $\alpha_m$ converges to a limiting value of 0.5. Figure 4 shows the convergence of the factor $A_m$ for the linear heat equation. Notice that the limiting value of $A_m$ is not universal; it depends on the initial profile function.

![Figure 3: Plot of $\alpha_m$ for $m = 0, 1, \ldots, 300$ for $\lambda = 0, \mu = 0$, for the initial profiles $f_{0,1}$ (red continuous line), $f_{0,2}$ (blue dotted line), $f_{0,3}$ (black dash-dotted line).]
Figure 4: Plot of $A_m$ for $m = 0, 1, ..., 300$ for $\lambda = 0, \mu = 0$, for the initial profiles $f_{0,1}$ (red continuous line), $f_{0,2}$ (blue dotted line), $f_{0,3}$ (black dash-dotted line).
4.2 Numerical results for equation (5)

In this Subsection we verify the correctness of the Conjecture for equation (5) without the perturbation function. We visualize in Figure 5 and 6 the convergence of respectively $\alpha_m, A_m$ for $\lambda = 0, \mu = 0.1$ and $g(x) = \cos(x)$. Again, in each of the simulations $\alpha_m$ converges to the value of 0.5.

Figure 5: Plot of $\alpha_m$ for $m = 0, 1, ..., 300$ for $\lambda = 0, \mu = 0.1$ and $g(x) = \cos(x)$ for the initial profile $f_{0,1}$ (red continuous line), $f_{0,2}$ (blue dotted line), $f_{0,3}$ (black dash-dotted line).
Figure 6: Plot of $A_m$ for $m = 0, 1, \ldots, 300$ for $\lambda = 0, \mu = 0.1$ and $g(x) = \cos(x)$ for the initial profile $f_{0,1}$ (red continuous line), $f_{0,2}$ (blue dotted line), $f_{0,3}$ (black dash-dotted line).
We verify the correctness of the scaling in the conjecture by the harmonic mean \( \sigma := H(g, \mu) \) by plotting \( -\log(f_n) \) vs \( -\log(\phi) \) for \( n = 300 \). First notice that \( -\log(\phi) \) is equal to \( \frac{\sigma^2}{4} \). If the conjecture holds then \( -\log(f_n) \) is (almost) equal to \( \frac{\sigma^2}{4} \). Notice that the value of \( \sigma \) is now equal to \( -\log(\phi) \) for \( -\log(f_n) = 1 \). In Figure 7 this plot is shown for initial data \( f_{0,1}, f_{0,2} \) and \( f_{0,3} \).

We remark just one line in Figure 7 which despite the convergence of the profile functions to a scaled Gaussian and shows that the \( f_n \) is universal with respect to the initial data. The fact that the dot in the Figure 7 is on the line indicates the correctness of the scaling in the conjecture by the harmonic mean.

Figure 7: Plot of \( -\log(f_n) \) vs \( -\log(\phi) \) for \( n = 300 \). The initial profile function is \( f_{0,1} \) (red continuous line), \( f_{0,2} \) (blue dotted line), \( f_{0,3} \) (black dash-dotted line). The dot is the \( \sigma \) calculated by the harmonic mean as in conjecture 12.
We study now the effect of the parameter $\mu$ on the convergence of $\alpha_m$ and $A_m$. We visualize in Figure 8 and 9 the convergence of respectively $\alpha_m$, $A_m$ for $\lambda = 0, \mu = 0, 0.1, 0.3$ and $g(x) = \cos(x)$. Also in these simulations the convergence of $\alpha_m$ (to 0.5) and $A_m$ was obtained.

Figure 8: Plot of $\alpha_m$ for $m = 0, 1, \ldots, 300$ with initial profile $f_{0,1}$ for $\mu = 0$ (red continuous line), $\mu = 0.1$ (blue dotted line), $\mu = 0.3$ (black dash-dotted line), $\mu = -0.3$ (green dash-dash line).
Figure 9: Plot of $A_m$ for $m = 0, 1, ..., 300$ with initial profile $f_{0.1}$ for $\mu = 0$ (red continuous line), $\mu = 0.1$ (blue dotted line), $\mu = 0.3$ (black dash-dotted line), $\mu = -0.3$ (green dash-dash line).
Also for this simulation, we check the correctness of the scaling in the conjecture by the harmonic mean. Again, the straight lines despite the convergence of the profile functions to a scaled Gaussian. The fact that the dots in the Figure 10 lie almost on the lines indicates the correctness of the scaling in the conjecture by the harmonic mean.

