Playing billiards: a wave-particle duality of surface waves

Niels van Hoof

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Abstract

Capillary surface waves on a vertically shaken container can support small droplets that appear to float on the surface. Through a symmetry-breaking bifurcation, these droplets turn into walkers: particles coexisting with their own wave packets, which exhibit the phenomena of quantum mechanics, such as diffraction of particles moving through slits.

The question is if these particles make a regular or chaotic quantum billiard. It is a question because collisions with the sides of such a billiard are mediated by waves.

In this internship we built an experiment, made walking droplets, and tracked them using video while they bounced around in a billiard. From the collected orbits the angles of the wall interactions are deduced. These angles differ from the classical ones, so that regular billiards may support chaotic orbits.
Introduction

The goal of this internship was to determine a boundary collision law for droplets that bounce on silicon oil. These droplets are also known as ‘walkers’ due to their property of moving in both the vertical and horizontal plane. These ‘walkers’ are propelled by surface effects which will be explained in this report.

To investigate the boundary interactions of these ‘walkers’ a square container is constructed. This square billiard ensures a large set of collisions with the boundary to analyse. The added value of using a square container is further discussed in this report.

The interactions are examined to check if this system exhibits a regularity in the angels of incidence and reflection. This kind of behaviour would point to a classical system and the regular square billiard would generate regular orbits. If there are more factors playing a role in these interactions the collisions could turn out to be non-classical. This experiment is designed to answer that very question.

Judging from literature these collisions are influenced by intrinsic properties as the viscosity and surface tension, but the liquid height in the container and the driving frequency of the exciter could also play a role. A second goal of this internship is to identify the main parameters which influence the behaviour of a ‘walker’ near a boundary.
Chapter 1: Billiards

In this experiment we will create so called 'walkers'. A 'walker' is a droplet that bounces on top of a vibrating substrate. A detailed description of these 'walkers' can be found further into this report. What's important at this point is that these droplets behave somewhat similar to billiard-balls. They have interactions with each other, and interaction with the boundaries. The boundary interaction will be the main topic throughout this whole report.

Classical

The goal is to let these previously described ‘Walkers’ inhibit a square billiard. Classically the end result would look rather dull. The angles of incidence are equal to angles of refraction and all the future boundary hitting points are known and show regularity, this is shown in figure 1(left).

If this billiard gets distorted a little bit the regularity will disappear and instead a chaotic spread of angles is observed. However the classical orbits still satisfy reflecting boundary conditions when colliding with a boundary. In this sense, playing billiards with ‘walkers’ may result in surprises, as ‘walkers’ have more complex boundary interactions 1(right).

Quantum

These rules of regularity and chaos explained in the previous part also hold in the quantum world. If a state (instead of a particle) occupies some kind of billiard. A regular (classical) billiard has a regular nodal pattern of the wave function which it supports. A chaotic classical billiard has a very irregular nodal pattern (and a continuous spectrum of eigenenergies). However, this nodal pattern still shows the traces of (unstable) periodic orbits of its classical counterpart, as Figure (2) illustrates.
Figure 1: (left) A regular square billiard, with one particle inside and starting position of $x=[0 \ 0]$ and an angle of 45 degrees. (right) A distorted square billiard, with one particle inside and starting position of $x=[0 \ 0]$ and an angle of 45 degrees.

Figure 2: Chaotic scars that are visible when a quantum state inhibits a stadium billiard, and its similarities with a classical billiard.
Chapter 2: Capillary waves

Capillary waves are a phenomenon that occur on the surface of a fluid, with dynamical behaviour dominated by surface tension. These waves can be found everywhere in our daily lives and are better known as ripples. For example single drops falling in a filled sink create these well known ripples.

These waves have been studied quite extensively and their inviscid dispersion relation is given by:

\[ \omega^2 = \tanh(kh) \left( gk + \frac{\sigma}{\rho} k^3 \right) \]  \hspace{1cm} (1)

With the wave number \( k \) the depth of the fluid layer \( h \), \( g \) the acceleration of gravity, \( g \), \( \sigma \) the surface tension and \( \rho \) the density of the fluid.

To better understand equation (1), we are going to cut it up in its different parameters and their implications on this experiment.