Figure 10: Plot of $-\log (f_n)$ vs $-\log (\phi)$ for $n = 300$. The initial profile function is $f_{0,1}$ and we have chosen for $\mu$: $\mu = 0$ (red continuous line), $\mu = 0.1$ (blue dotted line), $\mu = 0.3$ (black dash-dotted line), $\mu = -0.3$ (green dash-dash line). The dots are the $\sigma$’s calculated by the harmonic mean as in conjecture 12.
In Figure 11 and Figure 12 we look for the effect of the periodic function \( g(x) \) on the limiting value of \( \alpha_m \) and \( A_m \). We have chosen the following parameters: \( \lambda = 0, g(x) = \cos(3x) + \sin(2x) \) and initial profile function \( f_{0,1} \), \( \mu = 0.1 \), initial profile function \( f_{0,1} \), \( \mu = 0.3 \), initial profile function \( f_{0,2} \), \( \mu = 0.3 \). Also in these simulations the convergence of \( \alpha_m \) (to 0.5) and \( A_m \) was obtained.

Figure 11: Plot of \( \alpha_m \) for \( m = 0, 1, ..., 300 \) for \( \lambda = 0, g(x) = \cos(3x) + \sin(2x) \) and initial profile function \( f_{0,1} \), \( \mu = 0.1 \) (red continuous line), initial profile function \( f_{0,1} \), \( \mu = 0.3 \) (blue dotted line), initial profile function \( f_{0,2} \), \( \mu = 0.3 \) (black dash-dotted line).
Figure 12: Plot of $A_m$ for $m = 0, 1, ..., 300$ for $\lambda = 0, g(x) = \cos(3x) + \sin(2x)$ and initial profile function $f_{0,1}, \mu = 0.1$ (red continuous line), initial profile function $f_{0,1}, \mu = 0.3$ (blue dotted line), initial profile function $f_{0,2}, \mu = 0.3$ (black dash-dotted line).
Finally, we check the effect of the perturbation $F$ on the limiting value of $\alpha_m$ and $A_m$. We show in the Figure (13) and Figure (14) these sequences. We have chosen the following parameters: initial profile function $f_{0,2}, \lambda = 0.3, \mu = 0.1, g(x) = \cos (3x) + \sin (2x), a = 6, b = c = 0$ and initial profile function $f_{0,3}, \lambda = 0.1, \mu = 0.3, g(x) = \cos (x), a = 2, b = c = 1$ and initial profile function $f_{0,1}, \lambda = 0.1, \mu = -0.1, g(x) = \cos (x), a = 0, b = 2, c = 3$. Note that in these three simulations the parameters $a, b, c$ satisfy equation (11). Also, in these simulations, the convergence of $\alpha_m$ (to 0.5) and $A_m$ was obtained.

![Figure 13: Plot of $\alpha_m$ for $m = 0, 1, \ldots, 300$ for initial profile function $f_{0,2}, \lambda = 0.3, \mu = 0.1, g(x) = \cos (3x) + \sin (2x), a = 6, b = c = 0$ (red continuous line) and initial profile function $f_{0,3}, \lambda = 0.1, \mu = 0.3, g(x) = \cos (x), a = 2, b = c = 1$ (blue dotted line) and initial profile function $f_{0,1}, \lambda = 0.1, \mu = -0.1, g(x) = \cos (x), a = 0, b = 2, c = 3$ (black dash-dotted line).]
Figure 14: Plot of $A_m$ for $m = 0, 1, ..., 300$ for initial profile function $f_{0.2}, \lambda = 0.3, \mu = 0.1, g(x) = \cos(3x) + \sin(2x), a = 6, b = c = 0$ (red continuous line) and initial profile function $f_{0.3}, \lambda = 0.1, \mu = 0.3, g(x) = \cos(x), a = 2, b = c = 1$ (blue dotted line) and initial profile function $f_{0.1}, \lambda = 0.1, \mu = -0.1, g(x) = \cos(x), a = 0, b = 2, c = 3$ (black dash-dotted line).
5 Renormalization Group analysis for the porous-media-like equation

In this Section we apply the RG analysis for a particular PM-like equation. Consider the following problem

\[
\begin{aligned}
&u_t = \text{div}(u \nabla u) + \lambda F(u, \partial_x u, \partial_x^2 u), \quad x \in \mathbb{R}^d, d \in \mathbb{N}_+, t > 1, \\
u(x, 1) = f(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\tag{88}
\]

where \( F(u, v, w) \) is an analytic function of the parameters \( u, v, w \) around \( u = v = w = 0 \):

\[
F(u, v, w) = u^a v^b w^c + \sum_{\{i > a, j > b, k > c\}} c_{ijk} u^i v^j w^k,
\tag{89}
\]

where \( a, b, c \in \mathbb{N} \). Note that this PDE without the perturbation function \( F \) is in fact the famous ‘modified’ Porous media equation \( u_t = \Delta (u^2 / 2) \). We will do the RG analysis only for 1 dimensional version of this equation. We neglect the higher order terms of the perturbation function, since these are irrelevant in RG sense. Then we can write the problem now as

\[
\begin{aligned}
&u_t = [\partial_x^2 uu + (\partial_x u)^2] + \lambda u^a (\partial_x u)^b (\partial_x^2 u)^c, \quad x \in \mathbb{R}, t > 1, \\
u(x, 1) = f(x), \quad x \in \mathbb{R}.
\end{aligned}
\tag{90}
\]

It is essential that the PDE in question has a solution that behaves asymptotically as self-similar solution. This means, the solution behaves as (for large \( t \))

\[
u(x, t) \sim t^{-\alpha} \phi(x t^{-\beta}) = t^{-\alpha} \phi(\eta), \quad \text{with} \quad \eta = x t^{-\beta}.
\tag{91}
\]

We check if such a self-similar solution exists for the PDE without the perturbation function. To do so, we first calculate \( u_t, \partial_x u, \partial_x^2 u \):

\[
\begin{aligned}
u_t &= -\alpha t^{-\alpha - 1} \phi(\eta) t^{-\alpha} \phi'(\eta) x t^{-\beta - 1} \beta = t^{-\alpha - 1} (-\alpha \phi(\eta) - \beta \phi'(\eta) \eta),
\partial_x u &= t^{-\alpha - \beta} \phi'(\eta),
\partial_x^2 u &= t^{-\alpha - 2 \beta} \phi''(\eta).
\end{aligned}
\tag{92}
\]

Next, we plug those formulas in (90) and find

\[
t^{-\alpha - 1} [-\alpha \phi(\eta) - \beta \phi'(\eta) \eta] = t^{-2 \alpha - 2 \beta} \left[ (\phi'(\eta))^2 \phi'(\eta) + \phi(\eta) \phi''(\eta) \right].
\tag{93}
\]

We obtain a self-similar solution, if we choose \(-\alpha - 1 = -2\alpha - 2\beta \iff \alpha + 2\beta = 1\). Surprisingly, (90) has an explicit (source) solution. This solution is given by [7, p. 23-24]

\[
\phi(x, t) = t^{-\alpha} F \left( x t^{-\alpha} \right),
\tag{94}
\]
where \( F \) is defined as
\[
F(\xi) := (C - k\xi^2)_+,
\]
with \( \alpha = 1/3, \ k = \alpha/2 \). The parameter \( C \) can determined by [7, p. 23-24] \( M = qC^\gamma \) where \( M \) is the mass of the initial data
\[
M := \int_\mathbb{R} f_0(x)dx,
\]
d depends on \( k \)
\[
q := \int_0^\infty (1 - ky^2)_+ dy
\]
and \( \gamma = \frac{1}{2\alpha} \).

5.1 A conjecture on the large time behavior of the porous-media-like equation

In this subsection we present our conjecture for solutions of (90). This conjecture is inspired by the conjecture of Braga et al. [1].

**Conjecture.** Let \( u(x,t) \) be the solution of (90). If the lowest order exponents \( a, b, c \) of the perturbation function \( F \) satisfy the inequality
\[
a + 2b + 3c - 4 > 0
\]
and if \( f \in L^1(\mathbb{R}), \ f \geq 0 \), then there is a constant \( A \geq 0 \), which depends on \( f \) and \( F \) such that
\[
t^\alpha u(\sqrt{t}x,t) \to A(C - kx^2)_+ \quad \text{as} \quad t \to \infty
\]
with \( \alpha = 1/3, \ k = \alpha/2 \) and \( C \) depends on the choice of \( f \).

5.2 Renormalization Group analysis for the porous-media-like equation

We follow the same procedure as in Section 2.3 and Section 2.4. We define a sequence \( \{u_n\}_{n=0}^\infty \) as
\[
u_n(x,t) := L^{\alpha_n}u_{n-1}(L^{\beta_n}x, L^{\gamma_n}t) \quad n \in \mathbb{N}, n \geq 1,
\]
with \( L \in \mathbb{R}, L > 1 \) a fixed constant. The induction argument gives
\[
u_n(x,t) = L^\alpha + \ldots + \alpha_1 u_0(L^{\beta_\alpha + \ldots + \beta_1}x, L^{\gamma_\alpha + \ldots + \gamma_1}t) \quad \text{for all} \quad n \in \mathbb{N}.
\]
If \( u(x,t) \) is a solution of (90), then \( u_n(x,t) \) is a solution of the renormalized problem (90). We are now interested in the renormalized problem of (90). Let us first define \( \xi_n := L^{a_n+\ldots+\alpha_1}, \omega_n := L^{a_n+\ldots+\beta_1} \) and \( \zeta_n := L^{\gamma_n+\ldots+\gamma_1} \). To find a renormalized problem, we compute the partial derivatives \( \partial_x u_n(x,t), \partial_x^2 u_n(x,t) \). We use the transformation of variables \( y := \omega_n x, \tau := \zeta_n t \), we obtain

\[
\begin{align*}
\partial_x u_n(x,t) &= \partial_x \xi_n u(\omega_n x, \zeta_n t) \\
&= \xi_n \omega_n \partial_y u(y, \tau), \\
\partial_x^2 u_n(x,t) &= \xi_n \omega_n^2 \partial_y^2 u(y, \tau).
\end{align*}
\](102)