Depth

If we go to infinite depth the nature of equation (1), becomes surprisingly simple. In this case \( \tanh(k_0 h) \) can be approximated by one. In which case we can divide by \( \omega_0^2 \) to obtain:

\[ G + \Sigma = 1 \]  \hspace{1cm} (2)

With \( G = gk_0/\omega_0^2 \) as the contribution from gravity and \( \Sigma = \sigma k_0^3/\rho \omega_0^2 \) is the contribution from capillary forces. This makes it clear that the behaviour of the surface of a fluid is in beautiful balance between capillary and gravitational influences.

Furthermore, this simplification can be used to do a rough estimation of the forces causing the behaviour of the fluid. In our experiments, the fraction \( \Sigma/G = 4 \) shows that the capillary behaviour is far more important then the gravitational influence.

Depth Dependence of Faraday waves

Because in general the experiments are not done at an infinite depth, both factors \( G \) and \( \Sigma \) play a role in our experiment. The onset amplitude of Faraday waves is one of the most important depth dependent parameters in this experiment. The critical acceleration is shifted if the depth
is altered. This phenomenon is use to create the boundaries of the square billiard. Due to the
shifted critical driving amplitude ‘walkers’ can not exist in the boundary area. These ‘walkers’ are
reflected back before even reaching this area because the path of the ‘walker’ is mediated by the
waves surrounding it.

Other cases can be identified with their specific dispersion relations due to the specific assumptions
made in the process. But this would go beyond the scope of this report.
Chapter 3: Faraday Instability

In the 19th century Michael Faraday published a paper on the forms and states assumed by fluids on a vibrating plate. He noticed waves forming on the surface of a plate, when the plate was driven vertically with a specific amplitude and frequency. He also noticed that the wavelength of these waves was directly correlated to the driving frequency of the plate. More specifically he noticed that the frequency of the waves on the surface was exactly half the driving frequency.

The waves observed by Faraday are standing waves, which will be created if a fluid is driven at a frequency and amplitude dependent critical value (material specific). This is better known as the Faraday instability. The amplitude and eigenvalues of the fluid layer can be described (when linearised) by the Mathieu Equation. This was shown in 1954 by Benjamin and Ursell [2]. And is shown in equation (3).

\[ \ddot{n}_k + (\omega_k^2 + ak \sin(\Omega t)) n_k = 0 \]  (3)

Where \( n_k \) is the amplitude of a wave with wave-number \( k \), \( \Omega \) the driving frequency and \( a \) the driving amplitude.

Equation(3) is the basis for understanding the phenomenon of the Faraday waves. The implications of this equation under different assumptions can be found in [1].

The Mathieu equation has two solutions which can be split in: harmonic and sub-harmonic. The figure (4) shows the (sub)-harmonic tongues with frequencies of 40, 80 and 120 Hz in a neutral stability chart.
Figure 4: The neutral stability chart for the onset of Faraday waves [50 Hertz]. Below (above) the resonance tongues the flat surface is stable (unstable). Tongues labeled by odd (even) n correspond to a sub-harmonic (harmonic) time-dependence of the surface waves. Open symbols: for an inviscid fluid ($\nu = 0$) resulting from the undamped Mathieu oscillator. Solid symbols: for a viscous fluid ($\nu = 10\text{mm}^2/\text{s}$) resulting in a damped Mathieu oscillator. [4]

We are only interested in the first (Sub-harmonic) state. Therefore the critical acceleration $a_c$ and critical wave-number $k_c$ are of the most importance. At this critical point, parametrically forced waves of frequency $f_0/2$ form and cover the whole surface. Recent studies [5][6] have shown that just below this critical point an interesting phenomenon occurs.

Droplets created on the surface will emit local Faraday waves. As a result a bifurcation occurs and the droplet becomes 'self-propelled'(begins to walk). The droplet will travel with a constant horizontal velocity $V_W$ ranging from 1 up to 18 mm/s. The horizontal velocity is caused by the droplet (systematically) falling on the forward front of the wave that was emitted during the previous bounce. For such a cycle to complete itself on every bounce is mitigated though a delicate balance between the surface waves and the shaking action of vibrating plate.
Chapter 4: Walkers

To understand more about the behaviour of a ‘walker’, one must understand the principles and history behind its creation. A chronological chain of events will explain the phenomenon of a ‘Walker’ from its discovery up to the present knowledge about them.