The renormalized problem follows now from the computation of \( \partial_t u_n(x,t) \) where we use (102):

\[
\partial_t u_n(x,t) = \partial_t \xi_n u(\omega_n x, \zeta_n t)
\]

\[
= \xi_n \omega_n \partial_y u(y, \tau)
\]

\[
= \xi_n \omega_n \xi_n \omega_n \partial_y u(y, \tau) + \lambda u \partial_y u(y, \tau) \partial_y^2 u(y, \tau) + \xi_n \omega_n \partial_y \partial_y^2 u(y, \tau) + \xi_n \omega_n \partial_y^2 \partial_y^2 u(y, \tau)
\]

\[
\begin{align*}
&= \xi_n \omega_n \partial_y u(y, \tau) \\
&= \xi_n \omega_n \partial_y u(y, \tau) + \lambda u \partial_y u(y, \tau) \partial_y^2 u(y, \tau) + \xi_n \omega_n \partial_y \partial_y^2 u(y, \tau) + \xi_n \omega_n \partial_y^2 \partial_y^2 u(y, \tau)
\end{align*}
\]

So we obtain that \( u_n \) satisfies

\[
\partial_t u_n(x,t) = \rho_n \left( [\partial_x u_n(x,t)]^2 + \partial_x^2 u_n(x,t) u_n(x,t) \right)
\]

\[
+ \lambda_n (\omega_n(x,t))^a (\partial_x u_n(x,t))^b \partial_x^2 u_n(x,t)^c,
\]

with

\[
\rho_n := \xi_n \omega_n^{-2},
\]

\[
\lambda_n := \xi_n^{1-a-b-c} \omega_n \xi_n \omega_n^{1-c} \xi_n \omega_n^{-2}.
\]

If we choose \( \alpha_n \in \{0,1\}, \gamma_n \in \{0,1\}, \) and \( \omega_n \in \{0,1\}, \) to satisfy

\[
\gamma_n - \alpha_n - 2 \beta_n = 0,
\]

then we get \( \rho_n = 1 \), and hence the ’divergence’ part of (90) is invariant. We choose \( \alpha_n := \beta_n := 1/3 \) and \( \gamma_n := 1 \) for \( n \in \mathbb{N} \), which satisfies (106). Note that this choice also ensures the asymptotic self-similarity behavior of the solution. Furthermore, this choice of \( \alpha \) fits to the value of \( \alpha \) in (94). By this choice, (104) becomes

\[
\partial_t u_n(x,t) = \left( [\partial_x u_n(x,t)]^2 + \partial_x^2 u_n(x,t) u_n(x,t) \right)
\]

\[
+ L^{n/3} \left( \omega_n(x,t) \right)^a (\partial_x u_n(x,t))^b \partial_x^2 u_n(x,t)^c.
\]

Since we required \( a, b, c \) to satisfy (98), we expect that the effect of the perturbation function to be negligible for large \( n \). We proceed now by defining the RG-transformation for this problem. To do so, we first define the initial data sequence \( \{f_n\} \) as follows

\[
f_n(x) := L^{n/3} u(L^{n/3} x, L^n) \quad \forall n \in \mathbb{N}.
\]

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We define for problem (90) the Renormalization transformation as

\[ \mathcal{R}_{L,n} : \mathcal{L}^1(\mathbb{R}) \to \mathcal{L}^1(\mathbb{R}) \]
\[ \mathcal{R}_{L,n}(f_n(x)) := L^{1/3} u_n(L^{1/3} x, L) \quad \text{for all } n \in \mathbb{N}. \] \hfill (109)

Similarly as for the RG transformation of the diffusion equation, see (30), we can also write

\[ \mathcal{R}_{L,n}(f_n(x)) = \mathcal{R}_{L,n}(L^{n/3} u(L^{n/3} x, L^n)) = L^{(n+1)/3} u(L^{(n+1)/3} x, L^{n+1}) = f_{n+1}(x). \] \hfill (110)

Thus note that we have constructed \( \mathcal{R}_L \) such that the following holds.

\[ \mathcal{R}_L(f_n(x)) = f_{n+1}(x). \] \hfill (111)

Since \( L > 1 \) is a fixed constant, we can 'renormalize' it by \( L^n \). Using the induction argument, we obtain the semigroup property for \( \mathcal{R}_L \):

\[ \mathcal{R}_{L^n,0} = \mathcal{R}_L^n, \] \hfill (112)

where \( \mathcal{R}_L^n \) is defined as \( \mathcal{R}_L^n := \mathcal{R}_{L,n-1} \circ \ldots \circ \mathcal{R}_{L,0} \). This property allows us to use the numerical procedure of Section 6. We have to find fixed points of this Renormalization transformation \( \mathcal{R}_L \), since those are the long time asymptotics of solutions to (90). The explicit solution of equation (94) at time \( t = 1 \) is a fixed point of \( \mathcal{R}_L \):

\[ \mathcal{R}_L (\phi(x, 1)) = L^{1/3} \phi(L^{1/3} x, L) = L^{1/3} L^{-1/3} F(L^{1/3} x L^{-1/3}) = F(x) = \phi(x, 1). \] \hfill (113)

We use Fourier Transformations, as defined in (37), to obtain a relation between \( \mathcal{R}_{L,0} f_0 \) and \( f_0 \). We start off by the definition of \( \mathcal{R}_{L,0} f_0 \) and we get

\[ \mathcal{F} [\mathcal{R}_{L,0} f_0] (\omega) = \mathcal{F} \left[ L^{1/3} u \left( L^{1/3} x, L \right) \right] (\omega). \] \hfill (114)

The scaling property of the Fourier transform gives us

\[ \mathcal{F} [\mathcal{R}_{L,0} f_0] (\omega) = \mathcal{F} \left[ L^{1/3} u \left( x, L \right) \right] (\omega L^{-1/3}) \]
\[ = \mathcal{F} \left[ u \left( x, L \right) \right] (\omega L^{-1/3}). \] \hfill (115)

Since we can write the solution of the PM equation without perturbation as a convolution of the fundamental solution and the initial data [7, p. 23-24], we arrive at

\[ \mathcal{F} [\mathcal{R}_{L,0} f_0] (\omega) = \mathcal{F} [\phi(x, L - 1) * f_0(x)] (\omega L^{-1/3}). \] \hfill (116)

Next, we use the property of the Fourier transform which states that the Fourier transform of a convolution is the product of the Fourier transforms. This yields

\[ \mathcal{F} [\mathcal{R}_{L,0} f_0] (\omega) = \mathcal{F} [\phi(x, L - 1)] (\omega L^{-1/3}) \mathcal{F} [f_0(x)] (\omega L^{-1/3}). \] \hfill (117)

It is difficult to derive the Fourier transform of the fundamental solution. We use the computer algebra program Mathematica 8.0 to find this Fourier transform. We obtain

\[ \mathcal{F} [\phi(x, L - 1)] (\omega L^{-1/3}) = \frac{2L}{3(L - 1)^{2/3}} \left[ -\sqrt{6C} (\frac{L-1}{L})^{1/3} \omega \cos \left( \sqrt{6C} \left( \frac{L-1}{L} \right)^{1/3} \omega \right) \right. \]
\[ + \sin \left( \sqrt{6C} \left( \frac{L-1}{L} \right)^{1/3} \omega \right) \]. \hfill (118)
We set $T_\omega$ to be $T_\omega := \left(\frac{L-1}{L}\right)^{1/3} \omega \sqrt{6C}$. We arrive at

$$\mathcal{F}[\mathcal{R}_{L,0} f_0](\omega) = \frac{2T_\omega^3}{3(6C)^{3/2}} [-T_\omega \cos(T_\omega) + \sin(T_\omega)] \cdot \mathcal{F}[f_0](\omega L^{-1/3}).$$ (119)

Note that we have not discussed the basin of attraction of the fixed point (94) of Renormalization transformation. It requires further investigation to determine this basin of attraction. Equation (119) can be used in this investigation. Also, we have not proved the uniqueness of the fixed point (94) and we will not prove the convergence of numerical procedure as presented in Section 6. This also requires more investigation.
6 Numerical procedure for the porous-media-like equation (90)

In this Section we present the numerical procedure for finding fixed points of the Renormalization Transformation of solutions of equation (90). The Numerical procedure we have in mind here is almost the same as the procedure of Braga et al. we discussed in Section 3. So, the procedure constructs the sequence \( f_n \) as defined in equation (108). First, we let \( f_0 = f \), where \( f \) is the initial condition at \( t = 1 \). For \( m = 0, \ldots, n \) we have the following three steps:

1. Evolve \( f_m \) forward from \( t = 1 \) to \( t = L \) using a discretization method for the PDE (104), this yields \( u_m(x, L) \).

2. Set \( \beta_{m+1} = 1/3 \) and solve \( \alpha_{m+1} \) from the equation
\[
L^{\alpha_{m+1}} u_m(0,1) = u_m(0, L).
\]

3. Set \( f_{m+1} = L^{\alpha_{m+1}} u_m(L^{\beta_{m+1}} x, L) \).

Let us define the sequences \( \{A_n\}_{n=0}^\infty \) and \( \{B_n\}_{n=0}^\infty \) as in Section 3 via
\[
A_n := L^{n\alpha_n-(\alpha_n+\alpha_1)} \quad \text{and} \quad B_n := L^{n\beta_n-(\beta_n+\beta_1)}.
\]

The solution of (90) at time \( L^n \) can be found by a scaling of \( f_n \):
\[
u(x, L^n) = A_n L^{-n\alpha_n} f_n(B_n L^{-n\beta_n} x),
\]

Again, we expect
\[
L^{n\alpha_n} u(L^{n\beta_n} x, L^n) = \frac{A_n}{L^{(n\alpha_n-\alpha)} f_n \left( \frac{B_n x}{L^{n(\beta_n-\beta)}} \right)} \rightarrow A \phi(Bx),
\]

as \( n \rightarrow \infty \) if \( A_n \rightarrow A, B_n \rightarrow B, \alpha_n \rightarrow \alpha, \beta_n \rightarrow \beta \) and \( f_n \rightarrow \phi \) (as \( n \rightarrow \infty \)). In Section 7 we check those limiting procedures. The numerical implementation of the numerical procedure is the same as the implementation of the numerical procedure for the diffusion equation. Next, we provide some details of the steps of the Numerical procedure.