Discovery

One of the first scientific observations done on this topic was a simple but fundamental one, by Reynolds in 1886[9]. He observed that if a droplet was to collide with a surface, it’s actual coalescence with the surface (of the same substance) would be delayed a fraction of a second. He assumed this had something to do with the layer of air between the droplet and the surface. This cushion of air would slow down the movement of the droplet and eventually would break to let the droplet through and coalescence with the surface. He was completely right in this claim, but nothing really happened with his findings until roughly 100 years later.

In 1978 J. Walker discovered that droplets could in fact made to float on a vibrating plate. He used a vertical oscillator to accelerate a soapy solution and observed small droplets bouncing on it’s surface indefinitely [8]. He named this discovery after himself, but never understood why these droplets could exist so long. This would not have been possible without the fundamental insights from Michael Fraday, as explained in Chapter 2.

Several years later Y. Couder initiated a similar series of experiments using silicon oil instead of the soap solution J. Walker used. He discovered a range of interesting properties. One of them is the reason behind the prolonged existence of these droplets, a complete bifurcation.

Phase diagram

Figure (5) shows a phase diagram for a specific fluid ($\mu_L = 50$ mPa s) and a constant driving frequency $f_0$ of 80 Hz. Showing the behaviour of the droplets under variation of droplet size $D$ and the relative acceleration $\gamma_m/g$. Below a relative acceleration of $\gamma_m/g = 0.8$ no bouncing is observed and above $\gamma_m/g = 4.5$ Faraday waves form [Chapter 2]. These two accelerations are characteristic for this specific combination of fluid and driving frequency. For different fluids and/or driving frequencies this diagram will show subtle changes. However the walking behaviour is only observed for fluids of $\mu_L = 5 – 100$ mPa s [5]. Furthermore different regions are identified. The regions of most importance in this experiment are the simple bouncing (B), Period-doubled bouncing (PDB) and the Walking regime (W).
Figure 5: Phase diagram of the various behaviours of a drop as a function of its diameter $D$ and of the forcing acceleration $\gamma_m/g$, for a silicon oil with $\mu_L = 50 \text{ mPa s}$ oscillated at frequency $50\text{Hz}$. The behaviours in the various domains are the following. In B there is simple bouncing, in PDB period-doubling, in PDC transition to temporal chaos by a period doubling cascade, in Int the drop has an intermittent behaviour, W is the region of walkers and F the Faraday instability domain.

Bifurcations

As explained in the previous chapter a bouncing droplet on a vibrating fluid sends out Faraday waves. These waves are the reason for a bifurcation to occur. Protire et al. [5] have shown that if the amplitude of these waves becomes large enough, the vertical bouncing of the droplet becomes unstable. The droplet will spontaneously start moving with a well defined horizontal speed $V_W$ [6]. This is a local symmetry breaking bifurcation. If the droplet size has a specific value [7], another (period doubling) bifurcation completes the transition. The droplet will only hit the surface once every two periods, and thus a ‘walker’ has been created. These different transitions are identified in figure (5).

From bouncing to walking

The previous sections describe the underlying principles of the transition of a droplet from simple bouncing to walking. This section will describe the transition from a more phenomenologic point of view. The behaviour of a droplet is determined by its interaction with the surface. More precisely the interaction with the surface wave it has created during its previous bounce(s). When the droplet comes in contact with the surface it will form a capillary wave, described by equation (1. As the vertical acceleration amplitude is slightly smaller than the critical one for homogeneous Faraday waves, this Faraday wave will decay very slowly. These waves have a frequency of $f_0/2$. And will exist long after the droplet has collided with the surface once more. On the next collision a droplet falls on the slow-spreading Faraday wave and could end up on the forward front of its previous created wave, sending it off into a classical parabolic flight with a horizontal speed of $V_W$.