6.1 Numerical procedure for porous-media-like equation (step 1)

In step 1 we use again the Euler finite difference method. So we define \( \Delta x \) to be the mesh size and \( \Delta t \) to be the time step. With \( u_{j}^i := u(j \Delta x, i \Delta t) \) the approximation of (90) results in
\[
u_{j}^{i+1} = u_{j}^{i} + \Delta t \left[ u_{j}^{i} \frac{u_{j}^{i-1}+2u_{j}^{i}+u_{j}^{i+1}}{(\Delta x)^2} + \left( \frac{u_{j}^{i}-u_{j}^{i-1}}{\Delta x} \right)^2 \right] + \Delta t \lambda_n \left[ \frac{u_{j}^{i}-u_{j}^{i-1}}{\Delta x} \right]^a \left[ \frac{u_{j}^{i-1}+2u_{j}^{i}+u_{j}^{i+1}}{(\Delta x)^2} \right]^b \left[ \frac{u_{j}^{i}-u_{j}^{i-1}}{\Delta x} \right]^c
\]

\[\text{as } n \rightarrow \infty \] if \( A_n \rightarrow A, B_n \rightarrow B, \alpha_n \rightarrow \alpha, \beta_n \rightarrow \beta \) and \( f_n \rightarrow \phi \) (as \( n \rightarrow \infty \)). In Section 7 we check those limiting procedures. The numerical implementation of the numerical procedure is the same as the implementation of the numerical procedure for the diffusion equation. Next, we provide some details of the steps of the Numerical procedure.
6.2 Numerical procedure for the porous-media-like equation (step 2)

The choice of $\beta^{1/3}$ is explained in Section 5.2, while the choice of $\alpha$ comes from the self-similar behavior of the solution, for the necessary details see Section 3.2.

6.3 Numerical procedure for the porous-media-like equation (step 3)

We use the same rescaling trick as presented in Section 3.3.
7 Numerical results for the porous-media-like equation (90)

In this Section we use the numerical procedure of Section 6 to indicate the correctness of our conjecture as presented in Section 5.1. We expect the Conjecture to hold if the sequences \( \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{A_n\}_{n=0}^{\infty}, \{B_n\}_{n=0}^{\infty} \) and \( \{f_n\}_{n=0}^{\infty} \) converge. For the sequences \( \{\beta_n\}_{n=0}^{\infty} \) and \( \{B_n\}_{n=0}^{\infty} \) we have already obtained convergence since we choose \( \beta_n = 1/3 \) for all \( n \in \mathbb{N} \) in step 2 of the numerical procedure.

We use the same situation as in Section 4. So for the initial profile functions we have 27 grid points on the interval \([-5, 5]\) and we choose again the scaling factor \( L \) to be 1.021. We set the initial time step \( \Delta t \) to be 0.007. For the numerical simulation for will use the three initial profile functions \( f_{0,1}, f_{0,2}, f_{0,3} \) as defined in Section 4. To indicate the correctness of the conjecture by the numerical procedure, we first check it for the ‘divergence’ part of the equation (i.e. \( \lambda = 0 \)). Then, we add the perturbation function to the PDE. In all simulations, we obtain the convergence of the profile functions \( f_n \) to (a multiple of) the fundamental solution \( \phi(x, 1) \) as defined in (94). In Figure 15 and Figure 16 we show the profile function \( f_n \) for \( n = 450 \).

![Figure 15: Plot of profile function \( f_n \) for \( n = 450 \) for initial profile functions \( f_{0,1} \) and \( f_{0,2} \).]
7.1 Numerical results (without perturbation)

In this Subsection we verify the correctness of the Conjecture for $\lambda = 0$. For each of the initial profile functions we iterate the numerical procedure for $m = 0, 1, ..., 500$. In Figure 17 we indicate $\alpha_m$ ($m = 0, 1, ..., 500$). Indeed, we see that $\alpha_m$ converges to a limiting value of $1/3$. Figure 18 shows the convergence of the factor $A_m$. Note that the limiting value of $A_m$ is not universal and depends on the initial profile function.
Figure 17: Plot of $\alpha_m$ for $m = 0, 1, \ldots, 500$ for initial profile functions $f_{0,1}$ (red continuous line), $f_{0,2}$ (blue dotted line), $f_{0,3}$ (black dash-dotted line).

Figure 18: Plot of $A_m$ for $m = 0, 1, \ldots, 500$ for initial profile functions $f_{0,1}$ (red continuous line), $f_{0,2}$ (blue dotted line), $f_{0,3}$ (black dash-dotted line).
7.2 Numerical results (with perturbation)

We study the effect of the perturbation $F$ on the limiting value of $\alpha_m$ and $A_m$. In the Figure 19 and Figure 20 we show these sequences. We have chosen the following parameters: initial profile function $f_{0,2}$, parameters $\lambda = 0.3, a = 4, b = 2, c = 0$ and initial profile function $f_{0,1}$, parameters $\lambda = -0.3, a = b = c = 1$ and initial profile function $f_{0,3}$, parameters $\lambda = 0.5, a = 1, b = 2, c = 5$. Notice that in these three simulations the parameters $a, b, c$ satisfy (98). Also in these simulations the convergence of $\alpha_m$ (to $1/3$) and $A_m$ was obtained.

Figure 19: Plot of $\alpha_m$ for $m = 0, 1, ..., 500$ for $\lambda = 0.3, a = 4, b = 2, c = 0$ (red continuous line) and initial profile function $f_{0,1}$, parameters $\lambda = -0.3, a = b = c = 1$ (blue dotted line) and initial profile function $f_{0,3}$, parameters $\lambda = 0.5, a = 1, b = 2, c = 5$ (black dash-dotted line).
Figure 20: Plot of $A_m$ for $m = 0, 1, \ldots, 500$ for $\lambda = 0.3, a = 4, b = 2, c = 0$ (red continuous line) and initial profile function $f_{0,1}$, parameters $\lambda = -0.3, a = b = c = 1$ (blue dotted line) and initial profile function $f_{0,3}$, parameters $\lambda = 0.5, a = 1, b = 2, c = 5$ (black dash-dotted line).
8 Conclusion

In this bachelor thesis we used the Renormalization Group analysis technique to understand the large time behavior of partial differential equations. To understand this technique, we studied the paper of Braga and collaborators [1]. We have implemented their numerical scheme to find fixed points of the Renormalization Transformation. With this numerical scheme, we have recovered their results for a non-linear diffusion equation with a perturbation factor. These numerical results indicate the correctness of their conjecture about the asymptotic behavior of solutions of the diffusion equation. We have applied the Renormalization Group Analysis technique to a modified Porous Media equation. The modified Porous Media equation also includes a perturbation function. A small change in their numerical scheme made it suitable for our partial differential equation. With the numerical results of numerical scheme applied to the Porous Media, we formulated a conjecture. This conjecture is similar to the conjecture of Braga et al. [1] and is about the asymptotic behavior of solutions of the modified Porous Media equation. Remarkable is that most aspects of the limiting behavior of solutions of the modified Porous Media equation are universal. The scaling exponents and most aspects of the limiting profile function are independent of the initial conditions. It turns out that a certain class of perturbation functions does also not affect the limiting profile function.

The Renormalization Group Analysis for the modified Porous Media equation requires more investigation. For the non-linear diffusion equation the convergence of the numerical scheme is shown by a Banach fixed-point argument based on Fourier Transformations, see [2] and [3]. We expect that a similar approach can be used to show the convergence of the numerical scheme for the Porous Media equation. Also, we have not discussed the basin of attraction and the uniqueness of the fixed point of the Renormalization transformation. Also this requires further investigation. Additionally, the use of boundary conditions in the numerical scheme is not very clear and requires some extra research.
9 References


10 Appendix: MATLAB code for the Numerical Procedures of Section 3 and Section 6

In this Section we list the MATLAB code we have used to implement the numerical procedure of Section 3 and numerical procedure of Section 6. Subsection 10.1 and Subsection 10.2 show the MATLAB code for solving (5) and (90) respectively.

10.1 MATLAB code for the diffusion equation

function diffusion(maxiter, interval)

    close all;
    % time step and spatial step size
    dt = 0.0007;
    dx = 2*interval/26;

    % scaling factor L
    L = 1.021;
    % L^beta
    Lh = L^(0.5);

    % mesh points
    xvals = -interval:dx:interval;
    % number of mesh points
    J=length(xvals)

    % initialization of array of alpha’s
    alpha=zeros(100,1);

    % initial profile functions:
    %u0 = @(x) exp((-x.^2)/4);
    u0 = @(x) 1-abs(x./interval);
    %u0 = @(x) (1-sin((pi*x)./5))/2;
    %periodic functions g
    %g = @(x) cos(x);
    g= @(x) cos(3*x)+sin(2*x);

    %parameters mu, lambda from conjecture of Braga
    muu=0.1;

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lambda = 0.3;

% parameters a, b, c from conjecture of Braga
a = 6;
b = 0;
c = 0;

% initial profile function to array
v(:,1) = u0(xvals);

% figure of initial profile
figure(1)
plot(xvals,v(:,1));
grid on
hold on;

for p = 1:maxiter

    % show the iteration number
    p

    % new data points of solution, mesh size has changed
    if(p>1)
        J = 2*q+3;
        for i = 1:J
            v(i,1) = 0;
        end
        for i = (J-1)/2+1:(J+1)/2
            v(i,1) = u(i-(J-1)/2);
        end
    end