From this point onward the droplet size and precise driving frequency determine its behaviour. If these are in the walking region of figure(5) then the droplet will successively bounce on every last created Faraday wave thus creating a walker. The horizontal speed of the walker $V_W$ is in the
Realizing reflecting boundaries

A billiard needs reflecting boundaries. In our experiment these boundaries are made by areas of reduced depth, as sketched in figure 7. In this boundary area, the critical acceleration is higher then in the rest of the billiard and therefore no walkers can exist in that area. The waves surrounding a 'walker' are reflected back from these shallow areas. These areas together make up the edges of our billiard. In figure (6) one of these boundary interaction is shown. As seen in this figure the 'walker' does not reach the shallow area, but instead turns around some millimetres before the shallow area.

The interaction with the boundaries is non-classical in the sense that there is no sharp angle between inbound and outbound particle-wave packet. As shown in figure (6) a droplet moves in a smooth trajectory from boundary to boundary. At the boundary the path of the 'walker' is gradually curving as a result of the waves around the droplet reaching the boundary first. These waves reflecting this 'information about the boundary' back to the droplet. Thereby bending the trajectory gradually at the following bounce(s).

The boundaries don’t need to rise above the surface. But instead can be realized by constructing a shallow area. This shallow area acts as a boundary due to the fact that the threshold for Faraday waves shifts to a higher driving amplitude when the depth of the fluid decreases. This implies a higher driving amplitude to realize a droplet in a walker state. The characteristic driving amplitude for walking $\gamma_W$ is not met in the boundary area. Therefore a walker is unable to 'exist' in this area and is reflected. This behaviour is illustrated in figure (7).

It turns out that this boundary interaction displays the same probabilistic characteristics as exhibited in the quantum world [10]. For example a droplet close to a shallow barrier; the wave field

Figure 6: The path of a 'walker' near a shallow boundary area. The path of the 'walker' in this particular time frame is from right to left. The angles of incidence are therefore also defined as such. To put it all in perspective this cut out is approximately 6 cm wide. And is constructed from 30 frames of video at a rate of 10 fps.
Figure 7: (a) The measured thresholds for the Faraday instability $\gamma^F_m$ (triangles) and for the walking instability of $\gamma^W_m$ of a drop of diameter $D=0.78$ mm (filled circles) as a function of the depth $h$ of the liquid bath. (b) In cells formed of two regions of depths $h_0 = 4.1$ mm and $h_1 = 1.1$ mm, when the system is tuned to a value $\gamma^W_m(h_0) > \gamma_m > \gamma^W_m(h_1)$ (e.g., $\gamma_m/g = 3.75$), a drop forms a walker in the regions of large depth but remains motionless in the shallow zone. The continuous lines are simple interpolations[10].

of the droplet will be reflected and partly transmitted through the barrier. The transmitted wave is damped exponentially while on top of the shallow barrier and is eventually transmitted at the other side. There is a chance that a walker will cross the boundary. This chance is determined by its speed, angle of incidence and the horizontal and vertical dimensions of the boundary. These parameters have a similar function as energy and potential barrier thickness in the quantum world.
Chapter 5: Experimental set-up

This chapter is dedicated to explaining the capabilities and weaknesses of the set-up that will be used for replicating the experiments done by Yves Couder and Emmanuel Fort [3]. The basis of these experiments is build from the components that Mark-Tiele Westra [1] used in his experiments. For a deep analysis into the high order vibrations in this system I would also like to reference to his work [1].

Characteristics

These experiments were conducted on the surface of a fluid contained in a square cell which was shaken vertically using a vertical exciter. The fluid (silicon oil) used for these experiments differs from the fluid used by Couder & Emmanuel. The Brand name of the oil is Tegiloxan (3 mixed with 10.000) and is produced by Goldschmidt AG (Germany). It has viscosity $\nu = 30 mPas$, density $\rho = 900 kg m^{-3}$ and a surface tension of $\sigma = 0.02 N m^{-1}$. It has been created by mixing Tegiloxan 3 and Tegiloxan 10.000 in volume percentages of respectively 76% and 24%. The choice for using this particular low surface tension fluid is explained in [1] P.57.