    % new data points of periodic function, mesh size has changed
    d = g(xvals*L^(p/2));
    if p>1
        lambdan = lambda*L^((p-1)-(b+2*c)*(p-1)/2 + (1-a-b-c)*sum(alpha(1:p-1)));
    else
        lambdan = lambda;
    end

    % determine number of time steps
    N = round((L-1)/dt);

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% variable to evolve PDE
mu = dt/dx^2;

% evolve PDE from t=1 to t=L
for n = 1:N
    v(2:J-1,n+1) = mu*(1+muu*d(2:J-1))*v(3:J,n)
    + (1-2*mu*(1+muu*d(2:J-1)))*v(2:J-1,n)+mu*(1+muu*d(2:J-1))
    *v(1:J-2,n)+dt*lambdan*((v(2:J-1,n))^(a))
    *(((v(2:J-1,n)-v(1:J-2,n))/dx)^(b))
    *(((v(1:J-2,n)-2*v(2:J-1,n)+v(3:J,n))/(dx^2))^(c));
    v(1,n+1) = 0;
    v(J,n+1) = 0;
end
u=v(:,N+1);

% step 2 of the numerical procedure
f=v((J+1)/2,1)/u((J+1)/2);
alpha(p)=log(f)/log(L);

% show parameter alpha
alpha(p)

% step 3 of the numerical procedure
u=f*u;

% rescaling mesh size and time step
dx=dx/Lh;
dt=dt/L;

% plot profile function once for each 10 iterations
if mod(p,10)==0
    figure(1)
    plot(xvals,u)
    grid on
    hold on;
end

% parameter to determine how many points should
% be added to the interval
q=floor(interval/dx);

% current number of mesh points

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l=length(xvals);

% construction new mesh points
xvals=-q*dx:dx:q*dx;
end

xvals=union(union(xvals,interval),-interval);
10.2 MATLAB code for the porous-media-like equation

```matlab
function porousmedia(maxiter, interval)
    close all;
    % time step and spatial step size
dt = 0.007;
dx = 2*interval/26;

    % scaling factor L
L = 1.021;
% L^beta
Lh = L^(1/3);

    % mesh points
xvals = -interval:dx:interval;
% number of mesh points
J=length(xvals);

    % initialization of array of alpha's
alpha=zeros(100,1);

    % initial profile functions:
u0 = @(x) exp((-x.^2)/4);
%u0 = @(x) 1-abs(x./interval);
%u0 = @(x) (1-sin((pi*x)./5))/2;

    %parameters mu, lambda from conjecture
lambda=-0.3;

    %parameters a, b, c from conjecture
a=6;
b=0;
c=0;

    % initial profile function to array
v(:,1) = u0(xvals);

    % figure of initial profile
figure(1)
plot(xvals,v(:,1));
goto
hold on;
```

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for p=1:maxiter

    % show the iteration number
    p

    % new data points of solution, mesh size has changed
    if(p>1)
        J=2*q+3;
        for i=1:J
            v(i,1)=0;
        end
        for i=(J-l)/2+1:(J+l)/2
            v(i,1)=u(i-(J-l)/2);
        end
    end

    if p>1
        lambdan=lambda*L^((p-1)-(b+2*c)*(p-1)/3+(1-a-b-c)*sum(alpha(1:p-1)));
    else
        lambdan=lambda;
    end

    % determine number of time steps
    N=round((L-1)/dt);

    % evolve PDE from t=1 to t=L
    for n = 1:N
        v(2:J-1,n+1) = v(2:J-1,n)+
                        dt.*(((v(3:J,n)-v(1:J-2,n))/(2.*dx)).^2
                          +v(2:J-1,n).*((v(1:J-2,n)-2.*v(2:J-1,n)
                          +v(3:J,n))/(dx.^2)))+dt.*lambdan
                          .*((v(2:J-1,n)).^a).*(((v(2:J-1,n)
                          -v(1:J-2,n))/dx).^b).*(((v(1:J-2,n)-2
                          .*v(1:J-2,n)+v(3:J,n))/(dx^2)).^c);
        v(1,n+1) = 0;
        v(J,n+1) = 0;
    end
    u=v(:,N+1);

    % step 2 of the numerical procedure
    f=v(((J+1)/2,1)/u(((J+1)/2);
alpha(p)=log(f)/log(L);

% show parameter alpha
alpha(p)

% step 3 of the numerical procedure
u=f*u;

% rescaling mesh size and time step
dx=dx/Lh;
dt=dt/(L^(2/3));

% plot profile function once for each 10 iterations
if mod(p,10)==0
    figure(1)
    plot(xvals,u)
    grid on
    hold on;
end

% parameter to determine how many points should be added to the interval
q=floor(interval/dx);

% current number of mesh points
l=length(xvals);

% construction new mesh points
xvals=-q*dx:dx:q*dx;
xvals=union(union(xvals,interval),-interval);
end