The experiments are preformed in a square cell measuring 80 x 80 mm and a depth of $h_0 = 10 mm$ where the boundary is at a right angle with the substrate. This cell is embedded in a square cell (98 x 98 mm) of depth 25 mm where the boundary is slightly curved, as shown in figure (8). The cell has been designed specifically to reduce capillary waves from forming on the edges of the cell.

![Figure 8: A sketch of the vibrating plate as viewed from the side.](image)

A vertical exciter (Vibration Exciter Brel & Kjaer Type 4808) is used to induce a forcing frequency onto the cell and thus also the fluid in the cell. The cell is carefully levelled to ensure that the axis of vibration is exactly perpendicular to the surface of the fluid. The fine-tuning was done by checking that, at threshold, forced Faraday waves would form all over the surface simultaneously.

To obtain droplets to experiment with a small pin is dipped into the fluid and retracted swiftly. The breaking of the fluid bridge results in one (or several) small droplet(s). If a suitable droplet is created (in the size range of the walking region.)[7] this droplet will be kept and the rest will be carefully removed from the cell.
The movements of the droplet (with its wave packet) are captured using an AVT Dolphin (F-145B/F-145C) Camera. The movement of the droplet is our main concern, therefore the camera is focused at the plane of the droplet. The camera captures at 10 frames per second in sync with the 50 Hz from the function generator by placing a divisor module between the function generator and the camera. From a distance of 0.9 m above the surface the camera is mounted to be able to see the whole bath. The whole set-up is covered by blankets to ensure the 50 Hz TL-lighting doesn’t influence the illumination and to minimize any airflow over the surface. The surface is illuminated by 2 optical-fiber lamps placed at 45 deg, to minimize glare.

Figure 9 (left) shows five superimposed raw measurements produced using the described set-up. The glare-points from the lamps can be clearly identified on the sides of the oil bath. These lay outside our region of interest and therefore can be subtracted from the raw data. The picture is further enhanced by subtracting the background from the image, resulting in a dark picture 9 (right) where only the droplets remain as dark spots.

To track the movements of the droplets in these frames a simple particle tracking script was developed. This script identifies the position of the biggest ‘blob’ in every frame, in our case this is the droplet. This script was based on a code that is freely distributed online [11]. The positions are marked as the centers of these blobs and are accurate up to 0.5 mm.

The next step is to identify the angles of refraction and angles of incidence at the boundaries, as defined in figure (6). This is done by in the follow way. First a boundary interaction is detected by identifying a change in signs in the velocity \((\text{position}_n - \text{position}_{n-1})\) of a blob. These interactions have a characteristic time scale of about 0.5 sec. So the average angles of 0.25 sec before and after the event are calculated and labelled as angles of incidence and refraction respectively.
Chapter 6: Results

In this chapter the results of ‘walkers’ initialized inside a square billiard are presented. After their initial creation the walkers will undertake an undisturbed trajectory, resulting in histograms shown in figure(12) and derivative results. All results published in this chapter have been acquired under identical circumstances. With a forcing frequency of ($f_0 = 50\, Hz$), a forcing amplitude of ($\gamma_m = 3.25g$) a density of $\rho = 900kgm^{-3}$. The droplet diameter was almost constant for all the undertaken experiments.

Stabilization

A major obstacle in this mostly histogram based research is stabilizing the system. This process also includes levelling the system and checking if the surface is indeed perpendicular to gravity and the vibrational axis. This alignment is done in different stages. First the (adjustable) surface on which the exciter rests is levelled and thereby the exciter. Through tightening the screws on the edges of the substrate the surface itself is levelled even further, checking with an equal and centred onset of Faraday waves. These is shown in the figures below. The figures 10 and 11 illustrate this difference between proper and bad alignment.

Figure 10: Three images taken to check the horizontal positioning of the vibrational table. Table as shown from above. The vibration exciter is turned on at $t=0$ sec. Evolving Faraday waves are observed in the other two pictures ($t=0.7$ sec and $t=1.4$ sec). These waves should grow from all around the substrate if it would be aligned properly instantaneously.
Another concern was the airflow over the surface. This was damped by fixing blankets around the whole set-up. Nevertheless we have seen the effect of the airflow of the air-conditioner in the experiments, which introduces weird trajectories, inevitably destroying the measurement.

But by switching off the airco the temperature became uncontrollable and thus a slight difference in properties throughout the series of experiments where encountered. This effect was assumed to be neglectable against the airflow and alignment effects.

Even after careful alignment the walker has a preferable direction in our set-up. This makes the histogram study of these droplets very hard. Future experiments should get rid this preferable direction from the system before continuing.

Below two such a histogram images are shown, these images have a different amount of frames making up the histograms and a different fluid height $h_0$, therefore are not comparable, but give a nice impression of the path of a walker.

Figure 11: Three images taken to check the horizontal positioning of the vibrational table. Table as shown from above. The vibrational exciter is turned on at $t=0$ sec. Evolving Faraday waves are observed in the other two pictures ($t=0.7$ sec and $t=1.4$ sec). These waves should grow from all around the substrate if it would be aligned properly instantaneously.

Figure 12: (left) A histogram (9270 frames) of a walker in a square billiard with a liquid height $h_0$ of 10 mm. (right) A histogram (12400 frames) of a walker in a square billiard with a liquid height $h_0$ of 10.6 mm. The enlarged versions (figures 13 & 14) can be found below.
Figure 13: A histogram (9270 frames) of a walker in a square billiard with a liquid height $h_0$ of 10 mm.
Interaction law

The goal of this project is to determine the interaction of a walker with it’s boundary. The aim is to determine some kind of interaction law for these walkers. When these walkers hit a shallow boundary area there could be some relation linking the angle of incidence with the angle of reflection.

To probe for such a relation a simple script was used to distill the angles of incidence and refraction from the trajectories. (A brief description of the script can be found in chapter 5.) If this interaction would turn out to be classical a straight line would be found from plotting the angles of incidence against the angles of refraction. As already predicted by [5] this is not the case. A wild variety of angles is observed and is shown in the graphs below. These correspond to the histograms seen in figure (12).
Figure 15: Angles of Incidence plotted against their corresponding Angles of Refraction. This figure is made from the 456 collisions that occurred in the trajectory shown in figure (12)(left).
As seen in figures (15 and 16) there is little correlation between incidence and refraction. But what can be observed is that (almost) all the data is populating the upper regions of the graph. Thus the angle of refraction is (almost) always bigger than the angle of incidence. This means that the boundary interaction is diverting from ideal in some kind if dissipative way. This makes sense because the waves that are send out to the boundary are partly reflected and partly transmitted into the shallow area. This transmitted part stands for the energy loss at the boundary and could be a measure for how much these angles divert from the normal. But this is not investigated in this report. Furthermore the wide spread of these angles can be explained by the fact that these waves develop a path memory of the previous paths they have followed [12]. Each bounce sends out Faraday waves, these wave continue to exist for a long period in time. During this time they can have an interaction with the newly formed waves, resulting in a slightly altered path for the ‘walker’. At some point there will be enough altered Faraday waves in the surface to render the chaotic. And these ‘walkers’ start moving in weird curly trajectories even in they are in the middle of the substrate. For these reasons it is highly advisable to restrict measurements on such a system to only take up around 1-2 min.
Conclusions and Outlook

This thesis has presented an insight into a ‘walkers’ interaction with a shallow boundary. In the quest of a billiard showing particle-wave duality, we have studied the collision laws.

The results from the boundary interaction have not been sufficient to determine a simple interaction law for these ‘walker’. But a tendency of the ‘walker’ to lose some horizontal kinetic energy during its collision was observed. The work leading up to this final result have explained and highlighted some difficulties that can now be evaded in future experiments.

A quantum-like analogy has been identified and creates a lot food for thought for future experiments. These experiments should be carried out on a larger scale in contrast to this thesis. In which there should go careful thought into the control and stability of the surface. It would be interesting to test to which extent this analogy holds for single-photons. For example a photon-photon collision meeting in semi transparent mirror. In this analogy that could be two ‘walker’ meeting over a shallow area.

All in all it has been a great privilege to do this thesis at the Turbulentie en Vortexdynamica department. It has been a real challenge but under the careful supervision of Willem van de Water we have produced nice thesis. Therefore a word of gratitude to Willem, who supported me with guidance and a lot of positive thinking throughout this entire thesis.
Bibliography


